The 15$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 6$^{\text{th}}$ April 2005
Category I

Problem 1 *Let* $S_0 = \{z \in \mathbb{C} : |z| = 1, z \neq -1\}$ *and* $f(z) = \frac{\operatorname{Im} z}{1+\operatorname{Re} z}$. *Prove that* f *is a bijection between* S_0 *and* $\mathbb{R}$. *Find* f^{-1}.

Solution Using $z = e^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t)$, we can interpret our function as

$$f(t) = \frac{\sin(t)}{1+\cos(t)},\ t \in (-\pi, \pi).$$

We can find easily (using L'Hospital or trigonometrical identities) $\lim f(\pi_-) = \infty$ and $\lim f(\pi_+) = -\infty$, and by the continuity the surjectivity follows. The injectivity can be deduced by

$$f'(t) = \frac{1}{1+\cos(t)} > 0.$$

Since $f(0) = 0$, f maps $(0, \pi)$ to $\mathbb{R}^+$ and $(-\pi, 0)$ to $\mathbb{R}^-$. For $t > 0$, $\sin t > 0$ and $y = f(t) > 0$. Then $0 < y = f(t) = \sqrt{\frac{1-\cos t}{1+\cos t}}$. We get $\cos t = \frac{1-y^2}{1+y^2}$ and finally $t = \arccos(-1 + \frac{2}{1+y^2})$, and similarly for $t < 0$ and $y < 0$: $t = -\arccos(-1 + \frac{2}{1+y^2})$. $\square$

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Problem 2 *Let $f\colon A^3 \to A$, where A is a nonempty set and f satisfies:*

1. *for all $x, y \in A$, $f(x, y, y) = f(y, y, x) = x$ and*
2. *for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$,*

$$f\big(f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)\big) = \\ = f\big(f(x_1, y_1, z_1), f(x_2, y_2, z_2), f(x_3, y_3, z_3)\big).$$

Prove that for an arbitrary, fixed $a \in A$, the operation $x + y = f(x, a, y)$ is an Abelian group addition.

Solution Neutral element:

$$a + x = f(a, a, x) = x = f(x, a, a) = x + a.$$

Associativity:

$$(x + y) + z = f(f(x, a, y), a, z) = f(f(x, a, y), f(a, a, a), f(a, a, z)) = \\ = f(f(x, a, a), f(a, a, a), f(y, a, z)) = f(x, a, f(y, a, z)) = x + (y + z).$$

Inverse element: Define $-x$ to be $f(a, x, a)$. Now,

$$x + (-x) = f(x, a, f(a, x, a)) = f(f(x, a, a), f(a, a, a), f(a, x, a)) = \\ = f(f(x, a, a), f(a, a, x), f(a, a, a)) = f(x, x, a) = a.$$

Commutativity:

$$x + y = f(x, a, y) = f(f(a, a, x), f(a, a, a), f(y, a, a)) = \\ = f(f(a, a, y), f(a, a, a), f(a, a, x)) = f(y, a, x) = y + x.$$

□

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Problem 3 *Find all reals λ for which there exists a non-zero polynomial P with real coefficients such that*

$$\frac{P(1)+P(3)+P(5)+\cdots+P(2n-1)}{n}=\lambda P(n)$$

for all positive integers n. Find all such polynomials for $\lambda=2$.

Solution Let P be a polynomial satisfying the given equation. Then, for all positive integers n,

$$P(2n+1)=\lambda\big((n+1)P(n+1)-nP(n)\big)$$

Hence $P(2x+1)-\lambda\big((x+1)P(x+1)-xP(x)\big)$ is a polynomial with infinitely many zeros; therefore it must be the zero polynomial. Conversely, if P is a polynomial satisfying

$$P(2x+1)-\lambda\big((x+1)P(x+1)-xP(x)\big)=0, \tag{1}$$

we get (by putting $x=0$) $P(1)=\lambda P(1)$ and then by induction

$$P(1)+P(3)+\cdots+P(2n-1)=\lambda nP(n)$$

for all positive integers n. We can therefore equally well consider (1).

