Mathematical Excalibur

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Olympiad Corner

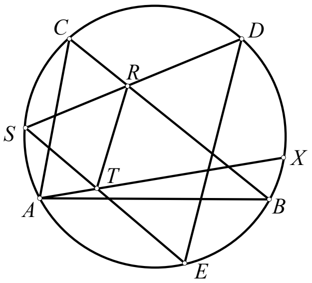
Below are the problems of the 2011 Chinese Math Olympiad, which was held on January 2011.

Problem 1. Let $a_1, a_2, \ldots, a_n$ ($n \ge 3$) be real numbers. Prove that

$$\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i a_{i+1} \le \left[\frac{n}{2}\right](M-m)^2,$$

where $a_{n+1} = a_1,\ M = \max\limits_{1\le i\le n} a_i,\ m = \min\limits_{1\le i\le n} a_i$, $[x]$ denotes the greatest integer not exceeding x.

Problem 2. In the figure, D is the midpoint of the arc BC on the circumcircle Γ of triangle ABC. Point X is on arc BD. E is the midpoint of arc AX. S is a point on arc AC. Lines SD and BC intersect at point R. Lines SE and AX intersect at point T. Prove that if $RT \parallel DE$, then the incenter of triangle ABC is on line RT.



(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 28, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Klamkin's Inequality

Kin Y. Li

In 1971 Professor Murray Klamkin established the following

Theorem. For any real numbers x,y,z, integer n and angles α,β,γ of any triangle, we have

$$x^2+y^2+z^2 \ge (-1)^{n+1}2(yz\cos n\alpha + zx\cos n\beta + xy\cos n\gamma).$$

Equality holds if and only if

$$\frac{x}{\sin n\alpha} = \frac{y}{\sin n\beta} = \frac{z}{\sin n\gamma}.$$

The proof follows immediately from expanding

$$\big(x+(-1)^n(y\cos n\gamma + z\cos n\beta)\big)^2 + (y\sin n\gamma - z\sin n\beta)^2 \ge 0.$$

There are many nice inequalities that we can obtain from this inequality. The following are some examples (see references [1] and [2] for more).

Example 1. For angles α,β,γ of any triangle, if n is an odd integer, then

$$\cos n\alpha + \cos n\beta + \cos n\gamma \le 3/2.$$

If n is an even integer, then

$$\cos n\alpha + \cos n\beta + \cos n\gamma \ge -3/2.$$

(This is just the case $x=y=z=1$.)

Example 2. For angles α,β,γ of any triangle,

$$\sqrt{3}\cos\alpha + 2\cos\beta + 2\sqrt{3}\cos\gamma \le 4.$$

(This is just the case $n = 1$, $x = \sin 90^\circ$, $y = \sin 60^\circ$, $z = \sin 30^\circ$.)

There are many symmetric inequalities in α,β,γ, which can be proved by standard identities or methods. However, if we encounter *asymmetric* inequality like the one in example 2, it may be puzzling in coming up with a proof.

Example 3. Let a,b,c be sides of a triangle with area Δ. If r,s,t are any real numbers, then prove that

$$\left(\frac{ar+bs+ct}{4\Delta}\right)^2 \ge \frac{st}{bc} + \frac{tr}{ca} + \frac{rs}{ab}.$$

Solution. Let α,β,γ be the angles of the triangle. We first observe that

$$4\Delta^2 = a^2b^2\sin^2\gamma = b^2c^2\sin^2\alpha = c^2a^2\sin^2\beta$$

and $\cos 2\theta = 1-2\sin^2\theta$. So we can try to set $n=2$, $x=ar$, $y=bs$, $z=ct$. Indeed, after applying Klamkin's inequality, we get the result.

Example 4. Let a,b,c be sides of a triangle with area Δ. Prove that

$$\left(\frac{a^2+b^2+c^2}{4\Delta}\right)^2 \ge \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

Comment: It may seem that we can use example 3 by setting $r=a$, $s=b$, $t=c$, but unfortunately

$$\frac{st}{bc} + \frac{tr}{ca} + \frac{rs}{ab} = 3 \ge \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}$$

holds only when $a=b=c$ by the AM-GM inequality.

