Mathematical Excalibur

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Olympiad Corner

The following were the problems of the 2009 Asia-Pacific Math Olympiad.

Problem 1. Consider the following operation on positive real numbers written on a blackboard: *Choose a number r written on the blackboard, erase that number, and then write a pair of real numbers a and b satisfying the condition $2r^2 = ab$ on the board.*

Assume that you start out with just one positive real number r on the blackboard, and apply this operation k^2-1 times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr.

Problem 2. Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:

$$\frac{a_1}{k^2+1}+\frac{a_2}{k^2+2}+\frac{a_3}{k^2+3}+\frac{a_4}{k^2+4}+\frac{a_5}{k^2+5}=\frac{1}{k^2}$$

for k = 1,2,3,4,5. Find the value of

$$\frac{a_1}{37}+\frac{a_2}{38}+\frac{a_3}{39}+\frac{a_4}{40}+\frac{a_5}{41}.$$

(Express the value in a single fraction.)

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 7, 2009***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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A Nice Identity

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There are many methods to prove inequalities. In this paper, we would like to introduce to the readers some applications of a nice identity for solving inequalities.

Theorem 0. Let *a, b, c* be real numbers. Then

$$(a+b)(b+c)(c+a)$$
$$= (a+b+c)(ab+bc+ca) - abc.$$

Proof. This follows immediately by expanding both sides.

Corollary 1. Let *a, b, c* be real numbers. If $abc = 1$, then

$$(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-1.$$

Corollary 2. Let *a, b, c* be real numbers. If $ab + bc + ca = 1$, then

$$(a+b)(b+c)(c+a)=a+b+c-abc\ .$$

Next we will give some applications of these facts. The first example is a useful well-known inequality.

Example 1. Let *a, b, c* be nonnegative real numbers. Prove that

$$(a+b)(b+c)(c+a)\ge\frac{8}{9}(a+b+c)(ab+bc+ca).$$

Solution. By the *AM-GM* inequality,

$$\frac{1}{9}(a+b+c)(ab+bc+ca)-abc$$
$$\ge\frac{1}{9}(3\sqrt[3]{abc})(3\sqrt[3]{a^2b^2c^2})-abc=0.$$

Using Theorem 0, we have

$$(a+b)(b+c)(c+a)\ge\frac{8}{9}(a+b+c)(ab+bc+ca).$$

The next example was a problem on the third team selection test of Romania for the Balkan Mathematical Olympiad 2005. Subsequently, it also appeared in the Croatian Team Selection Test 2006.

Example 2. (*Cezar Lupu, Romania 2005; Croatia TST 2006*) Let *a, b, c* be positive real numbers satisfying $(a+b)(b+c)(c+a) = 1$. Prove that

$$ab+bc+ca \le 3/4.$$

Solution. By the *AM-GM* inequality,

$$a+b+c=\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}$$
$$\ge 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}=\frac{3}{2}$$

and

$$abc=\sqrt{ab}\sqrt{bc}\sqrt{ca}$$
$$\le\frac{(a+b)(b+c)(c+a)}{8}=\frac{1}{8}.$$

Using Theorem 0, we get

$$1=(a+b)(b+c)(c+a)$$
$$=(a+b+c)(ab+bc+ca)-abc$$
$$\ge\frac{3}{2}(ab+bc+ca)-\frac{1}{8}.$$

Hence $ab+bc+ca\le\frac{3}{4}$.

The following example was taken from the Vietnamese magazine, Mathematics and Youth Magazine.

Example 3. (*Proposed by Tran Xuan Dang*) Let *a, b, c* be nonnegative real numbers satisfying $abc = 1$. Prove that

$$(a+b)(b+c)(c+a)\ge 2(1+a+b+c).$$

Solution. Using Corollary 1, this is equivalent to

$$(a+b+c)(ab+bc+ca-2)\ge 3.$$

We can obtain this by the *AM-GM* inequality as follows:

$$(a+b+c)(ab+bc+ca-2)$$
$$\ge(3\sqrt[3]{abc})(3\sqrt[3]{a^2b^2c^2}-2)=3.$$

The inequality in the next example is very hard. It was a problem in the Korean Mathematical Olympiad.