The set of all real λ for which there is a non-trivial solution of (1) is given by the set

$$\left\{\frac{2^k}{k+1}\right\}_k=\left\{1,\frac{4}{3},2,\dots\right\}$$

where k runs through all non-negative integers. To prove this, let the pair (P,λ) be a solution for (1), and write $P(x)=ax^k+Q(x)$ where $a\neq 0$ and Q is a polynomial of degree less than k. Then the polynomial $S(x)=(x+1)Q(x+1)-xQ(x)$ is of degree less than k as well, and hence

$$\begin{aligned}(x+1)P(x+1)-xP(x) &= a\big((x+1)^{k+1}-x^{k+1}\big)+S(x)\\ &= a(k+1)x^k+T(x)\end{aligned}$$

for some polynomial T of degree less than k. On the other hand, the degree-k-coefficient of $P(2x+1)$ is 2^ka, and therefore we conclude from (1)

$$2^ka=\lambda a(k+1)$$

which is the same as $\lambda=\frac{2^k}{k+1}$. Conversely, let λ be of the form $\frac{2^k}{k+1}$ for some non-negative integer k, and write P_n for the $(n+1)$-dimensional vector space of all real polynomials of degree at most n. By the computations above, we have a linear map

$$\begin{array}{ccl} P_k & \longrightarrow & P_{k-1}\\ P & \mapsto & P(2x+1)-\lambda\big((x+1)P(x+1)-xP(x)\big)\end{array}$$

Since the dimension of P_{k-1} is less than the dimension of P_k, this map must have non-trivial kernel. Hence there is some non-zero polynomial P satisfying (1).

Now let $\lambda=2$. If k is the degree of P, we have seen that $\frac{2^k}{k+1}=\lambda=2$, but this is only possible for $k=3$. Putting $x=1$ in (1) we get $P(1)=2P(1)$, so $P(1)=0$. For $x=-1$ we get

$$P(-1)-2\big(-P(-1)\big)=0,$$

which is $P(-1)=0$. Hence $P(x)=(x-1)(x+1)(ax+b)$ for some reals a,b. From $P(1)+P(3)=4P(2)$ we get $P(3)=4P(2)$, hence

$$2\cdot 4\cdot(3a+b)=4\cdot 3\cdot(2a+b),$$

that is, $b=0$. Thus we got $P(x)=a(x^3-x)$. The set of polynomials we are looking for form a non-trivial vector space contained in the one-dimensional vector space spanned by x^3-x. Therefore these are exactly the solutions of (1) for $\lambda=2$, and we are done. $\square$

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Problem 4 *Let $(x_n)_{n\geq 2}$ be a sequence of real numbers such that $x_2 > 0$ and $x_{n+1} = -1 + \sqrt[n]{1 + nx_n}$ for $n \geq 2$. Find*

1. $\lim\limits_{n\to\infty} x_n$;
2. $\lim\limits_{n\to\infty} nx_n$.

Solution

1. It follows by induction that the sequence $(x_n)_{n\geq 2}$ is correctly defined and that $x_n > 0$ for $n \geq 2$. From the Bernoulli inequality follows

$$1 + nx_n = (1 + x_{n+1})^n \geq 1 + nx_{n+1},$$

i.e. this sequence decrease (i.e. it is convergent). If $\lim\limits_{n\to\infty} x_n = x \geq 0$, then

$$1 + nx_n = (1 + x_{n+1})^n \geq (1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2} \cdot x^2,$$

i.e. $0 \leq x^2 \leq \frac{2(x_n - x)}{n-1}$. Because $\lim\limits_{n\to\infty}(x_n - x) = 0$ and $\lim\limits_{n\to\infty}(n-1) = \infty$, we get $x = 0$, i.e. $\lim\limits_{n\to\infty} x_n = 0$.

2. From the Stolz theorem, because $\left(\frac{1}{x_n}\right)_{n\geq 2}$ increase and $\lim\limits_{n\to\infty} \frac{1}{x_n} = \infty$ (after 1), we get

$$\limsup_{n\to\infty} nx_n \leq \limsup_{n\to\infty} \frac{1}{\frac{1}{x_n} - \frac{1}{x_{n+1}}} = \limsup_{n\to\infty} \frac{x_n x_{n+1}}{x_n - x_{n+1}}.$$

It follows

$$1 + nx_n = (1 + x_{n+1})^n \geq 1 + nx_{n+1} + \frac{n(n-1)}{2} \cdot x_{n+1}^2,$$

i.e. $x_n - x_{n+1} \geq \frac{n-1}{2} \cdot x_{n+1}^2$ and $\limsup\limits_{n\to\infty} nx_n \leq \limsup\limits_{n\to\infty} \frac{nx_n}{\frac{n(n-1)}{2} \cdot x_{n+1}}$. From the equality $\ln(1 + x_{n+1}) = \frac{\ln(1+nx_n)}{n}$ for $n \geq 2$ and inequality $x \leq (1 + x)\ln(1 + x)$ for $x \geq 0$ follows

$$\begin{aligned} 0 \leq \limsup_{n\to\infty} nx_n &\leq \limsup_{n\to\infty} \frac{(1 + nx_n)\ln(1 + nx_n)}{\frac{n(n-1)}{2} \cdot x_{n+1}} = \\ &= 2 \cdot \limsup_{n\to\infty} \left(\frac{1 + nx_n}{n - 1} \cdot \frac{\ln(1 + x_{n+1})}{x_{n+1}}\right) = 0\,. \end{aligned}$$

It follows $0 \leq \limsup\limits_{n\to\infty} nx_n \leq \limsup\limits_{n\to\infty} nx_n \leq 0$, i.e. $\lim\limits_{n\to\infty} nx_n = 0$.

