PROOF OF HUNGKHTN'S CONJECTURE

Bài toán 1 *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^7}{a^6+b^6}+\frac{b^7}{b^6+c^6}+\frac{c^7}{c^6+a^6}\ge \frac{a+b+c}{2}.$$

(Hungkhtn's conjecture)

LỜI GIẢI. The inequality is equivalent to

$$\left(\frac{a^7}{a^6+b^6}-\frac{a}{2}\right)+\left(\frac{b^7}{b^6+c^6}-\frac{b}{2}\right)+\left(\frac{c^7}{c^6+a^6}-\frac{c}{2}\right)\ge 0$$

$$\Leftrightarrow \frac{a(a^6-b^6)}{a^6+b^6}+\frac{b(b^6-c^6)}{b^6+c^6}+\frac{c(c^6-a^6)}{c^6+a^6}\ge 0$$

$$\Leftrightarrow \frac{(a-c+c)(a^6-b^6)}{a^6+b^6}+\frac{(b-c+c)(b^6-c^6)}{b^6+c^6}+\frac{c(c^6-a^6)}{c^6+a^6}\ge 0$$

$$\Leftrightarrow \frac{(a-c)(a^6-b^6)}{a^6+b^6}+\frac{(b-c)(b^6-c^6)}{b^6+c^6}+c\left(\frac{a^6-b^6}{a^6+b^6}+\frac{b^6-c^6}{b^6+c^6}+\frac{c^6-a^6}{c^6+a^6}\right)\ge 0$$

$$\Leftrightarrow \frac{(a-c)(a^6-b^6)}{a^6+b^6}+\frac{(b-c)(b^6-c^6)}{b^6+c^6}\ge \frac{c(a^6-b^6)(b^6-c^6)(c^6-a^6)}{(a^6+b^6)(b^6+c^6)(c^6+a^6)}$$

From this, we can easily check that it suffices for us to consider the inequality in the case $a\ge c\ge b$, but if $a=c$ or $b=c$, then the inequality is trivial. So we will consider the case $a>c>b$, then our inequality is equivalent to

$$\frac{(a-c)(c^6+b^6)}{c^6-b^6}+\frac{(c-b)(a^6+b^6)}{a^6-b^6}\ge \frac{c(a^6-c^6)}{a^6+c^6}$$

$$\Leftrightarrow \frac{(a-c)(c^6+b^6)}{c^6-b^6}+\frac{2(c-b)b^6}{a^6-b^6}+c-b\ge c-\frac{2c^7}{a^6+c^6}$$

$$\Leftrightarrow \frac{(a-c)(c^6+bc^6)}{c^6-b^6}+\frac{2(c-b)b^6}{a^6-b^6}+\frac{2c^7}{a^6+c^6}-b\ge 0$$

Notice that

$$\frac{2(c-b)b^6}{a^6-b^6}\ge 0$$

Hence, it suffices to prove that

$$f(a)=\frac{(a-c)(c^6+b^6)}{c^6-b^6}+\frac{2c^7}{a^6+c^6}-b\ge 0$$

We have

$$f'(a)=\frac{c^6+b^6}{c^6-b^6}-\frac{12c^7a^5}{(c^6+a^6)^2}$$

$$f''(a)=\frac{12c^7a^5(7a^6-5c^6)}{(a^6+c^6)^3}>0$$

Hence $f'(a)$ is increasing. Therefore, if $f'(c)\ge 0$, then $f'(a)>f'(c)\ge 0\Rightarrow f(a)$ is increasing, and thus

$$f(a)>f(c)=c-b>0$$

If $f'(c)<0$, then there exist a_0 (a_0 is the unique) such that $f'(a_0)=0$, that is

$$\frac{c^6+b^6}{c^6-b^6}=\frac{12c^7a_0^5}{(c^6+a_0^6)^2}$$

$$\Leftrightarrow b^6 = \frac{c^6\left[12c^7a_0^5 - (c^6+a_0^6)^2\right]}{(c^6+a_0^6)^2 + 12c^7a_0^5} \geq 0$$

From now, we can easily check that

$$\begin{aligned} f(a) &\geq f(a_0) = \frac{(a_0-c)(c^6+b^6)}{c^6-b^6} + \frac{2c^7}{a_0^6+c^6} - b \\ &= \frac{12(a_0-c)c^7a_0^5}{(c^6+a_0^6)^2} + \frac{2c^7}{a_0^6+c^6} - \sqrt[6]{\frac{c^6\left[12c^7a_0^5-(c^6+a_0^6)^2\right]}{(c^6+a_0^6)^2+12c^7a_0^5}} \\ &= \frac{2c^7(7a_0^6-6ca_0^5+c^6)}{(c^6+a_0^6)^2} - \sqrt[6]{\frac{c^6\left[12c^7a_0^5-(c^6+a_0^6)^2\right]}{(c^6+a_0^6)^2+12c^7a_0^5}} \end{aligned}$$

Hence, it suffices to prove that

$$\frac{2c^7(7a_0^6-6ca_0^5+c^6)}{(c^6+a_0^6)^2} \geq \sqrt[6]{\frac{c^6\left[12c^7a_0^5-(c^6+a_0^6)^2\right]}{(c^6+a_0^6)^2+12c^7a_0^5}}$$

$$\Leftrightarrow \frac{64c^{36}(7a_0^6-6ca_0^5+c^6)^6}{(c^6+a_0^6)^{12}} \geq \frac{12c^7a_0^5-(c^6+a_0^6)^2}{(c^6+a_0^6)^2+12c^7a_0^5}$$

$$\Leftrightarrow \frac{64(7t-6t^{5/6}+1)^6}{(t+1)^{12}} \geq \frac{12t^{5/6}-(t+1)^2}{(t+1)^2+12t^{5/6}}$$

where $t = \sqrt[6]{\frac{a_0}{c}} > 1$.

$$\Leftrightarrow \frac{64(7t-6t^{5/6}+1)^6}{(t+1)^{12}} + \frac{2(t+1)^2}{(t+1)^2+12t^{5/6}} \geq 1$$

I have checked that this inequality is valid for all $t > 1$ but I could not find any simple proofs for it. So I hope someone will helps me to do this thing. $\square$

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