Eight circles theorem-A generalization Brianchon's theorem

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Abstract

We give a result: Eight circles theorem

Theorem 1. *Let $A_1, A_2, A_3, A_4, A_5, A_6$ lie on a circle; $B_1, B_2, B_3, B_4, B_5, B_6$ lie on another circle. Such that A_1, A_2, B_1, B_2 lie on a circle (C_1), A_2, A_3, B_2, B_3 lie on a circle (C_2); A_3, A_4, B_3, B_4 lie on a circle (C_3); A_4, A_5, B_4, B_5 lie on a circle (C_4); A_5, A_6, B_5, B_6 lie on a circle (C_5); A_6, A_1, B_6, B_1 lie on a circle (C_6). Then C_1C_4, C_2C_5, C_3C_6 are concurrent.If two circles $(A_1A_2A_3)$ and $(B_1B_2B_3)$ have same center, then six points $C_1C_2C_3C_4C_5C_6$ are is a circumscribed hexagon*

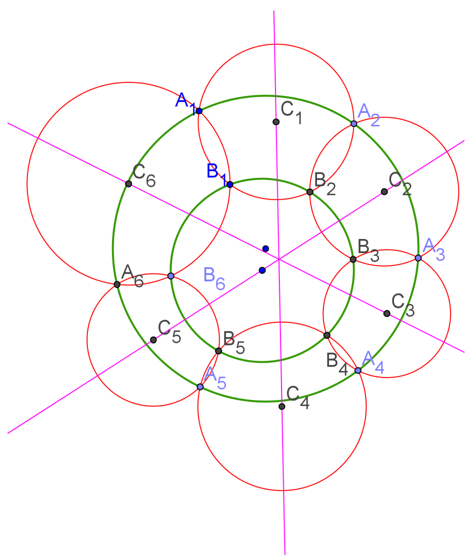

Figure 1: Eight circles theorem

Proof. Label (O_1) and (O_2) the circumcircles of the hexagons $A_1A_2A_3A_4A_5A_6$ and $B_1B_2B_3B_4B_5B_6$ respectively. Invert (O_1) and (O_2) into concentric circles through an inversion centered at one of their limiting points. In the new figure, by obvious symmetry, $A_1A_2B_2B_1$, $A_2A_3B_3B_2$, etc, become isosceles trapezoids with $A_1A_2 \parallel B_1B_2$, $A_2A_3 \parallel B_2B_3$, etc $\Longrightarrow$ cyclic hexagons $A_1A_2A_3A_4A_5A_6$ and $B_1B_2B_3B_4B_5B_6$ with corresponding parallel sides are then homothetic $\Longrightarrow A_1A_3 \parallel B_1B_3 \Longrightarrow A_1A_3B_3B_1$ is isosceles trapezoid due to symmetry, thus in the original figure $A_1A_3B_3B_1$ is cyclic. Hence, A_1B_1, A_2B_2, A_3B_3 concur at the radical center H of C_1, C_2, $\odot(A_1A_3B_3B_1) \Longrightarrow HA_1 \cdot HB_1 = HA_2 \cdot HB_2 = HA_3 \cdot HB_3$ $\Longrightarrow H$ is center of the direct inversion that swaps (O_1) and (O_2). By similar reasoning, A_4B_4, A_5B_5, A_6B_6 go through H.

Let O_1O_2 cut (O_1) and (O_2) at X, Y and U, V respectively (U is between H,X and V is between H,Y). Arbitrary ray issuing from H cuts $(O_1), (O_2)$ at A, B. $UXAB$ and $VYAB$ are then cyclic $\Longrightarrow$ perpendicular bisector ℓ of AB cuts perpendicular bisectors of UX, VY at their circumcenters I, J, respectively. If M, N, L are the midpoints of AB, UX, VY and P is the projection of O_2 on ℓ, then from cyclic $IMHN$ and $JMHL$, we have $\angle NPO_2 = \angle NIO_2$ and $\angle LPO_2 = \angle LJO_2$. But since O_2I and O_2J clearly bisect $\angle BO_2U$ and $\angle BO_2V$, the angle $\angle IO_2J$ is right $\Longrightarrow NIO_2 = \angle LO_2J \Longrightarrow \angle NPL = \angle NPO_2 + \angle LPO_2 = \angle LO_2J + \angle LJO_2 = 90^\circ \Longrightarrow P$ moves on the circle ω with diameter $NL \Longrightarrow \ell$ envelopes the ellipse $\mathcal{E}$ with focus O_2, O_1 and pedal circle ω.

As a result, perpendicular bisectors of A_1B_1, A_2B_2, etc, touch the ellipse $\mathcal{E} \Longrightarrow$ hexagon $C_1C_2C_3C_4C_5C_6$ is circumscribed to $\mathcal{E}$. Hence, by Brianchon theorem, C_1C_4, C_2C_5 and C_3C_6 concur. Obviously when (O_1) and (O_2) are concentric, $\mathcal{E}$ becomes a circle concentric with these.

$\square$

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