Problem . *Given a triangle with sides a,b,c satisfying $a^2+b^2+c^2=3$. Show that*

$$\frac{a+b}{\sqrt{a+b-c}}+\frac{b+c}{\sqrt{b+c-a}}+\frac{c+a}{\sqrt{c+a-b}}\geq 6.$$

Solution. Firstly, to prove the original inequality, we will show that[1]

$$ab+bc+ca\geq\frac{4a^3(b+c-a)}{(b+c)^2}+\frac{4b^3(c+a-b)}{(c+a)^2}+\frac{4c^3(a+b-c)}{(a+b)^2},$$

$$\sum_{cyc}\left[a^2-\frac{4a^3(b+c-a)}{(b+c)^2}\right]\geq a^2+b^2+c^2-ab-bc-ca,$$

$$\frac{a^2(2a-b-c)^2}{(b+c)^2}+\frac{b^2(2b-c-a)^2}{(c+a)^2}+\frac{c^2(2c-a-b)^2}{(a+b)^2}\geq a^2+b^2+c^2-ab-bc-ca,$$

Without loss of generality, we may assume that $a\geq b\geq c$, then

$$\frac{a^2}{(b+c)^2}\geq\frac{b^2}{(c+a)^2}\quad\text{and}\quad(2a-b-c)^2\geq(2b-c-a)^2.$$

Thus, using **Chebyshev's Inequality**, we have

$$\frac{a^2(2a-b-c)^2}{(b+c)^2}+\frac{b^2(2b-c-a)^2}{(c+a)^2}\geq\frac{1}{2}\left[\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}\right]\left[(2a-b-c)^2+(2b-c-a)^2\right].\qquad(1)$$

Notice that

$$\frac{1}{4}\left[(2a-b-c)^2+(2b-c-a)^2\right]-(a^2-ab+b^2)\geq\frac{1}{8}(a+b-2c)^2-\frac{1}{4}(a+b)^2,\qquad(2)$$

And

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}\geq\frac{2(a+b)^2}{(a+b+2c)^2}\geq\frac{1}{2},$$

which yieds that

$$\begin{aligned}&\frac{1}{2}\left[\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}-\frac{1}{2}\right]\left[(2a-b-c)^2+(2b-c-a)^2\right]\geq\\ \geq\ &\frac{1}{2}\left[\frac{2(a+b)^2}{(a+b+2c)^2}-\frac{1}{2}\right]\left[(2a-b-c)^2+(2b-c-a)^2\right]\\ \geq\ &\frac{1}{4}\left[\frac{2(a+b)^2}{(a+b+2c)^2}-\frac{1}{2}\right](a+b-2c)^2.\qquad(3)\end{aligned}$$

From $(1),(2)$ and (3), we obtain

$$\frac{a^2(2a-b-c)^2}{(b+c)^2}+\frac{b^2(2b-c-a)^2}{(c+a)^2}-(a^2-ab+b^2)\geq\frac{(a+b)^2(a+b-2c)^2}{2(a+b+2c)^2}-\frac{1}{4}(a+b)^2$$

Using this inequality, we have to prove

$$\frac{(a+b)^2(a+b-2c)^2}{2(a+b+2c)^2}-\frac{1}{4}(a+b)^2+\frac{c^2(a+b-2c)^2}{(a+b)^2}\geq c^2-c(a+b),$$

$$\frac{(a+b)^2(a+b-2c)^2}{2(a+b+2c)^2}+\frac{c^2(a+b-2c)^2}{(a+b)^2}\geq\frac{1}{4}(a+b-2c)^2,$$

$$\frac{(a+b)^2}{2(a+b+2c)^2}+\frac{c^2}{(a+b)^2}\geq\frac{1}{4},$$

which can be easily checked. Thus, the above statement is proved.
Now, turning back to our problem, using **Holder Inequality**, we have

$$\left(\sum_{cyc}\frac{b+c}{\sqrt{b+c-a}}\right)^2\left[\sum_{cyc}\frac{a^3(b+c-a)}{(b+c)^2}\right]\geq\left(\sum_{cyc}a\right)^3.$$

It follows that

$$\left(\sum_{cyc}\frac{b+c}{\sqrt{b+c-a}}\right)^2\geq\frac{\left(\sum\limits_{cyc}a\right)^3}{\sum\limits_{cyc}\frac{a^3(b+c-a)}{(b+c)^2}}\geq\frac{4\left(\sum\limits_{cyc}a\right)^3}{\sum\limits_{cyc}ab}.$$

Moreover, by **AM-GM Inequality**, we have

$$\begin{aligned}\sum_{cyc}ab &= \sqrt{\frac{1}{3}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)}\\ &\leq \sqrt{\frac{1}{3}\left(\frac{\sum\limits_{cyc}ab+\sum\limits_{cyc}ab+\sum\limits_{cyc}a^2}{3}\right)^3}=\frac{\left(\sum\limits_{cyc}a\right)^3}{9}.\end{aligned}$$

Hence

$$\left(\sum_{cyc}\frac{b+c}{\sqrt{b+c-a}}\right)^2\geq\frac{4\left(\sum\limits_{cyc}a\right)^3}{\sum\limits_{cyc}ab}\geq 36.$$

Our proof is completed. Equality holds if and only if $a=b=c=1$.