A Simple Proof of Poncelet’s Theorem
(on the occasion of its bicentennial)

Lorenz Halbeisen and Norbert Hungerbühler

Abstract. We present a proof of Poncelet’s Theorem in the real projective plane which relies only on Pascal’s Theorem.

1. INTRODUCTION In 1813, while Poncelet was in captivity as a war prisoner in the Russian city of Saratov, he discovered the following theorem.

Theorem 1.1 (Poncelet’s Theorem). Let $K$ and $C$ be nondegenerate conics in general position. Suppose there is an $n$-sided polygon inscribed in $K$ and circumscribed about $C$ such that none of its vertices belongs to $C$ (in the case when $K$ and $C$ intersect or meet). Further suppose there is an $(n - 1)$-sided polygonal chain with vertices on $K$ such that all its sides are tangent to $C$ and none of its vertices belongs to $C$. Then the side, which closes the polygonal chain, is also tangent to $C$.

An immediate consequence is the following more popular version of Poncelet’s Theorem.

Corollary 1.2. Let $K$ and $C$ be nondegenerate conics in general position which neither meet nor intersect. Suppose there is an $n$-sided polygon inscribed in $K$ and circumscribed about $C$. Then for any point $P$ of $K$, there exists an $n$-sided polygon, also inscribed in $K$ and circumscribed about $C$, which has $P$ as one of its vertices.

After his return to France, Poncelet published a proof in his book [17], which appeared in 1822. He derived the version displayed above from a more general statement, where the sides of the polygon are tangent to conics $C_i$ from a pencil containing $K$. He first proved the statement for a pencil of circles, thus generalizing theorems of Chapple [5] and Euler [9] for triangles and of Fuss [10] for bicentric polygons. In order to extend his main theorem from circles to conics, Poncelet then invoked a projection theorem which states that every pair of conics with no more than two intersections can be considered as the projective image of a pair of circles. Poncelet finished by arguing that the case of more than two intersections followed by the “principle of continuity.” Poncelet’s treatise was a milestone in the development of projective geometry, and his theorem is widely considered the deepest and most beautiful result about conics.

Poncelet’s Theorem gained immediately the attention of the mathematical community. Already in 1828, Jacobi gave in [13] an analytic proof for pairs of nested circles by using the addition theorem for elliptic functions. In the sequel, Cayley investigated algebraic conditions for two conics to be in Poncelet position. Using the theory of Abelian integrals, he formulated a criterion in [4]. Cayley actually published a series of papers dealing with Poncelet’s porism. In the early 20th century, Lebesgue revisited Cayley’s work and formulated the proof in the language of projective geometry and algebra, see [15]. He used an observation by Hart who gave in [12] an elegant argument for Poncelet’s Theorem for triangles. In recent times, Griffiths and Harris...
used Abel’s Theorem and the representation of elliptic curves by means of the Weierstrass $\wp$-function to establish the equivalence of Poncelet’s Theorem and the group structure on elliptic curves, see [11]. Poncelet’s Theorem has a surprising mechanical interpretation for elliptic billiards in the language of dynamical systems: see [8] or [7] for an overview of this facet. A common approach to all four classical closing theorems (the Poncelet porism, Steiner’s Theorem, the Zigzag Theorem, and Emch’s Theorem) has recently been established by Protasov in [18]. King showed in [14], that Poncelet’s porism is isomorphic to Tarski’s plank problem (a problem about geometric set-inclusion) and to Gelfand’s question (a number theoretic problem) via the construction of an invariant measure. However, according to Berger [1, p. 203], all known proofs of Poncelet’s Theorem are rather long and recondite.

The aim of this paper is to give a simple proof of Poncelet’s Theorem in the real projective plane. More precisely, we will show that Poncelet’s Theorem is a purely combinatorial consequence of Pascal’s Theorem. Before we give several forms of the latter, let us introduce some notation. For two points $a$ and $b$, let $a - b$ denote the line through $a$ and $b$, and for two lines $\ell_1$ and $\ell_2$, let $\ell_1 \wedge \ell_2$ denote the intersection point of these lines in the projective plane. In abuse of notation, we often write $a - b - c$ in order to emphasize that the points $a$, $b$, $c$ are collinear. In the sequel, points are often labeled with numbers, and lines with encircled numbers like ③.

