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2002 National Contests: Problems and Solutions

1.1 Belarus

Problem 1 We are given a partition of $\{1, 2, \dots, 20\}$ into nonempty sets. Of the sets in the partition, k have the following property: for each of the k sets, the product of the elements in that set is a perfect square. Determine the maximum possible value of k.

Solution:

Let $A_1, A_2 \dots A_k$ be the k disjoint subsets of $\{1, 2, \dots, 20\}$, and let A be their union. It is clear that 11, 13, 17, 19 $\notin A$. Therefore $\|A\| \le 16$. Because 1, 4, 9, 16 are the only perfect squares, if a set contains an element other than those 4 perfect squares, the size of that site is at least 2. Therefore, $k \le 4 + \frac{16-4}{2} = 10$, equality occurs when 1, 4, 9, 16 form their own set and the other 12 numbers are partitioned into 6 sets of 2 elements. This, however cannot be achieved because the only numbers that contain the prime 7 are 7 and 14, but 7×14 is not a perfect square. Therefore, $k \le 9$. This is possible: $\{1\}$, $\{4\}$, $\{9\}$, $\{16\}$, $\{3, 12\}$, $\{5, 20\}$, $\{8, 18\}$, $\{2, 7, 14\}$, $\{6, 10, 15\}$.

Problem 2 The rational numbers $\alpha_1, \dots, \alpha_n$ satisfy

$$\sum_{i=1}^{n} \{k\alpha_i\} < \frac{n}{2}$$

for any positive integer k. (Here, $\{x\}$ denotes the fractional part of x, the unique number in $[0, 1)$ such that $x - \{x\}$ is an integer.)

(a) Prove that at least one of $\alpha_1, \dots, \alpha_n$ is an integer.

(b) Do there exist $\alpha_1, \dots, \alpha_n$ that satisfy $\sum_{i=1}^{n} \{k\alpha_i\} \le \frac{n}{2}$, such that no α_i is an integer?

Solution:

(a) Assume the contrary. The problem would not change if we replace α_i with $\{\alpha_i\}$. So we may assume $0 < \alpha_i < 1$ for all $1 \le i \le n$. Because α_i is rational, let $\alpha_i = \frac{p_i}{q_i}$, and $D = \prod_{i=1}^{n} q_i$. Because $(D - 1)\alpha_i + \alpha_i = D\alpha_i$ is an integer, and α_i is not an integer, $\{(D - 1)\alpha_i\} + \{\alpha_i\} = 1\alpha_i$. Then

$$1 > \sum_{i=1}^{n}\{(D-1)\alpha_i\} + \sum_{i=1}^{n}\{\alpha_i\} = \sum_{i=1}^{n}\{(D-1)\alpha_i + \alpha_i\} = \sum_{i=1}^{n} 1 = n$$

contradiction. Therefore, one of the α_i has to be an integer.

(b) Yes. Let $\alpha_i = \frac{1}{2}$ for all i. Then $\sum_{i=1}^{n}\{k\alpha_i\} = 0$ when k is even and $\sum_{i=1}^{n}\{k\alpha_i\} = \frac{n}{2}$ when k is odd.

Problem 3 There are 20 cities in Wonderland. The company Wonderland Airways establishes 18 air routes between them. Each of the routes is a closed loop that passes through exactly five different cities. Each city belongs to at least three different routes. Also, for any two cities, there is at most one route in which the two cities are neighboring stops. Prove that using the airplanes of Wonderland Airways, one can fly from any city of Wonderland to any other city.

Solution:

We donate the 20 cities with 20 points, and connect two points with with a line if there is a direct flight between. We want to show that the graph is connected.

If, for the sake of contradiction, the graph is not connected. Because for each city, there are at least 3 loops passing through it, and therefore at least 6 cities next to it, and they all have to be distinct. Therefore, each connected graph consists of at least 7 points, but $3 \times 7 = 21 > 20$, we can only have 2 connected parts.

We call the two parts A and B, and assume the points in A is less or equal to that in B. Assume there are k points in A. If for all the points in A, they belong to exactly 3 loops, then we have $3k = 5l$, where l is the number of loops in A. (Because A and B are not connected, each loop lies entirely in one of them.) Because $7 \le k \le 10$ and 5 divides k, we have $k = 10$. If $k = 10$, then because there are $18 = 90$ direct connections established by the airlines, and at most $2 * \binom{10}{2} = 90$ possible direct flights, and each was counted at most once by the loops, we conclude that all the points are connected in A. Let A_i be the points in A, then A_1, A_2 are neighbors in $A_1A_2A_3A_4A_5$ and $A_1A_2A_3A_4A_6$, contradiction.

Otherwise, assume there is a city in A that is in 4 loops, then that city has 8 neighboring cities, and they are all distinct. Then there are 9 or 10 cities in A. We've done the case when it's 10, and now we assume it's 9. Because there are at most $\binom{9}{2} = 36$ direct flights in

A, A has at most 7 loops. Therefore, B has at least 11 loops. But $11 \times 5 = 55 = \binom{11}{2}$, we conclude that B is a complete graph, and a contradiction follows similar to the previous case.

Problem 4 Determine whether there exists a three-dimensional solid with the following property: for any natural $n \geq 3$, there is a plane such that the orthogonal projection of the solid onto the plane is a convex n-gon.

Problem 5 Prove that there exist infinitely many positive integers that cannot be written in the form

$$x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^{11}$$

for some positive integers x_1, x_2, x_3, x_4, x_5.

Solution:

For each integer N, we consider the number of integers in $[1, N]$ that can be written in the above form. Because $x_1 \leq N^{\frac{1}{3}}$, there are at most $N^{\frac{1}{3}}$ ways to choose x_1. Similar argument applies to the other x_is. Therefore, there are at most $N^{\frac{1}{3}} N^{\frac{1}{5}} N^{\frac{1}{7}} N^{\frac{1}{9}} N^{\frac{1}{11}} = N^{\frac{3043}{3465}}$ combinations. So there are at least $N - N^{\frac{3043}{3465}}$ integers not covered. It is easy to see that this value can be arbitrarily large as N approaches infinity. Therefore, there exist infinitely many positive integers that cannot be written in the form $x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^{11}$.

Problem 6 The altitude $\overline{CH}$ of the right triangle ABC ($\angle C = \pi/2$) intersects the angle bisectors $\overline{AM}$ and $\overline{BN}$ at points P and Q, respectively. Prove that the line passing through the midpoints of segments $\overline{QN}$ and $\overline{PM}$ is parallel to line AB.

Solution:

This problem can be solved by direct computation, but we shall provide a geometric solution.

Because $\angle CMQ = \angle MBA + \angle BAM = \angle ACQ + \angle QAC = \angle MQC$, triangle CQM is isosceles. Similarly, CPN is isosceles as well. Let R, T be the midpoints of QM and NP respectively, then $CR \perp AM$ and $CT \perp BN$. Therefore, C, R, Q, N is cyclic. Let CR and CT intersect AB at D and E respectively and let AM and BN intersect at I, the I is the incenter of $\triangle ABC$ and therefore CI is the angle bisector of $\angle C$. Therefore, $\angle CDA = \angle CBA + \angle DCB = \angle CBA + \angle DCB = 45\,\text{deg} + \angle CBN = \angle PCB + \angle CBP = \angle CPB = \angle CRN$. Therefore, NR is parallel to AB.

Problem 7 On a table lies a point X and several face clocks, not necessarily identical. Each face clock consists of a fixed center, and two hands (a minute hand and an hour hand) of equal length. (The hands rotate around the center at a fixed rate; each hour, a minute hand completes a full revolution while an hour hand completes $1/12$ of a revolution.) It is known that at some moment, the following two quantities are distinct:

- the sum of the distances between X and the end of each minute hand; and
- the sum of the distances between X and the end of each hour hand.

Prove that at some moment, the former sum is greater than the latter sum.

Problem 8 A set S of three-digit numbers formed from the digits $1, 2, 3, 4, 5, 6$ (possibly repeating one of these six digits) is called *nice* if it satisfies the following condition: for any two distinct digits from $1, 2, 3, 4, 5, 6$, there exists a number in S which contains both of the chosen digits. For each nice set S, we calculate the sum of all the elements in S; determine, over all nice sets, the minimum value of this sum.

1.2 Bulgaria

Problem 1 Let $a_1, a_2, \ldots$ be a sequence of real numbers such that

$$a_{n+1} = \sqrt{a_n^2 + a_n - 1}$$

for $n \geq 1$. Prove that $a_1 \notin (-2, 1)$.

Solution: Note that $a_n \geq 0$ for $n \geq 2$. Moreover, since $a_n^2 + a_n - 1 = a_{n+1}^2 \geq 0$, $a_n \geq r$ for $n \geq 2$, where $r = \frac{\sqrt{5}-1}{2}$. Also, the function $f(x) = \sqrt{x^2 + x - 1}$ is continuous on $[r, \infty)$. Now, suppose (for contradiction) that $a_1 \in (-2, 1)$. Then $a_2^2 = a_1^2 + a_1 - 1 = (a_1 + \frac{1}{2})^2 - \frac{5}{4} < (\frac{3}{2})^2 - \frac{5}{4} = 1$, so $a_2 \in [r, 1)$. Now, if $a_n \in [r, 1)$, we have $a_{n+1}^2 = a_n^2 + a_n - 1 < a_n^2$, so $a_{n+1} < a_n$. Thus (by induction) $a_2, a_3, \ldots$ is a decreasing sequence of real numbers in $[r, 1)$, and therefore $\lim_{n\to\infty} a_n$ exists and is in $[r, 1)$. Now, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$ (since f is continuous). But f has no fixed points in $[r, 1)$, so this is a contradiction, and therefore $a_1 \notin (-2, 1)$.

Problem 2 Consider the feet of the orthogonal projections of A, B, C of triangle ABC onto the external angle bisectors of angles BCA, CAB, and ABC, respectively. Let d be the length of the diameter of the circle passing through these three points. Also, let r and s be the inradius and semiperimeter, respectively, of triangle ABC. Prove that $r^2 + s^2 = d^2$.

Solution: Let $a = BC$, $b = CA$, and $c = AB$, and A, B, C be the measures of angles CAB, ABC, and BCA, respectively. Also let A_1, B_1, C_1 be (respectively) the feet of the orthogonal projections of A, B, C onto the external angle bisectors of angles BCA, CAB, and ABC. Similarly, let A_2, B_2, C_2 be (respectively) the feet of the orthogonal projections of A, B, C onto the external angle bisectors of angles ABC, BCA, and CAB. We claim that A_1, B_1, C_1, A_2, B_2, and C_2 all lie on a single circle. To show this, we calculate the square of the circumradius R of triangle $A_1C_2B_1$.

Since B_1 and C_2 both lie on the external angle bisector of angle CAB, $B_1C_2 = B_1A + AC_2 = (b + c)\sin\frac{A}{2}$. Also, the triangle A_1C_2C has circumcircle with diameter $\overline{AC}$, and $\angle A_1CC_2 = (\frac{\pi}{2} - \frac{C}{2}) - \frac{A}{2} = \frac{B}{2}$, so by the extended Law of Sines, $C_2A_1 = b\sin\frac{B}{2}$. Since quadrilateral

ACA_1C_2 is cyclic, $\angle B_1C_2A_1 = \angle AC_2A_1 = \pi - \angle ACA_1 = \pi - (\frac{\pi}{2} - \frac{C}{2}) = \frac{\pi}{2} + \frac{C}{2}$. Now, $R = \frac{B_1A_1}{2\sin(\frac{\pi}{2}+\frac{C}{2})}$, so $R^2 = \frac{(B_1A_1)^2}{4\cos^2\frac{C}{2}}$. By the Law of Cosines and our previous calculations, this gives

$$R^2 = \frac{(b+c)^2\sin^2\frac{A}{2} + b^2\sin^2\frac{B}{2} + 2b(b+c)\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4\cos^2\frac{C}{2}}.$$

Using the half angle formulas and the identity $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{4}(\cos A + \cos B + \cos C - 1)$, we can simplify this expression to

$$R^2 = \frac{b^2 + bc + c^2 - c(b+c)\cos A + bc\cos B + b(b+c)\cos C}{4(1+\cos C)},$$

and removing the cosines with the Law of Cosines simplifies this further to

$$R^2 = \frac{a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2 + abc}{4(a+b+c)}.$$

Since this expression for the square of the circumradius is symmetric in a, b, and c, this shows by symmetry that the circumradius is the same for each of the triangles $A_1C_2B_1$, $C_2B_1A_2$, $B_1A_2C_1$, $A_2C_1B_2$, $C_1B_2A_1$, and $B_2A_1C_2$. It is easily verified that this implies that A_1, B_1, C_1, A_2, B_2, C_2 form a cyclic hexagon. Thus triangle $A_1B_1C_1$ also has circumradius R, and so $d^2 = 4R^2$. Also, $s^2 = \frac{(a+b+c)^3}{4(a+b+c)}$, and $r^2 = \frac{(-a+b+c)(a-b+c)(a+b-c)}{4(a+b+c)}$ by Heron's formula for the area of the triangle and area $= rs$, so $d^2 = s^2 + r^2$, as desired.

Problem 3 Given are n^2 points in the plane, no three of them collinear, where $n = 4k+1$ for some positive integer k. Find the minimum number of segments that must be drawn connecting pairs of points, in order to ensure that among any n of the n^2 points, some 4 of the n chosen points are connected to each other pairwise.

Problem 4 Let I be the incenter of non-equilateral triangle ABC, and let T_1, T_2, T_3 be the tangency points of the incircle with sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Prove that the orthocenter of triangle $T_1T_2T_3$ lies on line OI, where O is the circumcenter of triangle ABC.

Solution: Let H' and G' be the orthocenter and centroid, respectively, of triangle $T_1T_2T_3$. Since I is the circumcenter of this triangle, H', G', and I are on the Euler line of triangle $T_1T_2T_3$ and thus are collinear. We want to show that O is also on this line, so it is sufficient to show that O, G', and I are collinear.

We will approach this problem using vectors, treating the plane as a vector space with O at the origin. Let $a = BC$, $b = CA$, and $c = AB$. For any point P, let $\vec{P}$ be the vector corresponding to P. First, note that $\vec{I} = x\vec{A} + y\vec{B} + z\vec{C}$ for unique real numbers x, y, z with $x + y + z = 1$. We must have $\frac{x\vec{A}+y\vec{B}}{x+y} = \vec{P_C}$, where P_C is the intersection point of the angle bisector through C with side $\overline{AB}$. By the angle bisector theorem, this gives $\frac{x}{y} = \frac{a}{b}$. Similarly, $\frac{y}{z} = \frac{b}{c}$, and thus $x = \frac{a}{a+b+c}$, $y = \frac{b}{a+b+c}$, and $z = \frac{c}{a+b+c}$, so

$$\vec{I} = \frac{a\vec{A} + b\vec{B} + c\vec{C}}{a+b+c}.$$

Also, $\vec{T_1} = \frac{T_1C\cdot\vec{B}+T_1B\cdot\vec{C}}{a} = \frac{(a+b-c)\vec{B}+(a+c-b)\vec{C}}{2a}$, and similar formulas hold cyclically for $\vec{T_2}$, $\vec{T_3}$, so $\vec{G'} = \frac{1}{3}(\vec{T_1} + \vec{T_2} + \vec{T_3}) = \sum_{\text{cyc}} \frac{(a+b-c)\vec{B}+(a+c-b)\vec{C}}{2a}$. Rearranging the terms gives

$$\vec{G'} = \frac{1}{6}\sum_{\text{cyc}} \frac{ab + ac + 2bc - b^2 - c^2}{bc}\vec{A}.$$

We now need the following lemma:

Lemma. *If O is the circumcenter of triangle ABC and $\vec{O} = \vec{0}$, then*

$$\sum_{cyc} a^2(b^2 + c^2 - a^2)\vec{A} = \vec{0}.$$

Proof. First, note that dividing by $2abc$ and then applying the Law of Cosines shows that it is equivalent to prove that $\sum_{\text{cyc}} (a\cos A)\vec{A} = \vec{0}$. Let (x, y, z) be the unique triplet of real numbers such that $x + y + z = 1$ and $x\vec{A} + y\vec{B} + z\vec{C} = \vec{0}$. Then $\frac{x\vec{A}+y\vec{B}}{x+y} = \vec{Q_C}$, where Q_C is the intersection point of $\overline{CO}$ with $\overline{AB}$. The Law of Sines gives $AQ_C = \frac{b\sin(\frac{\pi}{2}-B)}{\sin(\frac{\pi}{2}+B-A)} = \frac{b\cos B}{\cos(A-B)}$, and similarly $Q_CB = \frac{a\cos A}{\cos(A-B)}$. Therefore $\frac{x}{y} = \frac{Q_CB}{AQ_C} = \frac{a\cos A}{b\cos B}$, and similarly $\frac{y}{z} = \frac{b\cos B}{c\cos C}$. Thus $(a\cos A, b\cos B, c\cos C)$ is a multiple of (x, y, z), which proves the desired result. □

We are ready to show that a non-zero linear combination of $\vec{I}$ and $\vec{G'}$ equals $\vec{0}$, which then implies that I, G', and O are collinear, as desired. Let $\vec{X} = (-a^3-b^3-c^3+a^2b+a^2c+b^2a+c^2a+b^2c+bc^2+4abc)(a+b+c)\vec{I} - 6abc(a+b+c)\vec{G'}$. We claim that $\vec{X} = \sum_{\text{cyc}} a^2(b^2 + c^2 - a^2)\vec{A}$. To see this, it is sufficient to note that the coefficient of $\vec{A}$ on each side is the same; the rest follows from cyclic symmetry. Inspection easily shows that

$$(-a^3 - b^3 - c^3 + a^2b + a^2c + b^2a + c^2a + b^2c + bc^2 + 4abc)a$$
$$-a(a+b+c)(ab+ac+2bc-b^2-c^2) = a^2(b^2+c^2-a^2),$$

so the desired result has been proven.

Problem 5 Let b, c be positive integers, and define the sequence $a_1, a_2, \ldots$ by $a_1 = b$, $a_2 = c$, and

$$a_{n+2} = |3a_{n+1} - 2a_n|$$

for $n \geq 1$. Find all such (b, c) for which the sequence $a_1, a_2, \ldots$ has only a finite number of composite terms.

Solution: The only solutions are (p, p) for p not composite, $(2p, p)$ for p not composite, and $(7, 4)$.

The sequence $a_1, a_2, \ldots$ cannot be strictly decreasing because each a_n is a positive integer, so there exists a smallest $k \geq 1$ such that $a_{k+1} \geq a_k$. Define a new sequence $b_1, b_2, \ldots$ by $b_n = a_{n+k-1}$, so $b_2 \geq b_1$, $b_{n+2} = |3b_{n+1} - 2b_n|$ for $n \geq 1$, and $b_1, b_2, \ldots$ has only a finite number of composite terms. Now, if $b_{n+1} \geq b_n$, $b_{n+2} = |3b_{n+1} - 2b_n| = 3b_{n+1} - 2b_n = b_{n+1} + 2(b_{n+1} - b_n) \geq b_{n+1}$, so by induction $b_{n+2} = 3b_{n+1} - 2b_n$ for $n \geq 1$.

Using the general theory of linear recurrence relations (a simple induction proof also suffices), we have

$$b_n = A \cdot 2^{n-1} + B$$

for $n \geq 1$, where $A = b_2 - b_1$, $B = 2b_1 - b_2$. Suppose (for contradiction) that $A \neq 0$. Then b_n is an increasing sequence, and, since it contains only finitely many composite terms, $b_n = p$ for some prime $p > 2$ and some $n \geq 1$. However, then $b_{n+l(p-1)}$ is divisible by p and thus composite for $l \geq 1$, because $b_{n+l(p-1)} = A \cdot 2^{n-1} \cdot 2^{l \cdot (p-1)} + B \equiv A \cdot 2^{n-1} + B \equiv 0 \bmod p$ by Fermat's Little Theorem. This is a contradiction, so $A = 0$ and $b_n = b_1$ for $n \geq 1$. Therefore b_1 is not composite; let $b_1 = p$, where $p = 1$ or p is prime.

We now return to the sequence $a_1, a_2, \ldots$, and consider different possible values of k. If $k = 1$, we have $a_1 = b_1 = b_2 = a_2 = p$, so $b = c = p$ for p not composite are the only solutions. If $k > 1$, consider that $a_{k-1} > a_k$ by the choice of k, but $a_{k+1} = |3a_k - 2a_{k-1}|$, and $a_{k+1} = b_2 = b_1 = a_k$, so $a_{k+1} = 2a_{k-1} - 3a_k$, and thus $a_{k-1} = 2p$. For

$k=2$, this means that $b=2p, c=p$ for p not composite are the only solutions. If $k>2$, the same approach yields $a_{k-2}=\frac{3a_{k-1}+a_k}{2}=\frac{7}{2}p$, so $p=2$. For $k=3$, this gives the solution $b=7, c=4$, and because $\frac{3\cdot 7+4}{2}$ is not an integer, there are no solutions for $k>3$.

Problem 6 In a triangle ABC, let $a=BC$ and $b=CA$, and let ℓ_a and ℓ_b be the lengths of the internal angle bisectors from A and B, respectively. Find the smallest number k such that

$$\frac{\ell_a+\ell_b}{a+b}\le k$$

for all such triangles ABC.

Solution: The answer is $k=\frac{4}{3}$.

Let $c=AB$. We will derive an algebraic expression for ℓ_a in terms of a, b, and c by calculating the area of triangle ABC in two different ways: this area equals $\frac{1}{2}bc\sin A$, but it also equals the sum of the two triangles into which it is divided by the angle bisector from A, so it equals $\frac{1}{2}(b+c)\ell_a\sin\frac{A}{2}$. Thus $\ell_a=\frac{2bc}{b+c}\cos\frac{A}{2}$. Since $\cos\frac{A}{2}=\sqrt{\frac{1+cos A}{2}}=\sqrt{\frac{(b+c)^2-a^2}{4bc}}$ (by the Law of Cosines), this gives

$$\ell_a=\frac{\sqrt{bc(b+c-a)(b+c+a)}}{b+c}$$

and of course a similar expression exists for ℓ_b.

To see that there does not exist a smaller k with the desired property, let $f(\epsilon)$ equal the value of the expression $\frac{\ell_a+\ell_b}{a+b}$ for the triangle with $a=b=1+\epsilon,\ c=2$. Using the above formula for ℓ_a and ℓ_b yields $f(\epsilon)=\frac{4\sqrt{(1+\epsilon)(4+2\epsilon)}}{(3+\epsilon)(2+2\epsilon)}$. Thus $\lim_{\epsilon\to 0}f(\epsilon)=\frac{4\sqrt{4}}{3\cdot 2}=\frac{4}{3}$, so for any $k'<\frac{4}{3}$, there exists $\epsilon>0$ such that $f(\epsilon)>k'$. It remains only to show that the inequality holds with $k=\frac{4}{3}$.

Because a, b, and c are lengths of sides of a triangle, we can let $a=y+z$, $b=x+z$, and $c=x+y$, where x, y, and z are positive real numbers. This gives

$$\ell_a=\frac{2\sqrt{x(x+z)(x+y)(x+y+z)}}{2x+y+z}\le\frac{2(x+\frac{z}{2})(x+y+\frac{z}{2})}{2x+y+z}$$

by the AM-GM inequality on the numerator. It thus suffices to show that

$$\frac{\frac{(x+\frac{z}{2})(x+y+\frac{z}{2})}{2x+y+z}+\frac{(y+\frac{z}{2})(x+y+\frac{z}{2})}{2y+x+z}}{x+y+2z}\le\frac{2}{3}.$$

Cross-multiplying to eliminate all of the fractions transforms this into the equivalent form

$$12x^3+60x^2y+60xy^2+12y^3+36x^2z+84xyz+36y^2z+27xz^2+27yz^2+6z^3 \leq$$

$$16x^3+56x^2y+56xy^2+16y^3+56x^2z+128xyz+56y^2z+56xz^2+56yz^2+16z^3.$$

This simplifies to

$$4x^2y + 4xy^2 \leq 4x^3 + 4y^3 + \text{terms involving z},$$

where the terms involving z have positive coefficients. This is true because $4x^3 + 4y^3 = 4((\frac{2}{3}x^3 + \frac{1}{3}y^3) + (\frac{1}{3}x^3 + \frac{2}{3}y^3) \geq 4(x^2y + xy^2)$ by the weighted AM-GM inequality. Thus the original inequality is true with $k = \frac{4}{3}$.

1.3 Canada

Problem 1 Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c,$$

and determine when equality holds.

Solution: We can rewrite $a + b + c$ as follows:

$$\begin{aligned} a + b + c, \\ \sqrt[4]{a^4} + \sqrt[4]{b^4} + \sqrt[4]{c^4}, \\ \sqrt[4]{\frac{a^3}{bc}\frac{a^3}{bc}\frac{b^3}{ca}\frac{c^3}{ab}} + \sqrt[4]{\frac{a^3}{bc}\frac{b^3}{ca}\frac{b^3}{ca}\frac{c^3}{ab}} + \sqrt[4]{\frac{a^3}{bc}\frac{b^3}{ca}\frac{c^3}{ab}\frac{c^3}{ab}}. \end{aligned}$$

By the arithmetic-geometric mean inequality and some algebra,

$$\begin{aligned} a + b + c = \sqrt[4]{\frac{a^3}{bc}\frac{a^3}{bc}\frac{b^3}{ca}\frac{c^3}{ab}} + \sqrt[4]{\frac{a^3}{bc}\frac{b^3}{ca}\frac{b^3}{ca}\frac{c^3}{ab}} + \sqrt[4]{\frac{a^3}{bc}\frac{b^3}{ca}\frac{c^3}{ab}\frac{c}{a}} \\ \leq \frac{1}{4}\left(\frac{a^3}{bc} + \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}\right) + \frac{1}{4}\left(\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{b^3}{ca} + \frac{c^3}{ab}\right) + \frac{1}{4}\left(\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} + \frac{c^3}{ab}\right. \\ = \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \\ a + b + c \leq \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \end{aligned}$$

as desired.

Problem 2 Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with center B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let line CP meet Γ again at Q. Prove that $PQ = r$.

Solution: Let O be the center of Γ.

By the law of cosines,

$$\begin{aligned} AB^2 = OA^2 + OB^2 + 2OA \cdot OB cos\angle AOB, \\ AB^2 = 2r^2(1 - cos\angle AOB). \end{aligned}$$

Because $AB < \sqrt{3}r$,

$$2r^2(1 - cos\angle AOB) < 3r^2,$$
$$1 - cos\angle AOB < \frac{3}{2},$$
$$cos\angle AOB > -\frac{1}{2},$$

which in turn implies that $\angle AOB < 120^\circ$, because $0^\circ < \angle AOB \leq 180^\circ$, and $cosx$ is monotonically decreasing on this interval.

Thus AB subtends an arc that is less than one third the perimeter of Γ, and we can conclude that P lies within Γ.

Define $\angle OBA = \angle OAB = \theta$. Because $BC = BA$, $\triangle OBA \cong \triangle OBC$ and $\angle OBC = \theta$, thus $\angle ABC = 2\theta$ and $\angle PBC = \angle ABC - \angle ABP = 2\theta - 60^\circ$. Because $BP = BC$, $\angle BPC = \angle BCP = \frac{1}{2}(180 - \angle PBC) = 120^\circ - \theta$.

Because C, P, Q are collinear, $\angle QPA = 180 - \angle BPC - \angle APB = \theta$. Furthermore, because A, B, C, Q are cyclic, $\angle AQP = \angle AQC = 180 - \angle CBA = 180 - 2\theta$, which in turn implies that $\angle QAP = \theta$. Thus we can conclude that $\triangle QPA \cong \triangle OBA$, therefore $PQ = OB = r$.

Problem 3 Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all positive integers x, y.

Solution: The constant function $f(x) = k$, where k is any positive integer, is the only possible solution.

It is easy to see that the constant function satisfies the given condition. Suppose that a non-constant function satisfies the given condition. There must exist some two positive integers a and b such that $f(a) < f(b)$.

This implies that $(a + b)f(a) < af(b) + bf(a) < (a + b)f(b)$, which by the given condition is equivalent to $(a+b)f(a) < (a+b)f(a^2+b^2) < (a + b)f(b)$, which in turn is equivalent to $f(a) < f(a^2 + b^2) < f(b)$ because $a + b$ must be positive.

Thus, given any two different values $f(a)$ and $f(b)$, we can find another value of the function strictly between those two. We can repeat this process an arbitrary number of times, each time finding another different value of f strictly between $f(a)$ and $f(b)$. However, the function gives only positive integer values, so there is a finite number of positive integers between any two values of the function, which is a contradiction. Thus the function must be constant.

1.4 China

Problem 1 Let ABC be a triangle with $AC < BC$, and let D be a point on side BC such that segment AD bisects $\angle BAC$.

(a) Determine the necessary and sufficient conditions, in terms of angles of triangle ABC, for the existence of points E and F on sides AB and AC ($E \neq A, B$ and $F \neq A, C$), respectively, such that $BE = CF$ and $\angle BDE = \angle CDF$.

(b) Suppose that points E and F in part (a) exist. Express BE in terms of the side lengths of triangle ABC.

Problem 2 Let $\{P_n(x)\}_{n=1}^{\infty}$ be a sequence of polynomials such that $P_1(x) = x^2 - 1$, $P_2(x) = 2x(x^2 - 1)$, and

$$P_{n+1}(x)P_{n-1}(x) = (P_n(x))^2 - (x^2 - 1)^2$$

for $n \geq 2$. Let S_n denote the sum of the absolute values of the coefficients of $P_n(x)$. For each positive integer n, find the largest nonnegative integer k_n such that 2^{k_n} divides S_n.

Problem 3 In the soccer championship of Fatland, each of 18 teams plays exactly once with each other team. The championship consists of 17 rounds of games. In each round, nine games take place and each team plays one game. All games take place on Sundays, and games in the same round take place on the same day. (The championship lasts for 17 Sundays.) Let n be a positive integer such that for any possible schedule, there are 4 teams with exactly one game played among them after n rounds. Determine the maximum value of n.

Solution: The maximum value of n is 7.

We first show that if $n \leq 7$, there must exist some 4 teams with exactly one game played among them. We will consider the graph G whose vertices represent the teams and where vertices a and b are connected by an edge iff teams a and b have played each other in the first n rounds. Each vertex of G has degree n because each team has played exactly n other teams up to that point. What we wish to show is that there exist 4 vertices such that the subgraph induced by G on those vertices has exactly 1 edge.

We proceed by contradiction. Suppose that for no 4 vertices of G does the induced subgraph on those four vertices have exactly 1 edge. Let a, b be a pair of adjacent vertices such that the number of vertices adjacent to both a and b is maximal. Suppose there are exactly k vertices adjacent to both a and b. Then because the degree of a is n, there are $n-k$ vertices (including b) adjacent to a but not b, and similarly $n-k$ vertices adjacent to b but not a. So the number of vertices of G adjacent to neither a nor b is $18-k-2(n-k) = 18-2n+k$. Because $n \leq 7$, there are at least $k+4$ vertices that are adjacent to neither of a and b.

We now claim that for any pair of vertices c, d such that neither of c, d is adjacent to either of a or b, c and d must be adjacent to each other. For suppose not. Then among the four vertices a, b, c, d, a and b would be connected by an edge, but no other pair of those four would have an edge connecting them. Thus there would be only one edge of G among those four vertices, contradicting our assumption.

We proved above that we can find $k+4$ distinct vertices, call them $e_1, e_2, \ldots, e_{k+4}$ such that none of them is adjacent to either a or b. Then our claim shows that for any distinct i, j, $1 \leq i, j \leq k+4$, e_i is adjacent to e_j. Namely, e_1 and e_2 are adjacent, and any of the $k+2$ vertices $e_2, e_3, \ldots, e_{k+4}$ is adjacent to both e_1 and e_2. So e_1 and e_2 form a pair of adjacent vertices with $k+2 > k$ other vertices adjacent to both of them. This contradicts the maximality of the pair a, b.

We now show that for $n \geq 8$, it is possible to have a situation in which, after n rounds, no subset of 4 teams has had exactly one game played among them. For convenience, partition the set of teams into two "leagues" of size 9, call them $A = a_1, a_2, \ldots, a_9$ and $B = b_1, b_2, \ldots, b_9$. Call a pair of teams "friendly" if either they both belong to the same league, or one of them is team a_k and the other team b_k for the same value of k. If not, the pair is "unfriendly". Each team is friendly with exactly 9 other teams and unfriendly with exactly 8 other teams.

We claim that (a) it is possible for 8 rounds to take place in which all unfriendly pairs, and no friendly pairs, play each other, and (b) it is also possible for 9 rounds to take place in which exactly the friendly pairs play each other. Combining the two, in any order, will give a complete 17-round tournament.

We first show (a). For each i with $1 \leq i \leq 8$, in round i let team a_k play team b_{k+i} for $k = 1, 2, \ldots, 9$. (Indices are here taken mod 9.) Thus if a_k and b_j, $k \neq j$ are two unfriendly teams, they will get to play each other exactly in round $k - j \pmod 9$, and friendly pairs will never be matched with each other.

Part (b) is slightly more complicated. In round i, where $1 \leq i \leq 9$, let team a_i play team b_i, and for $k \neq i$, let team a_k play team a_{2i-k} and team b_k play team b_{2i-k} (indices again mod 9). This determines the team matchings for each round. No pair of unfriendly teams can play each other and each pair of the form a_k, b_k is matched up in round k. Finally, because 2 is relatively prime to 9, each pair of the form a_j, a_k or b_j, b_k is matched up in round $(j+k) \cdot 2^{-1} \pmod 9$.

We now apply this to the problem at hand. First we give a counterexample for $n = 8$. Let the tournament proceed so that in the first 8 rounds, the pairs that are matched up are exactly the unfriendly pairs (we showed above that we can finish the tournament by letting the friendly pairs play). We need to show that among any four teams, either no pair is unfriendly or at least two pairs are unfriendly. Note that if two teams belong to the same league, they cannot be unfriendly, and that any team in one league is unfriendly with all but one of the teams in the other league. If all four teams belong to the same league, all pairs are friendly. If three belong to one league, and one to the other, without loss of generality say that $a_i, a_j, a_k \in A$ and $b_l \in B$ then at most one of the pairs (a_i, b_l), (a_j, b_l) and (a_k, b_l) must be friendly, so at least two must be unfriendly. Finally, if our four teams are split with two in each league, say $a_i, a_j \in A$ and $b_k, b_l \in B$, then a_i is unfriendly with at least one of b_k, b_l, as must be a_j, again giving us two unfriendly pairs. This settles the $n = 8$ case.

To deal with $n \geq 9$, we claim that among any four teams, at least two pairs are friendly. Any two teams that belong to the same league must be friendly with each other, so if i of the 4 teams belong to A, and $4 - i$ of them belong to B, by Jensen's inequality on the convex function $\binom{x}{2} = \frac{x(x-1)}{2}$ there must be at least $\binom{i}{2} + \binom{j}{2} \geq 2$ friendly pairs among them. So let the tournament proceed with the friendly teams playing each other in the first 9 rounds, and the unfriendly pairs in the last 8. After the first 9 rounds, by the above, every set of four teams must have had at least two games played among them, and this will remain true after n rounds, so it is impossible to have four teams with only one game played among them after n rounds. This completes the proof that no value of $n \geq 8$ works.

Problem 4 Let P_1, P_2, P_3, P_4 be four distinct points on the plane. Determine the minimum value of

$$\frac{\sum_{1 \le i < j \le 4} P_iP_j}{\min\{P_iP_j, 1 \le i < j \le 4\}}.$$

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Problem 5 On the coordinate plane, a point is called *rational* if both of its coordinates are rational numbers. Prove that all the rational points can be partitioned into three sets A_1, A_2, A_3 such that

(i) inside any circle centered at a rational point there are points $P_i \in A_i$, $i = 1, 2, 3$;

(ii) on any line in the plane there is some i, $1 \le i \le 3$, such that there is no rational point in A_i lying on the line.

Solution: Any rational point can be written uniquely in the form $\left(\frac{a}{c}, \frac{b}{c}\right)$ where a, b, c are integers, $c > 0$ and $\gcd(a, b, c) = 1$. Let

$$A_1 = \left\{\left(\frac{a}{c}, \frac{b}{c}\right) \mid a \equiv 1 \pmod 2\right\},$$

$$A_2 = \left\{\left(\frac{a}{c}, \frac{b}{c}\right) \mid a \equiv 0 \pmod 2, b \equiv 1 \pmod 2\right\},$$

$$A_3 = \left\{\left(\frac{a}{c}, \frac{b}{c}\right) \mid a \equiv 0 \pmod 2, b \equiv 0 \pmod 2, c \equiv 1 \pmod 2\right\}.$$

Since a, b, and c cannot all be $0 \pmod 2$, this partitions the set of rational points.

We first show (i). Let ω be a circle centered at a rational point $P = \left(\frac{a}{c}, \frac{b}{c}\right)$ with arbitrary radius δ. Choose $N \in \mathbb{N}$ large enough so that $\frac{1}{2N} < \delta$ and $\frac{1}{2N}\frac{\sqrt{a^2+b^2}}{c^2} < \delta$. Let

$$P_1 = \left(\frac{2Na+1}{2Nc}, \frac{b}{c}\right)$$

$$P_2 = \left(\frac{a}{c}, \frac{2Nb+1}{2Nc}\right)$$

$$P_3 = \left(\frac{2Na}{2Nc+1}, \frac{2Nb}{2Nc+1}\right)$$

Then $P_1 \in A_1$, $P_2 \in A_2$ and $P_3 \in A_3$. Also, the distance from P to P_1 is $\frac{1}{2^n c} < \frac{1}{c} \le \delta$, so P_1 is contained inside ω. Similarly, P_2 is also contained in ω. As for P_3, by the Pythagorean theorem $d(P,P_3)^2 = \frac{(2Na)^2+(2Nb)^2}{(2Nc(2Nc+1))^2} < \frac{a^2+b^2}{4N^2c^4}$. Taking square roots, $d(P,P_3) < \frac{1}{2N}\frac{\sqrt{a^2+b^2}}{c^2} < \delta$, so P_3 is also inside the circle ω.

Now for part (ii). Any line passing through at most one rational point satisfies the condition trivially, so we may assume that our line l passes through at least two rational points. Then the equation of l will be of the form $px + qy + r = 0$ where p, q, and r are rational: multipying by a constant, we can assume they are integers with gcd 1. The rational point $\left(\frac{a}{c}, \frac{b}{c}\right)$ lies on l iff

$$pa + qb + rc = 0 \tag{1}$$

Suppose that this line l contains members of all three of the A_i. Because it contains a point in A_3, there are integers a_3, b_3, c_3 satisfying (1) and with $a_3, b_3 \equiv 0 \pmod 2$, $c_3 \equiv 1 \pmod 2$. Taking both sides of (1) mod 2 shows that $r \equiv 0 \pmod 2$. Now, l also contains a point in A_2, so we have integers a_2, b_2, c_2 also satisfying (1) and having $a_2 equiv 1 \pmod 2$, $b_2 \equiv 0 \pmod 2$. Again we take (1) mod 3 for this solution, and get $0 \equiv pa_2 + qb_2 + rc_2 \equiv q \pmod 2$ using that r is even. We do this one last time with integers a_1, b_1, c_1 satisfying (1) having $a_1 \equiv 1 \pmod 2$, now we get $0 \equiv pa_1 + qb_1 + rc_1 \equiv p \pmod 2$. So all of p, q, and r are divisible by 2, contradicting the fact that $\gcd(p, q, r) = 1$.

So it is impossible for any line in the plane to contain representative of all three of our subsets, and we are done.

Problem 6 Let c be a given real number with $\frac{1}{2} < c < 1$. Determine the smallest constant M such that for any positive integer $n \ge 2$ and real numbers $0 < a_1 \le a_2 \le \cdots \le a_n$, if

$$\frac{1}{n}\sum_{k=1}^{n} ka_k = c\sum_{k=1}^{n} a_k,$$

then

$$\sum_{k=1}^{n} a_k \le M \sum_{k=1}^{\lfloor cn \rfloor} a_k.$$

denotes the largest integer less than or equal to x.

Problem 7 Determine all integers $n > 1$ such that there exist real numbers $a_1, a_2, \ldots, a_n$ satisfying the property:

$$\{|a_i - a_j| \mid 1 \le i < j \le n\} = \left\{1, 2, \ldots, \frac{n(n-1)}{2}\right\}.$$

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Problem 8 Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{7, 8, \ldots, n\}$. For $i = 1, 2, \ldots, 20$, let $S_i = \{a_{i,1}, a_{i,2}, a_{i,3}, b_{i,1}, b_{i,2}\}$ such that $a_{i,1}, a_{i,2}, a_{i,3} \in A$, $b_{i,1}, b_{i,2} \in B$, and

$$|S_i \cap S_j| \le 2$$

for $1 \le i < j \le 20$. X Determine the minimum value of n.

Problem 9 Let $ABCD$ be a convex quadrilateral. Diagonals AC and BD intersect at point P. Lines AB and CD intersect at point E while lines AD and BC intersect at point F. Let O be a point on line EF such that $PO \perp EF$. Prove that

$$\angle AOD = \angle BOC.$$

Problem 10 Let k be an integer and let f be a function from the set of negative integers to the set of integers such that

$$f(n)f(n+1) = (f(n) + n - k)^2$$

for all integers $n < -1$. Determine an explicit expression for $f(n)$.

1.5 Czech and Slovak Republics

Problem 1 Find all integers x, y such that

$$\langle 4x\rangle_5 + 7y = 14,$$

$$\langle 2y\rangle_5 - \langle 3x\rangle_7 = 74,$$

where $\langle n\rangle_k$ denotes the multiple of k closest to the number n.

Solution: Looking at the first equation, we have $7y \equiv 14 \pmod 5$, which yields $y \equiv 2$. Thus, $\langle 2y\rangle_5 = 2y + 1$, and the second equation becomes $\langle 3x\rangle_7 = 2y - 73$. Rewrite the first equation as $\langle 4x\rangle_5 = 14 - 7y$. It is apparent that $3x - 3 \le 2y - 73$ and $4x - 2 \le 14 - 7y$, from which we obtain $\frac{16-4x}{7} \ge y \ge \frac{3x+70}{2}$. Solving the inequality $\frac{16-4x}{7} \ge \frac{3x+70}{2}$ gives $x \le -\frac{458}{29}$. Now, we use the inequalities $2y - 73 \le 3x + 3$ and $14 - 7y \le 4x + 2$ to obtain $\frac{3z+76}{2} \ge y \ge \frac{12-4x}{7}$. Using $\frac{3x+76}{2} \ge \frac{12-4x}{7}$ to solve for x, we get $x \ge -\frac{508}{29}$. Combining this with our upper bound for x, we see that the only possible values for x are -16 and -17. However, only -17 yields an integral value of y, and we get the unique solution $(x, y) = (-17, 12)$.

Problem 2 Let $ABCD$ be a square. Let KLM be an equilateral triangle such that K, L, M lie on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, respectively. Find the locus of the midpoint of segment $\overline{KL}$ for all such triangles KLM.

