Putnam Exam 2014 Solutions

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Abstract

These are my solutions to the 2014 Putnam Competition. This is not a list of all solutions, merely the ones that I handed in worthy of credit. Some proofs are more rigorous than others, and it should be noted that for problem A4, while I'm confident my answer is correct, I did not rigorously prove that $P(X \geq 4) = 0$, and merely assumed that that was the case based on intuitive reasoning. Thus, the solutions I present here are not guaranteed to be worthy of full credit, but they should provide valuable insights on how to approach some of these problems.

Problem A1. Prove that every nonzero coefficient of the Taylor series expansion of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

Proof. Let $f(x) = e^x(x^2 - x + 1)$. Then the Taylor series expansion for $f(x)$ about $x = 0$ is

$$T(x) = \sum_{i=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\begin{aligned}
f(x) =& e^x(x^2 - x + 1) \\
f'(x) =& e^x(x^2 + x + 0) \\
f''(x) =& e^x(x^2 + 3x + 1) \\
&\vdots \\
f^{(n)}(x) =& e^x(x^2 + (2n - 1)x + (n - 1)^2)
\end{aligned}$$

The general formula $f^{(n)}(x)$ is can be proved with induction.

$$\begin{aligned}
f^{(n+1)}(x) &= e^x(x^2 + (2n - 1)x + (n - 1)^2) + e^x(2x + (2n - 1) \\
f^{(n+1)}(x) &= e^x(x^2 + (2n + 1)x + n^2)
\end{aligned}$$

Thus,

$$f^{(n)}(0) = e^0(0^2 + (2n-1)(0) + (n-1)^2) = (n-1)^2$$

Therefore the coefficient n^{th} term in $T(x)$ is

$$\frac{(n-1)^2}{n!} = \frac{(n-1)}{n(n-2)!}$$

For $n = 0$, the coefficient is 1, and for $n = 1$, the coefficient is 0. Thereafter, the coefficient is either 1 or prime.

Proof by Cases:

1. $(n-1)$ is prime (we are done)
2. $(n-1) = p^2$ for some prime p

   Since $(n-1) = p^2$, $p < n-1$ so p is a term in $(n-2)!$ that can get canceled out. This leaves a single p on the numerator, so we are done.
3. $(n-1)$ is composite

   Since $(n-1)$ is composite, we can write it as $(n-1) = pq$ where $p \neq q$ and $p, q < n-1$. Clearly, both p and q are factors of $(n-2)!$, so they can both be canceled out leaving 1 on the numerator.

Thus, in every case the numerator of the reduced fraction is either 1 or prime. Therefore, every non-zero coefficient of $T(x)$ is either 1 or prime and we are done.

$\square$

Problem A2. Let A be the $n \times n$ matrix whose entry in the $i-th$ row and $j-th$ column is

$$\frac{1}{min(i,j)}$$

for $1 \leq i, j \leq n$. Compute $det(A)$.

Proof.

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} \end{bmatrix}$$

Now, we can compute the determinant about the n^{th} row. Notice that the sub matrix without the n^{th} row is of the following form

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \frac{1}{2} & \dots & \frac{1}{n-1} & \frac{1}{n-1} \end{bmatrix}$$

The n^{th} and $(n-1)^{th}$ columns are the same, indicating that the matrix is not full rank. Thus, for any sub matrix containing both the n^{th} and $(n-1)^{th}$ columns, the determinant is 0. Thus, we only need to consider the sub matrices that don't contain one of those columns. Further, notice that if n is even, our coefficients will be $[-+-...-+]$ and if n is odd, our coefficients will be $[+-+...-+]$. In either case, the determinant is going to be

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} \end{vmatrix} = \frac{-1}{n-1} \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & \frac{1}{2} & \dots & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{2} & \dots & \frac{1}{n-1} \end{vmatrix} + \frac{1}{n} \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & \frac{1}{2} & \dots & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{2} & \dots & \frac{1}{n-1} \end{vmatrix}$$

Let $D_n = det(A)$ for an $n \times n$ matrix A.

$$D_n = \left(\frac{1}{n} - \frac{1}{n-1}\right) D_{n-1}$$
$$D_n = \frac{-1}{n(n-1)} D_{n-1}$$

Further, we have the initial condition $D_1 = 1$ because $det(\begin{bmatrix}1\end{bmatrix}) = 1$.

$$D_n = \frac{-1}{n(n-1)} \frac{-1}{(n-1)(n-2)} \frac{-1}{(n-2)(n-3)} \cdots \frac{-1}{3 \cdot 2} \frac{-1}{2 \cdot 1}$$

$$\boxed{D_n = \frac{(-1)^{n+1}}{n[(n-1)!]^2} = \frac{(-1)^{n+1}}{n!(n-1)!}}$$

□

Problem A4. Suppose X is a random variable that takes on only non-negative integer values, with $E[X] = 1$, $E[X^2] = 2$, $E[X^3] = 5$. (Here $E[Y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X = 0$.

