Chapter 8

Final problem set

8.1 Applications

19. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\sqrt{\frac{a+b}{b+1}} + \sqrt{\frac{b+c}{c+1}} + \sqrt{\frac{c+a}{a+1}} \geq 3.$$

(*Vasile Cîrtoaje*, MC, 2005)

20. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\sqrt{\frac{a}{b+3}} + \sqrt{\frac{b}{c+3}} + \sqrt{\frac{c}{a+3}} \geq \frac{3}{2}.$$

(*Vasile Cîrtoaje*, MS, 2005)

21. Let a, b, c be non-negative numbers such that $a + b + c = 3$. Prove that

$$\frac{5-3bc}{1+a} + \frac{5-3ca}{1+b} + \frac{5-3ab}{1+c} \geq ab + bc + ca.$$

(*Vasile Cîrtoaje*, MS, 2005)

22. Let a, b, c, d be non-negative numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$(abc)^3 + (bcd)^3 + (cda)^3 + (dab)^3 \leq 4.$$

(*Vasile Cîrtoaje*, MS, 2004)

23. Let a, b, c be non-negative numbers, no two of which are zero. Then,

$$\sqrt{\frac{a}{4a+5b}}+\sqrt{\frac{b}{4b+5c}}+\sqrt{\frac{c}{4c+5a}} \le 1.$$

(*Vasile Cîrtoaje*, GM-A, 1, 2004)

24. Let $a_1, a_2, \dots, a_n$ be positive numbers. Prove that

$$(a) \qquad \frac{(a_1+a_2+\cdots+a_n)^2}{(a_1^2+1)(a_2^2+1)\dots(a_n^2+1)} \le \frac{(n-1)^{n-1}}{n^{n-2}};$$

$$(b) \qquad \frac{a_1+a_2+\cdots+a_n}{(a_1^2+1)(a_2^2+1)\dots(a_n^2+1)} \le \frac{(2n-1)^{n-\frac{1}{2}}}{2^n n^{n-1}}.$$

(*Vasile Cîrtoaje*, GM-B, 6, 1994)

25. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. Prove that

$$a_1b_1+\cdots+a_nb_n+\sqrt{(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)} \ge \frac{2}{n}(a_1+\cdots+a_n)(b_1+\cdots+b_n).$$

(*Vasile Cîrtoaje*, Kvant, 11, 1989)

26. Let k and n be positive integers with $k < n$, and let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 \le a_2 \le \cdots \le a_n$. Prove that

$$(a_1+a_2+\cdots+a_n)^2 \ge n(a_1a_{k+1}+a_2a_{k+2}+\cdots+a_na_k)$$

in the following cases:
(a) for $n = 2k$;
(b) for $n = 4k$.

(*Vasile Cîrtoaje*, CM, 5, 2005)

27. Let a, b, c, d be positive numbers such that $abcd = 1$. Prove that

$$\frac{1}{1+a+a^2+a^3}+\frac{1}{1+b+b^2+b^3}+\frac{1}{1+c+c^2+c^3}+\frac{1}{1+d+d^2+d^3} \ge 1.$$

(*Vasile Cîrtoaje*, GM-B, 11, 1999)

28. If a, b, c are non-negative numbers, then

$$9(a^4+1)(b^4+1)(c^4+1) \ge 8(a^2b^2c^2+abc+1)^2.$$

(*Vasile Cîrtoaje*, GM-B, 3, 2004)

29. If a, b, c, d are non-negative numbers, then

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} \geq \frac{1+abcd}{2}.$$

(*Vasile Cîrtoaje*, GM-B, 10, 2002)

30. Let a, b, c be non-negative numbers, no two of which are zero. Then,

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \geq \frac{9}{(a+b+c)^2}.$$

(*Vasile Cîrtoaje*, GM-B, 9, 2000)

31. Let a, b, c be positive numbers, and let

$$x = a + \frac{1}{b} - 1,\ y = b + \frac{1}{c} - 1,\ z = c + \frac{1}{a} - 1.$$

Prove that

$$xy + yz + zx \geq 3.$$

(*Vasile Cîrtoaje*, GM-B, 1, 1991)