Example 4. (*KMO Winter Program Test 2001*) Let *a, b, c* be positive real numbers. Prove that

$$\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)}$$
$$\ge abc+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}.$$

Solution. Dividing by abc, the given inequality becomes

$$\sqrt{\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{b}+\frac{b}{c}\right)}$$
$$\geq 1+\sqrt[3]{\left(\frac{a^2}{bc}+1\right)\left(\frac{b^2}{ca}+1\right)\left(\frac{c^2}{ab}+1\right)}.$$

After the substitution $x = a/b$, $y = b/c$ and $z = c/a$, we get $xyz = 1$. It takes the form

$$\sqrt{(x+y+z)(xy+yz+zx)}$$
$$\geq 1+\sqrt[3]{\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)}.$$

Using Corollary 1, the previous inequality becomes

$$\sqrt{(x+y)(y+z)(z+x)+1}$$
$$\geq 1+\sqrt[3]{\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)}.$$

Setting $t=\sqrt[3]{(x+y)(y+z)(z+x)}$, we need to prove that

$$\sqrt{t^3+1}\geq 1+t.$$

By the *AM-GM* inequality, we have

$$t=\sqrt[3]{(x+y)(y+z)(z+x)}$$
$$\geq \sqrt[3]{2\sqrt{xy}\,2\sqrt{yz}\,2\sqrt{zx}}=2.$$

Therefore,

$$\sqrt{t^3+1}=\sqrt{(t+1)(t^2-t+1)}$$
$$\geq \sqrt{(t+1)(2t-t+1)}=1+t.$$

In the next example, we will see a nice inequality. It was from a problem in the 2001 USA Math Olympiad Summer Program.

Example 5. (*MOSP 2001*) Let a, b, c be positive real numbers satisfying abc=1. Prove that

$$(a+b)(b+c)(c+a)\geq 4(a+b+c-1).$$

Solution. Using Corollary 1, it suffices to prove that

$$(a+b+c)(ab+bc+ca)-1$$
$$\geq 4(a+b+c-1)$$

or $$ab+bc+ca+\frac{3}{a+b+c}\geq 4.$$

We will use the inequality

$$(x+y+z)^2\geq 3(xy+yz+zx), \qquad (*)$$

which after expansion and cancelling common terms amounts to

$$x^2+y^2+z^2-xy-yz-zx$$
$$=\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right)\geq 0.$$

Using (*), it is easy to see that

$$(ab+bc+ca)^2\geq 3(ab\cdot bc+bc\cdot ca+ca\cdot ab)$$
$$=3(a+b+c). \qquad (**)$$

By the *AM-GM* inequality and (**),

$$ab+bc+ca+\frac{3}{a+b+c}$$
$$=3\left(\frac{ab+bc+ca}{3}\right)+\frac{3}{a+b+c}$$
$$\geq 4\sqrt[4]{\frac{3(ab+bc+ca)^3}{3^3(a+b+c)}}$$
$$\geq 4\sqrt[4]{\frac{3(3\sqrt[3]{a^2b^2c^2})(3(a+b+c))}{3^3(a+b+c)}}=4.$$

Next, we will show some nice trigonometric inequalities can also be proved using Theorem 0.

Example 6. For a triangle ABC, prove that

(i) $\sin A+\sin B+\sin C\leq 3\sqrt{3}/2.$

(ii) $\cos A+\cos B+\cos C\leq 3/2.$

Solution. By the substitutions

$a = \tan(A/2)$, $b = \tan(B/2)$, $c = \tan(C/2)$,

we get $ab+bc+ca = 1$.