In this terminology, Pascal’s Theorem and its equivalent forms read as follows.

Pascal’s Theorem (cf. [16])

Any six points 1, . . . , 6 lie on a conic

if and only if

$(1 - 2) \wedge (4 - 5)\]
$(2 - 3) \wedge (5 - 6)\]
$(3 - 4) \wedge (6 - 1)\] are collinear.

Carnot’s Theorem (cf. [3, no. 396])

Any six points 1, . . . , 6 lie on a conic

if and only if

$[(1 - 2) \wedge (3 - 4) - [(4 - 5) \wedge (6 - 1)]
$(2 - 5)\]
$(3 - 6)\] are concurrent.
Brianchon’s Theorem (cf. [2])

Any six lines \(\overline{1}, \ldots, \overline{6}\) are tangent to a conic

\[\begin{align*}
(\overline{1} \wedge \overline{2}) - (\overline{4} \wedge \overline{5}) \\
(\overline{2} \wedge \overline{3}) - (\overline{5} \wedge \overline{6}) \\
(\overline{3} \wedge \overline{4}) - (\overline{6} \wedge \overline{1})
\end{align*}\]

are concurrent.

Carnot’s Theorem

Any six lines \(\overline{1}, \ldots, \overline{6}\) are tangent to a conic

\[\begin{align*}
\left[(\overline{1} \wedge \overline{2}) - (\overline{3} \wedge \overline{4})\right] \wedge \left[(\overline{4} \wedge \overline{5}) - (\overline{6} \wedge \overline{1})\right]
\end{align*}\]

\[\begin{align*}
(\overline{2} \wedge \overline{3}) \\
(\overline{3} \wedge \overline{6})
\end{align*}\]

are collinear.

As a matter of fact, we would like to mention that if the conic is not degenerate, then the collinear points in Pascal’s Theorem are always pairwise distinct (the same applies to the concurrent lines in Brianchon’s Theorem).

Since the real projective plane is self-dual, Pascal’s Theorem and Brianchon’s Theorem are equivalent. Moreover Carnot’s Theorem and its dual Carnot’s Theorem* are just reformulations of Pascal’s Theorem and Brianchon’s Theorem by exchanging the points 3 and 5, and the lines 3 and 5, respectively. Recall that if two adjacent points, say 1 and 2, coincide, then the corresponding line \(1 - 2\) becomes a tangent with 1 as contact point. Similarly, if two lines, say \(\overline{1}\) and \(\overline{2}\), coincide, then \(\overline{1} \wedge \overline{2}\) becomes the contact point of the tangent \(\overline{1}\). As a last remark, we would like to mention that a conic is in general determined by five points, by five tangents, or by a combination like three tangents and two contact points of these tangents.

The paper is organized as follows. In Section 2, we prove Poncelet’s Theorem for the special case of triangles and at the same time we develop the kind of combinatorial arguments we shall use later. Section 3 contains the crucial tool which allows to show that Poncelet’s Theorem holds for an arbitrary number of edges. Finally, in Section 4, we use the developed combinatorial technics in order to prove some additional symmetry properties of Poncelet-polygons.

2. PONCELET’S THEOREM FOR TRIANGLES

In order to prove Poncelet’s Theorem for triangles, we will show that if the six vertices of two triangles lie on a conic \(K\), then the six sides of the triangles are tangents to some conic \(C\).

The crucial point in the proof of the following theorem (as well as in the proofs of the other theorems of this paper) is to find the suitable numbering of points and edges, and to apply some form of Pascal’s Theorem in order to find collinear points or concurrent lines.

**Theorem 2.1.** If two triangles are inscribed in a conic and the two triangles do not have a common vertex, then the six sides of the triangles are tangent to a conic.
Proof. Let $K$ be a conic in which two triangles $\triangle a_1 a_2 a_3$ and $\triangle b_1 b_2 b_3$ are inscribed where the two triangles do not have a common vertex.

First, we introduce the following three intersection points:

- $I := (a_1 - a_2) \land (b_1 - b_2)$,
- $X := (a_2 - b_3) \land (b_2 - a_3)$,
- $I' := (a_3 - a_1) \land (b_3 - b_1)$.