32. Let a, b, c be positive numbers, no two of which are zero. If n is a positive integer, then

$$\frac{2a^n - b^n - c^n}{b^2 - bc + c^2} + \frac{2b^n - c^n - a^n}{c^2 - ca + a^2} + \frac{2c^n - a^n - b^n}{a^2 - ab + b^2} \geq 0.$$

(*Vasile Cîrtoaje*, GM-B, 1, 2004)

33. Let $0 \leq a < b$ and let $a_1, a_2, \ldots, a_n \in [a, b]$. Prove that

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \ldots a_n} \leq (n-1)\left(\sqrt{b} - \sqrt{a}\right)^2.$$

(*Vasile Cîrtoaje* and *Gabriel Dospinescu*, MS, 2005)

34. Let a, b, c and x, y, z be positive numbers such that $x+y+z = a+b+c$. Prove that

$$ax^2 + by^2 + cz^2 + xyz \geq 4abc.$$

(*Vasile Cîrtoaje*, GM-A, 4, 1987)

35. Let a, b, c and x, y, z be positive numbers such that $x+y+z = a+b+c$. Prove that

$$\frac{x(3x+a)}{bc} + \frac{y(3y+a)}{ca} + \frac{z(3z+a)}{ab} \geq 12.$$

36. Let a, b, c be positive numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{9}{a+b+c}.$$

37. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{4n}{n + a_1 + a_2 + \cdots + a_n} \ge n + 2.$$

(*Vasile Cîrtoaje*, MS, 2005)

38. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$a_1 + a_2 + \cdots + a_n - n + 1 \ge \sqrt[n-1]{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n + 1}.$$

(*Vasile Cîrtoaje*, MS, 2006)

39. Let $r > 1$ and let a, b, c be non-negative numbers such that $ab+bc+ca=3$. Prove that

$$a^r(b+c) + b^r(c+a) + c^r(a+b) \ge 6.$$

40. Let a, b, c be positive real numbers such that $abc \ge 1$. Prove that

$$(a) \qquad a^{\frac{a}{b}} b^{\frac{b}{c}} c^{\frac{c}{a}} \ge 1;$$

$$(b) \qquad a^{\frac{a}{b}} b^{\frac{b}{c}} c^{c} \ge 1.$$

(*Vasile Cîrtoaje*, CM, 4, 2005)

41. Let a, b, c, d be non-negative numbers. Prove that

$$4(a^3 + b^3 + c^3 + d^3) + 15(abc + bcd + cda + dab) \ge (a + b + c + d)^3.$$

42. Let a, b, c be positive numbers such that

$$(a + b - c)\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) = 4.$$

Prove that

$$(a^4 + b^4 + c^4)\left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4}\right) \ge 2304.$$

(*Vasile Cîrtoaje*, MC, 2005)

43. Let a, b, c be positive numbers. Prove that

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab}>\frac{2}{ab+bc+ca}.$$

(*Vasile Cîrtoaje*, MS, 2005)

44. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab}\ge 1+\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

(*Vasile Cîrtoaje*, MS, 2006)

45. Let a, b, c be non-negative numbers, no two of which are zero. Then

$$\frac{(b+c)^2}{a^2+bc}+\frac{(c+a)^2}{b^2+ca}+\frac{(a+b)^2}{c^2+ab}\ge 6.$$

(*Peter Scholze* and *Darij Grinberg*, MS, 2005)

46. Let a, b, c be non-negative numbers, no two of which are zero. Then

$$\frac{b+c}{2a^2+bc}+\frac{c+a}{2b^2+ca}+\frac{a+b}{2c^2+ab}\ge\frac{6}{a+b+c}.$$

(*Vasile Cîrtoaje*, MS, 2006)

47. If a, b, c are non-negative numbers, then

$$a\sqrt{a^2+3bc}+b\sqrt{b^2+3ca}+c\sqrt{c^2+3ab}\ge 2(ab+bc+ca).$$

(*Vasile Cîrtoaje*, MS, 2005)

48. Let a, b, c be non-negative numbers, no two of which are zero. Then

$$\frac{a^2-bc}{\sqrt{a^2+bc}}+\frac{b^2-ca}{\sqrt{b^2+ca}}+\frac{c^2-ab}{\sqrt{c^2+ab}}\ge 0.$$

(*Vasile Cîrtoaje*, MS, 2005)

