

Problem 11045

(American Mathematical Monthly, Vol.110, December 2003)

Proposed by Manoj Prakash Singh, New Delhy, India.

Prove that when n is a sufficiently large positive integer there exists a finite set S of prime numbers such that the sum of $\lfloor n/p \rfloor$ over $p \in S$ is equal to n .

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

The proof is divided in two parts. In the first one, using Bertrand’s Postulate, we show that for any integer $n \geq 1$ there exists a finite set T of primes with $|T| \leq \log_2 n + 1$ such that

$$n = \sum_{p \in T} \lfloor n/p \rfloor + |T|.$$

In the second part, using Prime Number Theorem, we prove that for n sufficiently large there exists a set T' of $|T|$ primes contained in $(n/2, n] \setminus T$.

Therefore

$$\sum_{p \in T'} \lfloor n/p \rfloor = \sum_{p \in T'} 1 = |T'| = |T|$$

and letting $S = T \cup T'$ we have that for n sufficiently large

$$\sum_{p \in S} \lfloor n/p \rfloor = \sum_{p \in T} \lfloor n/p \rfloor + \sum_{p \in T'} \lfloor n/p \rfloor = \sum_{p \in T} \lfloor n/p \rfloor + |T| = n$$

with $|S| \leq 2(\log_2 n + 1)$.

1) We first note that

for any integer $k \in [1, n]$ there is a prime p such that $\lfloor n/p \rfloor + 1 \in (k/2, k]$.

In fact, by Bertrand’s Postulate, for any real number $x \geq 1$ there is a prime p such that $x < p \leq 2x$. Letting $x = n/k$ then

$$k/2 \leq n/p < k.$$

The first inequality implies that

$$k/2 \leq \lfloor 2n/p \rfloor / 2 \leq (2\lfloor n/p \rfloor + 1) / 2 < \lfloor n/p \rfloor + 1,$$

whereas the second one gives

$$\lfloor n/p \rfloor + 1 \leq k.$$

Let p_j be the j -th prime number for $j \geq 1$. Now we prove by induction on m that for any integer $m \in [1, n]$ there exist $j_1 < j_2 < \dots < j_N$ with $N \leq \log_2 m + 1$ such that

$$m = (\lfloor n/p_{j_1} \rfloor + 1) + (\lfloor n/p_{j_2} \rfloor + 1) + \dots + (\lfloor n/p_{j_N} \rfloor + 1).$$

If $m = 1$ then there is j_1 such that $\lfloor n/p_{j_1} \rfloor + 1 \in (m/2, m] = \{1\}$ that is

$$1 = \lfloor n/p_{j_1} \rfloor + 1.$$

Let $m \geq 2$ and assume that the statement is true for $1, 2, \dots, m-1$ then there is j_1 such that $\lfloor n/p_{j_1} \rfloor + 1 \in (m/2, m]$. Hence

$$0 \leq m - (\lfloor n/p_{j_1} \rfloor + 1) < m/2 \leq m-1$$

and, by induction hypothesis, there exist $j_2 < \dots < j_N$ with

$$N-1 \leq \log_2 (m - (\lfloor n/p_{j_1} \rfloor + 1)) + 1 < \log_2(m/2) + 1 = \log_2 m$$

such that

$$m - (\lfloor n/p_{j_1} \rfloor + 1) = (\lfloor n/p_{j_2} \rfloor + 1) + \dots + (\lfloor n/p_{j_N} \rfloor + 1).$$

Note that $j_1 < j_2$ because $\lfloor n/p_{j_1} \rfloor + 1 > m/2$ and therefore

$$(\lfloor n/p_{j_2} \rfloor + 1) \leq m - (\lfloor n/p_{j_1} \rfloor + 1) < (\lfloor n/p_{j_1} \rfloor + 1).$$

So we have established that for any integer $n \geq 1$ there exists a finite set T of primes such that

$$n = \sum_{p \in T} (\lfloor n/p \rfloor + 1) = \sum_{p \in T} \lfloor n/p \rfloor + |T|$$

with $|T| \leq \log_2 n + 1$.

2) The Prime Number Theorem says that

$$\lim_{n \rightarrow \infty} \frac{|\{p \in (1, n] : p \text{ is prime}\}|}{n/\log n} = 1.$$

Since $(n/2, n] = (1, n] \setminus (1, n/2]$ then

$$\lim_{n \rightarrow \infty} \frac{|\{p \in (n/2, n] : p \text{ is prime}\}|}{n/\log n} = 1/2,$$

and therefore, for n sufficiently large,

$$|\{p \in (n/2, n] : p \text{ is prime}\}| \geq \frac{n}{3 \log n} \geq 2(\log_2 n + 1) \geq 2|T|.$$

So it is possible to find $|T|$ primes in the interval $(n/2, n]$ which are different from those ones in T . This is the required set T' .