Mathematical Excalibur

Volume 9, Number 3 *August 2004 – September 2004*

Olympiad Corner

The 45th International Mathematical Olympiad took place on July 2004. Here are the problems.

Day 1 Time allowed: 4 hours 30 minutes.

Problem 1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of the side BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

Problem 2. Find all polynomials $P(x)$ with real coefficients which satisfy the equality

$$P(a-b)+P(b-c)+P(c-a) = 2P(a+b+c)$$

for all real numbers a, b, c such that $ab + bc + ca = 0$.

Problem 3. Define a *hook* to be a figure made up of six unit squares as shown in the diagram

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 20, 2004***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2004

T. W. Leung

The 45th International Mathematical Olympiad (IMO) was held in Greece from July 4 to July 18. Since 1988, we have been participating in the Olympiads. This year our team was composed as follows.

<u>*Members*</u>

Cheung Yun Kuen (Hong Kong Chinese Women's Club College)

Chung Tat Chi (Queen Elizabeth School)

Kwok Tsz Chiu (Yuen Long Merchant Association Secondary School)

Poon Ming Fung (STFA Leung Kau Kui College)

Tang Chiu Fai (HKTA Tang Hin Memorial Secondary School)

Wong Hon Yin (Queen's College)

Cesar Jose Alaban (Deputy Leader)

Leung Tat Wing (Leader)

I arrived at Athens on July 6. After waiting for a couple of hours, leaders were then delivered to Delphi, a hilly town 170 km from the airport, corresponding to 3 more hours of journey. In these days the Greeks were still ecstatic about what they had achieved in the Euro 2004, and were busy preparing for the coming Olympic Games in August. Of course Greece is a small country full of legend and mythology. Throughout the trip, I also heard many times that they were the originators of democracy, their contribution in the development of human body and mind and their emphasis on fair play.

After receiving the short-listed problems leaders were busy studying them on the night of July 8. However obviously some leaders had strong opinions on the beauty and degree of difficulty of the problems, so selections of all six problems were done in one day. Several problems were not even discussed in details of their own merits.

The following days were spent on refining the wordings of the questions and translating the problems into different languages.

The opening ceremony was held on July 11. In the early afternoon we were delivered to Athens. After three hours of ceremony we were sent back to Delphi. By the time we were in Delphi it was already midnight. Leaders were not allowed to talk to students in the ceremony.

Contests were held in the next two days. The days following the contests were spent on coordination, i.e. leaders and coordinators discussed how many points should be awarded to the answers of the students. This year the coordinators were in general very careful. I heard several teams spent more than three hours to go over six questions. Luckily coordination was completed on the afternoon of July 15. The final Jury meeting was held that night. In the meeting the cut-off scores were decided, namely 32 points for gold, 24 for silver and 16 for bronze. Our team was therefore able to obtain two silver medals (Kwok and Chung) and two bronze medals (Tang and Cheung). Other members (Poon and Wong) both solved at least one problem completely, thus received honorable mention. Unofficially our team ranked 30 out of 85. The top five teams in order were respectively China, USA, Russia, Vietnam and Bulgaria.

In retrospect I felt that our team was good and balanced, none of the members was particularly weak. In one problem we were as good as any strong team. Every team members solved problem 4 completely. Should we did better in the geometry problems our rank would be much higher. Curiously geometry is in our formal school curriculum while number theory and combinatorics are not. In this Olympiad we had two geometry problems, but fittingly so, after all, it was Greece.

Extending an IMO Problem

Hà Duy Hung

Dept. of Math and Informatics
Hanoi Univ. of Education

In this brief note we give a generalization of a problem in the 41st International Mathematical Olympiad held in Taejon, South Korea in 2000.

IMO 2000/5. Determine whether or not there exists a positive integer n such that n is divisible by exactly 2000 different prime divisors, and 2^n+1 is divisible by n.

