The $22^{\text{nd}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, $30^{\text{th}}$ March 2012
Category II

**Problem 1** *Let $f\colon [1,\infty) \to (0,\infty)$ be a non-increasing function such that*

$$\limsup_{n\to\infty} \frac{f(2^{n+1})}{f(2^n)} < \frac{1}{2}\,.$$

*Prove that*

$$\int_1^\infty f(x)\,\mathrm{d}x < \infty\,.$$

**Solution** Since

$$\limsup_{n\to\infty} \frac{2^{n+1} f(2^{n+1})}{2^n f(2^n)} < 1,$$

then by ratio test we obtain that the series

$$\sum_{n=1}^{\infty} 2^n f(2^n)$$

converges. Using Cauchy condensation test we obtain that

$$\sum_{n=1}^{\infty} f(n)$$

converges. Now, by integral test for convergence we have

$$\int_1^\infty f(x)\,\mathrm{d}x < \infty.$$

$\square$

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**Problem 2** *Let $M$ be the (tridiagonal) $10 \times 10$ matrix*

$$M = \begin{pmatrix} -1 & 3 & 0 & \cdots & \cdots & \cdots & 0 \\ 3 & 2 & -1 & 0 & & & \vdots \\ 0 & -1 & 2 & -1 & \ddots & & \vdots \\ \vdots & 0 & -1 & 2 & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

*Show that $M$ has exactly nine positive real eigenvalues (counted with multiplicities).*

**Solution** Let $x^T = (0, x_1, ..., x_9)$. Then the direct calculation shows that

$$x^T M x = x_1^2 + (x_2 - x_1)^2 + \cdots + (x_9 - x_8)^2 + x_9^2. \tag{1}$$

Let $\lambda_{\min} := \min\{\lambda \mid \lambda \in \sigma(M)\}$ (recall that if a matrix $M$ is symmetric then $\sigma(M) \subset \mathbb{R}$). Moreover, since $M$ is symmetric, there exists an orthogonal matrix $C$ such that $C^T MC = \operatorname{diag}\{\lambda_{\min}, \lambda_1 ..., \lambda_9\}$. Hence we infer that $y^T(\lambda_{\min} I - M)y \le 0$ for $y \in \mathbb{R}^{10}$. Let $y^T = (1, -1, 0, ..., 0)$. Then $2\lambda_{\min} \le y^T My = -5$. Thus $\lambda_{\min} < 0$.
Let $V_1 = \{(0, x_1, ..., x_9) \mid x_i \in \mathbb{R}\} \subset \mathbb{R}^{10}$. Then, in view of (1), we have

$$y^T M y \ge 0 \tag{2}$$

for any $y \in V_1$ and $y^T My = 0$ if and only if $y = 0$.

Suppose on the contrary that $M$ admits at least two nonpositive eigenvalues $\lambda_1, \lambda_2 \in \sigma(M)$. Consequently, there exist $y_1, y_2 \in \mathbb{R}^{10}$ such that $y_1 \perp y_2$, $y_1^T y_1 = y_2^T y_2 = 1$ and $My_i = \lambda_i y_i$ $(i = 1, 2)$. Put $V_2 := \operatorname{span}\{y_1, y_2\}$. Then for any $y = \alpha_1 y_1 + \alpha_2 y_2 \in V_2$ one has

$$y^T M y = \alpha_1^2 \cdot \lambda_1 + \alpha_2^2 \cdot \lambda_2 \le 0. \tag{3}$$

Finally, we obtain that

$$\dim V_1 + \dim V_2 = 9 + 2 = 11 > 10.$$

Therefore $V_1 \cap V_2 \ne \{0\}$. Take $0 \ne y \in V_1 \cap V_2$. Then, in view of (2), $y^T My > 0$. But (3) implies that $y^T My \le 0$ – a contradiction. $\square$

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**Problem 3** *Let $(A, +, \cdot)$ be a ring with unity, having the following property: for all $x \in A$ either $x^2 = 1$ or $x^n = 0$ for some $n \in \mathbb{N}$. Show that $A$ is a commutative ring.*

**Solution** Denote by $U(A)$ the multiplicative group of units of the ring $A$ ($U(A) = \{x \mid x \text{ is invertible}\}$). Note first that $(U(A), \cdot)$ is commutative, because if $x, y \in U(A)$, $(xy)^2 = 1 \Rightarrow xy \cdot xy = 1$, and multiplying by $x$ to the left and by $y$ to the right and using also the fact that $x^2 = 1 = y^2$, we get that

$$xy = yx. \tag{1}$$

We now show that if

$$x \notin U(A) \text{ then } 1 - x \in U(A).$$

Assume, by contradiction, that

$$\exists x \notin U(A) \text{ so } y = 1 - x \notin U(A). \tag{2}$$

By hypothesis,

$$\exists n \text{ and } m \in \mathbb{N} \text{ so } x^n = 0;\ y^m = 0$$

and as

$$xy = x(1-x) = x - x^2 = (1-x)x = yx$$

we get that

$$(x+y)^{n+m} = \sum_{i+j=n+m} C_{n+m}^i x^i y^j = 0,$$

Note that whenever $i + j = n + m$ we have

$$i \geq n \text{ or } j \geq m \text{ and so } x^i = 0 \text{ or } y^j = 0;$$

So

$$1 = x + y \notin U(A),$$

which is a contradiction; thus (2) is proved.

Commutativity in $A$ follows now from (1) and (2) with a case by case analysis: $x, y \in A$,

1. if $x \in U(A)$, $y \in U(A)$ then $(1) \Rightarrow xy = yx$;
2. if $x \in U(A)$, $y \notin U(A)$ then $(2) \Rightarrow 1 - y \in U(A)$ and from (1) we have $x(1-y) = (1-y)x \Rightarrow xy = yx$;
3. if $x \notin U(A)$, $y \in U(A)$ analogous to the case 2 and
4. if $x \notin U(A)$, $y \notin U(A)$ then $(2) \Rightarrow 1 - x,\ 1 - y \in U(A)$

and using

$$(1-x)(1-y) = (1-y)(1-x) \Leftrightarrow 1 - x - y + xy = 1 - y - x + xy \Leftrightarrow xy = yx.$$

Now cases $1 \to 4$ above show that $A$ is a commutative ring. $\square$

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**Problem 4** *Let $a, b, c, x, y, z, t$ be positive real numbers with $1 \le x, y, z \le 4$. Prove that*

$$\frac{x}{(2a)^t} + \frac{y}{(2b)^t} + \frac{z}{(2c)^t} \ge \frac{y+z-x}{(b+c)^t} + \frac{z+x-y}{(c+a)^t} + \frac{x+y-z}{(a+b)^t}.$$

**Solution** We will use the following variant of Schur's inequality.
**Lemma 1** *For arbitrary $A, B, C > 0$,*

$$x(A-B)(A-C) + y(B-A)(B-C) + z(C-A)(C-B) \ge 0.$$

**Proof** Without loss of generality we can assume $A \le B \le C$. Let $U = B - A$ and $V = C - B$. Then

$$LHS = xU(U+V) - yUV + z(U+V)V \ge U(U+V) - 4UV + (U+V)V = (U-V)^2 \ge 0.$$

□

**Lemma 2** *For every $p > 0$,*

$$\frac{1}{p^k} = \frac{1}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-pt} \mathrm{d}t.$$

**Proof** Substituting $u = pt$,

$$\int_0^\infty t^{k-1} e^{-pt}\,\mathrm{d}t = \frac{1}{p^k} \int_0^\infty u^{k-1} e^{-u}\,\mathrm{d}u = \frac{\Gamma(k)}{p^k}.$$

□

Now, applying Lemma 1 to $A = e^{-at}$, $B = e^{-bt}$, and $C = e^{-ct}$, the statement can be proved as

$$0 \le \int_0^\infty t^{k-1} \Big( x(e^{-at} - e^{-bt})(e^{-at} - e^{-ct}) + y(e^{-bt} - e^{-at})(e^{-bt} - e^{-ct}) + z(e^{-ct} - e^{-at})(e^{-ct} - e^{-bt}) \Big)\,\mathrm{d}t$$
$$= \int_0^\infty t^{k-1} \Big( xe^{-2at} + ye^{-2bt} + ze^{-2ct} - (y+z-x)e^{-(b+c)t} - (z+x-y)e^{-(c+a)t} - (x+y-x)e^{-(a+b)t} \Big)\,\mathrm{d}t$$
$$= \Gamma(k) \left( \frac{x}{(2a)^k} + \frac{y}{(2b)^k} + \frac{z}{(2c)^k} - \frac{y+z-x}{(b+c)^k} - \frac{z+x-y}{(c+a)^k} - \frac{x+y-z}{(a+b)^k} \right).$$

□