In Memory of Maryam Mirzakhani (1977-2017)

THERE ARE NO ODD PERFECT NUMBERS

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Abstract. While the general form of even perfect numbers is well-known, the existence or non-existence of odd perfect numbers is still an open problem. We address this problem and prove that if a natural number is odd, then it's not perfect.

1. introduction

The following is one of the ancient open problems in number theory, perhaps [R1] the oldest open problem in all mathematics, at the time of writing:

Definition. *(Perfect Number) A natural number n is said to be perfect if the sum of all its [positive] divisors, including n itself, is equal to $2n$.*

$$\sum_{d|n} d = 2n, \text{ equivalently: } \sum_{\substack{d|n \\ d<n}} d = n. \tag{1.1}$$

Example: 6, 28, 496, and 8128 are the first few perfect numbers. ◁

All even perfect numbers are completely determined [R1] by the following theorem:

Theorem. *Even Perfect Numbers*

A) *Euclid (300 B.C.) if $2^p - 1$ is prime, then $n = 2^{p-1}(2^p - 1)$ is a perfect number. (Elements, Book IX, Proposition 36, as cited in [R1]).*

B) *Euler (1707 – 1783) If n is even, then the converse of part (A) is also true. [R1] i.e. even perfect numbers *must* be of the form given by Euclid in part (A).*

The ancient open problem is whether or not any *odd* perfect numbers exist? We answer that question, as already suggested by many authors [R3], negatively; using only elementary tools.

A good account of previous work on this topic can be found, for example in [R2] and [R3].

2. Preliminaries

It's understood that 1 is not a perfect number. The sum of divisors of 1 is 1, which is not two times 1. Therefore, when we look for odd perfect numbers, $n > 1$, and it can be uniquely factorized as

$$n = \prod_{i=1}^{m} p_i^{a_i} \quad (p_i \text{ odd prime}) \ (a_i \text{ positive}).$$

The sum of divisors of x - usually denoted by $\sigma(x)$ - for a prime power is:

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$$\sigma(p^a) = \sum_{j=0}^{a} p^j,$$

and because σ is a multiplicative function ([R1]), for every n in general, it can be written as:

$$\sigma(n) = \prod_{i=1}^{m} \left(\sum_{j=0}^{a_i} p_i^j\right), \tag{2.1}$$

Example: For $n = 3^4$ we have,

$$\sigma(n) = \sigma(3^4) = \sum_{j=0}^{4} 3^j = 1 + 3 + 3^2 + 3^3 + 3^4 = 121,$$

and in other forms,

$$\sigma(3^4) = 1 + 3\sigma(3^{4-1}) = \sigma(3^{4-1}) + 3^4 = \frac{3^{4+1} - 1}{3 - 1} = 121.$$

as partial sum of a geometric series. $\lhd$

Therefore, using (1.1), our main equation is

$$\prod_{i=1}^{m} \left(\sum_{j=0}^{a_i} p_i^j\right) = 2\prod_{i=1}^{m} p_i^{a_i}$$

or, equivalently,

$$\prod_{i=1}^{m} \sigma(p_i^{a_i}) = 2\prod_{i=1}^{m} p_i^{a_i} \quad (p_i \neq 2, a_i > 0, m \geq 1). \tag{2.2}$$

To solve the problem we have to exhibit a literal odd number with this property, or prove that it's impossible.

If $\sigma(n) < 2n$, we say n is **deficient**. Every number with $m = 1$, that is every prime power, is deficient. Every *odd* number with two distinct primes is also known to be deficient (See [R2], Nocco's Theorem). If $\sigma(n) > 2n$, we say n is **abundant**.

For $m \geq 3$, looking back at 2.2, we note that only one $\sigma(p_i^{a_i})$ must be even, and of the form $2S$, with odd S. The corresponding p_i is usually called the "special prime". So we can give it a special index, like, $i = s$ (p_s is the special prime). Moreover, to make it possible to balance the equation 2.2, with the above conditions, a_s must be odd. Further investigation reveals [R1] that, whatever a_s might be, $p_s + 1$ divides $\sigma(p_s^{a_s})$; hence $\frac{p_s+1}{2}$ divides n. Put another way, there is at least one $p_i < p_s$.

Furthermore, it should be obvious that every a_i, other than a_s must be even.

Finally, the last result we need from [R1] is that, if n is perfect, every divisor d of n would be deficient.

3. Odd perfect numbers don't exist

Proposition. *If a natural number is odd, then it's not a perfect number.*

Proof. We argue by contradiction. Let $n > 1$ be an odd perfect number and assume, on the contrary to the statement above, that n is perfect. Hence, 2.2 holds.