In order to visualize the intersection points $I$, $X$, and $I'$, we break up the conic $K$ and draw it as two straight lines, one for each triangle as follows.

Now, we number the six points $a_1, a_2, a_3, b_1, b_2, b_3$ on the conic $K$ as shown by the following figure.
By Pascal’s Theorem we get that the three intersection points

\[(1 - 2) \wedge (4 - 5), \quad (2 - 3) \wedge (5 - 6), \quad \text{and} \quad (3 - 4) \wedge (6 - 1)\]

are collinear, which is the same as saying that the points \(I - X - I'\) are collinear.

In the next step, we label the sides of the triangles as shown in the following figure.

By Carnot’s Theorem\textsuperscript{*} we get that the six sides \(\mathfrak{b} \cdots \mathfrak{a}\) of the two triangles are tangents to a conic if and only if

\[
\left[ (\mathfrak{a} \wedge \mathfrak{b}) - (\mathfrak{c} \wedge \mathfrak{d}) \right] \wedge \left[ (\mathfrak{e} \wedge \mathfrak{f}) - (\mathfrak{g} \wedge \mathfrak{h}) \right],
\]

\[
(\mathfrak{a} \wedge \mathfrak{b}), \quad \text{and}
\]

\[
(\mathfrak{c} \wedge \mathfrak{d})
\]

are collinear. Now, this is the same as saying that the points \(X - I - I'\) are collinear, which, as we have seen above, is equivalent to \(a_1, a_2, a_3, b_1, b_2, b_3\) lying on a conic.

As an immediate consequence we get Poncelet’s Theorem for triangles.

\textbf{Corollary 2.2 (Poncelet’s Theorem for triangles).} Let \(K\) and \(C\) be nondegenerate conics. Suppose there is a triangle \(\triangle a_1 a_2 a_3\) inscribed in \(K\) and circumscribed about \(C\). Then for any point \(b_1\) of \(K\) for which two tangents to \(C\) exist, there is a triangle \(\triangle b_1 b_2 b_3\) which is also inscribed in \(K\) and circumscribed about \(C\).
Proof. Let $K$ and $C$ be two conics with a triangle $\triangle a_1a_2a_3$ inscribed in $K$ and circumscribed about $C$. Let $b_1$ be an arbitrary point on $K$ which is distinct from $a_1, a_2, a_3$, and let $b_2$ and $b_3$ be distinct points on $K$ such that $b_1 - b_2$ and $b_1 - b_3$ are two tangents to $C$. By construction, we get that five sides of the triangles $\triangle a_1a_2a_3$ and $\triangle b_1b_2b_3$ are tangents to $C$. On the other hand, by Theorem 2.1, we know that all six sides of these triangles are tangents to some conic $C'$. Now, since a conic is determined by five tangents, $C'$ and $C$ coincide, which implies that the triangle $\triangle b_1b_2b_3$ is circumscribed about $C$.

As a special case of Brianchon’s Theorem we get the following.

Fact 2.3. Let $C$ be a conic and let the triangle $\triangle a_1a_2a_3$ be circumscribed about $C$. Furthermore, let $t_1, t_2, t_3$ be the contact points of the three tangents $a_2 - a_3, a_3 - a_1, a_1 - a_2$. Then the three lines $a_1 - t_1, a_2 - t_2$, and $a_3 - t_3$ meet in a point.

Proof. Label the three sides of the triangles as follows:

\[
\begin{align*}
\text{①} &= a_2 - a_3 = \mathcal{E}, & \text{②} &= a_3 - a_1 = \mathcal{F}, & \text{③} &= a_1 - a_2 = \mathcal{G}.
\end{align*}
\]

Then $\mathcal{E} \land \mathcal{F} = t_1$, $\mathcal{F} \land \mathcal{G} = t_2$, $\mathcal{G} \land \mathcal{E} = t_3$, and by Brianchon’s Theorem we get that $a_1 - t_1, a_2 - t_2, a_3 - t_3$ meet in a point.

$q.e.d.$

In general, for arbitrary $n$-gons tangent to $C$ the analogous statement will be false. However, if $n$ is even and if the $n$-gon is at the same time inscribed in a conic $K$, a similar phenomenon occurs (see Theorem 4.2).

