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Another Synthetic Proof of Dao's Generalization of the Simson Line Theorem

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Abstract. We give a synthetic proof of Dao's generalization of the Simson line theorem.

In [3], Dao Thanh Oai published without proof a remarkable generalization of the Simson line theorem.

Theorem 1 (Dao). *Let ABC be a triangle with its orthocenter H, let P be an arbitrary point on the circumcircle. Let l be a line through the circumcenter and AP, BP, CP meet l at A_1, B_1, C_1, respectively. Denote A_2, B_2, C_2 the orthogonal projections of A_1, B_1, C_1 onto BC, CA, AB, respectively. Then A_2, B_2, C_2 are collinear and the line passing through A_2, B_2, C_2 bisects PH.*

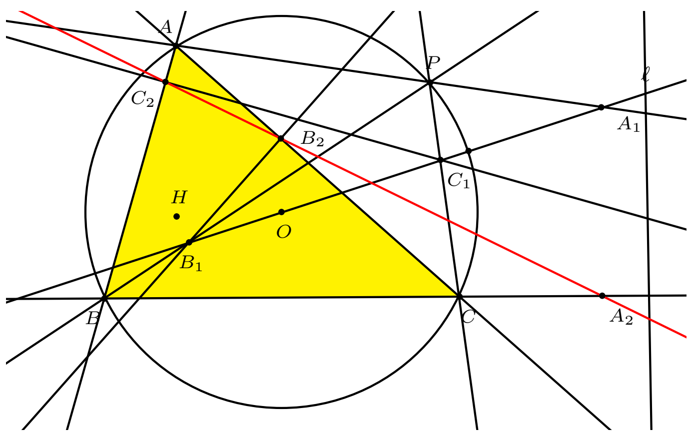

Figure 1. Dao's generalization of Simson line theorem

Note that when l passes through P, the line coincides with the simson line of P with respect to triangle ABC. Two proofs, by Telv Cohl and Luis Gonzalez, can be found in [2]. Nguyen Le Phuoc and Nguyen Chuong Chi have given a synthetic proof in [4]. In this note we give another synthetic proof of Theorem 1 by considering the reformulation.

Theorem 1′ *Let $ABCD$ be a quadrilateral inscribed in circle (O). An arbitrary line l through O intersects the lines AB, BC, CD, DA, AC, BD at X, Y, Z, T,*

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U, V, respectively. Denote by X_1, Y_1, Z_1, T_1, U_1, V_1 the orthogonal projections of X, Y, Z, T, U, V onto CD, AD, AB, BC, BD, AC respectively.
(a) *The six points X_1, Y_1, Z_1, T_1, U_1, V_1 all lie on a line $\mathcal{L}$.*
(b) *If H_a, H_b, H_c, H_d are the orthocenters of triangles BCD, CDA, DAB, ABC respectively, then AH_a, BH_b, CH_c, DH_d share a common midpoint K which lies on the line $\mathcal{L}$.*

We shall make use of two lemmas.

Lemma 2 ([1, Theorem 475]). *The locus of a point the ratio of whose powers with respect to two given circles is constant, both in magnitude and in sign, is a circle coaxal with the given circles.*

Lemma 3. *Let M, N, P, Q be the midpoints of AB, BC, CD, DA respectively, and d_M, d_N, d_P, d_Q the perpendiculars from M, N, P, Q to CD, DA, AB, BC respectively. The eight lines AH_a, BH_b, CH_c, DH_d, d_M, d_N, d_P, d_Q are concurrent.*

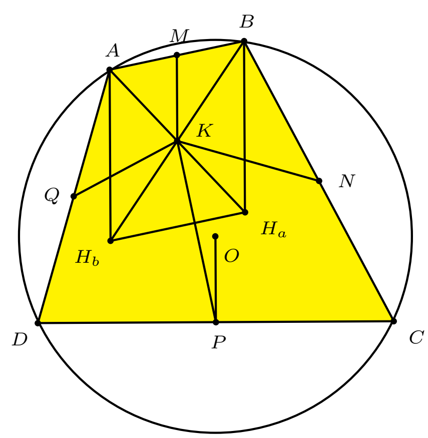

Figure 2. Lemma 3

Proof. Since the distance between one vertex of a triangle and its orthocenter is twice the one between circumcenter and the opposite side, we have $AH_b = 2OP = BH_a$. But $AH_b \parallel BH_a$ then AH_bH_aB is a parallelogram. This means AH_a and BH_b share a common midpoint K. The actually applies to every pair among the four segments AH_a, BH_b, CH_c and DH_d. Therefore, K is the common midpoint of the four segments. Moreover, MK is a midline of triangle ABH_a then $MK \parallel BH_a$, and is perpendicular to CD. It is the line d_M. Similarly, d_N, d_P, d_Q are the lines NK, PK, QK respectively. □