Solution: Let P be the midpoint of KM. Note that $\angle KPL + \angle KBL = \pi$. Thus, quadrilateral $KBLP$ is cyclic and $\angle PBA = \angle PBK = \angle PQK = \frac{\pi}{6}$, where Q is the midpoint of BC. Similarly, $\angle PCD = \frac{\pi}{6}$. This shows that P is a fixed point as triangle KLM varies.

Let R be the midpoint of KL. Note that $\angle PRL + \angle PCL = \angle PRL + \frac{\pi}{3} = \angle PRL + \angle PRK = \pi$. Hence, $PRLC$ is a cyclic quadrilateral, and so $\angle PCR = \angle PLR = \frac{\pi}{6}$. Therefore, $\angle BCR = \frac{\pi}{6}$, which implies that the locus of R is a line segment.

We wish to find the endpoints of this line segment. One endpoint occurs when $K = A$. Let X be the midpoint of KL for this particular triangle KLM. The other endpoint occurs when $L = C$. Let Y be the midpoint of KL for this particular triangle KLM. Then, the locus of KL is simply the segment XY.

Problem 3 Show that a given positive integer m is a perfect square if and only if for each positive integer n, at least one of the differences

$$(m+1)^2 - m, (m+2)^2 - m, \ldots, (m+n)^2 - m$$

is divisible by n.

Solution: First, assume that m is a perfect square. If $m = a^2$, then $(m+c)^2 - m = (m+c)^2 - a^2 = (m+c+a)(m+c-a)$. Clearly, there exists some c, with $1 \leq c \leq n$, for which $m + c + a$ is divisible by n. Thus, one of the given differences is divisible by n if m is a perfect square.

Now, we assume that m is not a perfect square and show that there exists n for which none of the given differences are divisible by n. Clearly, there exist a prime p and positive integer k such that p^{2k-1} is the highest power of p which divides m. We may let $m = bp^{2k-1}$, with b and p being relatively prime. Furthermore, pick $n = p^{2k}$. For the sake of contradiction, assume there exists a positive integer c for which $(m+c)^2 - m$ is divisible by n. By expanding $(m+c)^2 - m$, we note that

$$p^{2k} \mid (2bcp^{2k-1} + c^2 - bp^{2k-1})$$

If p^{2k} divides the quantity, then so does p^{2k-1}. Thus, $p^{2k-1} \mid c^2$ and so $p^k \mid c$. Let $c = rp^k$. Then, we have

$$p^{2k} \mid (2brp^{3k-1} + r^2p^{2k} - bp^{2k-1})$$

However, this implies that $p \mid b$, which contradicts the original assumption that b and p are relatively prime. Therefore, if m is not a perfect square, n may be chosen so that none of the given differences are divisible by n. This completes the proof.

Problem 4 Find all pairs of real numbers a, b such that the equation

$$\frac{ax^2 - 24x + b}{x^2 - 1} = x$$

has exactly two real solutions, and such that the sum of these two real solutions is 12.

Solution: Call the given equation $(*)$. We start by multiplying $(*)$ by $x^2 - 1$ and rearranging terms to get $p(x) = 0$, where $p(x) = x^3 - ax^2 + 23x - b$. Note that $p(x)$ has at least two roots because any root of $(*)$ is also a root of $p(x)$. Moreover, because $p(x)$ has degree

3, it must have three real roots (possibly repeated) which we will call r_1, r_2, and r_3. By Vieta's relations, we have the following equations:

$$a = r_1 + r_2 + r_3 \tag{1}$$

$$23 = r_1 r_2 + r_2 r_3 + r_3 r_1 \tag{2}$$

$$b = r_1 r_2 r_3 \tag{3}$$

Also note that any root of $p(x)$ corresponds to a root of $(*)$ as long as it is not equal to -1 or 1. Therefore, in order for the given equation to have exactly two roots, we must have one of the following cases:

Case 1: One of the roots of $p(x)$ is -1 and the other two roots are different and not equal to -1 or 1. WLOG, let r_1 = -1. Using the fact that $p(-1) = 0$, we obtain $a + b = -24$. From the problem statement, we find that $r_2 + r_3 = 12$. From (1), we get $a = 11$. Thus, $b = -24 - a = -35$. We note that $(a, b) = (11, -35)$ is valid because (*) only has two roots, namely, 5 and 7.

Case 2: One of the roots of $p(x)$ is 1 and the other two roots are different and not equal to -1 or 1. WLOG, let $r_1 = 1$. Because $p(1) = 0$, we find that $a + b = 24$. As in the previous case, $r_2 + r_3 = 12$. From (1), $a = 13$. Thus, $b = 11$. However, then $p(x) = (x-1)^3(x-11)$, which would contradict our original assumption that $r_2, r_3 \neq 1$. Hence, there is no valid pair (a, b) in this case.

Case 3: $r_1 = r_2$ and none of the roots of $p(x)$ are equal to -1 or 1 By the problem statement, $r_1 + r_3 = 12$. By rewriting (2) only in terms of r_1, we obtain $(r_1 - 1)(r_1 - 23) = 0$, which implies that $r_1 = 23$. Thus, $r_2 = 23$ and $r_3 = -11$. Now, we may use (1) and (3) to obtain $(a, b) = (35, -5819)$. It is easy to verify that this solution works.

Therefore, the only valid pairs (a, b) are (11, -35) and (35, -5819).

Problem 5 In the plane is given a triangle KLM. Point A lies on line KL, on the opposite side of K as L. Construct a rectangle $ABCD$ whose vertices B, C, and D lie on lines KM, KL, and LM, respectively.

Problem 6 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f(xf(y)) = f(xy) + x$$

for all positive reals x, y.

Solution: The only possible function is $f(x) = x + 1$. Suppose a is in the range of f and $f(t) = a$. Then, letting $x = 1$ and $y = t$ in the given equation shows that $f(a) = a + 1$. Now, any number $c > a$ is also in the range of f, which is seen by substituting $x = c - a$ and $y = \frac{t}{c-a}$ into the equation. Hence, $f(c) = c + 1$ for all $c \geq a$.

Now, suppose the equation

$$f(c) = c + 1 \tag{$*$}$$

is true for all $c \geq a$. Define a sequence $\{a_n\}_{n\geq 0}$ given by $a_0 = a$ and $a_k = \frac{a_{k-1}^2}{a_{k-1}+1}$, where $k \geq 1$. We now show that if $(*)$ holds for all $c \geq a_k$, then it also holds for all $c \geq a_{k+1}$. Assume it holds for $c \geq a_k$. If $a_{k+1} \leq c \leq a_k$, we may substitute $x = \frac{c}{a_k}$ and $y = a_k$ to obtain

$$\begin{aligned} f(c) &= f\left(\frac{c}{a_k} f(a_k)\right) - \frac{c}{a_k} \\ &= f\left(\frac{c}{a_k}(a_k + 1)\right) - \frac{c}{a_k} \\ &= \frac{c}{a_k}(a_k + 1) + 1 - \frac{c}{a_k} \\ &= c + 1 \end{aligned}$$

using the fact that $\frac{c}{a_k}(a_k + 1) \geq a_k$. Thus, $(*)$ holds for any $c \geq a_{k+1}$. Because $a_k \leq a\left(\frac{a}{a+1}\right)^k$, a_k can assume an arbitrarily small positive number for suitably large k. Thus, we may conclude that $(*)$ holds for all $c > 0$.

1.6 Germany

Problem 1 Determine all ordered pairs (a, b) of real numbers that satisfy

$$2a^2 - 2ab + b^2 = a$$
$$4a^2 - 5ab + 2b^2 = b.$$

Solution:

Clearly, $a = 0, b = 0$ is a solution. Also, from the two equations we can easily get that if one of a and b is 0, the other is also. So we suppose that neither a nor b is 0.

$$2a^2 - 2ab + b^2 = a$$
$$\Rightarrow 4a^2 - 4ab + 2b^2 = 2a$$

We are also given $4a^2-5ab+2b^2 = b$, so we subtract to get $ab = 2a-b$. Solving for b, this gives $b = \frac{2a}{a+1}$ Combining this with the first equation, we get:

$2a^2 - 2a\frac{2a}{a+1} + (\frac{2a}{a+1})^2 = a$

Since a is not 0, we can cancel a.

$2a - 2\frac{2a}{a+1} + \frac{4a}{(a+1)^2} = 1$

Multiply by $(a + 1)^2$ on both sides and simplify. This will yield:

$2a^3 - a^2 - 1 = 0$, which factors into $(a - 1)(2a^2 + a + 1) = 0$. $2a^2 + a + 1 = 0$ has no reals roots, so the only solution is $a = 1$. $b = \frac{2a}{a+1}$, so $b = 1$. This satisfies the equation. So the only two solutions are (a,b)=(0,0) or (1,1). □

Problem 2

(a) Prove that there exist eight points on the surface of a sphere with radius R, such that all the pairwise distances between these points are greater than $1.15R$.

(b) Do there exist nine points with this property?

Problem 3 Let p be a prime. Prove that

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^3}{p} \right\rfloor = \frac{(p-2)(p-1)(p+1)}{4}.$$

Proof. When $p = 2$, clearly it is true because both sides are 0.

So now assume $p \geq 3$. p is odd.

$$1^3 + 2^3 + \cdots + (p-1)^3 = \frac{p^2(p-1)^2}{4}.$$

So

$$\frac{1^3}{p} + \frac{2^3}{p} + \cdots + \frac{(p-1)^3}{p} = \frac{p(p-1)^2}{4}.$$

So I want to prove

$$\left\{\frac{1^3}{p}\right\} + \left\{\frac{2^3}{p}\right\} + \cdots + \left\{\frac{(p-1)^3}{p}\right\} = \frac{p(p-1)^2}{4} - \frac{(p-1)(p+1)(p-2)}{4} = \frac{p-1}{2},$$

where $\{x\}$ is the fraction part of $x := x - [x]$. Obviously, for $1 \leq i \leq p-1$, $1 > \left\{\frac{i^3}{p}\right\} > 0$. This is because p is not a factor of i, so p is not a factor of i^3. Also notice that

$$i^3 + (p-1)^3 = p^3 - 3p^2 i + 3pi^2 - i^3 + i^3 =$$

$$= p^3 - 3p^2 i + 3pi^2 - i^3 = p(p^2 - 3pi + 3i^2)$$

is a multiple of p. So $\frac{(p-i)^3}{p} + \frac{i^3}{p}$ is an integer. Hence

$$\left\{\frac{i^3}{p}\right\} + \left\{\frac{(p-i)^3}{p}\right\} \in \mathbb{Z}.$$

But

$$0 < \left\{\frac{i^3}{p}\right\} + \left\{\frac{(p-i)^3}{p}\right\} < 2.$$

So it is 1.

Since p is odd, $\frac{p-1}{2}, \frac{p+1}{2} \in \mathbb{Z}$.

$$\left\{\frac{1^3}{p}\right\} + \left\{\frac{(p-1)^3}{p}\right\} = 1$$

$$\left\{\frac{2^3}{p}\right\} + \left\{\frac{(p-2)^3}{p}\right\} = 1$$

$$\vdots$$

$$\left\{\frac{\left(\frac{p-1}{2}\right)^3}{p}\right\} + \left\{\frac{\left(\frac{p+1}{2}\right)^3}{p}\right\} = 1.$$

So

$$\left\{\frac{1^3}{p}\right\} + \left\{\frac{2^3}{p}\right\} + \cdots + \left\{\frac{(p-1)^3}{p}\right\} = \frac{p-1}{2}$$

as desired. □

Problem 4 Let a_1 be a positive real number, and define $a_2, a_3, \ldots$ recursively by setting $a_{n+1} = 1 + a_1 a_2 \cdots a_n$ for $n \geq 1$. In addition, define $b_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$ for all $n \geq 1$. Prove that $b_n < x$ holds for all n if and only if $x \geq \frac{2}{a_1}$.

Proof. First, we prove, for all $n \in \mathbb{N}$,

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq \frac{2}{a_1}$$

which is equivalent to

$$\frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n} \geq 0.$$

Lemma. For all $n \in \mathbb{N}$,

$$\frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n} \quad = \quad \frac{1}{a_1 a_2 \cdots a_n}\ .$$

We will prove this Lemma by induction.

First,

$$\frac{1}{a_1} - \frac{1}{a_2} = \frac{a_2 - a_1}{a_1 a_2} = \frac{(a_1 + 1) - a_1}{a_1 a_2} = \frac{1}{a_1 a_2}.$$

If

$$\frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_k} \quad = \quad \frac{1}{a_1 a_2 \cdots a_k}\ ,$$

then

$$\begin{aligned}
\frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_k} - \frac{1}{a_{k+1}} \quad &= \quad \frac{1}{a_1 a_2 \cdots a_n} - \frac{1}{a_{k+1}} \\
= \quad \frac{a_{k+1} - a_1 a_2 \cdots a_k}{a_1 a_2 \cdots a_k a_{k+1}} \quad &= \quad \frac{a_1 a_2 \cdots a_k + 1 - a_1 a_2 \cdots a_k}{a_1 a_2 \cdots a_k a_{k+1}} \\
= \quad \frac{1}{a_1 a_2 \cdots a_k a_{k+1}} \quad & \quad .
\end{aligned}$$

Thus completes the induction and the proof of the Lemma.

One can easily see by strong induction that $a_k > 0$ for any $k \in \mathbb{N}$. So

$$\frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n} = \frac{1}{a_1 a_2 \cdots a_n} \geq 0\ .$$

Thus we finished this part.

Now we prove that if $x < \frac{2}{a_1}$, then there exists an $n \in \mathbb{N}$ such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} > x$$

which is equivalent to: there exists an $n \in \mathbb{N}$ such that

$$x - \frac{2}{a_1} + \frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n} < 0$$

which is equivalent to: there exists an $n \in \mathbb{N}$ such that

$$\frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n} < \frac{2}{a_1} - x.$$

But $\frac{2}{a_1} - x > 0$, so we need only to prove that for any $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that

$$\frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n} < \epsilon\ .$$

Since

$$\frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n} > 0\ ,$$

we only need to prove that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\left|\frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n}\right| < \epsilon\ .$$

This is equivalent to prove that the sequence

$$c_n = \frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n}$$

has limit 0. By the above proved Lemma,

$$C_n = \frac{1}{a_1 a_2 \cdots a_n}.$$

So the sequence $\{c_n\}$ has limit 0 if and only if the sequence $d_n = a_1 a_2 \cdots a_n$ has limit ∞ which is equivalent to

$$\begin{aligned} &\lim_{n\to\infty} a_1 a_2 \cdots a_n = \infty \\ \Longleftrightarrow &\lim_{n\to\infty} (a_{n+1} - 1) = \infty \\ \Longleftrightarrow &\lim_{n\to\infty} a_{n+1} = \infty \\ \Longleftrightarrow &\lim_{n\to\infty} a_n = \infty\ . \end{aligned}$$

So it suffices to prove $\lim\limits_{n\to\infty} a_n = \infty$. This is true if we can prove that, for any $n \in \mathbb{N},\ a_{n+1} - a_n > a_1^2$. (Notice that $a_1 > 0$.)

For any $n \in \mathbb{N}$,

$$a_{n+1} - a_n = 1 + a_1a_2\cdots a_{n-1}(1 + a_1a_2\cdots a_{n-1}) - (1 + a_1a_2\cdots a_{n-1})$$
$$= a_1a_2\cdots a_{n-1}(a_1a_2\cdots a_{n-1} + 1 - 1) \quad = \quad (a_1a_2\cdots a_{n-1})^2.$$

But if $k \geq 2,\ a_k \geq 1$. This is because $a_{k-1} \geq 0,\ a_{k-2} \geq 0,\ \cdots,\ a_1 \geq 0$ and

$$a_k = 1 + a_{k-1}a_{k-2}\cdots a_1.$$

So

$$(a_2a_3\cdots a_{n-1})^2 \geq 1.$$

Ergo,

$$(a_1a_2a_3\cdots a_{n-1})^2 \geq a_1^2.$$

Therefore,

$$a_{n+1} - a_n = (a_1a_2a_3\cdots a_{n-1})^2 > a_1^2$$

and the sequence $\{a_n\}$ approaches ∞.

□

Problem 5 Prove that a triangle is a right triangle if and only if its angles α, β, γ satisfy

$$\frac{\sin^2\alpha + \sin^2\beta + \sin^2\gamma}{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 2.$$

Proof. If the triangle is a right triangle, WLOG, assume $\alpha = \frac{\pi}{2},\ \beta + \gamma = \frac{\pi}{2}$. Then $\cos\beta = \sin\gamma,\ \sin^2\alpha = 1$, and $\cos^2\alpha = 0$. Hence

$$\frac{\sin^2\alpha + \sin^2\beta + \sin^2\gamma}{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = \frac{1 + \sin^2\beta + \cos^2\beta}{0 + \sin^2\gamma + \cos^2\gamma} = \frac{1+1}{0+1} = 2.$$

If the triangle is obtuse, WLOG, assume $\alpha > \frac{\pi}{2},\ \beta + \gamma < \frac{\pi}{2}, 0 < \beta < \frac{\pi}{2} - \gamma$.

$\sin x$ increases in the interval $[0, \frac{\pi}{2}]$ which implies

$$0 \leq \sin\beta < \sin\left(\frac{\pi}{2} - \gamma\right) = \cos\gamma;$$

Similarly, that $\cos x$ decreases in the interval $[0, \frac{\pi}{2}]$ implies

$$\cos\beta > \cos\left(\frac{\pi}{2} - \gamma\right) = \sin\gamma \geq 0.$$

Also, since $\sin^2\alpha \le 1$ and $\cos^2\alpha \ge 0$, we have

$$\frac{\sin^2\alpha + \sin^2\beta + \sin^2\gamma}{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} < \frac{\sin^2\alpha + \cos^2\gamma + \sin^2\gamma}{\cos^2\alpha + \sin^2\gamma + \cos^2\gamma}$$

$$= \frac{\sin^2\alpha + 1}{\cos^2\alpha + 1} < \frac{1+1}{0+1} = 2.$$

If the triangle is acute, that is, all three angles α, β, γ are between 0 and $\frac{\pi}{2}$. We will prove that

$$\frac{\sin^2\alpha + \sin^2\beta + \sin^2\gamma}{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} > 2.$$

$$\iff \frac{\sin^2\alpha + \cos^2\alpha + \sin^2\beta + \cos^2\beta \sin^2\gamma + \cos^2\gamma}{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} > 2 + 1 = 3$$

$$\iff \frac{3}{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} > 3$$

$$\iff \cos^2\alpha + \cos^2\beta + \cos^2\gamma < 1.$$

$$\iff \sin^2\alpha + \sin^2\beta + \sin^2\gamma > 2.$$

So we need to prove that, for $0 < \alpha, \beta, \gamma < \frac{\pi}{2},\ \alpha + \beta + \gamma = \pi$,

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma > 2.$$

Observe that

$$\frac{d \sin\alpha}{d\alpha} = 2\sin\alpha\cos\alpha = \sin 2\alpha \qquad \text{and} \qquad \frac{d \sin 2\alpha}{d\alpha} = 2\cos 2\alpha.$$

So $\sin^2 x$ is concave up in $[0, \frac{\pi}{4}]$ and concave down in $[\frac{\pi}{4}, \frac{\pi}{2}]$. Clearly, since $\alpha + \beta + \gamma = \pi$, there are either two or three of α, β, γ in $(\frac{\pi}{4}, \frac{\pi}{2})$. Assume, WLOG, $\alpha \le \beta \le \gamma$. If two of α, β, γ are in $(\frac{\pi}{4}, \frac{\pi}{2})$, then $\alpha \le \frac{\pi}{4}$, and $\beta, \gamma > \frac{\pi}{4}$ and $\beta + \gamma \ge \frac{3\pi}{4}$. If we move γ towards $\frac{\pi}{2}$, fixing $\beta + \gamma = \pi - \alpha$, then β moves towards $\frac{\pi}{4}$. But $\beta + \gamma \ge \frac{3\pi}{4}$. So γ would reach $\frac{\pi}{2}$ first. Observe that as α, β, γ are moved, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is always decreasing, since $\beta, \gamma \in (\frac{\pi}{4}, \frac{\pi}{2})$, where $\sin^2 x$ is concave down and we are moving β and γ farther apart. Hence $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ will decrease until γ reaches $\frac{\pi}{2}$ in which case $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ will be 2. Therefore for the original α, β, γ, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma > 2$. For the case $\alpha, \beta, \gamma > \frac{\pi}{4}$ we will do the same thing: first fix β and move α and γ apart until α reaches $\frac{\pi}{4}$ or γ reaches $\frac{\pi}{2}$. If γ reaches $\frac{\pi}{2}$ first, we are done. Otherwise, we continue, like in the previous case, moving β and γ apart with α fixed. During these movements,

$\sin^2\alpha+\sin^2\beta+\sin^2\gamma$ is always decreasing. Continuing this procedure, at some point γ will reach $\frac{\pi}{2}$ and $\sin^2\alpha+\sin^2\beta+\sin^2\gamma$ will be 2. Therefore $\sin^2\alpha+\sin^2\beta+\sin^2\gamma$ is originally larger than 2. $\square$

Problem 6 Ralf Reisegern explains to his friend Markus, a mathematician, that he has visited eight EURO-counties this year. In order to motivate his five children to use the new Cent- and Euro-coins, he brought home five coins (not necessarily with distinct values) from each country. Because his children can use the new coins in Germany, Ralf made sure that among the 40 coins, each of the eight values (1, 2, 5, 10, 20, and 50 Cents; 1 and 2 Euros) appeared on exactly five coins. Now Ralf wonders whether he will be able to present each child eight coins, one from each country, such that the total value of the coins that each child receives is 3,88 Euro. (1 Euro equals 100 Cents, and 3,88 Euro equals 3 Euro and 88 Cents.) "That must be possible!" says Markus, without looking more carefully at the coins. Prove or disprove Markus' statement.

1.7 Iran

Problem 1 Find all functions f from the nonzero reals to the reals such that

$$xf\left(x+\frac{1}{y}\right)+yf(y)+\frac{y}{x}=yf\left(y+\frac{1}{x}\right)+xf(x)+\frac{x}{y}$$

for all nonzero reals x, y.

Problem 2 Let segment $\overline{AB}$ be a diameter of a circle ω. Let ℓ_a, ℓ_b be the lines tangent to ω at A and B, respectively. Let C be a point on ω such that line BC meets ℓ_a at a point K. The angle bisector of angle CAK meets line CK at H. Let M be the midpoint of arc CAB, and let S be the second intersection of line HM and ω. Let T be the intersection of ℓ_b and the line tangent to ω at M. Show that S, T, K are collinear.

Problem 3 Let $k \geq 0$ and $n \geq 1$ be integers, and let $a_1, a_2, \ldots, a_n$ be distinct integers such that there are at least $2k$ different integers modulo $n+k$ among them. Prove that there is a subset of $\{a_1, a_2, \ldots, a_n\}$ whose sum of elements is divisible by $n+k$.

Problem 4 The sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and

$$x_{n+1}=\left\lfloor x_n!\sum_{k=1}^{\infty}\frac{1}{k!}\right\rfloor.$$

Prove that $\gcd(x_m, x_n) = x_{\gcd(m,n)}$ for all positive integers m, n.

Problem 5 Distinct points B, M, N, C lie on a line in that order such that $BM = CN$. A is a point not on the same line, and P, Q are points on segments $\overline{AN}, \overline{AM}$, respectively, such that $\angle PMC = \angle MAB$ and $\angle QNB = \angle NAC$. Prove that $\angle QBC = \angle PCB$.

Problem 6 A *strip* of width w is the closed region between two parallel lines a distance w apart. Suppose that the unit disk $\{(x, y) \in \mathbb{R}^2, x^2+y^2 \leq 1\}$ is covered by strips. Show that the sum of the widths of these strips is at least 2.

Problem 7 Given a permutation $(a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, we call the permutation *quadratic* if there is at least one perfect square among the numbers $a_1, a_1+a_2, \ldots, a_1+a_2+\cdots+a_n$. Find all positive integers n such that every permutation of $1, 2, \ldots, n$ is quadratic.

Problem 8 A $10 \times 10 \times 10$ cube is divided into 1000 $1 \times 1 \times 1$ blocks. 500 of the blocks are black and the others are white. Show that there exists at least 100 unit squares which are a shared face of a black block and a white block.

Problem 9 Let ABC be a triangle. The incircle of triangle ABC touches side $\overline{BC}$ at A'. Let segment $\overline{AA'}$ meet the incircle again at P. Segments $\overline{BP}, \overline{CP}$ meet the incircle at M, N, respectively. Show that lines AA', BN, CM are concurrent.

Problem 10 Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^{n} x_i^2 = n$. Write $S = \sum_{i=1}^{n} x_i$. Show that for any real λ with $0 \le \lambda \le 1$, at least

$$\left\lceil \frac{S^2(1-\lambda)^2}{n} \right\rceil$$

of the x_i are greater than $\frac{\lambda S}{n}$.

Problem 11 Around a circular table sit n people labelled $1, 2, \ldots, n$. Some pairs of them are friends, where if A is a friend of B, then B is a friend of A. Each minute, one pair of neighbor friends exchanges seats. What is the necessary and sufficient condition about the friendship relations among the people, such that it is possible to form any permutation of the initial seating arrangement?

Problem 12 Circle ω_1 is internally tangent to the circumcircle of triangle ABC at point M. Assume that ω_1 is tangent to sides $\overline{AB}$ and $\overline{AC}$ as well. Let H be the point where the incircle of triangle ABC touches side $\overline{BC}$, and let A' be a point on the circumcircle for which we have $\overline{AA'} \parallel \overline{BC}$. Show that points M, H, A' are collinear.

1.8 Japan

Problem 1 On a circle ω_0 are given three distinct points A, M, B with $AM = MB$. Let P be a variable point on the arc AB not containing M. Denote by ω_1 the circle inscribed in ω_0 that is tangent to ω_0 at P and also tangent to chord $\overline{AB}$. Let Q be the point where ω_0 intersects chord $\overline{AB}$. Prove that $MP \cdot MQ$ is constant, independent of the choice of P.

Solution: We first claim that points P, Q, M are collinear.

Let C_0, C_1 be the centers of ω_0 and ω_1 respectively. Let r_0, r_1 be their radii.

$\overline{AB}$ is tangent to ω_1, so $\overline{C_1Q} \perp \overline{AB}$. Also, $\overline{C_0M} \perp \overline{AB}$ because M is the midpoint of $\widehat{AB}$. We thus have $\overline{C_1Q} \parallel \overline{C_0M}$.

Now the two circles ω_0 and ω_1 are tangent to each other at P, so P, C_0, and C_1 must be collinear. Then parallel lines $\overline{C_1Q}$ and $\overline{C_0M}$ intersect the same line, $\overline{C_0P}$. Thus, $\angle PC_1Q = \angle PC_0M$. But because

$$\frac{PC_1}{PC_0} = \frac{r_1}{r_0} = \frac{C_1Q}{C_0M}$$

,

we now have $\triangle PC_1Q \sim \triangle PC_0M$. Then $\angle C_1PQ = \angle C_0PM = \angle C_1PM$, implying that P, Q, M are collinear.

Notice that $\angle MAQ = \angle MAB = \angle MBA = \angle MPA$. Also, $\angle AMQ = \angle PMA$, so $\triangle MAQ \sim \triangle MPA$.

From these similar triangles, we obtain $\frac{MA}{MP} = \frac{MQ}{MA}$, which implies $MA^2 = MP \cdot MQ$. The length of $\overline{MA}$ is constant, so it follows that $MP \cdot MQ$ is constant.

Problem 2 There are $n \geq 3$ coins are placed along a circle, with one showing heads and the others showing tails. An *operation* consists of simultaneously turning over each coin that satisfies the following condition: among the coin and its two neighbors, there is an odd number of heads among the three.

(a) Prove that if n is odd, then the coins will never become all tails.

(b) For what values of n will the coins eventually show all tails? For those n, how many operations are required to make all the coins show tails?

Solution:

(a) We define H to be a coin on the circle showing heads and T to be a coin showing tails. Let α be the flip that turns the middle coin of the triple HTT, β the flip that turns the middle coin of THT, γ the flip that turns the middle coin of TTH, and δ the flip that turns the middle coin of HHH. It is easy to see that these four flips are the only possible flips in an operation.

Suppose that a finite sequence of operations on the odd number of coins can turn all the coins to T's.

We consider the configuration of coins, C, one operation before all coins are tails. The last operation must have consisted only of β flips and γ flips, since other flips introduce coins showing heads. So we have three cases.

Case 1: Suppose that all flips in the final operation were β flips. Then the configuration C must have been a series of alternating T's and H's. But that requires the number of T's to equal the number of H's, and hence n to be even. Because we assumed that n were odd, this is a contradiction.

Case 2: Suppose that all flips in the final operation were γ flips. Then the configuration C must have been a series of H's. Now consider the configuration C' that is one operation before C. Every flip in the operation that takes C' to C must have been either an α flip or a γ flip, since other flips introduce coins showing tails. So one of the two triples HTT and TTH exists in the circle. In both cases, the double TT exists. Let $\omega = TT$ exist somewhere in the circle. The coin to the right of ω cannot be T, because then we have TTT, a triple in which the middle T will not be flipped into an H after an operation. So the coin ω_0 immediately after ω is H. The coin after ω_0 cannot be T, since then we have THT, in which the middle coin is turned into a T after an operation. Thus, the coin after ω_0 is H. Continuing the reasoning shows that $HHTT$ must follow every instance of TT. Thus, C' must be composed of alternating HH's and TT's. But that implies that there are an even number of coins, a contradiction since there are n, an odd number of coins.

Case 3: Both β flips and γ flips were in the final operation. Label any coin in the circle as the starting point and going clockwise, find the first triple THT after which HHH occurs.

Then there exists the triple THH, which is a contradiction since after an operation, the middle coin H is not turned into a T.

Thus, our assumption was false and it follows that if n is odd, the coins will never become all tails. Notice that the fact that the n coins started out with only one showing heads was not used.

(b)

Problem 3 Let $n \geq 3$ be an integer. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers with

$$a_1 + a_2 + \cdots + a_n = 1 \quad \text{and} \quad b_1^2 + b_2^2 + \cdots + b_n^2 = 1.$$

Prove that

$$a_1(b_1 + a_2) + a_2(b_2 + a_3) + \cdots + a_n(b_n + a_1) < 1.$$

Solution: The left-hand side is equivalent to

$$\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} a_i a_{i+1},$$

where a_{n+1} is defined as a_1.

By the Cauchy-Schwarz inequality,

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 = \sum_{i=1}^{n} a_i^2,$$

where we have used the fact that $\sum_{i=1}^{n} b_i^2 = 1$.

The above implies

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2}.$$

Thus, it suffices to prove the following equation:

$$\sqrt{\sum_{i=1}^{n} a_i^2} + \sum_{i=1}^{n} a_i a_{i+1} < 1.$$

$\sum_{i=1}^{n} a_i = 1$, so squaring, we obtain the identity

$$\sum_{i=1}^{n} a_i^2 = 1 - 2 \sum_{1 \leq i,j \leq n, i \neq j} a_i a_j,$$

which reduces our inequality to proving the following (α):

$$\sqrt{1 - 2\sum_{1\le i,j\le n, i\ne j} a_i a_j} < 1 - \sum_{i=1}^{n} a_i a_{i+1}.$$

Now observe that

$$1 = \sum_{1\le i\le n} a_i^2 + 2\sum_{1\le i,j\le n, i\ne j} a_i a_j \ge \sum_{1\le i,j\le n, i\ne j} a_i a_j.$$

Also, notice the following (β):

$$\sum_{1\le i,j\le n, i\ne j} a_i a_j \ge \sum_{i=1}^{n} a_i a_{i+1},$$

which is true because the terms on the left-hand side contains the terms of the right-hand side.

From the above two observations, we have

$$1 \ge \sum_{i=1}^{n} a_i a_{i+1},$$

or equivalently,

$$1 - \sum_{i=1}^{n} a_i a_{i+1} \ge 0.$$

Then, both sides of α are nonnegative, so it is true if and only if the squared expression is true. After squaring and rearranging, we have

$$2(\sum_{i=1}^{n} a_i a_{i+1} - \sum_{1\le i,j\le n, i\ne j} a_i a_j) < (\sum_{i=1}^{n} a_i a_{i+1})^2$$

But all the a_i are positive, so the right-hand side is positive. From β, the left-hand side is not positive, so the conclusion easily follows.

Problem 4 A set S of 2002 distinct points in the xy-plane is chosen. We call a rectangle *proper* if its sides are parallel to the coordinate axes and if the endpoints of at least one diagonal lie in S. Find the largest N such that, no matter how the points of S are chosen, at least one proper rectangle contains $N + 2$ points on or within its boundary.

1.9 Korea

Problem 1 Let p be a prime of the form $12k+1$ for some positive integer k, and write $\mathbb{Z}_p\{0,1,2,\ldots,p-1\}$. Let $\mathbb{E}_p$ consist of all (a,b) such that $a,b \in \mathbb{Z}_p$ and $p \nmid (4a^3+27b^2)$. For $(a,b),(a',b') \in \mathcal{E}_p$, we say that (a,b) and (a',b') are *equivalent* if there is a nonzero element $c \in \mathbb{Z}_p$ such that

$$p \mid (a'-ac^4) \quad \text{and} \quad p \mid (b'-bc^6).$$

Find the maximal number of elements in $\mathbb{E}_p$ such that no two of the chosen elements are equivalent.

Solution: Answer: 32

By Fermat's Little Theorem, we know $a^{12} \equiv 1 \pmod{p = 12k+1}$

Since $a^{12k} \equiv 1 \pmod p$ has $12k$ solutions, $(c^4)^{3k} \equiv 1 \pmod p$ has $3k$ solutions, and $(c^6)^{2k} \equiv 1 \pmod p$ has $2k$ solutions. Let the $3k$ possible values for c^4 be $c_1, c_2, \ldots, c_{3k}$. Then, for some fixed a, we have

$a_1 \equiv ac_1, a_2 \equiv ac_2, \ldots, a_{3k} \equiv ac_{3k}$.

Lemma 1: If $a_i \equiv ac_i$, then there exists some c_j such that $a \equiv a_i c_j$.

Proof: First, let us show that $c_i c_j \equiv c_m$ for some $1 <= i,j,m <= 3k$. This is true since $(c_i)^{3k} \equiv 1 \pmod p$, $(c_j)^{3k} \equiv 1 \pmod p$, so $(c_i c_j)^(3k) \equiv 1 \pmod p$. Therefore $c_i c_j$ must be some possible value of c^4, for example, c_m. From this we see that $c_i c_j takesonthevaluesc_1, c_2, \ldots, c_(3k)$ as j ranges from 1 to $3k$. Notice that since $1^{3k} \equiv 1 \pmod p$, one of the c_j's is 1. Therefore, for any c_i, there exists some c_j so that $c_i c_j \equiv 1 \pmod p$. Now let us return to $a_i \equiv ac_i$. Take the c_j such that $c_i c_j \equiv 1 \pmod p$ and multiply it by both sides of the equation. Then we have $a_i c_j \equiv ac_i c_j \pmod p \equiv a \pmod p$. Hence our lemma is proved.

From this lemma, we can conclude that there exist four sets: $(a_1, a_2, \ldots, a_{3k})$, $(a_{3k+1}, a_{3k+2}, \ldots, a_{6k})$, $(a_{6k+1}, a_{6k+2}, \ldots, a_{9k})$, $(a_{9k+1}, a_{9k+2}, \ldots,$ where for any a_i, a_j in the same set, there exists some c_i such that $a_i c_i \equiv a_j \pmod p$. Using the same logic, there exist six such sets for the b_i's That means that for a_i, a_j in the same set, $p \mid a_i - a_j c_j$ for some c_j and for b_i, b_j in the same set, $p \mid b_i - b_j c_j$ for some c_j. Take some pair (a,b). If (a,b) is in $\mathcal{E}_p$, then for all a_i in the same set as a, and for all b_i in the same set as b, $p \mid a - a_i c^4$ and $p \mid b - b_i c^6$ will be satisfied for some c. Therefore, we can take at most one member from each of the four a sets and pair them with one member from each of the six b sets. That gives us 24 pairs (a, b) in $\mathcal{E}_p$.

However, this value fails to consider 0 being a part of $\mathcal{E}_p$. $(a, 0)$ for some a in each of the four a sets, and $(0, b)$ for some b in each of the six b sets can also be added to $\mathcal{E}_p$. That gives us a total of 34 sets.

However, we haven't considered the condition $p \not| (4a^3 + 27b^2)$

Problem 2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x - f(y)) = f(x) + xf(y) + f(f(y))$$

for all $x, y \in \mathbb{R}$.

Solution: Answers: $f(x) = -x^2/2$. $f(x) = 0$.

Let

$$f(y) = x.$$

From here, we see that

$$f(0) = 2f(x) + x^2 \to f(x) = -x^2/2 + f(0)/2$$

for x in the range of f.

Now, let

$$x = f(z).$$

Then we have

$$f(f(z) - f(y)) = f(f(z)) + f(z)f(y) + f(f(y))$$

Since $f(z)$ and $f(y)$ are both in the range of f, we can replace them with

$$-z^2/2 + f(0)/2$$

and

$$-y^2/2 + f(0)/2$$

respectively.

From here, we see that

$$f(f(z) - f(y)) = -(z^2 + y^2)/2 + f(0) + f(z)f(y)$$

which becomes

$$f(f(z) - f(y)) = -(f(z) - f(y))^2/2 + f(0)$$

Now, let

$$f(y) = a$$

where a is a constant.

Then we have

$$f(x-a) = f(x) + xa + f(a) \rightarrow f(x-a) - f(x) = xa + f(a)$$

Since x can be any real number,

$$xa + f(a) \in \mathbb{R} \rightarrow f(x-a) - f(x) \in \mathbb{R}$$

That means the difference of two outputs of f can be any real number. Since we already know

$$f(f(z) - f(y)) = -(f(z) - f(y))^2/2 + f(0)$$

That means

$$f(x) = -x^2/2 + f(0)$$

for any real x. Plugging this into the original function yields $f(0) = 0$. However, notice that in our proof we assumed that f has at last two values in its range. Assume f has only one value in its range. Then, $f(x) = k$ for some constant k. Plugging this into the original function yields $k = 0$.

Problem 3 Find the minimum value of n such that in any mathematics contest satisfying the following conditions, there exists a contestant who solved all the problems:

(i) The contest contains $n \geq 4$ problems, each of which is solved by exactly four contestants.

(ii) For each pair of problems, there is exactly one contest who solved both problems.

(iii) There are at least $4n$ contestants.

Solution: Answer: 13

First, let us present an arrangement for 13 problems.

Problem	*Solvers*
1	$ABCD$
2	$AEFG$
3	$AHIJ$
4	$AKLM$
5	$BEHK$
6	$BFIL$
7	$BGJM$
8	$CEJL$
9	$CFHM$
10	$CGIK$
11	$DEIM$
12	$DFJK$
13	$DGHL$

Lemma: If no person solves all the problems, then no person can solve more than four problems.

Proof: Assume some person A solves 5 problems. Then the solvers of those 5 problems would be:

Problem	*Solvers*
1	$ABCD$
2	$AEFG$
3	$AHIJ$
4	$AKLM$
5	$ANOP$

Assume A does not solve problem 6. However, only 4 people solved problem 6, and the solvers of problem 6 must pair up with the solvers of problems 1 through 5. Clearly A must have solved problem 6. Similarly, A must have solved every remaining problem. However, no person can solve all the problems. Hence our lemma is proved.

Let the solvers of problem 1 be persons A, B, C, and D. From our lemma we know that A, B, C, and D can each solve at most 3 more problems. That means there can be at most $3 * 4 = 12$ more pairs made between problem 1 and problems $2, 3, \ldots$ making 13 total. Hence our proof is complete.

Problem 4 Let $n \geq 3$ be an integer. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers, where the b_i are pairwisedistinct.

(a) Find the number of distinct real zeroes of the polynomial

$$f(x) = (x - b_1)(x - b_2) \cdots (x - b_n) \sum_{j=1}^{n} \frac{a_j}{x - b_j}.$$

(b) Writing $S = a_1 + a_2 + \cdots + a_n$ and $T = b_1 b_2 \cdots b_n$, prove that

$$\frac{1}{n-1} \sum_{j=1}^{n} \left(1 - \frac{a_j}{S}\right) b_j > \left(\frac{T}{S} \sum_{j=1}^{n} \frac{a_j}{b_j}\right)^{1/(n-1)}.$$

Solution: a) Answer: $n - 1$

Since addition and multiplication are commutative, let us rearrange

$$f(x) = (x - b_1)(x - b_2) \cdots (x - b_n) \sum_{j=1}^{n} \frac{a_j}{x - b_j}.$$

so that $b_1 < b_2 < \cdots < b_n$. By expanding f(x), we get

$(x-b_2)(x-b_3)\ldots(x-b_n)a_1 + (x-b_1)(x-b_3)\ldots(x-b_n)a_2 + \cdots + (x - b_1)(x - b_2)\ldots(x - b_(n - 1))$

Plug in $x = b_1$. Everything cancels out except for $(x - b_2)(x - b_3)\ldots(x - b_n)a_1$. Since $b_1 < b_2, b_3, \ldots, b_n$, if n is odd, then $(x - b_2)(x-b_3)\ldots(x-b_n)a_1$ is positive, and if n is even, then it's negative. Plug in $x = b_2$. Everything cancels out except for $(x - b_1)(x - b_3)\ldots(x - b_n)a_2$. Since $b_2 > b_1, b_2 < b_3, b_4, \ldots, b_n$, if n is odd, then $(x - b_1)(x - b_3)\ldots(x - b_n)a_2$ is negative, and if n is even, then it's positive. Repeat this for $x = b_3, b_4, \ldots, b_n$. Each time f alternates sign, for a total of $n-1$ sign changes. That means f has at least $n-1$ zeroes, but since f is a polynomial of degree $n-1$, it has at most $n-1$ zeroes, so it must have exactly $n - 1$ zeroes.

b) This inequality is equivalent to the AM of the roots of f being greater than the GM of the roots of f. The inequality is strict since the equality case of the AM-GM inequality requires all terms to be equal, whereas f has all distinct zeroes.

Problem 5 Let ABC be an acute triangle and let ω be itscircum-circle. Let the perpendicular from A to line BC meetω at D. Let P be a point on ω, and let Q bethe foot of the perpendicular from P to line AB. Suppose that$Q$ lies outside ω and that $2\angle QPB = \angle PBC$. Provethat D, P, Q are collinear.

Solution: Let $\angle BPQ$ be x. Since PQ is perpendicular to AB, $\angle PQB$ is $90°$, and $\angle PBQ$ is $90° - x$. Since the measure $\angle PBQ$ is given to be $2x$, the measure of $\angle ABC$ is $90° - x$. Since AD is perpendicular to BC, the measure of $\angle DAB$ is x. Since $ABPD$ is a cyclic quadrilateral, $\angle BPD = 180° - \angle DAB = 180° - x$, so we have $\angle BPD = 180° - x$ and $\angle BPQ = x$. Adding these together, we get $\angle DPQ = 180°$, so D, P, Q are collinear.

