Preface

This book is a continuation *Mathematical Olympiads 1995-1996: Olympiad Problems from Around the World*, published by the American Mathematics Competitions. It contains solutions to the problems from 25 national and regional contests featured in the earlier pamphlet, together with selected problems (without solutions) from national and regional contests given during 1997.

This collection is intended as practice for the serious student who wishes to improve his or her performance on the USAMO. Some of the problems are comparable to the USAMO in that they came from national contests. Others are harder, as some countries first have a national olympiad, and later one or more exams to select a team for the IMO. And some problems come from regional international contests ("mini-IMOs").

Different nations have different mathematical cultures, so you will find some of these problems extremely hard and some rather easy. We have tried to present a wide variety of problems, especially from those countries that have often done well at the IMO.

Each contest has its own time limit. We have not furnished this information, because we have not always included complete contests. As a rule of thumb, most contests allow a time limit ranging between one-half to one full hour per problem.

Thanks to Walter Mientka for his continuing support of this project, and to the students of the 1997 Mathematical Olympiad Summer Program for their help in preparing solutions.

Dr. Walter Mientka
Secretary, IMO Advisory Broad
1740 Vine Street
Lincoln, NE 68588-0658, USA.

Contents

1 1996 National Contests: Problems and Solutions

1.1 Bulgaria

1. Prove that for all natural numbers $n \geq 3$ there exist odd natural numbers x_n, y_n such that $7x_n^2 + y_n^2 = 2^n$.

   Solution: For $n = 3$ we have $x_3 = y_3 = 1$. Now suppose that for a given natural number n we have odd natural numbers x_n, y_n such that $7x_n^2 + y_n^2 = 2^n$; we shall exhibit a pair (X, Y) such that $7X^2 + Y^2 = 2^{n+1}$. In fact,

   $$7\left(\frac{x_n \pm y_n}{2}\right)^2 + \left(\frac{7x_n \mp y_n}{2}\right)^2 = 2(7x_n^2 + y_n^2) = 2^{n+1}.$$

   One of $(x_n + y_n)/2$ and $|x_n - y_n|/2$ is odd (as their sum is the larger of x_n and y_n, which is odd), giving the desired pair.

2. The circles k_1 and k_2 with respective centers O_1 and O_2 are externally tangent at the point C, while the circle k with center O is externally tangent to k_1 and k_2. Let ℓ be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k perpendicular to ℓ. Assume that O and A lie on the same side of ℓ. Show that the lines AO_2, BO_1, ℓ have a common point.

   Solution: Let r, r_1, r_2 be the respective radii of k, k_1, k_2. Also let M and N be the intersections of AC and BC with k. Since AMB is a right triangle, the triangle AMO is isosceles and

   $$\angle AMO = \angle OAM = \angle O_1CM = \angle CMO_1.$$

   Therefore O, M, O_1 are collinear and $AM/MC = OM/MO_1 = r/r_1$. Similarly O, N, O_2 are collinear and $BN/NC = ON/NO_2 = r/r_2$.

   Let P be the intersection of ℓ with AB; the lines AN, BM, CP concur at the orthocenter of ABC, so by Ceva's theorem, $AP/PB = (AM/MC)(CN/NB) = r_2/r_1$. Now let D_1 and D_2 be the intersections of ℓ with BO_1 and AO_2. Then $CD_1/D_1P = O_1C/PB = r_1/PB$, and similarly $CD_2/D_2P = r_2/PA$. Thus $CD_1/D_1P = CD_2/D_2P$ and $D_1 = D_2$, and so AO_2, BO_1, ℓ have a common point.

3. Let a, b, c be real numbers and let M be the maximum of the function $y = |4x^3 + ax^2 + bx + c|$ in the interval $[-1, 1]$. Show that $M \geq 1$ and find all cases where equality occurs.

Solution: For $a = 0, b = -3, c = 0$, we have $M = 1$, with the maximum achieved at $-1, -1/2, 1/2, 1$. On the other hand, if $M < 1$ for some choice of a, b, c, then

$$(4x^3 + ax^2 + bx + c) - (4x^3 + 3x)$$

must be positive at -1, negative at $-1/2$, positive at $1/2$, and negative at 1, which is impossible for a quadratic function. Thus $M \geq 1$, and the same argument shows that equality only occurs for $(a, b, c) = (0, -3, 0)$. (Note: this is a particular case of the minimum deviation property of Chebyshev polynomials.)

4. The real numbers $a_1, a_2, \dots, a_n$ ($n \geq 3$) form an arithmetic progression. There exists a permutation $a_{i_1}, a_{i_2}, \dots, a_{i_n}$ of $a_1, a_2, \dots, a_n$ which is a geometric progression. Find the numbers $a_1, a_2, \dots, a_n$ if they are all different and the largest of them is equal to 1996.

Solution: Let $a_1 < a_2 < \cdots < a_n = 1996$ and let q be the ratio of the geometric progression $a_{i_1}, \dots a_{i_n}$; clearly $q \neq 0, \pm 1$. By reversing the geometric progression if needed, we may assume $|q| > 1$, and so $|a_{i_1}| < |a_{i_2}| < \cdots < |a_{i_n}|$. Note that either all of the terms are positive, or they alternate in sign; in the latter case, the terms of either sign form a geometric progression by themselves.

There cannot be three positive terms, or else we would have a three-term geometric progression a, b, c which is also an arithmetic progression, violating the AM-GM inequality. Similarly, there cannot be three negative terms, so there are at most two terms of each sign and $n \leq 4$.

If $n = 4$, we have $a_1 < a_2 < 0 < a_3 < a_4$ and $2a_2 = a_1 + a_3$, $2a_3 = a_2 + a_4$. In this case, $q < -1$ and the geometric progression is either a_3, a_2, a_4, a_1 or a_2, a_3, a_1, a_4. Suppose the former occurs (the argument is similar in the latter case); then $2a_3q = a_3q^3 + a_3$ and $2a_3 + a_3q + a_3q^2$, giving $q = 1$, a contradiction.

We deduce $n = 3$ and consider two possibilities. If $a_1 < a_2 < 0 < a_3 = 1996$, then $2a_2 = a_2q^2 + a_2q$, so $q^2 + q - 2 = 0$ and

$q = -2$, yielding $(a_1, a_2, a_3) = (-3992, -998, 1996)$. If $a_1 < 0 < a_2 < a_3 = 1996$, then $2a_2 = a_2 q + a_2 q^2$, so again $q = -2$, yielding $(a_1, a_2, a_3) = (-998, 499, 1996)$.

5. A convex quadrilateral ABC is given for which $\angle ABC + \angle BCD < 180°$. The common point of the lines AB and CD is E. Prove that $\angle ABC = \angle ADC$ if and only if

$$AC^2 = CD \cdot CE - AB \cdot AE.$$

Solution: Let C_1 be the circumcircle of ADE, and let F be its second intersection with CA. In terms of directed lengths, we have $AC^2 = CD \cdot CE + AB \cdot AE$ if and only if

$$AB \cdot AE = AC^2 - CD \cdot CE = CA^2 - CA \cdot AF = AC \cdot AF,$$

that is, if and only if B, C, E, F are concyclic. But this happens if and only if $\angle EBC = \angle EFC$, and

$$\angle EFC = \angle EFA = \pi - \angle ADE = \angle CDA$$

(in directed angles modulo π), so B, C, E, F are concyclic if and only if $\angle ABC = \angle ADC$ (as undirected angles), as desired.

6. Find all prime numbers p, q for which pq divides $(5^p - 2^p)(5^q - 2^q)$.

Solution: If $p | 5^p - 2^p$, then $p | 5 - 2$ by Fermat's theorem, so $p = 3$. Suppose $p, q \neq 3$; then $p | 5^q - 2^q$ and $q | 5^p - 2^p$. Without loss of generality, assume $p > q$, so that $(p, q-1) = 1$. Then if a is an integer such that $2a \equiv 5 \pmod{q}$, then the order of a mod q divides p as well as $q - 1$, a contradiction.

Hence one of p, q is equal to 3. If $q \neq 3$, then $q | 5^3 - 2^3 = 9 \cdot 13$, so $q = 13$, and similarly $p \in \{3, 13\}$. Thus the solutions are $(p, q) = (3, 3), (3, 13), (13, 3)$.

7. Find the side length of the smallest equilateral triangle in which three discs of radii $2, 3, 4$ can be placed without overlap.

Solution: A short computation shows that discs of radii 3 and 4 can be fit into two corners of an equilateral triangle of side $11\sqrt{3}$ so as to just touch, and that a disc of radius 2 easily fits into the third corner without overlap. On the other hand, if the discs of radii 3 and 4 fit into an equilateral triangle without overlap, there exists a line separating them (e.g. a tangent to one perpendicular to their line of centers) dividing the triangle into a triangle and a (possibly degenerate) convex quadrilateral. Within each piece, the disc can be moved into one of the corners of the original triangle. Thus the two discs fit into the corners without overlap, so the side length of the triangle must be at least $11\sqrt{3}$.

8. The quadratic polynomials f and g with real coefficients are such that if $g(x)$ is an integer for some $x > 0$, then so is $f(x)$. Prove that there exist integers m, n such that $f(x) = mg(x) + n$ for all x.

   Solution: Let $f(x) = ax^2 + bx + c$ and $g(x) = px^2 + qx + r$; assume without loss of generality $p > 0$ and $q = 0$ (by the change of variable $x \to x - q/(2p)$). Let k be an integer such that $k > s$ and $t = \sqrt{(k-s)/p} > q/(2p)$. Since $g(t) = k$ is an integer, so is $f(t) = a(k-s)/p + bt + c$, as is

   $$f\left(\sqrt{\frac{k+1-s}{p}}\right) - f\left(\sqrt{\frac{k-s}{p}}\right) = \frac{b}{\sqrt{p}}\frac{1}{\sqrt{k+1-s} - \sqrt{k-s}} + \frac{a}{p}.$$

   This tends to a/p as k increases, so a/p must be an integer; moreover, b must equal 0, or else the above expression will equal a/p plus a small quantity for large k, which cannot be an integer. Now put $m = a/p$ and $n = c - ms$; then $f(x) = mg(x) + n$.

9. The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by

   $$a_1 = 1, \quad a_{n+1} = \frac{a_n}{n} + \frac{n}{a_n}, \quad n \geq 1.$$

   Prove that for $n \geq 4$, $\lfloor a_n^2 \rfloor = n$.