3. THE GENERAL CASE Let $K$ and $C$ be nondegenerate conics in general position. We assume that there is an $n$-sided polygon $a_1, \ldots, a_n$ which is inscribed in $K$ such that all its $n$ sides $a_1 - a_2, a_2 - a_3, \ldots, a_n - a_1$ are tangent to $C$ and none of its vertices belongs to $C$. Let us assume that $n$ is minimal with this property (thus, in particular, the points $a_1, \ldots, a_n$ are pairwise distinct). Further, let $b_1, \ldots, b_n$ be an $(n - 1)$-sided polygonal chain on $K$ where all $n - 1$ sides $b_1 - b_2, b_2 - b_3, \ldots, b_{n-1} - b_n$ are tangent to $C$ and none of its vertices is one of $a_1, \ldots, a_n$ or belongs to $C$. We do not yet know that $b_n - b_1$ is tangent to $C$ too. If we break up the conic $K$ and draw it as two straight lines, one for the polygon and one for the polygonal chain, we get the following situation.

\[
\begin{array}{cccccccc}
& & a_{n-1} & a_n & a_1 & a_2 & & & \\
& & b_{n-1} & b_n & b_1 & b_2 & & & \\
\end{array}
\]

In order to prove Poncelet’s Theorem, we have to show that $b_n - b_1$ is also tangent to $C$. This will follow easily from the following result.

Lemma 3.1. For $n \geq 4$, the three intersection points

\[
I := (a_1 - a_2) \land (b_1 - b_2),
\]

\[
X := (a_2 - b_{n-1}) \land (b_2 - a_{n-1}), \quad \text{and}
\]

\[
I' := (a_{n-1} - a_n) \land (b_{n-1} - b_n),
\]

© THE MATHEMATICAL ASSOCIATION OF AMERICA
and let

\[ I \]

are pairwise distinct and collinear, which is visualized above by the dashed line.

**Proof.** Depending on the parity of \( n \), we have one of the following anchorings, from which we will work step by step outwards.

- For even, with \( k = \frac{n}{2} \):
- For odd, with \( k = \frac{n+1}{2} \):

By Carnot’s Theorem\(^*\) we have that \( I - X - I' \) are collinear, which proves the lemma for \( n = 4 \).

For \( n \geq 5 \), the lemma will follow from the following two claims.

**Claim 1.** Let \( p \) and \( q \) be integers with \( 2 \leq p < q \leq n - 1 \). Further, let

\[
I_{p-1} := (a_{p-1} - a_p) \land (b_{p-1} - b_p), \quad I_p := (a_p - a_{p+1}) \land (b_p - b_{p+1}), \quad I_q := (a_{q-1} - a_q) \land (b_{q-1} - b_q), \quad I_{q+1} := (a_q - a_{q+1}) \land (b_q - b_{q+1}),
\]

and let

\[ X := (a_p - b_q) \land (b_p - a_q). \]

If \( I_{p-1} - X - I_{q+1} \) are pairwise distinct and collinear, then \( I_{p-1} - X - I_{q+1} \) are also pairwise distinct and collinear. This implication is visualized by the following figure.

[Diagram illustrating the relationships between points and segments, including lines labeled \( a_1, a_2, \ldots, a_{n-1}, a_n, b_1, b_2, \ldots, b_{n-1}, b_n \), and \( I, I', X \).]

\[ \downarrow \]

\[ \downarrow \]

January 2014] 
A SIMPLE PROOF OF PONCELET’S THEOREM 7
Proof of Claim 1.

(a) By Brianchon’s Theorem, the lines $\alpha, \gamma, \varepsilon$ are pairwise distinct and concurrent.

(b) The lines $\alpha, \beta, \gamma$ meet, by assumption, in $X$, and they are pairwise distinct. By (a), we have that $\alpha$ and $\gamma$ are distinct, and by symmetry also $\beta$ and $\gamma$ are distinct. Since a straight line meets a nondegenerate conic in at most two points, $\alpha$ and $\beta$ are also distinct.

