Problem.Let x, y, z be positive real number such that $xy + yz + zx = 1$. Prove that

$$\frac{27}{4}(x+y)(y+z)(z+x) \geq (\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x})^2 \geq 6\sqrt{3}$$

Solution:
From the constraint, we have

$$(x+y)(y+z) = y^2 + 1$$
$$(y+z)(z+x) = z^2 + 1$$
$$(z+x)(x+y) = x^2 + 1$$

so that the right inequality can be rewritten as

$$x + y + z + \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1} \geq 3\sqrt{3}(1)$$

Now,

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2 \geq xy + yz + zx + 2 = 3$$

hence

$$x + y + z \geq 3(2)$$

Also the function

$$f(t) = \sqrt{t^2+1}$$

is a convex function (its second derivative satisfies

$$f''(t) = (t^2+1)^{-3/2} > 0$$

.
Thus,

$$\sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1} \geq 3\sqrt{(\frac{x+y+z}{3})^2 + 1}$$

and using (2) we obitan

$$\sqrt{x^2+z} + \sqrt{y^2+1} + \sqrt{z^2+1} \geq 2\sqrt{3}$$

Adding (2) and (3) yields (1). Asfor the left inequality ,it is equivalent to

$$\frac{1}{x^2+1} + \frac{1}{y^2+1} + \frac{1}{z^2+1} \leq \frac{3\sqrt{3}}{2}(4)$$

The constraint allows us to write

$$x = tan\frac{a}{2}, y = tan\frac{b}{2}, z = tan\frac{c}{2}$$

where a, b, ca are the angles of a triangle . Then (4) can be rewritten as

$$cos\frac{a}{2} + cos\frac{b}{2} + cos\frac{c}{2} \leq \frac{3\sqrt{3}}{2},$$

which holds because from the concavity of cos on $(0, \frac{\pi}{2})$we have

$$cos\frac{a}{2} + cos\frac{b}{2} + cos\frac{c}{2} \leq 3cos\frac{a+b+c}{6} = \frac{3\sqrt{3}}{2}.$$

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