   Solution: We will show by induction that $\sqrt{n} \leq a_n \leq n/\sqrt{n-1}$ for $n \geq 1$, which will imply the claim. These inequalities clearly hold for $n = 1, 2, 3$. Now assume the inequality for some n. Let $f_n(x) = x/n + n/x$. We first have for $n \geq 3$,

   $$a_{n+1} = f_n(a_n) \geq f_n\left(\frac{n}{\sqrt{n-1}}\right) = \frac{n}{\sqrt{n-1}} > \sqrt{n+1}.$$

On the other hand, using that $a_n > (n-1)/\sqrt{n-2}$ (which we just proved), we get for $n \geq 4$,

$$a_{n+1} = f_n(a_n) < f_n\left(\frac{n-1}{\sqrt{n-2}}\right) = \frac{(n-1)^2 + n^2(n-2)}{(n-1)n\sqrt{n-2}} < \sqrt{n+2}.$$

10. The quadrilateral $ABCD$ is inscribed in a circle. The lines AB and CD meet at E, while the diagonals AC and BD meet at F. The circumcircles of the triangles AFD and BFC meet again at H. Prove that $\angle EHF = 90^\circ$.

Solution: (We use directed angles modulo π.) Let O be the circumcenter of $ABCD$; then

$$\angle AHB = \angle AHF + \angle FHB = \angle ADF + \angle FCB = 2\angle ADB = \angle AOB,$$

so O lies on the circumcircle of AHB, and similarly on the circumcircle of CHD. The radical axes of the circumcircles of AHB, CHD and $ABCD$ concur; these lines are AB, CD and HO, so E, H, O are collinear. Now note that

$$\angle OHF = \angle OHC + \angle CHF = \angle ODC + \angle CBF = \frac{\pi}{2} - \angle CAD + \angle CBD,$$

so $\angle EHF = \angle OHF = \pi/2$ as desired. (Compare IMO 1985/5.)

11. A 7×7 chessboard is given with its four corners deleted.

    (a) What is the smallest number of squares which can be colored black so that an uncolored 5-square (Greek) cross cannot be found?

    (b) Prove that an integer can be written in each square such that the sum of the integers in each 5-square cross is negative while the sum of the numbers in all squares of the board is positive.

Solution:

(a) The 7 squares

$$(2,5), (3,2), (3,3), (4,6), (5,4), (6,2), (6,5)$$

suffice, so we need only show that 6 or fewer will not suffice. The crosses centered at

$$(2,2),(2,6),(3,4),(5,2),(5,6),(6,4)$$

are disjoint, so one square must be colored in each, hence 5 or fewer squares do not suffice. Suppose exactly 6 squares are colored. Then none of the squares $(1,3),(1,4),(7,2)$ can be colored; by a series of similar arguments, no square on the perimeter can be colored. Similarly, $(4,3)$ and $(4,5)$ are not covered, and by a similar argument, neither is $(3,4)$ or $(5,4)$. Thus the center square $(4,4)$ must be covered.

Now the crosses centered at

$$(2,6),(3,3),(5,2),(5,6),(6,4)$$

are disjoint and none contains the center square, so each contains one colored square. In particular, $(2,2)$ and $(2,4)$ are not colored. Replacing $(3,3)$ with $(2,3)$ in the list shows that $(3,2)$ and $(3,4)$ are not colored. Similar symmetric arguments now show that no squares besides the center square can be covered, a contradiction. Thus 7 squares are needed.

(b) Write -5 in the 7 squares listed above and 1 in the remaining squares. Then clearly each cross has negative sum, but the total of all of the numbers is $5(-7)+(45-7)=3$.

1.2 Canada

1. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} + \frac{1-\gamma}{1+\gamma}.$$

Solution: The given quantity equals

$$2\left(\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1}\right) - 3.$$

Since $P(x) = x^3 - x - 1$ has roots α, β, γ, the polynomial $P(x-1) = x^3 - 3x^2 + 2x - 1$ has roots $\alpha+1, \beta+1, \gamma+1$. By a standard formula, the sum of the reciprocals of the roots of $x^3 + c_2x^2 + c_1x + c_0$ is $-c_1/c_0$, so the given expression equals $2(2) - 3 = 1$.

2. Find all real solutions to the following system of equations:

$$\begin{aligned} \frac{4x^2}{1+4x^2} &= y \\ \frac{4y^2}{1+4y^2} &= z \\ \frac{4z^2}{1+4z^2} &= x. \end{aligned}$$

Solution: Define $f(x) = 4x^2/(1+4x^2)$; the range of f is $[0,1)$, so x, y, z must lie in that interval. If one of x, y, z is zero, then all three are, so assume they are nonzero. Then $f(x)/x = 4x/(1+4x^2)$ is at least 1 by the AM-GM inequality, with equality for $x = 1/2$. Therefore $x \le y \le z \le x$, and so equality holds everywhere, implying $x = y = z = 1/2$. Thus the solutions are $(x, y, z) = (0,0,0), (1/2, 1/2, 1/2)$.

3. Let $f(n)$ be the number of permutations $a_1, \dots, a_n$ of the integers $1, \dots, n$ such that

   (i) $a_1 = 1$;

   (ii) $|a_i - a_{i+1}| \le 2$, $i = 1, \dots, n-1$.

Determine whether $f(1996)$ is divisible by 3.

Solution: Let $g(n)$ be the number of permutations of the desired form with $a_n = n$. Then either $a_{n-1} = n-1$ or $a_{n-1} = n-2$; in the latter case we must have $a_{n-2} = n-1$ and $a_{n-3} = n-3$. Hence $g(n) = g(n-1) + g(n-3)$ for $n \geq 4$. In particular, the values of $g(n)$ modulo 3 are $g(1) = 1, 1, 1, 2, 0, 1, 0, 0, \ldots$ repeating with period 8.

Now let $h(n) = f(n) - g(n)$; $h(n)$ counts permutations of the desired form where n occurs in the middle, sandwiched between $n-1$ and $n-2$. Removing n leaves an acceptable permutation, and any acceptable permutation on $n-1$ symbols can be so produced except those ending in $n-4, n-2, n-3, n-1$. Hence $h(n) = h(n-1) + g(n-1) - g(n-4) = h(n-1) + g(n-2)$; one checks that $h(n)$ modulo 3 repeats with period 24.

Since $1996 \equiv 4 \pmod{24}$, we have $f(1996) \equiv f(4) = 4 \pmod 3$, so $f(1996)$ is not divisible by 3.

4. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

Solution: Let $\alpha = \angle A, \beta = (\pi - \alpha)/4$ and assume $AB = 1$. Then by the Law of Sines,

$$BC = \frac{\sin \alpha}{\sin 2\beta}, \quad BD = \frac{\sin \alpha}{\sin 3\beta}, \quad AD = \frac{\sin \beta}{\sin 3\beta}.$$

Thus we are seeking a solution to the equation

$$\sin(\pi - 4\beta) \sin 3\beta = (\sin(\pi - 4\beta) + \sin \beta) \sin 2\beta.$$

Using the sum-to-product formula, we rewrite this as

$$\cos \beta - \cos 7\beta = \cos 2\beta - \cos 6\beta + \cos \beta - \cos 3\beta.$$

Cancelling $\cos \beta$, we have $\cos 3\beta - \cos 7\beta = \cos 2\beta - \cos 6\beta$, which implies

$$\sin 2\beta \sin 5\beta = \sin 2\beta \sin 4\beta.$$

Now $\sin 5\beta = \sin 4\beta$, so $9\beta = \pi$ and $\beta = \pi/9$.

5. Let $r_1, r_2, \ldots, r_m$ be a given set of positive rational numbers whose sum is 1. Define the function f by $f(n) = n - \sum_{k=1}^{m} \lfloor r_k n \rfloor$ for each positive integer n. Determine the minimum and maximum values of $f(n)$.

Solution: Of course $\lfloor r_k n \rfloor \leq r_k n$, so $f(n) \geq 0$, with equality for $n = 0$, so 0 is the minimum value. On the other hand, we have $r_k n - \lfloor r_k n \rfloor < 1$, so $f(n) \leq m - 1$. Here equality holds for $n = t - 1$ if t is the least common denominator of the r_k.

1.3 China

1. Let H be the orthocenter of acute triangle ABC. The tangents from A to the circle with diameter BC touch the circle at P and Q. Prove that P, Q, H are collinear.

   Solution: The line PQ is the polar of A with respect to the circle, so it suffices to show that A lies on the pole of H. Let D and E be the feet of the altitudes from A and B, respectively; these also lie on the circle, and $H = AD \cap BE$. The polar of the line AD is the intersection of the tangents AA and DD, and the polar of the line BE is the intersection of the tangents BB and EE. The collinearity of these two intersections with $C = AE \cap BD$ follows from applying Pascal's theorem to the cyclic hexagons $AABDDE$ and $ABBDEE$. (An elementary solution with vectors is also possible and not difficult.)

2. Find the smallest positive integer K such that every K-element subset of $\{1, 2, \ldots, 50\}$ contains two distinct elements a, b such that $a+b$ divides ab.

   Solution: The minimal value is $k = 39$. Suppose $a, b \in S$ are such that $a + b$ divides ab. Let $c = \gcd(a, b)$, and put $a = ca_1, b = cb_1$, so that a_1 and b_1 are relatively prime. Then $c(a_1 + b_1)$ divides $c^2 a_1 b_1$, so $a_1 + b_1$ divides $ca_1 b_1$. Since a_1 and b_1 have no common factor, neither do a_1 and $a_1 + b_1$, or b_1 and $a_1 + b_1$. In short, $a_1 + b_1$ divides c.

   Since $S \subseteq \{1, \ldots, 50\}$, we have $a + b \leq 99$, so $c(a_1 + b_1) \leq 99$, which implies $a_1 + b_1 \leq 9$; on the other hand, of course $a_1 + b_1 \geq 3$. An exhaustive search produces 23 pairs a, b satisfying the condition:

$a_1 + b_1 = 3$	$(6, 3), (12, 6), (18, 9), (24, 12),$ $(30, 15), (36, 18), (42, 21), (48, 24)$
$a_1 + b_1 = 4$	$(12, 4), (24, 8), (36, 12), (48, 16)$
$a_1 + b_1 = 5$	$(20, 5), (40, 10), (15, 10), (30, 20), (45, 30)$
$a_1 + b_1 = 6$	$(30, 6)$
$a_1 + b_1 = 7$	$(42, 7), (35, 14), (28, 21)$
$a_1 + b_1 = 8$	$(40, 24)$
$a_1 + b_1 = 9$	$(45, 36)$

Let $M = \{6, 12, 15, 18, 20, 21, 24, 35, 40, 42, 45, 48\}$ and $T = \{1, \ldots, 50\} - M$. Since each pair listed above contains an element of M, T does not have the desired property. Hence we must take $k \geq |T| + 1 = 39$.

