Inequalities from 2008 Mathematical Competition

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Happy new year 2009!

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Abbreviations

- **IMO** International mathematical Olympiad
- **TST** Team Selection Test
- **MO** Mathematical Olympiad
- **LHS** Left hand side
- **RHS** Right hand side
- **W.L.O.G** Without loss of generality
- $\sum$: $\sum_{cyclic}$

Contents

Chapter 1

Problems

Pro 1. ***(Vietnamese National Olympiad 2008)*** *Let x, y, z be distinct non-negative real numbers. Prove that*

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2} \geq \frac{4}{xy+yz+zx}.$$

∇

Pro 2. ***(Iranian National Olympiad (3rd Round) 2008)****. Find the smallest real K such that for each $x, y, z \in \mathbb{R}^+$:*

$$x\sqrt{y}+y\sqrt{z}+z\sqrt{x} \leq K\sqrt{(x+y)(y+z)(z+x)}$$

∇

Pro 3. ***(Iranian National Olympiad (3rd Round) 2008)****. Let $x, y, z \in \mathbb{R}^+$ and $x+y+z=3$. Prove that:*

$$\frac{x^3}{y^3+8}+\frac{y^3}{z^3+8}+\frac{z^3}{x^3+8} \geq \frac{1}{9}+\frac{2}{27}(xy+xz+yz)$$

∇

Pro 4. ***(Iran TST 2008.)*** *Let $a, b, c > 0$ and $ab+ac+bc=1$. Prove that:*

$$\sqrt{a^3+a}+\sqrt{b^3+b}+\sqrt{c^3+c} \geq 2\sqrt{a+b+c}$$

∇

Pro 5. ***(Macedonian Mathematical Olympiad 2008.)*** *Positive numbers a, b, c are such that $(a+b)(b+c)(c+a)=8$. Prove the inequality*

$$\frac{a+b+c}{3} \geq \sqrt[27]{\frac{a^3+b^3+c^3}{3}}$$

∇

Pro 6. ***(Mongolian TST 2008)*** *Find the maximum number C such that for any non-negative x, y, z the inequality*

$$x^3 + y^3 + z^3 + C(xy^2 + yz^2 + zx^2) \geq (C + 1)(x^2y + y^2z + z^2x).$$

holds.

∇

Pro 7. ***(Federation of Bosnia, 1. Grades 2008.)*** *For arbitrary reals x, y and z prove the following inequality:*

$$x^2 + y^2 + z^2 - xy - yz - zx \geq \max\{\frac{3(x-y)^2}{4}, \frac{3(y-z)^2}{4}, \frac{3(y-z)^2}{4}\}.$$

∇

Pro 8. ***(Federation of Bosnia, 1. Grades 2008.)*** *If a, b and c are positive reals such that $a^2 + b^2 + c^2 = 1$ prove the inequality:*

$$\frac{a^5 + b^5}{ab(a+b)} + \frac{b^5 + c^5}{bc(b+c)} + \frac{c^5 + a^5}{ca(a+b)} \geq 3(ab + bc + ca) - 2$$

∇

Pro 9. ***(Federation of Bosnia, 1. Grades 2008.)*** *If a, b and c are positive reals prove inequality:*

$$(1 + \frac{4a}{b+c})(1 + \frac{4b}{a+c})(1 + \frac{4c}{a+b}) > 25$$

∇

Pro 10. ***(Croatian Team Selection Test 2008)*** *Let x, y, z be positive numbers. Find the minimum value of:*

$$(a) \quad \frac{x^2 + y^2 + z^2}{xy + yz}$$

$$(b) \quad \frac{x^2 + y^2 + 2z^2}{xy + yz}$$

∇

Pro 11. ***(Moldova 2008 IMO-BMO Second TST Problem 2)*** *Let $a_1, \ldots, a_n$ be positive reals so that $a_1 + a_2 + \ldots + a_n \leq \frac{n}{2}$. Find the minimal value of*

$$A = \sqrt{a_1^2 + \frac{1}{a_2^2}} + \sqrt{a_2^2 + \frac{1}{a_3^2}} + \ldots + \sqrt{a_n^2 + \frac{1}{a_1^2}}$$

∇

Pro 12. ***(RMO 2008, Grade 8, Problem 3)*** *Let $a, b \in [0, 1]$. Prove that*

$$\frac{1}{1 + a + b} \leq 1 - \frac{a+b}{2} + \frac{ab}{3}.$$

∇

Pro 13. ***(Romanian TST 2 2008, Problem 1)*** *Let $n \geq 3$ be an odd integer. Determine the maximum value of*

$$\sqrt{|x_1 - x_2|} + \sqrt{|x_2 - x_3|} + \ldots + \sqrt{|x_{n-1} - x_n|} + \sqrt{|x_n - x_1|},$$

where x_i are positive real numbers from the interval $[0, 1]$

∇

Pro 14. ***(Romania Junior TST Day 3 Problem 2 2008)*** *Let a, b, c be positive reals with $ab + bc + ca = 3$. Prove that:*

$$\frac{1}{1 + a^2(b + c)} + \frac{1}{1 + b^2(a + c)} + \frac{1}{1 + c^2(b + a)} \leq \frac{1}{abc}.$$

∇

Pro 15. ***(Romanian Junior TST Day 4 Problem 4 2008)*** *Determine the maximum possible real value of the number k, such that*

$$(a + b + c)\left(\frac{1}{a + b} + \frac{1}{c + b} + \frac{1}{a + c} - k\right) \geq k$$

for all real numbers $a, b, c \geq 0$ with $a + b + c = ab + bc + ca$.

∇

Pro 16. ***(2008 Romanian Clock-Tower School Junior Competition)*** *For any real numbers $a, b, c > 0$, with $abc = 8$, prove*

$$\frac{a - 2}{a + 1} + \frac{b - 2}{b + 1} + \frac{c - 2}{c + 1} \leq 0$$

∇

Pro 17. ***(Serbian National Olympiad 2008)*** *Let a, b, c be positive real numbers such that $x + y + z = 1$. Prove inequality:*

$$\frac{1}{yz + x + \frac{1}{x}} + \frac{1}{xz + y + \frac{1}{y}} + \frac{1}{xy + z + \frac{1}{z}} \leq \frac{27}{31}.$$

∇

Pro 18. ***(Canadian Mathematical Olympiad 2008)*** *Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that*

$$\frac{a - bc}{a + bc} + \frac{b - ca}{b + ca} + \frac{c - ab}{c + ab} \leq \frac{3}{2}.$$

∇

Pro 19. ***(German DEMO 2008)*** *Find the smallest constant C such that for all real x, y*

$$1 + (x + y)^2 \le C \cdot (1 + x^2) \cdot (1 + y^2)$$

holds.

∇

Pro 20. ***(Irish Mathematical Olympiad 2008)*** *For positive real numbers a, b, c and d such that $a^2 + b^2 + c^2 + d^2 = 1$ prove that*

$$a^2b^2cd + +ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \le 3/32,$$

and determine the cases of equality.

∇

Pro 21. ***(Greek national mathematical olympiad 2008, P1)*** *For the positive integers $a_1, a_2, ..., a_n$ prove that*

$$\left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i}\right)^{\frac{kn}{t}} \ge \prod_{i=1}^n a_i$$

where $k = max\{a_1, a_2, ..., a_n\}$ and $t = min\{a_1, a_2, ..., a_n\}$. When does the equality hold?

∇

Pro 22. ***(Greek national mathematical olympiad 2008, P2)***
If x, y, z are positive real numbers with $x, y, z < 2$ and $x^2 + y^2 + z^2 = 3$ prove that

$$\frac{3}{2} < \frac{1 + y^2}{x + 2} + \frac{1 + z^2}{y + 2} + \frac{1 + x^2}{z + 2} < 3$$

∇

Pro 23. ***(Moldova National Olympiad 2008)*** *Positive real numbers a, b, c satisfy inequality $a + b + c \le \frac{3}{2}$. Find the smallest possible value for:*

$$S = abc + \frac{1}{abc}$$

∇

Pro 24. ***(British MO 2008)*** *Find the minimum of $x^2 + y^2 + z^2$ where $x, y, z \in \mathbb{R}$ and satisfy $x^3 + y^3 + z^3 - 3xyz = 1$*

∇

Pro 25. ***(Zhautykov Olympiad, Kazakhstan 2008, Question 6)*** *Let a, b, c be positive integers for which $abc = 1$. Prove that*

$$\sum \frac{1}{b(a + b)} \ge \frac{3}{2}.$$

∇

Pro 26. ***(Ukraine National Olympiad 2008, P1)*** *Let x, y and z are non-negative numbers such that $x^2+y^2+z^2=3$. Prove that:*

$$\frac{x}{\sqrt{x^2+y+z}}+\frac{y}{\sqrt{x+y^2+z}}+\frac{z}{\sqrt{x+y+z^2}}\leq\sqrt{3}$$

∇

Pro 27. ***(Ukraine National Olympiad 2008, P2)*** *For positive a,b,c,d prove that*

$$(a+b)(b+c)(c+d)(d+a)(1+\sqrt[4]{abcd})^4\geq 16abcd(1+a)(1+b)(1+c)(1+d)$$

∇

Pro 28. ***(Polish MO 2008, Pro 5)*** *Show that for all nonnegative real values an inequality occurs:*

$$4(\sqrt{a^3b^3}+\sqrt{b^3c^3}+\sqrt{c^3a^3})\leq 4c^3+(a+b)^3.$$

∇

Pro 29. ***(Brazilian Math Olympiad 2008, Problem 3).*** *Let x,y,z real numbers such that $x+y+z=xy+yz+zx$. Find the minimum value of*

$$\frac{x}{x^2+1}+\frac{y}{y^2+1}+\frac{z}{z^2+1}$$

∇

Pro 30. ***(Kiev 2008, Problem 1).*** *Let $a,b,c\geq 0$. Prove that*

$$\frac{a^2+b^2+c^2}{5}\geq min((a-b)^2,(b-c)^2,(c-a)^2)$$

∇

Pro 31. ***(Kiev 2008, Problem 2).*** *Let $x_1,x_2,\cdots,x_n\geq 0, n>3$ and $x_1+x_2+\cdots+x_n=2$ Find the minimum value of*

$$\frac{x_2}{1+x_1^2}+\frac{x_3}{1+x_2^2}+...+\frac{x_1}{1+x_n^2}$$

∇

Pro 32. ***(Hong Kong TST1 2009, Problem 1).*** *Let $\theta_1,\theta_2,\ldots,\theta_{2008}$ be real numbers. Find the maximum value of*

$$\sin\theta_1\cos\theta_2+\sin\theta_2\cos\theta_3+\ldots+\sin\theta_{2007}\cos\theta_{2008}+\sin\theta_{2008}\cos\theta_1$$

∇

Pro 33. ***(Hong Kong TST1 2009, Problem 5)***. *Let a, b, c be the three sides of a triangle. Determine all possible values of*

$$\frac{a^2+b^2+c^2}{ab+bc+ca}$$

∇

Pro 34. ***(Indonesia National Science Olympiad 2008)***. *Prove that for x and y positive reals,*

$$\frac{1}{(1+\sqrt{x})^2}+\frac{1}{(1+\sqrt{y})^2}\ge\frac{2}{x+y+2}.$$

∇

Pro 35. ***(Baltic Way 2008)***. *Prove that if the real numbers a, b and c satisfy $a^2+b^2+c^2=3$ then*

$$\sum\frac{a^2}{2+b+c^2}\ge\frac{(a+b+c)^2}{12}.$$

When does the inequality hold?