We arrange the primes in ascending order,

$$p_1 < p_2 < \cdots < p_m \tag{3.1}$$

relabeling the primes if necessary. Of course p_s (the special prime) apears in this chain of inequalities. We know that $p_s \neq p_1$, but we make no assumptions about its exact place.

We also have the list of divisors of n in descending order,

$$n > p_1^{a_1-1}\prod_{i=2}^{m} p_i^{a_i} > \cdots > p_1p_2 > \cdots > p_1 > 1. \tag{3.2}$$

We define,

$$\lambda = p_1^{a_1-1}\prod_{i=2}^{m} p_i^{a_i} = \frac{n}{p_1}, \tag{3.3}$$

and we note that (A) λ is the greatest [proper] divisor of n, and (B) $a_1 - 1 \neq 0$, because a_1 is even.

Finally, we divide $n-1$ by λ, and keep dividing the remainder by the next greatest divisor of n. The process will be described in detail.

Note. We wish to mention that the reader is already familiar with this idea. (In calculating the greates common divisor of two numbers, say, or converting a number from base 10 to base b, etc. The general idea is clear, but the difference is in several details, like "keep dividing the remainder" or "keep dividing the quotient" etc.) $\lhd$

Now, we write:

$$n - 1 = x\lambda + r \quad (0 \leq r < \lambda), \tag{3.4}$$

and solve for x. Of course, $n = p_1\lambda = (p_1 - 1)\lambda + \lambda$, implies that,

$$n - 1 = (p_1 - 1)\lambda + (\lambda - 1), \quad (x = p_1 - 1) \ \ (r = \lambda - 1). \tag{3.5}$$

(The division algorithm is carried out correctly.)

Next, we divide *the remainder* by $\frac{\lambda}{p_1} = \lambda_2$ which must be the next largest divisor of n. We obtain,

$$n - 1 = (p_1 - 1)\lambda + (p_1 - 1)\lambda_2 + (\lambda_2 - 1). \tag{3.6}$$

After several more steps, a_1-steps to be precise, the remainder will be $\frac{n}{p_1^{a_1}} - 1$, that is, the $p_1^{a_1}$ factor is vanished; and the way we arranged the primes, from smallest to largest, we are sure that the next greatest divisor of n is $p_2^{a_2-1}\prod_{i=3}^{m} p_i^{a_i}$, which goes into the last remainder $p_2 - 1$ times. This procedure goes on, until it reaches to the last two steps: $p_m^2 - 1 = (p_m - 1)p_m + (p_m - 1)$ and finally, $p_m - 1 = 1 \cdot (p_m - 1) + 0$. It takes $\sum_{i=1}^{m} a_i$ steps to finish. (We'll demonstrate this with a numerical example, shortly.) The result looks like,

$$n - 1 = (p_1 - 1)\lambda + (p_1 - 1)\lambda_2 + \cdots + (p_i - 1)\lambda_j + \cdots + (p_m - 1)p_m + (p_m - 1). \tag{3.7}$$

Note. What we've accomplished so far, is simply, converting the number $n - 1$ to the *mixed base* $p_1, p_2, ..., p_m$. We think of the quotients $p_1 - 1, p_2 - 1, \cdots, p_m - 1$ as *digits*, calculated in a **top-down** process. (But it's not necessary for the reader to adopt this point of view, as long as she agrees that the quotients are unique, and therefore the representation of $n - 1$ as above, is well defined.)

Now, in the usual algorithm for converting a number, for instance 523, to any base, let's say base 10, we first calculate the right-most digit of the result as $(523 \mod 10) = 3$. (This number is already written in base 10, but we are investigating how it was obtained.) If we call this the **bottom-up** process, we can define a top-down process, too, in which the *left-most* (i.e. the **most significant**) digit is calculated first. We start by observing that $10^2 < 523 < 10^3$ the we solve $523 = x10^2 + r$, hence $x = 5$. In this case, 10^2 is the greatest power of 10 which goes into 523.