On the other hand, from the 23 pairs mentioned above we can select 12 pairs which are mutually disjoint:

$$(6,3), (12,4), (20,5), (42,7), (24,8), (18,9),$$
$$(40,10), (35,14), (30,15), (48,16), (28,21), (45,36).$$

Any 39-element subset must contain both elements of one of these pairs. We conclude the desired minimal number is $k = 39$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x, y \in \mathbb{R}$,

$$f(x^3 + y^3) = (x + y)(f(x)^2 - f(x)f(y) + f(y)^2). \qquad (1)$$

Prove that for all $x \in \mathbb{R}$, $f(1996x) = 1996f(x)$.

Solution: Setting $x = y = 0$ in the given equation, we have $f(0) = 0$. Setting $y = 0$, we find $f(x^3) = xf(x)^2$, or equivalently,

$$f(x) = x^{1/3}f(x^{1/3})^2. \qquad (2)$$

In particular, x and $f(x)$ always have the same sign, that is, $f(x) \geq 0$ for $x \geq 0$ and $f(x) \leq 0$ for $x \leq 0$.

Let S be the set

$$S = \{a > 0 : f(ax) = af(x) \forall x \in \mathbb{R}\}.$$

Clearly $1 \in S$; we will show $a^{1/3} \in S$ whenever $a \in S$. In fact,

$$axf(x)^2 = af(x^3) = f(ax^3) = f((a^{1/3}x)^3) = a^{1/3}f(a^{1/3}x)^2$$

and so

$$[a^{1/3}f(x)]^2 = f(a^{1/3}x)^2.$$

Since x and $f(x)$ have the same sign, we conclude $f(a^{1/3}x) = a^{1/3}f(x)$.

Now we show that $a, b \in S$ implies $a + b \in S$:

$$\begin{aligned}
f((a+b)x) &= f((a^{1/3}x^{1/3})^3 + (b^{1/3}x^{1/3})^3) \\
&= (a^{1/3} + b^{1/3})[f(a^{1/3}x^{1/3})^2 - f(a^{1/3}x^{1/3})f(b^{1/3}x^{1/3}) + f(b^{1/3}x^{1/3})^2] \\
&= (a^{1/3} + b^{1/3})(a^{2/3} - a^{1/3}b^{1/3} + b^{2/3})x^{1/3}f(x^{1/3})^2 \\
&= (a+b)f(x).
\end{aligned}$$

By induction, we have $n \in S$ for each positive integer n, so in particular, $f(1996x) = 1996f(x)$ for all $x \in \mathbb{R}$.

4. Eight singers participate in an art festival where m songs are performed. Each song is performed by 4 singers, and each pair of singers performs together in the same number of songs. Find the smallest m for which this is possible.

   Solution: Let r be the number of songs each pair of singers performs together, so that

$$m\binom{4}{2} = r\binom{8}{2}$$

   and so $m = 14r/3$; in particular, $m \geq 14$. However, $m = 14$ is indeed possible, using the arrangement

$$\begin{array}{llll} \{1,2,3,4\} & \{5,6,7,8\} & \{1,2,5,6\} & \{3,4,7,8\} \\ \{3,4,5,6\} & \{1,3,5,7\} & \{2,4,6,8\} & \{1,3,6,8\} \\ \{2,4,5,7\} & \{1,4,5,8\} & \{2,3,6,7\} & \{1,4,6,7\} \\ \{1,2,7,8\} & \{2,3,5,8\}. & & \end{array}$$

5. Suppose $n \in \mathbb{N}$, $x_0 = 0$, $x_i > 0$ for $i = 1, 2, \ldots, n$, and $\sum_{i=1}^n x_i = 1$. Prove that

$$1 \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{1 + x_0 + \cdots + x_{i-1}} \cdot \sqrt{x_i + \cdots + x_n}} < \frac{\pi}{2}.$$

   Solution: The left inequality follows from the fact that

$$\sqrt{1 + x_0 + x_1 + \cdots + x_{i-1}}\sqrt{x_1 + \cdots + x_n} \leq \frac{1}{2}(1+x_0+\cdots+x_n) = 1,$$

   so that the middle quantity is at least $\sum x_i = 1$. For the right inequality, let

$$\theta_i = \arcsin(x_0 + \cdots + x_i) \quad (i = 0, \ldots, n)$$

   so that

$$\sqrt{1 + x_0 + x_1 + \cdots + x_{i-1}}\sqrt{x_i + \cdots + x_n} = \cos\theta_{i-1}$$

and the desired inequality is

$$\sum_{i=1}^{n} \frac{\sin\theta_i - \sin\theta_{i-1}}{\cos\theta_{i-1}} < \frac{\pi}{2}.$$

Now note that

$$\sin\theta_i - \sin\theta_{i-1} = 2\cos\frac{\theta_i + \theta_{i-1}}{2}\sin\frac{\theta_i - \theta_{i-1}}{2} < cos\theta_{i-1}(\theta_i - \theta_{i-1}),$$

using the facts that $\theta_{i-1} < \theta_i$ and that $\sin x < x$ for $x > 0$, so that

$$\sum_{i=1}^{n} \frac{\sin\theta_i - \sin\theta_{i-1}}{\cos\theta_{i-1}} < \sum_{i=1}^{n} \theta_i - \theta_{i-1} = \theta_n - \theta_0 < \frac{\pi}{2},$$

as claimed.

6. In triangle ABC, $\angle C = 90°$, $\angle A = 30°$ and $BC = 1$. Find the minimum of the length of the longest side of a triangle inscribed in ABC (that is, one such that each side of ABC contains a different vertex of the triangle).

Solution: We first find the minimum side length of an equilateral triangle inscribed in ABC. Let D be a point on BC and put $x = BD$. Then take points E, F on CA, AB, respectively, such that $CE = \sqrt{3}x/2$ and $BF = 1 - x/2$. A calculation using the Law of Cosines shows that

$$DF^2 = DE^2 = EF^2 = \frac{7}{4}x^2 - 2x + 1 = \frac{7}{4}\left(x - \frac{4}{7}\right)^2 + \frac{3}{7}.$$

Hence the triangle DEF is equilateral, and its minimum possible side length is $\sqrt{3/7}$.

We now argue that the minimum possible longest side must occur for some equilateral triangle. Starting with an arbitrary triangle, first suppose it is not isosceles. Then we can slide one of the endpoints of the longest side so as to decrease its length; we do so until there are two longest sides, say DE and EF. We now fix D, move E so as to decrease DE and move F at the same time so as to decrease EF; we do so until all three sides become equal in length. (It is fine if the vertices move onto the extensions of the sides, since the bound above applies in that case as well.)

Hence the mininum is indeed $\sqrt{3/7}$, as desired.

1.4 Czech and Slovak Republics

1. Prove that if a sequence $\{G(n)\}_{n=0}^{\infty}$ of integers satisfies

$$\begin{aligned} G(0) &= 0, \\ G(n) &= n - G(G(n)) \qquad (n = 1, 2, 3, \ldots), \end{aligned}$$

then

(a) $G(k) \geq G(k-1)$ for any positive integer k;

(b) no integer k exists such that $G(k-1) = G(k) = G(k+1)$.

Solution:

(a) We show by induction that $G(n) - G(n-1) \in \{0, 1\}$ for all n. If this holds up to n, then

$$G(n+1) - G(n) = 1 + G(G(n-1)) - G(G(n)).$$

If $G(n-1) = G(n)$, then $G(n+1) - G(n) = 1$; otherwise, $G(n-1)$ and $G(n)$ are consecutive integers not greater than n, so $G(G(n)) - G(G(n-1)) \in \{0, 1\}$, again completing the induction.

(b) Suppose that $G(k-1) = G(k) = G(k+1) + A$ for some k, A. Then

$$A = G(k+1) = k + 1 - G(G(k)) = k + 1 - G(A)$$

and similarly $A = k - G(A)$ (replacing $k+1$ with k above), a contradiction.

Note: It can be shown that $G(n) = \lfloor nw \rfloor$ for $w = (\sqrt{5} - 1)/2$.

2. Let ABC be an acute triangle with altitudes AP, BQ, CR. Show that for any point P in the interior of the triangle PQR, there exists a tetrahedron $ABCD$ such that P is the point of the face ABC at the greatest distance (measured along the surface of the tetrahedron) from D.

Solution: We first note that if S is the circumcircle of an acute triangle KLM, then for any point $X \neq S$ inside the triangle, we have

$$\min\{XK, XL, XM\} < SK = SL = SM,$$

since the discs centered at K, L, M whose bounding circles pass through S cover the entire triangle.

Fix a point V in the interior of the triangle PQR; we first assume the desired tetrahedron exists and determine some of its properties. Rotate the faces ABD, BCD, CAD around their common edges with face ABC into the plane ABC, so that the images D_1, D_2, D_3 of D lie outside of triangle ABC. We shall choose D so that triangle $D_1D_2D_3$ is acute, contains triangle ABC and has circumcenter V; this suffices by the above observation.

In other words, we need a point D such that AV is the perpendicular bisector of D_1D_3, BV that of D_1D_2, and CV that of D_2D_3. We thus need $\angle D_1D_2D_3 = \pi - \angle BVC$ and so on. Since V lies inside PQR, the angle BVC is acute, and so $\angle D_1D_2D_3$ is fixed and acute. We may then construct an arbitrary triangle $D_1'D_2'D_3'$ similar to the unknown triangle $D_1D_2D_3$, let V' be its circumcenter, and construct points A', B', C' on the rays from V through the midpoints of $D_3'D_1', D_1'D_2', D_2'D_3'$, respectively, so that triangles $A'B'C'$ and ABC are similar. We can also ensure that the entire triangle $A'B'C'$ lies inside $D_1'D_2'D_3'$. Then folding up the hexagon $A'D_1'B'D_2'C'D_3'$ along the edges of triangle $A'B'C'$ produces a tetrahedron similar to the required tetrahedron.

3. Given six three-element subsets of a finite set X, show that it is possible to color the elements of X in two colors such that none of the given subsets is all in one color.