Solution. To solve this one, we bring in the circumradius R of the triangle. We recall that $2\Delta = bc\sin\alpha$ and by extended sine law, $2R = a/(\sin\alpha)$. So $4\Delta R = abc$. Now we set $r=bcx$, $s=cay$ and $t=abz$. Then the inequality in example 3 becomes

$$(x+y+z)^2R^2 \ge yza^2 + zxb^2 + xyc^2. \quad (*)$$

Next, we set $yz=1/b^2$, $zx=1/c^2$, $xy=1/a^2$, from which we can solve for x,y,z to get

$$x = \frac{b}{ac} = \frac{b^2}{4\Delta R},\quad y = \frac{c}{4\Delta R},\quad z = \frac{a}{4\Delta R}.$$

Then (*) becomes

$$\left(\frac{a^2+b^2+c^2}{4\Delta}\right)^2 \ge \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

Example 5. (*1998 Korean Math Olympiad*) Postive real numbers a,b,c satisfy $a+b+c=abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}}\le\frac{3}{2}$$

and determine when equality holds.

Solution. Let $a=\tan u$, $b=\tan v$ and $c=\tan w$, where $u,v,w>0$. As $a+b+c=abc$,

$$\tan u+\tan v+\tan w=\tan u\tan v\tan w,$$

which can be written as

$$-\tan u=\frac{\tan v+\tan w}{1-\tan v\tan w}=\tan(v+w).$$

This implies $u+v+w=n\pi$ for some odd positive integer n. Let $\alpha=u/n$, $\beta=v/n$ and $\gamma=w/n$. Taking $x=y=z=1$ in Klamkin's inequality (as in example 1), we have

$$\cos n\alpha+\cos n\beta+\cos n\gamma\le 3/2,$$

which is the desired inequality. Equality holds if and only if $a=b=c=\sqrt{3}$.

For the next two examples, we will introduce the following

Fact: Three positive real numbers x,y,z satisfy the equation

$$x^2+y^2+z^2+xyz=4 \qquad (**)$$

if and only if there exists an acute triangle with angles α,β,γ such that

$$x=2\cos\alpha,\ y=2\cos\beta,\ z=2\cos\gamma.$$

Proof. If $x,y,z>0$ and $x^2+y^2+z^2+xyz=4$, then x^2, y^2, $z^2<4$. So $0<x,y,z<2$. Hence, there are positive $\alpha,\beta,\gamma<\pi/2$ such that

$$x=2\cos\alpha,\ y=2\cos\beta \text{ and } z=2\cos\gamma.$$

Substituting these into (**) and simplifying, we get $\cos\gamma=-\cos(\alpha+\beta)$, which implies $\alpha+\beta+\gamma=\pi$. We can get the converse by using trigonometric identities.

Example 6. (*1995 IMO Shortlisted Problem*) Let a,b,c be positive real numbers. Determine all positive real numbers x,y,z satisfying the system of equations

$$x+y+z=a+b+c,$$

$$4xyz-(a^2x+b^2y+c^2z)=abc.$$

Solution. We can rewrite the second equation as

$$\left(\frac{a}{\sqrt{yz}}\right)^2+\left(\frac{b}{\sqrt{zx}}\right)^2+\left(\frac{c}{\sqrt{xy}}\right)^2+\frac{abc}{xyz}=4.$$

By the fact, there exists an acute triangle with angles α,β,γ such that

$$\frac{a}{\sqrt{yz}}=2\cos\alpha,\frac{b}{\sqrt{zx}}=2\cos\beta,\frac{c}{\sqrt{xy}}=2\cos\gamma.$$

Then the first equation becomes

$$x+y+z=2(\sqrt{yz}\cos\alpha+\sqrt{zx}\cos\beta+\sqrt{xy}\cos\gamma).$$

This is the equality case of Klamkin's inequality. So

$$\frac{\sqrt{x}}{\sin\alpha}=\frac{\sqrt{y}}{\sin\beta}=\frac{\sqrt{z}}{\sin\gamma}.$$

As $\gamma+\beta=\pi-\alpha$, so $\sin(\gamma+\beta)/\sin\alpha=1$. Then

$$\frac{b}{2x}+\frac{c}{2x}=\frac{\sqrt{z}}{\sqrt{x}}\cos\beta+\frac{\sqrt{y}}{\sqrt{x}}\cos\gamma$$

$$=\frac{\sin\gamma\cos\beta+\sin\beta\cos\gamma}{\sin\alpha}=1.$$

So $x=(b+c)/2$. Similarly, $y=(c+a)/2$ and $z=(a+b)/2$.

