The 9th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 24th March 1999
Category I

Problem 1 *Find the limit*

$$\lim_{n\to\infty}(\mathrm{e}^{\frac{1999}{n}}-1)\ln\left(\prod_{k=1}^{n}\left(\frac{k}{k+n}\right)\right).$$

Solution

$$\begin{aligned}\lim_{n\to\infty}(\mathrm{e}^{\frac{1999}{n}}-1)\ln\left(\prod_{k=1}^{n}\left(\frac{k}{k+n}\right)\right) &= \lim_{n\to\infty}\frac{(\mathrm{e}^{\frac{1999}{n}}-1)}{\frac{1999}{n}}\frac{1999}{n}\ln\left(\prod_{k=1}^{n}\left(\frac{k}{k+n}\right)\right)\\ &= 1999\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{k+n}\right)\\ &= 1999\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{1}{1+\frac{n}{k}}\right)\\ &= 1999\int_0^1\ln\left(\frac{x}{x+1}\right)\mathrm{d}x = -3998\ln 2\end{aligned}$$

$\square$

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Problem 2 *Find all natural numbers $n \geq 1$ such that the following implication holds*

$$(a, b \text{ - natural, } 11 \mid a^n + b^n) \Rightarrow (11 \mid a \text{ and } 11 \mid b)\,.$$

Solution Observe that for odd natural number n we have $11 \mid 10^1 + 1^1$ and

$$10^{2k+1} + 1 = 9 \cdot 11 \cdot 10^{2k-1} + 10^{2k-1} + 1$$

for any natural number k. Hence our assertion follows easily by an induction argument. Consequently, the required implication is false odd natural numbers.

We prove it for all even neutral numbers. If $n = 2m$, then $a^n = (a^m)^2$ and $b^n = (b^m)^2$ are squares of natural numbers. Hence $a^m = 11e + f$ and $b^m = 11g + h$ for some natural numbers e, g and $f, h \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Therefore standard computation gives

$$a^n = 11A + C \quad \text{and} \quad b^n = 11B + D,$$

for some natural numbers A, B and $C, D \in \{0, 1, 3, 4, 5, 9\}$. By simple calculation one can check that $11 \mid C + D$ if and only if $C = D = 0$. This implies that $11 \mid a^n$ and $11 \mid b^n$. This yields the assertion. $\square$

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Problem 3 *Suppose that we have countable set A of balls and a unit cube in $\mathbb{R}^3$. Let us also assume that for every finite set B, which is a subset of A, it is possible to put all the balls from B into the cube in such a way that they have disjoint interiors. Show that it is possible to arrange all the balls in the cube that all have pairwise disjoint interiors.*

Solution

$\square$

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Problem 4 *Show that for complex numbers x, y the following implication folows:*
$x+y, x^2+y^3, x^3+y^3, x^4+y^4$ *are integers, then for all natural n the numbers x^n+y^n are also integers.*

Solution Notice that $(x+y)^2-(x^2+y^2)=2xy\in\mathbb{Z}$, moreover $-(x^4+y^4)+(x^2+y^2)^2=2x^2y^2\in\mathbb{Z}$.

So it follows that xy is of the shape $\frac{n}{2}$ for n an integer. From the second relation mentioned above we infer that $\frac{n^2}{2}$ is integer. Hence n is even and xy is integer.

So we arrive at $x+y, xy\in\mathbb{Z}$. The rest of the solution is by induction. Namely for the $n<5$ the validity is granted. So, assume that for some natural $k>4$ the numbers x^m+y^m are integers for all $m<k$. Now we consider x^k+y^k. If k is even then $x^k+y^k=(x^{\frac{k}{2}}+y^{\frac{k}{2}})^2-2(xy)^{\frac{k}{2}}$ is integer. Otherwise it k is odd then it is divisible by a prime p, and

$$x^k+y^k=(x^{\frac{k}{p}})^p+(y^{\frac{k}{p}})^p=(x^{\frac{k}{p}}+y^{\frac{k}{p}})(x^{kp(p-1)}-x^{kp(p-2)}y^{\frac{k}{p}}+\dots)=\dots$$

This ends the proof. $\square$

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Category II

Problem 1 *Find the minimal k such that every set of k different lines in $\mathbb{R}^3$ contains either 3 mutually parallel lines or 3 mutually intersecting lines or 3 mutually skew lines.*

Solution

1. Let us show that $k > 8$:
   Let $ABCDA'B'C'D'$ be a cube and let K, L, M, N be the centres of the edges $A'B', B'C', C'D', D'A'$, respectively. The lines $AB, BC, CD, DA, KL, LM, MN$ and NK form an 8-tuple which does not contain any triple either of parallel or intersecting or skew lines.

2. Let us show that $k \leq 9$:

   Lemma *If we have 5 different lines among which no 2 are parallel then choosing any other line causes that we will get 3 intersecting lines or 3 skew lines.*

   Proof Let us consider the graph $G = (V, E)$ and its colouring $f\colon E \to \{1, 2, 3\}$ having the following properties:
   $V = \{\text{the chosen lines}\}, E = \{[p_1, p_2]; p_i \in V\}$ and

$$\begin{aligned} f(p_1, p_2) &= 1 \quad \text{if } p_1 \text{ is parallel to } p_2 \\ f(p_1, p_2) &= 2 \quad \text{if } p_1 \text{ is intersecting to } p_2 \\ f(p_1, p_2) &= 3 \quad \text{if } p_1 \text{ is skew to } p_2 \end{aligned}$$

   If we have the 5-vertex complete graph coloured by two colours (2 and 3) then either it already contains a single-colour triangle (and we have 3 intersecting or 3 skew lines) or our graph is isomorphic to the following one:

   In this case, adding of a 6-th vertex causes that a triangle of colour 2 and 3 will appear. □

   Parallelness is a transitive property, i.e. if $f(p_1, p_2) = f(p_1, p_3) = 1$ then 3 parallel lines exist. Let us now consider a 9-vertex graph with the above mentioned colouring. Then it is clear that there is at least one edge of colour 1 in each 5-tuple. Since the maximal number of edges of colour 1 is four, the proof is finished.

□

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Problem 2 *Let $a, b \in \mathbb{R}, a \leq b$. Assume that $f\colon [a,b] \to [a,b]$ satisfies*

$$|f(x) - f(y)| \leq |x - y|$$

for every $x, y \in [a,b]$. Choose an $x_1 \in [a,b]$ and define

$$x_{n+1} = \frac{x_n + f(x_n)}{2}, \quad n = 1, 2, 3, \dots .$$

Show that $\{x_n\}_{n=1}^{\infty}$ converges to some fixed point of f.

Solution Let

$$W = \{w \in [a,b] : x_{n_k} \to w \text{ for some subsequence } \{n_k\}\} .$$

It is clear that W is nonempty, compact subset of $[a,b]$. Let $g\colon W \to \mathbb{R}^+$ be defined by

$$g(w) = |w - f(w)| .$$

First we show that $e = \inf\limits_{w \in W} g(w) = 0$. If not, let

$$W_1 = \{w \in W : g(w) = e\} .$$

Since W is compact and g is continuous, $W_1 \neq \emptyset$. Set

$$A = \{w \in W_1 : e = w - f(w)\}$$

and $B = W_1 \setminus A$. Suppose $A \neq \emptyset$. Let $w_A = \min A$. Put $w_1 = \frac{w_A + f(w_A)}{2}$. It is clear that $w_1 \in W$. Observe that

$$\begin{aligned} g(w_1) &\leq |f(w_1) - f(w_A)| + |f(w_A) - w_1| \leq \\ &\leq |w_1 - w_A| + \left|f(w_A) - \frac{w_A + f(w_A)}{2}\right| = w_A - f(w_A) = e . \end{aligned}$$

Consequently, $w_1 \in W$ and since $w_1 < w_A$, $w_1 \in B$. But then $f(w_A) < w_1 < w_A < f(w_1)$. Since f satisfied the Lipschitz condition, this leads to a contradiction. If $B \neq \emptyset$, taking $x_B = \max B$ and reasoning in the same manner, we get a contradiction.