(c) By Brianchon’s Theorem, the lines $\beta, \varepsilon, \delta$ are pairwise distinct and concurrent. By (a) & (b) we get that $\alpha, \beta$, and $\varepsilon$ meet in $X$, and by (c) we get that also $\alpha, \beta$, and $\delta$ meet in $X$, which implies that $I_{p-1} - X - I_{q+1}$ are collinear and pairwise distinct. If $I_{p-1} = I_{q+1}$, then the four lines $a_{p-1} - a_p, b_{p-1} - b_p, a_q - a_{q+1}, b_q - b_{q+1}$, which are all tangent to $C$, would be concurrent. But then these four lines are not pairwise distinct, and since the eight points $a_{p-1}, a_p, a_q, a_{q+1}, b_{p-1}, b_p, b_q, b_{q+1}$ are pairwise distinct (recall that $1 \leq p - 1 < q + 1 \leq n$), this contradicts our assumption that the
conic $K$ is nondegenerate. By similar arguments it follows that both $I_{p-1}$ and $I_{q+1}$ are distinct from $X$. q.e.d.

**Claim 2.** Let $I_{p-1}$, $I_{q+1}$, and $X$ be as above, and let

$$X' := (a_{p-1} - b_{q+1}) \land (b_{p-1} - a_{q+1}).$$

If $I_{p-1} - X - I_{q+1}$ are pairwise distinct and collinear, then $I_{p-1} - X' - I_{q+1}$ are pairwise distinct and collinear too. This implication is visualized by the following figure.

![Diagram](image)

**Proof of Claim 2.**

(a) By assumption, the points $I_{p-1} - X - I_{q+1}$ are pairwise distinct and collinear.

(b) By Pascal’s Theorem, the points $I_{p-1} - X - J$ are pairwise distinct and collinear.
(c) By Pascal’s Theorem, the points \(X' - J - I_{q+1}\) are pairwise distinct and collinear.

By (a) & (b) we get that \(I_{p-1} - J - I_{q+1}\) are collinear, and by (c) we get that \(X'\) lies on \(J - I_{q+1}\). Hence, \(I_{p-1} - X' - I_{q+1}\) are collinear. By (a), (c) and a symmetric version of (c), the three points \(I_{p-1}, X', I_{q+1}\) are pairwise distinct.

* q.e.d.

By an iterative application of Claim 1 & 2, we finally get the situation

\[
\begin{array}{c}
\cdots \ I \ a_1 \ a_2 \ a_3 \ \cdots \ a_{n-1} \ a_n \ b_1 \ b_2 \ \cdots \ b_{n-1} \ b_n \ \\
\cdots \ \\
\end{array}
\]

in which \(I - X - I'\) are pairwise distinct and collinear. \(q.e.d.\)

With similar arguments as in the proof of Poncelet’s Theorem for triangles (Corollary 2.2), we can now prove the general case of Poncelet’s Theorem (Theorem 1.1):

* Proof of Poncelet’s Theorem. * Let \(K\) and \(C\) be nondegenerate conics in general position. We assume that there is an \(n\)-sided polygon \(a_1, \ldots, a_n\) which is inscribed in \(K\) such that all its \(n\) sides \(a_1 - a_2, a_2 - a_3, \ldots, a_n - a_1\) are tangent to \(C\) and none of its vertices belongs to \(C\). Let us assume that \(n\) is minimal with this property. Further we assume that there is an \((n-1)\)-sided polygonal chain \(b_1, \ldots, b_n\) whose \(n-1\) sides are tangent to \(C\) and none of its vertices is one of \(a_1, \ldots, a_n\) or belongs to \(C\). We have to show that \(b_n - b_1\) is tangent to \(C\).

\[
\begin{array}{c}
\cdots \ b_{n-1} \ b_n \ b_1 \ b_2 \ \cdots \ b_{n-1} \ b_n \ b_1 \ b_2 \ \\
\cdots \ \\
\end{array}
\]

By Lemma 3.1 we know that \(I - X - I'\) are pairwise distinct and collinear, where \(I = (a_1 - a_2) \land (b_1 - b_2)\), \(I' = (a_{n-1} - a_n) \land (b_{n-1} - b_n)\), and \(X = (a_2 - b_{n-1}) \land (b_2 - a_{n-1})\). In order to show that \(b_n - b_1\) is tangent to \(C\), we have to introduce two more intersection points:

\[
\begin{align*}
J &= (a_{n-1} - a_1) \land (b_{n-1} - b_1), \\
X' &= (a_n - b_1) \land (b_n - a_1).
\end{align*}
\]

\(\copyright\ THE\ MATHEMATICAL\ ASSOCIATION\ OF\ AMERICA\ [Monthly\ 121] \)
Since, by Lemma 3.1, the five lines are collinear.