The answer to the question is positive. This intriguing problem made me recall a well-known theorem due to O. Reutter in [1] as follows.

Theorem 1. If a is a positive integer such that $a+1$ is not a power of 2, then a^n+1 is divisible by n for infinitely many positive integers n.

We frequently encounter the theorem in the case $a=2$. The theorem and the IMO problem prompted me to think of more general problem. Can we replace the number 2 in the IMO problem by other positive integers? The difficulty partly lies in the fact that the two original problems are solved independently. After a long time, I finally managed to prove a generalization as follows.

Theorem 2. Let s, a, b be given positive integers, such that a, b are relatively prime and $a+b$ is not a power of 2. Then there exist infinitely many positive integers n such that

- n has exactly s different prime divisors; and
- a^n+b^n is divisible by n.

We give a proof of Theorem 2 below. We shall make use of two familiar lemmas.

Lemma 1. Let n be an odd positive integer, and a, b be relative prime positive integers. Then

$$\frac{a^n+b^n}{a+b}$$

is an odd integer ≥ 1, equality if and only if $n=1$ or $a=b=1$.

The proof of Lemma 1 is simple and is left for the reader.

Also, we remind readers the usual notations $r \mid s$ means s is divisible by r and $u \equiv v \pmod{m}$ means $u-v$ is divisible by m.

Lemma 2. Let a, b be distinct and relatively prime positive integers, and p an odd prime number which divides $a+b$. Then for any non-negative integer k,

$$p^{k+1} \mid a^m+b^m,$$

where $m=p^k$.

Proof. We prove the lemma by induction. It is clear that the lemma holds for $k=0$. Suppose the lemma holds for some non-negative integer k, and we proceed to the case $k+1$.

Let $x=a^{p^k}$ and $y=b^{p^k}$. Since

$$x^p+y^p=(x+y)\sum_{i=0}^{p-1}(-1)^i x^{p-1-i}y^i,$$

it suffices to show that the whole summation is divisible by p. Since $x \equiv -y \pmod{p^{k+1}}$, we have

$$\begin{aligned}&\sum_{i=0}^{p-1}(-1)^i x^{p-1-i}y^i\\ &\equiv \sum_{i=0}^{p-1}(-1)^{2i}x^{p-1}\\ &\equiv px^{p-1} \qquad \pmod{p^{k+1}}\end{aligned}$$

completing the proof.

In the rest of this note we shall complete the proof of Theorem 2.

Proof of Theorem 2. Without loss of generality, let $a>b$. Since $a+b$ is not a power of 2, it has an odd prime factor p. For natural number k, set

$$x_k=a^{p^k}+b^{p^k},\ y_k=\frac{x_{k+1}}{x_k}.$$

Then y_k is a positive integer and

$$\begin{aligned}y_k&=\sum_{i=0}^{p-1}(-1)^i(a^{p^k})^{p-1-i}(b^{p^k})^i\\ &\equiv \sum_{i=0}^{p-1}(-1)^{2i}(a^{p^k})^{p-1}\\ &\equiv px^{p-1} \qquad \pmod{p^{k+1}}\end{aligned}$$

which implies that $\dfrac{y_k}{p}$ is a positive integer. Also, we have

$$\frac{y_k}{p} \equiv b^{p^k(p-1)} \quad \left(\bmod \frac{x_k}{p}\right),$$

so that

$$\gcd\left(\frac{x_k}{p},\frac{y_k}{p}\right)=1$$

for $k=1, 2, \ldots$ By Lemma 2, we also have

$$\gcd\left(\frac{y_k}{p},p^k\right)=1$$

for $k=1, 2, \ldots$. Moreover, we have $x_k \geq p^k$. This leads us to

$$\begin{aligned}y_k&=b^{p^k(p-1)}+\sum_{i=1}^{\frac{p-1}{2}}[(a^{p^k})^{2i}(b^{p^k})^{p-1-2i}\\ &\qquad -(a^{p^k})^{2i-1}(b^{p^k})^{p-2i}]\\ &>b^{p^k}+a^{p^k}\\ &=x_k\\ &\geq p^{k+1}\end{aligned}$$

It follows that

$$\frac{y_k}{p} \geq p^k > 1.$$

By Lemma 1, $\dfrac{y_k}{p}$ is an odd positive integer, so we can choose an odd prime divisor q_k of $\dfrac{y_k}{p}$.

We now have a sequence of odd prime numbers $\{q_k\}_{k=1}^{+\infty}$ satisfying the following properties

(i) $\gcd(x_k, q_k)=1$
(ii) $\gcd(p, q_k)=1$
(iii) $q_k \mid x_{k+1}$
(iv) $x_k \mid x_{k+1}$.

We shall now show that the sequence $\{q_k\}_{k=1}^{+\infty}$ consists of distinct prime numbers and is thus infinite. Indeed, if $k_0<k_1$ are positive integers and $q_{k_0}=q_{k_1}$, then

$$q_{k_1}=q_{k_0} \mid x_{k_0+1} \mid \cdots \mid x_{k_1}$$

by properties (iii) and (iv). But this contradicts property (i).

Next, set $n_0=p^s q_1 \ldots q_{s-1}$ and $n_{k+1}=pn_k$ for $k=0, 1, 2, \ldots$. It is evident that $\{n_k\}_{k=0}^{+\infty}$ is a strictly increasing sequence

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***October 20, 2004.***

Problem 206. (*Due to Zdravko F. Starc, Vršac, Serbia and Montenegro*) Prove that if a, b are the legs and c is the hypotenuse of a right triangle, then

$$(a+b)\sqrt{a}+(a-b)\sqrt{b}<\sqrt{2\sqrt{2}}\,c\sqrt{c}.$$

Problem 207. Let $A = \{0, 1, 2, ..., 9\}$ and $B_1, B_2, ..., B_k$ be nonempty subsets of A such that B_i and B_j have at most 2 common elements whenever $i \neq j$. Find the maximum possible value of k.

Problem 208. In ΔABC, $AB > AC > BC$. Let D be a point on the minor arc BC of the circumcircle of ΔABC. Let O be the circumcenter of ΔABC. Let E, F be the intersection points of line AD with the perpendiculars from O to AB, AC, respectively. Let P be the intersection of lines BE and CF. If $PB = PC + PO$, then find $\angle BAC$ with proof.

Problem 209. Prove that there are infinitely many positive integers n such that $2^n + 2$ is divisible by n and $2^n + 1$ is divisible by $n - 1$.

Problem 210. Let $a_1 = 1$ and

$$a_{n+1}=\frac{a_n}{2}+\frac{1}{a_n}$$

for $n = 1, 2, 3, ...$. Prove that for every integer $n > 1$,

$$\frac{2}{\sqrt{a_n^2-2}}$$

is an integer.

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Solutions

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Problem 201. (*Due to Abderrahim OUARDINI, Talence, France*) Find which nonright triangles ABC satisfy

$$\tan A \tan B \tan C > [\tan A] + [\tan B] + [\tan C],$$

where $[t]$ denotes the greatest integer less than or equal to t. Give a proof.

Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4). , **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece),

From

$$\begin{aligned}\tan C &= \tan(180^\circ - A - B)\\ &= -\tan(A+B)\\ &= -(\tan A + \tan B)/(1-\tan A\tan B),\end{aligned}$$

we get

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Let $x = \tan A$, $y = \tan B$ and $z = \tan C$. If $xyz \le [x]+[y]+[z]$, then $x+y+z \le [x] + [y] + [z]$. As $[t] \le t$, x, y, z must be integers.

If triangle ABC is obtuse, say $A > 90^\circ$, then $x < 0 < 1 \le y \le z$. This implies $1 \le yz = (x + y + z)/x = 1 + (y + z)/x < 1$, a contradiction. If triangle ABC is acute, then we may assume $1 \le x \le y \le z$. Now $xy = (x+y+z)/z \le (3z)/z = 3$. Checking the cases $xy = 1, 2, 3$, we see $x+y+z = xyz$ can only happen when $x=1$, $y=2$ and $z=3$. This corresponds to $A = \tan^{-1} 1$, $B = \tan^{-1} 2$ and $C = \tan^{-1} 3$. Reversing the steps, we see among nonright triangles, the inequality in the problem holds except only for triangles with angles equal $45^\circ = \tan^{-1} 1$, $\tan^{-1} 2$ and $\tan^{-1} 3$.

Problem 202. (*Due to LUK Mee Lin, La Salle College*) For triangle ABC, let D, E, F be the midpoints of sides AB, BC, CA, respectively. Determine which triangles ABC have the property that triangles ADF, BED, CFE can be folded above the plane of triangle DEF to form a tetrahedron with AD coincides with BD; BE coincides with CE; CF coincides with AF.

Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Observe that $ADEF$, $BEFD$ and $CFDE$ are parallelograms. Hence $\angle BDE = \angle BAC$, $\angle ADF = \angle ABC$ and $\angle EDF = \angle BCA$. In order for AD to coincide with BD in folding, we need to have $\angle BDE + \angle ADF > \angle EDF$. So we need $\angle BAC + \angle ABC > \angle BCA$. Similarly, for BE to coincide with CE and for CF to coincide with AF, we need $\angle ABC + \angle BCA > \angle BAC$ and $\angle BCA + \angle BAC > \angle ABC$. So no angle of ΔABC is 90 or more. Therefore, ΔABC is acute.

Conversely, if ΔABC is acute, then reversing the steps, we can see that the required tetrahedron can be obtained.

Problem 203. (*Due to José Luis DÍAZ-BARRERO, Universitat Politecnica de Catalunya, Barcelona, Spain*) Let a, b and c be real numbers such that $a + b + c \neq 0$. Prove that the equation

$$(a+b+c)x^2 + 2(ab+bc+ca)x + 3abc = 0$$

has only real roots.

Solution. **CHAN Pak Woon (**Wah Yan College, Kowloon, Form 6), **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Hoi Kit** (SKH Lam Kau Mow Secondary School, Form 7), **CHEUNG Yun Kuen** (HKUST, Math, Year 1), **Murray KLAMKIN** (University of Alberta, Edmonton, Canada), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

The quadratic has real roots if and only if its discriminant

$$\begin{aligned}D &= 4(ab+bc+ca)^2 - 12(a+b+c)abc\\ &= 4[(ab)^2+(bc)^2+(ca)^2-(a+b+c)abc]\\ &= 4[(ab-bc)^2+(bc-ca)^2+(ca-ab)^2]\end{aligned}$$

is nonnegative, which is clear.

Other commended solvers: **Jason CHENG Hoi Sing** (SKH Lam Kau Mow Secondary School, Form 7), **POON Ho Yin** (Munsang College (Hong Kong Island), Form 4) and **Anderson TORRES** (Universidade de Sao Paulo – Campus Sao Carlos).

Problem 204. Let n be an integer with $n > 4$. Prove that for every n distinct integers taken from $1, 2, \ldots, 2n$, there always exist two numbers whose least common multiple is at most $3n + 6$.

Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Let S be the set of n integers taken and k be the minimum of these integers. If $k \le n$, then either $2k$ is also in S or $2k$ is not in S. In the former case, $\mathrm{lcm}(k,2k) = 2k \le 2n < 3n+6$. In the latter case, we replace k in S by $2k$. Note this will not decrease the least common multiple of any pair of numbers. So if the new S satisfies the problem, then the original S will also satisfy the problem. As we repeat this, the new minimum will increase strictly so that we eventually reach either k and $2k$ both in S, in which case we are done, or the new S will consist of $n+1$, $n+2$, …, $2n$. So we need to consider the latter case only.

If $n > 4$ is even, then $3(n+2)/2$ is an integer at most $2n$ and $\text{lcm}(n+2, 3(n+2)/2) = 3n+6$. If $n > 4$ is odd, then $3(n+1)/2$ is an integer at most $2n$ and $\text{lcm}(n+1, 3(n+1)/2) = 3n+3$.

Problem 205. *(Due to HA Duy Hung, Hanoi University of Education, Vietnam)* Let a, n be integers, both greater than 1, such that $a^n - 1$ is divisible by n. Prove that the greatest common divisor (or highest common factor) of $a - 1$ and n is greater than 1.

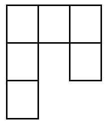
Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Let p be the smallest prime divisor of n. Then $a^n - 1$ is divisibly by p so that $a^n \equiv 1 \pmod p$. In particular, a is not divisible by p. Then, by Fermat's little theorem, $a^{p-1} \equiv 1 \pmod p$.

Let d be the smallest positive integer such that $a^d \equiv 1 \pmod p$. Dividing n by d, we get $n = dq + r$ for some integers q, r with $0 \le r < d$. Then $a^r \equiv (a^d)^q a^r = a^n \equiv 1 \pmod p$. By the definition of d, we get $r = 0$. Then n is divisible by d. Similarly, dividing $p - 1$ by d, we see $a^{p-1} \equiv 1 \pmod p$ implies $p - 1$ is divisible by d. Hence, $\gcd(n, p - 1)$ is divisible by d. Since p is the smallest prime dividing n, we must have $\gcd(n, p - 1) = 1$. So $d = 1$. By the definition of d, we get $a - 1$ is divisible by p. Therefore, $\gcd(a - 1, n) \ge p > 1$.

Olympiad Corner

(continued from page 1)



or any of the figures obtained by applying rotations and reflections to this figure.

Determine all $m \times n$ rectangles that can be covered with hooks so that

- the rectangle is covered without gaps and without overlaps;
- no part of a hook covers area outside the rectangle.

Day 2 Time allowed: 4 hours 30 minutes.

Problem 4. Let $n \ge 3$ be an integer. Let $t_1, t_2, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n) \times \left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \le i < j < k \le n$.

Problem 5. In a convex quadrilateral $ABCD$ the diagonal BD bisects neither the angle ABC not the angle CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Problem 6. We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.

Find all positive integers n such that n has a multiple which is alternating.

Extending an IMO Problem

(continued from page 2)

of positive integers and each term of the sequence has exactly s distinct prime divisors.

It remains to show that

$$n_k \mid a^{n_k} + b^{n_k}$$

for $k = 0, 1, 2, \ldots$ Note that for odd positive integers m, n with $m \mid n$, we have $a^m + b^m \mid a^n + b^n$. By property (iii), we have, for $0 \le k < s$,

$$q_k \mid x_{k+1} \mid x_s \mid a^{n_0} + b^{n_0} \mid a^{n_j} + b^{n_j}$$

for $j = 0, 1, 2, \ldots$. Now it suffices to show that

$$p^{k+s} \mid a^{n_k} + b^{n_k}$$

for $k = 0, 1, 2, \ldots$. But this follows easily from Lemma 2 since

$$p^{s+k} \mid x_{k+s} \mid a^{n_k} + b^{n_k}.$$

This completes the proof of Theorem 2.

References:

[1] O. Reutter, *Elemente der Math.*, 18 (1963), 89.

[2] W. Sierpinski, *Elementary Theory of Numbers*, English translation, Warsaw, 1964.

2004 Hong Kong team to IMO: From left to right, Cheung Yun Kuen, Poon Ming Fung, Tang Chiu Fai, Cesar Jose Alaban (Deputy Leader), Leung Tat Wing (Leader), Chung Tat Chi, Kwok Tsz Chiu & Wong Hon Yin.