Problem 6 Let $p_1 = 2, p_2 = 3, p_3 = 5, \dots$ be the sequence of primes in increasing order.

(a) Let $n \geq 10$ be a fixed integer. Let r be the smallest integer satisfying

$$2 \leq r \leq n-2 \quad \text{and} \quad n-r+1 < p_r.$$

For $s = 1, 2, \dots, p_r$, define $N_s = sp_1p_2\cdots p_{r-1} - 1$. Prove that there exists j, with $1 \leq j \leq p_r$, such thatnone of $p_1, p_2, \dots, p_n$ divides N_j.

(b) Using the result from (a), find all positive integers m for which

$$p_{m+1}^2 < p_1p_2\cdots p_m.$$

Solution: a) Clearly $p_1, p_2, \dots, p_{r-1} \not| \; jp_1p_2\dots p_{r-1} - 1$ for all j. The primes of concern are $p_r, p_{r+1}, \dots, p_n$, which is a total of $n-r+1$ primes.

Lemma: $p_k \mid jp_1p_2\dots p_{r-1} - 1$ for at most one j where $1 <= j <= p_r$. Proof: Assume there exist j_1, j_2 where $1 <= j_1 < j_2 <= p_r$ such that $p_k \mid j_1p_1p_2\dots p_{r-1} - 1$ and $p_k \mid j_2p_1p_2\dots p_{r-1} - 1$. From this we know that $p_k \mid (j_2 - j_1)(p_1p_2\dots p_{r-1})$. But $(j_2 - j_1), p_1, p_2, \dots, p_{r-1} < p_k$, and p_k is prime, so we have a contradiction. Therefore, each $p_k \mid jp_1p_2\dots p_{r-1} - 1$ for at most one j.

Additionally, we are concerned with $p_r, p_{r+1}, \dots, p_n$, a total of $n - r + 1 < p_r$ numbers. Since j can take on p_r values (from 1 to p_r), there will be some j for which $p_1, p_2, \dots p_n \not| \; jp_1p_2\dots p_{r-1} - 1$.

1.10 Poland

Problem 1 Determine all triples of positive integers a, b, c such that a^2+1, b^2+1 are prime and $(a^2+1)(b^2+1) = c^2+1$.

Solution: The only answers are $(a, b, c) = (1, 2, 3)$ or $(2, 1, 3)$.
Let us assume without loss of generality that $a \leq b$, as a and b are symmetric. Transforming the given, we have

$$a^2(b^2+1) = c^2 - b^2$$

which implies that

$$b^2 + 1 = \frac{c^2 - b^2}{a^2} = \frac{(c+b)(c-b)}{a^2}. \tag{1}$$

This means that we can find integers n and k such that $nk = a^2$, $n|(c+b)$, and $k|(c-b)$. Because b^2+1 is prime, either $\frac{c+b}{n} = 1$ or $\frac{c-b}{k} = 1$.
Case 1: $\frac{c+b}{n} = 1$.
From (1), we have

$$b^2 + 1 = \frac{c-b}{k}.$$

Substituting $c = n - b$ into the equation and rearranging, we have

$$k = \frac{n - 2b}{b^2 + 1}.$$

Because k is an integer, $n - 2b \geq b^2 + 1$. However, we assumed that $a \leq b$, so

$$n < a^2 + 1 \leq b^2 + 1.$$

Therefore, $n - 2b < n < b^2 + 1$, so we have a contradiction, and there are no solutions for this case.
Case 2: $\frac{c-b}{k} = 1$.
From (1), we have

$$b^2 + 1 = \frac{c+b}{n}.$$

Substituting $c = b + k$ into the equation and rearranging, we have

$$n = \frac{2b + k}{b^2 + 1}. \tag{2}$$

Because n is an integer, $2b + k \equiv 0 \pmod{b^2+1}$. We have

$$2b \leq b^2 + 1 \quad \text{and} \quad k < a^2 + 1 \leq b^2 + 1$$

where the first inequality is by the AM-GM inequality. Adding the two inequalities, we have

$$2b + k < 2(b^2 + 1).$$

Therefore, if n is an integer, $2b + k = b^2 + 1$, implying that $n = 1$. Because $n = 1$, we have $k = a^2$. Substituting this into (2) and rearranging, we have

$$\begin{aligned} b^2 - 2b + 1 &= a^2 \\ (b-1)^2 &= a^2 \\ b &= a + 1. \end{aligned}$$

Hence, both $a^2 + 1$ and $(a + 1)^2 + 1 = b^2 + 1$ are prime. They have different parities, so one of them must be 2, the only even prime. Because 2 is the smallest prime, we have $a^2 + 1 = 2$, as it is smaller than $(a + 1)^2 + 1$. Thus, we have $a = 1$, so $b = 2$ and $c = 3$, which is the only solution for this case.
We assumed that $a \leq b$, so $(a, b, c) = (2, 1, 3)$ is also a solution, as we can use analogous reasoning while assuming $b \leq a$. Thus, the only solutions are $(a, b, c) = (2, 1, 3) \quad \text{or} \quad (1, 2, 3)$.

Problem 2 On sides $\overline{AC}, \overline{BC}$ of acute triangle ABC are constructed rectangles $ACPQ$ and $BKLC$. The rectangles lie outside triangle ABC and have equal areas. Prove that a single line passes through C, the midpoint of segment $\overline{PL}$, and the circumcenter of triangle ABC.

Solution: Let O be the circumcenter. Let X and Y be the projections of O onto BC and AC, respectively. Let M be the midpoint of CP and let W and Z be the projections of M onto LC and PC, respectively. Then, we need to prove O, C, and M are colinnear.
Because rectangles $BCLK$ and $CPQA$ have equal areas, setting the areas equal and rearranging, we have $\frac{BC}{AC} = \frac{PC}{LC}$. Because $XC = \frac{BC}{2}$ and $CY = \frac{AC}{2}$, from the previous equation we have $\frac{XC}{CY} = \frac{PC}{LC}$. Now, because M is the midpoint of PL, triangles LMC and PMC have equal areas. Therefore, we have $PC \cdot MZ = LC \cdot MW$, which implies

$$\frac{MW}{MZ} = \frac{LC}{PC} = \frac{XC}{YC}. \qquad (*)$$

Because $\angle MWC = \angle MZC = 90°$, $MWCZ$ is cyclic, so

$$\angle WMZ = 180° - \angle WCZ = \angle XCY. \qquad (\dagger)$$

By $(*)$ and $(\dagger)$, $\triangle CXY \sim \triangle MWZ$. Hence, $\angle CYX = \angle MZW = \angle MCW$. Also, $OXCY$ is cyclic because $\angle OXC = \angle OYC = 90°$, so $\angle XCO = \angle XYO = 90° - \angle CYX$. Thus, we have

$$\angle OCM = \angle OCX + \angle XCW + \angle WCM$$

$$= 90° - \angle CYX + 90° + \angle MCW = 180°.$$

Therefore, because $\angle OCM = 180°$, we have that O, C, and M are colinnear, which is what we wanted.

Problem 3 On a board are written three nonnegative integers. Each minute, one erases two of the numbers k, m, replacing them with their sum $k+m$ and their positive difference $|k-m|$. Determine whether it is always possible to eventually obtain a triple of numbers such that at least two of them are zeroes.

Solution: Let us represent a combination of integers on the board as an ordered triple (a, b, c) where $a \geq b \geq c$. Then, let us first note that one can attain two zeros with the triple (ka, kb, kc) $(k \in \mathbb{Z})$ if and only if one can attain two zeros with the triple (a, b, c) because, after each minute, all of the numbers in the first triple will be k times the numbers in the second triple.

Now, let us suppose that we start with the triple $(2a, b, c)$. Then, we can get to the triple $(2a, b+c, b-c)$ by choosing b and c and then to the triple $(2a, 2b, 2c)$. Therefore, if one can reach two zeros with (a, b, c), one can reach it with $(2a, b, c)$. We now provide an algorithm to make a triple with two zeros.

If we start with the triple (a, b, c), we can use the previous fact repeatedly to transform the triple into (x, y, z), where x, y, and z are the largest odd factors of a, b, and c, respectively. Now, without loss of generality, let us assume that $x \geq y \geq z$. Then, replacing x and y, we can reach the triple $(x+y, x-y, z)$. Since x and y are odd, one can reach the desired triple from this triple if and only if one can reach it from $(\frac{x+y}{2}, \frac{x-y}{2}, z)$.

Now, note that the sum of the numbers in this triple is $x+z$, which is less than the sum of the numbers in (x, y, z). Therefore, the sum of the numbers in the triple is always decreasing if we use the algorithm unless the second smallest number is 0. Hence, we apply the algorithm repeatedly until the second smallest number is 0. Because the sum of the numbers is decreasing, the second smallest number will always eventually reach 0. Therefore, we can always reach a triple that contains at least two zeros, which is what we wanted.

Problem 4 Let $n \geq 3$ be an integer. Let $x_1, x_2, \ldots, x_n$ be positive integers, where indices are taken modulo n. Prove that one of the following inequalities holds:

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}} \geq \frac{n}{2} \quad \text{or} \quad \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + x_{i-2}} \geq \frac{n}{2}.$$

Solution: We show that

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}} + \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + x_{i-2}} = \sum_{i=1}^{n} \frac{x_i + x_{i+3}}{x_{i+1} + x_{i+2}} \geq n$$

which is equivalent to the problem statement because one of the sums must be greater than $\frac{n}{2}$ for the total to be greater than n.
Now, by the rearrangement inequality, we have

$$\sum_{i=1}^{n} \frac{x_i + x_{i+1}}{x_{i+1} + x_{i+2}} \geq n \tag{$*$}$$

and

$$\sum_{i=1}^{n} \frac{x_{i+2} + x_{i+3}}{x_{i+1} + x_{i+2}} \geq n. \tag{$\dagger$}$$

Adding ($*$) and ($\dagger$), we have

$$\sum_{i=1}^{n} \frac{x_i + x_{i+1} + x_{i+2} + x_{i+3}}{x_{i+1} + x_{i+2}} = n + \sum_{i=1}^{n} \frac{x_i + x_{i+3}}{x_{i+1} + x_{i+2}} \geq 2n.$$

Therefore, subtracting n from both sides, we get

$$\sum_{i=1}^{n} \frac{x_i + x_{i+3}}{x_{i+1} + x_{i+2}} \geq n$$

which is what we wanted.

Problem 5 In three-dimensional space are given a triangle ABC and a sphere ω, such that ω does not intersect plane (ABC). Lines AK, BL, CM are tangent to ω at K, L, M, respectively. There exists a point P on ω such that

$$\frac{AK}{AP} = \frac{BL}{BP} = \frac{CM}{CP}.$$

Prove that the circumsphere of tetrahedron $ABCP$ is tangent to ω.

Solution: Let O_1 be the circumcenter of tetrahedron $ABCP$ and O_2 be the center of ω. Let X, Y, and Z be the projections of O_2 onto the extensions of $\overrightarrow{AP}$, $\overrightarrow{BP}$, and $\overrightarrow{CP}$, respectively. Let T, U, and V be the projections of O_1 onto $\overrightarrow{AP}$, $\overrightarrow{BP}$, and $\overrightarrow{CP}$, respectively. Let a, b, and c equal AP, BP, and CP, respectively. Finally, let $k = \frac{AK}{AP} = \frac{BL}{BP} = \frac{CM}{CP}$.

Note that $TP = \frac{a}{2}$, $UP = \frac{b}{2}$, and $VP = \frac{c}{2}$ because the projections of the circumcenter bisect the segments of a tetrahedron. Now, because AK is tangent to ω, we have $AK \perp O_2K$. Hence, by the Pythagorean Theorem, we have $AO = \sqrt{AK^2 + OK^2} = \sqrt{k^2a^2 + OK^2}$. Again by the Pythagorean Theorem, we have

$$AX^2 + OX^2 = (a + PX)^2 + OX^2 = OA^2 = k^2a^2 + OK^2 \quad (1)$$

and

$$PX^2 + OX^2 = OP^2. \quad (2)$$

Therefore, substituting (2) into (1) and noting that $OK = OP$, we find that $PX = \frac{k^2-1}{2}a$. Likewise, we have $PY = \frac{k^2-1}{2}b$ and $PZ = \frac{k^2-1}{2}c$. Therefore, we have

$$\frac{AP}{PX} = \frac{BP}{PY} = \frac{CP}{PZ} = \frac{1}{k^2-1}. \quad (3)$$

By (3), tetrahedrons $ABCP$ and $XYZP$ are similar, since the the ratios of three of their corresponding lengths are equal and they have an equal vertex angle. Therefore, there exists a homothety about P with ratio $-\frac{1}{k^2-1}$ that takes each point in $ABCP$ to the cooresponding point in $XYZP$. Hence, the circumcenter of $ABCP$ is taken to the circumcenter of $XYZP$, so they are colinnear with P. The circumcenter of $XYZP$ and O_2 are related by a homothety with ratio 2 about P, and O_1 is the circumcenter of $ABCP$, so O_1, P, and O_2 are colinnear. Because O_1 and O_2 are colinnear with a point on the intersection of ω and the circumsphere of $ABCP$, no other intersection point can exist, and the two spheres are tangent, as desired.

Problem 6 Let k be a fixed positive integer. We define the sequence $a_1, a_2, \ldots$ by $a_1 = k + 1$ and the recursion $a_{n+1} = a_n^2 - ka_n + k$ for $n \geq 1$. Prove that a_m and a_n are relatively prime for distinct positive integers m and n.

Solution: We claim that $a_n = \prod_{i=0}^{n-1} a_i + k(n > 0)$, assuming that $a_0 = 1$. We prove this by induction on n. For the base case, $n = 1$. Because $a_1 = k+1 = a_0 + k$, the base case is true. Now, assume that $a_n = \prod_{i=0}^{n-1} a_i + k(n > 0)$ is true for all integers less than j. Then, we have

$$a_j = a_{j-1}^2 - ka_{j-1} + k = a_{j-1}(a_{j-1} - k) + k. \qquad (*)$$

By the inductive hypothesis, we have $a_{j-1} = \prod_{i=1}^{j-2} a_i + k$. Substituting this into $(*)$, we have

$$a_j = a_{j-1} \prod_{i=0}^{j-2} a_i + k = \prod_{i=0}^{j-1} a_i + k$$

which is what we wanted.

Therefore, we have that $a_n \equiv k \pmod{a_i}$ for $i < n$. Hence, if there exist integers $d > 1, x, y \geq 1$ such that $d \mid a_x$ and $d \mid a_y$, d divides k.

We now show that for $i > 0$, $a_i \equiv 1 \pmod{k}$ by induction on i. For the base case, $a_1 = k + 1 \equiv 1 \pmod{k}$. Now assume that $a_i \equiv 1 \pmod{k}$. Then, $a_{i+1} \equiv a_i^2 - ka_i + k \equiv a_i^2 \equiv 1 \pmod{k}$.

Thus, because all common divisors d of a_x and a_y must be divisors of k, we have $a_x \equiv 1 \pmod{d}$ and $a_y \equiv 1 \pmod{d}$. Therefore, no such divisors exist and a_i is relatively prime to a_j for all $i, j > 0$, as desired.

1.11 Romania

Problem 1 Find all pairs of sets A, B, which satisfy the conditions:

(i) $A \cup B = \mathbb{Z}$;

(ii) if $x \in A$, then $x - 1 \in B$;

(iii) if $x \in B$ and $y \in B$, then $x + y \in A$.

Solution: We will show that either $A = B = \mathbb{Z}$, or B must be the odd integers and A must be the even integers.

Lemma. *If* $0 \in B$*, then* $A = B = \mathbb{Z}$*.*

Proof. If $0 \in B$, then for all $x \in B$, $x + 0 = x \in A$. So $B \subseteq A$. This implies $0 \in A$, so $0 - 1 = -1 \in B$. Say $n \notin B$. Then because $A \cup B = \mathbb{Z}$, $n + 2 \in B$ or $n + 2 \in A$. If $n + 2 \in B$, then $n + 2 + (-1) = n + 1 \in A$. If $n + 2 \in A$, then $n + 1 \in B \subseteq A$. So $n + 1 \in A$, which implies $n \in B$, a contradiction. So $B = \mathbb{Z}$. Because $\mathbb{Z} = B \subseteq A \subseteq Z$, $A = \mathbb{Z}$. □

Now, if $1 \in A$, then $0 \in B$. So assume $1 \notin A$ and $0 \notin B$. We will prove that in this case B must be the odd integers and A must be the even integers. Since $A \cup B = \mathbb{Z}$, we must have $1 \in B$ and $0 \in A$. Notice that $0 \in A$ implies $-1 \in B$. If $-2n \in A$, then $-2n - 1 \in B$, and $-2 * n - 1 + -1 = -2 * n - 2 \in A$, so by a trivial induction A contains every negative even number and B contains every negative odd number.

In fact, this same reasoning shows that if $a \in A$, then $a - 2k \in A$ and $a - 2k - 1 \in B$ for all $k \geq 0$. Say a is odd, that $a > 1$, and that $a \in A$. Then $a - 1 \in B$. Since $-a + 2$ is negative and odd, $-a + 2 \in B$, so $-a + 2 + a - 1 = 1 \in A$, a contradiction. So A can contain no positive odd numbers. If B contains a positive even number a, then $a - 1$ is a positive odd number and $a - 1 \in A$, so B can contain no positive even numbers. Since $A \cup B = \mathbb{Z}$, B contains all odd numbers and A contains all even numbers. If B contains any even number a, then $-a + 1$ is odd, so $-a + 1 \in B$ and $-a + 1 + a = 1 \in A$, a contradiction. And if A contains any even number a, then $a - 1$ is odd and $a - 1 \in B$, a contradiction. So B must be the odd integers and A must be the even integers.

Problem 2 Let $a_0, a_1, a_2, \ldots$ be the sequence defined as follows: $a_0 = a_1 = 1$ and $a_{n+1} = 14a_n - a_{n-1}$ for any $n \geq 1$. Show that the number $2a_n - 1$ is a perfect square for all positive integers n.

Solution:

For $n \in \mathbb{N}$, define c_n by $c_0 = -1, c_1 = 1$, and $c_n = 4c_n - 1 - c_n - 2$ for $n \geq 2$.

Lemma. *We have*

$$c_n = \left(\frac{1+\sqrt{3}}{2}\right)\left(2+\sqrt{3}\right)^n + \left(\frac{1-\sqrt{3}}{2}\right)\left(2-\sqrt{3}\right)^n \qquad (*)$$

for each $n \in \mathbb{N}$.

Proof. Let d_n be the right-hand side of equation $(*)$. Because $2+\sqrt{3}$ and $2-\sqrt{3}$ are the roots of the characteristic polynomial x^2-4x+1 of c_n, we have that d_n satisfies the given recurrence. It is easily checked that $d_0 = -1$ and $d_1 = 1$, so we must have that $d_n = c_n$ for each $n \in \mathbb{N}$. $\square$

After squaring both sides of $(*)$ and collecting terms, we have that

$$\frac{c_n^2+1}{2} = \left(1 - \frac{\sqrt{3}}{2}\right)\left(7+4\sqrt{3}\right)^n + \left(1 + \frac{\sqrt{3}}{2}\right)\left(7-4\sqrt{3}\right)^n. \quad (**)$$

Let $f_n = \frac{c_n^2+1}{2}$. We claim that $f_n = a_n$ for each $n \in \mathbb{N}$. Because $7+4\sqrt{3}$ and $7-4\sqrt{3}$ are roots of the characteristic polynomial $x^2 - 14x + 1$ of a_n, we have that f_n satisfies the recurrence $f_n = 14f_{n-1} - f_{n-2}$ for all $n \geq 2$. It is easily checked that $f_n = a_n$ for $n = 0, 1$, so $f_n = a_n$ as claimed. Now expanding $a_n = \frac{c_n^2+1}{2}$ gives us $2a_n - 1 = c_n^2$ for each $n \in \mathbb{N}$ as desired.

Problem 3 Let ABC be an acute triangle. Segment $\overline{MN}$ is the midline of the triangle that is parallel to side $\overline{BC}$, and P is the orthogonal projection of point N onto side $\overline{BC}$. Let A_1 be the midpoint of segment $\overline{MP}$. Points B_1 and C_1 are constructed in a similar way. Show that if lines AA_1, BB_1, and CC_1 are concurrent, then triangle ABC has two congruent sides.

Solution:

First, let P be the foot of the altitude from A, and let Q be the midpoint of segment BC. Let $A_1' = AA_1 \cap BC$. Let $M' = AA_1 \cap MN$. Let D be the foot of the perpendicular from N to BC.

Lemma. *We have* $BA_1' : A_1'C = \frac{\tan B + 2\tan C}{2 tan B + tan C}$.

Proof. Without loss of generality, our triangle has circumradius $R = \frac{1}{2}$. Then we have $BC = 2R\sin A = \sin A$. Similarly, $CA = \sin B$ and $AB = \sin C$. Because $MN \parallel BC$, we have that $\triangle A_1M'M\ \triangle A_1A_1'D$ and that $\triangle AMM'\ \triangle ABA_1'$. This gives that $BA_1' : MM' = AB : AM = \frac{1}{2}$, and that $A_1'D : MM' = MA_1 : A_1D$. But $MA_1 : A_1D = 1$ because $MNDP$ is a rectangle, so we have $A_1'D : MM' = 1$ and $BA_1' : A_1'D = 2$. By right triangle NDC we have $DC = \frac{1}{2}AC\cos C = \frac{1}{2}\sin B cos C$. Therefore

$$BD = BC - DC = \sin A - \tfrac{1}{2}\sin B cos C$$
$$= (\sin B\cos C + \cos B\sin C) - \tfrac{1}{2}\sin B\cos C$$
$$= \sin C\cos B + \tfrac{1}{2}\sin B\cos C$$

and $BA_1' : BD = BA_1' : (BA_1' + A_1'D) = \frac{BA_1'}{A_1'D} : (\frac{BA_1'}{A_1'D} + 1) = \frac{2}{2+1} = \frac{2}{3}$. By a brief computation,

$$BA_1' = \frac{2}{3}\sin C\cos B + \frac{1}{3}\sin B\cos C$$

Thus

$$A_1'C = BD - BA_1' = \frac{1}{3}\sin C\cos B + \frac{2}{3}\sin B\cos C.$$

Taking the ratio $BA_1' : A_1'C$ now gives the desired result.

□

By Ceva's theorem, if AA_1, BB_1, CC_1 concur, then we must have

$$\frac{BA_1'}{A_1'C}\frac{CB_1'}{B_1'A}\frac{AC_1'}{C_1'B} = 1.$$

Applying the lemma three times gives us that

$$\prod_{cyc}(\tan B + 2\tan C) = \prod_{cyc}(2\tan B + \tan C).$$

Upon expanding and factoring, this gives $\prod_{cyc}(\tan B - \tan C) = 0$. Thus we must have two of $\tan A, \tan B, \tan C$ equal. Without loss of generality, we have $\tan A = \tan B$. By the itentity $\cot^2\theta + 1 = \csc^2\theta$, we have $\csc^2 A = \csc^2 B$. This implies $\sin^2 A = \sin^2 B$ Since $\sin A > 0$ for A an angle of a triangle, we must have $\sin A = \sin B$. By the law of signs, we then have $BC = CA$ and we are finished.

Problem 4 For any positive integer n, let $f(n)$ be the number of possible choices of signs $+$ or $-$ in the algebraic expression $\pm 1 \pm 2 \pm \cdots \pm n$, such that the obtained sum is zero. Show that $f(n)$ satisfies the following conditions:

(i) $f(n) = 0$, if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$;

(ii) $2^{\frac{n}{2}-1} \le f(n) < 2^n - 2^{\lfloor \frac{n}{2} \rfloor + 1}$, if $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$.

Solution:

Lemma. *Given $p \in [k, \frac{3k^2+k}{2}]$, there exists $S \subset \{k \ldots 2k\}$ such that $sum(S) = p$.*

Proof. Let $R = k+1 \ldots 2k$. The following algorithm begins at step (i) and produces a family T of subsets of R with the following property: Given $p \in [k, \frac{3k^2+k}{2}]$, there exists $S \subset R$ such that $sum(S) = p$. This proves the lemma.

(i) The algorithm maintains a set $S \subset R$. Initially, $S = k+1 \ldots 2k$. It also maintains a set T of subsets of R Go to step (ii).

(ii) Let $T = T \cup \{S\}$. If there exists $x \in R$ such that $x - 1 \notin S$ and $x \in S$, then go to step (iii). If no such x exists, then end the algorithm and return T if $|S| = 1$, and go to step (iv) if $|S| > 1$.

(iii) let $S = (S - \{x\}) \cup \{x - 1\}$, and then go to step (ii).

(iv) In this case, if $|S| = j$, then we claim $S = \{k, k+1 \ldots k+j-1\}$. Remove $k, k+1$ from S and add element $2k$ to S. Then go back to step (ii).

We must first prove our claim from step (iv). Suppose the algorithm enters step (iv). Clearly if m is the minimal element of S, then $m = k$; if not, then we could take $x = m$, contradicting the algorithm's choice to enter step (iv). Also, if $x \notin S$ then $x + 1 \notin S$. These properties require that S $= \{k \ldots k + j - 1\}$ for some j, proving the claim.

Now we must show the algorithm returns a set T with the required property. Note that after each repetition of step (ii) except for the last, $sum(S)$ decreases by 1 - the procedures in steps (iii) and (iv) decrease sum(S) by 1 and exactly one of (iii), (iv) is called after every repetition of step (ii). Since the algorithm can never produce $sum(S) < k$, the algorithm must eventually terminate. When the algorithm begins, we have

$$sum(S) = \sum_{j=k+1}^{2k} j = \sum_{j=1}^{k} (j+k) = k^2 + \sum_{j=1}^{k} j = \frac{3k^2+k}{2}$$

and when the algorithm terminates, we have $S = k$. Thus a set S with sum p is recorded in T for each $p \in [k, \frac{3k^2+k}{2}]$ as desired.

□

Call an assignment of + and - signs *good* if it produces a sum of 0, and call a subset P of $\{1 \dots n\}$ *good* if assigning + signs to members of P and - signs to the other numbers produces a good assignment. Given a good assignment, let P be the set of numbers assigned a + sign and Q be the set of numbers assigned a - sign. Clearly, we have $sum(P) = sum(Q)$, so P is a good set. Conversely, if P, Q is a partition of $\{1 \dots n\}$ such that $sum(P) = sum(Q)$, then assigning the members of P + signs and the members of Q - signs gives a good assignment. This gives a correspondence between good assignments and good sets.

We first prove no good sets exist for $n \equiv 1, 2 \pmod 4$. Suppose we have a good set P; as before, let $Q = 1...n - P$. Because $sum(P) + sum(Q) = 1 + \cdots + n = \frac{(n)(n+1)}{2}$, we must have $sum(P) = sum(Q) = \frac{(n)(n+1)}{4}$. If $n \equiv 1, 2 \pmod 4$, then $(n)(n+1)$ is not divisible by 4, so no such assignments exist.

We next prove the lower bound in part (ii). For the $n \equiv 0 \pmod 4$ case, we produce $2^{n/2-1}$ good subsets. The correspondence between good subsets and good assignments then proves the bound. Let $S_1 = \{1 \dots 2k-1\}$ and $S_2 = \{2k \dots 4k\}$. Let $k = \frac{n}{4}$. We claim that for each $A \subset S_1$, there exists a good subset R such that $R \cap \{1 \dots 2k-1\} = A$. Since there are $2^{n/2-1}$ choices of P, all of which yield distinct sets R, this will produce the required number of subsets. To prove this claim, note that if $A \subset S_1$, then $0 \leq sum(A) \leq \frac{4k^2+2k}{2}$. Thus $2k^2 \leq \frac{n^2+n}{4} - sum(A) \leq \frac{8k^2+2k}{2}$, which gives us

$$2k \leq \frac{n^2+n}{4} - sum(A) \leq \frac{3(2k)^2+2k}{2}$$

By the lemma, we can choose a set $B \subset S_2$ with $sum(B) = \frac{n^2+n}{4} - sum(A)$. Then $R = A \cup B$ has $R \cap \{1 \ldots 2k-1\} = A$ and $sum(R) = \frac{n^2+n}{4}$. This proves the $n \equiv 0 \pmod 4$ case.

For the $n \equiv 3 \pmod 4$ case, we can choose $n = 4k+3$. Then we can prove the lower bound in part (ii) by applying a similar argument with $S_1 = \{1 \ldots 2k\} \cup \{4k+3\}$, $S_2 = \{2k+1 \ldots 4k+2\}$.

For the upper bound in part (ii), we show the stronger upper bound $f(n) \leq 2^{n-1}$. We represent assignments as n-tuples $(x_1 \cdots x_n)$ of 1s and -1s with $x_i = 1$ representing a + and $x_i = -1$ representing a -, and call an n-tuple good if the corresponding assignment is good. Now let $g(x_1 \cdots x_n) = (-x_1, -x_2, x_3 \ldots x_n)$. We claim that g is an injection from the set A of good n-tuples to the set B of non-good n-tuples. To see this, let $(x_1, \cdots x_n)$ be a good n-tuple. Then we have

$$\begin{aligned} & -x_1 1 + -x_2 2 + x_3 3 + \cdots + x_n n \\ = {} & x_1 1 + x_2 2 + x_3 3 + \cdots + x_n n - 2(x_1 1 + x_2 2) \\ \equiv {} & 0 + 21 \equiv 1 \mod 2 \end{aligned}$$

which gives that $g(x_1 \cdots x_n)$ is not a satisfying n-tuple. Thus we have $|A| \leq |B|$. Because $|A|+|B| = 2^n$, we must have $2|A| \leq |A|+|B| = 2^n$, which gives us $|A| \leq 2^{n-1}$ as desired.

Problem 5 Let $ABCD$ be a unit square. For any interior points M, N such that line MN does not contain a vertex of the square, we denote by $s(M,N)$ the minimum area of all the triangles whose vertices lie in the set of points $\{A, B, C, D, M, N\}$. Find the least number k such that $s(M,N) \leq k$ for all such points M, N.

Solution: The answer is that $k = 1/8$. It is easy to generate an example where the smallest triangle is of area 1/8: Setting the coordinates of the squares corner's A, B, C, and D to (0, 0), (1, 0), (1, 1) and (0, 1) respectively, place M at (1/4, 1/2) and N at (3/4, 1/2). It is simple to check the areas in this case.

To show that it's impossible to do better, let $E, F, G, H, I, J, K, L, O, P, Q$, and R be the points at (1/4, 0), (3/4,0), (1,1/4), (1, 3/4), (3/4,1), (1/4,1), (0,3/4), (0,1/4), (1/2,1/4), (3/4,1/2), (1/2,3/4), and (1/4,1/2) respectively. Then any point in the interior of $ABCD$ must be within either of the hexagons $BGJDKF$ or $AEHCIL$, or one the triangles OEF, PGH, QIJ, or RKL.

If M is inside $BGJDKF$, then the distance from M to BD is bounded, because JG and FK are parallel to BD and are of distance $\frac{1}{4\sqrt{2}}$ to $\overline{BD}$. Because the length of $\overline{BD}$ is $\sqrt{2}$, the area of MBD is not more than

$$\frac{1}{2} \cdot \frac{1}{4}\sqrt{2} \cdot \sqrt{2} = \frac{1}{8}.$$

By similar reasoning, if M is inside $AEHCIL$, the area of MAC is not more than $\frac{1}{8}$. If M is inside of OEF, then M's distance to AB is not more than $1/4$, and the area of MAB is not more than $1/8$, and the cases for M being inside PGH, QIJ, or RKL are trivially the same with sides BC, CD, or DA. Therefore, no matter where M is placed, one of the triangles with vertices in the set $\{M, A, B, C, D\}$ will have an area of $1/8$ or less.

Problem 6 Let $p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$ and $q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$, where each coefficient a_i and b_i equals either 1 or 2002. Assuming that $p(x)$ divides $q(x)$, show that $m+1$ is a divisor of $n+1$.

Solution:

Lemma. *In any field F, we have that $gcd_{F[x]}(x^p - 1, x^q - 1) = x^{gcd(p,q)} - 1$*

Proof. : Let $g = gcd(p, q)$, and let R(x) $= gcd_{F[x]}(x^p - 1, x^q - 1)$. Then it is known that $a \mid b \Rightarrow x^a - 1 \mid x^b - 1$. There exist $a, b \in \mathbb{N}$ such that $ap = bq + g$ without loss of generality. Thus $r(x) \mid x^p - 1 \mid x^{ap} - 1$ and

$r(x) \mid x^q - 1 \mid x^{bq} - 1 \mid x^{bq+g} - x^g = x^{aP} - x^g$ Therefore $r(x) \mid (x^{ap} - 1) - (x^{aP} - x^g) = x^g - 1$. Trivially, we also have $x^g - 1 \mid r(x)$ since $x^g - 1 \mid x^p - 1, x^g - 1 \mid x^q - 1$, so we must have $r(x) = x^g - 1$. □

In the field F_3, we have that each coeficcient of $p(x)$ and $q(x)$ equals 1, since 2002 and 1 are both equivalent to $1 \pmod 3$. Thus in $F_3(x)$, we have that $p(x) = \frac{x^{m+1}-1}{x-1}$ and that $q(x) = \frac{x^{n+1}-1}{x-1}$. Since $p(x) \mid q(x)$ in $R(x)$, we have that $p(x) \mid q(x)$ in $F_3(x)$, giving that $x^{m+1} - 1 \mid x^{n+1} - 1$. This means that

$$gcd_{F[x]}(x^{m+1} - 1, x^{n+1} - 1) = x^{m+1} - 1.$$

By the lemma, this implies that $m + 1 \mid n + 1$ as desired.

Problem 7 Let a, b be positive real numbers. For any positive integer n, denote by x_n the sum of the digits of $an + b$ (written in its decimal representation). Show that $x_1, x_2, \cdots$ contains a constant (infinite) subsequence.

Solution:

Given a positive real $\alpha > 1$, define $F_\alpha = \{\lfloor \alpha n \rfloor\}_{n \in \mathbb{N}}$, with $\beta = \frac{\alpha}{\alpha - 1}$.

Lemma. *Given $\alpha > 1$, $\beta = \frac{\alpha}{\alpha-1}$, with α irrational, we have that F_α,F_β partition $\mathbb{N}$.*

Proof. It suffices to show that for each $n \in \mathbb{N}$, we have $|[n] \cap F_\alpha| + |[n] \cap F_\beta| = n$. We have that $|[n] \cap F_\alpha| = |\{k \in \mathbb{N} : \lfloor \alpha k \rfloor \leq n\}|$. But

$$\lfloor \alpha k \rfloor \leq n \iff \lfloor \alpha k \rfloor < n + 1 \iff \alpha k < n + 2 \iff k < \frac{n+1}{\alpha}.$$

Thus $\{k \in \mathbb{N} : \lfloor \alpha k \rfloor \leq n\} = \{k : 1 \leq k < \frac{n+1}{\alpha}\}$, and there are $\lfloor \frac{n+1}{\alpha} \rfloor$ elements in $[n] \cap F_\alpha$. After a similar argument for β, we must then show that $\lfloor \frac{n+1}{\alpha} \rfloor + \lfloor \frac{n+1}{\beta} \rfloor = n$. But by definition of β, we have $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Since neither of α, β is rational, we must then have $\{\frac{n+1}{\beta}\} + \{\frac{n+1}{\alpha}\} = 1$. Thus $\lfloor \frac{n+1}{\alpha} \rfloor + \lfloor \frac{n+1}{\beta} \rfloor = n + 1 - \left(\{\frac{n+1}{\beta}\} + \{\frac{n+1}{\alpha}\}\right) = n$ as desired. $\square$

Let $d(n)$ denote the sum of n's digits. Note that $d(a + b) \leq d(a) + d(b)$. It suffices to show that $d(x_n)$ has an infinite bounded subsequence, because than we have that some digit sum occurs infinitely often by the infinite pigeonhole principle. We first show the result for $\beta = 0$.

We first prove the statement for $y = 0$. The general case follows easily from d being bounded on $[b]$ and the triangle inequality for d. Case 1) If $\alpha \leq 1$, then $\{\lfloor \alpha n \rfloor\}_{n \in \mathbb{N}} = \mathbb{N}$ and the result is trivial.

Case 2) If $\alpha > 1$, and $\alpha = p/q, p, q \in \mathbb{N}$, then we have that $\{p10^k\}_{k \in \mathbb{N}}$ is a subsequence with constant digit sum.

Case 3) If $\alpha > 1$ and α is irrational, then:

Lemma. *Given α, F_α as before, we have $|F_\alpha \cap n \cdots n + r| < r + 1$, where $r = \lceil \frac{1}{\alpha - 1} \rceil$.*

Proof. We have $\lfloor \alpha k \rfloor = n \iff \frac{n-1}{p} < k \leq \frac{n}{p}$.

Suppose the statement is false; then we have integers $a ... a + r \in (\frac{n-1}{b}, \frac{n+r}{b}]$. Then the length of $(\frac{n-1}{b}, \frac{n+r}{b}]$ must be at least r, so

$\frac{n+r}{b} = \frac{n+r}{b} - \frac{n-1}{b} \geq r$. But by definition of r, we have $r > \frac{1}{b+1} \Rightarrow r > \frac{1+r}{b}$, a contradiction. Thus the statement holds. $\square$

Since F_b, F_y partition N by the first lemma, either F_b or F_y contains infinitely many members of the set $\{10^k\}$. If F_b does, then we have our desired subsequence. Otherwise if F_y does, then by lemma 2 F_y is missing infinitely many many members of the set $P = 10^k + j : 0 \leq j \leq r$ - where r is defined as in the second lemma - and thus F_b contains infinitely many members of P. Since $d(x)$ is bounded on $[r]$, and $d(a+b) \leq d(a) + d(b)$, we must have that $d(x)$ is bounded on the set P, and we have the desired conclusion.

Problem 8 At an international conference there are four official languages. Any two participants can talk to each other in one of these languages. Show that some language is spoken by at least 60% of the participants.

Solution: Let the languages spoken at the conference be A, B, C, and D. Given $X \subseteq \{A, B, C, D\}$, let $S(X)$ be the percent of participants who can only speak the languages in set X. If $S(\{A\}) > 0$, then there exists someone who can speak only A, and everyone must be able to communicate with him. So everyone must be able to speak A.

Assume $S(\{A\}) = S(\{B\}) = S(\{C\}) = S(\{D\}) = 0$. Call a participant billingual if he speaks only two languages. Say there is a billingual participant. Without loss of generallity,say he speaks A and B. Then $S(\{A, B\}) > 0$, so everyone must be able to speak either A or B. But there cannot be two billingual participants, one of whom speaks A but not B and one of whom speaks A but not B, because they would not be able to communicate with each other. So either all billingual participants speak A or all billingual participants speak B. Without loss of generallity, say they all speak A. Then

$$S(\{A,B\}) + S(\{A,C\}) + S(\{A,D\}) + S(\{A,B,C\}) +$$
$$S(\{A,B,D\}) + S(\{A,C,D\}) + S(\{B,C,D\}) + S(\{A,B,C,D\}) = 100$$

Notice that the only people who do not speak A are those who speak B, C, and D. So if $S(\{B, C, D\}) \leq 40$, then at least 60% of the participants can speak A. Say $S(\{B, C, D\}) > 40$. Define $b = S(\{A, B\}) + S(\{A, B, C\}) + S(\{A, B, D\})$, and let c and d be defined likewise.

Let $q = S(\{A, B\}) + S(\{A, C\}) + S(\{A, D\})$ and $r = S(\{A, B, C\}) + S(\{A, B, D\}) + S(\{A, C, D\}) + S(\{B, C, D\})$. Then $b + c + d = q + r$, so either b, c, or $d \geq (1/3)(q + r)$. Without loss of generallity, $b \geq (1/3)(q + r)$. So then, the percent of people who can speak B is

$$\begin{aligned} S(\{A, B\}) &+ S(\{A, B, C\}) + S(\{A, B, D\}) \\ &+ S(\{B, C, D\}) + S(\{A, B, C, D\}) \\ &= b + S(B, C, D) + S(A, B, C, D) \\ &\geq 1/3(q + r) + S(B, C, D) + S(A, B, C, D) \\ &\geq 1/3(100 - S(B, C, D)) + S(B, C, D) \\ &= 100/3 + (2/3)S(B, C, D) \\ &> 100/3 + (2/3)(40) \\ &= 60 \end{aligned}$$

If there are no bilingual people, then there is no difference in the proof, except that q = 0.

Problem 9 Let $ABCDE$ by a convex pentagon inscribed in a circle of center O, such that $\angle B = 120°$, $\angle C = 120°$, $\angle D = 130°$, and $\angle E = 100°$. Show that diagonals $\overline{BD}$ and $\overline{CE}$ meet at a point on diameter $\overline{AO}$.

Solution: $BCDE$ is cyclic, so $\angle BCD = 180° - \angle BED = 50$, so $\angle ABE = \angle ABC - \angle BED = 70°$. Similarly, $\angle BCD = 180° - \angle BED = 60°$, so $\angle AEB = \angle AED - \angle CED = 40°$. Also, $ABCE$ and $ABDE$ are cyclic, so $\angle AEC = 180 - \angle ABC = 60°$, and $\angle ABD = 180 - \angle AED = 80°$. Let X be the second intersection of AO with the circle, and let Y and Z be the intersections of CF and BE with AX respectively. Then $\angle FXA = \angle FBA = 70°$, and $\angle BXA = \angle AFB = 40°$. Then, because AFX and ABX are right triangles, $\angle FAX = 90° - \angle FXA = 20°$ and $\angle XAB = 90° - \angle AXB = 50°$, and so $\angle FYA = 180° - \angle YFA - \angle FAY = 100°$ and $\angle AZB = 180° - \angle ABY - \angle BAY = 50°$.

Now, we seek to prove that $\overline{AY} = \overline{AZ}$. To prove this, let R be the length of my $\overline{AX}$. Then the length of $\overline{AB}$ is simply $R\sin(40°)$, and so by the law of sines is $R\frac{\sin(40°)\sin(80°)}{\sin(50°)}$. Similarly, the length of $\overline{AF}$ is simply $R\sin(70^\circ)$, and so by the law of sines $R\frac{\sin(70^\circ)\sin(60^\circ)}{\sin(100^\circ)}$. So it is clear that proving the problem is equivalent to proving the following trigonometric identity:

$$\frac{\sin(40^\circ)\sin(80^\circ)}{\sin(50^\circ)} = \frac{\sin(70^\circ)\sin(60^\circ)}{\sin(100^\circ)}$$

However, this is an equality, as the following chain of equations shows:

$$\begin{aligned}
\cos(10^\circ) &= 2\cos(60^\circ)\cos(10^\circ) \\
\cos(10^\circ) &= \cos(70^\circ) + \cos(50^\circ) \\
(\cos(30^\circ) + \cos(10^\circ)) - (\cos(70^\circ) + \cos(50^\circ)) &= \frac{\sqrt{3}}{2} \\
\frac{1}{2}\cos(10^\circ)(\cos(20^\circ) - \cos(60^\circ)) &= \frac{\sqrt{3}}{8} \\
4\sin(80^\circ)\sin(20^\circ)\sin(40^\circ) &= \frac{\sqrt{3}}{2} \\
\sin(100^\circ)(2\sin(20^\circ)\cos(20^\circ))(2\sin(40^\circ)\cos(40^\circ)) &= \sin 60\cos(20^\circ)\cos(40^\circ) \\
\sin(100^\circ)\sin(40^\circ)\sin(80^\circ) &= \sin(70^\circ)\sin(60^\circ)\sin(50^\circ) \\
\frac{\sin(40^\circ)\sin(80^\circ)}{\sin(50^\circ)} &= \frac{\sin(70^\circ)\sin(60^\circ)}{\sin(100^\circ)}
\end{aligned}$$

Problem 10 Let $n \geq 4$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$. Show that

$$\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \cdots + \frac{a_n}{a_1^2+1} \geq \frac{4}{5}\left(a_1\sqrt{a_1} + a_2\sqrt{a_2} + \cdots + a_n\sqrt{a_n}\right)^2.$$

Solution:

Lemma. *given nonnegative variables $x_1 \cdots x_n$ with $x_1 + \cdots x_n = 1$, and $n \geq 4$, we have*

$$f(x) = \sum_{i=1}^{n} x_i x_{i+1} \leq \frac{1}{4}$$

with subscripts interpreted $\mod n$.