Proof. Notice that for large k, we want $P(X = k) = 0$ because if it were nonzero, then $P(X = 0)$ would have to increase to overcompensate. Thus, I will assume $P(X \geq 4) = 0$. There are 4 unknown variables, $P(X = 0), \dots, P(X = 3)$, and 4 constraints: the expected values must match and the probabilities must add to 1. Thus, I can setup and solve a 4×4 linear system.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \end{bmatrix}$$

where $P_k = P(X = k)$. Using Guassian elimination, we can easily solve this system:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 9 & 2 \\ 0 & 1 & 8 & 27 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 6 & 1 \\ 0 & 0 & 6 & 24 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 6 & 1 \\ 0 & 0 & 0 & 6 & 1 \end{array}\right]$$

Using back substitution, we get

$$P(X=3) = \frac{1}{6}$$
$$P(X=2) = \frac{1-6(\frac{1}{6})}{2} = 0$$
$$P(X=1) = 1 - 3\frac{1}{6} - 2(0) = \frac{1}{2}$$
$$P(X=0) = 1 - \frac{1}{6} - 0 - \frac{1}{2} = \frac{1}{3}$$

Thus, the minimum possible value for $P(X=0)$ is

$$\boxed{P(X=0) = \frac{1}{3}}$$

$\square$

Problem B1. A *base 10 over-expansion* of a positive integer N is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + ... + d_0 10^0$$

with $d_k \neq 0$ and $d_i \in \{0, 1, 2, ..., 10\}$ for all i. For instance, the integer N = 10 has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expression $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

Proof. **A positive integer N has a unique base 10 over-expansion if and only if $d_i \neq 0$ for all i in the usual base 10 expansion of N.**

To show this, I will break up the proof into two parts:

1. If N has a unique base 10 over-expansion, $d_i \neq 0$ for all i in the usual base 10 expansion of N.
2. If $d_i \neq 0$ for all i in the usual base 10 expansion of N, then N has a unique base 10 over-expansion.

Part 1: If N has a unique base 10 over-expansion, $d_i \neq 0$ for all i in the usual base 10 expansion of N.
Alternatively, I will show that if there is some i such that $d_i = 0$, then N does not have a unique base 10 over-expansion.
Assume there is some $d_i = 0$ in the usual base 10 expansion of N, and let d_j be the leftmost $d_i = 0$ (the $d_i = 0$ where i is maximized.) I can construct a base 10 over-expansion for N by letting $d'_j = 10$ and $d'_{j+1} = d_{j+1} - 1$. This new expression for N is a base 10 over-expansion, which means N has at least 2 base 10 over-expansions (this one and the usual one). Thus, N does not have a unique base 10 over-expansion.

Part 2: If $d_i \neq 0$ for all i in the usual base 10 expansion of N, then N has a unique base 10 over-expansion.

Alternatively, I will show that if N does not have a unique base 10 over-expansion, there exists some i such that $d_i = 0$.

If N does not have a unique base 10 over-expansion, then N has at least 2 representations in the base 10 over-expansion (the usual base 10 expansion, and at least one additional over-expansion). In the additional over-expansion, there is at least one $d_i = 10$, because if there wasn't, then it would be the usual expansion, not an over-expansion. Let d_j be the rightmost $d_i = 10$ (that is, the $d_i = 10$ where i is minimized). Then, to get to the usual base 10 expansion, we need to let $d'_i = 0$ and $d'_{i+1} = d'_{i+1} + 1$. Since it's possible for d'_{i+1} to equal 10 or 11, we need to propagate this transformation from right to left (small i to large i) until the new expression is in the usual representation. The final result will be the usual base 10 expression for N, and $d'_i = 0$, so the claim is true.

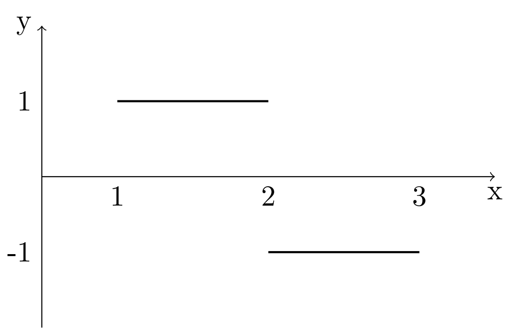
Thus, a positive integer N has a unique base 10 over-expansion if and only if $d_i \neq 0$ for all i in the usual base 10 expansion of N.

$\square$

Problem B2. Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}$ be?

Proof. The largest possible value for $\int_1^3 \frac{f(x)}{x}$ is $ln(4/3)$. The function corresponding to this value is

$$f(x) = \begin{cases} 1 & : 1 \leq x \leq 2 \\ -1 & : 2 < x \leq 3 \end{cases}$$



Clearly, this function satisfies the constraint $\int_1^3 f(x)dx = 0$ because the area of the two disjoint regions cancel each other out.

By symmetry, since we want the areas to cancel out, we should have $f(x) = -f(4-x)$ for all $x \in [1,2]$. If this were not the case, then either condition 1 wouldn't be satisfied, or the value of the definite integral wouldn't be optimal. Thus, let $y = f(x)$ denote the *optimal* value for y. We want to maximize the following function with respect to y (possibly in terms of x indicating that the optimal curve is non-constant).

$$g(y) = \frac{y}{x} - \frac{y}{4-x}$$
$$g'(y) = \frac{1}{x} - \frac{1}{4-x} = \frac{2(2-x)}{x(4-x)}$$

Thus, the derivative is constant (in terms of y), which indicates that there is no local extrema. Thus, we must check the endpoints for the possible values of y (1 and -1). Clearly, $g(1) > g(-1)$ under the constraint $x \in [1,2]$, so this function is maximized when $y = 1$. Thus, the $f(x)$ we defined above is optimal. Let's evaluate the integral

$$\int_1^3 \frac{f(x)}{x} = \int_1^2 \frac{1}{x} + \int_2^3 \frac{-1}{x}$$

$$[ln(2) - ln(1)] - [ln(3) - ln(2)]$$

$$\boxed{2ln(2) - ln(3) = ln(4/3)}$$

$\square$