Solutions to the 2013 Putnam Exam

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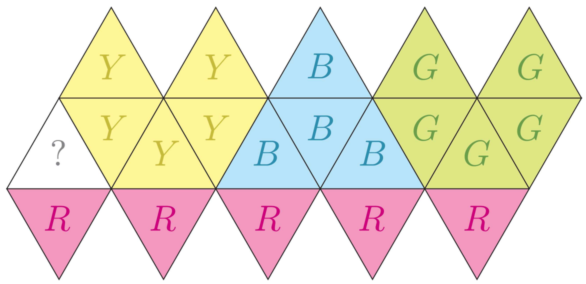
A1 Suppose that faces of the icosahedron shown below (after flattening) are labeled in such a way that no two faces sharing a vertex have the same label. For each integer c, there is at least one face not labeled c; we may suppose the face indicated by '?' is not labeled 'c'. Any two of the faces designated 'Y' share a common vertex, so at most one of these five faces is labeled c. Similarly, the label c appears on at most one of the faces designated 'B', and at most one of the faces labeled 'G', and at most one of the faces labeled 'R'. So every label appears on at most four faces of the icosahedron.



Now the smallest possible sum of labels is achieved when there are four faces with each of the labels 0,1,2,3,4; and so the minimum possible sum of labels on the faces is $4(0+1+2+3+4) = 40$. This is contrary to the hypothesis that the labels on the faces have total value 39.

A2 Suppose that f is not injective, so that $f(n) = f(n') = m$ for two integers $n, n' \in S$ with $n < n'$. There exist finite sets of integers A, A' with $n \in A \subseteq \{n, n+1, \ldots, m\}$ and $n' \in A \subseteq \{n', n'+1, \ldots, m\}$ such that both $\prod A$ and $\prod A'$ are perfect squares. Write $A = B \cup \Delta$ and $A' = B' \cup \Delta$ where $\Delta = A \cap A'$, $B = A \smallsetminus \Delta$, $B' = A' \smallsetminus \Delta$. Since

$$\left[\prod A\right]\left[\prod A'\right] = \left[\prod (B \cup B')\right]\left[\prod \Delta\right]^2$$

is a perfect square, $\prod(B \cup B')$ is also a perfect square. However, $B \cup B'$ has minimum element n so $f(n) \leq \max(B \cup B') < m$, a contradiction.

A3 Suppose that the polynomial defined by $f(y) = a_0 + a_1 y + a_2 y^2 + \cdots + a_n y^n$ has no root in the open interval $(0,1)$. Then by the Intermediate Value Theorem, $f(y)$ has constant sign, either positive or negative, on $(0,1)$. After replacing each a_i by $-a_i$ if necessary, we may further suppose that $f(y) > 0$ whenever $0 < y < 1$. In particular, $f(x^k) > 0$ for every $k \geq 1$, and

$$f(x^0) = f(1) = \lim_{y \to 1^-} f(y) \geq 0.$$

However,

$$\sum_{0\le i\le n} \frac{a_i}{1-x^{i+1}} = \sum_{0\le i\le n}\sum_{k\ge 0} a_k x^{(i+1)k} = \sum_{k\ge 0} x^k f(x^k) > 0$$

since the first term is nonnegative and all remaining terms are positive. This contradicts the hypothesis.

A4 For each arc w, denote the length of w by $L(w)$, so that $L(w) = Z(w) + N(w)$. We assume there are $\ell = z+n$ digits around the circle, consisting of z zeroes and n ones. Given an arc w of length $L = L(w)$ with $0 < L < \ell$, denote by $R^i w$ (for $i = 0, 1, 2, \ldots, \ell-1$) the arcs of the same length as w; their positions are the ℓ cyclic shifts of the position of w (let's say in the counterclockwise direction). If w is either empty or complete, i.e. of length 0 or ℓ respectively, then we may take $R^i w = w$. For every arc w, the ℓ cyclic shifts of w cover each zero $L(w)$ times, so $\sum_{i=0}^{\ell-1} Z(R^i w) = zL(w)$. Similarly, $\sum_{i=0}^{\ell-1} N(R^i w) = nL(w)$. Also for any arc w, we have

$$-\ell < \sum_{i=0}^{\ell-1}\big(Z(R^i w) - Z(w)\big) < \ell$$

since every term in the sum lies in $\{-1, 0, 1\}$, and at least one term (for $i = 0$) actually equals 0. Since $\sum_{i=0}^{\ell-1} Z(R^i w) = zL(w)$, we obtain