Proof. We take cases for n odd and n even. If n is even, then we have $f(x)$ a linear function in the variables x_{2i} . By elementary properties of linear functions, we then have that f takes on its maximum value when all but one of these variables is equal to 0. A similar argument for the variables x_{2i+1} gives that f takes on its maximum value when all but one of x_{2i+1} is equal to 0. This gives us that there exists a maximizing assignment of variables such that $f(x)$ contains at most one nonzero term $x_i x_{i+1}$. By the AM-GM inequality, we then have

$$f(x) \leq \left(\frac{x_i + x_{i+1}}{2}\right)^2 \leq \frac{1}{4}.$$

If n is odd, then $f(x)$ is a linear function in the variables x_1, x_3. Therefore f takes on its maximum value when one of $x_1, x_3 = 0$. Then we may apply the above argument to the remaining nonzero variables. $\square$

Now by Cauchy's inequality, we have

$$(\sum_{cyc}(\frac{a_1^3}{a_1^2 a_2^2 + a_1^2})(\sum_{cyc} a_1^2 a_2^2 + a_1^2) \geq (\sum_{cyc} a_1\sqrt{a_1})$$

But

$$\sum_{cyc} a_1^2 a_2^2 + a_1^2 = \sum_{cyc} a_1^2 a_2^2 + a_1^2 \leq \sum_{cyc} a_1^2 a_2^2 + 1 \leq \frac{5}{4}$$

with the last step proved by the lemma. Thus

$$\frac{5}{4}\sum_{cyc} \frac{a_1^3}{a_1^2 a_2^2 + a_1^2} \geq \sum_{cyc} a_1\sqrt{a_1}$$

as desired.

Problem 11 Let n be a positive integer. Let S be the set of all positive integers a such that $1 < a < n$ and $n \mid (a^{a-1} - 1)$. Show that if $S = \{n-1\}$, then n is twice a prime number.

Solution: Assume $n > 2$, since the case $n = 2$ trivially does not work. $n - 1 \in S$ implies that $n \mid (n-1)^{n-1} - 1$. So $(n-1)^{n-1} \equiv 1 \pmod{n}$. But the order of $n-1 \pmod{n}$ is simply 2, so $2 + \phi(n)k = n - 2$ for some $k \in \mathbb{Z}$. Therefore, $\phi(n) \mid n - 4$. Let $n = 2^x \prod_{i=1}^m {p_i}^{b_i}$ Notice that if $b_i > 1$, then $p_i \mid \phi(n)$ and $p_i \not\mid n - 4$. So $b_i = 1$ for all $i, 1 \leq i \leq m$. Let $l(p)$ be the highest power of two dividing $p \in \mathbb{N}$; so $2^{l(p)} \mid p$ and $2^{l(p)+1} \not\mid p$. We take the following cases based on $x = l(n)$:

Case 1: $x > 3$

In this case, $l(\phi(n)) = x - 1 > 2$ by definition of ϕ. But $l(n-4) \geq l(\phi(n))$ by divisibilty, and $l(n-4) = 2$. So $2 = l(n-4) \geq l(\phi(n)) = x - 1 > 2$, which is impossible. So $l(n) \neq 2$

Case 2: $x = 3$

In this case, we have $2 = l(n-4) \geq l(\phi(n)) = x - 1 \geq 2$ by the same reasoning as in case 1. Therefore, $l(\phi(n)) = 2$, so n can have no odd prime factors. Therefore $n = 8$. Yet $3^2 - 1 = 8$, so $3 \in S$. Therefore $l(n) \neq 3$.

Case 3: $x = 2$

We notice that

$$\left(2\prod_{i=1}^{m} p_i + 1\right)^{2\prod_{i=1}^{m} p_i} - 1 = \sum_{k=1}^{2\prod_{i=1}^{m} p_i} \binom{2\prod_{i=1}^{m} p_i}{k} \left(2\prod_{i=1}^{m} p_i\right)^k$$

It is trivial to note that eeach term of this expression is divisible by $4\prod_{i=1}^{m} p_i = n$. So $2\prod_{i=1}^{m} p_i = n/2 \in S$. Therefore, $l(n) \neq 2$.

Case 4: $x = 0$

This is clearly impossible, since $n - 4$ is odd if n is, yet $\phi(n)$ is always even, showing $\phi(n) \nmid n - 4$.

Case 5: $x = 1$

In this case, $n = 2\prod_{i=1}^{m} p_i$, so $\phi(n) = \prod_{i=1}^{m}(p_i - 1)$. Notice that $n - 4 = 2(\prod_{i=1}^{m} p_i - 2)$, so $l(n-4) = 1$. Yet $l(\phi(n)) = l(\prod_{i=1}^{m}(p_i - 1)) \geq m$, since each term $p_i - 1$ is even. Since $\phi(n) \mid n - 4$, this shows that only one odd prime can divide n. And some odd prime must do so: For $n \neq 2$, S is the null set because no a exists so that $1 < a < 2$.

So we must have $n = 2p$ for some odd prime p.

Problem 12 Show that there does not exist a function $f : \mathbb{Z} \to \{1, 2, 3\}$ satisfying $f(x) \neq f(y)$ for all $x, y \in \mathbb{Z}$ such that $|x - y| \in \{2, 3, 5\}$.

Solution:

We show that any such function f satisfies $f(x) = f(x+1)$. This results in a contradiction, because a constant function cannot satisfy the condition.

Without loss of generality, we have that $f(0) = 1, f(5) = 2$. Because $|5 - 2| = 3, |2 - 0| = 2$, we have that $f(2) = 3$. Then f(7) = 1 also, because $|7 - 2| = 5, |7 - 5| = 2$ give that $f(7)$ is different from $f(2), f(5)$. Finally, because $|0 - 3| = 3, |3 - 5| = 2$, we must have $f(3) = 3$. Hence $f(3) = f(2)$; translating this argument to $f(x)$ instead of $f(1)$ gives the desired result.

Problem 13 Let $a_1, a_2, \ldots$ be a sequence of positive integers defined as follows:

- $a_1 > 0$, $a_2 > 0$;
- a_{n+1} is the smallest prime divisor of $a_{n-1} + a_n$, for all $n \geq 2$.

The digits of the decimal representations of $a_1, a_2, \ldots$ are written in that order after a decimal point to form a real (decimal) number x. Prove that x is rational.

Solution: Say that $a_1 = 2$ and $a_2 = 2$. Then $a_1 + a_2 = 4$, so $a_3 = 2$. Proceeding trivially, we see that $x = \frac{2}{9}$. Say that $a_1 = x$, $a_2 = y$, where x, y are odd. Then $2 \mid x + y$, so $a_3 = 2$. Yet $y + 2$ is odd, so a_4 is odd. Similarly, a_5 is odd and a_6 is 2. In general, every third term is 2, and all other terms are odd. Let $b_1, b_2, \ldots, b_n, \ldots$ be the odd terms. Every b_n is the smallest prime divisor of $b_n - 1 + 2$. If $b_n = 3$, then by the recurrence $b_n + 1 = 5$, $b_n + 2 = 7$, $b_n + 3 = 3$, and the sequence repeats periodically. If b_n is not 3, we can show that either b_{n+1} or b_{n+2} must be smaller than b_n, by the following: If b_{n+2} is not prime, then $b_{n+1} \leq \sqrt{b_n + 2}$. But b_n is a positive odd prime, so ${b_n}^2 - b_n - 2 \geq 4 > 0$, so $\sqrt{b_n + 2} \leq b_n$. Therefore $b_n < b_{n-1}$. If $b_{n-1} + 2$ is prime, then $b_{n-1} + 4$ cannot be prime unless b_{n-1} is 3. Otherwise, $\sqrt{b_n + 4} \leq b_n$ by the same reasoning. So the odd terms must eventually decrease until $b_n = 3$ for some n, and after that n the decimal is repeating. Therefore, x is rational.

Problem 14 Let r be a positive number and let $A_1A_2A_3A_4$ be a unit square. Given any four discs $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ centered at A_1, A_2, A_3, A_4 with radii whose sum is r, we are given that there exists an equilateral triangle whose vertices lie in three of the four discs. (That is, there is an equilateral triangle BCD and three distinct discs $\mathcal{D}_i, \mathcal{D}_j, \mathcal{D}_k$ such that $B \in \mathcal{D}_i$, $C \in \mathcal{D}_j$, $D \in \mathcal{D}_k$.) Find the smallest positive number r with this property.

Solution: The required number is $\frac{8 \sin 15}{3}$. To show that this r is sufficient, given $r \geq \frac{8 \sin 15}{3}$, choose three discs D_p, D_q, D_r from $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ so that $p + q + r \geq \frac{3r}{4}$. Set up complex coordinates in the plane so that D_p is centered at 1, D_q at 0, and D_p at i without loss of generality. To see if points from D_p, D_q, D_r can form an equilateral triangle, choose points $1+a \in D_p, b \in D_q, c \in D_r$. Then it is well known that these points make an equilateral triangle if and only if

$$(i+c)+\omega(b)+\omega^2(1+a)=0,$$

if and only if

$$c+b\omega+a\omega^2=-i-\omega^2. \qquad (*)$$

We have that $-i-\omega^2$ has modulus

$$|\operatorname{cis}(90)+cis(240)|=2|\frac{\operatorname{cis}(75)+cis(-75)}{2}|=2\sin 15.$$

Because $|c|+|b|+|a| \geq \frac{3r}{4} \geq 2\sin 15$, we can choose a, b, c to be positive scalar multiples of $-i-\omega^2$ in a way that satisfies $*$. This proves sufficiency.

To show that $r \geq \frac{8\sin 15}{3}$ is necessary, give each of the four discs radius $\frac{r}{4}$. Then D_p, D_q, D_r in the above construction will have radius $\frac{r}{4}$. We can find the required equilateral triangle if and only if we can satisfy equation $*$, and if $*$ is satisfied then we must have

$$\frac{3r}{4}=|c|+|b|+|a| \geq c+b\omega+a\omega^2=|-i-\omega^2|=2\sin 15.$$

This gives the desired bound on r.

Problem 15 Elections occur and every member of parliament is assigned a positive number, his or her *absolute rating*. On the first day of parliament, the members are partitioned into groups. In each group, every member of that group receives a *relative rating*: the ratio if his or her absolute rating, to the sum of the absolute ratings of all members in that group. From time to time, a member of parliament decides to move to a different group, and immediately after the switch each member's relative rating changes accordingly. No two members can move to different groups at the same time. Show that only a finite number of moves to different groups can be made.

Solution: We proceed by induction. If there is only one group, then nobody can move. Say that this is true for $n-1$ or fewer groups. Let the politicians be $P_1, P_2, \ldots P_m$, with ratings $E_1, E_2, \ldots E_m$ respectively. Let the groups at the beginning be $G_1, G_2, \ldots G_n$. Let $R(G_i)$ be the sum of all ratings of the members of G_i. Note that the number of groups never changes: the number cannot increase trivially. If at some point it decreases, the politician that moved went from being in his own group (and having a rating of 1) to another nonempty group, giving him a rating strictly less than one. Let $M = Max(R(G_1)\dots R(G_n))$.

Lemma. *M can never increase.*

Proof. Suppose after some move of P_i from G_j to G_k, M increases to M'. Before the move, $R(G_j) \leq M$, so P_i's relative rating was $\frac{E_i}{R(G_j)} \geq \frac{E_i}{M}$. Afterwards, $R(G_k) \geq M' > M$, so $\frac{E_i}{R(G_k)} > \frac{E_i}{M}$. Therefore, E_i's move has decreased his realtive rating. □

If we can show that only finitly many moves can be made preserving M, we will be done. Without loss of generallity, say that $G_i \dots G_n$ are all the groups G_j with $R(G_j) = M$. Then only finitely many moves can be made within groups $G_1 \dots G_i - 1$ alone by inductive hypothesis, and trivially no one will move from one of the groups $G_1 \dots G_i - 1$ to $G_i \dots G_n$. So eventually someone will have to move out of one of the groups G_j of $G_i \dots G_n$, decreasing $R(G_j)$. Then $R(G_j) < M$. Continuing, eventualy all groups G_j will have to have $R(G_j) < M$.

Problem 16 Let m, n be positive integers of distinct parities such that $m < n < 5m$. Show that there exists a partition of $\{1, 2, \dots, 4mn\}$ into two-element subsets, such that the sum of the numbers in each pair is a perfect square.

Solution: Let $A = \{1, \dots, n^2 - 2mn + m^2 - 1\}$ and $B = \{n^2 - 2mn + m^2, \dots, 4mn\}$. Notice that this is possible because $n^2 - 2mn + m^2 \in \{1, \dots, 4mn\}$: $n^2 - 2mn + m^2 < 5nm - 2nm + nm = 4nm$. Notice also that the cardinalities of A and B are even, because if n and m are of different parities then $(n-m)^2$ is odd, so the cardinallity of A, $(n-m)^2 - 1$, is even, and the carinallity of B, $4mn - (n-m)^2 + 1$, is also even. Therefore we can seperate A completely into disjoint pairs by pairing each element $k \in A$ with $n^2 - 2mn + m^2 - k$. We can do the same for B by pairing k with $n^2 + 2mn + m^2 - k$. Each of these pairs has a sum of either $n^2 - 2mn + m^2 = (n-m)^2$ or $n^2 + 2mn + m^2 = (n+m)^2$, so they clearly satisfy the problem's requirements.

Problem 17 Let ABC be a triangle such that $AC \neq BC$ and $AB < AC$. Let Γ be its circumcircle. Let D be the intersection of line BC with the tangent line to Γ at A. Let Γ_1 be the circle tangent to Γ, segment $\overline{AD}$, and segment $\overline{BD}$. We denote by M the point where Γ_1 touches segment $\overline{BD}$. Show that $AC = MC$ if and only if line AM is the angle bisector of angle DAB.

Solution:

We first show that $AC = MC$ implies $\angle MAD = \angle BAM$. Let Γ_3 be the circle with center C and radius AC. Then the radius MC of Γ_3 is tangent to Γ_1, so the circles Γ_3 and Γ_1 are orthogonal. Let σ be the inversion through circle Γ_3. Then σ takes Γ_1 to itself, and therefore takes Γ_1 to the line AD tangent to Γ_1 at A. Because of this, we must have that σ take B to D, and that $\triangle DAC \sim \triangle ABC$. This gives that $\angle ABC = \angle CAD = 90$.

Now let $x = \angle DCA$. By tangent line AD we have $\angle BAD = \angle BCA = x$, and by isoceles triangle MCA we have $\angle CAM = 90 - \frac{x}{2}$. By right triangle ABC we have $\angle CAB = 90 - x$. Combining our information gives us

$$\angle CAD = 90$$

$$\angle CAM = 90 - \tfrac{x}{2}$$

$$\angle CAB = 90 - x$$

$$\angle MAD = \angle CAD - \angle CAM = \tfrac{x}{2}$$

$$\angle BAM = \angle CAM - \angle CAB = \tfrac{x}{2}$$

and $\angle MAD = \angle BAM$ as desired.

We now show the converse. Suppose we have $\angle MAD = \angle BAM$. Let $x = \angle DAB$. Then we have $\angle BCA = x$, $\angle DAC = 90$, and $\angle MAD = \angle BAM = \frac{x}{2}$. Now invert about A, and denote the image of point P by P'. We now have that D', B', C', A' are concyclic, $\angle D'A'C' = 90$, $\angle D'A'B' = x$, $\angle D'AM' = \angle M'AB' = \frac{x}{2}$, and $\angle B'AD' = x$. Therefore we have $D'A \parallel B'C'$.' Because of this and the right angles at B' and A, we have $\angle AD'B' = \angle B'C'A = 90$. We now have that

$$\angle M'B'C' = \angle M'B'A' + \angle D'B'C' = \angle M'AD' + 90 = 90 + \tfrac{x}{2}$$
$$\angle AD'M' = \angle B'D'M' + \angle AD'B' = 90 + \angle B'AM' = 90 + \tfrac{x}{2}$$

which gives that $\widehat{AM'} = \widehat{M'C'}$ in circle $AD'B'C'$, and that $AM'C'$ is isoceles. By inversion, we have that ACM is isoceles and that $AC = CM$ as desired.

Problem 18 There are $n \geq 2$ players who are playing a card game with np cards. The cards are colored in n colors, and there are p cards of each color, labelled $1, 2, \ldots, p$. They play a game according to the following rules:

- Each player receives p cards.
- During each round, one player throws a card (say, with the color c) on the table. Every other player also throws a card on the table; if it is possible to throw down a card of color c, then the player must do so. The winner is the player who puts down the card of color c labelled with the highest number.
- A person is randomly chosen to start the first round. Thereafter, the winner of each round starts the next round.
- All the cards thrown on the table during one round are removed from the game at the end of the round, and the game ends after p rounds.

At the end of the game, it turns out that all cards labeled 1 won some round. Prove that $p \geq 2n$.

Solution: Call the players $A_1 \ldots A_n$, and let the colors be $1 \ldots n$.

Lemma. *In the last round, each player must have a differently colored card.*

Proof. Let the card played in the last round by player A_i be C_i. Without loss of generallity, we will show that C_1's color is different from C_j's, for $i \neq j$. Say C_1 is of color c. If C_1 is labelled one, then C_1 must win the round, but any other card of color c would beat C_1. So not C_i, $i > 1$ is of color c. If C_1 is not labelled one, then say there is a $j > 1$ so that C_j is of color c. C_j obviously cannot be 1, for there is no way it could beat C_1 if it was. So the card X of color c, label one, must have won a previous round. But when X was played, player A_1 had C_1 in his hand, and player A_j had C_j in his hand, so one of the two was forced to play his card and beat X. Therefore C_j cannot be of color c. $\square$

Lemma. *Every card of label one was played by a different player.*

Proof. Let the color of C_i be i, and let the card of color i, label one be x_i. If the card x_i is played by someone other than A_i, A_i will be forced to play C_i and beat x_i. So x_i must be played by A_i. $\square$

Notice this also gives us that player A_i has x_i in his hand. Without loss of generallity, say the cards x_i are played in the order $x_1 \ldots x_n$. If A_1 has all the cards of color 1, then A_1 will win every round by the rules. But there cannot be any cards of color 1 in anyone else's hands when x_1 is played. So the must be at least one round before x_1 is played to eliminate the cards of color 1 from the hands of $A_2 \ldots A_n$. A_2 cannot win with x_2 unless he is the challenger, so there must be a round in between the round when x_1 wins and when x_2 wins. By the same reasoning, there must be a round in between the round when x_i wins and when x_{i+1} wins. So in total, there must be at least $2n$ rounds. Because there are p rounds in all, $p \geq 2n$ as desired.

1.12 Russia

Problem 1 Each cells in a 9×9 grid is painted either blue or red. Two cells are called *diagonal neighbors* if their intersection is exactly a point. Show that some cell has exactly two red neighbors, or exactly two blue neighbors, or both.

Problem 2 A monic quadratic polynomial f with integer coefficients attains prime values at three consecutive integer points. Show that it attains a prime value at some other integer point as well.

Problem 3 Let O be the circumcenter of an acute triangle ABC with $AB = AC$. Point M lies on segment $\overline{BO}$, and point M' is the reflection of M across the midpoint of side $\overline{AB}$. Point K is the intersection of lines $M'O$ and AB. Point L lies on side $\overline{BC}$ such that $\angle CLO = \angle BLM$. Show that O, K, B, L are concyclic.

Problem 4 There are $\lfloor \frac{4}{3}n \rfloor$ rectangles on the plane whose sides are parallel to the coordinate axes. It is known that any rectangle intersects at least n other rectangles. Show that one of the rectangles intersects all the other rectangles.

Problem 5 Around a circle are written the numbers $a_1, a_2, \ldots, a_{60}$, a permutation of the numbers $1, 2, \ldots, 60$. (All indices are taken modulo 60.) Is it possible that $2 \mid (a_n + a_{n+2})$, $3 \mid (a_n + a_{n+3})$, and $7 \mid (a_n + a_{n+7})$ for all n?

Problem 6 Let $ABCD$ be a trapezoid with $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \nparallel \overline{DA}$. Let A' be the point on the boundary of the trapezoid such that line AA' splits the trapezoid into two halves with the same area. The points B', C', D' are defined similarly. Let P be the intersection of the diagonals of quadrilateral $ABCD$, and let P' be the intersection of the diagonals of quadrilateral $A'B'C'D'$. Prove that P and P' are reflections of each other across the midpoint of the midline of trapezoid $ABCD$. (The midline of the trapezoid is the line connecting the midpoints of sides $\overline{BC}$ and $\overline{DA}$.)

Problem 7 18 stones are arranged on a line. It is known that there are 3 consecutive stones that weigh 99 grams each, whereas all the other stones weigh 100 grams each. You are allowed to perform the following operation twice: choose a subset of the 18 stones, then weigh that collection of stones. Describe a method for determining which three stones weigh 99 grams each.

Problem 8 What is the largest possible length of an arithmetic progression of positive integers $a_1, a_2, \ldots, a_n$ with difference 2, such that $a_k^2 + 1$ is prime for $k = 1, 2, \ldots, n$?

Problem 9 A convex polygon on the plane contains at least m^2+1 lattice points strictly in its interior. Show that one some $m+1$ lattice points strictly inside the polygon lie on the same line.

Problem 10 The perpendicular bisector of side $\overline{AC}$ of a triangle ABC meets side $\overline{BC}$ at a point M. The ray bisecting angle AMB intersects the circumcircle of triangle ABC at K. Show that the line passing through the incenters of triangles AKM and BKM is perpendicular to the angle bisector of angle AKB.

Problem 11

(a) The sequence $a_0, a_1, a_2, \ldots$ satisfies $a_0 = 0$ and $0 \le a_{k+1} - a_k \le 1$ for $k \ge 1$. Prove that

$$\sum_{k=0}^{n} a_k^3 \le \left(\sum_{k=0}^{n} a_k \right)^2 .$$

(b) If the sequence $a_0, a_1, a_2, \ldots$ instead satisfies $a_0 = 0$ and $a_{k+1} \ge a_k + 1$ for $k \ge 1$, prove the reverse of the inequality in (a).

Let $n \ge 3$ be an integer. On the x-axis have been chosen pairwise distinct points $X_1, X_2, \ldots, X_n$. Let $f_1, f_2, \ldots, f_m$ be the monic quadratic polynomials that have two distinct X_i as roots. Prove that $y = f_1(x) + \cdots + f_m(x)$ crosses the x-axis at exactly two points.

Problem 12 What is the largest number of colors in which one can paint all the squares of a 10×10 checkerboard so that each of its columns, and each of its rows, is painted in at most 5 different colors?

Problem 13 Real numbers x and y have the property that $x^p + y^q$ is rational for any distinct odd primes p, q. Prove that x and y are rational.

Problem 14 The altitude from S of pyramid $SABCD$ passes through the intersection of the diagonals of base $ABCD$. Let $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$, $\overline{DD_1}$ be the perpendiculars to lines SC, SD, SA, and SB, respectively (where A_1 lies on line SC, etc.). It is known that S, A_1, B_1, C_1, D_1 are distinct and lie on the same sphere. Show that lines AA_1, BB_1, CC_1, DD_1 are concurrent.

Solution: Let E denote the intersection of $\overline{AC}$ and $\overline{BD}$. Lines $\overline{SE}$, $\overline{AA_1}$, and $\overline{CC_1}$ pass through the orthocenter of $\triangle ASC$, which we denote H_1. Similarly, we denote H_2 as the orthocenter of $\triangle BSD$. Let R denote sphere $SABCD$.Now, H_1 must be on R due to cyclic quadrilateral $SA_1H_1C_1$. Similarly, H_2 is on R due to cyclic quadrilateral $SB_1H_2D_1$. However, since both H_1 and H_2 are on $\overline{SE}$, they must be the same point, and we are done.

Problem 15 The plane is divided into 1×1 cells. Each cell is colored in one of n^2 colors so that any $n \times n$ grid of cells contains one cell of each color. Show that there exists an (infinite) column colored in exactly n colors.

Solution: Counterexample?

Problem 15 - for colors 1a, 2a, 1b ... tessalate

1a 2a 3a 4a 1b 2b 3a 4a 1b 2b 3b 4b 1a 2a 3b 4b 1c 2c 3c 4c 1c 2c 3d 4d 1d 2d 3d 4d 1d 2d 3c 4c

1a 2a 4a 3a 1b 2b 4a 3a 1b 2b 4b 3b 1a 2a 4b 3b 2c 1c 3c 4c 2c 1c 3d 4d 2d 1d 3d 4d 2d 1d 3c 4c

Problem 16 Let $p(x)$ be a polynomial of odd degree. Show that the equation $p(p(x)) = 0$ has at least as many real roots as the equation $p(x) = 0$.

Solution: Because $p(x)$ is surjective, for every root r of $p$there exists an a such that $p(a) = r$, or $p(p(a)) = 0$.

Problem 17 There are $n > 1$ points on the plane. Two players choose in turn a pair of points and draw a vector from one to the other. It is forbidden to choose points already connected by a vector. If at a certain moment the sum of all drawn vectors is zero, then the second player wins. If at a certain moment it is impossible to draw a new vector and the sum of the existing vectors is not zero, then the first player wins. As a function of the choice of n points, which player has a winning strategy?

Solution: The first player wins in every configuration. For the first move she simply chooses the vector $\overrightarrow{AA'}$ with largest possible magnitude. Suppose for contradiction the second player can choose a $\overrightarrow{BB'}$ such that $\overrightarrow{AA'} + \overrightarrow{BB'} = 0$. Then $AA'BB'$ is a parallelogram and one of $\overrightarrow{AB'}, \overrightarrow{BA'}$ has a larger magnitude than $\overrightarrow{AA'}$, which is a contradiction. On the remaining moves, suppose the sum of the down vectors equals $\vec{V} \neq 0$ before the first player plays. The first player now chooses a vector such that:

(i) It has the largest possible magnitude

(ii) Itmakes an angle ≥ 90 with $\vec{V}$ when placed on its head.

Clearly, the second player can no longer choose a vector with magnitude large enough to make the sum of all the vectors equal zero. The first player simply continues the above strategy till it is impossible to draw more vectors.

Problem 18 Let $ABCD$ be a convex quadrilateral, and let $\ell_A, \ell_B, \ell_C, \ell_D$ be the bisectors of its external angles. Lines ℓ_A and ℓ_B meet at a point K, ℓ_B and ℓ_C meet at a point L, ℓ_C and ℓ_D meet at a point M, and ℓ_D and ℓ_A meet at a point N. Show that if the circumcircles of triangles ABK and CDM are externally tangent to each other, then the same is true for the circumcircles of triangles BCL and DAN.

Solution: Let lines $\ell'_A, \ell'_B, \ell'_C, \ell'_D$ be the internal angle bisectors of $ABCD$, and let ℓ'_A and ℓ'_B meet at K', with L', M', N' defined similarly. Let circle C_K be the circumcircle of triangle ABK, with circles C_L, C_M, C_N defined symmetrically. Let P be the intersection of AD and BC, with line ℓ_P the internal angle bisector of $\angle APB$. (If $AD \parallel BC$, P becomes a point on the line at infinity in the *projective plane* and line ℓ_P becomes a line parallel to AD and BC halfway between them.)Now, $\angle KAK' = \angle KBK' = 90$ so K' is on circle C_K and KK' is a diameter. Similarly, MM' is a diameter of C_M. Also, K, K' and M, M' are the incenters/excenters of triangles PAB and PDC respectively, so all four are on line ℓ_P. This implies that C_K and C_M are externally tangent if and only if K' and M' are the same point, which is true if and only if L' and N' are (that) same point as well, which lea to the desired result.
Note: A direct result is that C_K and C_M are tangent if and only if $ABCD$ is a rhombus.

Problem 19 Let n be a fixed integer between 2 and 2002, inclusive. On the segment $[0, 2002]$ are marked $n+1$ points with integer coordinates, including the two endpoints of the segment. These points divide $[0, 2002]$ into n segments, and we are given that the lengths of these segments are pairwise relatively prime. One is allowed to choose any segment whose endpoints are already marked, divide it into n equal parts, and mark the endpoints of all these parts — provided that these new marked points all have integer coordinates. (One is allowed to mark the same point twice.)

(a) By repeating this operation, is it always possible — for fixed n, but regardless of the choice of initial markings — to mark all the points on the segment with integer coordinates?

(b) Suppose that $n = 3$, and that when we divide any segment into 3 parts we must erase one of its endpoints. By repeating the modified operation, is it always possible — regardless of the choice of initial markings — to mark any given single point of $[0, N]$?

Problem 20 Distinct points O, B, C lie on a line in that order, and point A lies off the line. Let O_1 be the incenter of triangle OAB, and let O_2 be the excenter of triangle OAC opposite A. If $O_1A = O_2A$, show that triangle ABC is isosceles.

Problem 21 Six red, six blue, and six green points are marked on the plane. No three of these points are collinear. Show that the sum of the areas of those triangles whose vertices are marked points of the same color, does not exceed one quarter of the sum of the areas of *all* the triangles whose vertices are marked points.

Solution: Let us label the eighteen points $r_1, r_2, \ldots, r_6, b_1, \ldots, b_6, g_1, \ldots, g_6$ (r, b, g for red, blue, and green respectively). Let $[ABC]$ denote the area of triangle ABC. Using *symmetric sum* notation, we have

$$\sum_{sym_r} f(r_1, r_2, \ldots, r_6) = \sum_{\sigma(r_1, r_2, \ldots, r_6)} f(r_1, r_2, \ldots, r_6),$$

where we sum over permutations of the r_i on the right side. Let R denote the sum of the areas of the triangles formed by three red points, let R_b denote the sum of the areas of the triangles with two red and one blue point, and let $R_g, B, B_r \ldots$ be defined similarly. Now, observe that

$$[ABC] \leq [ABD] + [ACD] + [BCD].$$

From this, we have

$$[r_1r_2r_3] \leq [r_1r_2b_1] + [r_2r_3b_1] + [r_3r_1b_1].$$

Summing over both sides,

$$\sum_{sym_r}[r_1r_2r_3] \leq \sum_{sym_r}[r_1r_2b_1] + [r_2r_3b_1] + [r_3r_1b_1] = 3 \cdot \sum_{sym_r}[r_1r_2b_1].$$

Summing over b_i,

$$2 \cdot \sum_{sym_r}[r_1r_2r_3] \leq \sum_{x=1}^{6}\sum_{sym_r}[r_1r_2b_x].$$

Now,

$$R = \frac{\sum_{sym_r}[r_1r_2r_3]}{3! \cdot 3!} \quad \text{and} \quad R_b = \frac{\sum_{x=1}^{6}\sum_{sym_r}[r_1r_2b_x]}{2! \cdot 4!}, \quad \text{so}$$

$$2 \cdot 3! \cdot 3! \cdot R \leq 2! \cdot 4! \cdot R_b \quad \text{and}$$
$$\frac{3R}{2} \leq R_b.$$

It follows from symmetry that

$$R + \frac{3R}{2} + \frac{3R}{2} \leq R + R_b + R_g \quad \text{and again by symmetry}$$
$$4R + 4B + 4G \leq R + R_b + R_g + B + B_g + B_r + G + G_r + G_b$$

which yields the desired result.

Problem 22 A mathematical *hydra* consists of heads and necks, where any neck joins exactly two heads, and where each pair of heads is joined by exactly 0 or 1 necks. With a stroke of a sword, Hercules can destroy all the necks coming out of some head A of the hydra. Immediately after that, new necks appear joining A with all the heads that were not joined with A immediately before the stroke. To defeat a hydra, Hercules needs to chop it into two parts not joined by necks (that is, given any two heads, one from each part, they are not joined by a neck). Find the minimal N for which he can defeat any hydra with 100 necks by making at most N strokes.

Solution: The answer is 10. Let $c(X)$ denote a stroke of Hercules' sword around head X. Let the *degree* of a head denote the number of heads that are joined to it. We have the following lemma:

Lemma. *A hydra with h heads and H the head with least degree d can be defeated in* $\min\{d, h-d\}$ *strokes.*

Proof. Let H be joined to heads $H_1, H_2, \ldots, H_d$ and not joined to heads $H_{d+1}, H_{d+2}, \ldots, H_{h-1}$. The strokes $c(H_1), c(H_2), \ldots, c(H_d)$ will separate H from g in d strokes and the strokes $c(H), c(H_{d+1}), c(H_{d+2}), \ldots, c(H_{h-1})$ will separate H from g in $h-d$ strokes. $\square$

We will now prove that $N \leq 10$. Suppose by way of contradiction a hydra w ith h heads and 100 necks takes more than 10 strokes to die. By the lemma, each head has a degree of at least 11, so $\frac{11h}{2} \leq 100$ or $h \leq 18$. However, by the lemma this means the hydra can be defeated in at most $18 - 11 = 7$ strokes which is a contradiction.Now, let $K_{a,b}$ denote a hydra that can be partitioned into two sets of heads α and β of sizes a and b respectively such that

(i) every head in α is joined to every head in β,

(ii) no two heads in α are joined, and

(iii) no two heads in β are joined.

We claim that $K_{10,10}$ takes 10 strokes to die. Clearly, it is 100-necked, so if true our claim would prove that $N = 10$ and we'd be done. Let A be an element of α. Performing $c(A)$ would result in A being joined to every other element in α and being n joined to every element in β, resulting in $K_{a-1,b+1}$. Similarly, cutt ing a head from β would have resulted in $K_{a+1,b-1}$. A hydra of this form is defeated if and only if either a or b equals 0, which leads to the desired result.
Note: In graph theory $K_{i,j}$ is known as the *complete bipartite graph* or *complete bigraph* on i and j vertices.

Problem 23 There are 8 rooks on a chessboard, no two of which lie in the same column or row. We define the distance between two rooks to be the distance between the centers of the squares that they lie on. Prove that among all the distances between rooks, there are at least two distances that are equal.

Solution: We will first find an upper limit on the number of possible distances between two rooks. Without loss of generality, let the distance between two adjacent squares on the board equal 1. If two rooks are r rows and c columns apart, the distance between them is $\sqrt{r^2+c^2}$. We know that for any pair of rooks, $1 \leq r, c \leq 7$, giving at most $28 = 7 + \binom{7}{2}$ (for $r = c$ and $r \neq c$ respectively) different values for $\sqrt{r^2+c^2}$. However, we have double counted $\sqrt{50} = \sqrt{5^2+5^2} = \sqrt{1^2+7^2}$, implying at most 27 different possible distances.On the other hand, 8 rooks implies $\binom{8}{2} = 28 > 27$ different pairs of rooks, so two of them must share the same distance by the Pigeonhole Principle.

Problem 24 There are $k > 1$ blue boxes, one red box, and a stack of $2n$ cards numbered from 1 to $2n$. Originally, the cards in the stack are in some arbitrary order, and the stack is in the red box. One is allowed to take the top card from any box; say that the card's label is m. Then the card is put either (i) in an empty box, or (ii) in a box whose top card is labelled $m+1$. What is the maximal n for which it is possible to move all of the cards into one blue box?

Solution: The answer is $k-1$. We can label the boxes $R, B_1, B_2, \ldots, B_k$ and denote e cards in box X by $[X \mid c_1, c_2, \ldots, c_z]$ (c_1 is the top card). Let $(c : X, Y)$ denote moving card c from box X to box Y and be called a *move*.First, we will show that $n < k$. If $n \geq k$, the initial arrangement could be $[R \mid 1, 3, 5, \ldots, 2k-1, 2k\ldots]$. The first k moves would necessarily move the top card of R to an empty blue box. After this there are no more empty boxes and every subsequent move will involve moving card $2k-1$ between R and some B_x, leaving the problem statement unfulfilled.Now, we will provide an algorithm for moving the stack to a blue box when $n = k-1$. Let r denote the top card in R, and $d = \lceil r/2 \rceil$. (Note $1 \leq d \leq k-1$.) For the first $2k-2$ moves:

(i) If B_d is empty or if r is odd do $(r : R, B_d)$.

(ii) Otherwise, we have $[B_d \mid r-1]$. We now do the following: $(r-1 : B_d, B_k), (r : R, B_d)$, and then $(r-1 : B_k, B_d)$.

After these moves, R and B_k are empty and for $1 \leq j \leq k-1$ we have $[B_j \mid 2j, 2j-1]$. Now, for $j = 2, 3, \ldots, k-1$, we do $(2j : B_j, B_k), (2j-1 : B_j, B_1)$, and $(2j : B_k, B_1)$ in order, after which we have $[B_1 \mid 2k-2, 2k-3, \ldots, 1]$ and are done.

Problem 25 Let O be the circumcenter of triangle ABC. On sides $\overline{AB}$ and $\overline{BC}$ there have been chosen points M and N, respectively, such that $2\angle MON = \angle AOC$. Show that the perimeter of triangle MBN is at least AC.

Problem 26 Let $n \geq 1$ be an integer. $2^{2n-1}+1$ odd numbers are chosen from the interval $(2^{2n}, 2^{3n})$. Show that among these numbers, one can find two numbers a, b for which $a \not| b^2$ and $b \not| a^2$.

Solution: Let $P(c)$ denote the set of prime factors of c. Assume for contradiction there are no two numbers a, b such that $a \not| b^2$ and $b \not| a^2$. Clearly, $a \mid b^2$ only if $P(a) \subseteq P(b)$, so if $|P(a)| = |P(b)|$ we must have $P(a) = P(b)$ as well. This means w e can partition our $2^{2n-1}+1$ number into categories $C_0, C_1, C_2, \ldots$ where for any j and any two numbers ρ, σ in C_j we have $|P(\rho)| = |P(\sigma)| = j$ and $P(\rho) = P(\sigma)$.

Lemma. $|C_m| \leq \binom{2n-1}{m}$.

Proof. For integer r in C_m let $P(r)$ have elements $p_1, p_2, \ldots, p_m$. For some $a_i \geq 1$, we have

$$3^{\sum_{i=1}^m a_i} \leq p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m} = r < 2^{3n} < 3^{2n} \quad \text{implying}$$

$$\sum a_i < 2n.$$

The number of different ways for $a_i \geq 1$ and $\sum a_i = \omega$ is $\binom{\omega-1}{m-1}$. (Prove it!) Given $m \leq \omega \leq 2n-1$, we have

$$|C_m| \leq \sum_{\omega=m}^{2n-1} \binom{\omega-1}{m-1} = \binom{2n-1}{m}$$

by induction or Pascal's Triangle (Why?). □

We now have

$$2^{2n-1}+1 = \sum_i |C_i| \leq \sum_i \binom{2n-1}{i} = 2^{2n-1}$$

which is a contradiction and leads to the desired result.

Problem 27 Let p, q, r be polynomials with real coefficients, such that at least one of the polynomials has degree 2 and at least one of the polynomials has degree 3. Assume that

$$p^2 + q^2 = r^2.$$

Show that at least one of the polynomials both has degree 3 and has 3 (not necessarily distinct) real roots.

Solution: Define $\deg(p)$ to mean the degree of p, and likewise $\deg(q)$ and $\deg(r)$. Observe that $\max\{\deg(p), \deg(q)\} = \deg(r)$. This is because $\deg(r^2) = \deg(p^2+q^2) = \max\{\deg(p^2), \deg(q^2)\}$ (leading coefficients of p^2 and q^2 are positive and do not cancel), and $2 \cdot \deg(r) = \deg(r^2) = \max\{\deg(p^2), \deg(q^2)\} = 2 \cdot \max\{\deg(p), \deg(q)\}$. Thus one of p, q (without loss of generality, q) must have degree 2, and both p and r must have degree 3.

Now we write $p^2 = (r+q) \cdot (r-q)$. We know that both factors have degree 3. Since p has either 1 or 3 real roots, $r+q$ and $r-q$ either both have one real root or three real roots accordingly. Assume that $p, r+q$, and $r-q$ each have one real root; let the root of p be r_1. Since $r_1^2 | p^2$, this means that r_1 is a root of both $r+q$ and $r-q$, and is therefore a root of both r and q. Now let p', q', and r' be the three polynomials p, q, and r with the common root divided out. We have $(p')^2 + (q')^2 = (r')^2$. Then all three polynomials have real coefficients, p' and r' have degree 2, and q' has degree 1. We also know that p' has 0 real roots, q' has 1, and r' has 0 or 2. Since the latter case would imply that r has 3 real roots, we instead assume that r' has 0 real roots.

Writing $(q')^2 = (r'+p') \cdot (r'-p')$, we see that one of the factors must have degree 2 and the other degree 0. If the first factor has degree 2, we may write

$$p' = ax^2 + bx + c$$
$$r' = ax^2 + bx + d$$

for real a, b, c, d. Then $q' = (d-c) \cdot (2ax^2 + 2bx + c + d)$. Since p' and r' have no real roots, we know that $b^2 - 4ac < 0$ and $b^2 - 4ad < 0$. Adding the two inequalities and multiplying by 2, this means $4b^2 - 8a \cdot (c+d) < 0$. But then q' has no real roots, a contradiction.

Similarly, if $r' + p'$ has degree 0 and $r' - p'$ has degree 2, we write

$$p' = ax^2 + bx + c$$
$$r' = -ax^2 - bx + d.$$

Then $q' = -(c+d) \cdot (2ax^2 + 2b + (c-d))$. Again, p' and r' have no real roots, so $b^2 - 4ac < 0$ and $b^2 + 4ad < 0$. Adding and multiplying by 2 gives $4b^2 - 8a \cdot (c-d) < 0$, which again implies that q' has no real roots. Thus our original assumption is false, and either p or r must be a third-degree polynomial with 3 real roots.

Problem 28 Quadrilateral $ABCD$ is inscribed in circle ω. The line tangent to ω at A intersects the extension of side $\overline{BC}$ past B at a point K. The line tangent to ω at B meets the extension of side $\overline{AD}$ past A at a point M. If $AM = AD$ and $BK = BC$, show that quadrilateral $ABCD$ is a trapezoid.

