Problem (Nguyen Duy Tung)
Let x, y, z, t be positive real number such that $max(x, y, z, t) \leq \sqrt{5}min(x, y, z, t)$.
Prove that:

$$\frac{xy}{5x^2 - y^2} + \frac{yz}{5y^2 - z^2} + \frac{zt}{5z^2 - t^2} + \frac{tx}{5t^2 - x^2} \geq 1$$

Proof: From

$$max(x, y, z, t) \leq min\sqrt{5}min(x, y, z, t)$$

we have

$$5x^2 - y^2, 5y^2 - z^2, 5z^2 - t^2, 5t^2 - x^2 \geq 0.$$

Setting

$$a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{t}, d = \frac{t}{x}$$

we have $abcd = 1$
The inequality can rewrite:

$$\frac{a}{5a^2 - 1} + \frac{b}{5b^2 - 1} + \frac{c}{5c^2 - 1} + \frac{d}{5d^2 - 1} \geq 1.$$

We have:

$$\frac{a}{5a^2 - 1} \geq \frac{1}{a^3 + a^2 + a + 1} <=> \frac{(a - 1)^2(a^2 + 3a + 1)}{(5a^2 - 1)(a^3 + a^2 + a + 1)} \geq 0$$

(true)
We will prove that:

$$\frac{1}{(1 + a)(1 + a^2)} + \frac{1}{(1 + b)(1 + b^2)} + \frac{1}{(1 + c)(1 + c^2)} + \frac{1}{(1 + d)(1 + d^2)} \geq 1.$$

Without loss of generality assume that $a \geq b \geq c \geq d$
. Then from Chebyshev's inequality we have that

$$\frac{1}{(1 + a)(1 + a^2)} + \frac{1}{(1 + b)(1 + b^2)} + \frac{1}{(1 + c)(1 + c^2)} \geq$$

$$\geq \frac{1}{3}\left(\frac{1}{1 + a} + \frac{1}{1 + b} + \frac{1}{1 + c}\right)\left(\frac{1}{1 + a^2} + \frac{1}{1 + b^2} + \frac{1}{1 + c^2}\right).$$

Lemma (Vasile Cirtoaje): If$a \geq b \geq c \geq d$ and $abcd = 1$
then it holds that $\frac{1}{1 + a} + \frac{1}{1 + b} + \frac{1}{1 + c} \geq \frac{3}{1 + \sqrt[3]{abc}}$.
Proof of the Lemma: We know that
$\frac{1}{1 + a} + \frac{1}{1 + b} \geq \frac{2}{1 + \sqrt{ab}}$.
So, it suffices to prove that $\frac{1}{1 + c} + \frac{2}{1 + \sqrt{ab}} \geq \frac{3}{1 + \sqrt[3]{abc}}$.
Let us denote by

$$x = \sqrt{ab}, y = \sqrt[3]{abc} \Longrightarrow c = \frac{y^3}{x^2}$$

Substituting them to the above inequality we get that

$$\frac{1}{1 + c} + \frac{2}{1 + \sqrt{ab}} - \frac{3}{1 + \sqrt[3]{abc}} = \frac{x^2}{x^2 + y^3} + \frac{2}{1 + x} - \frac{3}{1 + y},$$

which reduces to the inequality

$$\frac{(x-y)^2\left[2y^2-y+x(y-2)\right]}{(1+x)(1+y)(x^2+y^3)}$$

, which is obvious since

$$2y^2-y+(y-2)x \geq 2y^2-y+(y-2)y^3 = y(y-1)(y^2-y+1) \geq 0.$$

Back to our inequality now, from the above lemma we deduce that:

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \geq \frac{3}{1+\sqrt[3]{abc}}$$

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2} \geq \frac{3}{1+\sqrt[3]{a^2b^2c^2}}.$$

For convenience denote by k the $\sqrt[3]{abc}$.
Therefore we have that

$$\frac{1}{(1+a)(1+a^2)}+\frac{1}{(1+b)(1+b^2)}+\frac{1}{(1+c)(1+c^2)} \geq \frac{3}{(1+k)(1+k^2)}.$$

Thus it remains to prove that

$$\frac{3}{(1+k)(1+k^2)}+\frac{1}{(1+d)(1+d^2)} \geq 1$$

But $abcd=1$.
So, the last fraction is of the form

$$\frac{1}{\left(1+\frac{1}{k^3}\right)\left(1+\frac{1}{k^6}\right)}.$$

After that we get

$$\frac{1}{\left(1+\frac{1}{k^3}\right)\left(1+\frac{1}{k^6}\right)}+\frac{3}{(1+k)(1+k^2)} \geq 1$$

Conclusion follows from the obvious inequality

$$\frac{(k-1)^2(2k^4+k^3+k+2)}{(k^3+1)(k^6+1)} \geq 0$$

Q.E.D The Enquality holds when $x=y=z=t=1$.

Remark: Let $x_1, x_2, ..., x_n$ be positive real number such that $max(x_1, x_2, .., x_n) \leq \sqrt{5}min(x_1, x_2, ..., x_n)$..Prove that:

$$\frac{x_1x_2}{5x_1^2-x_2^2}+\frac{x_2x_3}{5x_2^2-x_3^2}+...+\frac{x_nx_1}{5x_n^2-x_1^2} \geq 1$$

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