Problem. In any triangle ABC ,where m_a, m_b, m_c are the medians of a triangle ABC. show that

$$am_a + bm_b + cm_c \leq \sqrt{bc}m_a + \sqrt{ca}m_b + \sqrt{ab}m_c$$

Solution: We have to prove the inequality

$$am_a + bm_b + cm_c \leq \sqrt{bc}m_a + \sqrt{ca}m_b + \sqrt{ab}m_c$$

Since

$$\frac{2bc}{b+c} \leq \sqrt{bc}, \frac{2ca}{c+a} \leq \sqrt{ca}, \frac{2ab}{a+b} \leq \sqrt{ab}$$

by the HM-GM inequality, it will be enough to show the stronger inequality

$$am_a + bm_b + cm_c \leq \frac{2bc}{b+c}m_a + \frac{2ca}{c+a}m_b + \frac{2ab}{a+b}m_c$$

since then we will have

$$am_a + bm_b + cm_c \leq \frac{2bc}{b+c}m_a + \frac{2ca}{c+a}m_b + \frac{2ab}{a+b}m_c$$

$$\leq \sqrt{bc}m_a + \sqrt{ca}m_b + \sqrt{ab}m_c$$

and the initial inequality will be proven.
So in the following, we will concentrate on proving this stronger inequality.
Because the inequality we have to prove is symmetric, we can WLOG assume that $a \geq b \geq c$. Then, clearly, $bc \leq ca \leq ab$.
On the other hand, using the formulas

$$m_a^2 = \frac{1}{4}\left(2b^2 + 2c^2 - a^2\right)$$

And

$$m_b^2 = \frac{1}{4}\left(2c^2 + 2a^2 - b^2\right)$$

We can get as a result of a straightforward computation.

$$\left(\frac{m_a}{b+c}\right)^2 - \left(\frac{m_b}{c+a}\right)^2 = \frac{\left(3ac + 3bc + a^2 + b^2 + 4c^2\right)(a+b-c)(b-a)}{4(b+c)^2(c+a)^2}$$

Now, the fraction on the right hand side is ≤ 0, since $3ac+3bc+a^2+b^2+4c^2 \geq 0$ (this is trivial),

$a + b - c > 0$ (in fact, $a + b > c$ because of the triangle inequality) and $b - a \leq 0$ (since $a \geq b$). Hence,

$$\left(\frac{m_a}{b+c}\right)^2 - \left(\frac{m_b}{c+a}\right)^2 \leq 0$$

what yields

$$\left(\frac{m_a}{b+c}\right)^2 \leq \left(\frac{m_b}{c+a}\right)^2$$

and thus

$$\frac{m_a}{b+c} \leq \frac{m_b}{c+a}$$

. Similarly, using $b \geq c$, we can find

$$\frac{m_b}{c+a} \leq \frac{m_c}{a+b}.$$

Thus, we have

$$\frac{m_a}{b+c} \leq \frac{m_b}{c+a} \leq \frac{m_c}{a+b}$$

Since we have also $bc \leq ca \leq ab$, the sequences

$$\left(\frac{m_a}{b+c};\ \frac{m_b}{c+a};\ \frac{m_c}{a+b}\right)$$

and $(bc;\ ca;\ ab)$ are equally sorted. Thus, the Rearrangement Inequality yields

$$\frac{m_a}{b+c} \cdot bc + \frac{m_b}{c+a} \cdot ca + \frac{m_c}{a+b} \cdot ab \geq \frac{m_a}{b+c} \cdot ca + \frac{m_b}{c+a} \cdot ab + \frac{m_c}{a+b} \cdot bc$$

and

$$\frac{m_a}{b+c} \cdot bc + \frac{m_b}{c+a} \cdot ca + \frac{m_c}{a+b} \cdot ab \geq \frac{m_a}{b+c} \cdot ab + \frac{m_b}{c+a} \cdot bc + \frac{m_c}{a+b} \cdot ca$$

Summing up these two inequalities, we get

$$2\frac{m_a}{b+c} \cdot bc + 2\frac{m_b}{c+a} \cdot ca + 2\frac{m_c}{a+b} \cdot ab$$
$$\geq \frac{m_a}{b+c} \cdot (ca+ab) + \frac{m_b}{c+a} \cdot (ab+bc) + \frac{m_c}{a+b} \cdot (bc+ca)$$

This simplifies to

$$\frac{2bc}{b+c}m_a + \frac{2ca}{c+a}m_b + \frac{2ab}{a+b}m_c$$
$$\geq \frac{m_a}{b+c} \cdot a\,(b+c) + \frac{m_b}{c+a} \cdot b\,(c+a) + \frac{m_c}{a+b} \cdot c\,(a+b)$$
$$=> \frac{2bc}{b+c}m_a + \frac{2ca}{c+a}m_b + \frac{2ab}{a+b}m_c \geq am_a + bm_b + cm_c$$

Thus, we have

$$am_a + bm_b + cm_c \leq \frac{2bc}{b+c}m_a + \frac{2ca}{c+a}m_b + \frac{2ab}{a+b}m_c$$

And the proof is complete. Equality holds only if the triangle ABC is equilateral.

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