Using the facts $\sin 2x= (2\tan x) / (1+\tan^2 x)$ and $1 + a^2 = a^2 + ab + bc + ca = (a+b)(a+c)$, inequality (i) becomes

$$\frac{a}{1+a^2}+\frac{b}{1+b^2}+\frac{c}{1+c^2}\leq\frac{3\sqrt{3}}{4},$$

which is the same as

$$\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)}\leq\frac{3\sqrt{3}}{4}.$$

Clearing the denominators, this simplifies to $(a+b)(b+c)(c+a)\geq 8\sqrt{3}/9.$

To prove this, use the *AM-GM* inequality to get

$$1=ab+bc+ca\geq 3\sqrt[3]{a^2b^2c^2},$$

which is

$$abc\leq\sqrt{3}/9. \qquad (***)$$

Next, by (*),

$$a+b+c\geq\sqrt{3(ab+bc+ca)}=\sqrt{3}. \qquad (****)$$

Finally, by Corollary 2,

$$(a+b)(b+c)(c+a)=a+b+c-abc$$
$$\geq\sqrt{3}-\frac{\sqrt{3}}{9}=\frac{8\sqrt{3}}{9}.$$

Next, using $\cos 2x=(1-\tan^2 x)/(1+\tan^2 x)$, inequality (ii) becomes

$$\frac{1-a^2}{1+a^2}+\frac{1-b^2}{1+b^2}+\frac{1-c^2}{1+c^2}\leq\frac{3}{2}.$$

Using $1 + a^2 = a^2 + ab + bc + ca = (a+b)(a+c)$ in the denominators, doing the addition on the left and applying Corollary 2 in the common denominator, we can see the above inequality is the same as

$$\frac{2(a+b+c)-[a^2(b+c)+b^2(c+a)+c^2(a+b)]}{a+b+c-abc}\leq\frac{3}{2}.$$

Observe that $a^2(b+c)+b^2(c+a)+c^2(a+b) = (a+b+c)(ab+bc+ca)-3abc = a+b+c - 3abc$. So the inequality becomes

$$\frac{2(a+b+c)-(a+b+c-3abc)}{a+b+c-abc}\leq\frac{3}{2},$$

which simplifies to $a+b+c\geq 9abc$. This follows easily from (***) and (****).

Finally, we have some exercises for the readers.

Exercise 1. (*Due to Nguyen Van Ngoc*) Let a, b, c be positive real numbers. Prove that

$$abc(a+b+c)\leq\frac{3\left((a+b)(b+c)(c+a)\right)^{4/3}}{16}.$$

Exercise 2. (*Due to Vedula N. Murty*) Let a, b, c be positive real numbers. Prove that

$$\frac{a+b+c}{3}\leq\frac{1}{4}\sqrt[3]{\frac{(a+b)^2(b+c)^2(c+a)^2}{abc}}.$$

Exercise 3. (*Carlson's inequality*) Let a, b, c be positive real numbers. Prove that

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}\geq\sqrt{\frac{ab+bc+ca}{3}}.$$

Exercise 4. Let ABC be a triangle. Prove that

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\geq\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\sqrt{3}.$$

References

[1] Hojoo Lee, Topics in Inequalities -Theorems and Techniques, 2007.

[2] Pham Kin Hung, Secrets in Inequalities (in Vietnames), 2006.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***May 7, 2009.***

Problem 321. Let AA', BB' and CC' be three non-coplanar chords of a sphere and let them all pass through a common point P inside the sphere. There is a (unique) sphere S_1 passing through A, B, C, P and a (unique) sphere S_2 passing through A', B', C', P.

If S_1 and S_2 are externally tangent at P, then prove that $AA'=BB'=CC'$.

Problem 322. (*Due to Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam*) Let a, b, c be positive real numbers satisfying the condition $a+b+c=3$. Prove that

$$\frac{a^2(b+1)}{a+b+ab}+\frac{b^2(c+1)}{b+c+bc}+\frac{c^2(a+1)}{c+a+ca}\geq 2.$$

Problem 323. Prove that there are infinitely many positive integers n such that 2^n+2 is divisible by n.