$$(*) \qquad -1 \ < \ \frac{z}{\ell}L(w) \ - \ Z(w) \ < \ 1.$$

The same argument applied to w_j for $j = 1, 2, \ldots, k$ yields

$$-1 \ < \ \frac{z}{\ell}L(w_j) \ - \ Z(w_j) \ < \ 1.$$

Averaging over $j \in \{1, 2, \ldots, k\}$ yields

$$-1 \ < \ \frac{z}{k\ell}\sum_{j=1}^{k} L(w_j) \ - \ Z \ < \ 1.$$

Since $\frac{1}{k}\sum_{j=1}^{k} L(w_j) = \frac{1}{k}\sum_{j=1}^{k}\big(Z(w_j) + N(w_j)\big) = Z + N$, this yields

$$-1 \ < \ \frac{z}{\ell}\big(Z + L\big) \ - \ Z \ < \ 1.$$

Henceforth, let w be any arc of length $Z + N$. Rewriting (*) gives

$$(**) \qquad -1 \ < \ \frac{z}{\ell}\big(Z + N\big) \ - \ Z(w) \ < \ 1.$$

Now the open interval $I = \big(\frac{z}{\ell}(Z+N)-1, \frac{z}{\ell}(Z+N)-1\big)$ of length 2 contains both the integers $Z(w)$ and Z. If there exists another arc w' of the same length $Z + N$ but with a different number of zeroes, then $Z(w)$ and $Z(w')$ are the only two integers in the interval I, so either w or w' has exactly Z zeroes, and hence also the same number of ones, and we are done. Otherwise all arcs of length $Z+N$ have a constant number of zeroes. In the latter case, $Z(R^i w) = \frac{z}{\ell}\big(Z+N\big)$ for all i and, since this is the unique integer in the interval I, we must have $Z(w) = Z$.

A5 If $u \in \mathbb{R}^n$ is a random vector uniformly distributed over S^{n-1} (the set of all vectors of Euclidean norm 1), then there exists a positive constant C_n such that for every fixed vector $x \in \mathbb{R}^n$, the expected value of $|x \cdot u|$ is $C_n\|x\|$. For example for $n = 2,3$ we have, respectively,

$$E_{u\in S^1}(|x\cdot u|) = \frac{1}{2\pi}\int_0^{2\pi} \|x\||\cos\theta|\,d\theta = \tfrac{2}{\pi}\|x\|;$$

$$E_{u\in S^2}(|x\cdot u|) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \|x\||\cos\phi|\sin\phi\,d\phi\,d\theta = \tfrac{1}{2}\|x\|$$

so that $C_2 = \frac{2}{\pi}$ and (more directly relevant for our problem) $C_3 = \frac{1}{2}$. Now the key idea is embodied in the following:

(*) Let $\alpha_i \in \mathbb{R}$ and $x_i \in \mathbb{R}^n$ for $i = 1,2,\ldots,k$. If $\sum_{i=1}^k \alpha_i |x_i \cdot u| \geq 0$ for all $u \in \mathbb{R}^n$, then $\sum_{i=1}^k \alpha_i \|x_i\| \geq 0$.

To prove (*), assuming $\sum_{i=1}^k \alpha_i |x_i\cdot u| \geq 0$ for every value of the random vector $u \in S^{n-1}$, with the uniform distribution, taking the expected value yields $C_n\sum_{i=1}^k \alpha_i\|x_i\| \geq 0$ also, so (*) holds.

Now suppose $\{a_{ijk}\}_{i,j,k}$ is any list of area definite numbers for $\mathbb{R}^2$, and let $A_1, A_2, \ldots, A_m$ be points in $\mathbb{R}^3$. For every triple ijk with $1 \leq i < j < k \leq m$, let $v_{ijk} \in \mathbb{R}^3$ be a vector of norm Area$(\Delta A_iA_jA_k)$ with v_{ijk} normal to the plane of $\Delta A_iA_jA_k$; for this purpose it suffices to take $v_{ijk} = \frac{1}{2}\overrightarrow{A_iA_j}\times\overrightarrow{A_iA_k}$. For every unit vector $u \in \mathbb{R}^3$, $|v_{ijk}\cdot u|$ is the area of the projection of $\Delta A_iA_jA_k$ into the plane $u^\perp$; and so by hypothesis, $\sum_{ijk} a_{ijk}|v_{ijk}\cdot u| \geq 0$. By (*), it follows that $\sum_{ijk} a_{ijk}\|v_{ijk}\| \geq 0$, so the list of numbers $\{a_{ijk}\}_{ijk}$ is area definite for $\mathbb{R}^3$.