Solution: We reflect A over B to point A'. Then $\triangle A'KB \cong \triangle ACB$, and $\angle A'KB = \angle ACB = \angle ABM = \angle ADB = \angle BAK = \alpha$. Looking at triangles $AA'K$ and DMB, we have two triangles of the form XYZ with median ZP such that $\angle PXZ = \angle PZY = \alpha$. For such a triangle, let $\angle XZP = \theta$, $\angle ZYP = \phi$, and $\angle ZPX = \beta$. Then $\theta + \phi = 180 - 2\alpha$ is constant. By the law of sines applied to triangles XPZ and YPZ, we have $\frac{\sin\theta}{\sin\alpha} = \frac{\sin\alpha}{\sin\phi}$, so that $\sin\theta\sin\phi = \frac{1}{2}\cdot(\cos(\theta-\phi) - \cos(\theta+\phi))$ is also constant. Thus $\cos(\theta-\phi)$ is equal to a constant, and $\theta - \phi = \pm k$ for some constant k (since $0 < \theta, \phi < \pi$).

Furthermore, $180 - \alpha - \theta = \beta = \alpha + \phi$. Thus $\beta = \frac{1}{2}\cdot(180 - \alpha - \theta + \alpha + \phi) = 90 - \frac{1}{2}\cdot(\theta - \phi) = 90 \mp \frac{k}{2}$. Looking back at the original figure, β corresponds to $\angle KBA$ and $\angle BAD$, so that $\angle KBA = \angle BAD$ or $\angle KBA + \angle BAD = 180$. The first case gives a trapezoid with $AD \parallel BC$, and the second case gives a trapezoid with $AB \parallel CD$.

Problem 29 Show that for any positive integer $n > 10000$, there exists a positive integer m that is a sum of two squares and such that $0 < m - n < 3\sqrt[4]{n}$.

Solution: We have $a^2 < n \le (a+1)^2$ for some integer $a \ge 100$. If we write $n = a^2 + k$, this means that $k \le 2a + 1$. We want $m = a^2 + b^2$ for some integer b. The condition $0 < m - n < 3\sqrt[4]{n}$ becomes $k < b^2 < k + 3\sqrt[4]{a^2 + k}$. We will show that

$$b = \begin{cases} \sqrt{k} + 1 & \text{if } k \text{ is a perfect square} \\ \lceil\sqrt{k}\rceil & \text{if } k \text{ is not a perfect square} \end{cases}$$

will work.

Note that in both cases, $k < b^2 \le (\sqrt{k}+1)^2$. Thus we want

$$\begin{aligned}(\sqrt{k}+1)^2 &< k + 3\sqrt[4]{a^2+k}\\ 2\sqrt{k}+1 &< 3\sqrt[4]{a^2+k}\\ 4k+4\sqrt{k}+1 &< 9\sqrt{a^2+k}.\end{aligned}$$

Since $k \le 2a+1$, it is sufficient to prove

$$\begin{aligned}4\cdot(2a+1)+4\sqrt{2a+1}+1 &< 9a\\ a &> 4\sqrt{2a+1}+5\\ a^2-10a+25 &> 16\cdot(2a+1)\\ a^2-42a+9 &> 0.\end{aligned}$$

By a simple application of the quadratic formula, this last inequality is clearly true for $a \ge 100$.

Problem 30 Once upon a time, there were 2002 cities in a kingdom. The only way to travel between cities was to travel between two cities that are connected by a (two-way) road. In fact, the road system was such that even if it had been forbidden to pass through any one of the cities, it would still have been possible to get from any remaining city to any other remaining city. One year, the king decided to modify the road system from this initial set-up. Each year, the king chose a loop of roads that did not intersect itself, and then ordered:

(i) to build a new city,

(ii) to construct roads from this new city to any city on the chosen loop, and

(iii) to destroy all the roads of the loop, as they were no longer useful.

As a result, at a certain moment there no longer remained any loops of roads. Show that at this moment, there must have been at least 2002 cities accessible by exactly one road.

Solution: We represent the kingdoms by vertices and the roads by edges of a graph. We will show that at any point in the process, any two distinct edges emanating from one of the original vertices are always part of a loop. Thus the process will continue as long as one of the 2002 vertices is connected by more than one edge. Since every vertex is always connected by at least one edge, in the end all 2002 original vertices will be accessible by exactly one edge.

First note that the claim is true in the beginning. Consider one of the original vertices with two distinct edges emanating from it; call the vertex A and its neighbors B and C. Since it is still possible to travel from B to C without passing through A, there must be a path connecting B and C that does not include A. If we combine this path with edges AB and AC, we have a loop which passes through A and includes AB and AC.

Now we use induction. Assume that the claim is true for all steps in the process up to a certain point. After the next step, let A be the vertex in question (one of the original 2002 vertices with two distinct edges emanating from it) and again call its neighbors B and C. We have three cases:

Case i: None of the edges emanating from A were affected in the step. Edges AB and AC were part of a loop before the step, so we follow the loop until we reach an edge that was removed (such an edge may or may not exist). Suppose we have a path from A through B that terminates at D, where the next edge in the loop (emanating from D) has been removed, and a path from A through C that likewise terminates at E. Then in the step, we must have created vertex F such that DF and EF are edges of the graph. Thus, the paths from A to D and A to E, together with the edges DF and EF, form a loop which passes through A and includes AB and AC.

Case ii: Two edges emanating from A were removed; AB was created in the step, but AC was not. Then we know that B must have been the vertex created in the step. Furthermore, A had at least three distinct edges emanating from it before the step (AC and the two removed), so there must have been a loop through A containing AC. As before, we follow the loop from A through C until we reach an edge which was removed, which emanates from D. Then edge BD must also have been created in the step, and the path from A to D, together with edges AB and BD, forms a loop that passes through A and includes AB and AC.

Case iii: Two edges emanating from A were removed; neither AB nor AC were created in the step. This is identical to the first case, because we may follow the loop which previously contained AB and AC and passed through A, in both directions to D and E, which are connected to the new vertex F. Then the paths from A to D and

A to E, together with edges DF and EF, form a loop which passes through A and includes AB and AC.

Thus we have considered all cases, concluding the proof.

Problem 31 Let a, b, c be positive numbers with sum 3. Prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca.$$

Solution: Observe that

$$ab + bc + ca = \frac{(a+b+c)^2 - (a^2+b^2+c^2)}{2} = \frac{9}{2} - \frac{1}{2} \cdot (a^2+b^2+c^2),$$

so that the inequality in question is equivalent to

$$\sqrt{a} + \sqrt{b} + \sqrt{c} + \frac{1}{2} \cdot (a^2+b^2+c^2) \geq \frac{9}{2}.$$

By weighted AM-GM, we have

$$\frac{2}{3} \cdot \sqrt{a} + \frac{1}{3} \cdot a^2 \geq (a^{1/2})^{2/3} \cdot (a^2)^{1/3} = a.$$

Summing cyclically over a, b, and c, multiplying by $\frac{3}{2}$, and using the equality $a + b + c = 3$ produces the desired result.

Problem 32 The excircle of triangle ABC opposite A touches side $\overline{BC}$ at A'. Line ℓ_A passes through A' and is parallel to the angle bisector of angle CAB. The lines ℓ_B and ℓ_C are defined similarly. Prove that ℓ_A, ℓ_B, ℓ_C are concurrent.

Solution: Let A_1, B_1, and C_1 be the centers of the excircles opposite A, B, and C, respectively. It is a well-known fact that triangle $A_1B_1C_1$ circumscribes ABC, and AA_1, BB_1, and CC_1 are the altitudes of $A_1B_1C_1$.

Now let X be the intersection of ℓ_A and ℓ_B. We want to prove that ℓ_C passes through X. Using vectors, we have

$$(\mathbf{X} - \mathbf{A'}) \cdot (\mathbf{B_1} - \mathbf{C_1}) = 0$$

and

$$(\mathbf{X} - \mathbf{B'}) \cdot (\mathbf{C_1} - \mathbf{A_1}) = 0,$$

and we want

$$(\mathbf{X} - \mathbf{C'}) \cdot (\mathbf{A_1} - \mathbf{B_1}) = 0.$$

Rewriting the three equations, we have

$$\mathbf{X}\cdot(\mathbf{B_1}-\mathbf{C_1})=\mathbf{A'}\cdot(\mathbf{B_1}-\mathbf{C_1}), \quad (*)$$

$$\mathbf{X}\cdot(\mathbf{C_1}-\mathbf{A_1})=\mathbf{B'}\cdot(\mathbf{C_1}-\mathbf{A_1}), \quad (**)$$

and want

$$\mathbf{X}\cdot(\mathbf{A_1}-\mathbf{B_1})=\mathbf{C'}\cdot(\mathbf{A_1}-\mathbf{B_1}). \quad (***)$$

We use the identity

$$\cos A_1\sin(B_1-C_1)+\cos B_1\sin(C_1-A_1)+\cos C_1\sin(A_1-B_1)=0,$$

or equivalently,

$$\cot A_1\sin A_1\sin(B_1-C_1)+\cot B_1\sin B_1\sin(C_1-A_1)$$
$$+\cot C_1\sin C_1\sin(A_1-B_1)=0.(\dagger)$$

By angle chasing, we see that $\triangle A_1A'B\sim\triangle A_1CH$, where H is the orthocenter of $A_1B_1C_1$. Then

$$A_1A'=\frac{A_1B}{AH}\cdot A_1C=\sin C_1\cdot AC_1\cos A$$
$$=\sin C_1\cdot 2R\sin B_1\cdot\cos A=2R\sin A\sin B\sin C\cdot\cot A,$$

where R is the circumradius of $A_1B_1C_1$. We can obtain similar expressions for B_1B' and C_1C', so that $A_1A':B_1B':C_1C'=\cot A:\cot B:\cot C$. Furthermore, $B_1C_1:C_1A_1:A_1B_1=\sin A:\sin B:\sin C$, so that $(\dagger)$ becomes

$$A_1A'\cdot B_1C_1\cdot\sin(B_1-C_1)+B_1B'\cdot C_1A_1\cdot\sin(C_1-A_1)$$
$$+C_1C'\cdot A_1B_1\cdot\sin(A_1-B_1)=0.$$

Then $\angle A'A_1A=\angle A'A_1C-\angle AA_1B_1=(90-C_1)-(90-B_1)=B_1-C_1$, so that the cosine of the angle between $\mathbf{A_1}-\mathbf{A'}$ and $\mathbf{B_1}-\mathbf{C_1}$ is equal to $\sin(B_1-C_1)$. Hence our equation is actually

$$(\mathbf{A_1}-\mathbf{A'})\cdot(\mathbf{B_1}-\mathbf{C_1})+(\mathbf{B_1}-\mathbf{B'})\cdot(\mathbf{C_1}-\mathbf{A_1})+(\mathbf{C_1}-\mathbf{C'})\cdot(\mathbf{A_1}-\mathbf{B_1})=0.$$

Subtracting the identity

$$\mathbf{A_1}\cdot(\mathbf{B_1}-\mathbf{C_1})+\mathbf{B_1}\cdot(\mathbf{C_1}-\mathbf{A_1})+\mathbf{C_1}\cdot(\mathbf{A_1}-\mathbf{B_1})=0$$

and taking the inverse of both sides, we then obtain

$$\mathbf{A'}\cdot(\mathbf{B_1}-\mathbf{C_1})+\mathbf{B'}\cdot(\mathbf{C_1}-\mathbf{A_1})+\mathbf{C'}\cdot(\mathbf{A_1}-\mathbf{B_1})=0,$$

or

$$\mathbf{C}' \cdot (\mathbf{A_1} - \mathbf{B_1}) = -\mathbf{A}' \cdot (\mathbf{B_1} - \mathbf{C_1}) - \mathbf{B}' \cdot (\mathbf{C_1} - \mathbf{A_1}).$$

Adding $(*)$ and $(**)$, we then have

$$\mathbf{X} \cdot (\mathbf{B_1} - \mathbf{A_1}) + \mathbf{C}' \cdot (\mathbf{A_1} - \mathbf{B_1}) = 0,$$

which is exactly $(* * *)$.

Problem 33 A finite number of red and blue lines are drawn on the plane. No two of the lines are parallel to each other, and through any point where two lines of the same color meet, there also passes a line of the other color. Show that all the lines have a common point.

Problem 34 Some points are marked on the plane in such a way that for any three marked points, there exists a Cartesian coordinate system in which these three points are lattice points. (A Cartesian coordinate system is a coordinate system with perpendicular coordinate axes with the same scale.) Show that there exists a Cartesian coordinate system in which *all* the marked points have integer coordinates.

Solution: We begin with a lemma.

Lemma. *Given a Cartesian coordinate system with origin O and lattice point A, there exists a coordinate system with the same origin and axis OA such that all lattice points of the original coordinate system are also lattice points of the new system.*

Proof. We use vectors. Let the coordinates of O in the original system be $(0,0)$ and the coordinates of A be (m,n), where m and n are integers. If $P = (x,y)$ is any lattice point of the original system, we need to prove that $\operatorname{proj}_{x\mathbf{i}+y\mathbf{j}}(m\mathbf{i}+n\mathbf{j}) = \frac{k_1}{\|m\mathbf{i}+n\mathbf{j}\|}$ and $\operatorname{comp}_{x\mathbf{i}+y\mathbf{j}}(m\mathbf{i}+n\mathbf{j}) = \frac{k_2}{\|m\mathbf{i}+n\mathbf{j}\|}$ for integers k_1 and k_2. If we scale the new system so that each unit is of length $\frac{1}{\|m\mathbf{i}+n\mathbf{j}\|}$, the coordinates of P in the new system will then be the lattice point (k_1, k_2). Since

$$\operatorname{proj}_{x\mathbf{i}+y\mathbf{j}}(m\mathbf{i}+n\mathbf{j}) = \frac{(x\mathbf{i}+y\mathbf{j}) \cdot (m\mathbf{i}+n\mathbf{j})}{\|m\mathbf{i}+n\mathbf{j}\|} = \frac{xm+yn}{\|m\mathbf{i}+n\mathbf{j}\|}$$

and

$$\begin{aligned}\operatorname{comp}_{x\mathbf{i}+y\mathbf{j}}(m\mathbf{i}+n\mathbf{j}) &= \operatorname{proj}_{x\mathbf{i}+y\mathbf{j}}(-n\mathbf{i}+m\mathbf{j}) \\ &= \frac{(x\mathbf{i}+y\mathbf{j}) \cdot (-n\mathbf{i}+m\mathbf{j})}{\|m\mathbf{i}+n\mathbf{j}\|} = \frac{-xn+ym}{\|m\mathbf{i}+n\mathbf{j}\|},\end{aligned}$$

the result follows immediately. □

Now we apply the lemma to the original problem. We use induction on the number of points, n, where the base case $n = 3$ is given. (The cases $n = 1$ and $n = 2$ are trivial.) Now assume that there exists a coordinate system for every k points with the desired property. We want to show that if we instead have $k + 1$ points, we can find a coordinate system where they are all lattice points. Choose k of the points, call two of them A and B, and let C be the remaining point. We know by the assumption that some coordinate system S_1 includes the first set of k points as lattice points. Note that we may translate the axes of S_1 by any integral number of units to create a new coordinate system where all of the lattice points of S_1 are still lattice points of the new system. Thus we may create a new coordinate system, S_2, where the original k points are all lattice points and A is the origin. By the lemma, there then exists a coordinate system S_3 with one axis AB and the origin at A. Since B is a lattice point of the system, this means that a unit of S_3 must be of length $\frac{1}{m_1}$, where m_1 is an integer.

By the given condition, we also know that there exists a coordinate system where A, B, and C are lattice points. As before, we translate the axes to make A the origin, then use the lemma to create a new coordinate system, S_4, where AB is an axis and A is the origin. Since B is again a lattice point, the units of S_4 must be have length $\frac{1}{m_2}$, where m_2 is an integer. If we now create a coordinate system S_5 with A as the origin, axis AB, and units of length $\frac{1}{k_1 \cdot k_2}$, we see that all of the lattice points of S_3 and S_4 are among the lattice points of S_5. In particular, the original k points, along with C, are lattice points of S_5, completing the induction.

Problem 35 Show that

$$2|\sin^n x - \cos^n x| \leq 3|\sin^m x - \cos^m x|$$

for all $x \in (0, \pi/2)$ and for all positive integers $n > m$.

Solution: When $x \in [\frac{\pi}{4}, \frac{\pi}{2})$, the inequality becomes

$$2 \cdot (\sin^n x - \cos^n x) \leq 3 \cdot (\sin^m x - \cos^m x). \qquad (*)$$

If we replace x by $\frac{\pi}{2} - y$, we obtain the original inequality for $y \in (0, \frac{\pi}{4}]$. Thus it suffices to show that $(*)$ is true.

First consider the case $n = m + 2$ for $m > 1$. We will prove that

$$\sin^n x - \cos^n x \leq \sin^m x - \cos^m x \qquad (**)$$

for such n. Since $\cos x \leq \sin x$ in the given interval,

$$\cos^{m-2} x \leq \sin^{m-2} x$$

$$\cos^m x \cdot (1 - \cos^2 x) \leq \sin^m x \cdot (1 - \sin^2 x)$$

$$\sin^{m+2} x - \cos^{m+2} x \leq \sin^m x - \cos^m x,$$

as we want. By combining these inequalities, we have $(**)$ whenever $n = m + 2k$ for $m > 1$ and positive integers k, so that $(*)$ is certainly true.

In order to prove $(*)$ for m and n of different parity, we show that the inequality is true when $n = m + 1$ and $m > 1$. Observe that $\sin x + \cos x = \sqrt{2} \cdot \sin\left(x + \frac{\pi}{4}\right) \leq 1$, so that $\sin x + \cos x \leq \frac{1}{\sqrt{2}} < \frac{3}{2}$. Then

$$\begin{aligned}
&\sin^m x + \sin^{m-1} x \cdot \cos x + \cdots + \cos^m x \\
&\leq (\sin x + \cos x) \cdot (\sin^{m-1} x + \sin^{m-2} x \cdot \cos x + \cdots + \cos^{m-1} x) \\
&\leq \frac{3}{2} \cdot (\sin^{m-1} x + \sin^{m-2} x \cdot \cos x + \cdots + \cos^{m-1} x).
\end{aligned}$$

Multiplying both sides of the above inequality by $2 \cdot (\sin x - \cos x)$, we obtain $(*)$ for $n = m + 1$. This fact combined with the previous result proves $(*)$ for all $n > m > 1$.

Now all that remains is the case $m = 1$. Note that we need only prove the inequalities

$$2 \cdot (\sin^2 x - \cos^2 x) \leq 3 \cdot (\sin x - \cos x), \qquad (1)$$

$$2 \cdot (\sin^3 x - \cos^3 x) \leq 3 \cdot (\sin x - \cos x), \qquad (2)$$

then apply $(**)$ repeatedly for even and odd n, respectively. We have already shown that $\sin x + \cos x \leq \frac{3}{2}$; multiplying both sides by $2 \cdot (\sin x - \cos x)$ gives (1). Furthermore, $\sin x \cdot \cos x = \frac{1}{2} \cdot \sin 2x \leq \frac{1}{2}$, so that $\sin^2 x + \sin x \cdot \cos x + \cos^2 x \leq \frac{3}{2}$. Again multiplying both sides by $2 \cdot (\sin x - \cos x)$, we obtain (2). Thus $(*)$ holds for all positive integers $n > m$.

Problem 36 In a certain city, there are several squares. All streets are one-way and start or terminate only in squares; any two squares are connected by at most one road. It is known that there are exactly two streets that go out of any given square. Show that one can divide the city into 1014 districts so that (i) no street connects two cities in the same district, and (ii) for any two districts, all the streets that connect them have the same direction (either all the streets go from the first district to the second, or vice versa).

Problem 37 Find the smallest positive integer which can be written both as (i) a sum of 2002 positive integers (not necessarily distinct), each of which has the same sum of digits; and (ii) as a sum of 2003 positive integers (not necessarily distinct), each of which has the same sum of digits.

Solution: The answer is 10010. First observe that this is indeed a solution: $10010 = 2002 \cdot 5 = 1781 \cdot 4 + 222 \cdot 13$, so may express 10010 as the sum of 2002 fives or of 1781 fours and 222 thirteens, where $1781 + 222 = 2003$. To prove minimality, observe that a number is congruent modulo 9 to the sum of its digits, so two positive integers with the same digit sum are in the same residue class modulo 9. Let k_1 be the digit sum of the 2002 numbers and k_2 the digit sum of the 2003 numbers. Then $4 \cdot k_1 \equiv 2002 \cdot k_1 \equiv 2003 \cdot k_2 \equiv 5 \cdot k_2 \pmod 9$. If $k_1 \geq 5$, the sum of the 2002 numbers is at least 10010; if $k_2 \geq 5$, the sum of the 2003 numbers is greater than 10010. However, the solutions $k_1 \equiv 1, 2, 3, 4 \pmod 9$ give $k_2 \equiv 8, 7, 6, 5$, respectively, so that at least one of k_1 or k_2 is greater than or equal to 5, and the minimal integer is 10010.

Problem 38 Let $ABCD$ be a quadrilateral inscribed in a circle, and let O be the intersection point of diagonals $\overline{AC}$ and $\overline{BD}$. The circumcircles of triangles ABO and COD meet again at K. Point L has the property that triangles BLC and AKD are similar (with the similarity respecting this order of vertices). Show that if quadrilateral $BLCK$ is convex, then it is circumscribed about some circle.

Solution: Note that $BLCK$ is circumscribable about a circle if and only if $BL + CK = CL + BK$. Expressing the area of $ABCD$ in two ways, we have

$$\sin \angle ABC \cdot (AB \cdot BC + AD \cdot CD) = \sin \angle BAD \cdot (AB \cdot AD + BC \cdot CD).$$

Furthermore, by the law of sines applied to triangles ABC and ABD and the fact that $\angle ACB = \angle ADB$,

$$\frac{\sin\angle ABC}{AC} = \frac{\sin\angle ACB}{AB} = \frac{\sin\angle ADB}{AB} = \frac{\sin\angle BAD}{BD}.$$

Thus our equation becomes

$$AC \cdot (AB \cdot BC + AD \cdot CD) = BD \cdot (AB \cdot AD + BC \cdot CD).$$

Now note that $\angle KCA = \angle KDB$ and $\angle KAC = \angle KBD$, so that $\triangle KAC \sim \triangle KBD$ and $\frac{AK}{AC} = \frac{BK}{BD}$, and our equation turns into

$$AK \cdot (AB \cdot BC + AD \cdot CD) = BK \cdot (AB \cdot AD + BC \cdot CD). \quad (*)$$

Since triangles KAC and KBD are related by a spiral similarity, triangles KCD and KAB must also be similar. Then $\frac{AB}{CD} = \frac{AK}{CK} = \frac{BK}{DK}$. Dividing both sides of $(*)$ by AB and using this fact, we obtain

$$AK \cdot BC + CK \cdot AD = BK \cdot AD + DK \cdot BC$$

$$AD \cdot (BK - CK) = BC \cdot (AK - DK).$$

Since BLC and AKD are similar, we also have $BC \cdot (AK - DK) = AD \cdot (BL - CL)$. Combining this with the above equation, dividing through by AD, and rearranging terms, we have the equation that we want.

Problem 39 Show that there are infinitely many positive integers n for which the numerator of the irreducible fraction equal to $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is not a positive integer power of a prime number.

Solution: We write $S(a,b) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{b}$ for positive integers a and b. Furthermore, whenever we refer to the numerator and denominator of a fraction, we assume that the fraction is irreducible. We will show that for each odd prime p, there exists an n such that the numerator of $S(1,n)$ is divisible by p but is not a prime power.

First observe that $S(1, p^{k+1} - 1) = \frac{1}{p} \cdot S(1, p^k - 1) + S_p(1, p^{k+1})$, where the latter quantity denotes the sum of the reciprocals of the numbers between 1 and p^{k+1} that are coprime to p. We use a lemma:

Lemma. *The numerator of $S_p(1, p^{k+1})$ is divisible by p^{k+1}.*

Proof. We multiply the whole quantity by $(p^{k+1} - 1)!$ to clear denominators, then prove that the new expression is congruent to 0 modulo p^{k+1}, where reciprocals are multiplicative inverses of numbers. Then the reciprocals of the numbers between 1 and p^{k+1} coprime to p are simply the same numbers in a different order. Hence if g is a primitive root of p^{k+1},

$$S_p(1, p^{k+1}) \equiv 1 + g + g^2 + \cdots + g^{\phi(p^{k+1})-1} = \frac{g^{\phi(p^{k+1})} - 1}{g - 1}.$$

Furthermore, g is not congruent to 0 or 1 modulo p, so that the numerator is congruent to 0 modulo p^{k+1} but the denominator is not, making $S_p(1, p^{k+1}) \equiv 0 \pmod{p^{k+1}}$, as wanted. $\square$

Now observe that the numerator of $S(1, p^k - 1)$ is either always divisible by p or exactly divisible by p for some k. We use induction: For the base case, $k = 1$, we add $S(1, p-1)$ forward and backward term by term to obtain $2 \cdot S(1, p-1) = p \cdot \left(\frac{1}{1(p-1)} + \frac{1}{2(p-2)} + \cdots + \frac{1}{(p-1)1}\right)$, which clearly has a numerator divisible by p. Since p is odd, this means that the numerator of $S(1, p-1)$ is also divisible by p. Now consider the case when the numerator of $S(1, p^k - 1)$ is never exactly divisible by p, and assume that p divides the numerator of $S(1, p^m - 1)$, where $m > 1$. Observe that $S(1, p^{m+1} - 1) = \frac{1}{p} \cdot S(1, p^m - 1) + S_p(1, p^{m+1})(*)$. Since p divides the numerator of the first quantity by the assumption and the numerator of the second quantity by the lemma, this means that p must divide the numerator of $S(1, p^{m+1} - 1)$, as well. We assume that the numerator of $S(1, p^k - 1)$ is always a prime power, or else we would have found a value of n for this value of p.

Next we will show that the numerator of $S(1, p^k - 1)$ can never be exactly divisible by p (and hence be equal to p). Then $S(1, p^k - 1) = S(1, p^k - p) + S(p^k - 1, p^k - (p-1))$. We see that the numerator of the second quantity has a factor of p^2: As in the previous argument, we add the sum forward and backward to obtain

$$2 \cdot S(p^k - 1, p^k - (p-1)) = (2p^k - p) \cdot \left(\frac{1}{(p^k - 1)(p^k - (p-1))} + \cdots + \frac{1}{(p^k - (p-1))(p^k}\right.$$

dividing by p and looking at the quantity modulo p, we then have $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(p-1)^2}$. The multiplicative inverses of the residues of p are again the residues in a different order, so that the expression is congruent to $1^2 + 2^2 + \cdots + (p-1)^2 = \frac{(p-1)(p)(2p-1)}{6} \equiv 0 \pmod{p}$, as wanted. Since $S(1, p^k - p) = S(1, p^k - 1) - S(p^k - 1, p^k - (p-1))$, the numerator of $S(1, p^k - p)$ must also be exactly divisible by p. Again, we assume that the numerator of $S(1, p^k - p)$ is a prime power, so it is equal to p. Now we have $S(1,p^k-1)=\frac{p}{a}$ and $S(1,p^k-p)=\frac{p}{b}$ for some integers a and b, where $a,b<p$ ($S(1,n)>1$ for $n>1$). But then $S(p^k-1,p^k-(p-1))=p\cdot\frac{b-a}{ab}$, and the numerator clearly cannot be divisible by p^2, a contradiction.

If for some m, the number of factors of p in the numerator of $S(1,p^m-1)$ is f, where $f\le m+1$, we will have $S(1,p^{m+1}-1)$ expressed as the sum of two fractions in $(*)$, one of which has a numerator less than or equal to m and the other greater than or equal to $m+1$, by the lemma. This means that the number of factors of p in the numerator of $S(1,p^{m+1}-1)$ is also equal to f, making the numerator exactly equal to p^f.

Repeating the process, we have $S(1,p^{m+2}-1)=\frac{1}{p}\cdot S(1,p^{m+1}-1)+S_p(1,p^{m+2})$, so that $S(1,p^{m+2}-1)$ is expressed as the sum of two fractions with numerators divisible by p^{f-1} and p^{m+2}, respectively. Since $f-1\le m+2$, this means that the numerator of $S(1,p^{m+2}-1)$ is exactly divisible by p^{f-1} and therefore equal to it. Hence we may continue to reduce the power of p in the numerator of the sum in question. Eventually, we come to the equation $S(1,p^e-1)=\frac{1}{p}\cdot S(1,p^{e-1}-1)+S_p(1,p^e)$, where the numerator of $S(1,p^e-1)$ is equal to p. But by the previous argument, this leads to a contradiction.

The only remaining case is when the numerator of $S(1,p^k-1)$ is divisible by p^k for every positive integer k. Let c be the number of factors of p in the numerator of $S(1,p-1)$. If we add the sums $S(1,p-1)$ and $S(p^k-(p-1),p^k-1)$ (the second in reverse order), we obtain the fraction $p^k\cdot\left(\frac{1}{1(p^k-1)}+\frac{1}{2(p^k-2)}+\cdots+\frac{1}{(p-1)(p^k-(p-1))}\right)$, which has a numerator divisible by p^k. For $k>c$, this means that the largest power of p in the numerator of $S(p^k-1,p^k-(p-1))$ is equal to c. But then $S(1,p^k-p)=S(1,p^k-1)-S(p^k-(p-1),p^k-1)$, so that the largest power of p in the numerator of $S(1,p^k-p)$ is also equal to c, making the numerator equal to p^c (again assuming that it is a prime power). This cannot be true for every k, since the harmonic series is unbounded as k tends to infinity. Hence this last case produces a contradiction, completing the proof.

1.13 Taiwan

Problem 1 For each n, determine all n-tuples of nonnegative integers $x_1, x_2, \ldots, x_n$ such that

$$\sum_{i=1}^{n} x_i^2 = 1 + \frac{4}{4n+1}\left(\sum_{i=1}^{n} x\right)^2.$$

Solution: Without loss of generality, suppose that n-tuples are always listed in descending order.

The value $n = 2$ yields the double $(2, 1)$, and $n = 6$ yields the 6-tuple $(1, 1, 1, 1, 1, 0)$. All other values of n yield no n-tuples with the given property.

Multiplying both sides of the equation by $\frac{4n+1}{4}$ and rearranging terms yields

$$n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 = n - \frac{1}{4}\left(\sum_{i=1}^{n} x_i^2 - 1\right). \qquad (*)$$

We can rearrange the left side of this equation into a single double summation as follows.

$$\frac{1}{2}\left(n\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j + n\sum_{j=1}^{n} x_j^2\right),$$

$$\frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} 2x_i x_j + \sum_{i=1}^{n}\sum_{j=1}^{n} x_j^2\right),$$

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i^2 - 2x_i x_j + x_j^2),$$

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2.$$

If $x_1 = x_2 = \cdots = x_n$, then this is equal to 0. However, we see that this is impossible by supposing it to be true. Setting the right hand side of $(*)$ equal to 0 and supposing that all x_i are equal, we find:

$$
\begin{aligned}
0 = n - \frac{1}{4}\left(\sum_{i=1}^{n} x_i^2 - 1\right), \\
\sum_{i=1}^{n} x_i^2 - 1 = 4n, \\
nx_1^2 - 1 = 4n, \\
n(x_1^2 - 4) = 1.
\end{aligned}
$$

However, n divides the left side but not the right, and we have a contradiction. Thus the x_i cannot be equal. Suppose that k of the x_i have the same value as x_1. There will thus be at least $k(n-k)$ pairs (i, j) such that $x_i \neq x_j$, thus $(x_i - x_j)^2 \geq 1$. Because this means $i \neq j$, there must be at least $2k(n-k)$ *ordered* pairs (i, j) with this property. Clearly $2k(n-k)$ is minimized when $k = 1$, so there are at least $2(n-1)$ ordered pairs (i, j) such that $(x_i - x_j)^2 \geq 1$. From this we can conclude that $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2 \geq n - 1$. Thus:

$$
\begin{aligned}
n - \frac{1}{4}\left(\sum_{i=1}^{n} x_i^2 - 1\right) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2 \geq n - 1, \\
n - \frac{1}{4}\left(\sum_{i=1}^{n} x_i^2 - 1\right) \geq n - 1, \\
1 \geq \frac{1}{4}\left(\sum_{i=1}^{n} x_i^2 - 1\right), \\
5 \geq \sum_{i=1}^{n} x_i^2.
\end{aligned}
$$

Notice also that the right side of $(*)$ is an integer if and only if $\sum_{i=1}^{n} x_i^2 \equiv 1 (mod 4)$. Thus we must have $\sum_{i=1}^{n} x_i^2 = 1$ or 5. In the former case, the n-tuple would have to consist of 1 and some number of 0s. In this case, the given equation would lead to $0 = 4$, which is a contradiction, so some of these n-tuples are valid. The only n-tuples such that $\sum_{i=1}^{n} x_i^2 = 5$ are $(2, 1)$, $(1, 1, 1, 1, 1)$, and either of these with some number of 0s appended. Plugging in the appropriate values for the summations and solving for n to determine the number of 0s that must be appended, we find that $(2,1)$ and $(1,1,1,1,1,0)$ are the only valid n-tuples for any n.

Problem 2 We call a lattice point X in the plane *visible* from the origin O if the segment $\overline{OX}$ does not contain any other lattice points besides O and X. Show that for any positive integer n, there exists an square of n^2 lattice points (with sides parallel to the coordinate axes) such that none of the lattice points inside the square is visible from the origin.

Solution: Suppose that the lower-left lattice point of such a square has coordinates (x_1, y_1). We shall show that it is possible to select (x_1, y_1) such that the square of lattice points with (x_1, y_1) at it's corner and n points on a side contains only invisible points. This can be accomplished by ensuring that each point has both coordinates divisible by some prime number; this would imply that by dividing both coordinates by this prime we could find another lattice point that is between the origin and this point.

Select n^2 distinct prime numbers and call them $p_{i,j}|1 \le i, j \le n$. Now find x_1 satisfying the following congruences:

$$x_1 \equiv 0 (mod p_{1,1} p_{1,2} \dots p_{1,n}),$$
$$x_1 + 1 \equiv 0 (mod p_{2,1} p_{2,2} \dots p_{2,n}),$$
$$\dots$$
$$x_1 + n - 1 \equiv 0 (mod p_{n,1} p_{n,2} \dots p_{n,n}).$$

Likewise select y_1 satisfying:

$$y_1 \equiv 0 (mod p_{1,1} p_{2,1} \dots p_{n,1}),$$
$$y_1 + 1 \equiv 0 (mod p_{1,2} p_{2,2} \dots p_{n,2}),$$
$$\dots$$
$$y_1 + n - 1 \equiv 0 (mod p_{1,n} p_{2,n} \dots p_{n,n}).$$

Both values must exist by the Chinese Remainder Theorem. Thus we have demonstrated that it is possible to determine a position for (x_1, y_1) such that every point in the square of n^2 lattice points with (x_1, y_1) at it's lower left corner is associated with some prime by which both of its coordinates are divisible, thus all points in this square are not visible from the origin.

Problem 3 Let $x, y, z, a, b, c, d, e, f$ be real numbers satisfying

$$\max\{a, 0\} + \max\{b, 0\} < x + ay + bz < 1 + \min\{a, 0\} + \min\{b, 0\},$$
$$\max\{c, 0\} + \max\{d, 0\} < cx + y + dz < 1 + \min\{c, 0\} + \min\{d, 0\},$$
$$\max\{e, 0\} + \max\{f, 0\} < ex + fy + z < 1 + \min\{e, 0\} + \min\{f, 0\}.$$

Show that $0 < x, y, z < 1$.

Solution: Define $m = \min\{x, 1 - x, y, 1 - y, z, 1 - z\}$. We wish to show that $m > 0$, which will in turn imply that $0 < x, y, z < 1$.

Observe that, for any (k, h) if $k > h$, $k + h + |k - h| = 2k$, and if $k \leq h$, $k + h + |k - h| = 2h$. Thus, we can conclude that $\max\{k, h\} = (k + h + |k - h|)/2$, and similar logic shows that $\min\{k, h\} = (k + h - |k - h|)/2$.

Using this last observation, and the fact that $-|x| \leq x \leq |x|$ for any x, notice that:

$$|a|min\{y, 1 - y\} = |a|\frac{1 - |2y - 1|}{2} = \frac{|a| - |a(2y - 1)|}{2}$$
$$\leq \frac{|a| - a(2y - 1)}{2} = \frac{|a| + a}{2} - ay = \max\{a, 0\} - ay,$$

$$|a|min\{y, 1 - y\} = \frac{|a| - |a(2y - 1)|}{2}$$
$$\leq \frac{|a| + a(2y - 1)}{2} = \frac{|a| - a}{2} + ay = ay - \min\{a, 0\}.$$

Because this does not depend on any given conditions, the same result holds for any pair of variables.

Combining this with the first given equation, we see that:

$$\begin{aligned}
x + ay + bz &> \max\{a, 0\} + \max\{b, 0\}, \\
x &> \max\{a, 0\} - ay + \max\{b, 0\} - bz, \\
x &> |a| \min\{y, 1 - y\} + |b| + |b| \min\{z, 1 - z\}, \\
x &> (|a| + |b|)m,
\end{aligned}$$

$$\begin{aligned}
1 + \min\{a, 0\} + \min\{b, 0\} &> x + ay + bz, \\
1 - x &> ay - \min\{a, 0\} + bz - \min\{b, 0\}, \\
1 - x &> |a| \min\{y, 1 - y\} + |b| + |b| \min\{z, 1 - z\}, \\
1 - x &> (|a| + |b|)m.
\end{aligned}$$

Combining this with similar results for y and z we see that

$$\begin{aligned}
x, 1 - x &> (|a| + |b|)m, \\
y, 1 - y &> (|c| + |d|)m, \\
z, 1 - z &> (|e| + |f|)m.
\end{aligned}$$

Notice that, from the first given inequality:

$$\begin{aligned}
\max\{a, 0\} + \max\{b, 0\} &< 1 + \min\{a, 0\} + \min\{b, 0\}, \\
\max\{a, 0\} - \min\{a, 0\} + \max\{b, 0\} - \min\{b, -\} &< 1, \\
|a| + |b| &< 1.
\end{aligned}$$

Suppose for contradiction that $m \leq 0$. This would imply that $(|a| + |b|)m > m$ and similarly $(|c| + |d|)m > m$ and $(|e| + |f|)m > m$, thus $x, 1 - x, y, 1 - y, z, 1 - z > m$. However, this is clearly a contradiction since at least one of these values is equal to m. Thus $m > 0$ and therefore $0 < x, y, z < 1$.

Problem 4 Suppose that $0 < x_1, x_2, x_3, x_4 \leq \frac{1}{2}$. Prove that

$$\frac{x_1 x_2 x_3 x_4}{(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4)}$$

is less than or equal to

$$\frac{x_1^4 + x_2^4 + x_3^4 + x_4^4}{(1-x_1)^4 + (1-x_2)^4 + (1-x_3)^4 + (1-x_4)^4}.$$

Problem 5 The 2002 real numbers $a_1, a_2, \ldots, a_{2002}$ satisfy

$$\begin{aligned}
\frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_{2002}}{2003} &= \frac{4}{3},\\
\frac{a_1}{3} + \frac{a_2}{4} + \cdots + \frac{a_{2002}}{2004} &= \frac{4}{5},\\
&\ \vdots\\
\frac{a_1}{2003} + \frac{a_2}{2004} + \cdots + \frac{a_{2002}}{4004} &= \frac{4}{4005}.
\end{aligned}$$

Evaluate

$$\frac{a_1}{3} + \frac{a_2}{5} + \frac{a_3}{7} + \cdots + \frac{a_{2002}}{4005}.$$

Problem 6 Given three fixed points A, B, C in a plane, let D be a variable point different from A, B, C such that A, B, C, D are concylic. Let ℓ_A be the Simson line of A with respect to triangle BCD, and define ℓ_B, ℓ_C, ℓ_D analogously. (It is well known that if W is a point on the circumcircle of triangle XYZ, then the feet of the perpendiculars from W to lines XY, YZ, ZX lie on a single line. This line is called the *Simson line* of W with respect to triangle BCD.) As D varies, find the locus of all possible intersections of some two of $\ell_A, \ell_B, \ell_C, \ell_D$.

1.14 Vietnam

Problem 1 Let ABC be a triangle such that angle BCA is acute. Let the perpendicular bisector of side $\overline{BC}$ intersect the rays that trisect angle BAC at K and L, so that $\angle BAK = \angle KAL = \angle LAC = \frac{1}{3}\angle BAC$. Also let M be the midpoint of side $\overline{BC}$, and let N be the foot of the perpendicular from A to line BC. Find all such triangles ABC for which $AB = KL = 2MN$.

There are no such triangles.

We first prove that $\angle BCA = 2\angle BAC$. Let A', N', be the reflections of A and N in the perpendicular bisector of $\overline{BC}$. Note that this reflection fixes both the circumcircle of ABC, call it ω and BC, and thus A' lies on ω and N' on BC. Also, $NN' = NM+MN' = 2MN = AB$.

Now $\overline{AN}$ and $\overline{A'N'}$ are both perpendicular to line BC (the first by definition, and the second being the reflection of the first in the perpendicular bisector of $\overline{BC}$), $\overline{AA'}$ is parallel to line BC (because it is perpendicular to the perpendicular bisector of $\overline{BC}$) and $\overline{NN'}$ lies on line BC. This means that quadrilateral $AA'N'N$ is a rectangle. Hence $AA' = NN' = 2MN = AB$, and triangle $AA'B$ is isosceles.

We now use the fact that quadrilateral $A'ABC$ is cyclic and obtain

$$\begin{aligned} 2\angle ACB &= 2\angle AA'B \\ &= \angle A'AB + \angle A'BA \\ &= (180^\circ - \angle A'AB) \\ &= \angle A'CB \\ &= \angle ABC, \end{aligned}$$

the final step holding because triangles $A'CB$ and ABC are congruent by reflection. At this point, the problem becomes nearly identical to USA TSE 2001 problem 5, and the assumption that angle BCA is acute implies that $AB < KL$, meaning that there are no such triangles.

Problem 2 A positive integer is written on a board. Two players alternate performing the following operation until 0 appears on the board: the current player erases the existing number N from the board and replaces it with either $N-1$ or $\lfloor N/3 \rfloor$. Whoever writes the number 0 on the board first wins. Determine who has the winning strategy when the initial number equals (a) 120, (b) $(3^{2002}-1)/2$, and (c) $(3^{2002}+1)/2$.

The answers to the three parts are a) the second player, b) the first player, and c) the second player.

Define an integer n to be *winning* if when n is written on the board, the player to move has a winning strategy. If not, n is losing. Now, n is winning if and only if the player to move can make a move that puts the second player in a position without a winning strategy. Hence n is winning, if and only if at least one of the first player's options, $n-1$ and $\lfloor \frac{n}{3} \rfloor$ is losing. Conversely, n is losing if and only if both n and $\lfloor \frac{n}{3} \rfloor$ are winning (or n = 0, which implies that the player last to move has just won and there is nothing to be done).

We will solve this problem by induction upon a general result.

Lemma. *For all $n \geq 2$,*

- i) $\frac{3^n-5}{2}$ is winning.
- ii) $\frac{3^n-3}{2}$ is losing.
- iii) $\frac{3^n-1}{2}$ is winning.
- iv) $\frac{3^n+1}{2}$ is losing.