∇

Pro 36. ***(Turkey NMO 2008 Problem 3)***. *Let $a.b.c$ be positive reals such that their sum is 1. Prove that*

$$\frac{a^2b^2}{c^3(a^2-ab+b^2)}+\frac{b^2c^2}{a^3(b^2-bc+c^2)}+\frac{a^2c^2}{b^3(a^2-ac+c^2)}\ge\frac{3}{ab+bc+ac}$$

∇

Pro 37. ***(China Western Mathematical Olympiad 2008)***. *Given $x,y,z\in(0,1)$ satisfying that*

$$\sqrt{\frac{1-x}{yz}}+\sqrt{\frac{1-y}{xz}}+\sqrt{\frac{1-z}{xy}}=2.$$

Find the maximum value of xyz.

∇

Pro 38. ***(Chinese TST 2008 P5)***. *For two given positive integers $m,n>1$, let $a_{ij}(i=1,2,\cdots,n,j=1,2,\cdots,m)$ be nonnegative real numbers, not all zero, find the maximum and the minimum values of f, where*

$$f=\frac{n\sum_{i=1}^{n}(\sum_{j=1}^{m}a_{ij})^2+m\sum_{j=1}^{m}(\sum_{i=1}^{n}a_{ij})^2}{(\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij})^2+mn\sum_{i=1}^{n}\sum_{i=j}^{m}a_{ij}^2}$$

∇

Pro 39. ***(Chinese TST 2008 P6)*** *Find the maximal constant M, such that for arbitrary integer $n\ge3$, there exist two sequences of positive real number $a_1,a_2,\cdots,a_n$, and $b_1,b_2,\cdots,b_n$, satisfying*
(1):$\sum_{k=1}^{n}b_k=1,2b_k\ge b_{k-1}+b_{k+1},k=2,3,\cdots,n-1;$
(2):$a_k^2\le1+\sum_{i=1}^{k}a_ib_i,k=1,2,3,\cdots,n,a_n\equiv M.$

∇

Chapter 2

Solutions

Problem 1. ***(Vietnamese National Olympiad 2008)*** *Let x, y, z be distinct non-negative real numbers. Prove that*

$$\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} \geq \frac{4}{xy+yz+zx}.$$

Proof. *(Posted by* ***Vo Thanh Van****).* Assuming $z = min\{x, y, z\}$. We have

$$(x-z)^2 + (y-z)^2 = (x-y)^2 + 2(x-z)(y-z)$$

So by the **AM-GM inequality**, we get

$$\begin{aligned}
\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} &= \frac{1}{(x-y)^2} + \frac{(x-y)^2}{(y-z)^2(z-x)^2} + \frac{2}{(x-z)(y-z)} \\
&\geq \frac{2}{(x-z)(y-z)} + \frac{2}{(x-z)(y-z)} = \frac{4}{(x-z)(y-z)} \\
&\geq \frac{4}{xy+yz+zx}
\end{aligned}$$

Q.E.D. □

Proof. *(Posted by* ***Altheman****).* Let $f(x, y, z)$ denote the **LHS** minus the **RHS**. Then $f(x+d, y+d, z+d)$ is increasing in d so we can set the least of $x+d, y+d, z+d$ equal to zero (WLOG $z = 0$). Then we have

$$\frac{1}{(x-y)^2} + \frac{1}{x^2} + \frac{1}{y^2} - \frac{4}{xy} = \frac{(x^2+y^2-3xy)^2}{x^2y^2(x-y)^2} \geq 0$$

□

∇

Problem 2. ***(Iranian National Olympiad (3rd Round) 2008).*** *Find the smallest real K such that for each $x, y, z \in \mathbb{R}^+$:*

$$x\sqrt{y} + y\sqrt{z} + z\sqrt{x} \leq K\sqrt{(x+y)(y+z)(z+x)}$$

Proof. *(Posted by **nayel**).* By the **Cauchy-Schwarz inequality**, we have

$$\begin{aligned} LHS = \sqrt{x}\sqrt{xy} + \sqrt{y}\sqrt{yz} + \sqrt{z}\sqrt{zx} &\le \sqrt{(x+y+z)(xy+yz+zx)} \\ &\le \frac{3}{2\sqrt{2}}\sqrt{(x+y)(y+z)(z+x)} \end{aligned}$$

where the last inequality follows from

$$8(x+y+z)(xy+yz+zx) \le 9(x+y)(y+z)(z+x)$$

which is well known. □

Proof. *(Posted by **rofler**).* We want to find the **smallest** K. I claim $K = \dfrac{3}{2\sqrt{2}}$. The inequality is equivalent to

$$8(x\sqrt{y} + y\sqrt{z} + z\sqrt{x})^2 \le 9(x+y)(y+z)(z+x)$$

$$\Longleftrightarrow 8x^2y + 8y^2z + 8z^2x + 16xy\sqrt{yz} + 16yz\sqrt{zx} + 16xz\sqrt{xy} \le 9\sum_{sym} x^2y + 18xyz$$

$$\Longleftrightarrow 16xy\sqrt{yz} + 16yz\sqrt{zx} + 16xz\sqrt{xy} \le x^2y + y^2z + z^2x + 9y^2x + 9z^2y + 9x^2z + 18xyz$$

By the **AM-GM inequality**, we have

$$z^2x + 9y^2x + 6xyz \ge 16\sqrt[16]{z^2x \cdot y^{18}x^9 \cdot x^6y^6z^6} = 16xy\sqrt{xz}$$

Sum up cyclically. We can get equality when $x = y = z = 1$, so we know that K cannot be any smaller. □

Proof. *(Posted by **FelixD**).* We want to find the smallest K such that

$$(x\sqrt{y} + y\sqrt{z} + z\sqrt{x})^2 \le K^2(x+y)(y+z)(z+x)$$

But

$$\begin{aligned} (x\sqrt{y} + y\sqrt{z} + z\sqrt{x})^2 &= \sum_{cyc} x^2y + 2(\sum_{cyc} xy\sqrt{yz}) \\ &\le \sum_{cyc} x^2y + 2(\sum_{cyc} \frac{xyz + xy^2}{2}) \\ &= (x+y)(y+z)(z+x) + xyz \\ &\le (x+y)(y+z)(z+x) + \frac{1}{8}(x+y)(y+z)(z+x) \\ &= \frac{9}{8}(x+y)(y+z)(z+x) \end{aligned}$$

Therefore,

$$K^2 \ge \frac{9}{8} \to K \ge \frac{3}{2\sqrt{2}}$$

with equality holds if and only if $x = y = z$. □

∇

Problem 3. ***(Iranian National Olympiad (3rd Round) 2008)***. *Let $x, y, z \in \mathbb{R}^+$ and $x + y + z = 3$. Prove that:*

$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} \geq \frac{1}{9} + \frac{2}{27}(xy + xz + yz)$$

Proof. (Posted by ***rofler****).* By the **AM-GM inequality**, we have

$$\frac{x^3}{(y+2)(y^2-2y+4)} + \frac{y+2}{27} + \frac{y^2-2y+4}{27} \geq \frac{x}{3}$$

Summing up cyclically, we have

$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} + \frac{x^2+y^2+z^2-(x+y+z)+6*3}{27}$$

$$\geq 1 \geq \frac{1}{3} + \frac{1}{9} - \frac{x^2+y^2+z^2}{27}$$

Hence it suffices to show that

$$\frac{1}{3} - \frac{x^2+y^2+z^2}{27} \geq \frac{2}{27}(xy + xz + yz)$$
$$\iff 9 - (x^2+y^2+z^2) \geq 2(xy + xz + yz)$$
$$\iff 9 \geq (x+y+z)^2 = 9$$

Q.E.D. □

∇

Problem 4. ***(Iran TST 2008.)*** *Let $a, b, c > 0$ and $ab + ac + bc = 1$. Prove that:*

$$\sqrt{a^3+a} + \sqrt{b^3+b} + \sqrt{c^3+c} \geq 2\sqrt{a+b+c}$$

Proof. (Posted by ***Albanian Eagle****).* It is equivalent to:

$$\sum_{cyc} \frac{a}{\sqrt{a(b+c)}} \geq 2\sqrt{\frac{(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)}}$$

Using the **Jensen inequality**, on $f(x) = \dfrac{1}{\sqrt{x}}$, we get

$$\sum_{cyc} \frac{a}{\sqrt{a(b+c)}} \geq \frac{a+b+c}{\sqrt{\dfrac{\sum_{sym} a^2b}{a+b+c}}}$$

So we need to prove that

$$(a+b+c)^2\left(\sum_{sym} a^2b + 2abc\right) \geq 4(ab+bc+ca)\left(\sum_{sym} a^2b\right)$$

Now let c be the smallest number among a, b, c and we see we can rewrite the above as

$$(a-b)^2(a^2b + b^2a + a^2c + b^2c - ac^2 - bc^2) + c^2(a+b)(c-a)(c-b) \geq 0$$

□

Proof. *(Posted by* ***Campos****).* The inequality is equivalent to

$$\sum \sqrt{a(a+b)(a+c)} \geq 2\sqrt{(a+b+c)(ab+bc+ca)}$$

After squaring both sides and canceling some terms we have that it is equivalent to

$$\sum a^3 + abc + 2(b+c)\sqrt{bc(a+b)(a+c)} \geq \sum 3a^2b + 3a^2c + 4abc$$

From the **Schur's inequality** we have that it is enough to prove that

$$\sum (b+c)\sqrt{(ab+b^2)(ac+c^2)} \geq \sum a^2b + a^2c + 2abc$$

From the **Cauchy-Schwarz inequality** we have

$$\sqrt{(ab+b^2)(ac+c^2)} \geq a\sqrt{bc} + bc$$

so

$$\sum (b+c)\sqrt{(ab+b^2)(ac+c^2)} \geq \sum a(b+c)\sqrt{bc} + bc(b+c) \geq \sum a^2b + a^2c + 2abc$$

as we wanted to prove. □

Proof. *(Posted by* ***anas****).* Squaring the both sides , our inequality is equivalent to:

$$\sum a^3 - 3\sum ab(a+b) - 9abc + 2\sum \sqrt{a(a+b)(a+c)}\sqrt{b(b+c)(b+a)} \geq 0$$

But, by the **AM-GM inequality**, we have:

$$\begin{aligned} &a(a+b)(a+c) \cdot b(b+c)(b+a) \\ &\qquad = (a^3 + a^2c + a^2b + abc)(ab^2 + b^2c + b^3 + abc) \\ &\qquad \geq (a^2b + abc + ab^2 + abc)^2 \end{aligned}$$

So we need to prove that:

$$a^3 + b^3 + c^3 - ab(a+b) - ac(a+c) - bc(b+c) + 3abc \geq 0$$

which is clearly true by the **Schur inequality** □

∇

Problem 5. ***Macedonian Mathematical Olympiad 2008.*** *Positive numbers a, b, c are such that $(a+b)(b+c)(c+a) = 8$. Prove the inequality*

$$\frac{a+b+c}{3} \geq \sqrt[27]{\frac{a^3+b^3+c^3}{3}}$$

Proof. *(Posted by* ***argady****).* By the **AM-GM inequality**, we have

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 24 = a^3 + b^3 + c^3 + 3 + \cdots + 3 \geq 9\sqrt[9]{(a^3+b^3+c^3) \cdot 3^8}$$

Q.E.D. □

*Proof. (Posted by **kunny**).* The inequality is equivalent to

$$(a+b+c)^{27} \geq 3^{26}(a^3+b^3+c^3) \ \cdots [*]$$

Let $a+b=2x,\ b+c=2y,\ c+a=2z$, we have that

$$(a+b)(b+c)(c+a)=8 \Longleftrightarrow xyz=1$$

and

$$2(a+b+c)=2(x+y+z) \Longleftrightarrow a+b+c=x+y+z$$

$$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a) \Longleftrightarrow a^3+b^3+c^3=(x+y+z)^3-24$$

