Putnam 2014

–	A
1	Prove that every nonzero coefficient of the Taylor series of $(1-x+x^2)e^x$ about $x=0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.
2	Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is $$\frac{1}{\min(i,j)}$$ for $1 \le i,j \le n$. Compute $\det(A)$.
3	Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \ge 1$. Compute $$\prod_{k=0}^{\infty}\left(1-\frac{1}{a_k}\right)$$ in closed form.
4	Suppose X is a random variable that takes on only nonnegative integer values, with $E[X]=1$, $E[X^2]=2$, and $E[X^3]=5$. (Here $E[Y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X=0$.
5	Let $P_n(x) = 1+2x+3x^2+\cdots+nx^{n-1}$. Prove that the polynomials $P_j(x)$ and $P_k(x)$ are relatively prime for all positive integers j and k with $j \ne k$.
6	Let n be a positive integer. What is the largest k for which there exist $n \times n$ matrices $M_1,\dots,M_k$ and $N_1,\dots,N_k$ with real entries such that for all i and j, the matrix product M_iN_j has a zero entry somewhere on its diagonal if and only if $i \ne j$?
–	B
1	A *base* 10 *over-expansion* of a positive integer N is an expression of the form $N = d_k10^k + d_{k-1}10^{k-1} + \cdots + d_0 10^0$ with $d_k \ne 0$ and $d_i \in \{0,1,2,\dots,10\}$ for all i. For instance, the integer $N=10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

2 Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?

3 Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m+n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

4 Show that for each positive integer n, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

5 In the 75th Annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:

(1) A player cannot choose an element that has been chosen by either player on any previous turn.

(2) A player can only choose an element that commutes with all previously chosen elements.

(3) A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy?

6 Let $f : [0,1] \to \mathbb{R}$ be a function for which there exists a constant $K > 0$ such that $|f(x) - f(y)| \leq K|x-y|$ for all $x, y \in [0,1]$. Suppose also that for each rational number $r \in [0,1]$, there exist integers a and b such that $f(r) = a + br$. Prove that there exist finitely many intervals $I_1, \ldots, I_n$ such that f is a linear function on each I_i and $[0,1] = \bigcup_{i=1}^n I_i$.