Proof of Theorem $1'$

Denote Z_1', X_1' the intersections of Y_1T_1 with AB, CD, respectively.

We will show that the ratios of powers of four points Z_1', X, X_1', Z with respect to (O) and the circle with diameter YT are equal.

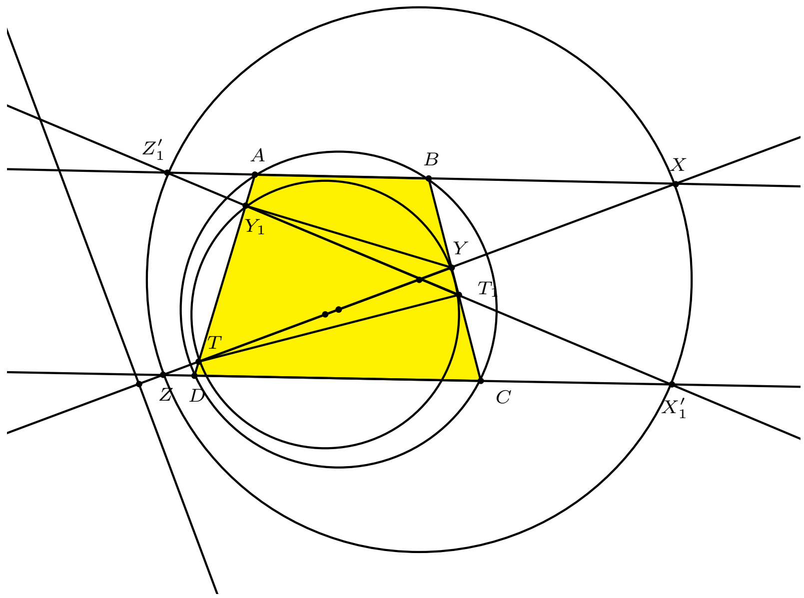

Figure 3. Proof of Theorem $1'$(a)

By simple angle chasing, we have
(i) $\angle Z_1'Y_1A = \angle TY_1T_1 = \angle TYT_1 = \angle BYX$,
(ii) $\angle Z_1'AY_1 + \angle XAT = 180^\circ$,
(iii) $\angle Z_1'T_1B = \angle ATX$,
(iv) $\angle Z_1'BT_1 + \angle YBX = 180^\circ$.
From these,

$$\begin{aligned}
&\frac{\sin \angle Z_1'Y_1A}{\sin \angle Z_1'AY_1} \cdot \frac{\sin \angle Z_1'T_1B}{\sin \angle Z_1'BT_1} = \frac{\sin \angle XTA}{\sin \angle XAT} \cdot \frac{\sin \angle XYB}{\sin \angle XBY} \\
\Longrightarrow\ &\frac{Z_1'A \cdot Z_1'B}{Z_1'Y_1 \cdot Z_1'T_1} = \frac{XA \cdot XB}{XY \cdot XT} \\
\Longrightarrow\ &\frac{\mathscr{P}_{(O)}(Z_1')}{\mathscr{P}_{(YT)}(Z_1')} = \frac{\mathscr{P}_{(O)}(X)}{\mathscr{P}_{(YT)}(X)}.
\end{aligned}$$

The same reasoning actually gives

$$\frac{\mathscr{P}_{(O)}(Z_1')}{\mathscr{P}_{(YT)}(Z_1')} = \frac{\mathscr{P}_{(O)}(X)}{\mathscr{P}_{(YT)}(X)} = \frac{\mathscr{P}_{(O)}(X_1')}{\mathscr{P}_{(YT)}(X_1')} = \frac{\mathscr{P}_{(O)}(Z)}{\mathscr{P}_{(YT)}(Z)}.$$

By Lemma 2, the four points X, Z, X_1', Z_1' lie on a circle ω which is coaxal with (O) and the circle with diameter YT. The center of ω obviously lies on l. Therefore, XZ is a diameter of ω. It follows that Z_1' and X_1' are the orthogonal projections of Z, X onto AB and CD respectively. This means X_1' and Z_1' coincide with X_1 and Z_1 respectively. Hence, X_1, Y_1, Z_1, T_1 are collinear on a line $\mathcal{L}$. By a similar reasoning the same line $\mathcal{L}$ also contains U_1 and V_1.