49. If a, b, c are non-negative numbers, then

$$(a^2-bc)\sqrt{a^2+4bc}+(b^2-ca)\sqrt{b^2+4ca}+(c^2-ab)\sqrt{c^2+4ab}\ge 0.$$

(*Vasile Cîrtoaje*, MS, 2005)

50. If a, b, c are positive numbers, then

$$\frac{a^2 - bc}{\sqrt{8a^2 + (b+c)^2}} + \frac{b^2 - ca}{\sqrt{8b^2 + (c+a)^2}} + \frac{c^2 - ab}{\sqrt{8c^2 + (a+b)^2}} \geq 0.$$

(*Vasile Cîrtoaje*, MS, 2006)

51. If a, b, c are non-negative numbers, then

$$\sqrt{a^2 + bc} + \sqrt{b^2 + ca} + \sqrt{c^2 + ab} \leq \frac{3}{2}(a + b + c).$$

(*Pham Kim Hung*, MS, 2005)

52. Let a, b, c be non-negative numbers such that $a^2 + b^2 + c^2 = 3$. Then,

$$21 + 18abc \geq 13(ab + bc + ca).$$

(*Vasile Cîrtoaje*, MS, 2005)

53. Let a, b, c be non-negative numbers such that $a^2 + b^2 + c^2 = 3$. Then

$$\frac{1}{5 - 2ab} + \frac{1}{5 - 2bc} + \frac{1}{5 - 2ca} \leq 1.$$

(*Vasile Cîrtoaje*, MS, 2005)

54. Let a, b, c be non-negative numbers such that $a^2 + b^2 + c^2 = 3$. Then,

$$(2 - ab)(2 - bc)(2 - ca) \geq 1.$$

(*Vasile Cîrtoaje*, MS, 2005)

55. Let a, b, c be non-negative numbers such that $a + b + c = 2$. Prove that

$$\frac{bc}{a^2 + 1} + \frac{ca}{b^2 + 1} + \frac{ab}{c^2 + 1} \leq 1.$$

(*Pham Kim Hung*, MS, 2005)

56. Let a, b, c be non-negative numbers, no two of which are zero. Then,

$$\frac{a^3 + 3abc}{(b+c)^2} + \frac{b^3 + 3abc}{(c+a)^2} + \frac{c^3 + 3abc}{(a+b)^2} \geq a + b + c.$$

(*Vasile Cîrtoaje*, MS, 2005)

57. Let a, b, c be positive numbers such that $a^4 + b^4 + c^4 = 3$. Then,

$$a)\qquad \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3;$$

$$b)\qquad \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{3}{2}.$$

(*Alexey Gladkich*, MS, 2005)

58. If a, b, c are positive numbers, then

$$\frac{a^3 - b^3}{a+b} + \frac{b^3 - c^3}{b+c} + \frac{c^3 - a^3}{c+a} \leq \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{8}.$$

(*Marian Tetiva* and *Darij Grinberg*, MS, 2005)

59. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a^2}{(2a+b)(2a+c)} + \frac{b^2}{(2b+c)(2b+a)} + \frac{c^2}{(2c+a)(2c+b)} \leq \frac{1}{3}.$$

(*Tigran Sloyan*, MS, 2005)

60. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{1}{5(a^2+b^2) - ab} + \frac{1}{5(b^2+c^2) - bc} + \frac{1}{5(c^2+a^2) - ca} \geq \frac{1}{a^2+b^2+c^2}.$$

(*Vasile Cîrtoaje*, MS, 2006)

61. Let a, b, c be non-negative real numbers such that $a^2+b^2+c^2 = 1$. Prove that

$$\frac{bc}{a^2+1} + \frac{ca}{b^2+1} + \frac{ab}{c^2+1} \leq \frac{3}{4}.$$

(*Pham Kim Hung*, MS, 2005)

62. Let a, b, c be non-negative numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{1}{3+a^2-2bc} + \frac{1}{3+b^2-2ca} + \frac{1}{3+c^2-2ab} \leq \frac{9}{8}.$$

(*Vasile Cîrtoaje* and *Wolfgang Berndt*, MS, 2006)

63. If a, b, c are positive numbers, then

$$\frac{4a^2-b^2-c^2}{a(b+c)}+\frac{4b^2-c^2-a^2}{b(c+a)}+\frac{4c^2-a^2-b^2}{c(a+b)} \le 3.$$