Problem 324. $ADPE$ is a convex quadrilateral such that $\angle ADP = \angle AEP$. Extend side AD beyond D to a point B and extend side AE beyond E to a point C so that $\angle DPB = \angle EPC$. Let O_1 be the circumcenter of ΔADE and let O_2 be the circumcenter of ΔABC.

If the circumcircles of ΔADE and ΔABC are not tangent to each other, then prove that line O_1O_2 bisects line segment AP.

Problem 325. On a plane, n distinct lines are drawn. A point on the plane is called a *k-point* if and only if there are exactly k of the n lines passing through the point. Let $k_2, k_3, \ldots, k_n$ be the numbers of 2-points, 3-points, …, n-points on the plane, respectively.

Determine the number of regions the n lines divided the plane into in terms of $n, k_2, k_3, \ldots, k_n$.

(*Source: 1998 Jiangsu Province Math Competition*)

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Solutions

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Problem 316. For every positive integer $n>6$, prove that in every n-sided convex polygon $A_1A_2\ldots A_n$, there exist $i\neq j$ such that

$$|\cos\angle A_i-\cos\angle A_j|<\frac{1}{2(n-6)}.$$

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5).

Note the sum of all angles is

$$(n-2)180^\circ = 6\times120^\circ+(n-6)180^\circ.$$

So there are at most five angles less than 120°. The remaining angles are in $[120^\circ, 180^\circ)$ and their cosines are in $(-1,-1/2]$. Divide $(-1,-1/2]$ into $n-6$ left open, right closed intervals with equal length. By the pigeonhole principle, there exist two of the cosines in the same interval, which has length equal to $1/(2n-12)$. The desired inequality follows.

Problem 317. Find all polynomial $P(x)$ with integer coefficients such that for every positive integer n, 2^n-1 is divisible by $P(n)$.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5).

First we prove a fact: for all integers p and n and all polynomials $P(x)$ with integer coefficients, p divides $P(n+p)-P(n)$. To see this, let $P(x)=a_kx^k+\cdots+a_0$. Then

$$P(n+p)-P(n)=\sum_{i=1}^{k}a_i\left[(n+p)^i-n^i\right]$$

$$=\sum_{i=1}^{k}a_i p\left[\sum_{j=0}^{i-1}(n+p)^j n^{i-1-j}\right].$$

Now we claim that the only polynomials $P(x)$ solving the problem are the constant polynomials 1 and -1.

Assume $P(x)$ is such a polynomial and $P(n)\neq\pm1$ for some integer $n>1$. Let p be a prime which divides $P(n)$, then p divides 2^n-1. So p is odd and $2^n\equiv1 \pmod p$.

By the fact above, p also divides $P(n+p)-P(n)$. Hence, p divides $P(n+p)$. Since $P(n+p)$ divides $2^{n+p}-1$, p also divides $2^{n+p}-1$. Then $2^p\equiv2^n2^p=2^{n+p}\equiv1 \pmod p$.

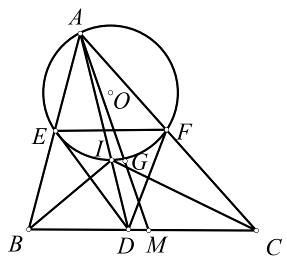
By Fermat's little theorem, $2^p\equiv2 \pmod p$. Thus, $1\equiv2 \pmod p$. This leads to p divides $2-1=1$, which is a contradiction. Hence, $P(n)=1$ or -1 for every integer $n>1$. Then $P(x)-1$ or $P(x)+1$ has infinitely many roots, i.e. $P(x)\equiv1$ or -1.

Comments: Two readers pointed out that this problem appeared earlier as Problem 252 in vol. 11, no. 2.

Problem 318. In ΔABC, side BC has length equal to the average of the two other sides. Draw a circle passing through A and the midpoints of AB, AC. Draw the tangent lines from the centroid of the triangle to the circle. Prove that one of the points of tangency is the incenter of ΔABC.

(*Source: 2000 Chinese Team Training Test*)

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5).