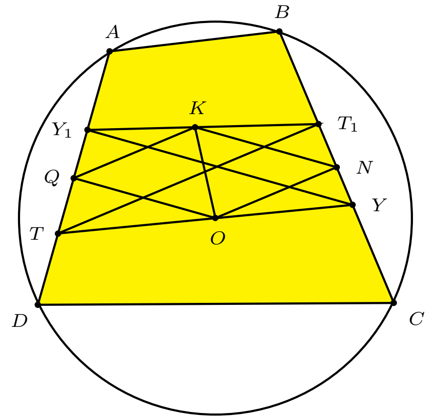

Figure 4. Proof of Theorem $1'$(b)

On the other hand, by Lemma 3, QK is parallel to ON, and NK is parallel to OQ. Thus, $ONKQ$ is a parallelogram. From this, $\frac{KN}{Y_1Y} = \frac{OQ}{Y_1Y} = \frac{OT}{TY} = \frac{T_1N}{T_1Y}$. By Thales' theorem, T_1, K, Y_1 are collinear. Therefore, the line $\mathcal{L}$ containing the six points X_1, Y_1, Z_1, T_1, U_1, V_1 also passes through K. This completes the proof of Theorem $1'$.

The Simson line theorem has a well-known property which states that *the angle between the Simson lines of two point P and P' is half the angle of the arc PP'*. In Theorem 1, if we choose another point P' on (O) and define A_2', B_2', C_2' analogously to A_2, B_2, C_2 respectively, then the angle between the lines through A_2, B_2, C_2 and A_2', B_2', C_2' is also half the angle of the arc PP'.

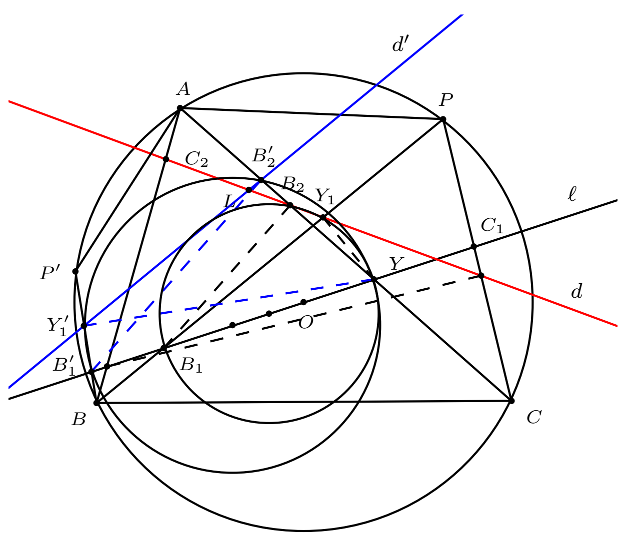

Figure 5. Another property of the generalization of Simson line

Proof. Let Y be the intersection of l and AC, Y_1, Y_1' be the orthogonal projections of Y onto $PB, P'B$, respectively; d and d' the lines through A_2, B_2, C_2 and A_2', B_2', C_2', respectively. Let d meets d' at L.

From the second form of Theorem 1, Y_1 lies on d and Y_1' lies on d'.

We have the directed angle between the lines d and d' given by

$$\begin{aligned}(d, d') &= \angle B_2' L B_2 \\ &= 180^\circ - \angle L B_2 B_2' - \angle L B_2' B_2 \\ &= \angle Y_1' B_1' B_1 - \angle Y_1 B_2 Y \\ &= \angle Y_1' B_1' B_1 - \angle Y_1 B_1 Y \\ &= \angle B_1' B B_1 \\ &= \angle P' B P,\end{aligned}$$

which is half the angle of the arc PP'. □

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