Solution: Let $A_1, \dots, A_6$ be the subsets; we induct on the number n of elements of X, and there is no loss of generality in assuming $n \geq 6$. If $n = 6$, since $\binom{6}{3} = 20 > 2 \cdot 6$, we can find a three-element subset Y of X not equal to any of $A_1, \dots, A_6$ or their complements; coloring the elements of Y in one color and the other elements in the other color meets the desired condition.

Now suppose $n > 6$. There must be two elements u, v of X such that $\{u, v\}$ is not a subset of any A_i, since there are at least $\binom{7}{2} = 21$ pairs, and at most $6 \times 3 = 18$ lie in an A_i. Replace all occurrences of u and v by a new element w, and color the resulting elements using the induction hypothesis. Now color the original set by giving u and v the same color given to w.

4. An acute angle XCY and points A and B on the rays CX and CY, respectively, are given such that $|CX| < |CA| = |CB| < |CY|$. Show how to construct a line meeting the ray CX and the segments AB, BC at the points K, L, M, respectively, such that

$$KA \cdot YB = XA \cdot MB = LA \cdot LB \neq 0.$$

Solution: Suppose K, L, M have already been constructed. The triangles ALK and BYL are similar because $\angle LAK = \angle YBL$ and $KA/LA = LB/YB$. Hence $\angle ALK = \angle BYL$. Similarly, from the similar triangles ALX and BML we get $\angle AXL = \angle MLB$. We also have $\angle MLB = \angle ALK$ since M, L, K are collinear; we conclude $\angle LYB = \angle AXL$. Now

$$\angle XLY = \angle XLB + \angle BLY = \angle XAL + \angle AXL + \angle ABM - \angle LYB = 2\angle ABC.$$

We now construct the desired line as follows: draw the arc of points L such that $\angle XLY = 2\angle ABC$, and let L be its intersection with AB. Then construct M on BC such that $\angle BLM = \angle AXL$, and let K be the intersection of LM with CA.

5. For which integers k does there exist a function $f : \mathbb{N} \to \mathbb{Z}$ such that

   (a) $f(1995) = 1996$, and

   (b) $f(xy) = f(x) + f(y) + kf(\gcd(x, y))$ for all $x, y \in \mathbb{N}$?

Solution: Such f exists for $k = 0$ and $k = -1$. First take $x = y$ in (b) to get $f(x^2) = (k + 2)f(x)$. Applying this twice, we get

$$f(x^4) = (k + 2)f(x^2) = (k + 2)^2 f(x).$$

On the other hand,

$$\begin{aligned} f(x^4) &= f(x)+f(x^3)+kf(x) = (k+1)f(x)+f(x^3) \\ &= (k+1)f(x)+f(x)+f(x^2)+kf(x) = (2k+2)f(x)+f(x^2) \\ &= (3k+4)f(x). \end{aligned}$$

Setting $x = 1995$ so that $f(x) \neq 0$, we deduce $(k+2)^2 = 3k+4$, which has roots $k = 0, -1$. For $k = 0$, an example is given by

$$f(p_1^{e_1} \cdots p_n^{e_n}) = e_1 g(p_1) + \cdots + e_n g(p_n),$$

where $g(5) = 1996$ and $g(p) = 0$ for all primes $p \neq 5$. For $k = 1$, an example is given by

$$f(p_1^{e_1} \cdots p_n^{e_n}) = g(p_1) + \cdots + g(p_n).$$

6. A triangle ABC and points K, L, M on the sides AB, BC, CA, respectively, are given such that

$$\frac{AK}{AB} = \frac{BL}{BC} = \frac{CM}{CA} = \frac{1}{3}.$$

Show that if the circumcircles of the triangles AKM, BLK, CML are congruent, then so are the incircles of these triangles.

Solution: We will show that ABC is equilateral, so that AKM, BLK, CML are congruent and hence have the same inradius. Let R be the common circumradius; then

$$KL = 2R\sin A, \quad LM = 2R\sin B, \quad MK = 2R\sin C,$$

so the triangles KLM and ABC are similar. Now we compare areas:

$$[AKM] = [BLK] = [CLM] = \frac{2}{9}[ABC],$$

so $[KLM] = \frac{1}{3}[ABC]$ and the coefficient of similarity between KLM and ABC must be $\sqrt{1/3}$. By the law of cosines applied to ABC and AKM,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ \frac{1}{3}a^2 &= \left(\frac{2p}{3}\right)^2 + \left(\frac{c}{3}\right)^2 - 2\frac{2b}{3}\frac{c}{3}\cos A. \end{aligned}$$

From these we deduce $a^2 = 2b^2 - c^2$, and similarly $b^2 = 2c^2 - a^2$, $c^2 = 2a^2 - b^2$. Combining these gives $a^2 = b^2 = c^2$, so ABC is equilateral, as desired.

1.5 France

1. Let ABC be a triangle and construct squares $ABED, BCGF, ACHI$ externally on the sides of ABC. Show that the points D, E, F, G, H, I are concyclic if and only if ABC is equilateral or isosceles right.

   Solution: Suppose D, E, F, G, H, I are concyclic; the perpendicular bisectors of DE, FG, HI coincide with those of AB, BC, CA, respectively, so the center of the circle must be the circumcenter O of ABC. By equating the distances OD and OF, we find

   $$(\cos B + 2\sin B)^2 + \sin^2 B = (\cos C + 2\sin C)^2 = \sin^2 C.$$

   Expanding this and cancelling like terms, we determine

   $$\sin^2 B + \sin B \cos B = \sin^2 C + \sin C \cos C.$$

   Now note that

   $$2(\sin^2\theta + \sin\theta\cos\theta) = 1 - \cos 2\theta + \sin\theta = 1 + \sqrt{2}\sin(2\theta - \pi/4).$$

   Thus we either have $B = C$ or $2B - \pi/4 + 2C - \pi/4 = \pi$, or $B + C = 3\pi/4$. In particular, two of the angles must be equal, say A and B, and we either have $A = B = C$, so the triangle is equilaterla, or $B + (\pi - 2B) = 3\pi/4$, in which case $A = B = \pi/4$ and the triangle is isosceles right.

2. Let a, b be positive integers with a odd. Define the sequence $\{u_n\}$ as follows: $u_0 = b$, and for $n \in \mathbb{N}$,

   $$u_{n+1} = \begin{cases} \frac{1}{2}u_n & \text{if } u_n \text{ is even} \\ u_n + a & \text{otherwise.} \end{cases}$$

   (a) Show that $u_n \le a$ for some $n \in \mathbb{N}$.

   (b) Show that the sequence $\{u_n\}$ is periodic from some point onwards.

   Solution:

   (a) Suppose $u_n > a$. If u_n is even, $u_{n+1} = u_n/2 < u_n$; if u_n is odd, $u_{n+2} = (u_n + a)/2 < u_n$. Hence for each term greater than

a, there is a smaller subsequent term. These form a decreasing subsequence which must eventually terminate, which only occurs once $u_n \leq a$.

(b) If $u_m \leq a$, then for all $n \geq m$, either $u_n \leq a$, or u_n is even and $u_n \leq 2a$, by induction on n. In particular, $u_n \leq 2a$ for all $m \geq n$, and so some value of u_n eventually repeats, leading to a periodic sequence.

choose

3. (a) Find the minimum value of x^x for x a positive real number.

(b) If x and y are positive real numbers, show that $x^y + y^x > 1$.

Solution:

(a) Since $x^x = e^{x \log x}$ and e^x is an increasing function of x , it suffices to determine the minimum of $x \log x$. This is easily done by setting its derivative $1 + \log x$ to zero, yielding $x = 1/e$. The second derivative $1/x$ is positive for $x > 0$, so the function is everywhere convex, and the unique extremum is indeed a global minimum. Hence x^x has minimum value $e^{-1/e}$.

(b) If $x \geq 1$, then $x^y \geq 1$ for $y > 0$, so we may assume $0 < x, y < 1$. Without loss of generality, assume $x \leq y$; now note that the function $f(x) = x^y + y^x$ has derivative $f'(x) = x^y \log x + y^{x-1}$. Since $y^x \geq x^x \geq x^y$ for $x \leq y$ and $1/x \geq -\log x$, we see that $f'(x) > 0$ for $0 \leq x \leq y$ and so the minimum of f occurs with $x = 0$, in which case $f(x) = 1$; since $x > 0$, we have strict inequality.

4. Let n be a positive integer. We say a positive integer k satisfies the condition C_n if there exist $2k$ distinct positive integers $a_1, b_1, \ldots,$ a_k, b_k such that the sums $a_1 + b_1, \ldots, a_k + b_k$ are all distinct and less than n.

(a) Show that if k satisfies the condition C_n, then $k \leq (2n-3)/5$.

(b) Show that 5 satisfies the condition C_{14}.

(c) Suppose $(2n-3)/5$ is an integer. Show that $(2n-3)/5$ satisfies the condition C_n.

(a) If k satisfies the condition C_n, then

$$1 + 2 + \cdots + 2k \leq (n-1) + (n-2) + \cdots + (n-k),$$

or $k(2k+1) \leq k(2n-k-1)/2$, or $4k+2 \leq 2n-k-1$, or $5k \leq 2n-3$.

(b) We obtain the sums $9, 10, 11, 12, 13$ as follows:

$$9 = 7+2, 10 = 6+4, 11 = 10+1, 12 = 9+3, 13 = 8+5.$$

(c) Imitating the above example, we pair $2k$ with 1, $2k-1$ with 3, and so on, up to $2k-(k-1)/2$ with k (where $k = (2n-3)/5$), giving the sums $2k+1, \ldots, n-1$. Now we pair $2k-(k+1)/2$ with 2, $2k-(k+3)/2$ with 4, and so on, up to $k+1$ with $k-1$, giving the sums from $(5k+1)/2$ to $2k$.

1.6 Germany

1. Starting at $(1,1)$, a stone is moved in the coordinate plane according to the following rules:

   (i) From any point (a,b), the stone can move to $(2a,b)$ or $(a,2b)$.

   (ii) From any point (a,b), the stone can move to $(a-b,b)$ if $a>b$, or to $(a,b-a)$ if $a<b$.

   For which positive integers x,y can the stone be moved to (x,y)?

   Solution: It is necessary and sufficient that $\gcd(x,y)=2^s$ for some nonnegative integer s. We show necessity by noting that $\gcd(p,q)=\gcd(p,q-p)$, so an odd common divisor can never be introduced, and noting that initially $\gcd(1,1)=1$.

