Problem 11776
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Proposed by D. Beckwith (USA).

Given urns $U_1, \ldots, U_n$ in a line, and plenty of identical blue and identical red balls, let a_n be the number of ways to put balls into the urns subject to the conditions that
(i) each urn contains at most one ball,
(ii) any urn containing a red ball is next to exactly one urn containing a blue ball, and
(iii) no two urns containing a blue ball are adjacent.
(a) Show that

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1+x+2x^2}{1-x-x^2-3x^3}.$$

(b) Show that

$$a_n = \sum_{j\geq 0}\sum_{m\geq 0} 4^j \left[\binom{n-2m}{j}\binom{m}{j} + \binom{n-2m-1}{j}\binom{m}{j} + 2\binom{n-2m}{j}\binom{m-1}{j}\right].$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Two non-empty urns belong to the same *connected component* if also the urns between them are non-empty. Let c_k be the number of ways to put the balls in a connected component of k urns. It is easy to see that when $k \equiv 1 \pmod 3$ then $c_k = 1$:

$$B, BRRB, BRRBRRB, BRRBRRBRRB, BRRBRRBRRBRRB, \ldots$$

Moreover, if $k \equiv 2 \pmod 3$ then a disposition can be obtained from one of size $k-1$ by adding a ball R to the left or to the right, so $c_k = 2$. On the other hand, if $k \equiv 0 \pmod 3$ then a disposition can be obtained from one of size $k-2$ by adding a two balls R to the left and to the right, so $c_k = 1$. It follows that

$$C(x) = \sum_{k\geq 1} c_k x^k = \frac{x}{1-x} + \frac{x^2}{1-x^3} = \frac{x(1+x)^2}{1-x^3}.$$

Let us assume that a disposition has j connected components of total size k. Let t_i be the number of empty urns between the $(i-1)$th and the ith components for $i = 1, \ldots, j+1$. Then $t_i \geq 1$ for $i = 2, \ldots, j$ and the number of dispositions of this kind is

$$\binom{n-k+1}{j} \sum_{k_1+k_2+\cdots+k_j=k} \prod_{i=1}^{j} c_{k_i} = [x^k]\binom{n-k+1}{j} C^j(x)$$

Hence

$$\begin{aligned}
a_n &= \sum_{k\geq 0} [x^k] \sum_{j\geq 0} \binom{n-k+1}{j} C^j(x) = \sum_{k=0}^{n} [x^k](1+C(x))^{n-k+1} \\
&= [x^n] \sum_{k=0}^{n} (1+C(x))^{n-k+1} x^{n-k} = [x^n] \sum_{k\geq 0} (1+C(x))^{k+1} x^k \\
&= [x^n] \frac{1+C(x)}{1-x(1+C(x))} = [x^n] \frac{1+x+2x^2}{1-x-x^2-3x^3}
\end{aligned}$$

and (a) is proved.

As regards (b), we observe that for $r \geq 0$,

$$\sum_{n\geq 0} \binom{n-r}{j} x^n = \frac{x^{r+j}}{(1-x)^{j+1}}.$$

Therefore, for $a, b \geq 0$,

$$\begin{aligned}
\sum_{n\geq 0}\sum_{j\geq 0}\sum_{m\geq 0} 4^j \binom{n-2m-a}{j}\binom{m-b}{j} x^n &= \sum_{m\geq 0}\sum_{j\geq 0}\binom{m-b}{j} 4^j \sum_{n\geq 0}\binom{n-2m-a}{j} x^n \\
&= \frac{x^a}{1-x}\sum_{m\geq 0} x^{2m}\sum_{j\geq 0}\binom{m-b}{j}\left(\frac{4x}{1-x}\right)^j \\
&= \frac{x^a}{1-x}\sum_{m\geq b} x^{2m}\left(1+\frac{4x}{1-x}\right)^{m-b} \\
&= \frac{x^{a+2b}}{1-x}\sum_{m\geq b}\left(\frac{x^2(1+3x)}{1-x}\right)^{m-b} \\
&= \frac{x^{a+2b}}{1-x}\cdot\frac{1}{1-\frac{x^2(1+3x)}{1-x}} = \frac{x^{a+2b}}{1-x-x^2-3x^3}.
\end{aligned}$$

Thus (b) follows from (a):

$$[x^n]\sum_{n\geq 0}\sum_{j\geq 0}\sum_{m\geq 0}\Big[\dots\Big] = [x_n]\frac{1+x+2x^2}{1-x-x^2-3x^3} = a_n.$$

$\square$