Example 7. (*2007 IMO Chinese Team Training Test*) Positive real numbers u,v,w satisfy the equation $u+v+w+\sqrt{uvw}=4$.

Prove that

$$\sqrt{\frac{vw}{u}}+\sqrt{\frac{uw}{v}}+\sqrt{\frac{uv}{w}}\ge u+v+w.$$

Solution. By the fact, there exists an acute triangle with angles α,β,γ such that

$$\sqrt{u}=2\cos\alpha,\sqrt{v}=2\cos\beta,\sqrt{w}=2\cos\gamma.$$

The desired inequality becomes

$$\frac{2\cos\beta\cos\gamma}{\cos\alpha}+\frac{2\cos\gamma\cos\alpha}{\cos\beta}+\frac{2\cos\alpha\cos\beta}{\cos\gamma}$$

$$\ge 4(\cos^2\alpha+\cos^2\beta+\cos^2\gamma).$$

Comparing with Klamkin's inequality, all we have to do is to take $n=1$ and

$$x=\sqrt{\frac{2\cos\beta\cos\gamma}{\cos\alpha}},\qquad y=\sqrt{\frac{2\cos\gamma\cos\alpha}{\cos\beta}}$$

$$z=\sqrt{\frac{2\cos\alpha\cos\beta}{\cos\gamma}}.$$

Example 8. (*1988 IMO Shortlisted Problem*) Let n be an integer greater than 1. For $i=1,2,\ldots,n$, $\alpha_i>0$, $\beta_i>0$ and

$$\sum_{i=1}^n\alpha_i=\sum_{i=1}^n\beta_i=\pi.$$

Prove that $\displaystyle\sum_{i=1}^n\frac{\cos\beta_i}{\sin\alpha_i}\le\sum_{i=1}^n\cot\alpha_i.$

Solution. For $n=2$, we have equality

$$\frac{\cos\beta_1}{\sin\alpha_1}+\frac{\cos\beta_2}{\sin\alpha_2}=\frac{\cos\beta_1}{\sin\alpha_1}-\frac{\cos\beta_1}{\sin\alpha_1}$$

$$=0=\cot\alpha_1+\cot\alpha_2.$$

For $n=3$, α_1, α_2, α_3 are angles of a triangle, say with opposite sides a,b,c. Let Δ be the area of the triangle. Now $2\Delta=bc\sin\alpha_1=ca\sin\alpha_2=ab\sin\alpha_3$. Combining with the cosine law, we get

$$\cot\alpha_1=\frac{\cos\alpha_1}{\sin\alpha_1}=\frac{b^2+c^2-a^2}{4\Delta}$$

and similarly for $\cot\alpha_2$ and $\cot\alpha_3$. By Klamkin's inequality,

$$\sum_{i=1}^n\frac{4\Delta\cos\beta_i}{\sin\alpha_i}=2(bc\cos\beta_1+ca\cos\beta_2+ab\cos\beta_3)$$

$$\le a^2+b^2+c^2=\sum_{i=1}^3 4\Delta\cot\alpha_i.$$

Cancelling 4Δ, we will finish the case $n=3$. For the case $n>3$, suppose the case $n-1$ is true. We have

$$\sum_{i=1}^n\frac{\cos\beta_i}{\sin\alpha_i}=\left[\frac{\cos\beta_1}{\sin\alpha_1}+\frac{\cos\beta_2}{\sin\alpha_2}-\frac{\cos(\beta_1+\beta_2)}{\sin(\alpha_1+\alpha_2)}\right]$$

$$+\left[\sum_{i=3}^n\frac{\cos\beta_i}{\sin\alpha_i}+\frac{\cos(\beta_1+\beta_2)}{\sin(\alpha_1+\alpha_2)}\right]$$

$$=\left[\frac{\cos\beta_1}{\sin\alpha_1}+\frac{\cos\beta_2}{\sin\alpha_2}+\frac{\cos(\pi-(\beta_1+\beta_2))}{\sin(\pi-(\alpha_1+\alpha_2))}\right]$$

$$+\left[\sum_{i=3}^n\frac{\cos\beta_i}{\sin\alpha_i}+\frac{\cos(\beta_1+\beta_2)}{\sin(\alpha_1+\alpha_2)}\right]$$

$$\le\left[\cot\alpha_1+\cot\alpha_2+\cot(\pi-(\alpha_1+\alpha_2))\right]$$

$$+\left[\sum_{i=3}^n\cot\alpha_i+\cot(\alpha_1+\alpha_2)\right]$$

$$=\sum_{i=1}^n\cot\alpha_i.$$

This finishes the induction.