(*Vasile Cîrtoaje*, MS, 2006)

64. If a, b, c are positive numbers such that $abc = 1$, then

$$a^2+b^2+c^2+6 \ge \frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

(*Vasile Cîrtoaje*, MS, 2006)

65. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1+a_2+\cdots+a_n = n$. Prove that

$$a_1a_2\ldots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n+3\right) \le 3.$$

(*Vasile Cîrtoaje*, MS, 2004)

66. Let a, b, c be the side lengths of a triangle. If $a^2+b^2+c^2 = 3$, then

$$ab+bc+ca \ge 1+2abc.$$

(*Vasile Cîrtoaje*, MS, 2005)

67. Let a, b, c be the side lengths of a triangle. If $a^2+b^2+c^2 = 3$, then

$$a+b+c \ge 2+abc.$$

(*Vasile Cîrtoaje*, MS, 2005)

68. If a, b, c are the side lengths of a non-isosceles triangle, then

$$a) \qquad \left|\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}\right| > 5;$$

$$b) \qquad \left|\frac{a^2+b^2}{a^2-b^2}+\frac{b^2+c^2}{b^2-c^2}+\frac{c^2+a^2}{c^2-a^2}\right| > 3.$$

(*Vasile Cîrtoaje*, GM-B, 3, 2003)

69. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$a^2\left(\frac{b}{c}-1\right)+b^2\left(\frac{c}{a}-1\right)+c^2\left(\frac{a}{b}-1\right) \ge 0.$$

(*Vasile Cîrtoaje*, Moldova TST, 2006)

70. Let a, b, c be the lengths of the sides of an triangle. Prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

(Vietnam TST, 2006)

71. If $a_1, a_2, a_3, a_4, a_5, a_6 \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$, then

$$\frac{a_1-a_2}{a_2+a_3}+\frac{a_2-a_3}{a_3+a_4}+\cdots+\frac{a_6-a_1}{a_1+a_2} \geq 0.$$

(*Vasile Cîrtoaje*, AJ, 7-8, 2002)

72. Let a, b, c be positive numbers such that $a^2+b^2+c^2 \geq 3$. Prove that

$$\frac{a^5-a^2}{a^5+b^2+c^2}+\frac{b^5-b^2}{a^2+b^5+c^2}+\frac{c^5-c^2}{a^2+b^2+c^5} \geq 0.$$

(*Vasile Cîrtoaje*, MS, 2005)

73. Let a, b, c be positive numbers such that $x+y+z \geq 3$. Then,

$$\frac{1}{x^3+y+z}+\frac{1}{x+y^3+z}+\frac{1}{x+y+z^3} \leq 1.$$

(*Vasile Cîrtoaje*, MS, 2005)

74. Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 x_2 \ldots x_n \geq 1$.
If $\alpha > 1$, then

$$\sum \frac{x_1^\alpha}{x_1^\alpha+x_2+\cdots+x_n} \geq 1.$$

(*Vasile Cîrtoaje*, CM, 2, 2006)

75. Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 x_2 \ldots x_n \geq 1$.
If $n \geq 3$ and $\frac{-2}{n-2} \leq \alpha < 1$, then

$$\sum \frac{x_1^\alpha}{x_1^\alpha+x_2+\cdots+x_n} \leq 1.$$

(*Vasile Cîrtoaje*, CM, 2, 2006)

76. Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1x_2 \ldots x_n \ge 1$.
If $\alpha > 1$, then

$$\sum \frac{x_1}{x_1^\alpha + x_2 + \cdots + x_n} \le 1.$$

(*Vasile Cîrtoaje*, CM, 2, 2006)

77. Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1x_2 \ldots x_n \ge 1$.
If $-1 - \dfrac{2}{n-2} \le \alpha < 1$, then

$$\sum \frac{x_1}{x_1^\alpha + x_2 + \cdots + x_n} \ge 1.$$

(*Vasile Cîrtoaje*, CM, 2, 2006)

78. Let $n \ge 3$ be an integer and let p be a real number such that $1 < p < n-1$.
If $0 < x_1, x_2, \ldots, x_n \le \dfrac{pn - p - 1}{p(n-p-1)}$ such that $x_1x_2 \ldots x_n = 1$, then

$$\frac{1}{1+px_1} + \frac{1}{1+px_2} + \cdots + \frac{1}{1+px_n} \ge \frac{n}{1+p}.$$

