30 characterizations of the symmedian line

Luis González

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Abstract

The symmedians of a triangle are defined as the isogonal conjugates of its medians, i.e. they are obtained reflecting the medians over the corresponding angle bisectors. Thus they are concurrent at the isogonal conjugate of its centroid (symmedian point or Lemoine point). Starting with a scalene triangle labeled $\triangle ABC$, we present 30 different characterizations of the symmedian line issuing from vertex A. Proofs are left to the reader.

1) The tangents to the circumcircle of $\triangle ABC$ at B and C meet at S. AS is the A-symmedian.

2) X is a point on $\overline{BC}$ verifying that $XB : XC = AB^2 : AC^2$. AX is the A-symmedian

3) Let E and F be points on AC and AB, such that EF is antiparallel to BC with respect to AB, AC, i.e. $\angle AEF = \angle ABC$. If M is the midpoint of $\overline{EF}$, then AM is the A-symmedian.

4) The A-symmedian is the locus of the point P on the plane verifying that the ratio of its distances to AB and AC equals $\frac{AB}{AC}$.

5) P is a point on the plane. Parallel from P to AC cuts BC, BA at D, E and the parallel from P to AB cuts CB, CA at F, G. If D, E, F, G are concyclic, then AP is the A-symmedian.

6) P is a point on the plane. Antiparallel to AC through P cuts BC, BA at D, E and the antiparallel to AB through P cuts CB, CA at F, G. If D, E, F, G are concyclic, then AP is the A-symmedian.

7) Internal and external bisectors of $\angle BAC$ cut BC at U and V. The circle with diameter $\overline{UV}$ cuts the circumcircle (O) of $\triangle ABC$ again at S. AS is the A-symmedian.

8) Let E and F be the projections of B and C on AC and AB. M is the midpoint of $\overline{BC}$ and AM cuts EF at P. If X is the projection of P on BC, then AX is the A-symmedian.

9) Squares $ACPQ$ and $ABRS$ are constructed outwardly. If $X \equiv PQ \cap RS$, then AX is the A-symmedian.

10) The perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect the A-altitude at P and Q. O is the circumcenter of $\triangle ABC$ and J is the circumcenter of $\triangle OPQ$. AJ is the A-symmedian.

11) Parallel to BC cuts AB and AC at Y and Z, respectively. $P \equiv BZ \cap CY$ and the circles $\odot(PBY)$ and $\odot(PCZ)$ meet again at Q. AQ is the A-symmedian.

12) The two circles passing through A and touching BC at B and C meet again at P. If Q is the reflection of P on BC, then AQ is the A-symmedian.

13) The circle passing through A, B tangent to AC and the circle passing through A, C tangent to AB meet again at P. AP is the A-symmedian and moreover if AP cuts the circumcircle of $\triangle ABC$ again at Q, then P is midpoint of $\overline{AQ}$.

14) Assume that $\triangle ABC$ is acute. O is circumcenter and D is the projection of A on BC. Circle ω_A has center on $\overline{AD}$, passes through A and touches $\odot(OBC)$ externally at X. AX is the A-symmedian.

15) Let E and F be the projections of B and C on AC and AB. M is the midpoint of $\overline{BC}$. AM cuts CF at X and the parallel from X to AC cuts BE at Y. AY is the A-symmedian.

16) Let N and L be the midpoints of $\overline{AC}$ and $\overline{AB}$, respectively and let D be the projection of A on BC. Circles $\odot(BDL)$ and $\odot(CDN)$ meet again at P. AP is the A-symmedian.

17) Internal bisector of $\angle BAC$ cuts BC at D and M is the midpoint of the arc BAC of the circumcircle (O). If MD cuts (O) again at X, then AX is the A-symmedian.

18) Let M be the midpoint of $\overline{BC}$. The perpendicular bisectors of $\overline{AC}$ and $\overline{AB}$ cut AM at B' and C', respectively. If $A' \equiv CB' \cap BC'$, then AA' is the A-symmedian.

19) D, E, F are the projections of A, B, C on BC, CA, AB and M, L are the midpoints of BC, BA. If $X \equiv DE \cap ML$, then AX is the A-symmedian and moreover $BX \parallel EF$.

20) Assume that $\triangle ABC$ is acute with orthocenter H. M is the midpoint of BC and AM cuts $\odot(HBC)$ at P (P is between A and M). BP and CP cut AC and AB at U and V. N is the midpoint of $\overline{UV}$ and X is the projection of N on BC. AX is the A-symmedian.

21) Let M be the midpoint of $\overline{BC}$ and let D be the projection of A on BC. A' is the reflection of A on M. P is the projection of A on $A'B$ and Q is the reflection of P on B. If $T \equiv CQ \cap DA'$, then AT is the A-symmedian.

22) Ω_2 is the second Brocard point of $\triangle ABC$ (i.e. the point Ω_2 inside $\triangle ABC$ verifying $\angle \Omega_2 CB = \angle \Omega_2 BA = \angle \Omega_2 AC$) and L is midpoint of $\overline{AB}$. If $P \equiv CL \cap B\Omega_2$, AP is the A-symmedian.

23) The mixtilinear incircles ω_B and ω_C againts B and C touch the circumcircle (O) at B' and C', respectively. BB' and CC' cut ω_B and ω_C again at B'' and C''. Tangents of ω_B and ω_C at B'' and C'', respectively, meet at P. AP is the A-symmedian.

24) P is a variable point on BC. Parallels fom P to AB and AC cut AC and AB at B', C'. Circles $\odot(AB'C')$ go through to fixed points A and Q. AQ is the A-symmedian.

25) P is an arbitrary point on the circumcircle (O) of $\triangle ABC$. AP cuts the tangents of (O) through B, C at M, N, respectively. $S \equiv CM \cap BN$ and PS cuts BC at X. AX is the A-symmedian.

26) Let E and F be the projections of B and C on AC and AB and let Y and Z be the midpoints of $\overline{CE}$ and $\overline{BF}$, respectively. P is an arbitrary point on the perpendicular bisector of $\overline{BC}$. Perpendiculars from B and C to PZ and PY, respectively, meet at Q. AQ is the A-symmedian.

27) U and V are two points on $\overline{BC}$, such that AU and AV are isogonals with respect to $\angle BAC$. A circle passes through U, V and touches the circumcircle (O) of $\triangle ABC$ at X (X and A are on different sides of the line BC). AX is the A-symmedian.

28) U and V are two isotomic points with respect to B, C (i.e. $\overline{UV}$ and $\overline{BC}$ have the same midpoint). The isogonals of AU, AV with respect to $\angle BAC$ hit the circumcircle (O) of $\triangle ABC$ again at X, Y and the tangents of (O) at X, Y meet at S. AS is the A-symmedian.
Corollary: The A-mixtilinear incircle and the A-mixtilinear excircle touch (O) at X, Y. If the tangents of (O) at X, Y meet at S, then AS is the A-symmedian.

29) P is a variable point on BC. Parallels fom P to AB and AC cut AC and AB at B', C'. The perpendicular from P to $B'C'$ envelopes a parabola with focus F. AF is the A-symmedian.

30) $\mathcal{H}_A$ is the rectangular hyperbola that passes through A, B, C and whose center is the midpoint of $\overline{BC}$. The tangent of $\mathcal{H}_A$ at A is the A-symmedian.

References

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