Therefore

$$[*] \Longleftrightarrow (x+y+z)^{27} \geq 3^{26}\{(x+y+z)^3-24\}.$$

Let $t=(x+y+z)^3$, by **AM-GM inequality**, we have that

$$x+y+z \geq 3\sqrt[3]{xyz} \Longleftrightarrow x+y+z \geq 3$$

yielding $t \geq 27$.
Since $y=t^9$ is an increasing and concave up function for $t>0$, the tangent line of $y=t^9$ at $t=3$ is $y=3^{26}(t-27)+3^{27}$.We can obtain

$$t^9 \geq 3^{26}(t-27)+3^{27}$$

yielding $t^9 \geq 3^{26}(t-24)$, which completes the proof. □

*Proof. (Posted by **kunny**).* The inequality is equivalent to

$$\frac{(a+b+c)^{27}}{a^3+b^3+c^3} \geq 3^{26}.$$

Let $x=(a+b+c)^3$, by the **AM-GM inequality**, we have:

$$8=(a+b)(b+c)(c+a) \leq \left(\frac{2(a+b+c)}{3}\right)^3$$

so $a+b+c \geq 3$ The left side of the above inequality

$$f(x):=\frac{x^9}{x-24} \Longrightarrow f'(x)=\frac{8x^8(x-27)}{(x-24)^2} \geq 0$$

We have $f(x) \geq f(27)=3^{26}$. □

∇

Problem 6. ***(Mongolian TST 2008)*** *Find the maximum number C such that for any nonnegative x,y,z the inequality*

$$x^3+y^3+z^3+C(xy^2+yz^2+zx^2) \geq (C+1)(x^2y+y^2z+z^2x).$$

holds.

*Proof. (Posted by **hungkhtn**).* Applying **CID** (**C**yclic **I**nequality of **D**egree 3) [1] theorem, we can let $c = 0$ in the inequality. It becomes

$$x^3 + y^3 + cx^2y \geq (c+1)xy^2.$$

Thus, we have to find the minimal value of

$$f(y) = \frac{y^3 - y^2 + 1}{y^2 - y} = y + \frac{1}{y(y-1)}$$

when $y > 1$. It is easy to find that

$$f'(y) = 0 \Leftrightarrow 2y - 1 = (y(y-1))^2 \Leftrightarrow y^4 - 2y^3 + y^2 - 2y + 1 = 0.$$

Solving this symmetric equation gives us:

$$y + \frac{1}{y} = 1 + \sqrt{2} \Rightarrow y = \frac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2}$$

Thus we found the best value of C is

$$y + \frac{1}{y(y-1)} = \frac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2} + \frac{1}{\sqrt{\sqrt{2} + \sqrt{2\sqrt{2} - 1}}} \approx 2.4844$$

□

∇

Problem 7. ***(Federation of Bosnia, 1. Grades 2008.)*** *For arbitrary reals x, y and z prove the following inequality:*

$$x^2 + y^2 + z^2 - xy - yz - zx \geq \max\{\frac{3(x-y)^2}{4}, \frac{3(y-z)^2}{4}, \frac{3(y-z)^2}{4}\}.$$

*Proof. (Posted by **delegat**).* Assume that $\frac{3(x-y)^2}{4}$ is max. The inequality is equivalent to

$$4x^2 + 4y^2 + 4z^2 \geq 4xy + 4yz + 4xz + 3x^2 - 6xy + 3y^2$$

$$\Leftrightarrow x^2 + 2xy + y^2 + z^2 \geq 4yz + 4xz$$

$$\Leftrightarrow (x + y - 2z)^2 \geq 0$$

so we are done. □

∇

Problem 8. ***(Federation of Bosnia, 1. Grades 2008.)*** *If a, b and c are positive reals such that $a^2 + b^2 + c^2 = 1$ prove the inequality:*

$$\frac{a^5 + b^5}{ab(a+b)} + \frac{b^5 + c^5}{bc(b+c)} + \frac{c^5 + a^5}{ca(a+b)} \geq 3(ab + bc + ca) - 2$$

[1] You can see here: `http://www.mathlinks.ro/viewtopic.php?p=1130901`

*Proof. (Posted by **Athinaios**).* Firstly, we have

$$(a+b)(a-b)^2(a^2+ab+b^2) \geq 0$$

so

$$a^5+b^5 \geq a^2b^2(a+b).$$

Applying the above inequality, we have

$$LHS \geq ab+bc+ca$$

So we need to prove that

$$ab+bc+ca+2 \geq 3(ab+bc+ca)$$

or

$$2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$$

Which is clearly true. □

*Proof. (Posted by **kunny**).* Since $y = x^5$ is an increasing and downwards convex function for $x > 0$, by the **Jensen inequality** we have

$$\frac{a^5+b^5}{2} \geq \left(\frac{a+b}{2}\right)^5 \Longleftrightarrow \frac{a^5+b^5}{ab(a+b)} \geq \frac{1}{16}\cdot\frac{(a+b)^4}{ab} = \frac{1}{16}(a+b)^2\cdot\frac{(a+b)^2}{ab}$$

$$\geq \frac{1}{16}(a+b)^2\cdot 4$$

(because $(a+b)^2 \geq 4ab$ for $a > 0,\ b > 0$)
Thus for $a > 0,\ b > 0,\ c > 0$,

$$\begin{aligned}\frac{a^5+b^5}{ab(a+b)}+\frac{b^5+c^5}{bc(b+c)}+\frac{c^5+a^5}{ca(c+a)} &\geq \frac{1}{4}\{(a+b)^2+(b+c)^2+(c+a)^2\}\\ &= \frac{1}{2}(a^2+b^2+c^2+ab+bc+ca)\\ &\geq ab+bc+ca\end{aligned}$$

Then we are to prove

$$ab+bc+ca \geq 3(ab+bc+ca)-2$$

which can be proved by

$$ab+bc+ca \geq 3(ab+bc+ca)-2 \Leftrightarrow 1 \geq ab+bc+ca \Leftrightarrow a^2+b^2+c^2 \geq ab+bc+ca$$

Q.E.D. □

Comment
We can prove the stronger inequality:

$$\frac{a^5+b^5}{ab(a+b)}+\frac{b^5+c^5}{bc(b+c)}+\frac{c^5+a^5}{ca(a+c)} \geq 6-5(ab+bc+ca).$$

Proof. (Posted by ***HTA****)*. It is equivalent to

$$\sum \frac{a^5+b^5}{ab(a+b)} - \sum \frac{1}{2}(a^2+b^2) \geq \frac{5}{2}\left(\sum (a-b)^2\right)$$

$$\sum (a-b)^2 \left(\frac{2a^2+ab+2b^2}{2ab} - \frac{5}{2}\right) \geq 0$$

$$\sum \frac{(a-b)^4}{ab} \geq 0$$

which is true. □

∇

Problem 9. ***(Federation of Bosnia, 1. Grades 2008.)*** *If a, b and c are positive reals prove inequality:*

$$(1+\frac{4a}{b+c})(1+\frac{4b}{a+c})(1+\frac{4c}{a+b}) > 25$$

Proof. (Posted by ***polskimisiek****)*. After multiplying everything out, it is equivalent to:

$$4(\sum_{cyc} a^3) + 23abc > 4(\sum_{cyc} a^2(b+c))$$

which is obvious, because by the **Schur inequality**, we have:

$$(\sum_{cyc} a^3) + 3abc \geq \sum_{cyc} a^2(b+c)$$

So finally we have:

$$4(\sum_{cyc} a^3) + 23abc > 4(\sum_{cyc} a^3) + 12abc \geq 4\sum_{cyc} a^2(b+c)$$

Q.E.D □

∇

Problem 10. ***(Croatian Team Selection Test 2008)*** *Let x, y, z be positive numbers. Find the minimum value of:*

$$(a) \quad \frac{x^2+y^2+z^2}{xy+yz}$$

$$(b) \quad \frac{x^2+y^2+2z^2}{xy+yz}$$

Proof. (Posted by ***nsato****)*.
(a) The minimum value is $\sqrt{2}$. Expanding

$$\left(x - \frac{\sqrt{2}}{2}y\right)^2 + \left(\frac{\sqrt{2}}{2}y - z\right)^2 \geq 0,$$

we get $x^2+y^2+z^2-\sqrt{2}xy-\sqrt{2}yz \geq 0$, so

$$\frac{x^2+y^2+z^2}{xy+yz} \geq \sqrt{2}.$$

Equality occurs, for example, if $x=1$, $y=\sqrt{2}$, and $z=1$.

(b) The minimum value is $\sqrt{8/3}$. Expanding

$$\left(x-\sqrt{\frac{2}{3}}y\right)^2+\frac{1}{3}\left(y-\sqrt{6}z\right)^2 \geq 0,$$

we get $x^2+y^2+2z^2-\sqrt{8/3}xy-\sqrt{8/3}yz \geq 0$, so

$$\frac{x^2+y^2+z^2}{xy+yz} \geq \sqrt{\frac{8}{3}}.$$

Equality occurs, for example, if $x=2$, $y=\sqrt{6}$, and $z=1$. □

∇

Problem 11. ***(Moldova 2008 IMO-BMO Second TST Problem 2)*** *Let $a_1,\dots,a_n$ be positive reals so that $a_1+a_2+\ldots+a_n \leq \frac{n}{2}$. Find the minimal value of*

$$A=\sqrt{a_1^2+\frac{1}{a_2^2}}+\sqrt{a_2^2+\frac{1}{a_3^2}}+\ldots+\sqrt{a_n^2+\frac{1}{a_1^2}}$$

*Proof. (Posted by **NguyenDungTN**).* Using **Minkowski** and **Cauchy-Schwarz** inequalities we get

$$A \geq \sqrt{(a_1+a_2+\ldots+a_n)^2+\left(\frac{1}{a_1}+\frac{1}{a_2}\ldots+\frac{1}{n}\right)^2}$$
$$\geq \sqrt{(a_1+a_2+\ldots+a_n)^2+\frac{n^4}{(a_1+a_2+\ldots+a_n)^2}}$$

By the **AM-GM inequality**:

$$(a_1+a_2+\ldots+a_n)^2+\frac{\left(\frac{n}{2}\right)^4}{(a_1+a_2+\ldots+a_n)^2} \geq \frac{n^2}{2}$$

Because $a_1+a_2+\ldots+a_n \leq \frac{n}{2}$ so

$$\frac{\frac{15n^4}{16}}{(a_1+a_2+\ldots+a_n)^2} \geq \frac{15n^2}{4}$$

We obtain

$$A \geq \sqrt{\frac{n^2}{2}+\frac{15n^2}{4}} = \frac{\sqrt{17}n}{2}$$

□

*Proof. (Posted by **silouan**).* Using **Minkowski** and **Cauchy-Schwarz** inequalities we get

$$A \geq \sqrt{(a_1 + a_2 + \ldots + a_n)^2 + \left(\frac{1}{a_1} + \frac{1}{a_2} \ldots + \frac{1}{n}\right)^2}$$

$$\geq \sqrt{(a_1 + a_2 + \ldots + a_n)^2 + \frac{n^4}{(a_1 + a_2 + \ldots + a_n)^2}}$$

Let $a_1 + ... + a_n = s$. Consider the function $f(s) = s^2 + \frac{n^4}{s^2}$
This function is decreasing for $s \in \left(0, \frac{n}{2}\right]$. So it attains its minimum at $s = \frac{n}{2}$ and we are done . □

*Proof. (Posted by **ddlam**).* By the **AM-GM inequality**, we have

$$a_1^2 + \frac{1}{a_2^2} = a_1^2 + \frac{1}{16a_2^2} + ... + \frac{1}{16a_2^2} \geq 17 \sqrt[17]{\frac{a_1^2}{(16a_2^2)^{16}}}$$

so

$$A \geq \sqrt{17} \sum_{i=1}^{n} \sqrt[34]{\frac{a_i^2}{16^{16} a_{i+1}^{32}}} \quad (a_{i+1} = a_1)$$

By the **AM-GM inequality** again:

$$\sum_{i=1}^{n} \sqrt[34]{\frac{a_i^2}{16^{16} a_{i+1}^{32}}} \geq \frac{n}{\left(\prod_{i=1}^{n} 16^{16n} x_i^{30}\right)^{34n}}$$

But

$$\prod i = 1^n x_i^n \leq \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^n \leq \frac{1}{2^n}$$

So

$$A \geq \frac{\sqrt{17} n}{2}$$

□

∇

Problem 12. ***(RMO 2008, Grade 8, Problem 3)*** *Let $a, b \in [0, 1]$. Prove that*

$$\frac{1}{1 + a + b} \leq 1 - \frac{a + b}{2} + \frac{ab}{3}.$$

*Proof. (Posted by **Dr Sonnhard Graubner**).* The given inequality is equivalent to

$$3(1 - a)(1 - b)(a + b) + ab(1 - a + 1 - b) \geq 0$$

which is true because of $0 \leq a \leq 1$ and $0 \leq b \leq 1$. □

*Proof. (Posted by **HTA**).* Let

$$f(a,b) = 1 - \frac{a+b}{2} + \frac{ab}{3} - \frac{1}{1+a+b}$$

Consider the difference between $f(a,b)$ and $f(1,b)$ we see that

$$f(a,b) - f(1,b) = \frac{1}{6}\frac{(b-1)(a+2a(b+1)+3b+2b(b+1))-3a}{(1+a+b)(2+b)} \geq 0$$

it is left to prove that $f(1,b) \geq 0$ which is equivalent to

$$\frac{-1}{6}\frac{b(b-1)}{2+b} \geq 0$$

Which is true . □

∇

Problem 13. ***(Romanian TST 2 2008, Problem 1)*** *Let $n \geq 3$ be an odd integer. Determine the maximum value of*

$$\sqrt{|x_1 - x_2|} + \sqrt{|x_2 - x_3|} + \ldots + \sqrt{|x_{n-1} - x_n|} + \sqrt{|x_n - x_1|},$$

where x_i are positive real numbers from the interval $[0,1]$

*Proof. (Posted by **Myth**).* We have a continuous function on a compact set $[0,1]^n$, hence there is an optimal point $(x_1, ..., x_n)$. Note now that

1. impossible to have $x_{i-1} = x_i = x_{i+1}$;
2. if $x_i \leq x_{i-1}$ and $x_i \leq x_{i+1}$, then $x_i = 0$;
3. if $x_i \geq x_{i-1}$ and $x_i \geq x_{i+1}$, then $x_i = 1$;
4. if $x_{i+1} \leq x_i \leq x_{i-1}$ or $x_{i-1} \leq x_i \leq x_{i+1}$, then $x_i = \dfrac{x_{i-1} + x_{i+1}}{2}$.

It follows that $(x_1, ..., x_n)$ looks like

$$(0, \frac{1}{k_1}, \frac{2}{k_1}, ..., 1, \frac{k_2 - 1}{k_2}, ..., \frac{2}{k_2}, \frac{1}{k_2}, 0, \frac{1}{k_3}, ..., \frac{1}{k_l}),$$

where k_1, k_2, ..., k_l are natural numbers, $k_1 + k_2 + ... + k_l = n$, l is even clearly. Then the function is this point equals

$$S = \sqrt{k_1} + \sqrt{k_2} + ...\sqrt{k_l}.$$

Using the fact that l is even and $\sqrt{k} < \sqrt{k-1} + 1$ we conclude that maximal possible value of S is $n - 2 + \sqrt{2}$ ($l = n-1$, $k_1 = k_2 = ... = k_{l-1} = 1$, $k_l = 2$ in this case). □

*Proof. (Posted by **Umut Varolgunes**).* Since n is odd, there must be an i such that both x_i and x_{i+1} are both belong to $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$. without loss of generality let $x_1 \le x_2$ and x_1, x_2 belong to $[0, \frac{1}{2}]$. We can prove that

$$\sqrt{x_2 - x_1} + \sqrt{Ix_3 - x_2I} \le \sqrt{2}$$

If $x_3 > x_2$, $\sqrt{x_2 - x_1} + \sqrt{x_3 - x_2} \le 2 \cdot \sqrt{\frac{x_3 - x_1}{2}} \le \sqrt{2}$;
else x_1, x_2, x_3 are all belong to $[0, \frac{1}{2}]$.
Hence, $\sqrt{x_2 - x_1} + \sqrt{Ix_3 - x_2I} \le \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}}$. Also all of the other terms of the sum are less then or equal to 1. summing them gives the desired result.
Example is $(0, \dfrac{1}{2}, 1, 0, 1, \dots, 1)$
Note: all the indices are considered in modulo n □

∇

Problem 14. ***(Romania Junior TST Day 3 Problem 2 2008)*** *Let a, b, c be positive reals with $ab + bc + ca = 3$. Prove that:*

$$\frac{1}{1 + a^2(b + c)} + \frac{1}{1 + b^2(a + c)} + \frac{1}{1 + c^2(b + a)} \le \frac{1}{abc}.$$

*Proof. (Posted by **silouan**).* Using the **AM-GM inequality**, we derive $\dfrac{ab + bc + ca}{3} \ge \sqrt[3]{(abc)^2}$. Then $abc \le 1$. Now

$$\sum \frac{1}{1 + a^2(b + c)} \le \sum \frac{1}{abc + a^2(b + c)} = \sum \frac{1}{3a} = \frac{1}{abc}$$

□

∇

Problem 15. ***(Romanian Junior TST Day 4 Problem 4 2008)*** *Determine the maximum possible real value of the number k, such that*

$$(a + b + c)\left(\frac{1}{a + b} + \frac{1}{c + b} + \frac{1}{a + c} - k\right) \ge k$$

for all real numbers $a, b, c \ge 0$ with $a + b + c = ab + bc + ca$.

Proof. (Original solution). Observe that the numbers $a = b = 2, c = 0$ fulfill the condition $a + b + c = ab + bc + ca$. Plugging into the givent inequality, we derive that $4\left(\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - k\right) \ge k$ hence $k \le 1$.
We claim that the inequality hold for $k = 1$, proving that the maximum value of k is 1. To this end, rewrite the inequality as follows

$$(ab + bc + ca)\left(\frac{1}{a + b} + \frac{1}{c + b} + \frac{1}{a + c} - 1\right) \ge 1$$

$$\Leftrightarrow \sum \frac{ab + bc + ca}{a + b} \ge ab + bc + ca + 1$$

$$\Leftrightarrow \sum \frac{ab}{a+b} + c \geq ab + bc + ca + 1 \Leftrightarrow \sum \frac{ab}{a+b} \geq 1$$

Notice that $\frac{ab}{a+b} \geq \frac{ab}{a+b+c}$, since $a, b, c \geq 0$. Summing over a cyclic permutation of a, b, c we get

$$\sum \frac{ab}{a+b} \geq \sum \frac{ab}{a+b+c} = \frac{ab+bc+ca}{a+b+c} = 1$$

as needed. □

Proof. (Alternative solution). The inequality is equivalent to the following

$$S = \frac{a+b+c}{a+b+c+1}\left(\frac{1}{a+b} + \frac{1}{c+b} + \frac{1}{a+c}\right)$$

Using the given condition, we get

$$\begin{aligned}
\frac{1}{a+b} + \frac{1}{c+b} + \frac{1}{a+c} &= \frac{a^2+b^2+c^2+3(ab+bc+ca)}{(a+b)(b+c)(c+a)} \\
&= \frac{a^2+b^2+c^2+2(ab+bc+ca)+(a+b+c)}{(a+b)(b+c)(c+a)} \\
&= \frac{(a+b+c)(a+b+c+1)}{(a+b+c)^2 - abc}
\end{aligned}$$

hence

$$S = \frac{(a+b+c)^2}{(a+b+c)^2 - abc}$$

It is now clear that $S \geq 1$, and equality hold iff $abc = 0$. Consequently, $k = 1$ is the maximum value. □

∇

Problem 16. ***(2008 Romanian Clock-Tower School Junior Competition)*** *For any real numbers $a, b, c > 0$, with $abc = 8$, prove*

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leq 0$$

Proof. (Original solution). We have:

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leq 0 \Leftrightarrow 3 - 3\sum \frac{1}{a+1} \leq 0 \Leftrightarrow 1 \leq \sum \frac{1}{a+1}$$

We can take $a = 2\dfrac{x}{y}, b = 2\dfrac{y}{z}, c = 2\dfrac{z}{x}$ to have

$$\sum \frac{1}{a+1} = \sum \frac{y^2}{2xy+y^2} \geq \frac{(x+y+z)^2}{x^2+y^2+z^2+2(xy+yz+zx)} = 1$$

(by the **Cauchy-Schwarz** inequality) as needed. □

∇

Problem 17. ***(Serbian National Olympiad 2008)*** *Let a, b, c be positive real numbers such that $x+y+z=1$. Prove inequality:*

$$\frac{1}{yz+x+\frac{1}{x}}+\frac{1}{xz+y+\frac{1}{y}}+\frac{1}{xy+z+\frac{1}{z}}\le\frac{27}{31}.$$

Proof. (Posted by ***canhang2007****).* Setting $x=\frac{a}{3}$,$y=\frac{b}{3}$,$z=\frac{c}{3}$. The inequality is equivalent to

$$\sum_{cyc}\frac{a}{3a^2+abc+27}\le\frac{3}{31}$$

By the **Schur Inequality**, we get $3abc\ge 4(ab+bc+ca)-9$. It suffices to prove that

$$\sum\frac{3a}{9a^2+4(ab+bc+ca)+72}\le\frac{3}{31}$$

$$\Leftrightarrow\sum\left(1-\frac{31a(a+b+c)}{9a^2+4(ab+bc+ca)+72}\right)\ge 0$$

$$\Leftrightarrow\sum\frac{(7a+8c+10b)(c-a)-(7a+8b+10c)(a-b)}{a^2+s}\ge 0$$

(where $s=\dfrac{4(ab+bc+ca)+72}{9}$.)

$$\Leftrightarrow\sum(a-b)^2\frac{8a^2+8b^2+15ab+10c(a+b)+s}{(a^2+s)(b^2+s)}\ge 0$$

which is true. □

∇

Problem 18. ***(Canadian Mathematical Olympiad 2008)*** *Let a, b, c be positive real numbers for which $a+b+c=1$. Prove that*

$$\frac{a-bc}{a+bc}+\frac{b-ca}{b+ca}+\frac{c-ab}{c+ab}\le\frac{3}{2}.$$

Proof. (Posted by ***Altheman****).* We have $a+bc=(a+b)(a+c)$, so apply that, etc. The inequality is

$$\sum(b+c)(a^2+ab+ac-bc)\le\frac{3}{2}(a+b)(b+c)(c+a)$$

$$\Longleftrightarrow\sum_{cyc}a^2b+b^2a\ge 6abc$$

which is obvious by the **AM-GM inequality**. □

∇

Problem 19. ***(German DEMO 2008)*** *Find the smallest constant C such that for all real x, y*

$$1 + (x+y)^2 \leq C \cdot (1+x^2) \cdot (1+y^2)$$

holds.

Proof. *(Posted by **JBL**).* The inequality is equivalent to

$$\frac{x^2+y^2+2xy+1}{x^2+y^2+x^2y^2+1} \leq C$$

The greatest value of **LHS** helps us find C in which all real numbers x, y satisfies the inequality.
Let $A = x^2 + y^2$, so

$$\frac{A+2xy+1}{A+x^2y^2+1} \leq C$$

To maximize the **LHS**, A needs to be minimized, but note that

$$x^2 + y^2 \geq 2xy.$$

So let us set $x^2 + y^2 = 2xy = a \Rightarrow x^2y^2 = \frac{a^2}{4}$
So the inequality becomes

$$L = \frac{8a+4}{(a+2)^2} \leq C$$

$$\frac{dL}{dx} = \frac{-8a+8}{(a+2)^3} = 0 \Rightarrow a = 1$$

It follows that $max(L) = C = \frac{4}{3}$ □

∇

Problem 20. ***(Irish Mathematical Olympiad 2008)*** *For positive real numbers a, b, c and d such that $a^2 + b^2 + c^2 + d^2 = 1$ prove that*

$$a^2b^2cd + +ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \leq 3/32,$$

and determine the cases of equality.

Proof. *(Posted by **argady**).* We have

$$a^2b^2cd + ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 = abcd(ab+ac+ad+bc+bd+cd)$$

By the **AM-GM inequality**,

$$a^2 + b^2 + c^2 + d^2 \geq 4\sqrt{abcd}$$

and

$$\frac{a^2+b^2+a^2+c^2+a^2+d^2+b^2+c^2+b^2+d^2+c^2+d^2}{2} \geq (ab+ac+ad+bc+bd+cd)$$

so $abcd \le \dfrac{1}{16}$ and $ab + ac + ad + bc + bd + cd \le \dfrac{3}{2}$
Multiplying we get

$$a^2b^2cd + ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \le \frac{1}{16} \cdot \frac{3}{2} = \frac{3}{32}.$$

The equality occurs when $a = b = c = d = \dfrac{1}{2}$. □

∇

Problem 21. ***(Greek national mathematical olympiad 2008, P1)*** *For the positive integers $a_1, a_2, ..., a_n$ prove that*

$$\left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i}\right)^{\frac{kn}{t}} \ge \prod_{i=1}^n a_i$$

where $k = max\{a_1, a_2, ..., a_n\}$ and $t = min\{a_1, a_2, ..., a_n\}$. When does the equality hold?

*Proof. (Posted by **rofler**).* By the **AM-GM** and **Cauchy-Schwarz** inequalities, we easily get that

$$\sqrt[2]{\frac{\sum a_i^2}{n}} \ge \frac{\sum a_i}{n}$$

$$\sum a_i^2 \ge \frac{(\sum a_i)^2}{n}$$

$$\frac{\sum a_i^2}{\sum a_i} \ge \frac{\sum a_i}{n} \ge \sqrt[n]{\prod_{i=1}^n a_i}$$

$$(\frac{\sum a_i^2}{\sum a_i})^n \ge \prod_{i=1}^n a_i$$

Now, $\dfrac{\sum a_i^2}{\sum a_i} \ge 1$
So therefore since $\frac{k}{t} \ge 1$

$$(\frac{\sum a_i^2}{\sum a_i})^{\frac{kn}{t}} \ge (\frac{\sum a_i^2}{\sum a_i})^n$$

Now, the direct application of **AM-GM** required that all terms are equal for equality to occur, and indeed, equality holds when all a_i are equal. □

∇

Problem 22. ***(Greek national mathematical olympiad 2008, P2)***
If x, y, z are positive real numbers with $x, y, z < 2$ and $x^2 + y^2 + z^2 = 3$ prove that

$$\frac{3}{2} < \frac{1 + y^2}{x + 2} + \frac{1 + z^2}{y + 2} + \frac{1 + x^2}{z + 2} < 3$$

*Proof. (Posted by **tchebytchev**).* From $x < 2, y < 2$ and $z < 2$ we find

$$\frac{1+y^2}{x+2}+\frac{1+z^2}{y+2}+\frac{1+x^2}{z+2} > \frac{1+y^2}{4}+\frac{1+z^2}{4}+\frac{1+x^2}{4} = \frac{3}{2}$$

and from $x > 0, y > 0$ and $z > 0$ we have

$$\frac{1+y^2}{x+2}+\frac{1+z^2}{y+2}+\frac{1+x^2}{z+2} < \frac{1+y^2}{2}+\frac{1+z^2}{2}+\frac{1+x^2}{2} = 3.$$

□

*Proof. (Posted by **canhang2007**).* Since with $x^2+y^2+z^2 = 3$, then we can easily get that $x, y, z \le \sqrt{3} < 2$. Also, we can even prove that

$$\sum \frac{x^2+1}{z+2} \ge 2$$

Indeed, by the **AM-GM** and **Cauchy Schwarz** inequalities, we have

$$\sum \frac{x^2+1}{z+2} \ge \frac{x^2+1}{\frac{z^2+1}{2}+2} = 2\sum \frac{x^2+1}{z^2+5} \ge \frac{2(x^2+y^2+z^2+3)^2}{\sum(x^2+1)(z^2+5)} = \frac{72}{\sum x^2y^2+33}$$

$$\ge \frac{72}{\frac{1}{3}(x^2+y^2+z^2)^2+33} = \frac{72}{3+33} = 2$$

□

∇

Problem 23. ***(Moldova National Olympiad 2008)*** *Positive real numbers a, b, c satisfy inequality $a+b+c \le \frac{3}{2}$. Find the smallest possible value for:*

$$S = abc + \frac{1}{abc}$$

*Proof. (Posted by **NguyenDungTN**).* By the AM-GM inequality, we have

$$\frac{3}{2} \ge a+b+c \ge 3\sqrt[3]{abc}$$

so $abc \le \frac{1}{8}$. By the AM-GM inequality again,

$$S = abc + \frac{1}{abc} = abc + \frac{1}{64abc} + \frac{63}{64abc} \ge 2\sqrt{abc.\frac{1}{64abc}} + \frac{63}{64abc} \ge \frac{1}{4} + \frac{63}{8} = \frac{65}{8}$$

□

∇

Problem 24. ***(British MO 2008)*** *Find the minimum of $x^2+y^2+z^2$ where $x, y, z \in \mathbb{R}$ and satisfy $x^3+y^3+z^3-3xyz = 1$*

*Proof. (Posted by **delegat**).* Condition of problem may be rewritten as:

$$(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=1$$

and since second bracket on LHS is nonnegative we have $x+y+z>0$.
Notice that from last equation we have:

$$x^2+y^2+z^2=\frac{1+(xy+yz+zx)(x+y+z)}{x+y+z}=\frac{1}{x+y+z}+xy+yz+zx$$

and since

$$xy+yz+zx=\frac{(x+y+z)^2-x^2-y^2-z^2}{2}$$

The last equation implies:

$$\begin{aligned}\frac{3(x^2+y^2+z^2)}{2}&=\frac{1}{x+y+z}+\frac{(x+y+z)^2}{2}\\&=\frac{1}{2(x+y+z)}+\frac{1}{2(x+y+z)}+\frac{(x+y+z)^2}{2}\\&\geq\frac{3}{2}\end{aligned}$$

This inequality follows from $AM\geq GM$ so $x^2+y^2+z^2\geq 1$ so minimum of $x^2+y^2+z^2$ is 1 and triple $(1,0,0)$ shows that this value can be achieved. □

Proof. (**Original solution**). Let $x^2+y^2+z^2=r^2$. The volume of the parallelpiped in R^3 with one vertex at $(0,0,0)$ and adjacent vertices at $(x,y,z),(y,z,x),(z,x,y)$ is $|x^3+y^3+z^3-3xyz|=1$ by expanding a determinant. But the volume of a parallelpiped all of whose edges have length r is clearly at most r^3 (actually the volume is $r^3\cos\theta\sin\varphi$ where θ and φ are geometrically significant angles). So $1\leq r^3$ with equality if, and only if, the edges of the parallepiped are perpendicular, where $r=1$. □

Proof. (**Original solution**). Here is an algebraic version of the above solution.

$$\begin{aligned}1&=(x^3+y^3+z^3-3xyz)^2=\left(x(x^2-yz)+y(y^2-zx)+z(z^2-xy)\right)^2\\&\leq(x^2+y^2+z^2)\left((x^2-yz)^2+(y^2-zx)^2+(z^2-xy)^2\right)\\&=(x^2+y^2+z^2)\left(x^4+y^4+z^4+x^2y^2+y^2z^2+z^2x^2-2xyz(x+y+z)\right)\\&=(x^2+y^2+z^2)\left(\left(x^2+y^2+z^2\right)^2-(xy+yz+zx)^2\right)\\&\leq\left(x^2+y^2+z^2\right)^3\end{aligned}$$

□

∇

Problem 25. ***(Zhautykov Olympiad, Kazakhstan 2008, Question 6)*** *Let a,b,c be positive integers for which $abc=1$. Prove that*

$$\sum\frac{1}{b(a+b)}\geq\frac{3}{2}.$$

*Proof. (Posted by **nayel**).* Letting $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$ implies

$$LHS = \sum_{cyc} \frac{x^2}{z^2 + xy} \geq \frac{(x^2 + y^2 + z^2)^2}{x^2y^2 + y^2z^2 + z^2x^2 + x^3y + y^3z + z^3x}$$

Now it remains to prove that

$$2(x^2 + y^2 + z^2)^2 \geq 3\sum_{cyc} x^2y^2 + 3\sum_{cyc} x^3y$$

Which follows by adding the two inequalities

$$x^4 + y^4 + z^4 \geq x^3y + y^3z + z^3x$$

$$\sum_{cyc}(x^4 + x^2y^2) \geq \sum_{cyc} 2x^3y$$

□

∇

Problem 26. ***(Ukraine National Olympiad 2008, P1)*** *Let x, y and z are non-negative numbers such that $x^2 + y^2 + z^2 = 3$. Prove that:*

$$\frac{x}{\sqrt{x^2 + y + z}} + \frac{y}{\sqrt{x + y^2 + z}} + \frac{z}{\sqrt{x + y + z^2}} \leq \sqrt{3}$$

*Proof. (Posted by **nayel**).* By Cauchy Schwarz we have

$$(x^2 + y + z)(1 + y + z) \geq (x + y + z)^2$$

so we have to prove that

$$\frac{x\sqrt{1 + y + z} + y\sqrt{1 + x + z} + z\sqrt{1 + x + y}}{x + y + z} \leq \sqrt{3}$$

But again by the **Cauchy Schwarz inequality** we have

$$x\sqrt{1 + y + z} + y\sqrt{1 + x + z} + z\sqrt{1 + x + y} = \sum \sqrt{x}\sqrt{x + xy + xz}$$

$$\leq \sqrt{(x + y + z)(x + y + z + 2(xy + yz + zx)}$$

and also

$$\sqrt{(x + y + z)(x + y + z + 2(xy + yz + zx)}$$

$$\leq \sqrt{(x + y + z)(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx)} = s\sqrt{s}$$

where $s = x + y + z$ so we have to prove that $\sqrt{s} \leq \sqrt{3}$ which is trivially true so QED □

Proof. *(Posted by **argady**).* We have

$$\sum_{cyc}\frac{x}{\sqrt{x^2+y+z}}=\sum_{cyc}\frac{x}{\sqrt{x^2+(y+z)\sqrt{\frac{x^2+y^2+z^2}{3}}}}\leq\sum_{cyc}\frac{x}{\sqrt{x^2+(y+z)\frac{x+y+z}{3}}}$$

Thus, it remains to prove that

$$\sum_{cyc}\frac{x}{\sqrt{x^2+(y+z)\frac{x+y+z}{3}}}\leq\sqrt{3}.$$

Let $x+y+z=3$. Hence,

$$\sum_{cyc}\frac{x}{\sqrt{x^2+(y+z)\frac{x+y+z}{3}}}\leq\sqrt{3}\Leftrightarrow\sum_{cyc}\left(\frac{1}{\sqrt{3}}-\frac{x}{\sqrt{x^2-x+3}}\right)\geq 0\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{1}{\sqrt{3}}-\frac{x}{\sqrt{x^2-x+3}}+\frac{5(x-1)}{6\sqrt{3}}\right)\geq 0\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\frac{(x-1)^2(25x^2+35x+3)}{((5x+1)\sqrt{x^2-x+3}+6\sqrt{3}x)\sqrt{x^2-x+3}}\geq 0.$$

□

∇

Problem 27. ***(Ukraine National Olympiad 2008, P2)*** *For positive a,b,c,d prove that*

$$(a+b)(b+c)(c+d)(d+a)(1+\sqrt[4]{abcd})^4\geq 16abcd(1+a)(1+b)(1+c)(1+d)$$

Proof. *(Posted by **Yulia**).* Let's rewrite our inequality in the form

$$\frac{(a+b)(b+c)(c+d)(d+a)}{(1+a)(1+b)(1+c)(1+d)}\geq\frac{16abcd}{(1+\sqrt[4]{abcd})^4}$$

We will use the following obvious lemma

$$\frac{x+y}{(1+x)(1+y)}\geq\frac{2\sqrt{xy}}{(1+\sqrt{xy})^2}$$

By lemma and Cauchy-Schwarz

$$\frac{a+b}{(1+a)(1+b)}\frac{c+d}{(1+c)(1+d)}(b+c)(a+d)\geq\frac{4\sqrt{abcd}(\sqrt{ab}+\sqrt{cd})^2}{(1+\sqrt{ab})^2(1+\sqrt{cd})^2}\geq\frac{16abcd}{(1+\sqrt[4]{abcd})^4}$$

Last one also by lemma for $x=\sqrt{ab}, y=\sqrt{cd}$ □

Proof. *(Posted by **argady**).* The inequality equivalent to
$(a+b)(b+c)(c+d)(d+a)-16abcd+$
$+4\sqrt[4]{abcd}\left((a+b)(b+c)(c+d)(d+a)-4\sqrt[4]{a^3b^3c^3d^3}(a+b+c+d)\right)+$
$+2\sqrt{abcd}\left(3(a+b)(b+c)(c+d)(d+a)-8\sqrt{abcd}(ab+ac+ad+bc+bd+cd)\right)$

$+4\sqrt[4]{a^3b^3c^3d^3}((a+b)(b+c)(c+d)(d+a)-4\sqrt[4]{abcd}(abc+abd+acd+bcd))\geq 0,$
which obvious because

$$(a+b)(b+c)(c+d)(d+a)-16abcd\geq 0$$

is true by AM-GM;

$$(a+b)(b+c)(c+d)(d+a)-4\sqrt[4]{a^3b^3c^3d^3}(a+b+c+d)\geq 0$$

is true since,

$$(a+b)(b+c)(c+d)(d+a)\geq(abc+abd+acd+bcd)(a+b+c+d)\Leftrightarrow(ac-bd)^2\geq 0$$

and

$$abc+abd+acd+bcd\geq 4\sqrt[4]{a^3b^3c^3d^3}$$

is true by AM-GM;

$$(a+b)(b+c)(c+d)(d+a)\geq 4\sqrt[4]{abcd}(abc+abd+acd+bcd)$$

is true because

$$a+b+c+d\geq 4\sqrt[4]{abcd}$$

is true by AM-GM;

$$3(a+b)(b+c)(c+d)(d+a)\geq 8\sqrt{abcd}(ab+ac+ad+bc+bd+cd)$$

follows from three inequalities:

$$(a+b)(b+c)(c+d)(d+a)\geq(abc+abd+acd+bcd)(a+b+c+d);$$

by Maclaren we obtain:

$$\frac{a+b+c+d}{4}\geq\sqrt{\frac{ab+ac+bc+ad+bd+cd}{6}}$$

and

$$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}{4}\geq\sqrt{\frac{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{ad}+\frac{1}{bc}+\frac{1}{bd}+\frac{1}{cd}}{6}},$$

which equivalent to

$$abc+abd+acd+bcd\geq\sqrt{\frac{8}{3}(ab+ac+bc+ad+bd+cd)abcd}.$$

□

∇

Problem 28. ***(Polish MO 2008, Pro 5)*** *Show that for all nonnegative real values an inequality occurs:*

$$4(\sqrt{a^3b^3}+\sqrt{b^3c^3}+\sqrt{c^3a^3})\leq 4c^3+(a+b)^3.$$

*Proof. (Posted by **NguyenDungTN**).* We have:

$$RHS - LHS = \left(\sqrt{a^3} + \sqrt{b^3} - 2\sqrt{c^3})^2 + 3ab(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$$

Thus we are done. Equality occurs for $a = b = c$ or $a = 0, b = \sqrt[3]{4}c$ or $a = \sqrt[3]{4}c, b = 0$ □

∇

Problem 29. ***(Brazilian Math Olympiad 2008, Problem 3).*** *Let x, y, z real numbers such that $x + y + z = xy + yz + zx$. Find the minimum value of*

$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1}$$

*Proof. (Posted by **crazyfehmy**).* We will prove that this minimum value is $-\dfrac{1}{2}$. If we take $x = y = -1$, $z = 1$, the value is $-\dfrac{1}{2}$.
Let's prove that

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} + \frac{1}{2} \geq 0$$

We have

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} + \frac{1}{2} = \frac{(1+x)^2}{2+2x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \geq \frac{y}{1+y^2} + \frac{z}{1+z^2}$$

$$\frac{y}{1+y^2} + \frac{z}{1+z^2} < 0$$

then $(y+z)(yz+1) < 0$ and by similar way $(x+z)(xz+1) < 0$ and $(y+z)(yz+1) < 0$.
Let all of x , y , z are different from 0.

- All of $x+y$, $y+z$, $x+z$ is ≥ 0 Then $x+y+z \geq 0$ and $xy+yz+xz \leq -3$.It's a contradiction.
- Exactly one of the $(x+y)$, $y+z$, $x+z$ is < 0.
  W.L.O.G, Assuming $y+z < 0$.
  Because $x+z > 0$ and $x+y > 0$ so $x > 0$.
  $xz+1 < 0$ and $xy+1 < 0$ hence y and z are < 0.
  Let $y = -a$ and $z = -b$. $x = \dfrac{ab+a+b}{a+b+1}$ and $x > a > \dfrac{1}{x}$ and $x > b > \dfrac{1}{x}$.
  So $x > 1$ and $ab > 1$.
  Otherwise because of $x > a$ and $x > b$ hence $\frac{ab+a+b}{a+b+1} > a$ and $\frac{ab+a+b}{a+b+1} > b$.
  So $b > a^2$ and $a > b^2$. So $ab < 1$. It's a contradiction.
- Exactly two of them $(x+y)$,$(y+z)$,$(x+z)$ are < 0.
  W.L.O.G, Assuming $y+z$ and $x+z$ are < 0.
  Because $x+y > 0$ so $z < 0$. Because $xy < 0$ we can assume $x < 0$ and $y > 0$.
  Let $x = -a$ and $z = -c$ and because $xz+1 > 0$ and $xy+1 < 0$ so $c < \frac{1}{y}$ and $a > \frac{1}{y}$.
  Because $y+z < 0$ and $x+y > 0$ hence $a < y < c$ and so $\frac{1}{y} < a < y < c < \frac{1}{y}$. It's a contradiction.

- All of them are < 0 .So $x + y + z < 0$ and $xy + yz + xz > 0$. It's a contradiction.
- Some of x , y , z are $= 0$
  W.L.O.G, Assuming $x = 0$. So $y + z = yz = K$ and

$$\frac{y}{1+y^2} + \frac{z}{1+z^2} = \frac{K^2+K}{2K^2-2K+1} \geq -\frac{1}{2} \Longleftrightarrow 4K^2+1 \geq 0$$

  which is obviously true.

The proof is ended. □

∇

Problem 30. ***(Kiev 2008, Problem 1)****. Let $a, b, c \geq 0$. Prove that*

$$\frac{a^2+b^2+c^2}{5} \geq min((a-b)^2, (b-c)^2, (c-a)^2)$$

Proof. (Posted by ***canhang2007****).* Assume that $a \geq b \geq c$, then

$$\min\left\{(a-b)^2, (b-c)^2, (c-a)^2\right\} = \min\{(a-b)^2, (b-c)^2\}$$

If $a + c \geq 2b$, then $(b-c)^2 = \min\{(a-b)^2, (b-c)^2\}$, we have to prove

$$a^2 + b^2 + c^2 \geq 5(b-c)^2$$

which is true because

$$a^2+b^2+c^2-5(b-c)^2 \geq (2b-c)^2+b^2+c^2-5(b-c)^2 = 3c(2b-c) \geq 0$$

If $a + c \leq 2b$, then $(a-b)^2 = \min\{(a-b)^2, (b-c)^2\}$, we have to prove

$$a^2 + b^2 + c^2 \geq 5(a-b)^2$$

which is true because

$$a^2 + b^2 - 5(a-b)^2 = 2(2a-b)(2b-a) \geq 0$$

This ends the proof. □

∇

Problem 31. ***(Kiev 2008, Problem 2)****. Let $x_1, x_2, \cdots, x_n \geq 0, n > 3$ and $x_1 + x_2 + \cdots + x_n = 2$ Find the minimum value of*

$$\frac{x_2}{1+x_1^2} + \frac{x_3}{1+x_2^2} + ... + \frac{x_1}{1+x_n^2}$$

Proof. (Posted by ***canhang2007****).* By AM-GM Inequality, we have that

$$\frac{x_2}{x_1^2+1} = x_2 - \frac{x_1^2 x_2}{x_1^2+1} \geq x_2 - \frac{1}{2}x_1x_2$$

Apply this for the similar terms and adding them up to obtain

$$LHS \geq 2 - \frac{1}{2}(x_1x_2 + x_2x_3 + \cdots + x_nx_1)$$

Moreover, we can easily show that

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1 \leq x_k(x_1 + \cdots + x_{k-1} + x_{k+1} + \cdots + x_n) \leq 1$$

for k is a number such that $x_k = \max\{x_1, x_2, \ldots, x_n\}$. Hence

$$LHS \geq 2 - \frac{1}{2} = \frac{3}{2}$$

□

∇

Problem 32. ***(Hong Kong TST1 2009, Problem 1)****Let $\theta_1, \theta_2, \ldots, \theta_{2008}$ be real numbers. Find the maximum value of*

$$\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_3 + \ldots + \sin\theta_{2007}\cos\theta_{2008} + \sin\theta_{2008}\cos\theta_1$$

*Proof. (Posted by **brianchung11**).* By the AM-GM Inequality, we have

$$\sin\theta_1\cos\theta_2+\sin\theta_2\cos\theta_3+\ldots+\sin\theta_{2007}\cos\theta_{2008}+\sin\theta_{2008}\cos\theta_1 \leq \frac{1}{2}\sum(\sin^2\theta_i+\cos^2\theta_{i+1}) = 1004$$

Equality holds when θ_i is constant. □

∇

Problem 33. ***(Hong Kong TST1 2009, Problem 5)****. Let a, b, c be the three sides of a triangle. Determine all possible values of*

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca}$$

*Proof. (Posted by **Hong Quy**).* We have

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

and $|a - b| < c$ then $a^2 + b^2 - c^2 < 2ab$.
Thus,

$$a^2 + b^2 + c^2 < 2(ab + bc + ca)$$

$$1 \leq F = \frac{a^2 + b^2 + c^2}{ab + bc + ca} < 2$$

□

∇

Problem 34. ***(Indonesia National Science Olympiad 2008)*** *Prove that for x and y positive reals,*

$$\frac{1}{(1+\sqrt{x})^2} + \frac{1}{(1+\sqrt{y})^2} \geq \frac{2}{x+y+2}.$$

Proof. (Posted by ***Dr Sonnhard Graubner****).* This inequality is equivalent to

$$2+2x+2y+x^2+y^2-2\sqrt{x}y-2x\sqrt{y}+2x^{\frac{3}{2}}+2y^{\frac{3}{2}}-8\sqrt{x}\sqrt{y}\geq 0$$

We observe that the following inequalities hold

1. $x+y\geq 2\sqrt{xy}$
2. $x+y^2\geq 2y\sqrt{x}$
3. $y+x^2\geq 2x\sqrt{y}$
4. $2+2y^{\frac{3}{2}}+2x^{\frac{3}{2}}\geq 6\sqrt{xy}$.