(*Vasile Cîrtoaje*, GM-A, 1, 2005)

79. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \ge 1.$$

(*Pham Van Thuan*, MS, 2006)

80. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$a^2 + b^2 + c^2 + 9(ab + bc + ca) \ge 10(a + b + c).$$

81. Let a, b, c be non-negative numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{a(b^2+c^2)}{a^2+bc} + \frac{b(c^2+a^2)}{b^2+ca} + \frac{c(a^2+b^2)}{c^2+ab} \ge 3.$$

(*Pham Huu Duc*, MS, 2006)

82. If a, b, c are positive numbers, then

$$a + b + c + \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge \frac{6(a^2+b^2+c^2)}{a+b+c}.$$

(*Pham Huu Duc*, MS, 2006)

83. If a, b, c are positive numbers, then

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{3(a^3+b^3+c^3)}{2(a^2+b^2+c^2)}.$$

(*Pham Huu Duc*, MS, 2006)

84. If a, b, c are given non-negative numbers, find the minimum value $E(a, b, c)$ of the expression

$$E = \frac{ax}{y+z} + \frac{by}{z+x} + \frac{cz}{x+y}$$

for any positive numbers x, y, z.

(*Vasile Cîrtoaje*, MS, 2006)

85. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^2 + b^2 + c^2.$$

(*Vasile Cîrtoaje*, Romania TST, 2006)

86. Let a, b, c be non-negative real numbers such that $a + b + c = 3$. Prove that

$$(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \leq 12.$$

(*Pham Kim Hung*, MS, 2006)

87. Let a, b, c be non-negative real numbers such that $a + b + c = 1$. Prove that

$$\sqrt{a+b^2} + \sqrt{b+c^2} + \sqrt{c+a^2} \geq 2.$$

(*Phan Thanh Nam*)

88. If a, b, c are non-negative real numbers, then

$$a^3 + b^3 + c^3 + 3abc \geq \sum bc\sqrt{2(b^2+c^2)}.$$

89. If a, b, c are non-negative real numbers, then

$$(1+a^2)(1+b^2)(1+c^2) \geq \frac{15}{16}(1+a+b+c)^2.$$

(*Vasile Cîrtoaje*, MS, 2006)

90. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \geq (a + b + c + d)^2.$$

(*Pham Kim Hung*, MS, 2006)

91. If $x_1, x_2, \ldots, x_n$ are non-negative numbers, then

$$x_1 + x_2 + \cdots + x_n \geq (n - 1)\sqrt[n]{x_1 x_2 \ldots x_n} + \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}.$$

(*Vasile Cîrtoaje*, MS, 2006)

92. If k is a real number and $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$(n-1)\left(x_1^{n+k}+x_2^{n+k}+\cdots+x_n^{n+k}\right)+x_1x_2\ldots x_n\left(x_1^k+x_2^k+\cdots+x_n^k\right)\geq$$
$$\geq (x_1+x_2+\cdots+x_n)\left(x_1^{n+k-1}+x_2^{n+k-1}+\cdots+x_n^{n+k-1}\right).$$

(*Gjergji Zaimi* and *Keler Marku*, MS, 2006

93. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a^4}{a^3 + b^3} + \frac{b^4}{b^3 + c^3} + \frac{c^4}{c^3 + a^3} \geq \frac{a + b + c}{2}.$$

8.2 Solutions

1. *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\sqrt{\frac{a + b}{b + 1}} + \sqrt{\frac{b + c}{c + 1}} + \sqrt{\frac{c + a}{a + 1}} \geq 3.$$

Solution. By AM-GM Inequality, it follows that

$$\sqrt{\frac{a + b}{b + 1}} + \sqrt{\frac{b + c}{c + 1}} + \sqrt{\frac{c + a}{a + 1}} \geq 3\sqrt[6]{\frac{(a + b)(b + c)(c + a)}{(b + 1)(c + 1)(a + 1)}}.$$

Thus, we still have to show that

$$(a + b)(b + c)(c + a) \geq (a + 1)(b + 1)(c + 1).$$

Let $A = a + b + c$ and $B = ab + bc + ca$. The AM-GM Inequality yields $A \geq 3$ and $B \geq 3$. Since

$$(a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca) - abc = AB - 1$$