Let G be the centroid and I be the incenter of ΔABC. Let line AI intersect side BC at D. Let E and F be the midpoints of AB and AC respectively. Let O be the circumcenter of ΔAEF. Let M be the midpoint of side BC.

We claim I is the circumcenter of ΔDEF. To see this, note I is on line AD. So

$$\frac{DB}{2EB}=\frac{DB}{AB}=\frac{DI}{AI}=\frac{DC}{AC}=\frac{DC}{2FC}=\lambda.$$

Also, $DB+DC=BC=(AB+AC)/2=EB+FC=2\lambda(DB+DC)$ implies $\lambda=1/2$. Then $DB=EB$ and $DC=FC$. So lines BI and CI are the perpendicular bisectors of DE and DF respectively.

Now we show I is on the circumcircle of ΔAEF. To see this, we compute

$\angle EIF = 2\angle EDB$
$= 2(180^\circ-\angle BDE-\angle CDF)$
$= (180^\circ-2\angle BDE)+(180^\circ-2\angle CDF)$
$= \angle DBE+\angle DCF$
$= 180^\circ-\angle EAF.$

Finally, we show $OI\perp IG$. Since $IE=IF$, $OI\perp EF$. Since $EF\|BC$, we just need to show $IG\|BC$, which follows from $DI/AI=1/2=MG/AG$.

Problem 319. For a positive integer n, let S be the set of all integers m such

that $|m| < 2n$. Prove that whenever $2n+1$ elements are chosen from S, there exist three of them whose sum is 0.
(*Source: 1990 Chinese Team Training Test*)

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery) and **Fai YUNG**.

For $n = 1$, $S = \{-1,0,1\}$. If 3 elements are chosen from S, then they are $-1,0,1$, which have zero sum.

Suppose case n is true. For the case $n+1$, S is the union of $S'=\{m: -2n+1 \le m \le 2n-1\}$ and $S''=\{-2n-1,-2n,2n,2n+1\}$. Let T be a $2n+3$ element subset of S.

Case 1: (T contains at most 2 elements of S''). Then T contains $2n+1$ elements of S'. By case n, T has 3 elements with zero sum.

Case 2: (T contains exactly 3 elements of S''.) There are 4 subcases:

Subcase 1: ($\pm 2n$ and $2n+1$ are in T.) If 0 is in T, then $\pm 2n$ and 0 are in T with zero sum. If -1 is in T, then $2n+1, -2n, -1$ are in T with zero sum.

Otherwise, the other $2n$ numbers of T are among $1, \pm 2, \pm 3, \ldots, \pm(2n-1)$, which (after removing n) can be divided into the $2n-2$ pairs $\{1, 2n-1\}$, $\{2, 2n-2\}, \ldots, \{n-1, n+1\}, \{-2, -2n+1\}, \{-3, -2n+2\}, \ldots, \{-n, -n-1\}$. By the pigeonhole principle, the two numbers in one of the pairs must both be in T. Since the sums for these pairs are either $2n$ or $-2n-1$, we can add the pair to $-2n$ or $2n+1$ to get three numbers in T with zero sum.

Subcase 2: ($2n$ and $\pm(2n+1)$ are in T.) If 0 is in T, then $\pm(2n+1)$ and 0 are in T with zero sum. If 1 is in T, then $-2n-1$, $2n$, 1 are in T with zero sum.

Otherwise, the other $2n$ numbers of T are among $-1, \pm 2, \pm 3, \ldots, \pm(2n-1)$, which (after removing $-n$) can be divided into the $2n-2$ pairs $\{2, 2n-1\}$, $\{3, 2n-2\}, \ldots, \{n, n+1\}, \{-1, -2n+1\}, \{-2, -2n+2\}, \ldots, \{-n+1, -n-1\}$. By the pigeonhole principle, the two numbers in one of the pairs must both be in T. Since the sums for these pairs are either $2n+1$ or $-2n$, we can add the pair to $-2n-1$ or $2n$ to get three numbers in T with zero sum.