References

[1] M.S.Klamkin, "*Asymetric Triangle Inequalities*," Publ.Elektrotehn. Fak. Ser. Mat. Fiz. Univ. Beograd, No. 357-380 (1971) pp. 33-44.

[2] Zhu Hua-Wei, From Mathematical Competitions to Competition Mathematics, Science Press, 2009 (in Chinese).

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 28, 2011.***

Problem 361. Among all real numbers a and b satisfying the property that the equation $x^4+ax^3+bx^2+ax+1=0$ has a real root, determine the minimum possible value of a^2+b^2 with proof.

Problem 362. Determine all positive rational numbers x,y,z such that

$$x+y+z,\ \ xyz,\ \ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

are integers.

Problem 363. Extend side CB of triangle ABC beyond B to a point D such that $DB=AB$. Let M be the midpoint of side AC. Let the bisector of $\angle ABC$ intersect line DM at P. Prove that $\angle BAP=\angle ACB$.

Problem 364. Eleven robbers own a treasure box. What is the least number of locks they can put on the box so that there is a way to distribute the keys of the locks to the eleven robbers with no five of them can open all the locks, but every six of them can open all the locks? The robbers agree to make enough duplicate keys of the locks for this plan to work.

Problem 365. For nonnegative real numbers a,b,c satisfying $ab+bc+ca=1$, prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{1}{a+b+c}\geq 2.$$

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Solutions

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Problem 356. A and B alternately color points on an initially colorless plane as follow. A plays first. When A takes his turn, he will choose a point not yet colored and paint it red. When B takes his turn, he will choose 2010 points not yet colored and paint them blue. When the plane contains three red points that are the vertices of an equilateral triangle, then A wins. Following the rules of the game, can B stop A from winning?

Solution. **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Anna PUN Ying** (HKU Math) and **The 7B Mathematics Group** (Carmel Alison Lam Foundation Secondary School).

The answer is negative. In the first $2n$ moves, A can color n red points <u>on a line</u>, while B can color $2010n$ blue points. For each pair of the n red points A colored, there are two points (on the perpendicular bisector of the pair) that can be chosen as vertices for making equilateral triangles with the pair. When $n > 2011$, we have

$$2\binom{n}{2}=n(n-1)>2010n.$$

Then B cannot stop A from winning.

Other commended solvers: **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Andy LOO** (St. Paul's Co-ed College),**Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy) and **Lorenzo PASCALI** (Università di Roma "La Sapienza", Roma, Italy), **WONG Sze Nga** (Diocesan Girls' School).

Problem 357. Prove that for every positive integer n, there do not exist four integers a, b, c, d such that $ad=bc$ and $n^2 < a < b < c < d < (n+1)^2$.

Solution. **U. BATZORIG** (National University of Mongolia) and **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

We first prove a useful

<u>*Fact (Four Number Theorem)*</u>: Let a,b,c,d be positive integers with $ad=bc$, then there exists positive integers p,q,r,s such that $a=pq,\ b=qr,\ c=ps,\ d=rs$.

To see this, let $p=\gcd(a,c)$, then $p|a$ and $p|c$. So $q=a/p$ and $s=c/p$ are positive integers. Now $p=\gcd(a,c)$ implies $\gcd(q,s)=1$. From $ad=bc$, we get $qd=sb$. Then $s|d$. So $r=d/s$ is a positive integer and $a=pq,\ b=qr,\ c=ps,\ d=rs$.

For the problem, assume a,b,c,d exist as required. Applying the fact, since $d > b > a$, we get $s>q$ and $r>p$. Then $s\geq q+1$, $r\geq p+1$ and we get

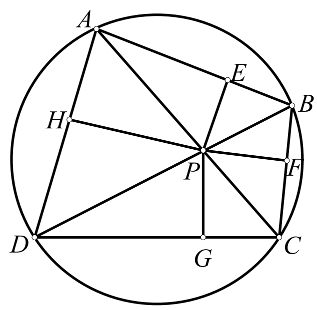
$$d=rs\geq(p+1)(q+1)\geq(\sqrt{pq}+1)^2$$
$$=(\sqrt{a}+1)^2>(n+1)^2,$$

a contradiction.