We now apply Pascal’s Theorem twice as illustrated below.

(a) By Pascal’s Theorem, the points \( I \) – \( X \) – \( J \) are pairwise distinct and collinear.

(b) By Pascal’s Theorem, the points \( I' \) – \( J' \) – \( X' \) are pairwise distinct and collinear. Since, by Lemma 3.1, \( I \) – \( X \) – \( I' \) are pairwise distinct and collinear, by (a) we get that \( I - X - J - I' \) are collinear, and by (b) we finally get that \( I \) – \( X' \) – \( I' \) are collinear.

For the last step, we apply Carnot’s Theorem\(^*\).

Since \( I \) – \( X' \) – \( I' \) are collinear, by Carnot’s Theorem\(^*\) we get that the six lines \( \odot, \ldots, \odot \) are tangent to some conic \( C' \). Now, since a conic is determined by five tangents, and the five lines \( \odot, \odot, \odot, \odot, \odot \) are tangent to \( C, C', \) and \( C \) coincide. This implies that \( \odot \) is tangent to \( C \), which is what we had to show.

\( q.e.d. \)
4. SYMMETRIES IN PONCELET-POLYGONS In this section we present some
symmetries in 2n-sided polygons which are inscribed in some conic K
and circumscribed about another conic C. To keep the terminology short,
we shall call such a polygon a 2n-Poncelet-polygon with respect to K & C.

Theorem 4.1. Let K and C be nondegenerate conics in general position
which neither meet nor intersect and let \(a_1, \ldots, a_{2n}\) be the vertices
of a 2n-Poncelet-polygon with respect to K & C. Further let \(t_1, \ldots, t_{2n}\)
be the contact points of the tangents \(a_1 - a_2, \ldots, a_{2n} - a_1\).

(a) All the \(n\) diagonals \(a_1 - a_{n+1}, a_2 - a_{n+2}, \ldots, a_n - a_{2n}\)
meet in a point \(H_0\).

(b) All the \(n\) lines \(t_1 - t_{n+1}, t_2 - t_{n+2}, \ldots, t_n - t_{2n}\)
meet in the same point \(H_0\).

Proof. (a) By the proof of Lemma 3.1, we get that the three points \(a_1, a_{n+1}\), and
\((a_2 - a_{n+2}) \land (a_n - a_{2n})\) are collinear.

This is the same as saying that the three diagonals \(a_1 - a_{n+1}, a_2 - a_{n+2},\) and \(a_n - a_{2n}\)
meet in a point, say \(H_0\). Now, by cyclic permutation we get that all \(n\) diagonals
meet in \(H_0\).

(b) By the proof of Lemma 3.1, we get that the three points \(t_1 - H_0 - t_{n+1}\) are
collinear.

Thus, by cyclic permutation we get that all \(n\) lines \(t_1 - t_{n+1}, t_2 - t_{n+2}, \ldots, t_n - t_{2n}\)
pass through \(H_0\), which implies that all \(n\) lines meet in \(H_0\). q.e.d.

In the last result, we show that the point \(H_0\) is independent of the particular 2n-
Poncelet-polygon (compare with Poncelet’s results no. 570 & 571 in [17]).

Theorem 4.2. Let K and C be nondegenerate conics in general position
which neither meet nor intersect and let \(a_1, \ldots, a_{2n}\) and \(b_1, \ldots, b_{2n}\)
be the vertices of two 2n-Poncelet-polygons with respect to K & C. Further let \(t_1, \ldots, t_{2n}\)
and \(t'_1, \ldots, t'_{2n}\)
be the contact points of the Poncelet-polygons. Then all \(4n\) lines \(a_1 - a_{n+1}, \ldots,\)
\(t_1 - t_{n+1}, \ldots, b_1 - b_{n+1}, \ldots, t'_1 - t'_{n+1}\) meet in a point \(H_0\). Moreover, opposite
sides of the Poncelet-polygons meet on a fixed line \(h\), where \(h\) is the polar of \(H_0\), both
with respect to C and K.