Adding $(1),(2),(3)$ and (4) we get the desired result. □

Proof. (Posted by ***limes123****).* We have

$$(1+xy)(1+\frac{x}{y})\geq(1+x)^2 \iff \frac{1}{(1+x)^2}\geq\frac{1}{1+xy}\cdot\frac{y}{x+y}$$

and analogously

$$\frac{1}{(1+y)^2}\geq\frac{1}{1+xy}\cdot\frac{x}{x+y}$$

as desired. □

∇

Problem 35. ***(Baltic Way 2008)****. Prove that if the real numbers a, b and c satisfy $a^2+b^2+c^2=3$ then*

$$\sum\frac{a^2}{2+b+c^2}\geq\frac{(a+b+c)^2}{12}.$$

When does the inequality hold?

Proof. (Posted by ***Raja Oktovin****).* By the Cauchy-Schwarz Inequality, we have

$$\frac{a^2}{2+b+c^2}+\frac{b^2}{2+c+a^2}+\frac{c^2}{2+a+b^2}\geq\frac{(a+b+c)^2}{6+a+b+c+a^2+b^2+c^2}.$$

So it suffices to prove that

$$6+a+b+c+a^2+b^2+c^2\leq 12.$$

Note that $a^2+b^2+c^2=3$, then we only need to prove that

$$a+b+c\leq 3$$

But

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)\leq a^2+b^2+c^2+2(a^2+b^2+c^2)=3(a^2+b^2+c^2)=9.$$

Hence $a+b+c\leq 3$ which completes the proof. □

∇

Problem 36. ***(Turkey NMO 2008 Problem 3)***. *Let a.b.c be positive reals such that their sum is 1. Prove that*

$$\frac{a^2b^2}{c^3(a^2-ab+b^2)}+\frac{b^2c^2}{a^3(b^2-bc+c^2)}+\frac{a^2c^2}{b^3(a^2-ac+c^2)}\ge\frac{3}{ab+bc+ac}$$

Proof. (Posted by ***canhang2007****).* The inequality is equivalent to

$$\sum\frac{a^2b^2}{c^3(a^2-ab+b^2)}\ge\frac{3(a+b+c)}{ab+bc+ca}$$

Put $x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}$, then the above inequality becomes

$$\sum\frac{z^3}{x^2-xy+y^2}\ge\frac{3(xy+yz+zx)}{x+y+z}$$

This is a very known inequality. □

Proof. (Posted by ***mehdi cherif****).* The inequality is equivalent to :

$$\sum\frac{(ab)^2}{c^3(a^2-ab+b^2)}\ge\frac{3(a+b+c)}{ab+ac+bc}$$

$$\iff\sum\frac{(ab)^5}{a^2-ab+b^2}\ge\frac{3(abc)^3(a+b+c)}{ab+ac+bc}$$

But

$$3(abc)^3(a+b+c)=3abc(\sum a)(abc)^2\le(\sum ab)^2(abc)^2(AM-GM)$$

Hence it suffices to prove that :

$$\sum\frac{(ab)^5}{a^2-ab+b^2}\ge(abc)^2(\sum ab)$$

$$\iff\sum\frac{(ab)^3}{c^2(a^2-ab+b^2)}\ge\sum ab$$

$$\iff\sum\frac{(ab)^3}{c^2(a^2-ab+b^2)}+\sum c(a+b)\ge3\sum ab$$

$$\iff\sum\frac{(ab)^3+(bc)^3+(ca)^3}{c^2(a^2-ab+b^2)}\ge3\sum ab$$

On the other hands,

$$\sum\frac{(ab)^3+(bc)^3+(ca)^3}{c^2(a^2-ab+b^2)}\ge9\frac{(ab)^3+(ac)^3+(bc)^3}{2\sum(ab)^2-abc(\sum a)}$$

It suffices to prove that :

$$9\frac{(ab)^3+(ac)^3+(bc)^3}{2\sum(ab)^2-abc(\sum a)}\ge3\sum ab$$

Denote that $x=ab,y=ac$ and $z=bc$

$$3\frac{x^3+y^3+z^3}{2(x^2+y^2+z^2)-xy+yz+zx}\ge x+y+z$$

$$\iff\sum x^3+3xyz\ge\sum xy(x+y)$$

which is Schur inequality ,and we have done. □

∇

Problem 37. ***(China Western Mathematical Olympiad 2008)****. Given $x, y, z \in (0,1)$ satisfying that*

$$\sqrt{\frac{1-x}{yz}}+\sqrt{\frac{1-y}{xz}}+\sqrt{\frac{1-z}{xy}}=2.$$

Find the maximum value of xyz.

Proof. (Posted by ***Erken****).* Let's make the following substitution: $x=\sin^2\alpha$ and so on... It follows that

$$2\sin\alpha\sin\beta\sin\gamma=\sum\cos\alpha\sin\alpha$$

But it means that $\alpha+\beta+\gamma=\pi$, then obviously

$$(\sin\alpha\sin\beta\sin\gamma)^2\le\frac{27}{64}$$

□

Proof. (Posted by turcas$_c$). We have that

$$2\sqrt{xyz}=\frac{1}{\sqrt{3}}\sum\sqrt{x(3-3x)}\le\frac{1}{\sqrt{3}}\sum\frac{x+3(1-x)}{2}=$$

$$=\frac{3\sqrt{3}}{2}-\frac{1}{\sqrt{3}}\sum x.$$

So $2\sqrt{xyz}\le\frac{3\sqrt{3}}{2}-\frac{1}{\sqrt{3}}\sum x\le\frac{3\sqrt{3}}{2}-\sqrt{3}\cdot\sqrt[3]{xyz}$.

If we denote $p=\sqrt[6]{xyz}$ we get that $2p^3\le\frac{3\sqrt{3}}{2}-\sqrt{3}p^2$. This is equivalent to

$$4p^3+2\sqrt{3}p^2-3\sqrt{3}\le 0\Rightarrow(2p-\sqrt{3})(2p^2+2\sqrt{3}p+3)\le 0,$$

then $p\le\frac{\sqrt{3}}{2}$. So $xyz\le\frac{27}{64}$. The equality holds for $x=y=z=\frac{3}{4}$. □

∇

Problem 38. ***(Chinese TST 2008 P5)*** *For two given positive integers $m, n>1$, let $a_{ij}(i=1,2,\cdots,n, j=1,2,\cdots,m)$ be nonnegative real numbers, not all zero, find the maximum and the minimum values of f, where*

$$f=\frac{n\sum_{i=1}^{n}(\sum_{j=1}^{m}a_{ij})^2+m\sum_{j=1}^{m}(\sum_{i=1}^{n}a_{ij})^2}{(\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij})^2+mn\sum_{i=1}^{n}\sum_{i=j}^{m}a_{ij}^2}$$

Proof. (Posted by ***tanpham****).* We will prove that the maximum value of f is 1.

- For $n=m=2$. Setting $a_{11}=a, a_{21}=b, a_{12}=x, a_{21}=y$. We have

$$f=\frac{2\left((a+b)^2+(x+y)^2+(a+x)^2+(b+y)^2\right)}{(a+b+x+y)^2+4\left(a^2+b^2+x^2+y^2\right)}\le 1$$

$$\Leftrightarrow=(x+b-a-y)^2\ge 0$$

as needed.

- For $n = 2, m = 3$. Using the similar substitution:

$$(x, y, z), (a, b, c)$$

We have

$$f = \frac{2(a+b+c)^2 + 2(x+y+z)^2 + 3(a+x)^2 + 3(b+y)^2 + 3(c+z)^2}{6(a^2+b^2+c^2+x^2+y^2+z^2) + (a+b+c+x+y+z)^2} \leq 1$$

$$\Leftrightarrow (x+b-y-a)^2 + (x+c-z-a)^2 + (y+c-b-z)^2 \geq 0$$

as needed.

- For $n = 3, m = 4$. With

$$(x, y, z, t), (a, b, c, d), (k, l, m, n)$$

The inequality becomes

$$(x+b-a-y)^2 + (x+c-a-z)^2 + (x+d-a-t)^2 + (x+l-k-y)^2 +$$

$$+ (x+m-k-z)^2 + (x+n-k-t)^2 + (y+c-b-z)^2 + (y+d-b-t)^2 +$$

$$+ (y+m-l-z)^2 + (y+n-l-t)^2 + \cdots \geq 0$$

as needed.
By induction, the inequality is true for every integer numbers $m, n > 1$

□

∇

Chapter 3

The inequality from IMO 2008

In this chapter, we will introduce 11 solutions for the inequality from **IMO 2008**.
Problem.
(i). If x, y and z are three real numbers, all different from 1 , such that $xyz = 1$, then prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

With the sign $\sum$ for cyclic summation, this inequality could be rewritten as

$$\sum \frac{x^2}{(x-1)^2} \geq 1$$

(ii). Prove that equality is achieved for infinitely many triples of rational numbers x, y and z.
Solution.