Other commended solvers: **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Anna PUN Ying** (HKU Math), **The 7B Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **WONG Sze Nga** (Diocesan Girls' School).

Problem 358. $ABCD$ is a cyclic quadrilateral with AC intersects BD at P. Let E, F, G, H be the feet of perpendiculars from P to sides AB, BC, CD, DA respectively. Prove that lines EH, BD, FG are concurrent or are parallel.

Solution. **U. BATZORIG** (National University of Mongolia), **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Abby LEE Shing Chi** (SKH Lam Woo Memorial Secondary School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Anna PUN Ying** (HKU Math), **Anderson TORRES** (São Paulo, Brazil) and **WONG Sze Nga** (Diocesan Girls' School).



Since $ABCD$ is cyclic, $\angle BAC=\angle CDB$ and $\angle ABD=\angle DCA$, which imply ΔAPB and ΔDPC are similar. As E and G are feet of perpendiculars from P to these triangles (and similarity implies the corresponding segments of triangles are proportional), we get $AE/EB=DG/GC$. Similarly, we get $AH/HD=BF/FC$.

If $EH \parallel BD$, then $AE/EB = AH/HD$, which is equivalent to $DG/GC=BF/FC$, and hence $FG \parallel BD$.

Otherwise, lines EH and BD intersect at some point I. By Menelaus theorem and its converse, we have

$$\frac{\overrightarrow{AE}}{\overrightarrow{EB}}\cdot\frac{\overrightarrow{BI}}{\overrightarrow{ID}}\cdot\frac{\overrightarrow{DH}}{\overrightarrow{HA}}=-1,$$

which is equivalent to

$$\frac{\overrightarrow{BI}}{\overrightarrow{ID}} \cdot \frac{\overrightarrow{DG}}{\overrightarrow{GC}} \cdot \frac{\overrightarrow{CF}}{\overrightarrow{FB}} = -1,$$

and lines BD and FG also intersect at I.

Other commended solvers: **Lorenzo PASCALI** (Università di Roma "La Sapienza", Roma, Italy).

Problem 359. (*Due to Michel BATAILLE*) Determine (with proof) all real numbers x,y,z such that $x+y+z \geq 3$ and

$$x^3+y^3+z^3+x^4+y^4+z^4 \leq 2(x^2+y^2+z^2).$$

Solution. **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy) and **Terence ZHU** (Affilated High School of South China Normal University).

Let x,y,z be real numbers satisfying the conditions. For all real w, $w^2+3w+3 \geq (w+3/2)^2$ implies $(w^2+3w+3)(w-1)^2 \geq 0$. Expanding, we get (*) $w^4+w^3-2w^2 \geq 3w-3$. Applying (*) to $w=x,y,z$ and adding, then using the conditions on x,y,z, we get

$$0 \geq x^3+y^3+z^3+x^4+y^4+z^4-2(x^2+y^2+z^2)$$
$$\geq 3(x+y+z)-9 \geq 0.$$

Thus, for such x,y,z, we must have equalities in the (*) inequality for x,y,z. So $x = y = z = 1$ is the only solution.

Comments: For the idea behind this solution, we refer the readers to the article on the tangent line method (see *Math Excalibur*, *vol.* 10, *no.* 5, *page* 1). For those who do not know this method, we provide the

Proposer's solution. Suppose (x,y,z) is a solution. Let $s=x+y+z$ and $S=x^2y+y^2z+z^2x+xy^2+yz^2+zx^2$. By expansion, we have $s(x^2+y^2+z^2)-S= x^3+y^3+z^3$. Hence,

$$s(x^2+y^2+z^2)-S+x^4+y^4+z^4 \leq 2(x^2+y^2+z^2),$$

which is equivalent to

$$(s-2)(x^2+y^2+z^2)+x^4+y^4+z^4 \leq S. \quad (*)$$

Since S is the dot product of the vectors $v=(x^2,y^2,z^2,x,y,z)$ and $w=(y,z,x,y^2,z^2,x^2)$, by the Cauchy Schwarz inequality,

$$S \leq x^2+y^2+z^2+x^4+y^4+z^4. \quad (**)$$

Combining (*) and (**), we conclude $(s-3)(x^2+y^2+z^2) \leq 0$. Since $s \geq 3$, we get $s=3$ and (*) and (**) are equalities. Hence, vectors v and w are scalar multiple of each other. Since x,y,z are not all zeros, simple algebra yields $x=y=z=1$. This is the only solution.

Comments: Some solvers overlooked the possibility that x or y or z may be negative in applying the Cauchy Schwarz inequality!

Other commended solvers: **U. BATZORIG** (National University of Mongolia) and **Shaarvdorj** (11th High School of UB, Mongolia), **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Thien NGUYEN** (Luong The Vinh High School, Dong Nai, Vietnam), **Anna PUN Ying** (HKU Math), **The 7B Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **WONG Sze Nga** (Diocesan Girls' School).

Problem 360. (*Due to Terence ZHU, Affiliated High School of Southern China Normal University*) Let n be a positive integer. We call a set S of at least n distinct positive integers a *n-divisible* set if among every n elements of S, there always exist two of them, one is divisible by the other.

Determine the least integer m (in terms of n) such that every n-divisible set S with m elements contains n integers, one of them is divisible by all the remaining $n-1$ integers.

Solution. **Anna PUN Ying** (HKU Math) and the proposer independently.

The smallest m is $(n-1)^2+1$. First choose distinct prime numbers $p_1, p_2, \ldots, p_{n-1}$. For i from 1 to $n-1$, let

$$A_i = \{p_i, p_i^2, \ldots, p_i^{n-1}\}$$

and let A be any nonempty subset of their union. Then A is n-divisible because among every n of the elements, by the pigeonhole principle, two of them will be in the same A_i, then one is divisible by the other. However, among n elements, two of them will also be in different A_i's and neither one is divisible by the other. So $m \leq (n-1)^2$ will not work.

If $m \geq (n-1)^2+1$ and S is a n-divisible set with m elements, then let k_1 be the largest element in S and let B_1 be the subset of S consisted of all the divisors of k_1 in S. Let k_2 be the largest element in S and not in B_1. Let B_2 be the subset of S consisted of all the divisors of k_2 in S and not in B_1. Repeat this to get a partition of S.

Assume there are at least n of these B_i set. For i from 1 to n, let j_i be the largest element in B_i. However, by the definition of the B_i sets, $\{j_1,j_2,\ldots,j_n\}$ contradicts the n-divisiblity of S. So there are at most $n-1$ B_i's.

Since $m \geq (n-1)^2+1$, one of the B_i must have at least n elements. Then for S, we can choose n elements from <u>this</u> B_i with k_i included so that k_i is divisible by all the remaining $n-1$ integers. Therefore, the least m is $(n-1)^2+1$.

Other commended solvers: **WONG Sze Nga** (Diocesan Girls' School).

Olympiad Corner

(continued from page 1)

Problem 3. Let A be a finite set of real numbers. $A_1,A_2,\ldots,A_n$ are nonempty subsets of A satisfying the following conditions:

(1) the sum of all elements in A is 0;
(2) for every $x_i \in A_i$ ($i=1,2,\ldots,n$), we have $x_1+x_2+\cdots+x_n > 0$.

Prove that there exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that

$$\left|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}\right| < \frac{k}{n}|A|.$$

Here $|X|$ denotes the number of elements in the finite set X.

Problem 4. Let n be a positive integer, set $S = \{1,2,\ldots,n\}$. For nonempty finite sets A and B of real numbers, find the minimum of $|A \,\Delta\, S|+|B \,\Delta\, S|+|C \,\Delta\, S|$, where $C = A+B = \{a+b \mid a \in A, b \in B\}$, $X \,\Delta\, Y = \{x \mid x$ belongs to exactly one of X or $Y\}$, $|X|$ denotes the number of elements in the finite set X.

Problem 5. Let $n \geq 4$ be a given integer. For nonnegative real numbers $a_1,a_2,\ldots,a_n$, $b_1,b_2,\ldots,b_n$ satisfying $a_1+a_2+\cdots+a_n = b_1+b_2+\cdots+b_n > 0$, find the maximum of

$$\frac{\sum_{i=1}^{n} a_i(a_i+b_i)}{\sum_{i=1}^{n} b_i(a_i+b_i)}.$$

Problem 6. Prove that for every given positive integers m,n, there exist infinitely many pairs of coprime positive integers a,b such that

$$a+b \mid am^a+bn^b.$$