Proof. By Theorem 4.1 we know that the 2n lines \(a_1 - a_{n+1}, \ldots, t_1 - t_{n+1}\) meet
in a point \(H_0\). First, we show that the polar \(h\) of the pole \(H_0\) with respect to C is
the same as the polar \(h'\) of \(H_0\) with respect to K, and then we show that the point \(H_0\) is
independent of the choice of the 2n-Poncelet-polygon.
First notice that in the figure above, $H_0$ is on the polar $p$ of $P$ with respect to the conic $C$ and that $H_0$ is also on the polar $p'$ of $P$ with respect to the conic $K$ (see for example Coxeter and Greitzer [6, Theorem 6.51]). Thus, $P$ lies on the polar $h$ of $H_0$ with respect to $C$, as well as on the polar $h'$ of $H_0$ with respect to $K$. Since the same applies to the point $Q$, the polars $h$ and $h'$ coincide, which shows that the pole $H_0$ has the same polar with respect to both conics.

The fact that $H_0$ is independent of the choice of the $2n$-Poncelet-polygon is just a consequence of the following.

**Claim.** Let $H_0$ be as above and let $h$ be the polar of $H_0$ (with respect to $K$ or $C$). Choose an arbitrary point $P$ on $h$. Let $s_1$ & $s_2$ be the two tangents from $P$ to $C$ and let $A$ & $A'$ and $B$ & $B'$ be the intersection points of $s_1$ and $s_2$ with $K$.

Then $H_0 = (A - B') \wedge (B - A')$.

**Proof of Claim.** By a projective transformation, we may assume that $h$ is the line at infinity. Then, the pole $H_0$ becomes the common center of both conics and the claim follows by symmetry. q.e.d.
Now, let $a_1, \ldots, a_{2n}$ and $b_1, \ldots, b_{2n}$ be the vertices of two $2n$-Poncelet-polygons with respect to $K$ & $C$. Furthermore, let $H_0 = (a_1 - a_{n+1}) \wedge (a_2 - a_{n+2})$ and $H_0' = (b_1 - b_{n+1}) \wedge (b_2 - b_{n+2})$, and let $h$ and $h'$ be their respective polars. Choose any point $P$ which lies on both $h$ and $h'$, and draw the two tangents from $P$ to $C$ which intersect $K$ in the points $A, A', B, B'$. If the conics $K$ and $C$ do not meet (what we assume), then these points are pairwise distinct and by the Claim we get $H_0 = (A - B') \wedge (B - A') = H_0'$.

q.e.d.

Notice, that for $n = 3$, $H_0$ is the Brianchon point with respect to $C$ of the Poncelet-hexagon, and $h$ its Pascal line with respect to $K$. So, for $n > 3$, the point $H_0$ is the generalized Brianchon point with respect to $C$ of the $2n$-Poncelet-polygon, and $h$ its generalized Pascal line with respect to $K$.

REFERENCES

5. W. Chapple, An essay on the properties of triangles inscribed in and circumscribed about two given circles, Miscellanea Curiosa Mathematica (1746) 117–124.
9. L. Euler, Solutio facilis problematum quorundam geometricorum difficilliorum, Novi Commentarii academiae scientiarum imperialis Petropolitanae (1767) 103–123.

LORENZ HALBEISEN (lorenz.halbeisen@math.ethz.ch) received his PhD from the ETH Zürich in 1994. After research positions in France, Spain, and California, he became lecturer at Queen’s University Belfast and is now senior scientist at the ETH Zürich. His interests range across set theory, finite and infinite combinatorics, geometry, and number theory.

Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland

NORBERT HUNGERBÜHLER (norbert.hungerbuehler@math.ethz.ch) received his PhD at ETH Zürich in 1994. After research positions in Germany and the US he became assistant professor at the University of Alabama at Birmingham and later professor at the University of Fribourg. Currently he is professor at ETH. His interests range across analysis, geometry, discrete mathematics and number theory.

Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland