THE 29th ANNUAL (2007) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. Let x be my son's age now. My age is $3x$. Five years ago, my age was $3x - 5$ and his was $x - 5$. Therefore, $3x - 5 = 4(x - 5)$, which yields $x = 15$. The answer is **(c)**.

2. Write the numbers as powers of 2: $2^{1/2}, 2^{2/4}, 2^{3/8}, 2^{4/16}, 2^{5/32}$. The exponent 5/32 is smallest, so the answer is **(e)**.

3. Substitute $y = 8x + b$ into $y = x^2$ to obtain $x^2 - 8x - b = 0$. This has exactly one root when $b = -16$ since then $x^2 - 8x + 16 = (x - 8)^2$. The answer is **(a)**.

4. Let s be the speed of the car. Take the starting point of the car as the origin. At time t, the car is in position st, Rover is in position $-100 + 25t$, and Spot is in position $-200 + 30t$. Rover and Spot are at the same point when $-100 + 25t = -200 + 30t$, so when $t = 20$. This position is $-100 + 25t = 400$. At $t = 20$, the car is therefore at $400 = 20s$, so $s = 20$. The answer is **(d)**.

5. There are 8 possibilities for the total: $15, 20, 25, 30, 35, 40, 45, 50, 60$. The answer is **(c)**.

6. One try is using eight 2's, which yields a product of 256. However, replacing three 2's by two 3's is better, since $9 > 8$. Doing this twice gives four 3's and two 2's, which yields a product of 324. Trying a few more comparisons shows that this is the best that can be done. The answer is **(d)**.

7. We have $\log 1000000 = 6$, $\log 100 = 2$, $\log 1000 = 3$, $\log 10000 = 4$. The fraction becomes $(6/2)/(3/4) = 4$ (so Newton misunderstood when Leibniz said "Feuer"). The answer is **(c)**.

8. The small car uses 1 gallon of gas per day, so it is driven 60 miles per day. The large car uses 3 gallons to travel these 60 miles each day. In 40 days, it uses 120 gallons. The answer is **(e)**.

9. Let n be the number of 4-inch cookies. Then $144\pi(3/2)^2 = n\pi(2)^2$. This yields $n = 81$. The answer is **(c)**.

10. We have $40 = 2p + 3q$. This implies that q is even, Since it is prime, $q = 2$. Therefore, $p = 17$. The total number of pigs is $p + q = 19$. The answer is **(d)**.

11. If Goofy sleeps, then Lassie is not sleeping, which implies that Pluto is not eating. This implies that Goofy is not sleeping, contradicting the original assumption. Therefore, Goofy never sleeps, so Aristotle is correct. The situation where Lassie sleeps, Pluto does not eat, and Goofy does not sleep is consistent with the implications in the problem and shows that Socrates and Plato are wrong. Therefore, only Aristotle has made a correct deduction. The answer is **(a)**.

12. We have $f(1/100)+f(99/100) = 10, \ldots, f(49/100)+f(51/100) = 10$. Therefore, the sum of these terms yields 490. The remaining term, $f(50/100)$, equals 5 since $f(50/100) + f(50/100) = 10$. Therefore, the sum is 495. The answer is **(b)**.

13. After paying taxes, you have $x - (x/100)x = -(1/100)x^2 + x$. When $a < 0$, the maximum value of ax^2+bx+c occurs when $x = -b/2a$ (this can be verified by completing the square). Therefore, the maximum occurs at $x = 50$. The answer is **(d)**.

14. The length of the ribbon was 5000. Therefore, the circumference of the semicircle was $\pi r = 5000$. The area of the semicircle was $(1/2)\pi r^2 = 25000000/2\pi \approx 4000000$. The answer is **(e)**.

15. There are 5 choices for the first stripe, 4 for the second stripe, and 4 for the third stripe (to avoid the color of the second stripe). This yields $5 \times 4 \times 4 = 80$ choices. The answer is **(b)**.

16. In 6 hours, Richard makes 6 chairs, Karen makes 3 chairs, and Harold makes 2 chairs, so they make a total of 11 chairs. To make 99 chairs, it takes 54 hours. The answer is **(d)**.

17. $1/2 = 2\sin\theta\cos\theta = \sin(2\theta)$. Since $-90 \le 2\theta \le 90$, we have $2\theta = 30$, so $\theta = 15$. The answer is **(b)**.

18. The nth child sits for the minute ending at time t if t is a multiple of n. Therefore, we need t to be a multiple of $1, 2, 3, 4, 5, 6, 7$. The least common multiple of these numbers is $420 = 2^2 \cdot 3 \cdot 5 \cdot 7$. Therefore, school ends at 3:00pm, which is after 420 minutes. The answer is **(e)**.

19. Let x be the length of the side that the triangle shares with the rectangle, and let y be the other side of the rectangle. The Pythagorean theorem yields $(x/2)^2 + y^2 = x^2$, where the hypotenuse is a side of the triangle inside the rectangle. This yields $y = (\sqrt{3}/2)x$. The perimeter of the triangle is $3x$ and the perimeter of the rectangle is $2x + 2y = (2 + \sqrt{3})x$. The ratio is $(2 + \sqrt{3})/3$. Using $\sqrt{3} \approx 1.7$ yields $3.7/3 \approx 1.23$. The answer is **(c)**.

20. Computing $(a + b + c)^3$ and experimenting yields the identity

$$6abc = (a + b + c)^3 + 2(a^3 + b^3 + c^3) - 3(a + b + c)(a^2 + b^2 + c^2).$$

Therefore, $6abc = -3/8$, so $abc = -1/16$. (The values of a, b, c are $1/2, (2 \pm \sqrt{6})/4$.) The answer is **(b)**.

21. Triangle ADE is isosceles and angle EAD is 150°. Therefore, angle AED is 15°. Similarly, angle BEC is 15°. Therefore, angle CED is 30°. The answer is **(d)**.

22. If Mary erases the numbers $1, 2, \ldots, 44$, then all numbers remaining are at least 45, so the product of any two is at least $45^2 > 2007$. Therefore, this set is a possibility for the problem. Therefore, $m \le 44$ (in fact, it can be shown that $m = 44$). The answer is **(a)**.

23. Let N denote a couple sitting next to each other and let A denote a couple sitting across from each other. The 8 possible patterns on one side of the table are $AAAAA$, $AAAN$, $AANA$, $ANAA$, $NAAA$, ANN, NAN, ANN. The other side of the table must match. For each of these patterns, there are $5! = 120$ ways to assign couples to letters and then $2^5 = 32$ ways to switch the orders of the partners within the letter. Therefore, the total number of arrangements is $8 \times 120 \times 32 = 30720$. The answer is **(e)**.

24. Write

$$2S = (x_1 + \cdots + x_{2007})^2 - (x_1^2 + \cdots + x_{2007}^2).$$

If we make $x_1 + \cdots + x_{2007} = 0$, there are an even number of $x_i \ne 0$, so $x_1^2 + \cdots + x_{2007}^2$ is at most 2006. If we make $x_1 + \cdots + x_{2007} = \pm 1$, then we can have $x_1^2 + \cdots + x_{2007}^2 = 2007$. In both cases, we get $S = -1003$. If we make $|x_1 + \cdots + x_{2007}|$ larger than 1, then $2S > -2006$. Therefore, $S = -1003$ is the smallest possible value of S. The answer is **(a)**.

25. Divide the first equation by $\sqrt{x}$ and the second by $\sqrt{y}$ to get $336/\sqrt{x} = x^{3/2} + y^{3/2} = 112/\sqrt{y}$. Therefore, $x = 9y$. The second equation in the problem yields $y^2 + 27y^2 = 112$, so $y = 2$. This implies that $x = 18$, so $x + y = 20$. The answer is **(e)**.