3. From 1 and 2 and $\lim\limits_{t\to 0} \frac{\ln t}{t} = 1$ follows

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{x_{n+1}}{\ln(1 + x_{n+1})} \cdot \frac{\ln(1 + nx_n)}{nx_n} = 1.$$

□

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Problem 1 *For an arbitrary square matrix M, define*

$$\exp(M) = I + \frac{M}{1!} + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots .$$

Construct 2×2 matrices A and B such that $\exp(A+B) \neq \exp(A)\exp(B)$.

Solution Note that if A and B commute then obviously $\exp(A+B) = \exp(A)\exp(B)$. So A and B should not commute.

Let $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then $A^2 = B^2 = 0$,

$$\exp(A) = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \exp(B) = I + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$
$$\exp(A)\exp(B) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

On the other hand,

$$(A+B)^k = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & k \quad \text{is even} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & k \quad \text{is odd} \end{cases}$$

so

$$\exp(A+B) = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{(2k)!} & \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \\ \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} & \sum_{k=0}^{\infty} \frac{1}{(2k)!} \end{pmatrix} = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}.$$

$\square$

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Problem 2 *Let $(a_{i,j})_{i,j=1}^n$ be a real matrix such that $a_{i,i}=0$ for $i=1,\,2,\,\dots,\,n$. Prove that there exists a set $\mathcal{J}\subset\{1,2,\dots,n\}$ of indices such that*

$$\sum_{\substack{i\in\mathcal{J}\\ j\notin\mathcal{J}}} a_{i,j} + \sum_{\substack{i\notin\mathcal{J}\\ j\in\mathcal{J}}} a_{i,j} \ge \frac{1}{2}\sum_{i,j=1}^n a_{i,j}.$$

Solution For every $\mathcal{I}\subset\{1,2,\dots,n\}$ let $F(\mathcal{I})$ be defined as follows:

$$F(\mathcal{I}) = \sum_{i\in\mathcal{I},j\notin\mathcal{I}} a_{i,j} + \sum_{i\notin\mathcal{I},j\in\mathcal{I}} a_{i,j}.$$

Let $\mathcal{J}\subset\{1,2,\dots,n\}$ satisfy the condition $F(\mathcal{J})=\max\{F(\mathcal{I}):\mathcal{I}\subset\{1,2,\dots,n\}\}$. We aim to prove, that $F(\mathcal{J})\ge\frac{1}{2}\sum_{i,j=1}^n a_{i,j}$.

For $k\in\mathcal{J}$ we have

$$0\le F(\mathcal{J})-F(\mathcal{J}\setminus\{k\}) = \sum_{j\notin\mathcal{J}} a_{k,j} + \sum_{i\notin\mathcal{J}} a_{i,k} - \sum_{j\in\mathcal{J}} a_{k,j} - \sum_{i\in\mathcal{J}} a_{i,k}.$$

Summing the above inequalities with $k\in\mathcal{J}$ we obtain

$$0\le F(\mathcal{J}) - 2\sum_{\substack{i\in\mathcal{J}\\ j\in\mathcal{J}}} a_{i,j} \quad \text{hence} \quad \sum_{\substack{i\in\mathcal{J}\\ j\in\mathcal{J}}} a_{i,j} \le \frac{1}{2}F(\mathcal{J}).$$

Similarily, for $k\notin\mathcal{J}$ we have

$$0\le F(\mathcal{J})-F(\mathcal{J}\cup\{k\}) = \sum_{j\in\mathcal{J}} a_{k,j} + \sum_{i\in\mathcal{J}} a_{i,k} - \sum_{j\notin\mathcal{J}} a_{k,j} - \sum_{i\notin\mathcal{J}} a_{i,k}.$$

Summing the above inequalities with $k\notin\mathcal{J}$ we obtain

$$0\le F(\mathcal{J}) - 2\sum_{\substack{i\notin\mathcal{J}\\ j\notin\mathcal{J}}} a_{i,j} \quad \text{hence} \quad \sum_{\substack{i\notin\mathcal{J}\\ j\notin\mathcal{J}}} a_{i,j} \le \frac{1}{2}F(\mathcal{J}).$$