   As for sufficiency, suppose $\gcd(x,y)=2^s$. Of those pairs (p,q) from which (x,y) can be reached, choose one to minimize $p+q$. Neither p nor q can be even, else one of $(p/2,q)$ or $(p,q/2)$ is an admissible pair. If $p>q$, then (p,q) is reachable from $((p+q)/2,q)$, a contradiction; similarly $p<q$ is impossible. Hence $p=q$, but $\gcd(p,q)$ is a power of 2 and neither p nor q is even. We conclude $p=q=1$, and so (x,y) is indeed reachable.

2. Suppose S is a union of finitely many disjoint subintervals of $[0,1]$ such that no two points in S have distance $1/10$. Show that the total length of the intervals comprising S is at most $1/2$.

   Solution: Cut the given segment into 5 segments of length $1/5$. Let AB be one of these segments and M its midpoint. Translate each point of AM by the vector $\vec{MB}$. No colored point can have a colored image, so all of the colored intervals of AB can be placed in MB without overlap, and their total length therefore does not exceed $1/10$. Applying this reasoning to each of the 5 segments gives the desired result.

3. Each diagonal of a convex pentagon is parallel to one side of the pentagon. Prove that the ratio of the length of a diagonal to that of its corresponding side is the same for all five diagonals, and compute this ratio.

Solution: Let CE and BD intersect in S, and choose T on AB with $CT \parallel BD$. Clearly S lies inside the pentagon and T lies outside. Put $d = AB, c = AE$, and $s = SC/AB$; then the similar triangles SCD and ABE give $SC = sd$ and $SD = sc$. The parallelograms $ABSE$, $ATCE$, $BTCS$ give $SE = d$, $TC = c$, $BT = sd$. From the similar triangles ESD and ATC we get $SD/TC = SE/TA$, and so $sc/c = d/(d+sd)$. We conclude s is the positive root of $s(1+s) = 1$, which is $s = (\sqrt{5} - 1)/2$.

Finally, we determine $EC = d(1+s)$ and the ratio $EC/AB = 1+s = (1 + \sqrt{5})/2$, and the value is clearly the same for the other pairs.

4. Prove that every integer $k > 1$ has a multiple less than k^4 whose decimal expansion has at most four distinct digits.

Solution: Let n be the integer such that $2^{n-1} \leq k < 2^n$. For $n \leq 6$ the result is immediate, so assume $n > 6$.

Let S be the set of nonnegative integers less than 10^n whose decimal digits are all 0s or 1s. Since $|S| = 2^n > k$, we can find two elements $a < b$ of S which are congruent modulo k, and $b - a$ only has the digits 8, 9, 0, 1 in its decimal representation. On the other hand,

$$b - a \leq b \leq 1 + 10 + \cdots + 10^{n-1} < 10^n < 16^{n-1} \leq k^4,$$

hence $b - a$ is the desired multiple.

1.7 Greece

1. In a triangle ABC the points D, E, Z, H, Θ are the midpoints of the segments BC, AD, BD, ED, EZ, respectively. If I is the point of intersection of BE and AC, and K is the point of intersection of $H\Theta$ and AC, prove that

   (a) $AK = 3CK$;

   (b) $HK = 3H\Theta$;

   (c) $BE = 3EI$;

   (d) the area of ABC is 32 times that of $E\Theta H$.

   Solution: Introduce oblique coordinates with $B = (0, 0)$, $C = (24, 0)$, $A = (0, 24)$. We then compute $D = (12, 0)$, $E = (6, 12)$, $Z = (6, 0)$, $H = (9, 6)$, $\Theta = (6, 6)$, $I = (8, 16)$, $K = (18, 6)$, from which the relations $AK = 3CK$, $HK = 3H\Theta$, $BE = 3EI$ are evident. As for $E\Theta H$, it has base ΘH whose length is half that of ZD, and ZD is 1/4 as long as BC, so $\Theta H = 1/8BC$. The altitude from E to ΘH is 1/4 the altitude from A to BC, so we conclude the area of $E\Theta H$ is 1/32 times that of ABC.

2. Let ABC be an acute triangle, AD, BE, CZ its altitudes and H its orthocenter. Let $AI, A\Theta$ be the internal and external bisectors of angle A. Let M, N be the midpoints of BC, AH, respectively. Prove that

   (a) MN is perpendicular to EZ;

   (b) if MN cuts the segments $AI, A\Theta$ at the points K, L, then $KL = AH$.

   Solution:

   (a) The circle with diameter AH passes through Z and E, and so $ZN = ZE$. On the other hand, MN is a diameter of the nine-point circle of ABC, and Z and E lie on that circle, so $ZN = ZE$ implies that $ZE \perp MN$.

   (b) As determined in (a), MN is the perpendicular bisector of segment ZE. The angle bisector AI of $\angle EAZ$ passes through the midpoint of the minor arc EZ, which clearly lies on MN; therefore this midpoint is K. By similar reasoning, L is the midpoint of the major arc EZ. Thus KL is also a diameter of circle EAZ, so $KL = MN$.

3. Given 81 natural numbers whose prime divisors belong to the set $\{2, 3, 5\}$, prove there exist 4 numbers whose product is the fourth power of an integer.

   Solution: It suffices to take 25 such numbers. To each number, associate the triple (x_2, x_3, x_5) recording the parity of the exponents of 2, 3, and 5 in its prime factorization. Two numbers have the same triple if and only if their product is a perfect square. As long as there are 9 numbers left, we can select two whose product is a square; in so doing, we obtain 9 such pairs. Repeating the process with the square roots of the products of the pairs, we obtain four numbers whose product is a fourth power. (See IMO 1985/4.)

4. Determine the number of functions $f : \{1, 2, \ldots, n\} \to \{1995, 1996\}$ which satsify the condition that $f(1) + f(2) + \cdots + f(1996)$ is odd.

   Solution: We can send $1, 2, \ldots, n-1$ anywhere, and the value of $f(n)$ will then be uniquely determined. Hence there are 2^{n-1} such functions.

1.8 Iran

1. Prove the following inequality for positive real numbers x, y, z:

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

Solution: After clearing denominators, the given inequality becomes

$$\sum_{\text{sym}} 4x^5y - x^4y^2 - 3x^3y^3 + x^4yz - 2x^3y^2z + x^2y^2z^2 \geq 0,$$

where the symmetric sum runs over all six permutations of x, y, z. (In particular, this means the coefficient of x^3y^3 in the final expression is -6, and that of $x^2y^2z^2$ is 6.)

Recall Schur's inequality:

$$x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \geq 0.$$

Multiplying by $2xyz$ and collecting symmetric terms, we get

$$\sum_{\text{sym}} x^4yz - 2x^3y^2z + x^2y^2z^2 \geq 0.$$

On the other hand,

$$\sum_{\text{sym}} (x^5y - x^4y^2) + 3(x^5y - x^3y^3) \geq 0$$

by two applications of AM-GM; combining the last two displayed inequalities gives the desired result.

2. Prove that for every pair m, k of natural numbers, m has a unique representation in the form

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t},$$

where

$$a_k > a_{k-1} > \cdots > a_t \geq t \geq 1.$$

Solution: We first show uniqueness. Suppose m is represented by two sequences $a_k, \ldots, a_t$ and $b_k, \ldots, b_t$. Find the first position in which they differ; without loss of generality, assume this position is k and that $a_k > b_k$. Then

$$m \leq \binom{b_k}{k} + \binom{b_k - 1}{k-1} + \cdots + \binom{b_k - k + 1}{1} < \binom{b_k + 1}{k} \leq m,$$

a contradiction.

To show existence, apply the greedy algorithm: find the largest a_k such that $\binom{a_k}{k} \leq m$, and apply the same algorithm with m and k replaced by $m - \binom{a_k}{k}$ and $k - 1$. We need only make sure that the sequence obtained is indeed decreasing, but this follows because by assumption, $m < \binom{a_k+1}{m}$, and so $m - \binom{a_k}{k} < \binom{a_k}{k-1}$.

3. In triangle ABC, we have $\angle A = 60^\circ$. Let O, H, I, I' be the circumcenter, orthocenter, incenter, and excenter opposite A, respectively, of ABC. Let B' and C' be points on the segments AC and AB such that $AB = AB'$ and $AC = AC'$. Prove that:

   (a) The eight points $B, C, H, O, I, I', B', C'$ are concyclic.

   (b) If OH intersects AB and AC at E and F, respectively, the perimeter of triangle AEF equals $AB + AC$.

   (c) $OH = |AB - AC|$.

Solution:

(a) The circle through B, C, H consists of all points P such that $\angle BPC = \angle BHC = 180^\circ - \angle CAB = 120^\circ$ (as directed angles mod 180°). Thus O lies on this circle, as does I because $\angle BIC = 90^\circ + \frac{1}{2}\angle A = 30^\circ$. Note that the circle with diameter II' passes through B and C (since internal and external angle bisectors are perpendicular). Hence I' also lies on the circle, whose center lies on the internal angle bisector of A. This means reflecting B and C across this bisector gives two more points B', C' on the circle.

(b) Let R be the circumradius of triangle ABC. The reflection across AI maps B and C to B' and C', and preserves I. By (a), the circle $BCHO$ is then preserved, and hence H maps to O. In other words, AHO is isosceles with $AH = AO = R$ and $\angle HAO = |\beta - \gamma|$, writing β for $\angle B$ and γ for $\angle C$.

In particular, the altitude of AHO has length $R \cos \beta - \gamma$ and so the equilateral triangle AEF has perimeter

$$\sqrt{3}R\cos(\beta-\gamma) = 2R\sin(\beta+\gamma)\cos(\beta-\gamma) = 2R(\sin\beta+\sin\gamma) = AB+AC.$$

(c) We use a, b, c to denote the lengths of BC, CA, AB. By a standard computation using vectors, we find $OH^2 = 9R^2-(a^2+b^2+c^2)$, but since $a = 2R\sin 60°$, we have $OH^2 = 2a^2 - b^2 - c^2$. By the Law of Cosines, $a^2 = b^2 + c^2 - bc$, so $OH^2 = b^2 + c^2 - 2bc = (b - c)^2$, and so $OH = |b - c|$.

4. Let ABC be a scalene triangle. The medians from A, B, C meet the circumcircle again at L, M, N, respectively. If $LM = LN$, prove that $2BC^2 = AB^2 + AC^2$.