Consequently $e = 0$. Take any $w \in W_1$. Then

$$|x_{n+1} - w| = |x_{n+1} - \frac{w + f(w)}{2}| \leq |x_n - w| .$$

Hence $x_n \to w$ and w is obviously a fixed point of f. $\square$

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Problem 3 *Suppose that we have a countable set A of balls and a unit cube in $\mathbb{R}^3$. Assume that for every finite subset B of A it is possible to put all balls of B into the cube in such a way that they have disjoint interiors. Show that it is possible to arrange all the balls in the cube so that all of them have pairwise disjoint interiors.*

Solution We number the balls (as the set A is countable) as $P_1, P_2, \dots$. It is possible for every n to arrange the balls $P_1, \dots, P_n$ in the cube in such a way that they have disjoint interiors. Let the $p_{1,n}, \dots, p_{n,n}$ be the centres of the balls $P_1, \dots, P_n$ in the mentioned arrangement. So we have following sequence:

$$p_{1,1} p_{1,2} p_{2,2} p_{1,3} p_{2,3} p_{3,3} \cdots\cdots\cdots$$

Now as the points $p_{1,n}$ lie in a compact set there is a subsequence $p_{1,n_1}, p_{1,n_2}, \dots$ convergent to some point R_1. Then we choose the subsequence $p_{2,n_{j_1}}, p_{2,n_{j_2}}, \dots$ converging to a point R_2, and so forth. In this way we get a sequence of points $R_1, R_2, \dots$. If we put the ball P_i in a position with the centre in R_i then this arrangement satisfies the requirements concerning the interiors. $\square$

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Problem 4 *Let $u_1, u_2, \ldots, u_n \in C([0,1]^n)$ be nonnegative and continuous functions, and let u_j do not depend on the j-th variable for $j = 1, \ldots, n$. Show that*

$$\left(\int_{[0,1]^n} \prod_{j=1}^{n} u_j\right)^{n-1} \le \prod_{j=1}^{n} \int_{[0,1]^n} u_j^{n-1}.$$

Solution

$$\left(\int_0^1\int_0^1\int_0^1 u_1(y,z)u_2(z,x)u_3(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^2 =$$

$$\left(\int_0^1\int_0^1 u_1(y,z)\left(\int_0^1 u_2(z,x)u_3(x,y)\right)\mathrm{d}y\,\mathrm{d}z\right)^2 \le$$

$$\left(\int_0^1\int_0^1 u_1^2(y,z)\,\mathrm{d}y\,\mathrm{d}z\right)\left(\int_0^1\int_0^1\left(\int_0^1 u_2(z,x)u_3(x,y)\right)^2\mathrm{d}y\,\mathrm{d}z\right) \le$$

$$\left(\int_0^1\int_0^1 u_1^2(y,z)\,\mathrm{d}y\,\mathrm{d}z\right)\left(\int_0^1\int_0^1\left(\int_0^1 u_2^2(z,x)\,\mathrm{d}x\right)\left(\int_0^1 u_3^2(x,y)\,\mathrm{d}x\right)\mathrm{d}y\,\mathrm{d}z\right) =$$

$$\left(\int_0^1\int_0^1 u_1^2(y,z)\,\mathrm{d}y\,\mathrm{d}z\right)\left(\int_0^1\int_0^1\int_0^1 u_2^2(z,x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\left(\int_0^1\int_0^1\int_0^1 u_3^2(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right) =$$

$$\left(\int_0^1\int_0^1 u_1^2(y,z)\,\mathrm{d}y\,\mathrm{d}z\right)\left(\int_0^1\int_0^1 u_2^2(z,x)\,\mathrm{d}z\,\mathrm{d}y\right)\left(\int_0^1\int_0^1 u_3^2(x,y)\,\mathrm{d}x\,\mathrm{d}y\right).$$

Both inequalities use the Hölder inequality for $p = 2$. Then we use Fubini's theorem (which is elementary for continuous functions on a compact interval). When passing to the last line, we use the fact that an integral over the interval $[0,1]$ with integrand not depending on the integration variable can be omitted. □