A6 It suffices to show that $A(S) \geq 6|S|$ for every finite $S \subset \mathbb{Z}\times\mathbb{Z}$. The claim holds when S is empty, since $A(S) = 0$ in that case. We proceed by induction on $|S|$. Because $w(-z) = w(z)$ for all $z \in \mathbb{Z}\times\mathbb{Z}$, we may write $A(S) = 12|S| + 2\sum_{s\in S} f_S(s)$ where $f_S : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ is defined by $f_S(s) = \sum_{s'\in S: s'\neq s} w(s-s')$. Since $A(S\cup\{s'\}) = A(S) + 12 + 2f_S(s')$ whenever $s' \in (\mathbb{Z}\times\mathbb{Z}) \smallsetminus S$, it suffices to show that every nonempty finite set $S \subset \mathbb{Z}\times\mathbb{Z}$ satisfying $A(S) \geq 6|S|$ contains an element s such that $f_S(s) \geq -3$. Suppose, on the contrary, that some finite set $S \subset \mathbb{Z}\times\mathbb{Z}$ satisfies $A(S) \geq 6|S|$ and $f_S(s) \leq -4$ for every $s \in S$. Then

$$A(S) = 12|S| + 2\sum_{s\in S} f_S(s) \;\le\; 12|S| - 8|S| \;=\; 4|S|,$$

a contradiction.

B1
$$\begin{aligned}\sum_{n=1}^{2013} c(n)c(n+2) &= c(1)c(3) + \sum_{m=1}^{1006}\bigl[c(2m)c(2m{+}2) + c(2m{+}1)c(2m{+}3)\bigr]\\ &= 1(-1) + \sum_{m=1}^{1006}\bigl[c(m)c(m{+}1) + (-1)^m c(m)(-1)^{m+1}c(m{+}1)\bigr]\\ &= -1 + \sum_{m=1}^{1006} 0 \;=\; -1.\end{aligned}$$

B2 For every real value of x and integer n not divisible by 3, we have

$$\cos 2\pi nx + \cos 2\pi n(x{+}\tfrac{1}{3}) + \cos 2\pi n(x{+}\tfrac{2}{3}) = \Re\bigl(e^{2\pi nxi}(1+\omega^n+\omega^{2n})\bigr) = \Re(0) = 0$$

where $\omega = e^{2\pi i/3}$. From this we obtain $f(0) + f\bigl(\frac{1}{3}\bigr) + f\bigl(\frac{2}{3}\bigr) = 3$, so $f(0) \le 3$. This upper bound is attained by the function in C_2 given by

$$f(x) = 1 + \tfrac{4}{3}\cos(2\pi x) + \tfrac{2}{3}\cos(4\pi x) = \tfrac{1}{3}\bigl(2\cos(2\pi x) - 1\bigr)^2 \ge 0.$$

B3 Yes, f must have the indicated form. Given $i \in \bigcup P$, there is a unique minimal member of P containing i (since if $S, S' \in P$ both contain i, then so does $S \cap S' \in P$); and we denote this minimal member by $S_i \in P$. Recall that there exists $T \in P$ such that $T \subset S_i$ and $|T| = |S_i| - 1$; and by minimality of S_i, we must have $S_i \smallsetminus \{i\} = T \in P$. So we may define

$$f_i = f(S_i) - f(S_i \smallsetminus \{i\}).$$

For any $i \in [n]$ with $i \notin \bigcup P$, the values f_i may be chosen arbitrarily (for example, $f_i = 0$). Now for each $S \in P$, define

$$g(S) = \sum_{i\in S} f_i\,.$$

We will show that $g(S) = f(S)$ for all $S \in P$, proceeding by induction on $|S|$. Note that $g(\varnothing) = f(\varnothing) = 0$ as required. Now for any nonempty member $S \in P$, let $T \in P$ such that $|T| = |S| - 1$. We may assume that $f(T) = g(T)$. There is a unique $i \in [n]$ such that $S = \{i\} \cup T$. We consider two cases. If $\{i\} \in P$, then $S_i = \{i\}$ and

$$f(S) = f(\{i\}) + f(T) - f(\{i\}\cap T) = F(S_i) + g(T) - f(\varnothing) = f_i + g(T) - 0 = g(S)$$

as required. Otherwise $\{i\} \notin P$ and

$$\begin{aligned} f(S) &= f(T \cup S_i) = f(T) + f(S_i) - f(T \cap S_i) = g(T) + f(S_i) - f(S_i \setminus \{i\}) \\ &= g(T) + f_i = g(S). \end{aligned}$$

The result $g(S) = f(S)$ follows by induction on $|S|$.

B4 We will use the following double integral formulation of $\mathrm{Var}(f)$:

$$\begin{aligned} 2\,\mathrm{Var}(f) &= 2\int_0^1 \big(f(t) - \mu_f\big)^2 dt \;=\; 2\int_0^1 f(t)^2 dt - 4\mu_f \int_0^1 f(t)\,dt + 2\mu_f^2 \\ &= 2\int_0^1 f(t)^2 dt - 2\mu_f^2 \;=\; \int_0^1 f(s)^2 ds + \int_0^1 f(t)^2 dt - 2\int_0^1 f(s)\,ds \int_0^1 f(t)\,dt \\ &= \int_0^1 \int_0^1 \big(f(s) - f(t)\big)^2 ds\,dt. \end{aligned}$$

Suppose first that $|f(t)|, |g(t)| \le 1$ for all $t \in [0,1]$; we prove the required inequality first in this special case. By Cauchy-Schwarz we have

$$\begin{aligned} \big[f(s)g(s) - f(t)g(t)\big]^2 &= \big[g(t)\big(f(s) - f(t)\big) + f(s)\big(g(s) - g(t)\big)\big]^2 \\ &\le \big[g(t)^2 + f(s)^2\big]\big[\big(f(s) - f(t)\big)^2 + \big(g(s) - g(t)\big)^2\big] \\ &\le 2\big(f(s) - f(t)\big)^2 + 2\big(g(s) - g(t)\big)^2. \end{aligned}$$

Integrating over $0 \le s, t \le 1$, our double integral formulation for variance yields

$$\mathrm{Var}(fg) \le 2\,\mathrm{Var}(f) + 2\,\mathrm{Var}(g),$$

still just in the special case $|f|, |g| \le 1$. In the general case we have $f(t) = M(f)\widetilde{f}(t)$, $g(t) = M(g)\widetilde{g}(t)$ where the continuous functions $\widetilde{f}, \widetilde{g} : [0,1] \to \mathbb{R}$ satisfy $|\widetilde{f}|, |\widetilde{g}| \le 1$; then

$$\begin{aligned} \mathrm{Var}(fg) &= M(fg)^2\,\mathrm{Var}\big(\widetilde{f}\widetilde{g}\big) \;\le\; M(f)^2 M(g)^2\,\mathrm{Var}\big(\widetilde{f}\widetilde{g}\big) \\ &\le 2M(f)^2\,\mathrm{Var}\big(\widetilde{f}\big) M(g)^2 + 2M(g)^2\,\mathrm{Var}\big(\widetilde{g}\big) M(f)^2 \\ &= 2\,\mathrm{Var}(f) M(g)^2 + 2\,\mathrm{Var}(g) M(f)^2. \end{aligned}$$

B5 Our proof uses the following encoding of permutations as strings. Let S be any nonempty set of symbols. Every permutation σ of S may be expressed first as a product of $\ell \ge 1$ disjoint cycles as

$$\sigma = (s_{1,1}, s_{1,2}, \cdots, s_{1,t_1})(s_{2,1}, s_{2,2}, \cdots, s_{2,t_2}) \cdots (s_{\ell,1}, s_{\ell,2}, \cdots, s_{\ell,t_\ell})$$

where $s_{i,j} \in S$ and $t_i \geq 1$ is the length of the i-th cycle. This cycle decomposition is unique if we require that (i) the least element of the i-th cycle is its last term s_{i,t_i} (as we may assume, after appropriately cycling terms in the i-th cycle); and (ii) the ℓ cycles are permuted in such a way that $s_{1,t_1} < s_{2,t_2} < \cdots < s_{\ell,t_\ell}$. Every such permutation σ, written in standard form, is uniquely determined by the sequence formed by stripping away all but the outermost parentheses:

$$s_\sigma = (s_{1,1}, s_{1,2}, \cdots, s_{1,t_1},\ s_{2,1}, s_{2,2}, \cdots, s_{2,t_2},\ \cdots,\ s_{\ell,1}, s_{\ell,2}, \cdots, s_{\ell,t_\ell}).$$

Our proof uses the fact that the correspondence $\sigma \leftrightarrow s_\sigma$ is bijective. The details of this argument are omitted; but the argument is very similar to Bóna [1, Lemma 6.15], who uses a slightly different canonical cycle form for expressing permutations.

Our main argument is a modification of Joyal's proof of Cayley's Theorem as found in Bóna [1, p.219]. For $1 \leq k \leq n$, denote by $\mathcal{T}_{n,k}$ the collection of all trees with vertex set $[n] = \{1, 2, 3, \ldots, n\}$, having one vertex in $[n]$ marked as 'Start', and one vertex in $[k]$ marked as 'Finish'. (The vertices marked as Start and Finish may coincide.) By Cayley's Theorem, the number of labeled trees with vertex set $[n]$ is n^{n-2}; and for each such tree there are n choices for Start and k choices for Finish, so $|\mathcal{T}_{n,k}| = kn^{n-1}$. We will exhibit a one-to-one correspondence between $\mathcal{T}_{n,k}$ and the collection (which we denote by $\mathcal{F}_{n,k}$) of all functions $f : [n] \to [n]$ such that for all $i \in [n]$, there exists $j \geq 0$ such that $f^{(j)}(i) \leq k$.

Given $f \in \mathcal{F}_{n,k}$, let $S \subseteq [n]$ be the (nonempty) set of all elements permuted in cycles, and let σ be the permutation induced on S by f. Construct a labeled tree $T = T(f) \in \mathcal{T}_{n,k}$ as follows: the elements of S form a path in $T(f)$ with vertices listed in order as given by the sequence $s_\sigma = (s_{1,1}, \ldots, s_{\ell,t_\ell})$, with the terminal vertices $s_{1,1}$ and s_{ℓ,t_ℓ} designated as the Start and Finish respectively. Since $f \in \mathcal{F}_{n,k}$, the least elements in the orbits of σ satisfy

$$s_{1,t_1} < s_{2,t_2} < \cdots < s_{\ell,t_\ell} \leq k$$

so that the vertex marked as Finish lies in $[k]$ as required. To complete the tree $T(f)$, add an edge from vertex i to vertex $f(i)$ for each $i \in [n]$ with $i \notin S$. The correspondence $\mathcal{F}_{n,k} \leftrightarrow \mathcal{T}_{n,k}$ given by $f \leftrightarrow T(f)$ is bijective. The essential details of this argument are as described in Bóna [1, Theorem 10.7]; our construction of $T(f)$ differs from his only in the particular choice of ordering of vertices on the distinguished path from Start to Finish.

B6 The case $n = 1$ is trivial, with a guaranteed win for Alice. Henceforth we assume that $n \geq 3$. It is convenient to represent all possible positions of play by bitstrings of length n, i.e. sequences $w \in \{0,1\}^n$ of 0's and 1's representing unoccupied and occupied spaces respectively. The *weight* of any bitstring w, denoted $\mathrm{wt}(w) \in \{0, 1, 2, \ldots, n\}$, is the number of 1's (i.e. stones). Denote by $W_{n,k}$ the set of bitstrings of length n and weight k.

Play begins with the initial sequence 0^n; thereafter the weight changes by -1, 0 or 1 at every turn. The first player to reach a position in $W_{n,n-1}$ loses, because then all remaining play positions form a set $W_{n,n-1} \cup W_{n,n}$ of even cardinality $n+1$, and any position in this set $W_{n,1} \cup W_{n,0}$ is attainable from any other in a single move.

We will show that Alice has the advantage iff on her first turn she places a stone in any space other than one of the two end spaces ('terminal' spaces). Thereafter her winning strategy is as follows. We assume it is Bob's turn to play. By induction, we may assume that the position has odd weight $\ell \in \{1, 3, 5, \dots, n-2\}$ and that the terminal spaces are either both occupied or both unoccupied.

(i) If Bob places a stone in a non-terminal space, then so does Alice.

(ii) If Bob places a stone in a terminal space, then Alice places a stone in the other terminal space.

(iii) If Bob removes a stone from one terminal space (and inserts a stone in an interior space), then Alice removes the stone in the other terminal space (and also inserts a stone in an interior space.

Each of the cases (i)–(iii) describes a consecutive pair of turns which preserves the symmetry of the terminal spaces (i.e. either both occupied or both unoccupied). Cases (i) and (ii) add 2 to the weight, whereas case (iii) leaves the weight unchanged; but if case (iii) occurs, then the next two turns will be described by (i) or (ii), thus incrementing the weight by 2. This strategy means that the weight is always odd after Alice's turn, so that the first move to attain a weight of $n-1$ is by Bob, who therefore must lose.

If instead Alice places a stone in a terminal space on her first move, then Bob responds by removing that stone and placing a stone in the adjacent space. This way Bob leaves an odd weight position on his first turn. Now the roles of Alice and Bob are reversed, and Bob uses the strategy described above to ensure that he leaves positions of weight $1, 3, 5, \dots, n-2$ after his turns, and that Alice is the first to attain weight $n-1$, so that Alice loses.

Reference

[1] M. Bóna, *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory,* 3rd ed., World Scientific, Singapore, 2011.