Proof. By induction.

Base Case: $n = 2$. First of all, we note that 1 and 2 are trivially winning: in either case the first player has only to move to $\lfloor \frac{1}{3} \rfloor = \lfloor \frac{2}{3} \rfloor = 0$. Now $2 = \frac{3^2-5}{2}$ and so i) holds. However, 3 is losing, for the player to move has only the options $3 - 1 = 2$ and $\lfloor \frac{3}{3} \rfloor = 1$ both of which are winning. This shows part ii), because $3 = \frac{3^2-3}{2}$.

In addition, $4 = \frac{3^2-1}{2}$ is winning because the first player can move from 4 to the losing number 3, and iii) is proved. As for iv), from $5 = \frac{3^2+1}{2}$ one can only move to the winning numbers 4 and 1, making 5 losing. This establishes our base case.

Inductive Step. Suppose that i) – iv) all hold for $n = k$. We must show they hold in addition that they hold for $n = k+1$. We establish our three propositions in order.

- i) From the number $\frac{3^{k+1}-5}{2}$, the player can use the second option to move to $\lfloor \frac{3^{k+1}-5}{6} \rfloor = \frac{3^k-3}{2}$. By part ii) of the induction hypothesis, that option is losing. Hence the original number $\frac{3^{k+1}-5}{2}$ is winning.
- ii) We must show that the player's two options from the number $\frac{3^{k+1}-3}{2}$ both give the other player a winning number. These

options are $\frac{3^{k+1}-3}{2} - 1 = \frac{3^{k+1}-5}{2}$ which was just shown to be winning, and $\lfloor \frac{3^{k+1}-3}{6} \rfloor = \frac{3^k-1}{2}$ which is winning by part iii) of the inductive hypothesis.

- iii) $\frac{3^{k+1}-1}{2}$ is winning. From $\frac{3^{k+1}-1}{2}$ the player to move can subtract 1 to produce $\frac{3^{k+1}-3}{2}$ which is a losing position. This establishes that $\frac{3^{k+1}-1}{2}$ is winning.
- iv) $\frac{3^{k+1}+1}{2}$ is losing. We again need to show that its two options are both winning. Now, $\frac{3^{k+1}+1}{2} - 1 = \frac{3^{k+1}-1}{2}$ was proved winning in the previous case, and $\lfloor \frac{3^{k+1}+1}{6} \rfloor = \frac{3^k-1}{2}$ which is winning by part iii) of the inductive hypothesis.

Our induction has been established, and so all four claims are true for all integer $n \geq 2$. □

We now apply the above result to the problem at hard. For part a), we note that $120 = \frac{3^5-3}{2}$, and so by ii) it is losing, that is to say, the second player to move wins. Letting $n = 2002$ in parts iii) and iv) shows that the first player wins in b) and the second in c).

Problem 3 The positive integer m has a prime divisor greater than $\sqrt{2m}+1$. Find the smallest positive integer M such that there exists a finite set T of distinct positive integers satisfying: (i) m and M are the least and greatest elements, respectively, in T, and (ii) the product of all the numbers in T is a perfect square.

Write $m = kp$ where $p > \sqrt{2m} + 1$ is prime. We claim that M $=$ $(k+1)p$. First we show that $M \geq (k+1)p$. For if not, m would be the only multiple of p in T, and the product of all elements in T could not be a square being divisible by p but not p^2.

We now give a construction for $M = (k+1)p$. Now,

$$2k < \frac{2m}{\sqrt{2m}+1} < \sqrt{2m} < p - 1.$$

Now, $p > 2k + 1 > 3$ is an odd prime, making $\frac{p+1}{2}$ and $\frac{p-1}{2}$ both integral.

Repeatedly using the key inequality $2k < p-1$ (which implies that

$k < \frac{p-1}{2} < p - 1 < p$) we find in turn that

$$
\begin{aligned}
m &= kp < k(p+1) = kp + k < kp + p = M \\
m &= kp < kp + p - 1 + k = (k+1)(p-1) < (k+1)p = M \\
m &= kp < kp - k + \frac{p}{2} - 1 < (2k+1)\frac{p-1}{2} < (k+1)p = M \\
m &= kp < (2k+1)\frac{p+1}{2} = kp + k + \frac{p+1}{2} < kp + \frac{p-1}{2} + \frac{p+1}{2} = M.
\end{aligned}
$$

Hence all of the quantities $k(p+1)$, $(k+1)(p-1)$, $(2k+1)\frac{p-1}{2}$, $(2k+1)\frac{p+1}{2}$ are integers strictly between m and M. Let A be the multiset $\{m, M, k(p+1), (k+1)(p-1), (2k+1)\frac{p-1}{2}, (2k+1)\frac{p+1}{2}\}$ which may contain repeated elements. We claim that the product of all the elements of A (with repetitions counted multiple times) is a square. For that product equals

$$
\begin{aligned}
&kp(k+1)pk(p+1)(k+1)(p-1)(2k+1)\frac{p-1}{2}(2k+1)\frac{p-1}{2} \\
&\qquad\qquad = \frac{kp(k+1)(p+1)(p-1)(2k+1)}{2}^2
\end{aligned}
$$

If A has no repetitions, A is the desired set T. If it does, none of its repeated elements can equal m or M for all other elements of A are strictly between those two. Hence if x appears more than once in A, both repetitions of x may be removed from A to form a new multiset A' with two fewer elements whose product is still a square. This can be done until all repetitions are removed to produce such a set T. Thus the lower bound $M = p(k+1)$ proved above can indeed be attained.

Problem 4 On an $n \times 2n$ rectangular grid of squares ($n \geq 2$) are marked n^2 of the $2n^2$ squares. Prove that for each $k = 2, 3, \ldots, \lfloor n/2 \rfloor + 1$, there exists k rows of the board and

$$\lceil \frac{k!(n-2k+2)}{(n-k+1)(n-k+2)\cdots(n-1)} \rceil$$

columns, such that the intersection of each chosen row and each chosen column is a marked square.

Label the $2n$ columns of the grid $C_1, C_2 \ldots C_{2n}$. Let a_j, $1 \leq j \leq 2n$ be the number of marked squares in the jth column. Then the number of sets of k squares on the grid that are all in the same column is

$$\sum_{1\le j\le 2n} \binom{C_j}{k}. \tag{$*$}$$

In order to bound the quantity in $(*)$ using Jensen's inequality, we consider the function $f_k : \mathbb{R} \to \mathbb{R}$ defined piecewise by $f_k(x) = 0$ for $x \le k-1$ and $f_k(x) = \binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k}$ for $x \ge k-1$. This function is important because $f_k(m) = \binom{m}{k}$ for any nonnegative integer m. In addition, it is continuous everywhere and differentiable except at $x = k-1$. We show f_k is (weakly) convex by showing that f_k', where defined, is nondecreasing. This is clear on the interval $(-\infty, k-1)$ where it is identically 0. For $(k-1, +\infty)$ we prove the equivalent statement that $g_k'(x)$ is increasing on $(0,\infty)$ where $g_k(x) = f_k(x+k-1) = \frac{x(x+1)\cdots(x+k+1)}{(k+1)!}$. Indeed $g_k(x)$ has all coefficents positive, as does $g_k'(x)$, so the latter is positive increasing on $(0,+\infty)$. Hence $f_k'(x)$ is positive and increasing on $(k-1,+\infty)$. Putting the pieces together shows that $f_k(x)$ is a convex function.

We now apply this result to bound the quantity in $(*)$. Because all the C_j are nonnegative integers, we may convert the binomial coefficients into f_k's, and apply Jensen's inequality:

$$\begin{aligned}
\sum_{1\le j\le 2n} \binom{C_j}{k} &= \sum_{1\le j\le 2n} f_k(j) \\
&\ge 2nf_k\left(\frac{1}{2n}\left(\sum_{1\le j\le 2n} C_j\right)\right) \\
&= 2nf_k\left(\frac{n}{2}\right) \\
&= 2n\binom{\frac{n}{2}}{k}.
\end{aligned}$$

because $\sum_{1\le j\le 2n} C_j$ is the total number of marked squares, or n^2. This establishes that there are at least $2n\binom{\frac{n}{2}}{k}$ subsets containing k marked squares all in one column.

We now wish to use the pigeonhole principle to find an M for which some union of k rows must contains M distinct such sets of k marked squares all in a column, or equivalently that the intersection of some k rows and M columns contains only marked squares. There are n rows in total, hence there are $\binom{n}{k}$ sets containing k rows. Each of the at least $2n\binom{\frac{n}{2}}{k}$ sets of k marked squares all in a column is contained in exactly one set of k rows. By the pigeonhole principle, some set of k rows must contain at least

$$M = \left\lceil \frac{2n\binom{\frac{n}{2}}{k}}{\binom{n}{k}} \right\rceil$$

distinct sets of k marked squares all in a column.

This shows that there exist k rows and M columns whose intersection contains only marked squares. However, our bound M is better than the bound the problem requires, which is

$$N = \left\lceil \frac{k!(n-2k+2)}{(n-k+1)(n-k+2)\cdots(n-1)} = \frac{n(n-2k+2)}{\binom{n}{k}} \right\rceil .$$

However,

$$\binom{\frac{n}{2}}{k} = \left(\frac{n-2k+2}{2}\right)\left(\frac{n-2k+4}{4}\right)\cdots\left(\frac{n}{2k}\right) \geq \frac{n-2k+2}{2}$$

for all the other terms in the product are ≥ 1 because $k \leq \frac{n}{2}+1$. Making use of this result,

$$\begin{aligned} M &= \left\lceil \frac{2n\binom{\frac{n}{2}}{k}}{\binom{n}{k}} \right\rceil \\ &\geq \left\lceil \frac{2n\frac{n-2k+2}{2}}{\binom{n}{k}} \right\rceil \\ &= \left\lceil \frac{n(n-2k+2)}{\binom{n}{k}} \right\rceil = N \end{aligned}$$

We have shown there are k rows and M columns whose intersection contains only marked squares. Therefore there are definitely k rows and $N \leq M$ columns whose intersection contains only marked squares.

Problem 5 Find all polynomials $p(x)$ with integer coefficients such that

$$q(x) = (x^2+6x+10)(p(x))^2 - 1$$

is the square of a polynomial with integer coefficients.

Define the sequence $\{A_n(y)\}$ of polynomials in y recursively by $A_0(x) = 0$, $A_1(x) = y$ and for $n \geq 2$, $A_n(y) = (4y^2+2)A_{n-1}(y) -$

$A_{n-2}(y)$. We claim that the required polynomials $p(x)$ are exactly those of the form $p(x) = \pm\frac{A_{n+1}(x+3)-A_n(x+3)}{2(x+3)}$.

In order to simplify the calculations, we make the substitution $y = x+3$, so that there are integer polynomials $r(y)$, $s(y)$ in y such that $p(x) = r(x+3)$, and $q(x) = (s(x+3))^2$. Then because $x^2+6x+10 = (x+3)^2+1$ our equation turns into

$$(s(y))^2 = (y^2+1)(r(y))^2 - 1. \tag{1}$$

Now, let $u(y) = r(y) + ys(y)$ and $v(y) = r(y) - ys(y)$, so that $r(y) = \frac{u(y)+v(y)}{2}$ and $s(y) = \frac{u(y)-v(y)}{2y}$. Then we can rewrite (1) as

$$\left(\frac{(u(y)+v(y))}{2}\right) = (y^2+1)\left(\frac{(u(y)-v(y)}{2y}\right) - 1. \tag{2}$$

After multiplying through by $4y^2$ and expanding, (2) becomes

$$(u(y))^2 + (v(y))^2 - (4y^2+2)u(y)v(y) - 4y^2 = 0. \tag{3}$$

We claim that if $u(y)$, $v(y)$ are polynomials in y that satisfy (3), then the set $\{u(y), v(y)\}$ must be either $\{A_n, A_{n+1}\}$ or $\{-A_n, -A_{n+1}\}$. We proceed by induction on $\max(\deg(u), \deg(v))$.

Base Case: $\max(\deg(u), \deg(v)) \le 1$. That is to say, $u(y)$ and $v(y)$ are both at most linear. This means that,

$$\begin{aligned} 0 &= (u(y))^2 + (v(y))^2 - (4y^2+2)u(y)v(y) - 4y^2 \\ &= -4y^2u(y)v(y) + Q(y) \end{aligned}$$

where $Q(y) = u((y))^2 + v((y))^2 - 2u(y)v(y) - 4y^2$ is at most a quadratic. Therefore $4y^2u(y)v(y)$ cannot have degree > 2. This can only happen if both of $u(y)$, $v(y)$ are constant or at least one is zero.

If both are constant – $u(y) = U$ and $v(y) = V$ for constants U and V, then by (3) $U^2 + V^2 - 2UV - (4+4UV)y^2 = 0$. Equating coefficients, $0 = U^2+V^2-2UV = (U-V)^2$ and $8UV = 4$. The first equation implies that $U = V$ and the second that $-1 = UV = U^2$, giving no integer solutions for U and V. We are left with the possibility that one of $u(y)$, $v(y)$ is zero. Without loss of generality we may let $u(y) = 0 = A_0(y)$. Then by (3), $(v(y))^2 - 4y^2 = 0$. This has the solutions $v(y) = \pm y = \pm A_1(y)$, and the base case holds. In addition, the only solution to (3) with one of $u(y)$, $v(y)$ zero is the one found above, and so we can assume in our inductive step that $u(y)$, $v(y)$ are nonzero.

Inductive Step: Assume that the claim holds for all $u(y)$, $v(y)$ with $\max(\deg(u), \deg(v)) < k$. Let $u(y)$. $v(y)$ be polynomials satisfying (3) with $\max(\deg(u), \deg(v)) = k \geq 2$. We count degrees in order to establish an infinite descent. Consider the left hand side of (3). It is composed of four terms whose sum is zero, and so among those four terms there cannot be a unique term of highest degree. Without loss of generality, assume $\deg(u) = k$, so $\deg(v) = j \leq k$. Then $\deg((u(y))^2) = 2k$, $\deg((v(y))^2) = 2j$, $\deg(-(4y^2+2)u(y)v(y)) = 2 + j + k$, and $\deg(-4y^2) = 2$. Then $\deg(-4y^2) = 2 < 2k = \deg((u(y))^2)$ so $-4y^2$ cannot be a term of highest degree. In addition $\deg((v(y)^2)) = 2j < 2 + j + k \deg(-(4y^2+2)u(y)v(y))$ cannot be a term of highest degree either. Thus $2k = \deg((u(y))^2) = \deg(-(4y^2+2)u(y)v(y)) = 2 + j + k$, or $j = k - 2$.

Now, we rewrite (3) as a quadratic in $u(y)$:

$$(u(y))^2 - ((4y^2+2)v(y))u(y) + (v(y))^2 - 4y^2 = 0$$

Thus the quadratic $z^2 - ((4y^2+2)v(y))z + (v(y))^2 - 4y^2$ has the root $z = u(y)$. Hence it also has another root, call it $z = w(y)$. Then $w(y) = (4y^2+2)v(y) - u(y) = \frac{(v(y))^2 - 4y^2}{u(y)}$, and $w(y)$ has integer coefficients. We deduce from the relactionship between the coefficients of the quadratic and its roots that $w(y) = \frac{(v(y))^2 - 4y^2}{u(y)}$ that $\deg w = 2(k-2) - k = k - 4$. Thus $\max \deg v, \deg w < n$ and by the definition of $w(y)$, $(w(y))^2 + (v(y))^2 - (4y^2+2)w(y)v(y) - 4y^2 = 0$. Hence by the induction hypothesis, $\{v(y), w(y)\} = \{A_n, A_{n+1}\}$ or $\{-A_n, -A_{n+1}\}$. Because $v(y)$ has the greater degree and $\deg A_n$ is strictly increasing (by induction), it must be that $v(y) = \pm A_{n+1}$.

Without loss of generality we may assume $v(y) = +A_{n+1}$. Then

$$u(y) = (4y^2+2)v(y) - w(y) = (4y^2+2)A_{n+1} - A_n = A_{n+2},$$

similarly if $v(y) = -An + 1$, $u(y) = -A_{n+2}$. This proves that the inductive step works.

Hence by induction all solutions to (3) are of the form $\{A_n, A_{n+1}\}$ or $\{-A_n, -A_{n+1}\}$. We have also shown that if $\{u(y), v(y)\}$ solves (3), $\{(4y^2+2)v(y), u(y)\}$ does too. Because $\{A_0, A_1\}$ and its negative are solutions to (3), by an induction argument using the recurrence for A_n, $\{u(y), v(y)\} = \{A_n, A_{n+1}\}$ or $\{-A_n, -A_{n+1}\}$ are all solutions to (3) as well, and we have just shown that they are the only solutions.

Retracing our steps, we find that any solution to the original equation must be of the form $p(x) = \pm\frac{A_{n+1}(x+3) - A_n(x+3)}{2(x+3)}$ and $q(x) =$

$\left(\frac{A_{n+1}(x+3)+A_n(x_3)}{2}\right)^2$. If we can show that for any n, the values $p(x)$ defined by those polynomials are integer polynomials, we will have proved that those values of $p(x)$ are exactly those integer polynomials that make $q(x)$ a square.

Essentially, we must prove that $2y \mid A_n(y) - A_{n+1}(y)$ and $2 \mid A_n(y) + A_{n-1}(y)$. But we know that $u(y) = A_n(y)$, $v(y) = A_{n+1}(y)$ is a solution to (3), so

$$\begin{aligned} 0 &= (A_n(y))^2 + (A_{n+1}(y))^2 - (4y^2+2)A_n(y)A_{n+1}(y) - 4y^2 \\ &= (A_n(y) - A_{n+1}(y))^2 - 4y^2(A_n(y)A_{n+1}(y) - 1). \end{aligned}$$

This means that $4y^2 \mid (A_n(y) - A_{n+1}(y))^2$ and $2y \mid A_n(y) - A_{n+1}(y)$. Thus also $2 \mid A_n(y) - A_{n+1}(y) + 2(A_{n+1}(y)) = A_n(y) + A_{n-1}(y)$. This establishes that $p(x) = \pm\frac{A_{n+1}(x+3)-A_n(x+3)}{2(x+3)}$ is always an integer polynomial, and that those polynomials $p(x)$ are exactly the values of $p(x)$ that make $q(x)$ a square.

Problem 6 Prove that there exists an integer $m \geq 2002$ and m distinct positive integers $a_1, a_2, \ldots, a_m$ such that

$$\prod_{i=1}^{m} a_i^2 - 4\sum_{i=1}^{m} a_i^2$$

is a perfect square.

We proceed by solving the problem with the condition of distinctness relaxed, and then proceed to make the a_i distinct, one by one.

For if we need not have the a_i distinct, we first choose a_1, a_2 to be odd and large enough so that $a_1^2a_2^2 - 4(a_1^2 + a_2^2) > 4 \cdot 2000$. (This can be done because $a_1^2a_2^2 - 4(a_1^2 + a_2^2) = (a_1^2 - 4)(a_2^2 - 4) - 16$.) Because a_1^2, a_2^2 are both $1 \mod 4$, $a_1^2a_2^2 - 4(a_1^2 + a_2^2) = 4k + 1$ for some $k \geq 2000$. Let $m = k + 2 \geq 2002$ and let $a_3, a_4, \ldots, a_m$ all equal 1.

Then

$$\prod_{i=1}^{m} a_i^2 - 4\sum_{i=1}^{m} a_i^2 = a_1^2a_2^2 - 4a_1^2 - 4a_2^2 - 4k = 1$$

is square.

We now must modify our sequence to make all the terms distinct. To do so, we induct upon the following claim.

Claim: For any integer $1 \leq j \leq m$, there exists a sequence $a_1, a_2, \ldots, a_m$ of positive integers satisfying

$$\prod_{i=1}^{m} a_i^2 - 4\sum_{i=1}^{m} a_i^2 = b^2 \tag{$*$}$$

for some integer b and such that $a_i < a_{i+1}$ for all i with $1 \leq i < j$.

Proof of Claim: By induction.

Base Case: $j = 1$. In this case, the condition $a_i < a_{i+1}$ for all $1 \leq i < j$ holds vacuously. We have already constructed a sequence $a_1, a_2, \ldots, a_n$ which satisfies all other conditions. Thus the base case is established.

Inductive Step: Suppose there exists such a sequence $a_1, a_2, \ldots a_n$ of positive integers satisfying $(*)$ such that $a_i < a_{i+1}$ for all i with $1 \leq i < j$. We must show there is such a sequence $a'_1, a'_2, \ldots, a'_n$ with $a'_i < a'_{i+1}$ for all i with $1 \leq i < j+1$.

Consider the Diophantine equation

$$\left(\prod_{i \neq j+1} a_i^2 - 4\right) p^2 - 4\sum_{i \neq j+1} a_i^2 = q^2$$

with p, q positive integers. This is a Pell equation of the form $Ap^2 - B = q^2$. A basic result in the theory of Pell equations states that if such an equation has a solution in positive integers, it has infinitely many solutions in positive integers. However, by $(*)$, the above equation has the solution $p = a_{j+1}$, $q = b$. Hence it has infinitely many positive integer solutions, namely, it has a positive integer solution with $p > a_j$. Let that solution be $p = a'_{j+1} > a_j$, $q = b'$.

We claim that the sequence $a'_1, a'_2, \ldots, a'_n$ with a'_{j+1} defined as above and $a'_i = a_i$, $i \neq j+1$ is our desired sequence. For by the induction hypothesis $a'_i \leq a'_{i+1}$ for $1 \leq i < j$ and $a'_j < a'_{j+1}$ by construction. To show that $(*)$ holds with $b = b'$, we use the definition of a'_{j+1}, b' to show

$$\prod_{i=1}^{m} {a'_i}^2 - 4\sum_{i=1}^{m} {a'_i}^2 = \left(\prod_{i \neq j+1} a_i^2 - 4\right) a'_{j+1} - 4\sum_{i \neq j+1} a_i^2 = b'^2$$

This shows that if our induction hypothesis holds for j, it holds in addition for $j+1$. Hence by mathematical induction it is true for all positive integer $j \leq m$. If we let $j = m$, then our sequence $a_1, a_2, \ldots, a_n$ has all terms distinct and satisfies $(*)$, as desired.

2
2002 Regional Contests: Problems and Solutions

2.1 Asian Pacific Mathematics Olympia d

Problem 1 Let $a_1, a_2, \ldots, a_n$ be a sequence of non-negative integers, where n is a positive integer. Let

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Prove that

$$a_1!a_2! \cdots a_n! \geq \left(\lfloor A_n \rfloor !\right)^n,$$

and determine when equality holds. (Here, $\lfloor A_n \rfloor$ denotes the greatest integer less than or equal to A_n, $a! = 1 \times 2 \times \cdots \times a$ for $a \geq 1$, and $0! = 1$.)

Solution: Let's prove the following statement.

Lemma. *Let a, b be nonnegative integers such that $a - b > 1$. Then $a!b! > (a-1)!(b+1)!$.*

Proof. Dividing both sides by $(a-1)!b!$, we obtain an equivalent inequality $a > b + 1$ that is obvious. □

The number of sequences $a_1, a_2, \ldots, a_n$ of nonnegative integers with constant sum is finite. Hence, there exists such sequence $b_1, b_2, \ldots, b_n$ with minimal $b_1!b_2! \cdots b_n!$ over all such sequences. It is sufficient to solve the problem only for that sequence.

Let's prove that any two of b_i differs on at most 1. Otherwise there exist $i \neq j$ such that $b_i - b_j > 1$. Let's substitute b_i and b_j in the sequence by $b_i - 1$ and $b_j + 1$. Then the sum of numbers will not change but, as a consequence of Lemma, the product of factorials of the numbers will decrease. It makes a contradiction with our choice of the numbers.

Therefore, all numbers in the sequence take at most two values, say x and $x + 1$ for some nonnegative integer x. Let $k < n$ of them equal to $x + 1$ and other $n - k$ equal to x. We have

$$\left\lfloor \frac{b_1 + b_2 + \cdots + b_n}{n} \right\rfloor = \left\lfloor \frac{k(x+1) + (n-k)x}{n} \right\rfloor = \left\lfloor \frac{nx + k}{n} \right\rfloor = x.$$

It makes the inequality obvious: $b_1!b_2! \cdots b_n! \geq (x!)^n$ because each of b_i is at least x.

Problem 2 Find all positive integers a and b such that

$$\frac{a^2+b}{b^2-a} \quad \text{and} \quad \frac{b^2+a}{a^2-b}$$

are both integers.

Solution: The answers are $(2,2), (3,3), (1,2), (2,3), (2,1)$ and $(3,2)$

Consider the following cases.

First case: $a = b$. In this case $\frac{a^2+a}{a^2-a} = \frac{a+1}{a-1}$ is an integer. It implies that a-1 divides a+1 so

$$(a-1) \mid \gcd(a+1, a-1) = \gcd(a-1, a+1-(a-1)) = \gcd(a-1, 2) \mid 2.$$

Hence, $a-1$ is either 1 or 2 because a is a positive integer. If $a-1 = 1$, then $a = 2$ and $\frac{a+1}{a-1} = 3$ is an integer. If $a - 1 = 2$, then $a = 3$ and $\frac{a+1}{a-1} = 2$ is an integer too. So we have only two answers $(2, 2)$ and $(3, 3)$ in this case.

Second case: $a > b$. Look at $\frac{b^2+a}{a^2-b}$. It is an integer and it is positive because $a^2 > b$. So $\frac{b^2+a}{a^2-b} \geq 1$ and $b^2 + a \geq a^2 - b$ that implies $a + b \geq a^2 - b^2 = (a - b)(a + b)$ that holds if and only if $a - b = 1$. So it is sufficient to find all such positive integers b, that $\frac{a^2+b}{b^2-a} = \frac{(b+1)^2+b}{b^2-(b+1)} = \frac{b^2+3b+1}{b^2-b-1} = 1 + \frac{4b+2}{b^2-b-1}$ is an integer so $\frac{4b+2}{b^2-b-1}$ is an integer too. If $b \geq 6$, then $b^2 \geq 6b = 5b + b > 5b + 3$. Therefore, $b^2 - b - 1 > 4b + 3$ and $\frac{4b+2}{b^2-b-1}$ can not be an integer. Now we should check all b less than 6. For $b = 1, 2, 3, 4$ and 5 $\frac{4b+2}{b^2-b-1}$ equals to $-6, 10, \frac{12}{5}, \frac{18}{11}$ and $\frac{22}{19}$ respectively. So the only answers in this case are $(1, 2)$ and $(2, 3)$.

The third case $a < b$ is similar to second one with transposition of a and b, and the answers are $(2, 1)$ and $(3, 2)$.

Problem 3 Let ABC be an equilateral triangle. Let P be a point on side $\overline{AC}$ and let Q be a point on side $\overline{AB}$ so that both triangles ABP and ACQ are acute. Let R be the orthocenter of triangle ABP and let S be the orthocenter of triangle ACQ. Let T be the intersection of segments $\overline{BP}$ and $\overline{CQ}$. Find all possible values of $\angle CBP$ and $\angle BCQ$ such that triangle TRS is equilateral.

Solution: Angles $\angle CBP$ and $\angle BCQ$ are both equal to $15°$. We will present quite technical solution using barycentric coordinates.

Let O be the center of triangle ABC and B_1 and C_1 be the midpoints of $\overline{AC}$ and $\overline{AB}$ respectively. Assume that barycentric coordinates of T in triangle ABC are (x, y, z). Points P and Q lie on segments $\overline{CB_1}$ and $\overline{BC_1}$ respectively because both triangles ABP and ACQ are acute.

We have $\frac{CP}{B_1A} = \frac{x}{z}$ and $PC = \frac{x}{x+z}$. So $B_1P = \left(\frac{1}{2} - \frac{x}{x+z}\right) AC$ because B_1 is the midpoint of AC. Point R lies on $\overline{BB_1}$ and $PR \perp AB$. It means that RP is parallel to CO and, as a consequence, triangles B_1PR and B_1CO are similar. Therefore, $\frac{B_1R}{B_1O} = \frac{B_1P}{B_1C} = 1 - \frac{2x}{x+z}$ and $\frac{OR}{B_1O} = 1 - \frac{B_1R}{B_1O} = \frac{2x}{x+z}$. We have $OB = 2OB_1$ so $\overrightarrow{OR} = -\frac{x}{x+z}\overrightarrow{OB}$. In the same manner we can prove that $\overrightarrow{OS} = -\frac{x}{x+y}\overrightarrow{OC}$ using similarity of triangles RC_1S and BC_1O.

Let's find vectors $\overrightarrow{TS}$ and $\overrightarrow{TR}$. We have $\overrightarrow{OT} = x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC}$ because (x, y, z) are barycentric coordinates of T. Using $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0$ we get

$$\begin{aligned}
\overrightarrow{TS} = \overrightarrow{OS} - \overrightarrow{OT} &= -\frac{x}{x+y}\overrightarrow{OC} - (x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC}) \\
&= -\frac{x}{x+y}\overrightarrow{OC} - (x\overrightarrow{OA} + y(-\overrightarrow{OA} - \overrightarrow{OC}) + z\overrightarrow{OC}) \\
&= \left(y - z - \frac{x}{x+y}\right)\overrightarrow{OC} + (y - x)\overrightarrow{OA}
\end{aligned}$$

and

$$\begin{aligned}
\overrightarrow{TR} = \overrightarrow{OR} - \overrightarrow{OT} & \\
&= -\frac{x}{x+z}\overrightarrow{OB} - (x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC}) \\
&= -\frac{x}{x+z}\overrightarrow{OB} - (x\overrightarrow{OA} + y\overrightarrow{OB} + z(-\overrightarrow{OA} - \overrightarrow{OB})) \\
&= (z - y - \frac{x}{x+z})\overrightarrow{OB} + (z - x)\overrightarrow{OA}.
\end{aligned}$$

Let $\vec{v}$ be the image of $\overrightarrow{TS}$ after counterclockwise rotation on $60°$. Vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{AC}$ rotated on $60°$ equal to $-\overrightarrow{OB}, -\overrightarrow{OC}$ and $-\overrightarrow{OA}$ respectively. It means that $\vec{v} = (z - y + \frac{x}{x+y})\overrightarrow{OA} + (x - y)\overrightarrow{OB}$.

Points S and R lie on segments $\overline{OC_1}$ and $\overline{OB_1}$ respectively and T lies in the interior of triangle OBC. Thus, $\overrightarrow{TR}$ equals to $\vec{v}$ and the following equation holds if and only if triangle RTS is equilateral: $(z - y - \frac{x}{x+z})\overrightarrow{OB} + (z - x)\overrightarrow{OA} = (z - y + \frac{x}{x+y})\overrightarrow{OA} + (x - y)\overrightarrow{OB}$. Vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are not parallel so $z - x = z - y + \frac{x}{x+y}$ and $z - y - \frac{x}{x+z} = x - y$. Multiplying by $x + y$ and $x + z$ respectively we obtain $y^2 - x^2 = x$ and $z^2 - x^2 = x$. Coordinates z and y are equal because $z^2 = y^2$ and they are positive. The sum of x, y, z is 1, so $z = \frac{1-x}{2}$ because $y = z$. Substituting z in $z^2 - x^2 = x$ we get $\left(\frac{1-x}{2}\right)^2 - x^2 = x$ and $3x^2 + 6x - 1 = 0$. For positive x there is only one solution, $x = \frac{2\sqrt{3}}{3} - 1$. Thus, triangle RST is equilateral if and only if $x = \frac{2\sqrt{3}}{3} - 1$ and $y = z = \frac{1-x}{2} = 1 - \frac{\sqrt{3}}{3}$. Let's prove that it is equivalent to $\frac{B_1P}{PC} = \frac{B_1B}{BC}$. The right-hand side equals to $\sin 60° = \frac{\sqrt{3}}{2}$. The left-hand side equals to $\frac{\frac{1}{2} - \frac{x}{x+z}}{\frac{x}{x+z}} = \frac{z-x}{2x}$. It is equal to $\frac{\sqrt{3}}{2} = \frac{(2-\sqrt{3})\sqrt{3}}{2(2-\sqrt{3})} = \frac{(2-\sqrt{3})\sqrt{3}}{2\sqrt{3}(\frac{2\sqrt{3}}{3}-1)} = \frac{2-\sqrt{3}}{2(\frac{2\sqrt{3}}{3}-1)}$ if and only if $x = \frac{2\sqrt{3}}{3} - 1$. The equality $\frac{B_1P}{PC} = \frac{B_1B}{BC}$ holds if and only if CQ is the bisector of $\angle B_1BC$ and $\angle PBC = 15°$. Therefore, triangle RST is equilateral if and only if $y = z$ and $\angle PBC = 15°$ (and $\angle QCB = 15°$ because $y = z$ and T lies on line AO).

Problem 4 Let x, y, z be positive numbers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Show that

$$\sqrt{x + yz} + \sqrt{y + zx} + \sqrt{z + xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

Solution: Let's denote $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ by a, b, c respectively. The sum of a, b, c is 1. So the inequality can be rewritten as

$$\sqrt{\frac{1}{a} + \frac{1}{bc}} + \sqrt{\frac{1}{b} + \frac{1}{ca}} + \sqrt{\frac{1}{c} + \frac{1}{ab}} \geq \sqrt{\frac{1}{abc}} + \sqrt{\frac{1}{a}} + \sqrt{\frac{1}{b}} + \sqrt{\frac{1}{c}}.$$

or

$$\sqrt{a + bc} + \sqrt{b + ca} + \sqrt{c + ab} \geq 1 + \sqrt{bc} + \sqrt{ca} + \sqrt{ab} \qquad (*)$$

after multiplying by $\sqrt{abc}$. We can factorize

$$a = (\sqrt{a + bc})^2 - (\sqrt{bc})^2 = (\sqrt{a + bc} - \sqrt{bc})(\sqrt{a + bc} + \sqrt{bc}) \qquad (\dagger)$$

The next step consists of proving that $\sqrt{a + bc} + \sqrt{bc} \leq 1$. From $a + b + c = 1$ we get $a = 1 - b - c$ and

$$\sqrt{a + bc} + \sqrt{bc} = \sqrt{1 - b - c + bc} + \sqrt{bc} = \sqrt{(1 - b)(1 - c)} + \sqrt{bc}.$$

Applying Cauchy-Schwarz inequality we obtain

$$\begin{aligned}1 &= (b+(1-b))(c+(1-c)) \\ &= ((\sqrt{b})^2+(\sqrt{(1-b)})^2)((\sqrt{c})^2+(\sqrt{(1-c)})^2) \\ &\geq (\sqrt{bc}+\sqrt{(1-b)(1-c)})^2\end{aligned}$$

Therefore, $\sqrt{a+bc}+\sqrt{bc} \leq 1$ and $a \geq \sqrt{a+bc}-\sqrt{bc}$ by (†). Similarly one can prove that $b \geq \sqrt{b+ca}-\sqrt{ca}$, $c \geq \sqrt{c+ab}-\sqrt{ab}$. These three inequalities add up to

$$a+b+c \geq \sqrt{a+bc}+\sqrt{b+ca}+\sqrt{c+ab}-\sqrt{bc}-\sqrt{ca}-\sqrt{ab}$$

that is the same as $(*)$ because $a+b+c=1$.

Problem 5 Find all functions $f:\mathbb{R}\to\mathbb{R}$ with the following properties:

(i) there are only finitely many s in $\mathbb{R}$ such that $f(s)=0$, and

(ii) $f(x^4+y)=x^3f(x)+f(f(y))$ for all $x,y\in\mathbb{R}$.

Solution: The only answer is $f(x)\equiv x$. The first our goal is to prove additivity of f. We have

$$f(x^4+y)=x^3f(x)+f(f(y))$$

Substituting x by 0, we obtain $f(y)=f(f(y))$ for all $y\in\mathbb{R}$. Therefore,

$$f(x^4+y)=x^3f(x)+f(y) \qquad (*)$$

Substituting y in $(*)$ by 0, we have $f(x^4)=x^3f(x)+f(0)$. When $x=1$ $f(1)=f(1)+f(0)$ so $f(0)=0$ and $f(x^4)=x^3f(x)$. Applying this to $(*)$ gives us $f(x^4+y)=f(x^4)+f(y)$ or in other words

$$f(a+y)=f(a)+f(y) \quad \text{for all} \quad a\geq 0, \qquad (\dagger)$$

because for all $a\geq 0$ there exists such x that $x^4=a$. When $y=-a$ $f(a)=-f(-a)$. Therefore, f is additive because $f(x+y)=f(x)+f(y)$ holds when x or y is nonnegative as a consequence of (†) and holds when x and y are negative, because

$$f(x+y)=-f(-x-y)=-f(-x)-f(-y)=f(x)+f(y).$$

We will present two approaches for the final part of solution.

First approach. As proved above, $f(f(y))=f(y)$, so from additivity of f we obtain that $f(f(y)-y)=0$ which means that there exist only finitely many values of $f(y)-y$. Suppose that there exist such z, that $f(z)-z=c\neq 0$. Then $f(kz)-kz=f(z)+\cdots+f(z)-kz=kc$ for all positive integers k that gives a contradiction with finiteness of values of $f(y)-y$. It proves that $f(z)=z$ for all $z\in\mathbb{R}$. Checking that $f(z)\equiv z$ really works we get full solution.

Second approach. For any function $g(x)$ let's denote $g(x)$ by $\Delta_g^0(x)$, $\Delta_g^0(x+1)-\Delta_g^0(x)$ by $\Delta_g^1(x)$, $\Delta_g^1(x+1)-\Delta_g^1(x)$ by $\Delta_g^2(x)$ and so on.

Let $f(1)=c$. As was proved above $f(x^4)=x^3f(x)$. Now we will find $\Delta_r^3(x)$ and $\Delta_s^3(x)$ where $r(x)=f(x^4)$ and $s(x)=x^3f(x)$, expanding powers of $x+1$ by Binomial Theorem and applying additivity:

$$
\begin{aligned}
\Delta_r^1 &= f((x+1)^4)-f(x^4)=f(x^4+4x^3+6x^2+4x+1)-f(x^4)\\
&= 4f(x^3)+6f(x^2)+4f(x)+c,\\
\Delta_r^2 &= 4(f(x+1)^3-f(x^3))+6(f(x+1)^2-f(x^2))+4(f(x+1)-f(x))\\
&= 4(3f(x^2)+3f(x)+c)+6(2f(x)+c)+4c\\
&= 12f(x^2)+24f(x)+10c,\\
\Delta_r^3 &= 12f((x+1)^2-x^2)+24(f(x+1)-f(x))\\
&= 24f(x)+12c+24c=24f(x)+36c,\\
\Delta_s^1 &= (x+1)^3f(x+1)-x^3f(x)\\
&= ((x+1)^3-x^3)f(x)+c(x+1)^3\\
&= (3x^2+3x+1)f(x)+c(x+1)^3,\\
\Delta_s^2 &= f(x+1)(3(x+1)^2+3(x+1)+1)+c(x+2)^3\\
&\quad - f(x)(3x^2+3x+1)-c(x+1)^3\\
&= f(x)(3(x+1)^2+3(x+1)-3x^2-3x)\\
&\quad + c(3(x+1)^2+3(x+1)+1)+c((x+2)^3-(x+1)^3)\\
&= (6x+6)f(x)+c(3x^2+6x+3+3x+3+1)+c(3x^2+9x+7)\\
&= (6x+6)f(x)+c(6x^2+18x+14),\\
\Delta_s^3 &= (6(x+1)+6)f(x+1)+c(6(x+1)^2+18(x+1)+14)\\
&\quad -(6x+6)f(x)-c(6x^2+18x+14)\\
&= 6f(x)+(6(x+1)+6)c+c(6(x+1)^2+18(x+1)-6x^2-18x)\\
&= 6f(x)+18xc+36c.
\end{aligned}
$$

We have $\Delta_s^3(x) = \Delta_r^3(x)$ for all x because $r(x) = s(x)$. Hence, $24f(x) + 36c = 6f(x) + 18xc + 36c$ that implies $18f(x) = 18xc$ and $f(x) = cx$. From $f(f(y)) = f(y)$ we obtain $cx = c^2x$ and $c = 0$ or 1. If $c = 1$, then $f(x) = cx$ for all x. If $c = 0$, then $f(x) = 0$ for all x that does not agree with statement (ii).

Notice that this is the only place where (ii) is used with the second approach.

2.2 Austrian-Polish Mathematics Olympiad

Problem 1 Let $A = \{2, 7, 11, 13\}$. A polynomial f with integer coefficients has the property that for each integer n, there exists $p \in A$ such that $p \mid f(n)$. Prove that there exists $p \in A$ such that $p \mid f(n)$ for all integers n.

Solution: Suppose that there is no value such that it divides $f(x)$ for all x. Then, there exist integers a, b, c, and d such that 2 does not divide $f(a)$, 7 does not divide $f(b)$, 11 does not divide $f(c)$, and 13 does not divide $f(d)$. Thus, we know that $f(a+7\cdot 11\cdot 13) = f(a)+k\cdot 7\cdot 11\cdot 13$, where k is some integer. Similarly for b, c, and d. Thus, we can use the Chinese Remainder Theorem to find a value r such that $r = a+r_1\cdot 7\cdot 11\cdot 13 = b+r_2\cdot 2\cdot 11\cdot 13 = c+r_3\cdot 2\cdot 7\cdot 13 = d+r_4\cdot 2\cdot 7\cdot 11$ where r_1, r_2, r_3, and r_4 are integers. Thus, 2, 7, 11, and 13 do not divide $f(r)$.

Problem 2 The diagonals of a convex quadrilateral $ABCD$ intersect at the point E. Let triangle ABE have circumcenter U and orthocenter H. Similarly, let triangle CDE have circumcenter V and orthocenter K. Prove that E lies on line UK if and only if it lies on line VH.

Problem 3 Find all functions $f : \mathbb{Z}^+ \to \mathbb{R}$ such that $f(x + 22) = f(x)$ and $f(x^2y) = (f(x))^2 f(y)$ for all positive integers x and y.

Solution: Look at what f(0) is. $f(0^2x) = f(0)^2 f(x)$, so f(0) must equal -1, 0, or 1. However, if f(0) = -1, then f(x) must also equal -1. Also, if f(0) = 1 then f(x) = 1. Each of these simple cases is self-consistent, and we will now look at the case where f(0) = 0. We know that $f(1^2\cdot 1) = f(1)^2 f(1)$, so f(1) is either 1, 0, or -1. If f(1) = 0, then $f(1^2\cdot x) = 0$ for all x. However, if f(1) is either ± 1, then we know that $f(1 + 44) = f(3)^2 f(5)$. $f(3) = f(5)^2 f(1)$, so $f(1) = f(5)^5 f(1)^2$, so $f(5) = f(1)$ as does f(3). Similarly, $f(9) = f(3)^2 f(1) = f(1)$. $f(15+66) = f(9)^2 f(1) = f(1)$. Thus we have f(1) = f(3) = f(5) = f(9) = f(15). However, we know that f(5 + 44) = f(7)2f(1), so f(7) = f(1). Similarly, $f(72\cdot 7) = f(13) = \pm f(1)$. We can continue this argument to find that $f(7) = f(13) = f(17) = f(19) = f(21) = \pm f(1)$. $f(2 + 66) = f(2)^2 f(17) = \pm f(1)$ or 0. If f(2) = 0, then since we can write every even value as $4 \cdot k$, where k is some odd value or previously determined even value, then f of every even value is 0. If f(2) does not equal 0, then since f(22) = f(0) = 0 then f(11) = 0. However, if f(2) = 0, then f(11 + 110) = $f(11)^2 f(1)$, so f(11) = f(1).

Problem 4 Determine the number of real solutions of the system

$$x_1 = \cos x_n, \quad x_2 = \cos x_1, \quad \ldots, \quad x_n = \cos x_{n-1}.$$

Solution: We will show that unless $x_1 = \cos(x_1)$, we cannot have such a chain for any value of n. To do this, we will do a case analysis by quadrants. Notice that since cos(0) = 1 and cos(p/2) = 1 and cosine is continuously decreasing in the first quadrant, there must be only one positive value x such that cos(x) = x. Now notice that cos(k) will be greater than x if k is less than x but will be less than x if k is greater than x (this is because cosine is a continuously decreasing function). Now, we must show that cos(x + r) is greater than x - r. Since the derivative of cosine is never greater than 1, we can see that this is true. Thus, if we begin with a point in the first quadrant and continue taking cosines of it, we will become closer and closer to the value of x. If we choose a point in the second or third quadrants, its absolute value will be greater than 1 and thus it will be impossible to ever obtain it. If we choose a point in the fourth quadrant and take cosine of it, we will come up with a positive value between 0 and 1, which is in the first quadrant. Taking successive cosines will not allow us to obtain a negative value.

Problem 5 For every real number x, let $F(x)$ be the family of real sequences $a_1, a_2, \ldots$ satisfying the recursion

$$a_{n+1} = x - \frac{1}{a_n}$$

for $n \geq 1$. The family $F(x)$ has *minimal period* p if (i) each sequence in $F(x)$ is periodic with period p, and (ii) for each $0 < q < p$, some sequence in $F(x)$ is not periodic with period q. Prove or disprove the following claim: for each positive integer P, there exists a real number x such that the family $F(x)$ has minimal period $p > P$.

2.3 Balkan Mathematical Olympiad

Problem 1 Let $A_1, A_2, \ldots, A_n$ $(n \geq 4)$ be points in the plane such that no three of them are collinear. Some pairs of distinct points among $A_1, A_2, \ldots, A_n$ are connected by line segments, such that every point is directly connected to at least three others. Prove that from among these points can be chosen an even number of distinct points $X_1, X_2, \ldots, X_{2k}$ $(k \geq 1)$ such that X_i is directly connected to X_{i+1} for $i = 1, 2, \ldots, 2k$. (Here, we write $X_{2k+1} = X_1$.)

Solution:

First, assume that the problem intends $(k \geq 2)$ in the second to last sentence, because otherwise any two connected points would satisfy the conditions, trivializing the problem. Then, assume the proposition is untrue. Choose a contradictory $A_1, A_2, \ldots, A_n$ $(n \geq 4)$ such that n is at a minimum. Let a *cycle* of length k denote a sequence of distinct points $X_1, X_2, \ldots, X_k$ $(k \geq 3)$ such that X_i is directly connected to X_{i+1} for $i = 1, 2, \ldots, k$ and $X_{k+1} = X_1$. Let a *path* of length k from A to B denote a sequence of points $A, X_2, \ldots, X_{k-1}, B$ such that X_i is connected to X_{i+1} and $X_1 = A, X_k = B$. By assumption, there are no cycles of even length. We will show that there exists a cycle. Starting from one point, move to a point connected to it. Continue moving to connected points– since each point has at least three adjacent points, we can continue without backtracking until we repeat a point. Then the set of points from the first to the second visitation of that point would constitute a cycle, which we will denote C, consisting of $X_1, X_2, \ldots, X_k$.

Now, note that for any X_i and X_j in the C, $i < j - 1$, there can be no path connecting X_i to X_j which is completely separate from C. This is because if that were the case, there would be at least three distinct paths from X_i to X_j: this path, the path around one side of the C, and the path around the other side of C. Then some two of these must have the same parity, so the two together would form a cycle of even length, a contradiction. Therefore no two points on C have a path between them independent of C. Hence, no two points on C have a path between them independent of C in any portion, because some portion of that path would be completely independent.

Consider the graph obtained when C is shrunk to a point, with all external lines moved to the point. Since C has at least three points and each point on it has at least one external line, this point has at least three lines passing through it. Also, no cycles have changed in length beyond C, since any such cycle must contain at least two of the points in C, and then would have a path between those two points not entirely on C, which would cannot happen, as shown above. In addition, the number of edges from any point not in C has not changed. Since C has at least three points, the new graph has at least four; the point formed from C and at least one external to each point in C. Therefore the new graph satisfies all the conditions for being a graph and if the original graph does not have an even cycle, neither does the new graph. Since we assumed that the original graph has the smallest number of points for a contradictory example and the new graph has fewer points, this is a contradiction. Therefore all such graphs have even cycles, QED.

Problem 2 The sequence $a_1, a_2, \dots$ is defined by the initial conditions $a_1 = 20$, $a_2 = 30$ and the recursion $a_{n+2} = 3a_{n+1} - a_n$ for $n \geq 1$. Find all positive integers n for which $1 + 5a_n a_{n+1}$ is a perfect square.

Solution: The only solution is $n = 3$. We can check that $20 \cdot 30 \cdot 5 + 1 = 3001$ and $30 \cdot 70 \cdot 5 + 1 = 10501$ are not perfect squares, while $70 \cdot 180 \cdot 5 + 1 = 63001 = 251^2$ is a perfect square. Then we must only prove that $1 + 5a_n a_{n+1}$ is not a perfect square for $n \geq 4$. First, we will prove a lemma:

Lemma. *For any integer $n \geq 2$,*

$$a_n^2 + 500 = a_{n-1}a_{n+1}.$$

Proof. We will prove this by induction on n. In the base case, $30^2 + 500 = 1400 = 20 \cdot 70$. Now assume that $a_n^2 + 500 = a_{n-1}a_{n+1}$. Then $a_n a_{n+2} = (3a_{n+1} - a_n)(a_n) = 3a_{n+1}a_n - a_n^2 = 3a_{n+1}a_n - (a_{n-1}a_{n+1} - 500) = 500 + a_{n+1}(3a_n - a_{n-1}) = 500 + a_{n+1}^2$, proving the inductive step. Therefore the desired statement is true from induction.

□

Now, for $n \geq 4$, $(a_n + a_{n+1})^2 = a_n^2 + a_{n+1}^2 + 2a_n a_{n+1}$. But $a_{n+1}^2 = 9a_n^2 + a_{n-1}^2 - 6a_{n-1}a_n$, so $(a_n + a_{n+1})^2 = 2a_n a_{n+1} + 3a_n(3a_n - a_{n-1}) + a_{n-1}^2 + a_n^2 - 3a_n a_{n-1} = 5a_n a_{n+1} + a_{n-1}^2 - a_n a_n - 2 =$

$5a_na_{n+1} + a_{n-1}^2 - (a_{n-1}^2 + 500) = 5a_na_{n+1} - 500$, by the lemma and definition of a.

Therefore $(a_n + a_{n+1})^2 = 5a_na_{n+1} - 500 < 5a_na_{n+1} + 1$. Since a_n is increasing and $n \geq 4$, $a_n + a_{n+1} \geq 180 + 470 = 650$, so $(a_n + a_{n+1} + 1)^2 = (a_n + a_{n+1})^2 + 2(a_n + a_{n+1}) + 1 > (a_n + a_{n+1})^2 + 501 = 5a_na_{n+1} + 1$. Because two adjacent integers have squares above and below $5a_na_{n+1} + 1$, that value is not a perfect square for $n \geq 4$, QED.

Problem 3 Two circles with different radii intersect at two points A and B. The common tangents of these circles are segments $\overline{MN}$ and $\overline{ST}$, where M, S lie on one circle while N, T lie on the other. Prove that the orthocenters of triangles AMN, AST, BMN, and BST are the vertices of a rectangle.

Solution: Because you can separate $\angle MAN$ and $\angle SAT$ by the line AB,

$$\angle MAN = \frac{1}{2}(\widehat{MB} + \widehat{NB})$$

$$\angle SAT = \frac{1}{2}(\widehat{BS} + \widehat{BT})$$

$$\angle MAN + \angle SAT = \frac{1}{2}(\widehat{NAT} + \widehat{MAS})$$

Therefore since $O_1S \parallel O_2T$ and $O_1M \parallel O_2N$, we have $\widehat{NAT} + \widehat{MAS} = 2\pi$, so

$$\angle MAN + \angle SAT = \pi$$

Let C, D, EF be the orthocenters of MAN, SAT, SBT, and MBN, respectively. Since SBT, MBN are reflections of MAN, MBN over O_1O_2, E, F are reflections of C, D over O_1O_2, so $CE \parallel AB \parallel DF$. Therefore if $CD \perp AB$, $CDFE$ would form a rectangle, and we would be done. Reflect TSD over AB to $T'S'D'$. Since $D'D \perp AB$, it would be sufficient to prove that $D' = C$. Note that $T'S' \parallel MN$ and $T'S' = MN$, so we can translate $T'AS'D'$ to $MA'ND''$. Since $\angle MA'N + \angle MAN = \angle SAT + \angle MAN = \pi$, $MA'NA$ is cyclic. To proceed we will prove a lemma:

Lemma. *Let $ABCD$ be a cyclic quadrilateral, and E, F be the orthocenters of ABC, ACD respectively. Then $\overrightarrow{BD} = \overrightarrow{EF}$.*

Proof. Let O be the circumcenter of $ABCD$. Then $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{AC}$ and $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OD}$, so $\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \overrightarrow{OD} - \overrightarrow{OB} = \overrightarrow{BD}$, QED. $\square$

By applying the lemma to $MA'NA$, $\overrightarrow{AA'} = \overrightarrow{CD''}$. But since $MA'ND''$ was translated from $T'ASD'$, $\overrightarrow{AA'} = \overrightarrow{D'D''}$. Therefore $\overrightarrow{D'D''} = \overrightarrow{CD''}$, so $C = D$. Thus we have $CDEF$ is a rectangle, QED.

Problem 4 Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2003$$

for all positive integers n.

Solution: $f(n) = n + 667$ or $n + 668$, where the set S of n for which $f(n) = n + 668$ is such that no two elements differ by 668. First, we will show that $f(n) = n + 667$ or $n + 668$ for all n. Assume the opposite, and let n be a value for which $f(n) > n + 668$ or $f(n) < n + 667$. Then, define the sequence $a_0 = n$, $a_i = f(a_{i-1})$ for all integer $i > 0$. Notice that if $a_i - a_{i-1} \geq 668 + k$, $k > 0$ and by the given $a_i + a_{i+1} \leq 2a_{i-1} + 2003$, we have $a_{i+1} - a_i \leq 2a_{i-1} + 2003 - 2a_i \leq 2003 - 2(668 + k) = 667 - 2k$. Similarly, if $a_i - a_{i-1} \leq 667 - k$, $k > 0$ then $a_i + a_{i+1} \geq 2a_{i-1} + 2001$, so $a_{i+1} - a_i \geq 667 + 2k$. Therefore, if $a_{i+1} - a_i = 668 + k$, $k > 0$, then $a_{i+2} - a_{i+1} \leq 667 - 2k$, so $a_{i+3} - a_{i+2} \geq 667 + 4k \geq 668 + 3k$. Since either $a_1 - a_0 < 667$, in which case $a_2 - a_1 > 668$, or $a_1 - a_0 > 668$, there must be some x for which $a_x - a_{x-1} = 668 + k$, $k > 0$. But then $a_{x+2y} - a_{x+2y-1} \geq 668 + 3^y k$. Since this increases as y does, for some m, $a_n - a_{n-1} \geq 668 + 1336$ for all $n \geq m$ with $n - m$ even. Now notice that if $a_{i+1} - a_i = 668 + k$, then $a_{i+2} - a_i = a_{i+2} - a_{i+1} + a_{i+1} - a_i \leq 667 - 2k + 668 + k = 1335 - k$. So for all the $n \geq m$, m , $a_{n+2} - a_n \leq -1$. So the sequence $a_m, a_{m+2}, a_{m+4}, \ldots$ is strictly decreasing. Therefore for some n, $a_n < 0$. But by definition, $f(x) \geq 0$ for all x, a contradiction. Therefore $f(n) = n + 667$ or $n + 668$ for all n.

Let S be the set of n for which $f(n) = n + 668$. Clearly, if two elements in S differ by 668, then if n is the lower of the two $f(n) + f(f(n)) = n + 668 + n + 668 + 668 = 2n + 2004 > 2n + 2003$, a contradiction. So no two elements in S differ by 668, QED.

2.4 Baltic Team Contest

Problem 1 A spider is sitting on a cube. A fly lands on the cube, hoping to maximize the length of the shortest path to the spider along the surface of the cube. Can the fly guarantee doing so by choosing the point directly opposite the spider (i.e., the point that is the reflection of the spider's position across the cube's center)?

Solution: The answer is no. Let $A = (0, 0, 0), B = (1, 0, 0), C = (1, 1, 0), D = (0, 1, 0)$ be one face of the cube and let $EFGH$ be the opposite face at $z = 1$. Consider the midpoint M of GH and its opposite, N, the midpoint of AB. We will compute the shortest distance d from M to points $P = (x, 0, 0)$ on AB, $0 < x \le \frac{1}{2}$ ($x = \frac{1}{2}$ corresponds to $P = N$). All possible paths are equivalent to one of the following.

- Path through a point on HD, a point on AD, then to P.
  Then $d^2 = (1 + \frac{1}{2})^2 + (1 + x)^2$, so $d > 2$ if $x > \frac{\sqrt{7}}{2} - 1 \approx 0.32$.
- Path through a point on CD, then to P.
  Then $d^2 = 2^2 + (\frac{1}{2} - x)^2$, so $d > 2$ for every $x < \frac{1}{2}$ and $d = 2$ if $x = \frac{1}{2}$.
- Path through a point on EH, a point on AD, then to P.
  Then $d^2 = 1^2 + (\frac{3}{2} + x)^2$, so $d > 2$ if $x > \sqrt{3} - \frac{3}{2} \approx 0.23$.

Hence, the shortest distance from M to N along the cubic surface is $d = 2$, while if $\frac{\sqrt{7}}{2} - 1 < x < \frac{1}{2}$, the shortest distance between M and P is always greater than 2. Therefore, the fly cannot guarantee maximizing the length of the shortest path to the spider by choosing the point directly opposite the spider.

Problem 2 Find all nonnegative integers m such that $(2^{2m+1})^2 + 1$ is divisible by at most two different primes.

Solution: We claim $m = 0, 1, 2$ are the only such integers. It is easy to check that these values of m satisfy the requirement. Suppose some $m \ge 3$ works. Write

$$\begin{aligned}(2^{2m+1})^2 + 1 &= (2^{2m+1} + 1)^2 - 2 \cdot 2^{2m+1} \\ &= (2^{2m+1} + 2^{m+1} + 1)(2^{2m+1} - 2^{m+1} + 1).\end{aligned}$$

The two factors are both odd, and their difference is 2^{m+2}; hence, they are relatively prime. It follows that each is a prime power. We also know that $(2^{2m+1})^2 = 4^{2m+1} \equiv -1 \pmod{5}$, so one of the factors $2^{2m+1} \pm 2^{m+1} + 1$ must be a power of 5. Let $2^{2m+1} + 2^{m+1}s + 1 = 5^k$, where $s = \pm 1$ is the appropriate sign.

Taking the above equation modulo 8, and using the assumption $m \geq 3$, we obtain $5^k \equiv 1 \pmod{8}$, so that k is even. Writing $k = 2l$, we have

$$2^{m+1}(2^m + s) = (5^l - 1)(5^l + 1).$$

The factor $5^l + 1 \equiv 2 \pmod{4}$, so $5^l - 1 = 2^m a$ for some odd integer a. But if $a = 1$, then

$$2 = (5^l + 1) - (5^l - 1) = 2(2^m + s) - 2^m = 2^m + 2s \geq 2^3 - 2,$$

a contradiction, whereas if $a \geq 3$, then $5^l - 1 \geq 3 \cdot 2^m$ while $5^l + 1 \leq 2(2^m + s)$, another contradiction.

Problem 3 Show that the sequence

$$\binom{2002}{2002}, \binom{2003}{2002}, \binom{2004}{2002}, \ldots,$$

considered modulo 2002, is periodic.

Solution: We will show that the sequence, taken modulo 2002, has period $m = 2002 \cdot 2002!$. Indeed,

$$\begin{aligned}
\binom{x+m}{2002} &= \frac{(x+m)(x-1+m)\cdots(x-2001+m)}{2002!} \\
&= \frac{x(x-1)\cdots(x-2001) + km}{2002!} \\
&= \frac{x(x-1)\cdots(x-2001)}{2002!} + 2002k \\
&\equiv \binom{x}{2002} \pmod{2002}.
\end{aligned}$$

Problem 4 Find all integers $n > 1$ such that any prime divisor of $n^6 - 1$ is a divisor of $(n^3 - 1)(n^2 - 1)$.

Solution: We show that $n = 2$ is the only such integer. It is clear that $n = 2$ satisfies the conditions. For $n > 2$, write $n^6 - 1 = (n^3 - 1)(n^3 + 1) = (n^3 - 1)(n + 1)(n^2 - n + 1)$; hence, all prime factors of n^2-n+1 must divide n^3-1 or $n^2-1=(n-1)(n+1)$. Note, however, that $(n^2-n+1, n^3-1) \le (n^3+1, n^3-1) \le 2$; on the other hand, $n^2-n+1 = n(n-1)+1$ is odd, so all prime factors of n^2-n+1 must divide $n+1$. But $n^2-n+1=(n+1)(n-2)+3$, so we must have $n^2-n+1=3^k$ for some k. Because $n>2$, we have $k \ge 2$. Now $3 \mid (n^2-n+1)$ gives $n \equiv 2 \pmod 3$; but for each of the cases $n \equiv 2, 5, 8 \pmod 9$, we have $n^2-n+1 \equiv 3 \pmod 9$, a contradiction.

Problem 5 Let n be a positive integer. Prove that the equation

$$x+y+\frac{1}{x}+\frac{1}{y}=3n$$

does not have solutions in positive rational numbers.

Solution: Suppose $x=\frac{a}{b}, y=\frac{c}{d}$ satisfies the given equation, where $(a,b)=(c,d)=1$. Clearing denominators,

$$(a^2+b^2)cd+(c^2+d^2)ab=3nabcd.$$

Thus, $ab \mid (a^2+b^2)cd$ and $cd \mid (c^2+d^2)ab$. Now $(a,b)=1$ implies $(a, a^2+b^2)=(a,b^2)=1$, so $ab \mid cd$; likewise, $cd \mid ab$, and together these give $ab=cd$. Thus,

$$a^2+b^2+c^2+d^2=3nab.$$

Now each square on the left is congruent to either 0 or 1 modulo 3. Hence, either all terms are divisible by 3 or exactly one is. The first case is impossible by the assumption $(a,b)=(c,d)=1$, and the second is impossible because $ab=cd$.

Problem 6 Does there exist an infinite, non-constant arithmetic progressions, each term of which is of the form a^b where a and b are positive integers with $b \ge 2$?

First Solution: Consider the arithmetic progression $c, c+k, c+2k, \ldots$. Let $d=(c,k)$. Then by Dirichlet's Theorem, the arithmetic progression

$$\frac{c}{d}, \frac{c}{d}+\frac{k}{d}, \frac{c}{d}+\frac{2k}{d}, \ldots$$

contains infinitely many primes. Choose such a prime $p>d$. Then pd is a term of the original progression that is not of the form a^b.

Second Solution: It is enough to show that the numbers of the form a^b become arbitrarily sparse. That is, letting $p(n)$ denote the number of such numbers less than or equal to n, we show that $p(n)/n$ can be made arbitrarily small. Indeed, let $n = 2^k$. Then the exponents b that contribute terms a^b counted in $p(n)$ satisfy $b \le k$. The number of terms each exponent contributes is at most $2^{k/2}$, so we have

$$\frac{p(2^k)}{2^k} \le \frac{k \cdot 2^{k/2}}{2^k} = \frac{k}{2^{k/2}},$$

which clearly approaches 0 as $k \to \infty$.

2.5 Czech-Polish-Slovak Mathematical C ompetition

Problem 1 Let a and b be distinct real numbers, and let k and m be positive integers with $k+m=n\geq 3$, $k\leq 2m$, and $m\leq 2k$. We consider n-tuples $(x_1, x_2, \ldots, x_n)$ with the following properties:

(i) k of the x_i are equal to a, and in particular $x_1 = a$;

(ii) m of the x_i are equal to b, and in particular $x_n = b$;

(iii) no three consecutive terms of $x_1, x_2, \ldots, x_n$ are equal.

Determine all possible values of the sum

$$x_n x_1 x_2 + x_1 x_2 x_3 + \cdots + x_{n-1} x_n x_1.$$

Solution: Due to (iii) and $x_1 = a, x_n = b$, in the expression $x_n x_1 x_2 + x_1 x_2 x_3 + \cdots + x_{n-1} x_n x_1$, all of the terms will either equal $a^2 b$ or $b^2 a$. Furthermore, if we write them as aab or bba, then the number of a's will be $3k$, since each x_i appears 3 times. Similarly, the number of b's would be $3m$.

I want to find how many are $a^2 b$ and how many are $b^2 a$. Each term $a^2 b$ has two a's, each term $b^2 a$ has one a. Let s be the number of $a^2 b$'s and t be the number $b^2 a$'s. Then

$$2s + t = 3k(\text{the total number of } a\text{'s })$$

$$2t + s = 3m(\text{the total number of } b\text{'s })$$

Solving, we get

$$s = 2k - m, \qquad t = 2m - k.$$

So the value of the sum $x_n x_1 x_2 + x_1 x_2 x_3 + \cdots + x_{n-1} x_n x_1$ is

$$(2k-m)a^2 b + (2m-k)b^2 a = ab(2ka - ma + 2mb - kb).$$

Problem 2 Given is a triangle ABC with side lengths $BC = a \leq CA = b \leq AB = c$ and area S. Let P be a variable point inside triangle ABC, and let D, E, F be the intersections of rays AP, BP, CP with the opposite sides of the triangle. Determine (as a function of a, b, c, and S) the greatest number u and the least number v such that $u \leq PD + PE + PF \leq v$ for all such P.

Solution: The answer is $u = \dfrac{2S}{c}$, $v = c$.

Part One $u = \dfrac{2S}{c}$.

Proof of Part One: Construct D' on BC, E' on AC, F' on AB such that $PD' \perp BC$, $PE' \perp AC$, $PF' \perp AB$. Then

$$PD+PE+PF \geq PD'+PE'+PF' \quad \text{and} \quad PD' \cdot BC+PE' \cdot AC+PF' \cdot AB = 2S.$$

Since AB is the longest side,

$$(PD' + PE' + PF')AB \geq PD' \cdot BC + PE' \cdot AC + PF' \cdot AB = 2S.$$

So

$$(PD' + PE' + PF') \geq \frac{2S}{c}.$$

Now we will prove that there is no larger u satisfying the condition. Since $PD+PE+PF$ is converges to $\frac{2S}{c}$ as P is approaching C along the altitude from C, there is no larger u satisfying the condition.

Part Two $v = c$.

Proof of Part Two: Construct X on BC and Y on AC such that XY passes P and $XY//AB$. Since AB is the largest side, $\angle BCA \geq \angle ABC$ and $\angle BCA \geq \angle CAB$. $\angle DXP = \angle ABC$ because $XY//AB$. $\angle DPX = \angle DAB < \angle CAB$ because D is on the segment BC. Hence $\angle XDP \geq \angle BCA \geq \angle DXP$ and $\angle XDP \geq \angle BCA \geq \angle DPX$. Therefore XP is the largest side of $\triangle DXP$.

Similarly, PY is the largest side of $\triangle PEY$. So $PD + PE \leq XY$, and $PD + PE + PF \leq XY + PF$.

We will prove $XY + PF \leq AB$.

First, note that CF is less than or equal to one of AC and BC. This is because at least one of $\angle CFA$ and $\angle CFB$ must be larger than or equal $90°$. It must be the largest angle in any triangle that is is a part of so the side corresponding to it in such a triangle must also be the largest side. So through $\triangle CFB$ and $\triangle CFA$ we can get $CF \leq BC$ or $CF \leq AC$. Hence, $CF \leq \max\{a, b\} \leq c$.

Next, note that if $CP = s$ and $PF = t \cdot s$, then $CF = (1 + t)s$ and $CP : CF = 1 : (1 + t)$. Hence if $XY = l$, then $AB = (1 + t)l$ because

$XY//AB$ and $XY : AB = 1 : (1+t)$.

$$XY + PF = l + ts$$

$$AB = (1+t)l = l + tl.$$

Since $CF < AB$, we conclude $CP \leq XY$. Hence $s < l$. So $l + ts \leq l + tl$ and $XY + PF \leq AB$. Thus we have proved that $PD + PE + PF \leq AB$.

When P is on AB, clearly $PD + PE + PF = AB$. So as P approaches to AB, $PD + PE + PF$ can be as close to AB as we want. Thus there is no smaller value of v satisfying the condition.

Problem 3 Let n be a given positive integer, and let $S = \{1, 2, \ldots, n\}$. How many functions $f : S \to S$ are there such that $x + f(f(f(f(x)))) = n + 1$ for all $x \in S$?

Solution: Clearly, if

$$f(a) = b, \quad f(b) = c, \quad f(c) = d,$$

then

$$\begin{aligned}
&f(d) = f(f(f(f(a)))) = f^4(a) = n + 1 - a,\\
&f(n+1-a) = f(f(f(f(b)))) = f^4(b) = n + 1 - b\\
&f(n+1-b) = f(f(f(f(c)))) = f^4(c) = n + 1 - c\\
&f(n+1-c) = f(f(f(f(d)))) = f^4(d) = n + 1 - d\\
&f(n+1-d) = n + 1 - (n + 1 - a) = a\\
&f^8(a) = a.
\end{aligned}$$

So clearly the function f is made of loops

$$a \to b \to c \to d \to n+1-a \to n+1-b \to n+1-c \to n+1-d \to a$$

and every $x \in S$ must be a part of one of these loops. If it is a part of two loops, then clearly these two loops are the same.

<u>*Lemma*</u> If two terms of the same loop are equal, then this loop must be

$$\frac{n+1}{2} \longrightarrow \frac{n+1}{2} \longrightarrow \frac{n+1}{2} \longrightarrow \cdots .$$

Proof of the Lemma: Between the two terms, there can be 0, 1, 2, or 3 terms. We divide the Lemma into four cases.

Case One. 0 term.

The loop must be

$$a \to a \to a \to a \to a \to a \to a \to a\ .$$

So $a = n+1-a,\ a = \frac{n+1}{2}$.

Case Two. 1 term.

The loop must be

$$a \to b \to a \to b \to a \to b \to a \to b\ .$$

So $a = n+1-a,\ b = n+1-b,\ a = b = \frac{n+1}{2}$.

Case Three. 2 terms.

The loop must be

$$a \to b \to c \to a \to b \to c \to a \to b\ .$$

Clearly the eighth term goes to the first term in the loop, $b \to a$. But $b \to c$ also. So $a = c$. Also, $a \to b$ and $a = c \to a$, so $b = a$. Thus $a = b = c$. This case is same as Case One.

Case Four. 3 terms.

$$a \to b \to c \to d \to a \to b \to c \to d\ .$$

Here $a = n+1-a,\ b = n+1-b,\ c = n+1-c,\ d = n+1-d,$, so $a = b = c = d = \frac{n+1}{2}$. Thus the Lemma is proven.

Call the loop $\frac{n+1}{2} \to \frac{n+1}{2} \to \cdots$ a repeating loop. Notice that if the function does not have a repeating loop, then the numbers in S must be divided into sets of 8. There is a number $k \in \mathbb{N}$ such that $n = 8k$. If there is a repeating loop, then other than $\frac{n+1}{2}$, the rest of the numbers are divided into sets of 8. So $n = 8k+1$. Clearly, if n is not one of the numbers $8k$ and $8k+1$, then there are no such functions.

If $n = 8k$, then there cannot be a repeating loop. To find the number of functions, we must find the number of ways to make the set of loops $a \to b \to c \to d \to n+1-a \to n+1-b \to n+1-c \to n+1-d$.

Let T be the set of unordered pairs,

$$T = \{(1, 4k+1),\ (2, 4k+2),\ \cdots,\ (4k, 8k)\}.$$

Every number in S is in exactly one of the pairs in T. Also, each loop in the function is formed by exactly four elements in T.

First let us consider the number of ways to divide T into k sets of four elements. There are $\dfrac{\binom{4k}{4}\binom{4k-4}{4}\cdots\binom{4}{4}}{k!}$ to do so. In each of the four element sets,

$$(a, n+1-a),\ (b, n+1-b),\ (c, n+1-c),\ (d, n+1-d),$$

we will see how many different loops can be created. Without loss of generality, assume the first term is a. Then the fifth term would be $n+1-a$. So we now have six choices for the second term. Once the second term is decided, the sixth term is also decided. We now have four choices for the third term. Similarly, once the third term is decided, the seventh term is also decided. And then there are two choices for the fourth term, after which the eighth term is decided. Hence there are $6 \times 4 \times 2 = 48$ ways to make a loop of each set. So the number of ways to make a set of k loops out of each set of k sets is $(48)^k$. So the number of functions must be

$$\frac{\binom{4k}{4}\binom{4k-4}{4}\cdots\binom{4}{4}}{k!} \cdot (48)^k\ .$$

If $n = 8k+1$, then there must be one repeating loop. So the number of functions is the number of ways to make the set of loops $a \to b \to c \to d \to n+1-a \to n+1-b \to n+1-c \to n+1-d$ out of the other $8k$ numbers. This is the same as the previous case. So the answer is also

$$\frac{\binom{4k}{4}\binom{4k-4}{4}\cdots\binom{4}{4}}{k!} \cdot (48)^k\ .$$

In conclusion, if $n \neq 8k$ and $n \neq 8k+1$ for any $k \in \mathbb{N}$, there are no functions. If $n = 8k$ or $n = 8k+1$ for some $k \in \mathbb{N}$, there are

$$\frac{\binom{4k}{4}\binom{4k-4}{4}\cdots\binom{4}{4}}{k!} \cdot (48)^k$$

functions.

Problem 4 Let n, p be integers such that $n > 1$ and p is a prime. If $n \mid (p-1)$ and $p \mid (n^3 - 1)$, show that $4p - 3$ is a perfect square.

Solution: $n|(p-1)$, so $p - 1 \geq n$ and $p > n$.

$$p|(n^3 - 1) = (n-1)(n^2+n+1) \text{ and } p \mid (n-1)$$

$$\Longrightarrow p|(n^2+n+1) \Longrightarrow \exists k \in \mathbb{N}, \quad p \cdot k = n^2+n+1$$

$$n|(p-1) \Longrightarrow p \equiv 1 \mod n \Longrightarrow p \cdot k \equiv k \mod n$$

$$\Longrightarrow n^2+n+1 \equiv k \mod n \Longrightarrow k \equiv 1 \mod n$$

There are integers $a > 0, b \geq 0$ such that $p = an+1,\ k = bn+1$.

$$(an+1)(bn+1) = n^2+n+1$$

$$abn^2 + (a+b)n + 1 = n^2+n+1, \quad n > 0$$

$$abn + (a+b) = n+1.$$

If $b \geq 1$, then $abn + (a+b) \geq n+2 > n+1$. So $b = 0, k = 1, p = n^2+n+1$. Therefore

$$4p - 3 = 4n^2 + 4n + 4 - 3 = 4n^2 + 4n + 1 = (2n+1)^2.$$

Problem 5 In acute triangle ABC with circumcenter O, points P and Q lie on sides $\overline{AC}$ and $\overline{BC}$, respectively. Suppose that

$$\frac{AP}{PQ} = \frac{BC}{AB} \quad \text{and} \quad \frac{BQ}{PQ} = \frac{AC}{AB}.$$

Show that O, P, Q, and C are concyclic.

Solution: Construct the circle circumscribed about $\triangle ABC$. Let $BC = a, CA = b, AB = c$. Without loss of generality, assume $a \geq b$.

From the given ratios, clearly there exists x such that $AP = ax$, $BQ = bx, PQ = cx$. I want to find such x. Observe that $CP = b - ax$ and $CQ = a - bx$.

By the Law of Cosines,

$$\begin{aligned} & PQ^2 = CP^2 + CQ^2 - 2(CP)(CQ)\cos\gamma \\ \iff\ & c^2x^2 = (b-ax)^2 + (a-bx)^2 - 2(b-ax)(a-bx)\cos\gamma \\ \iff\ & c^2x^2 = a^2x^2 + b^2x^2 + a^2 + b^2 - 4abx - 2ab\cos\gamma - 2abx^2\cos\gamma + 2(a^2+b^2) \\ \iff\ & 0 = c^2 - 4abx + 2(a^2+b^2)x\cos\gamma\ . \end{aligned}$$

So

$$x = \frac{c^2}{4ab - 2(a^2+b^2)\cos\gamma} = \frac{c^2}{4ab + \frac{2(a^2+b^2)(c^2-a^2-b^2)}{2ab}}$$

$$= \frac{c^2ab}{4a^2b^2 - (a^4+b^4+2a^2b^2) + a^2c^2 + b^2c^2} = \frac{c^2ab}{(a^2+b^2)c^2 - (a^2-b^2)^2}\ .$$

Lemma:

$$\frac{ax - \frac{b}{2}}{\frac{a}{2} - bx} = \frac{\sqrt{R^2 - (\frac{b}{2})^2}}{\sqrt{R^2 - (\frac{a}{2})^2}} .$$

Proof of the Lemma. Let $y = (a^2+b^2)c^2-(a^2-b^2)^2$. Then $x = \frac{c^2ab}{y}$ and

$$\begin{aligned} 2a^2c^2 - y &= 2a^2c^2 - (a^2+b^2)c^2 + (a^2-b^2)^2 \\ &= (a^2-b^2)c^2 + (a^2-b^2)^2 \\ &= (a^2-b^2)(c^2+a^2-b^2). \end{aligned}$$

So,

$$ax - \frac{b}{2} = \frac{b}{2}(\frac{2a^2c^2}{y} - 1) = \frac{b}{2}\frac{(a^2-b^2)(c^2+a^2-b^2)}{y}.$$

Symmetrically,

$$\frac{a}{2} - bx = -(bx - \frac{a}{2}) = \frac{a}{2}\frac{(a^2-b^2)(c^2+b^2-a^2)}{y}.$$

$$\text{left side} = \frac{b}{a}\left(\frac{c^2+a^2-b^2}{c^2+b^2-a^2}\right)$$

$$\text{right side} = \frac{\sqrt{R^2 - (\frac{b}{2})^2}}{\sqrt{R^2 - (\frac{a}{2})^2}} = \frac{\sqrt{\left(\frac{abc}{4(ABC)}\right)^2 - \frac{b^2}{4}}}{\sqrt{\left(\frac{abc}{4(ABC)}\right)^2 - \frac{a^2}{4}}}$$

$$= \frac{b}{a}\sqrt{\frac{\frac{a^2c^2}{4(ABC)^2} - 1}{\frac{b^2c^2}{4(ABC)^2} - 1}} = \frac{b}{a}\sqrt{\frac{a^2c^2 - 4(ABC)^2}{b^2c^2 - 4(ABC)^2}}$$

$$= \frac{b}{a}\sqrt{\frac{a^2c^2 - 4(\frac{ac\sin\beta}{2})^2}{b^2c^2 - 4(\frac{bc\sin\alpha}{2})^2}} = \frac{b}{a}\sqrt{\frac{a^2c^2 - a^2c^2\sin^2\beta}{b^2c^2 - b^2c^2\sin^2\alpha}} = \frac{b}{a}\sqrt{\frac{a^2c^2\cos^2\beta}{b^2c^2\cos^2\alpha}}$$

$$= \frac{b}{a}\frac{ac\cos\beta}{bc\cos\alpha} = \frac{b}{a}\frac{ac\frac{c^2+a^2-b^2}{2ac}}{bc\frac{c^2+b^2-a^2}{2bc}} = \frac{b}{a}\frac{(c^2+a^2-b^2)}{(c^2+b^2-a^2)} = \text{left side} .$$

In the above, we use the fact that $cos\beta > 0$ and $cos\alpha > 0$ which follows from that the triangle is acute.

Let E be the midpoint of AC and D be the midpoint of BC. Using the Lemma, I want to prove that $\angle POQ = \angle EOD$.

Notice that $OE = \sqrt{R^2 - \left(\frac{b}{2}\right)^2}$, by the Pythagorean Theorem. Similarly, $OD = \sqrt{R^2 - \left(\frac{a}{2}\right)^2}$. Since $\triangle ABC$ is acute and OD and

OE are both nonzero, if one of $ax - \frac{b}{2}$ and $\frac{a}{2} - bx$ is zero, the other must be zero. Next, notice that $PE = |ax - \frac{b}{2}|$ and $DQ = |\frac{a}{2} - bx|$. If they are zero, then P coincides with E and D coincides with Q, leaving the fact that $\angle POQ = \angle EOD$ obvious. So assume they are not zero. Then notice that

$$\frac{ax - \frac{b}{2}}{\frac{a}{2} - bx} = \frac{\sqrt{R^2 - (\frac{b}{2})^2}}{\sqrt{R^2 - (\frac{a}{2})^2}} > 0,$$

$ax - \frac{b}{2}$ has the same sign as $\frac{a}{2} - bx$.

If $ax > \frac{b}{2}$, then $\frac{a}{2} > bx$. If $ax < \frac{b}{2}$, then $\frac{a}{2} < bx$. So P and Q are on opposite sides of DE. Ir follows from the lemma and the fact that OE and OD bisect chords AC and CB respectively, $\frac{PE}{DQ} = \frac{OE}{OD}$ and $\angle PEO = \angle QDO = 90°$. So $\triangle OEP \sim \triangle ODQ$ and they have the same orientation. So $\angle EOP = \angle DOQ$. So $\angle EOD = \angle POQ$.

Let X be the point diametrically opposite to C on the circle. Then $CE = \frac{CA}{2}, CO = \frac{CX}{2}$. So $EO//AX$. Similarly, $DO//BX$. So $\angle EOD = \angle AXB, \angle POQ = \angle EOD = \angle AXB$.

Since $ACBX$ is cocyclic, $\angle AXB = 180° - \angle ACB$. Therefore, $\angle POQ = 180° - \angle ACB = 180° - \angle POQ$. So $PCQO$ is cocyclic.

Problem 6 Let $n \geq 2$ be a fixed even integer. We consider polynomials of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$$

with real coefficients, having at least one real root. Determine the least possible value of the sum $a_1^2 + \cdots + a_{n-1}^2$.

Solution: The answer is $4/(n-1)$. This can be achieved. When $a_1 = a_2 = a_3 = \cdots = a_{n-1} = -\frac{2}{n-1}$, the polynomial has root $x = 1$. Here, $a_1^2 + a_2^2 + \cdots + a_{n-1}^2 = (n-1)\left(\frac{4}{(n-1)^2}\right) = \frac{4}{n-1}$.

Now I want to prove that it cannot be less. There exists real number r such that

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + 1 = 0,$$

where r is obviously nonzero. In this case, $a_{n-1}r^{n-1} + \cdots + a_1 r = -1 - r^n$. By the Cauchy-Schwartz Inequality,

$$(-1-r^n)^2 = (a_{n-1}r^{n-1} + \cdots + a_1 r)^2 \leq (a_1^2 + a_2^2 + \cdots + a_{n-1}^2)(r^2 + r^4 + \cdots + r^{2n-2}).$$

So
$$(a_1^2 + a_2^2 + \cdots + a_{n-1}^2) \geq \frac{(-1 - r^n)^2}{r^2 + r^4 + \cdots + r^{2n-2}}.$$
So I only need to prove
$$\frac{(r^{2n} + r^n + r^n + 1)}{r^2 + r^4 + \cdots + r^{2n-2}} \geq \frac{4}{n-1},$$
which is equivalent to
$$(r^{2n} + r^n + r^n + 1)(n-1) \geq 4r^2 + 4r^4 + \cdots + 4r^{2n-2}.$$

To prove this, first observe that n is even, and therefore every term (in the inequality that we want to prove) involving r is raised to an even power. So if the inequality is true for r, it is also true for $-r$. Without loss of generality, assume $r > 0$.

Next, observe that if $a > c > d > b \geq 0$ and $a + b = c + d$, then $a - c = d - b$ and $(r^{a-c} - 1)r^c \geq (r^{d-b} - 1)r^b$. This is because

$$\text{if } r > 1, \text{ then } r^c > r^b \text{ and } r^{a-c} - 1 = r^{b-d} - 1 > 0;$$

$$\text{if } r < 1, \text{ then } r^c < r^b \text{ and } r^{a-c} - 1 = r^{b-d} - 1 < 0;$$

$$\text{if } r = 1, \text{ then both sides are zero,}$$

where we applied the fact that $c > b$.

So $r^a - r^c \geq r^d - r^b$ and $r^a + r^b \geq r^d + r^c$. Thus if $a > c > d > b \geq 0$ and $a + b = c + d$, then $r^a + r^b \geq r^d + r^c$.

Observe the $\frac{n}{2} - 1$ inequalities
$$\begin{array}{rcl} 2(r^{2n} + r^n) & \geq & 2(r^{2n-2} + r^{n+2}) \\ 2(r^{2n} + r^n) & \geq & 2(r^{2n-4} + r^{n+4}) \\ & \vdots & \\ 2(r^{2n} + r^n) & \geq & 2(r^{n+2} + r^{2n-2}) \end{array}$$
and the $\frac{n}{2} - 1$ inequalities
$$\begin{array}{rcl} 2(r^n + r^0) & \geq & 2(r^{n-2} + r^2) \\ 2(r^n + r^0) & \geq & 2(r^{n-4} + r^4) \\ & \vdots & \\ 2(r^n + r^0) & \geq & 2(r^2 + r^{n-2}) \end{array}$$
By AMGM,
$$(r^{2n} + 2r^n + 1) \geq 4r^n.$$

So adding all of these $2(\frac{n}{2}-1)$ inequalities and the last inequality, we get

$$2(\frac{n}{2}-1)(r^{2n}+r^n)+2(\frac{n}{2}-1)(r^n+1)+(r^{2n}+2r^n+1) \geq 4(r^2+r^4+\cdots+r^{2n-2}).$$

So

$$\begin{aligned}(r^{2n}+r^n+r^n+1)(n-1) &= 2(\frac{n}{2}-1)(r^{2n}+r^n)+2(\frac{n}{2}-1)(r^n+1)+(r^{2n}+2r^n+1)\\ &\geq 4(r^2+r^4+\cdots+r^{2n-2}),\end{aligned}$$

as desired.

Find all positive integers x, y such that $y \mid (x^2+1)$ and $x^2 \mid (y^3+1)$.