Solution: Let G be the centroid of triangle ABC; then triangles NLG and AGL are similar, so $LN/AC = LG/CG$. Similarly $LM/AB = GL/BG$. Thus if $LM = LN$, then $AB/AC = BG/CG$. Using Stewart's theorem to compute the lengths of the medians, we have

$$\frac{AB^2}{AC^2} = \frac{2AB^2 + 2BC^2 - AC^2}{2AC^2 + 2BC^2 - AB^2}$$

which reduces to $(AC^2 - AB^2)(2BC^2 - AB^2 - AC^2) = 0$. Since the triangle is scalene, we conclude $2BC^2 = AB^2 + AC^2$.

5. The top and bottom edges of a chessboard are identified together, as are the left and right edges, yielding a torus. Find the maximum number of knights which can be placed so that no two attack each other.

Solution: The maximum is 32 knights; if the chessboard is alternately colored black and white in the usual fashion, an optimal arrangement puts a knight on each black square. To see that this cannot be improved, suppose that k knights are placed. Each knight attacks 8 squares, but no unoccupied square can be attacked by more than 8 knights. Therefore $8k \leq 8(64 - k)$, whence $k \leq 32$.

6. Find all nonnegative real numbers $a_1 \leq a_2 \leq \ldots \leq a_n$ satisfying

$$\sum_{i=1}^{n} a_i = 96, \quad \sum_{i=1}^{n} a_i^2 = 144, \quad \sum_{i=1}^{n} a_i^3 = 216.$$

Solution: Adding or removing zeroes has no effect, so we may assume the a_i are positive. By Cauchy-Schwarz,

$$(a_1 + \cdots + a_n)(a_1^3 + \cdots + a_n^3) \geq (a_1^2 + \cdots + a_n^2).$$

Since $96 \cdot 216 = 144^2$, we have equality, so the sequences $a_1, \cdots, a_n$ and $a_1^3, \cdots, a_n^3$ are proportional, so that $a_1 = \cdots = a_n = a$. Now $na = 96, na^2 = 144$ so that $a = 3/2, n = 32$.

7. Points D and E lie on sides AB and AC of triangle ABC such that $DE||BC$. Let P be an arbitrary point inside ABC. The lines PB and PC intersect DE at F and G, respectively. If O_1 is the circumcenter of PDG and O_2 is the circumcenter of PFE, show that $AP \perp O_1O_2$.

Solution: (Note: angles are directed modulo π.) Let M be the second intersection of AB with the circumcircle of DPG, and let N be the second intersection of N with the circumcircle of EPF. Now $\angle DMP = \angle DGP$ by cyclicity, and $\angle DGP = \angle BCP$ by parallelism, so $\angle DMP = \angle BCP$ and the points B, C, P, M are concyclic. Analogously, B, C, P, N are concyclic. Therefore the points B, C, M, N are concyclic, so $\angle DMN = \angle BCN$. Again by parallels, $\angle BCN = \angle DEN$, so the points D, E, M, N are concyclic.

We now apply the radical axis theorem to the circumcircles of DGP, EPF, and $DEMN$ to conclude that $DM \cap EN = A$ lies on the radical axis of the circles PDG and PEF, so $AP \perp O_1O_2$ as desired.

8. Let $P(x)$ be a polynomial with rational coefficients such that $P^{-1}(\mathbb{Q}) \subseteq \mathbb{Q}$. Show that P is linear.

Solution: By a suitable variable substitution and constant factor, we may assume $P(x)$ is monic and has integer coefficients; let $P(0) = c_0$. If p is a sufficiently large prime, the equation $P(x) = p + c_0$ has a single real root, which by assumption is rational and which we may also assume is positive (since P has positive leading coefficient). However, by the rational root theorem, the only rational roots of $P(x) - p - c_0$ can be ± 1 and $\pm p$. Since the root must be positive and cannot be 1 for large p, we have $P(p) - p - c_0 = 0$ for infinitely many p, so $P(x) = x + c_0$ is linear.

9. For $S = \{x_1, x_2, \ldots, x_n\}$ a set of n real numbers, all at least 1, we count the number of reals of the form

$$\sum_{i=1}^{n} \epsilon_i x_i, \quad \epsilon_i \in \{0, 1\}$$

lying in an open interval I of length 1. Find the maximum value of this count over all I and S.

Solution: The maximum is $\binom{n}{\lfloor n/2 \rfloor}$, achieved by taking $x_i = 1 + i/(n+1)$. To see that this cannot be improved, note that for any permutation σ of $\{1, \ldots, n\}$, at most one of the sets $\{\sigma(1), \ldots, \sigma(i)\}$ for $i = 1, \ldots, n$ has sum lying in I. Thus if T is the set of subsets whose sum lies in I, we have

$$\sum_{t \in T} t!(n-t)! \leq n! \Leftrightarrow \sum_{t \in T} \binom{n}{t}^{-1} \leq 1.$$

In particular, we have $|T| \leq \binom{n}{\lfloor n/2 \rfloor}$.

1.9 Ireland

1. For each positive integer n, find the greatest common divisor of $n!+1$ and $(n+1)!$.

   Solution: If $n+1$ is composite, then each prime divisor of $(n+1)!$ is a prime less than n, which also divides $n!$ and so does not divide $n!+1$. Hence $f(n) = 1$. If $n+1$ is prime, the same argument shows that $f(n)$ is a power of $n+1$, and in fact $n+1|n!+1$ by Wilson's theorem. However, $(n+1)^2$ does not divide $(n+1)!$, and thus $f(n) = n+1$.

2. For each positive integer n, let $S(n)$ be the sum of the digits in the decimal expansion of n. Prove that for all n,

   $$S(2n) \leq 2S(n) \leq 10S(2n)$$

   and show that there exists n such that $S(n) = 1996S(3n)$.

   Solution: It is clear that $S(a+b) \leq S(a) + S(b)$, with equality if and only if there are no carries in the addition of a and b. Therefore $S(2n) \leq 2S(n)$. Similarly $S(2n) \leq 5S(10n) = 5S(n)$. An example with $S(n) = 1996S(3n)$ is $133\cdots 35$ (with 5968 threes).

3. Let $f : [0,1] \to \mathbb{R}$ be a function such that

   (i) $f(1) = 1$,

   (ii) $f(x) \geq 0$ for all $x \in [0,1]$,

   (iii) if x, y and $x+y$ all lie in $[0,1]$, then $f(x+y) \geq f(x) + f(y)$.

   Prove that $f(x) \leq 2x$ for all $x \in K$.

   Solution: If $y > x$, then $f(y) \geq f(x) + f(y-x)$, so f is increasing. We note that $f(2^{-k}) \leq 2^{-k}$ by induction on k (with base case $k = 0$), as $2f(2^{-k}) \leq f(2^{-(k-1)})$. Thus for $x > 0$, let k be the positive integer such that $2^{-k} < x < 2^{-(k-1)}$; then $f(x) \leq f(2^{-(k-1)}) \leq 2^{-(k-1)} < 2x$. Since $f(0) + f(1) \leq f(1)$, we have $f(0) = 0$ and so $f(x) \leq 2x$ in all cases.

4. Let F be the midpoint of side BC of triangle ABC. Construct isosceles right triangles ABD and ACE externally on sides AB and AC with the right angles at D and E, respectively. Show that DEF is an isosceles right triangle.

   Solution: Identifying A, B, C with numbers on the complex plane, we have $F = (B + C)/2$, $D = B + (A - B)r$, $E = A + (C - A)r$, where $r = (1 + i)/2$. Then $E - F = A(1 - i)/2 - B/2 + Ci/2$ and $D - F = A(1 + i)/2 - Bi/2 - C/2$; in particular, $D - F = i(E - F)$ and so DEF is an isosceles right triangle.

5. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area.

   Solution: We dissect a 7×7 square into a 2×2 square A, a 3×3 square B, and three pieces C, D, E which form a 6×6 square, as shown below.

C	C	C	C	C	A	A
C	C	C	C	C	A	A
C	C	C	C	C	D	D
C	C	C	C	C	D	D
C	C	C	C	B	B	B
C	C	C	C	B	B	B
E	E	E	E	B	B	B

6. Let F_n denote the Fibonacci sequence, so that $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Prove that

   (i) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers n" is true if $k = 60$ and false for any positive integer $k < 60$;

   (ii) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers n" is true if $t = 300$ and false for any positive integer $t < 300$.

   Solution: A direct computation shows that the Fibonacci sequence has period 3 modulo 2 and 20 modulo 5 (compute terms until the initial terms $0, 1$ repeat, at which time the entire sequence repeats), yielding (a). As for (b), one computes that the period mod 4 is 6. The period mod 25 turns out to be 100, which is awfully many terms to compute by hand, but knowing that the period must be a multiple of 20 helps, and verifying the recurrence $F_{n+8} = tF_{n+4} + F_n$, where t is an integer congruent to 2 modulo 5, shows that the period divides 100; finally, an explicit computation shows that the period is not 20.

7. Prove that for all positive integers n,

$$2^{1/2} \cdot 4^{1/4} \cdots (2^n)^{1/2^n} < 4.$$

Solution: It suffices to show $\sum_{n=1}^{\infty} n/2^n = 2$:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2.$$

8. Let p be a prime number and a, n positive integers. Prove that if

$$2^p + 3^p = a^n,$$

then $n = 1$.

Solution: If $p = 2$, we have $2^2 + 3^2 = 13$ and $n = 1$. If $p > 2$, then p is odd, so 5 divides $2^p + 3^p$ and so 5 divides a. Now if $n > 1$, then 25 divides a^n and 5 divides

$$\frac{2^p + 3^p}{2 + 3} = 2^{p-1} - 2^{p-2} \cdot 3 + \cdots + 3^{p-1} \equiv p2^{p-1} \pmod{5},$$

a contradiction if $p \neq 5$. Finally, if $p = 5$, then $2^5 + 3^5 = 753$ is not a perfect power, so $n = 1$ again.

9. Let ABC be an acute triangle and let D, E, F be the feet of the altitudes from A, B, C, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C to EF, FD, DE, respectively. Prove that the lines AP, BQ, CR are concurrent.

Solution: It is a routine exercise to show that each of AP, BQ, CR passes through the circumcenter of ABC, so they all concur.

10. On a 5×9 rectangular chessboard, the following game is played. Initially, a number of discs are randomly placed on some of the squares, no square containing more than one disc. A turn consists of moving all of the discs subject to the following rules:

(i) each disc may be moved one square up, down, left, or right;

(ii) if a disc moves up or down on one turn, it must move left or right on the next turn, and vice versa;

(iii) at the end of each turn, no square can contain two or more discs.