Subcase 3: ($\pm 2n$ and $-2n-1$ are in T.) This can be handled as in subcase 1.

Subcase 4: ($-2n$ and $\pm(2n+1)$ are in T.) This can be handled as in subcase 2.

Case 3: (T contains S''.) If 0 is in T, then $-2n$, $2n$, 0 are in T with zero sum. If 1 is in T, then $-2n-1$, $2n$, 1 are in T with zero sum. If -1 is in T, then $2n+1$, $-2n$, -1 are in T with zero sum.

Otherwise, the other $2n-1$ numbers of T are among $\pm 2, \pm 3, \ldots, \pm(2n-1)$, which can be divided into the $2n-2$ pairs $\{2, 2n-1\}$, $\{3, 2n-2\}, \ldots, \{n, n+1\}, \{-2, -2n+1\}, \{-3, -2n+2\}, \ldots, \{-n, -n-1\}$. By the pigeonhole principle, the two numbers in one of the pairs must both be in T. Since the sums for these pairs are either $2n+1$ or $-2n-1$, we can add the pair to $-2n-1$ or $2n+1$ to get three numbers in T with zero sum.

This completes the induction and we are done.

Problem 320. For every positive integer $k > 1$, prove that there exists a positive integer m such that among the rightmost k digits of 2^m in base 10, at least half of them are 9's.
(*Source: 2005 Chinese Team Training Test*)

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5) and **G.R.A. 20 Problem Solving Group** (Roma, Italy).

We claim $m=2\times 5^{k-1}+k$ works. Let $f(k)=2\times 5^{k-1}$. We check by induction that

$$2^{f(k)} \equiv -1 \pmod{5^k}. \qquad (*)$$

First $f(2)=10$, $2^{10}=1024 \equiv -1 \pmod{5^2}$. Next, suppose case k is true. Then $2^{f(k)} = -1 + 5^k n$ for some integer n. We get

$$\begin{aligned} 2^{f(k+1)} &= (-1+5^k n)^5 \\ &= \sum_{j=0}^{5} \binom{5}{j} (-1)^{5-j} 5^{kj} n^j \\ &\equiv -1 \pmod{5^{k+1}}, \end{aligned}$$

completing the induction.

By (*), we get $2^m \equiv -2^k \pmod{5^k}$. Also, clearly $2^m \equiv 0 \equiv -2^k \pmod{2^k}$. Hence,

$$2^m \equiv -2^k \equiv 10^k - 2^k \pmod{10^k}.$$

This implies the k rightmost digits in base 10 of 2^m and $10^k - 2^k$ are the same. For $k > 1$, $2^k < 10^{(k-1)/2}$. So

$$10^k - 1 \ge 10^k - 2^k > 10^k - 10^{(k-1)/2}.$$

The result follows from the fact that the k-digit number $10^k - 10^{(k-1)/2}$ in base 10 has at least half of its digits are 9's on the left.

Olympiad Corner

(continued from page 1)

Problem 3. Let three circles $\Gamma_1, \Gamma_2, \Gamma_3$, which are non-overlapping and mutually external, be given in the plane. For each point P in the plane, outside the three circles, construct six points $A_1, B_1, A_2, B_2, A_3, B_3$ as follows: For each $i = 1,2,3$, A_i, B_i are distinct points on the circle Γ_i such that the lines PA_i and PB_i are both tangents to Γ_i. Call the point P *exceptional* if, from the construction, three lines A_1B_1, A_2B_2, A_3B_3 are concurrent. Show that every exceptional point of the plane, if exists, lies on the same circle.

Problem 4. Prove that for any positive integer k, there exists an arithmetic sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_k}{b_k}$$

of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \ldots, k$, such that the positive integers $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are all distinct.

Problem 5. Larry and Bob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every l kilometer driving from the start; Rob makes a 90° right turn after every r kilometer driving from the start, where l and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction.

Let the car start from Argovia facing towards Zillis. For which choices of the pair (l, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?