Let x, y, a be real numbers such that

$$x + y = x^3 + y^3 = x^5 + y^5 = a.$$

Determine all positive values of a.

Solution: The only values for a are 1 and 2.

First, we consider the case $x = y$. Then $2x = 2x^3$ implies $x = 0, 1$, or -1. $x = 0$ or -1 does not give a positive value for a, so x must equal 1. Then $a = 2$. So assume that $x > y$. x cannot be 0 since then $a = x + y = y < x = 0$. If $y = 0$, then $x = x^3$ implies $x = 0, 1$, or -1. Clearly, only $x = 1$ gives a positive value for a. So we additionally assume that x and y are nonzero.

Case I: $y > 0$.

Then since both x and y are nonnegative, we apply the Cauchy-Schwarz inequality to obtain the following:

$$(x^5 + y^5)(x + y) \geq (x^3 + y^3)^2.$$

But $(x^5 + y^5)(x + y) = a^2 = (x^3 + y^3)^2$, so equality holds in the above inequality. By the equality condition of the Cauchy-Schwarz inequality, $x^5/x = y^5/y$ implies $x = y$ or $x = -y$. $x = -y$ cannot occur because we require both x and y to be positive. $x = y$ cannot occur because of we assumed $x > y$. So there are no solutions for a in this case.

Case II: $y < 0$.

Clearly, $x > 0$, since otherwise, a would be negative. Let $y' = -y$. Then we have $x - y' = x^3 - y'^3 = x^5 - y'^5 = a$, where x and y' are positive. We claim that the following is true (*):

$$(x^5 - y'^5)(x - y') \leq (x^3 - y'^3)^2$$

with equality iff $x = y'$ or $x = -y'$.

We expand * to obtain the equivalent inequality:

$$x^6 + y'^6 - x^5y' - xy'^5 \leq x^6 + y'^6 - 2x^3y'^3,$$

or

$$x^5y' + xy'^5 - 2x^3y'^3 \geq 0.$$

Because $x, y' > 0$, we may divide by xy' to obtain the following:

$$x^4 + y'^4 - 2x^2y'^2 \geq 0,$$

which is trivial because the left-hand side is $(x^2 - y'^2)^2$, a nonnegative value, with equality iff $x = y'$ or $x = -y'$.

So our claim * is proven.

Because equality does indeed hold in *, $x = y'$ or $x = -y'$. Then by the reasoning in Case I, no positive solutions for a exist.

Thus, the ordered pairs $(x, y) = (1, 0), (0, 1)$ yield $a = 1$, and the ordered pair $(x, y) = (1, 1)$ yield $a = 2$. These two values are the only values attainable by a.

Let ABC be an acute triangle. Let M and N be points on the interiors of sides $\overline{AC}$ and $\overline{BC}$, respectively, and let K be the midpoint of segment $\overline{MN}$. The circumcircles of triangles CAN and BCM meet at C and at a second point D. Prove that line CD passes through the circumcircle of triangle ABC if and only if the perpendicular bisector of segment $\overline{AB}$ passes through K.

2002. numbers satisfying $a^2+b^2+c^2 = 1$. Prove that $\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \geq \frac{3}{4}\left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right)^2$ holds.

Solution: By the Cauchy-Schwarz inequality,

$$\sum \frac{a}{b^2+1} \sum (b^2+1)a^2 \geq (\sum a^{3/2})^2,$$

and multiplying $\frac{3}{4}$,

$$\frac{3}{4}\sum \frac{a}{b^2+1} \sum (b^2+1)a^2 \geq \frac{3}{4}(\sum a^{3/2})^2$$

So it suffices to prove that

$$\sum \frac{a}{b^2+1} \geq \frac{3}{4}\sum \frac{a}{b^2+1} \sum (b^2+1)a^2.$$

or

$$\frac{4}{3} \geq \sum a^2b^2 + \sum a^2,$$

which is equivalent to

$$1 \geq 3\sum a^2b^2,$$

where we have used the fact that $\sum a^2 = 1$.

We substitute $1 = (\sum a^2)^2 = \sum a^4 + 2\sum a^2b^2$ into the inequality above to obtain the following:

$$\sum a^4 + 2\sum a^2b^2 \geq 3\sum a^2b^2,$$

or

$$\sum a^4 \geq \sum a^2b^2.$$

The above inequality is true because after moving all terms to the left-hand side, we obtain

$$\sum \frac{(a^2-b^2)^2}{2} \geq 0.$$

From here it is easy to see that equality holds iff a = b = c.

2.6 St. Petersburg City Mathematical Olympiad (Russia)

Problem 1 Positive numbers a, b, c, d, x, y, and z satisfy $a + x = b + y = c + z = 1$. Prove that

$$(abc + xyz)\Big(\frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx}\Big) \geq 3.$$

Solution: Let us note that, by the rearrangement inequality, we have

$$\Big(\frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx}\Big) \geq \Big(\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}\Big).$$

Therefore, we only need prove that

$$(abc + xyz)\Big(\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}\Big) \geq 3.$$

We do so by smoothing. Now, suppose that we wish to smooth a to $\frac{1}{2}$. Then, we want

$$\frac{bc}{x} + \frac{ca}{y} + \frac{ab}{z} + \frac{xy}{c} + \frac{xz}{b} + \frac{yz}{a} \geq 2bc + \frac{c}{2y} + \frac{b}{2z} + \frac{y}{2c} + \frac{z}{2b} + 2yz.$$

So, if we have that $a \geq x, c \geq y, b \geq z, bc \geq yz$ or $x \geq a, y \geq c, z \geq b, yz \geq bc$, we can smooth a to $\frac{1}{2}$. Likewise, if $b \geq y, c \geq x, a \geq z, ac \geq xz$ or $y \geq b, x \geq c, z \geq a, xz \geq ac$, we can smooth b to $\frac{1}{2}$ and if $c \geq z, b \geq x, a \geq y, ab \geq xy$ or $z \geq c, x \geq b, y \geq a, xy \geq ab$, we can smooth c to $\frac{1}{2}$. Therefore, we can always smooth one of a, b, c to $\frac{1}{2}$. Because the variables are symmetric on the left side of the inequality, we can smooth any of a, b, or c to $\frac{1}{2}$. Thus, without loss of generality, smooth a to $\frac{1}{2}$. Then, applying the same reasoning, we can smooth b to $\frac{1}{2}$ and c to $\frac{1}{2}$. Therefore, because the expression equals 3 when $a = b = c = \frac{1}{2}$, we have

$$(abc + xyz)\Big(\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}\Big) \geq 3$$

as desired.

Problem 2 Let $ABCD$ be a convex quadrilateral such that $\angle ABC = 90°$, $AC = CD$, and $\angle BCA = \angle ACD$. Let E be the midpoint of segment AD, and L be the intersection point of segments BE and AC. Prove that $BC = CL$.

Problem 3 One can make the following operations on a positive integer:

(i) raise it to any positive integer power;

(ii) cut out the last two digits of the integer, multiply the obtained two-digit number by 3, and add it to the number formed by the remaining digits of the initial integer. (For example, from 3456789 one can get $34567 + 3 \cdot 89$.)

Can one obtain 82 from 81 by using operations (i) and (ii)?

Solution: We claim that 82 cannot be reached from 81. Let us consider the situation modulo 299. Suppose that we apply the second operation to a number x. Then, we get the number $\lfloor \frac{x}{100} \rfloor + 3\big(x - 100\lfloor \frac{x}{100} \rfloor\big) = 3x - 299\lfloor \frac{x}{100} \rfloor$. Therefore, given a number x, we obtain a number congruent to $3x$ modulo 299. Now, note that starting with a number that is congruent to 3^k modulo 299 and applying the second operation, we get another number that is also congruent to 3^k modulo 299. Now, if we start with such a number and apply the first operation, we again get a number that is congruent to 3^k modulo 299. Therefore, regardless of the operations we apply, we can only obtain numbers that are congruent to 3^k modulo 299 from 81. Now, note that numbers congruent to 3^k modulo 299 must be congruent to a number in the set $\{1, 3, 9, 27, 81, 243, 131, 94, 282, 248, 146, 139, 118, 55, 165, 196, 289, 269$ $209, 29, 87, 261, 185, 256, 170, 211, 35, 105, 16, 48, 144, 133, 100\}$, where each number is the residue modulo 299 of three times the previous and $1 \equiv 3 \cdot 100$ modulo 299. Since 82 is not in this set, it cannot be reached from 81, as desired.

Problem 4 Points M and N are marked on the diagonals AC and BD of cyclic quadrilateral $ABCD$. Given that $\frac{BN}{DN} = \frac{AM}{CM}$ and $\angle BAD = \angle BMC$, prove that $\angle ANB = \angle ADC$.

Problem 5 A country consists of no fewer that 100,000 cities, where 2001 paths are outgoing from each city. Each path connects two cities, and every two cities are connected by no more than one path. The government decides to close some of the paths (at least one but not all) so that the number of paths outgoing from each city is the same. Is this always possible?

Problem 6 Let ABC be a triangle and let I be the center of its incircle ω. The circle Γ passes through I and is tangent to AB and AC at points X and Y, respectively. Prove that segment $\overline{XY}$ is tangent to ω.

Solution: Extend line segment AI so that it meets with line segment XY at a point M. Since AX is tangent to Γ at X, $\angle AXI = \angle IYM$. Since AX and AY are both tangents to Γ at X and Y, their lengths must be equal. Therefore, triangle XAY is isosceles. Because AI is the angle bisector of $\angle BAC$, AM must be the angle bisector of $\angle XAY$. We just showed that triangle XAY is isosceles, so AM is also the altitude and median of triangle XAY. Thus, IM = IM, $\angle IMX = \angle IMY$, and XM = YM and so triangles IMX and IMY are congruent by the SAS congruence theorem. As a consequence, all corresponding parts are congruent, in particular, $\angle IYM = \angle IXM = \angle IXY$. Therefore:

$\angle AXI = \angle IYM = \angle IXM = \angle IXY$ and so XI is the angle bisector of $\angle AXY$.

This combined with the fact that AI is the angle bisector of $\angle XAY$, so conclude that I is the incenter of triangle AXY and therefore ω is the incircle of triangle AXY.

Problem 7 Several 1×3 rectangles and 100 L-shaped figures formed by three unit squares ("corners") are situated on a grid plane. It is known that these figures can be shifted parallel to themselves so that the resulting figure is a rectangle. A student Olya can translate 96 corners to form 48 2×3 rectangles. Prove that the remaining four corners can be translated to form two additional 2×3 rectangles.

Problem 8 The sequence $\{a_n\}$ is given by the following relation:

$$a_{n+1} = \begin{cases} (a_n - 1)/2, & \text{if } a_n \geq 1; \\ 2a_n/(1 - a_n), & \text{if } a_n < 1. \end{cases}$$

Given that a_0 is a positive integer, $a_n \neq 2$ for each $n = 1, 2, ..., 2001$, and $a_{2002} = 2$. Find a_0.

Solution: Answer: $a_0 = 3 \cdot 2^2 002 - 1$.

We will first show that this value actually satisfies the condition $a_2 002 = 2 and a_i \neq 2$ for any $i < 2002$. Applying the first rule,

$a_(n+1) = ((a_n - 1)/22002$ times will show that a_2002 is in fact 2 and a_i is actually greater than 2 for all $i < 2002$.

Lemma 1 For $n \leq 2000$, if a_n is not an integer, $a_n = p_n/q_n$, where p_n and q_n are positive odd integers, $(p, q) = 1$ and $q > 1$.

Proof We proceed by using induction. Base Case: $n = 2000$. a_2000 is either 5/7 or 1/5. Suppose $a_(n + 1) = p/q$. Then:

Case 1: $a_n = 2 \cdot p/q + 1$. Then $a_n = (2p + q)/q$. Suppose $d \mid (2p + q)$ and $d \mid q$. Then $d \mid (2p + q - q)$ or $d \mid 2p$. Since q is odd, d must also be odd. So $(d, 2) = 1$ and therefore $d \mid p$. Because $d \mid p$, $d \mid q$, and $(p, q) = 1$, d must equal 1. Therefore $(2p + q, q) = 1$ and since $2p + q$ and q are both odd as well, a_n satisfies the conditions stated in the lemma, thus completing the inductive step.

Case 2: $a_n = (p/q)/(2 + p/q) = p/(2q + p)$ Again suppose d divides both the numerator p and denominator $2q + p$. Then $d \mid (2q + p - p)$ or $d \mid 2q$. But since p is odd, d is odd, and so $d \mid q$. Because $(p, q) = 1$, d is equal to 1 and so $(p, 2q + p) = 1$. Since p and $2q + p$ are odd and are relatively prime, a_n satisfies the conditions and completes the proof.

We must now only consider the case where $a_2001 = 1/2$. In this case, a_2000 is either 2 or 5/7. If it is 2, the conditions of the problem are violated. The lemma says that from the 5/7, we will never see an integer value for any previous term.

Problem 9 There are two 2-pan balances in a zoo for weighing animals. An elephant is located on a pan of the first balance and a camel is on a pan of the second balance. The weight of the elephant, as well as the weight of the camel, expressed in kilograms, is a whole number. The total weight of both the elephant and the camel does not exceed 2 tons (2000 kilograms). A set of weights, total of 2 tons, had been delivered to the zoo, where each weight weighs a whole number of kilograms. It turned out that no matter what the elephant's and the camel's weights are, one can distribute some of the weights over the 4 balances' pans so that both balances are in the equilibrium. Find the minimal possible number of weights that can be delivered to the zoo.

Problem 10 An integer $N = \overline{a0a0 \ldots a0b0c0c0 \ldots c0}$, where the digits a and c are written 1001 times each) is divisible by 37. Prove that $b = a + c$.

Problem 11 Let $ABCD$ be a trapezoid the length of the lateral side AB equals the sum of lengths of the base AD and base BC.

Prove that the angle bisectors of the angles A and B meet at a point belonging to the side CD.

Problem 12 Can the sum of the pairwise distances between the vertices of a 25-vertex tree be equal to 1225?

Problem 13 The integers from 5 to 10 are written on a blackboard. One time a minute, Kolya erases three or four smallest integers and write down seven or eight consecutive integers followed by the biggest integer. Prove that the sum of all the integers on the blackboard at an arbitrary moment of time is not a power of 3.

Problem 14 Find the maximal value of $\alpha > 0$ for which any set of eleven real numbers,

$$0 = a_1 \leq a_2 \leq ... \leq a_{11} = 1 \ ,$$

can be split into two disjoint subsets with the following property: the arithmetical mean of the numbers in the first subset differs from the arithmetical mean of the numbers in the second subset by less than α.

Problem 15 Let O be the circumcenter of an acute scalene triangle ABC, point C_1 be the point symmetric to C with respect to O, D be the midpoint of side AB, and K be the circumcenter of $\triangle ODC_1$. Prove that point O divides into two equal halves the segment of line OK that lies inside the angle ACB.

Problem 16 Polygon F, any three vertices of which are not collinear, can be dissected in two different ways into triangles by drawing its non-intersecting diagonals. Prove that there some four F's vertices form a convex quadrilateral lying entirely inside F.

Problem 17 Let p be a prime number. Given that the equation

$$p^k + p^l + p^m \ = \ n^2$$

has an integer solution, prove that $p + 1$ is divisible by 8.

Problem 18 Prove that any 13 numbers

$$0 = x_1 \leq x_2 \leq ... \leq x_{12} \leq x_{13} = 1$$

can be divided into two disjoint groups such that the arithmetic mean of the first group differ from the arithmetic mean of the second group by no less than 13/24.

Problem 19 An alchemist has 50 different substances. He can convert any 49 substances taken in equal quantities into the remaining (i.e. the 50th) substance without changing the total mass. Prove that, after finite number of manipulations, the alchemist can obtain the same amount of each of the 50 substances.

Problem 20 Let $ABCD$ be a cyclic quadrilateral. Points X and Y are marked on the sides AB and BC such that $XBYD$ is a parallelogram. Points M and N are the midpoints of diagonals AC and BD, and the lines AC and XY meet at point L. Prove that points M, N, L, and D lie on the same circle.

Problem 21 Two players play the following game. There are 64 vertices on the plane at the beginning. The first player each time picks any two vertices that are not connected yet and connect them by an edge, and the second player orient this edge arbitrarily (i.e. introduce a direction on this edge). The second player wins if the graph obtained after 1959 moves is connected, otherwise the first player wins. Which player wins in the true game?

Problem 22 The shape of a lakeside is a convex centrally-symmetric 100-gon $A_1A_2 \ldots A_{100}$ with the center of symmetry O. There is the island $B_1B_2 \ldots B_{100}$ inside the lake whose vertices B_i are the midpoints of segments OA_i for each $i = 1, 2, \ldots, 100$. There is a jail on the island surrounded with a high fence along its perimeter. Two security guards are situated at the opposite points of the lakeside. Prove that they observe together the whole lakeside entirely.

Problem 23 A secret code to any of the FBI's safe is a positive integer from 1 to 1700. Two spies learned two different codes, each one his own, and decided to exchange their information. Coordinated their action, they have met at the river's shore nearby a pile of 26 rocks. At the beginning, the first spy threw several rocks into the river water, then the second , then the first, and so on until all the rocks were used. The spies went away after that. How can the information be transmitted? (The spies said no word to each other.)

Problem 24 A flea jumps along integer points of the real line, starting from the origin. The length of each its jump is 1. During each jump, the flea sings one of the $(p-1)/2$ known songs, where p is an odd prime number. Consider all the flea's musical paths from the origin back to the origin each of which consists of no more than $p-1$ jumps. Prove that the number of all such musical paths is divisible by p.

Problem 25 Let $ABCD$ be a quadrilateral, and O be the center of the circle inscribed into $ABCD$. A line ℓ passes through O and meet side AB at point X and side CD at point Y. Given that $\angle AXY = \angle DYX$, prove that $AX/BX = CY/DY$.

Problem 26 Let $a_n = F_n^n$ be the sequence, where F_n are the Fibonacci numbers ($F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$). Is the sequence $b_n = \sqrt{a_1 + \sqrt{a_2 + \ \ldots\ \sqrt{a_n}}}$ bounded from the above?

Problem 27 Let a and b be positive integers such that $2a+1$ and $2b+1$ are relatively prime integers. Find all possible values of $g.c.d.(2^{2a+1} + 2^{2a+1} + 1,\ 2^{2b+1} + 2^{2b+1} + 1)$.

Problem 28 Let O be the center of the circle inscribed into $\triangle ABC$. It is tangent to the sides BC, CA, and AB at points A_1, B_1, and C_1, respectively. Line ℓ passes through point A_1, is perpendicular to segment AA_1, and meets line B_1C_1 at point X. Prove that line BC passes through the midpoint of segment AX.

Problem 29 A positive integer is written on a blackboard. Dima and Sasha play the following game. Dima calls some positive integer x, and Sasha changes the number on the blackboard by either adding x to it, or by subtracting x from it. They repeat this procedure many times. Dima's goal is to get eventually a power of a particular positive integer k on the board (including also $k^0 = 1$). Find all possible values of k for which Dima will be able to do this independently of the initial number written on the board.

Problem 30 Find all continuous functions $f : (0, \infty) \longrightarrow (0, \infty)$ such that for any positive x and y

$$f(x)f(y) = f(xy) + f(\frac{x}{y}) \ .$$

3
2003 Selected Problems from Around the World

3.1 Algebra

Problem 1 Find a monic polynomial $f(x)$ with integer coefficients, and degree at least three, such that there exist non-constant polynomials $g(x)$ and $h(x)$ with integer coefficients for which

$$(f(x))^3 - 2 = g(x)h(x).$$

Problem 2 Find all triples of positive real numbers (x, y, z), if any, that satisfy the system of equations

$$x + y + z = x^3 + y^3 + z^3$$
$$x^2 + y^2 + z^2 = xyz.$$

Problem 3 Let a, b, and c be the side lengths of a triangle with perimeter 2. Prove that

$$1 < ab + bc + ca - abc \leq \frac{28}{27}.$$

Problem 4 Find all angles α for which the three-element set

$$S = \{\sin\alpha, \sin 2\alpha, \sin 3\alpha\}$$

is equal to the set

$$T = \{\cos\alpha, \cos 2\alpha, \cos 3\alpha\}.$$

Problem 5 Suppose we have a polynomial

$$f(x) = a_{2003}x^{2003} + a_{2002}x^{2002} + \cdots + a_1x + a_0,$$

and there are positive integers p, q, and r with $p < q < r$ such that $f(p) = q$, $f(q) = r$, and $f(r) = p$. Prove that at least one coefficient a_i is not a integer $(0 \leq i \leq 2003)$.

Problem 6 Find the number of triples of real numbers (x, y, z) that satisfy

$$x + y + z = 3xy$$
$$x^2 + y^2 + z^2 = 3xz$$
$$x^3 + y^3 + z^3 = 3yz.$$

Problem 7 Let $P(x)$ be a polynomial with integer coefficients such that $P(n) > n$ for all positive integers n. Suppose that for each positive integer m, there is a term in the sequence

$$P(1),\ P(P(1)),\ P(P(P(1))),\ \ldots$$

which is divisible by m. Show that $P(x) = x + 1$.

Problem 8 Let p and q be positive integers, and let

$$P(x) = (x+1)^p(x-3)^q = x^n + a_1x^{n-1} + \cdots + a_n.$$

(a) Suppose that $a_1 = a_2$. Prove that $3n$ is a perfect square.

(b) Prove that there exist infinitely many pairs of positive integers (p, q) such that $a_1 = a_2$.

Problem 9 Let n be an integer greater than 1, and let

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + 1$$

be a polynomial with nonnegative integer coefficients such that $a_k = a_{n-k}$ for $1 \le k \le n-1$. Prove that there exist infinitely many pairs of positive integers a and b for which a divides $p(b)$ and b divides $p(a)$.

Problem 10 Find all functions $f:\ \mathbb{Q} \to \mathbb{R}$ such that for all rational numbers x and y,

(a) $f(x+y) - yf(x) - xf(y) = f(x)f(y) - x - y + xy$

(b) $f(x) = 2f(x+1) + x + 2$

(c) $f(1) + 1 > 0$.

Problem 11 Let a, b, and c be the lengths of the sides of a triangle with $a + b + c = 1$. Prove that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2},$$

for all integers n greater than 1.

Problem 12 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(x)f(y) = f(x) + f(y) + f(xy),$$

for all pairs of real numbers x and y.

Problem 13 Let a, b, and c be positive real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that $a + b + c \leq 3$.

Problem 14 Determine the smallest positive number a such that there exists a positive real number b for which

$$\sqrt{1+x} + \sqrt{1-x} \leq 2 - \frac{x^a}{b},$$

for all real numbers x in the interval $[0, 1]$.

Problem 15 Let $\mathbb{R}^+$ denote the set of positive real numbers, and let F be the set of all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(3x) \geq f(f(2x)) + x$$

for all positive real numbers x. Determine the maximum value of constant c such that $f(x) \geq cx$ for all f in the set F and all positive real numbers x.

Problem 16 Determine if there exists a surjective function $f : \mathbb{R} \to \mathbb{R}$, such that the expression $f(x + y) - f(x) - f(y)$ takes exactly two values 0 and 1 for all pairs of real numbers x and y.

Problem 17 Let x, y, and z be nonnegative real numbers such that

$$x^2 + y^2 + z^2 + x + 2y + 3z = \frac{13}{4}.$$

(a) Determine the maximum value of $x + y + z$.

(b) Prove that

$$x + y + z \geq \frac{\sqrt{22} - 3}{2}.$$

Problem 18 Let k be an integer, and let $\{y_n\}_{n \geq 1}$ be a sequence of real numbers such that $y_1 = y_2 = 1$, and

$$y_{n+2} = (4k - 5)y_{n+1} - y_n + 4 - 2k$$

for all $n \geq 1$. Determine all integers k such that every term of the sequence is a square of an integer.

Problem 19 Let a, b, c, d be positive real numbers with $ab+cd = 1$. For $i = 1, 2, 3, 4$, points $P_i = (x_i, y_i)$ are on the unit circle. Prove that

$$(ay_1 + by_2 + cy_3 + dy_4)^2 + (ax_4 + bx_3 + cx_2 + dx_1)^2$$
$$\leq 2\left(\frac{a^2+b^2}{ab} + \frac{c^2+d^2}{cd}\right).$$

Problem 20 Determine all polynomials $P(x)$ with integer coefficients such that for any positive integer n the equation $P(x) = 2^n$ has at least one integer root.

Problem 21 For every real number x, define $\langle x \rangle = \min(\{x\}, \{1 - x\})$, where $\{x\}$ denotes the fractional part of x. Prove that for every irrational number α and every positive real number ϵ there exists a positive integer n such that

$$\langle n^2\alpha \rangle < \epsilon.$$

3.2 Combinatorics

Problem 1 Eight different positive integers are used to label the vertices of a cube. Each edge of the cube is then labeled with the unsigned difference of the labels of its endpoints. What is the smallest number of distinct edge labels that could be produced in this way?

Problem 2 Consider the function f defined on the set of positive integers

$$f(n) = \begin{cases} n+1, & \text{if } x \text{ is odd,} \\ \frac{n}{2}, & \text{if } x \text{ is even.} \end{cases}$$

For a positive integer k with $k \geq 2$, let $f^k(n) = f(f^{k-1}(n))$. We say n has *characteristic* k if k is the smallest positive integer such that $f^k(n) = 1$. For each positive integer k, determine the smallest positive integer m_k with characteristic k.

Problem 3 Nine chairs, indistinguishable except for color, are to be placed around a circular table. Three of the chairs are red, three are green, and three are blue. How many different arrangements are possible? (Two arrangements are considered the same if and only if one can be obtained from the other by a rotation.)

Problem 4 Let n be a positive integer. Alex and Chris play the following game. Alex writes down n different positive integers. Chris then deletes some numbers (possibly none, but not all), puts the signs $+$ and $-$ in front of each of the remaining numbers, and sums them up. If the result is divisible by 2003, Chris wins the game. Otherwise, Alex wins. Who has an winning strategy?

Problem 5 Let $A = \{1, 2, \ldots, 2002\}$ and $M = \{1001, 2003, 3005\}$. A subset B of A is called M-*free* if the sum $b_1 + b_2$ of any pairs of elements b_1 and b_2 in B is not in M. An ordered pair of subsets (A_1, A_2) is called a M-*partition* of A if A_1 and A_2 form a partition of A and both A_1 and A_2 are M-free. Determine the number of M-partitions of A.

Problem 6 Let $x_1, x_2, \ldots, x_5$ be real numbers. Find the least positive integer n with the following property: if some n distinct sums of the form $x_p + x_q + x_r$ $(1 \leq p < q < r \leq 5)$ are equal to 0, then $x_1 = x_2 = \cdots = x_5 = 0$.

Problem 7 Let n be a positive integer. Prove that one can partition the set of all sides and diagonals of a convex 3^n-sided polygon into groups of three segments, such that in each group the three segments form a triangle.

Problem 8 Determine if it is possible to color each positive integer in either red or blue such that

(a) there are infinitely many integers in each color; and

(b) the sum of $n = 2002$ distinct red integers is red and the sum of $n = 2002$ distinct blue integers is blue.

What if $n = 2003$?

Problem 9 Let m, n, and k be positive integers with $m > n > k$, and let $a_1, a_2, \ldots, a_m$ be a binary sequence (that is, $a_i = 0$ or 1 for all $1 \leq i \leq m$) such that the sum of every n consecutive terms is equal to k. Let $s = a_1 + a_2 + \cdots + a_m$, and let M_1 and M_2 be the maximum and minimum value of s, respectively. Determine the value of $M_1 - M_2$, in terms of m, n, and k.

Problem 10 Find the smallest positive integer n such that:

> For every finite set of points in the plane, if for every n points of this set, there exist two lines covering all n points, then there exists two lines covering all the points in the set.

Problem 11 Ten people are applying for a job. The job selection committee decides to interview the candidates one by one. The order of candidates being interviewed is random. Assume that all the candidates have distinct abilities. The following policies are set up within the committee:

(i) The first three candidates interviewed will not be fired;

(ii) For $4 \leq i \leq 9$, if the i^{th} candidate interviewed is more capable than all the previously interviewed candidates, then this candidate is hired and the interview process is terminated;

(iii) The 10^{th} candidate interviewed will be hired.

For $1 \leq k \leq 10$, let P_k denote the probability that the k^{th} most able person is hired under the selection policies. Show that,

1. $P_1 > P_2 > \cdots > P_8 = P_9 = P_{10}$

2. There is more than 70% chance that one of the three most able candidates is hired and there is no more than 10% chance that one of the three least able candidates is hired.

Problem 12 Given two positive integers m and n, determine the smallest positive integer k (in terms of n) such that among any k people, either there are $2m$ of them who form m pairs of mutually acquainted people or there are $2n$ of them forming n pairs of mutually unacquainted people. (Acquaintance between two people is a mutual relation.)

Problem 13 Let m and n be relatively prime odd integers. A rectangle $ABCD$ with $AB = m$ and $AD = n$ is partitioned into mn unit squares. Starting from $A_1 = A$ denote by $A_1, A_2, \ldots A_k = C$ the consequent intersecting points of the diagonal AC with the sides of the unit squares. Prove that

$$\sum_{j=1}^{k-1} (-1)^{j+1} A_j A_{j+1} = \frac{\sqrt{m^2+n^2}}{mn}.$$

Problem 14 Let n be a positive integer, and let $A_1, A_2, \ldots A_{n+1}$ be nonempty subsets of the set $\{1, 2, \ldots, n\}$. Prove that there exist nonempty and disjoint index sets $I_1 = \{i_1, i_2, \ldots, i_k\}$ and $I_2 = \{j_1, j_2, \ldots, j_m\}$ such that

$$A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} = A_{j_1} \cup A_{j_2} \cup \cdots \cup A_{j_m}.$$

Problem 15 Let n be a positive integer such that one can place n distinct points in the plane with no three collinear, and color them with either red, green, or yellow so that

(i) inside each triangle with all red vertices, there is at least one green point;

(ii) inside each triangle with all green vertices, there is at least one yellow point; and

(iii) inside each triangle with all yellow vertices, there is at least one red point.

Determine the maximum value of n.

Problem 16 We have partitioned a rectangular region into x smaller rectangular regions (not necessarily distinct in size) in such a way that the sides of the smaller rectangles are parallel to the sides of the original rectangular region. A point is a *cross point* if it is on the boundary of four smaller regions. Suppose that there are y cross points. A side of a region is *maximal* if there is no other side of a region that properly contains it. Suppose that there are z maximal sides. Compute $x - y - z$.

Problem 17 A set S of positive integers is *good* if for every integer k there exist distinct elements a and b in S, such that the numbers $a+k$ and $b+k$ are not relatively prime. Prove that if the sum of the elements of a good set S equals 2003, then there exists an element s in S such that $S \setminus \{s\}$ is also good.

Problem 18 Let S be a set such that

1. Each element of S is a positive integer no greater than 100;
2. For any two distinct elements a and b in S, there exists an element c in S such that a and c are relatively prime, as are b and c; and
3. For any two distinct elements a and b in S, there exists a third element d in S such that a and d are not relatively prime, nor are b and d.

Determine the maximum number of elements S can have.

Problem 19 Let set $S = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R},\ 1 \le i \le n\}$, and let A be a finite subset of S. For any pair of elements $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in A, define $d(a,b) = (|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_n - b_n|)$ and $D(A) = \{d(a,b) \mid a, b \in A\}$. Prove that the set $D(A)$ contains more elements than the set A does.

Problem 20 A set A of positive integers is *uniform* if after any one of its elements is removed, the remaining ones can be partitioned into two subsets with equal sums of their elements. Find the smallest positive integer n with $n > 1$ such that there exists a uniform set with n elements.

Problem 21 Let n and k be two positive integers with $2 \le k \le n$. Denote by P the set of all permutations of the set $\{1, 2, \ldots, n\}$. Let F be a subset of P such that for every pair of elements f and g in P, there exists a positive integer t with $1 \le t \le n$ satisfying this property: the block $(t, t+1, \ldots, t+k-1)$ is contained in both f and g. Prove that the set F contains at most 2^{n-k} elements.

3.3 Geometry

Problem 1 Prove that when three circles share the same chord AB, every line through A, other from AB, determines the same ratio $XY : YZ$, where X is an arbitrary point different from B on the first circle while Y and Z are the points where the line AX intersects the other two circles. (The points are labled in such a way that Y is between X and Z.)

Problem 2 In triangle ABC, $AB = 5$ and $BC = 8$. Let I be the incenter of triangle ABC, and let A_1, B_1, C_1 are the reflections of I across lines BC, CA, AB, respectively. Suppose that B lies on the circumcircle of triangle $A_1B_1C_1$. Find CA.

Problem 3 In parallelogram $ABCD$, the bisector of $\angle BAC$ intersects side BC at E. Given that $BE + BC = BD$, compute BD/BC.

Problem 4 Triangle PEA has a right angle at E, and $PE < EA$. Point X is on segment PE, and point Y is on segment AX, situated so that $\angle XYP = \angle YPX = \angle EAP$. Calculate the ratio AY/XE.

Problem 5 Suppose $ABCD$ is a square piece of cardboard with side length a. On a plane are two parallel lines ℓ_1 and ℓ_2, which are also a units apart. The square $ABCD$ is placed on the plane so that sides AB and AD intersect line ℓ_1 at E and F, respectively, and sides CB and CD intersect line ℓ_2 at G and H, respectively. Let the perimeters of triangles AEF and CGH be m_1 and m_2, respectively. Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

Problem 6 Let ABC be a acute triangle with H as its orthocenter. Points M and N lie on line AC with $MN = AC$. Let D and E be the feet of perpendiculars from M to line BC and N to line AB, respectively.

(a) Prove that B, D, E, H lie on a circle.

(b) Prove that the midpoint of segment AN is symmetric to B with respect to the line passing through the circumcenters of triangles ABC and BED.

Problem 7 Let AKS be a triangle with $\angle AKS > 90°$. Construct a triangle ABC so that side BC lies on line KS in such a way that segment AS and line AK are the median and angle bisector of triangle ABC, respectively.

Problem 8 Let ABC be a fixed acute-angled triangle, and let S be a point on side AB. Determine the minimum value of $[SXY]$, where X and Y are the respective circumcenters of triangles ACS and BCS.

Problem 9 Point P lies inside triangle ABC. Line BP meets segment AC at Q, and line CP meets segment AB at R. Suppose that $AR = RB = CP$ and $CQ = PQ$. Determine $\angle BRC$.

Problem 10 Points K, L, and M lie, in that order, on a line in the plane. Find the locus of all the points C of squares $ABCD$ in the plane such that K, L, and M lie on segments AB, BD, and CD, respectively.

Problem 11 Let S be a set of n points in the plane such that any two points in S are at least 1 unit apart. Prove that there is a subset T of S with at least $\lceil \frac{n}{7} \rceil$ points such that any two points of T are at least $\sqrt{3}$ units apart.

Problem 12 Let ABC be a triangle with $AB \neq AC$, and let D be a point on line BC such that line AD is tangent to the circumcircle of triangle ABC. Point E lies on the perpendicular bisector of segment AB such that $EB \perp CB$. Point F lies on the perpendicular bisector of segment AC such that $FC \perp BC$. Prove that points D, E, F are collinear.

Problem 13 Let ABC be an acute triangle with I and H as its incenter and orthocenter, respectively. Let B_1 and C_1 be the midpoints of $\overline{AC}$ and $\overline{AB}$ respectively. Ray B_1I intersects $\overline{AB}$ at $B_2 \neq B$. Ray C_1I intersects ray AC at C_2 with $C_2A > CA$. Let K be the intersection of $\overline{BC}$ and $\overline{B_2C_2}$. Prove that triangles BKB_2 and CKC_2 have the same area if and only if A, I, A_1 are collinear, where A_1 is the circumcenter of triangle BHC.

Problem 14 Suppose that the incircle of triangle ABC is tangent to sides AB, BC, and CA at points P, Q, and R, respectively. Prove that

$$\frac{BC}{PQ} + \frac{CA}{QR} + \frac{AB}{RP} \geq 6.$$

Problem 15 Point D lies on side AC of triangle ABC such that $BD = CD$. Point E lie on side BC. Line ℓ passes through E and is parallel to line BD. Lines AB and ℓ meet at F. Let G denote the intersection of segments AE and BD. Prove that $\angle BCG = \angle BCF$.

Problem 16 Given a rhombus $ABCD$ with $\angle A < 90°$, its two diagonals AC and BD meet at point M. A point O on the segment MC is taken so that $O \neq M$ and $OB < OC$. The circle centered at O passing through B and D meets line AB at B and a point X ($X = B$ when line AB is tangent to the circle), and meets line BC at B and Y. Lines DX and DY meet the segments AC at P and Q, respectively. Suppose that $t = \frac{MA}{MO}$. Express $\frac{OQ}{OP}$ in terms of t.

Problem 17 Let A be a point outside circle ω. Lines AB and AC are tangent to circle ω at B and C, respectively. Line ℓ is tangent to circle ω, and it meets lines AB and AC at P and Q, respectively. Point R lies on line BC such that $PR \parallel AC$. Prove that as ℓ varies, line QR passes through a fixed point.

Problem 18 Let ABC be a triangle with ω and I as its incircle and incenter, respectively. Circle ω is tangent to sides AB and AC at points X and Y, respectively. Line XI meets ω again at M, and line CM meets side AB at Z. Point L lies on segment CZ such that $ZL = CM$. Prove that points A, I, and L are collinear if and only if $AB = AC$.

Problem 19 Let H be an arbitrary point on the altitude CP of the acute-angled triangle ABC. Lines AH and BH intersect sides BC and AC at M and N, respectively.

(a) Prove that $\angle NPC = \angle MPC$.

(b) Let lines MN and CP meet at O, and let ℓ be a line passing through O. Line ℓ intersects the sides of of quadrilateral $CNHM$ at D and E. Prove that $\angle EPC = \angle DPC$.

Problem 20 Let ABC be an acute triangle, and let D be a point on side BC such that $\angle BAD = \angle CAD$. Points E and F are the feet of perpendiculars from D to sides AC and AB, respectively. Let H be the intersection of segments BE and CF. The circumcircle of triangle AFH meets line BE again at G. Prove that segments BG, GE, BF can be the sides of a right triangle.

Problem 21 Triangle ABC is inscribed in circle ω. Circle ω_a is tangent to sides AB and AC and circle ω. Denote by r_a the radius of circle ω_a. Define r_b and r_c analogously. Let r be the inradius of triangle ABC. Prove that

$$r_a + r_b + r_c \geq 4r.$$

3.4 Number Theory

Problem 1 We have two distinct positive integers a and b, with b a multiple of a. Written in decimal, each of a and b consists of $2n$ digits. Furthermore, the n left-most digits of a are identical to the n right-most digits of b, and vice versa. (For example, $n = 2$, $a = 1234$, and $b = 3412$, although this example does not meet another condition that b be a multiple of a.) Determine a and b.

Problem 2 Find the last three digits of the number $2003^{2002^{2001}}$.

Problem 3 For each positive integer n, let $f(n)$ be the smallest positive number M for which $M!$ is divisible by n. Determine all positive integers n for which

$$\frac{f(n)}{n} = \frac{2}{3}.$$

Problem 4 Find all the ordered triples (a, m, n) of positive integers such that $a \geq 2$, $m \geq 2$, and $a^m + 1$ divides $a^n + 203$.

Problem 5 Let k be an integer greater then 13, and let p_k be the largest prime less than k. You may assume that $p_k \geq \frac{3k}{4}$. Let n be a composite integer. Prove that

(a) if $n = 2p_k$, then n does not divide $(n-k)!$

(b) if $n > 2p_k$, then n divides $(n-k)!$

Problem 6 Find all real numbers a such that

$$4\lfloor an \rfloor = n + \lfloor a \lfloor an \rfloor \rfloor$$

for every positive integer n. (We denote by $\lfloor x \rfloor$ the greatest integer not exceeding x.)

Problem 7 Find all ordered triples (a, b, c) of positive integers such that

(a) $a \leq b \leq c$

(b) $\gcd(a, b, c) = 1$

(c) $a^3 + b^3 + c^3$ is divisible by each of the numbers $a^2 b, b^2 c$, and $c^2 a$.

Problem 8 Find all permutations $(a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$ such that

$$(i+1) \mid 2(a_1 + a_2 + \cdots + a_i)$$

for all $1 \leq i \leq n$.

Problem 9 Determine if there are infinitely many pairs of ordered positive integers (a, b) such that

(1) a and b are relatively prime;

(2) a is a divisor of $b^2 - 5$; and

(3) b is a divisor of $a^2 - 5$.

Problem 10 Find the greatest positive integer n such that the system of equations

$$(x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = \cdots = (x+n)^2 + y_n^2$$

has an integer solution $(x, y_1, y_2, \dots, y_n)$.

Problem 11 Determine if it is possible to find 2002 distinct positive integers $a_1, a_2, \dots, a_{2002}$ such that $|a_i - a_j| = \gcd(a_i, a_j)$ for all $1 \le i < j \le 2002$.

Problem 12 A positive integer is *normal* if it can be represented as an arithmetic mean of some (not necessarily) distinct integers each being a nonnegative power of 2. A positive integer is *interesting* if it can be represented as an arithmetic mean of some distinct integers each being a nonnegative power of 2. Prove that all positive integers are normal and that there are infinitely not interesting positive integers.

Problem 13 The sequence $\{a_k\}_{k\ge 0}$ is defined as $a_0 = 2$, $a_1 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for all $n \ge 1$. Prove that if p is a prime factor of $a_{2k} - 2$, then it is a factor of $a_{2k+1} - 1$.

Problem 14 The set $\{1, 2, \dots, 3n\}$ is partitioned into three sets A, B, and C with each set containing n numbers. Determine with proof if it is always possible to choose one number out of each set so that one of these numbers is the sum of the other two.

Problem 15 Let m be a positive integer.

(a) Prove that if $2^{m+1} + 1$ divides $3^{2^m} + 1$, then $2^{m+1} + 1$ is a prime.

(b) Is the converse of (a) true?

Problem 16 Let A be a subset of the set $\{1, 2, \dots, 29\}$ such that for any integer k and any elements a and b in A (a and b are not necessarily distinct), $a + b + 30k$ is not the product of two consecutive integers. Find the maximum number of elements A can have.

Problem 17 Determine all ordered triples of integers (x, y, z) with $x \neq 0$ such that

$$2x^4 + 2x^2y^2 + y^4 = z^2.$$

Problem 18 Let a, b, c be rational numbers such that both $a+b+c$ and $a^2 + b^2 + c^2$ are equal integers. Prove that there exist relative prime integers m and n such that

$$abc = \frac{m^2}{n^3}.$$

Problem 19 Let n be a positive integer greater than 1, and let p be a prime such that n divides $p-1$ and p divides $n^3 - 1$. Prove that $4p - 3$ is a perfect square.

Problem 20 Determine all positive integers n such that

$$\frac{1}{n+1} \cdot \binom{2n}{n}$$

is odd.

Problem 21 Determine all functions from the set of positive integers to the set of real numbers such that

(a) $f(n+1) \geq f(n)$ for all positive integers n; and

(b) $f(mn) = f(m)f(n)$ for all relatively prime positive integers m and n.