In the above argument, we knew for sure that λ was the greatest divisor of n, and we knew in each step, what was exactly the next greatest divisor. In other words, in a concise manner, we just converted $n - 1$ to a mixed base. $\lhd$

Example. Perform the above algorithm for $n = 2205$.

solution. We have $n = 3^2 \cdot 5 \cdot 7^2$, in that order of primes. Then,

$$2205 = 2(3^1 \cdot 5 \cdot 7^2) + (3^1 \cdot 5 \cdot 7^2),$$

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$$2205 = 2(3^1 \cdot 5 \cdot 7^2) + 2(5 \cdot 7^2) + 4(7^2) + 7^2,$$

$$2205 = 2(3^1 \cdot 5 \cdot 7^2) + 2(5 \cdot 7^2) + 4(7^2) + 6(7) + 7.$$

The result is:

$$2205 - 1 = 2(3^1 \cdot 5 \cdot 7^2) + 2(5 \cdot 7^2) + 4(7^2) + 6(7) + 6.$$

Note that by adding 1 to both sides of this equation, all steps will run in *reversed* order. We explain that, if we think of $2, 2, 4, 6, 6$ as digits, as having a *carry*, and all the digits that are carried to the next position above, will add up to the number 2205, which plays a role similar to 10^n in base 10. By analogy, if desired, think of adding 1 to 9999 in base 10.

Again, the proof avoids any explicit use of these concepts, for simplicity. ◁

We expand the sum of divisors of n, as:

$$2n = \sum_{d|n} d \tag{3.8}$$

then,

$$n - 1 = \lambda + \lambda_2 + \cdots + p_2 p_1 + \cdots + p_1. \tag{3.9}$$

We like to compare this form, with the result of the previous calculation for $n-1$.

Given the uniqueness of the steps taken above, to obtain,

$$n - 1 = (p_1 - 1)\lambda + (p_1 - 1)\lambda_2 + \cdots + (p_i - 1)\lambda_j + \cdots + (p_m - 1)p_m + (p_m - 1), \tag{3.10}$$

We realize,

$$(\sum_{\substack{d|n \\ d<p_m}} d) - 1 = p_m - 1 \tag{3.11}$$

Note. The sum runs over all divisors of n (less than p_m), not all divisors of p_m. ◁

Otherwise, $\sum_{\substack{1<d|n \\ d<p_m}} \neq p_m - 1$, and either the last remainder or the next to the last remainder wouldn't be $p_m - 1$, which is a contradiction, if the above procedure is well-defined.

Similarly,

$$(\sum_{\substack{d|n \\ d<p_m^2}} d) - 1 = p_m^2 - 1, \tag{3.12}$$

up to,

$$(\sum_{\substack{d|n \\ d<p_m^{a_m}}} d) - 1 = p_m^{a_m} - 1. \tag{3.13}$$

We notice that if we add 1 to both sides of $n-1$ in

$$n - 1 = (p_1 - 1)\lambda + (p_1 - 1)\lambda_2 + \cdots + (p_i - 1)\lambda_j + \cdots + (p_m - 1)p_m + (p_m - 1), \tag{3.14}$$

we get a cascading effect of sums, (a chain reaction, if you like) that runs backwards, and in each step we get a divisor of n, from p_m^1 to $p_m^{a_m}$, then from $p_{(m-1)}^1 p_m^{a_m}$ to $p_{(m-1)}^{a_{(m-1)}} p_m^{a_m}$, all the way back to λ. For each step, we can deduce an equality like 3.12 and 3.13. The last of them being,

$$(\sum_{\substack{d|n\\ d<\lambda}} d) - 1 = \lambda - 1. \tag{3.15}$$

or,

$$\sum_{\substack{d|n\\ d<\lambda}} d = \lambda. \tag{3.16}$$

which works just fine for the last reversed step (to be added to $(p_1-1)\lambda$ and get n), but comparing this result with,

$$n = \sum_{\substack{d|n\\ d<n}} d \tag{3.17}$$

and it's expanded form, we notice that we must account for λ and everything else less than λ, in the right-hand side.

Note. We can't assume that every divisor d of n has the property $d|\lambda$. It holds for *some* divisors of n. (There are divisors with $p_1^{a_1}|d$, too.) But we know that for every divisor d of n we have $d<\lambda$, except of course for λ itself, and n. $\triangleleft$

But that simply leads to,

$$n = \sum_{\substack{d|n\\ d<n}} d = \lambda + \sum_{\substack{d|n\\ d<\lambda}} d = 2\lambda \tag{3.18}$$

Therefore n is odd and divisible by 2. From this contradiction we conclude that an odd number n cannot be perfect. This completes the proof. $\square$

(The proof could be represented as a direct argument, rather than an ad absurdum.)

4. Conclusion

We proved that perfect numbers are always even, and therefore, always related to Mersenne primes. Whether or not the set of Mersenne primes is infinite, is another interesting open problem [R3].

References

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