The game stops if it becomes impossible to complete another turn. Prove that if initially 33 discs are placed on the board, the game must eventually stop. Prove also that it is possible to place 32 discs on the board so that the game can continue forever.

Solution: If 32 discs are placed in an 8×4 rectangle, they can all move up, left, down, right, up, etc. To show that a game with 33 discs must stop, label the board as shown:

1	2	1	2	1	2	1	2	1
2	3	2	3	2	3	2	3	2
1	2	1	2	1	2	1	2	1
2	3	2	3	2	3	2	3	2
1	2	1	2	1	2	1	2	1

Note that a disc on 1 goes to a 3 after two moves, a disc on 2 goes to a 1 or 3 immediately, and a disc on 3 goes to a 2 immediately. Thus if k discs start on 1 and $k > 8$, the game stops because there are not enough 3s to accommodate these discs. Thus we assume $k \leq 8$, in which case there are at most 16 squares on 1 or 3 at the start, and so at least 17 on 2. Of these 17, at most 8 can move onto 3 after one move, so at least 9 end up on 1; these discs will not all be able to move onto 3 two moves later, so the game will stop.

1.10 Italy

1. Among triangles with one side of a given length ℓ and with given area S, determine all of those for which the product of the lengths of the three altitudes is maximum.

   Solution: Let A, B be two fixed points with $AB = \ell$, and vary C along a line parallel to AB at distance $2S/\ell$. The product of the altitudes of ABC is $8S^3$ divided by the lengths of the three sides, so it suffices to minimize $AC \cdot BC$, or equivalently to maximize $\sin C$. Let D be the intersection of the perpendicular bisector of AB with the line through C. If $\angle D$ is not acute, the optimal triangles are clearly those with a right angle at C.

   Suppose $\angle D$ is acute and $C \neq D$, and assume C is on the same side of the perpendicular bisector of AB as B: we show $\angle D \geq \angle C$, and so the optimal triangle is ABD. The triangles DAC and DBC have equal base and height, so equal altitude. However, $AC > BC$ since $\angle CAB > \angle CBA$, so $\sin\angle DAC < \sin\angle DBC$, and since the former is acute, we have $\angle DAC < \angle DBC$. Adding $\angle CAB + \angle ABD$ to both sides, we get $\angle DAB + \angle DBA < \angle CAB + \angle CBA$, and so $\angle ADB > \angle ACB$, as claimed.

2. Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions $\{a, b, c\}$.

   Solution: Let a be any odd number, let $b = (a^2 - 5)/2$ and $c = (a^2 - 1)/2$. Then

   $$c^2 - b^2 = (c + b)(c - b) = a^2 - 3.$$

3. Let A and B be opposite vertices of a cube of edge length 1. Find the radius of the sphere with center interior to the cube, tangent to the three faces meeting at A and tangent to the three edges meeting at B.

   Solution: Introduce coordinates so that $A = (0, 0, 0), B = (1, 1, 1)$ and the edges are parallel to the coordinate axes. If r is the radius of the sphere, then (r, r, r) is its center, and $(r, 1, 1)$ is the point of tangency of one of the edges at B. Therefore $r^2 = 2(1 - r)^2$, giving

$r^2 - 4r + 2 = 0$ and so $r = 2 - \sqrt{2}$ (the other root puts the center outside of the cube).

4. Given an alphabet with three letters a, b, c, find the number of words of n letters which contain an even number of a's.

   Solution: If there are $2k$ occurences of a, these can occur in $\binom{n}{2k}$ places, and the remaining positions can be filled in 2^{n-2k} ways. So the answer is $\sum_k \binom{n}{2k} 2^{n-2k}$. To compute this, note that

$$(1+x)^n + (1-x)^n = 2\sum_k \binom{n}{2k} x^{2k},$$

   so the answer is

$$\frac{1}{2} 2^n [(1+1/2)^n + (1-1/2)^n] = \frac{1}{2}(3^n + 1).$$

5. Let C be a circle and A a point exterior to C. For each point P on C, construct the square $APQR$, where the vertices A, P, Q, R occur in counterclockwise order. Find the locus of Q as P runs over C.

   Solution: Take the circle to be the unit circle in the complex plane. Then $(Q - P)i = A - P$, so $Q = A + (1 - i)P$. We conclude the locus of Q is the circle centered at A whose radius is the norm of $1 - i$, namely $\sqrt{2}$.

6. Whas is the minimum number of squares that one needs to draw on a white sheet in order to obtain a complete grid with n squares on a side?

   Solution: It suffices to draw $2n-1$ squares: in terms of coordinates, we draw a square with opposite corners $(0, 0)$ and (i, i) for $1 \leq i \leq n$ and a square with opposite corners (i, i) and (n, n) for $1 \leq i \leq n-1$.

   To show this many squares are necessary, note that the segments from $(0, i)$ to $(1, i)$ and from $(n-1, i)$ to (n, i) for $0 < i < n$ all must lie on different squares, so surely $2n-2$ squares are needed. If it were possible to obtain the complete grid with $2n-2$ squares, each of these segments would lie on one of the squares, and the same would hold for the segments from $(i, 0)$ to $(i, 1)$ and from $(i, n-1)$ to (i, n) for $0 < i < n$. Each of the aforementioned horizontal segments shares a square with only two of the vertical segments, so the only possible arrangements are the one we gave above without the square with corners $(0, 0)$ and (n, n), and the $90°$ rotation of this arrangement, both of which are insufficient. Hence $2n - 1$ squares are necessary.

1.11 Japan

1. Consider a triangulation of the plane, i.e. a covering of the plane with triangles such that no two triangles have overlapping interiors, and no vertex lies in the interior of an edge of another triangle. Let A, B, C be three vertices of the triangulation and let θ be the smallest angle of the triangle $\triangle ABC$. Suppose no vertices of the triangulation lie inside the circumcircle of $\triangle ABC$. Prove there is a triangle σ in the triangulation such that $\sigma \cap \triangle ABC \neq \emptyset$ and every angle of σ is greater than θ.

   Solution: We may assume $\theta = \angle A$. The case where ABC belongs to the triangulation is easy, so assume this is not the case. If BC is an edge of the triangulation, one of the two triangles bounded by BC has common interior points with ABC, and this triangle satisfies the desired condition. Otherwise, there is a triangle BEF in the triangulation whose interior intersects BC. Since EF crosses BC at an interior point, $\angle BEF < \angle BAF < \angle BAC$, so triangle BEF satisfies the desired condition.

2. Let m and n be positive integers with $\gcd(m, n) = 1$. Compute $\gcd(5^m + 7^m, 5^n + 7^n)$.

   Solution: Let $s_n = 5^n + 7^n$. If $n \geq 2m$, note that

   $$s_n = s_m s_{n-m} - 5^m 7^m s_{n-2m},$$

   so $\gcd(s_m, s_n) = \gcd(s_m, s_{n-2m})$.. Similarly, if $m < n < 2m$, we have $\gcd(s_m, s_n) = \gcd(s_m, s_{2m-n})$. Thus by the Euclidean algorithm, we conclude that if $m + n$ is even, then $\gcd(s_m, s_n) = \gcd(s_1, s_1) = 12$, and if $m+n$ is odd, then $\gcd(s_m, s_n) = \gcd(s_0, s_1) = 2$.

3. Let $x > 1$ be a real number which is not an integer. For $n = 1, 2, 3, \ldots$, let $a_n = \lfloor x^{n+1} \rfloor - x\lfloor x^n \rfloor$. Prove that the sequence $\{a_n\}$ is not periodic.

   Solution: Assume, on the contrary, that there exists $p > 0$ such that $a_{p+n} = a_n$ for every n. Since $\lfloor x^n \rfloor \to \infty$ as $n \to \infty$, we have

$\lfloor x^{n+p} \rfloor - \lfloor x^n \rfloor > 0$ for some n; then setting $a_{n+p} = a_n$ and solving for x, we get

$$x = \frac{\lfloor x^{n+p+1} \rfloor - \lfloor x^{n+1} \rfloor}{\lfloor x^{n+p} \rfloor - \lfloor x^n \rfloor}$$

and so x is rational.

Put $y = x^p$ and

$$b_m = \sum_{k=0}^{p-1} x^{p-k-1} a_{mp+k} = \lfloor x^{mp+p} \rfloor - x^p \lfloor x^m r \rfloor = \lfloor y^{m+1} \rfloor - y \lfloor y^m \rfloor.$$

Since $a_{p+n} = a_p$, we have $b_{m+1} = b_m$, and y is also a rational number which is not an integer. Now put $c_m = \lfloor y^{m+1} - y^m \rfloor$; then $c_{m+1} = yc_m = y^m c_1$. This means c_m cannot be an integer for large m, a contradiction.

4. Let θ be the maximum of the six angles between the edges of a regular tetrahedron and a given plane. Find the minimum value of θ over all positions of the plane.

   Solution: Assume the edges of the tetrahedron $\Gamma = ABCD$ have length 1. If we place the tetrahedron so that AC and BC are parallel to the horizontal plane H, we obtain $\theta = 45°$, and we shall show this is the minimum angle.

   Let a, b, c, d be the projections of A, B, C, D to the horizontal plane H, and $\ell_1, \dots, \ell_6$ the projections of the edges $L_1, \dots, L_6$. Since the angle between L_i and H has cosine ℓ, it suffices to consider the shortest ℓ_i.

   If a, b, c, d form a convex quadrilateral with largest angle at a, then one of ab or ad is at most $1/\sqrt{2}$ since $bd \leq 1$. Otherwise, it is easily shown that one of the ℓ_i originating from the vertex inside the convex hull has length at most $1/\sqrt{3}$.

5. Let q be a real number with $(1 + \sqrt{5})/2 < q < 2$. For a number n with binary representation

$$n = 2^k + a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2 + a_0$$

   with $a_i \in \{0, 1\}$, we define p_n as follows:

$$p_n = q^k + a_{k-1} q^{k-1} + \cdots + a_1 q + a_0.$$

Prove that there exist infinitely many positive integers k for which there does not exist a positive integer l such that $p_{2k} < p_l < p_{2k+1}$.

Solution: Define the sequence a_n as follows:

$$a_{2m} = \sum_{k=0}^{m} 2^{2k}, \qquad a_{2m+1} = \sum_{k=0}^{m} 2^{2k+1}.$$

We will show that $k = a_n$ satisfies the given condition by induction on n. The cases $n = 0, 1$ follow by noting

$$1 < q < q + 1 < q^2 < q^2 + 1 < q^2 + q < q^2 + q + 1$$

and $p_l \geq q^p \geq q^3 > q^2 + q = p_6$ for $l \geq 8$.