Finally, we have:

$$\begin{aligned}\frac{1}{2}\sum_{i,j=1}^n a_{i,j} &= \frac{1}{2}\Big(\sum_{\substack{i\in\mathcal{J}\\ j\in\mathcal{J}}} a_{i,j} + \sum_{\substack{i\notin\mathcal{J}\\ j\notin\mathcal{J}}} a_{i,j} + F(\mathcal{J})\Big) \le \\ &\le \frac{1}{2}\Big(\frac{1}{2}F(\mathcal{J}) + \frac{1}{2}F(\mathcal{J}) + F(\mathcal{J})\Big) = F(\mathcal{J}).\end{aligned}$$

$\square$

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Problem 3 *Let $f\colon [0,1]\times[0,1]\to\mathbb{R}$ be a continuous function. Find the limit*

$$\lim_{n\to\infty}\left(\frac{(2n+1)!}{(n!)^2}\right)^2\int_0^1\int_0^1\big(xy(1-x)(1-y)\big)^n f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Solution Answer: $f\big(\frac{1}{2},\frac{1}{2}\big)$.

Proof: Set

$$L_n(f)=\Big(\frac{(2n+1)!}{(n!)^2}\Big)^2\iint\limits_Q\big(xy(1-x)(1-y)\big)^n f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Step 1. $\lim\limits_{n\to\infty}L_n(x^ky^l)=\frac{1}{2^{k+l}}=\big(\frac{1}{2}\big)^k\big(\frac{1}{2}\big)^l$. This is a straightforward calculation:

$$\iint\limits_Q\big(xy(1-x)(1-y)\big)^n x^ky^l\,\mathrm{d}x\,\mathrm{d}y=\int_0^1 x^{n+k}(1-x)^n\,\mathrm{d}x\int_0^1 y^{n+l}(1-y)^n\,\mathrm{d}y=$$

$$\text{(integrate by parts)}\qquad=\frac{(n+k)!n!}{(2n+k+1)!}\cdot\frac{(n+l)!n!}{(2n+l+1)!}\,.$$

Thus

$$\lim_{n\to\infty}L_n(x^ky^l)=\lim_{n\to\infty}\frac{(n+1)(n+2)\dots(n+k)}{(2n+2)(2n+3)\dots(2n+k+1)}\cdot$$
$$\cdot\frac{(n+1)(n+2)\dots(n+l)}{(2n+2)(2n+3)\dots(2n+l+1)}=\frac{1}{2^{k+l}}\,.$$

Step 2. The desired result is satisfied for every polynomial $P(x,y)$. Indeed, the limit and L_n are linear operators.

Step 3. Fix an arbitrary $\varepsilon>0$. A polynomial $P(x,y)$ can be chosen such that $|f(x,y)-P(x,y)|<\varepsilon$ for every $(x,y)\in Q$. Then

$$|L_n(f)-L_n(P)|\le L_n(|f-P|)<L_n(\varepsilon\cdot\mathbb{I})=\varepsilon\,,$$

where $\mathbb{I}(x,y)=1$, for every $(x,y)\in Q$.

According to step 2 there exists n_0 such that $\big|L_n(P)-P\big(\frac{1}{2},\frac{1}{2}\big)\big|<\varepsilon$ for $n\ge n_0$. For these integers

$$\big|L_n(f)-f\Big(\frac{1}{2},\frac{1}{2}\Big)\big|\le|L_n(f)-L_n(P)|+\big|L_n(P)-P\Big(\frac{1}{2},\frac{1}{2}\Big)\big|+\big|f\Big(\frac{1}{2},\frac{1}{2}\Big)-P\Big(\frac{1}{2},\frac{1}{2}\Big)\big|<3\varepsilon\,,$$

which concludes the proof. $\square$

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Problem 4 *Let R be a finite ring with the folowing property: for any $a, b \in R$, there exists an element $c \in R$ (depending on a and b) such that $a^2 + b^2 = c^2$. Prove that for any $a, b, c \in R$, there exists an element $d \in R$ such that $2abc = d^2$.*

(Here $2abc$ denotes $abc + abc$. The ring R is assumed to be associative, but not necessarily commutative and not necessarily containing a unit.)

Solution Let us denote $S = \{x^2 : x \in R\}$. The property of R can be rewritten as $S + S \subseteq S$. For each $y \in S$ the function $S \to S$, $x \mapsto x + y$ is injective, but since S is finite it is indeed bijective. Therefore, S is also closed under subtraction, so S is an additive subgoup of R.

Now for any $x, y \in R$ we have $xy + yx = (x + y)^2 - x^2 - y^2$, so

$$xy + yx \in S\,.$$

We take arbitrary $a, b, c \in R$ and substitute:

$$x = a, y = bc \Rightarrow abc + bca \in S \tag{1}$$
$$x = c, y = ab \Rightarrow cab + abc \in S \tag{2}$$
$$x = ca, y = b \Rightarrow cab + bca \in S \tag{3}$$

If we add (1), (2) and subtract (3), we shall obtain $2abc \in S$. $\square$