*Proof. (Posted by **vothanhvan**).* We have

$$\sum_{cyc}\left(1-\frac{1}{y}\right)^2\left(1-\frac{1}{z}\right)^2 \geq \left(1-\frac{1}{x}\right)^2\left(1-\frac{1}{y}\right)^2\left(1-\frac{1}{z}\right)^2 \Leftrightarrow \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-3\right)^2 \geq 0$$

We conclude that

$$\sum_{cyc} x^2(y-1)^2(z-1)^2 \geq (x-1)^2(y-1)^2(z-1)^2 \Leftrightarrow \frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2} \geq 1$$

Q.E.D $\square$

∇

*Proof. (Posted by **TTsphn**).* Let

$$a = \frac{x}{1-x}, b = \frac{y}{1-y}, c = \frac{z}{1-z}$$

Then we have :

$$bc = (a+1)(b+1)(c+1) = \frac{1}{(x-1)(y-1)(z-1)} \Leftrightarrow ab+ac+bc+a+b+c+1 = 0$$

Therefore :

$$a^2+b^2+c^2 = a^2+b^2+c^2+2(ab+ac+bc)+2(a+b+c)+2 \Leftrightarrow a^2+b^2+c^2 = (a+b+c+1)^2+1 \geq 1$$

So problem a claim .
The equality hold if and only if $a+b+c+1 = 0$.
This is equivalent to

$$xy + zx + zx = 3$$

From $x = \frac{1}{yz}$ we have

$$\frac{1}{z} + \frac{1}{y} + yz = 3 \Leftrightarrow z^2y^2 - y(3z-1) + z = 0$$

$$\Delta = (3z-1)^2 - 4z^3 = (z-1)^2(1-4z)$$

We only chose $z = \dfrac{1-m^2}{4}$, $|m| > 0$ then the equation has rational solution y. Because $x = \dfrac{1}{yz}$ so it also a rational .
Problem claim . □

∇

*Proof. (Posted by **Darij Grinberg**).* We have

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} - 1 = \frac{(yz+zx+xy-3)^2}{(x-1)^2(y-1)^2(z-1)^2}$$

For part (ii) you are looking for rational x, y, z with $xyz = 1$ and $x+y+z = 3$.
In other words, you are looking for rational x and y with $x + y + \frac{1}{xy} = 3$. This rewrites as $y^2 + (x-3)\,y + \frac{1}{x} = 0$, what is a quadratic equation in y. So for a given x, it has a rational solution y if and only if its determinant $(x-3)^2 - 4 \cdot \frac{1}{x}$ is a square. But $(x-3)^2 - 4 \cdot \frac{1}{x} = \frac{x-4}{x}(x-1)^2$, so this is equivalent to $\frac{x-4}{x}$ being a square.
Parametrize... □

∇

*Proof. (Posted by **Erken**).* Let $a = 1 - \frac{1}{x}$ and so on... Then our inequality becomes:

$$a^2b^2 + b^2c^2 + c^2a^2 \geq a^2b^2c^2$$

while $(1-a)(1-b)(1-c) = 1$.
Second condition gives us that:

$$a^2b^2 + b^2c^2 + c^2a^2 = a^2b^2c^2 + (a+b+c)^2 \geq a^2b^2c^2$$

□

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*Proof. (Posted by **Sung-yoon Kim**).* First letting $x = \frac{q}{p}, y = \frac{r}{q}, z = \frac{p}{r},$. We have to show that

$$\sum \frac{q^2}{(p-q)^2} \geq 1$$

Define $f(t)$ to be

$$\sum \frac{(t+q)^2}{(p-q)^2} = (\sum \frac{1}{(p-q)^2})t^2 + 2(\sum \frac{q}{(p-q)^2})t + \sum \frac{q^2}{(p-q)^2} = At^2 + 2Bt + C$$

This is a quadratic function of t and we know that this has minimum at t_0 such that $At_0 + B = 0$.
Hence,

$$f(t) \geq f(t_0) = At_0^2 + 2Bt_0 + C = Bt_0 + C = \frac{AC - B^2}{A}$$

Since

$$AC - B^2 = (\sum \frac{1}{(p-q)^2})(\sum \frac{q^2}{(p-q)^2}) - (\sum \frac{q}{(p-q)^2})^2$$

and we have

$$(a^2+b^2+c^2)(d^2+e^2+f^2) - (ad+be+cf)^2 = \sum (ae-bd)^2,$$

We obtain

$$AC - B^2 = \sum (\frac{r-q}{(p-q)(q-r)})^2 = \sum \frac{1}{(p-q)^2} = A$$

This makes $f(t) \geq 1$, as desired.
The second part is trivial, since we can find (p, q, r) with fixed $p - q$ and any various $q - r$, which would give different (x, y, z) satisfying the equality. □

∇

*Proof. (Posted by **Ji Chen**).* We have

$$\begin{aligned}
&\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} - 1 \\
&\equiv \frac{a^6}{(a^3-abc)^2} + \frac{b^6}{(b^3-abc)^2} + \frac{c^6}{(c^3-abc)^2} - 1 \\
&= \frac{(bc+ca+ab)^2 (b^2c^2 + c^2a^2 + a^2b^2 - a^2bc - b^2ca - c^2ab)^2}{(a^2-bc)^2 (b^2-ca)^2 (c^2-ab)^2} \geq 0
\end{aligned}$$

□

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Proof. (Posted by ***kunny****).* By $xyz = 1$, we have

$$\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}-\left\{\frac{xy}{(x-1)(y1)}+\frac{yz}{(y-1)(z-1)}+\frac{zx}{(z-1)(x-1)}\right\}$$
$$=\frac{x(y-1)(z-1)+y(z-1)(x-1)+z(x-1)(y-1)-xy(z-1)-yz(x-1)-zx(y-1)}{(x-1)(y-1)(z-1)}$$
$$=\frac{x(y-1)(z-1-z)+y(z-1)(x-1-x)+zx(y-1-y)}{(x-1)(y-1)(z-1)}$$
$$=\frac{x+y+z-(xy+yz+zx)}{(x-1)(y-1)(z-1)}$$
$$=\frac{x+y+z-(xy+yz+zx)+xyz-1}{(x-1)(y-1)(z-1)}$$
$$=\frac{(x-1)(y-1)(z-1)}{(x-1)(y-1)(z-1)}=1$$

(Because $x \neq 1,\ y \neq 1,\ z \neq 1$).)
Therefore

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}$$
$$=\left(\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}\right)^2-2\left\{\frac{xy}{(x-1)(y-1)}+\frac{yz}{(y-1)(z-1)}+\frac{zx}{(z-1)(x-1)}\right\}$$
$$=\left(\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}\right)^2-2\left(\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}-1\right)$$
$$=\left(\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}-1\right)^2+1\geq 1$$

The equality holds when

$$\boxed{\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}=1 \Longleftrightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3}$$

□

∇

Proof. (Posted by ***kunny****).* Let $x+y+z=a,\ xy+yz+zx=b,\ xyz=1,\ x,\ y,\ z$ are the roots of the cubic equation :

$$t^3-at^2+bt-1=0$$

If $t=1$ is the roots of the equation, then we have

$$1^3-a\cdot 1^2+b\cdot 1-1=0 \Longleftrightarrow a-b=0$$

Therefore $t \neq 1 \Longleftrightarrow a - b \neq 0$.
Thus the cubic equation with the roots $\alpha = \frac{x}{x-1},\ \beta = \frac{y}{y-1},\ \gamma = \frac{z}{z-1}$ is

$$\begin{aligned}&\left(\frac{t}{t-1}\right)^3 - a\left(\frac{t}{t-1}\right) + b \cdot \frac{t}{t-1} - 1 = 0\\ \Longleftrightarrow\ &(a-b)t^3 - (a-2b+3)t^2 - (b-3)t - 1 = 0 \ \cdots [*]\end{aligned}$$

Let $a - b = p \neq 0,\ b - 3 = q$, we can rerwite the equation as

$$pt^3 - (p-q)t^2 - qt - 1 = 0$$

By Vieta's formula, we have

$$\alpha + \beta + \gamma = \frac{p-q}{p} = 1 - \frac{q}{p},\ \alpha\beta + \beta\gamma + \gamma\alpha = -\frac{q}{p}$$

Therefore

$$\begin{aligned}&\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} = \alpha^2 + \beta^2 + \gamma^2\\ &= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\\ &= \left(1 - \frac{q}{p}\right)^2 - 2\left(-\frac{q}{p}\right)\\ &= \left(\frac{q}{p}\right)^2 + 1 \geq 1,\end{aligned}$$

The equality holds when $\frac{q}{p} = 0 \Longleftrightarrow q = 0 \Longleftrightarrow b = 3$, which completes the proof. □

∇

*Proof. (Posted by **kunny**).* Since $x,\ y,\ z$ aren't equal to 1, we can set $x = a+1,\ y = b+1,\ z = c+1\ (abc \neq 0)$.

$$\begin{aligned}&\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} = \frac{(a+1)^2}{a^2} + \frac{(b+1)^2}{b^2} + \frac{(c+1)^2}{c^2}\\ &= 3 + 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\\ &= 3 + \frac{2(ab+bc+ca)}{abc} + \frac{(ab+bc+ca)^2 - 2abc(a+b+c)}{(abc)^2} \cdots [*]\end{aligned}$$

Let

$$a + b + c = p,\ ab + bc + ca = q,\ abc = r \neq 0$$

we have $xyz = 1 \Longleftrightarrow p + q + r = 0$, since $r \neq 0$, we have

$$\begin{aligned}&[*] = 3 + \frac{2q}{r} + \frac{q^2 - 2rp}{r^2} = 3 + \frac{2q}{r} + \left(\frac{q}{r}\right)^2 - 2\frac{p}{r}\\ &= 3 + \frac{2q}{r} + \left(\frac{q}{r}\right)^2 + 2\frac{q+r}{r} = \left(\frac{q}{r}\right)^2 + 4\frac{q}{r} + 5\\ &= \left(\frac{q}{r} + 2\right)^2 + 1 \geq 1.\end{aligned}$$

The equality holds when $q = -2r$ and $p+q+r = 0$ $(r > 0) \Longleftrightarrow p : q : r = 1 : (-2) : 1$.
Q.E.D. □

∇

*Proof. (Posted by **Allnames**).* The inequality can be rewritten in this form

$$\sum \frac{1}{(1-a)^2} \geq 1$$

or

$$\sum (1-a)^2(1-b)^2 \geq ((1-a)(1-b)(1-c))^2$$

where $x = \dfrac{1}{a}$ and $abc = 1$.
We set $a + b + c = p, ab + bc + ca = q, abc = r = 1$. So the above inequality is equivalent to

$$(p-3)^2 \geq 0$$

which is clearly true. □

∇

*Proof. (Posted by **tchebytchev**).* Let $x = \frac{1}{a}, y = \frac{1}{b}$ and $z = \frac{1}{c}$. We have

$$\begin{aligned}
&\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \\
&= \frac{1}{(1-a)^2} + \frac{1}{(1-b)^2} + \frac{1}{(1-c)^2} \\
&= \left[\frac{1}{(1-a)} + \frac{1}{(1-b)} + \frac{1}{(1-c)}\right]^2 - 2\left[\frac{1}{(1-a)(1-b)} + \frac{1}{(1-b)(1-c)} + \frac{1}{(1-c)(1-a)}\right] \\
&= \left[\frac{3-2(a+b+c)+ab+bc+ca}{ab+bc+ca-(a+b+c)}\right]^2 - 2\left[\frac{3-(a+b+c)}{ab+bc+ca-(a+b+c)}\right] \\
&= \left[1 + \frac{3-(a+b+c)}{ab+bc+ca-(a+b+c)}\right]^2 - 2\left[\frac{3-(a+b+c)}{ab+bc+ca-(a+b+c)}\right] \\
&= 1 + \left[\frac{3-(a+b+c)}{ab+bc+ca-(a+b+c)}\right]^2 \geq 1
\end{aligned}$$

□

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★★★★★

Glossary

1. **AM-GM inequality**
   For all non-negative real number $a_1, a_2, \cdots, a_n$ then
   $$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1 a_2 \cdots a_n}$$

2. **Cauchy-Schwarz inequality**
   For all real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ then
   $$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + n_n^2\right) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)$$

3. **Jensen Inequality**
   If f is convex on $\mathbb{I}$ then for all $a_1, a_2, \cdots, a_n \in \mathbb{I}$ we have
   $$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

4. **Schur Inequality**
   For all non-negative real numbers a, b, c and positive real number numbers r
   $$a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geq 0$$
   Moreover, if a, b, c are positive real numbers then the above results still holds for all **real** number r

5. **The extension of Schur Inequality (We often call 'Vornicu-Schur inequality')**
   For $x \geq y \geq z$ and $a \geq b \geq c$ then
   $$a(x-y)(x-z) + b(y-z)(y-x) + c(z-x)(z-y) \geq 0$$

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