Now suppose $n \geq 2$, assume the induction hypothesis, and suppose by way of contradiction that there exists l such that $p_{2a_n} < p_l < p_{2a_n+1}$. The argument falls into six cases, which we summarize in a table. The first column gives the conditions of the case, the second gives a lower bound for p_{2a_n}, the third is always equal to p_l, and the fourth gives an upper bound for p_{2a_n+1}; from these a contradiction to the induction hypothesis will become evident.

n even, $l = 2r + 1$	$qp_{2a_{n-1}} + 1$	$qp_r + 1$	$qp_{2a_{n-1}+1} + 1$
n even, $l = 4r$	$q^2 p_{2a_{n-2}}$	$q^2 p_r$	$q^2 p_{2a_{n-1}+1}$
n even, $l = 4r + 2$	$q^2 p_{2a_{n-2}} + q$	$q^2 p_r + q$	$q^2 p_{2a_{n-2}+1} + q$
n odd, $l = 2r$	$qp_{2a_{n-1}}$	qp_r	$qp_{2a_{n-1}+1}$
n even, $l = 4r + 1$	$q^2 p_{2a_{n-2}} + 1$	$q^2 p_r + 1$	$q^2 p_{2a_{n-2}+1} + 1$
n even, $l = 4r + 3$	$q^2 p_{2a_{n-2}} + q + 1$	$q^2 p_r + q + 1$	$q^2 p_{2a_{n-2}+1} + q + 1.$

1.12 Poland

1. Find all pairs (n, r), with n a positive integer and r a real number, for which the polynomial $(x+1)^n - r$ is divisible by $2x^2 + 2x + 1$.

   Solution: Let $t = (-1+i)/2$ be one of the roots of $2x^2 + 2x + 1$; then $(x+1)^n - r$ is divisible by $2x^2 + 2x + 1$ for r real if and only if $(t+1)^n = r$. Since the argument of $t+1$ is $\pi/4$, this is possible if and only if $n = 4m$, in which case $(t+1)^4m = (-4)^m$. Hence $(4m, (-4)^m)$ are the only solutions.

2. Let ABC be a triangle and P a point inside it such that $\angle PBC = \angle PCA < \angle PAB$. The line PB cuts the circumcircle of ABC at B and E, and the line CE cuts the circumcircle of APE at E and F. Show that the ratio of the area of the quadrilateral $APEF$ to the area of the triangle ABP does not depend on the choice of P.

   Solution: Note that $\angle AEP = \angle AEB = \angle ACB = \angle CBP$, so the lines AE and CP are parallel. Thus $[APE] = [ACE]$ and $[APEF] = [ACF]$. Now note that $\angle AFC = \pi - \angle EPA = \angle APB$ and $\angle ACF = \angle ACE = \angle ABE$. Therefore triangles ACF and ABP are similar and $[ACF]/[AB] = (AC/AB)^2$ independent of the choice of P.

3. Let $n \geq 2$ be a fixed natural number and let $a_1, a_2, \ldots, a_n$ be positive numbers whose sum is 1. Prove that for any positive numbers $x_1, x_2, \ldots, x_n$ whose sum is 1,

   $$2\sum_{i<j} x_i x_j \leq \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i},$$

   and determine when equality holds.

   Solution: The left side is $1 - \sum_i x_i^2$, so we can rewrite the desired result as

   $$\frac{1}{n-1} \leq \sum_{i=1}^{n} \frac{x_i^2}{1-a_i}.$$

   By Cauchy-Schwarz,

   $$\left(\sum_{i=1}^{n} \frac{x_i^2}{1-a_i}\right)\left(\sum_{i=1}^{n}(1-a_i)\right) \geq \left(\sum_{i=1}^{n} x_i\right)^2 = 1.$$

Since $\sum_i (1-a_i) = n-1$, we have the desired result.

4. Let $ABCD$ be a tetrahedron with $\angle BAC = \angle ACD$ and $\angle ABD = \angle BDC$. Show that edges AB and CD have the same length.

   Solution: Assume $AB \neq CD$. Draw the plane through AC bisecting the dihedral angle formed by the planes ABC and ACD, then draw a line ℓ in that plane perpendicular to AC through the midpoint O of AC. Now let B' and D' be the images of B and D, respectively, under the half-turn around the line ℓ; by assumption, $B' \neq D$ and $D' \neq B$. Since $\angle BAC = \angle ACD$, B' lies on CD and D' lies on AB. Now note that the quadrilateral $BB'D'D$ has total angular sum 2π. However, a nonplanar quadrilateral always has total angular sum less than 2π (divide it into two triangles, which each have angular sum π, and apply the spherical triangle inequality $\angle ABC + \angle CBD > \angle ABD$), so the lines AB and CD are coplanar, contradicting the assumption that $ABCD$ is a tetrahedron.

5. For a natural number k, let $p(k)$ denote the smallest prime number which does not divide k. If $p(k) > 2$, define $q(k)$ to be the product of all primes less than $p(k)$, otherwise let $q(k) = 1$. Consider the sequence

$$x_0 = 1, \qquad x_{n+1} = \frac{x_n p(x_n)}{q(x_n)} \qquad n = 0, 1, 2, \ldots.$$

   Determine all natural numbers n such that $x_n = 111111$.

   Solution: An easy induction shows that, if $p_0, p_1, \ldots$ are the primes in increasing order and n has base 2 representation $c_0 + 2c_1 + 4c_2 + \cdots$, then $x_n = p_0^{c_0} p_1^{c_1} \cdots$. In particular, $111111 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 = p_1 p_3 p_4 p_5 p_{10}$, so $x_n = 111111$ if and only if $n = 2^{10} + 2^5 + 2^4 + 2^3 + 2^1 = 1082$.

6. From the set of all permutations f of $\{1, 2, \ldots, n\}$ that satisfy the condition

$$f(i) \geq i - 1 \qquad i = 1, 2, \ldots, n,$$

   one is chosen uniformly at random. Let p_n be the probability that the chosen permutation f satisfies

$$f(i) \leq i + 1 \qquad i = 1, 2, \ldots, n.$$

Find all natural numbers n such that $p_n > 1/3$.

Solution: We have $p_n > 1/3$ for $n \leq 6$. Let c_n be the number of permutations of the first type. For such a permutation, either $f(1) = 1$, or $f(2) = 1$. In the first case, ignoring 1 gives a valid permutation of $\{2, \ldots, n\}$; in the latter case, we get a valid permutation of $\{2, \ldots, n\}$ by identifying 1 and 2 together. Hence $c_n = 2c_{n-1}$ and so $c_n = 2^{n-1}$ since $c_1 = 1$.

Let d_n be the number of permutations of the second type. For such a permutation, either $f(n) = n$ or $f(n) = n - 1$. In the first case, ignoring n gives a valid permutation of $\{1, \ldots, n-1\}$. In the latter case, we must have $f(n-1) = n$, so ignoring n and $n-1$ gives a valid permutation of $\{1, \ldots, n-2\}$. Thus $d_n = d_{n-1} + d_{n-2}$, and the initial conditions $d_1 = 1, d_2 = 2$ yield $d_n = F_{n+1}$, the $n+1$-st Fibonacci number.

It is easily shown (using the formula for F_n or by induction) that $c_n/d_n < 1/3$ for $n \geq 7$. Hence the desired n are $1, \ldots, 6$.

1.13 Romania

1. Let $n > 2$ be an integer and $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for any regular n-gon $A_1A_2 \dots A_n$,

$$f(A_1) + f(A_2) + \cdots + f(A_n) = 0.$$

Prove that f is the zero function.

Solution: We identify $\mathbb{R}^2$ with the complex plane and let $\zeta = e^{2\pi i/n}$. Then the condition is that for any $z \in \mathbb{C}$ and any positive real t,

$$\sum_{j=1}^{n} f(z + t\zeta^j) = 0.$$

In particular, for each of $k = 1, \dots, n$, we have

$$\sum_{j=1}^{n} f(z - \zeta^k + \zeta^j) = 0.$$

Summing over k, we have

$$\sum_{m=1}^{n} \sum_{k=1}^{n} f(z - (1 - \zeta^m)\zeta^k) = 0.$$

For $m = n$ the inner sum is $nf(z)$; for other m, the inner sum again runs over a regular polygon, hence is 0. Thus $f(z) = 0$ for all $z \in \mathbb{C}$.

2. Find the greatest positive integer n for which there exist n nonnegative integers $x_1, x_2, \dots, x_n$, not all zero, such that for any sequence $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ of elements of $\{-1, 0, 1\}$, not all zero, n^3 does not divide $\epsilon_1 x_1 + \epsilon_2 x_2 + \ldots + \epsilon_n x_n$.

Solution: The statement holds for $n = 9$ by choosing $1, 2, 2^2, \dots, 2^8$, since in that case

$$|\epsilon_1 + \cdots + \epsilon_9 2^8| \leq 1 + 2 + \cdots + 2^8 < 9^3.$$

However, if $n = 10$, then $2^{10} > 10^3$, so by the pigeonhole principle, there are two subsets A and B of $\{x_1, \dots, x_{10}\}$ whose sums are congruent modulo 10^3. Let $\epsilon_i = 1$ if x_i occurs in A but not in B, -1 if x_i occurs in B but not in A, and 0 otherwise; then $\sum \epsilon_i x_i$ is divisible by n^3.

3. Let x, y be real numbers. Show that if the set

$$\{\cos(n\pi x) + \cos(n\pi y) | n \in \mathbb{N}\}$$

is finite, then $x, y \in \mathbb{Q}$.

Solution: Let $a_n = \cos n\pi x$ and $b_n = \sin n\pi x$. Then

$$(a_n + b_n)^2 + (a_n - b_n)^2 = 2(a_n^2 + b_n^2) = 2 + (a_{2n} + b_{2n}).$$

If $\{a_n + b_n\}$ is finite, it follows that $\{a_n - b_n\}$ is also a finite set, and hence that $\{a_n\}$ is finite, since

$$a_n = \frac{1}{2}[(a_n + b_n) + (a_n - b_n)],$$

and similarly $\{b_n\}$ is finite. In particular, $a_m = a_n$ for some $m < n$, and so $(n - m)\pi x$ is an integral multiple of π. We conclude x and y are both rational.

4. Let $ABCD$ be a cyclic quadrilateral and let M be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (for a total of 16 points). Show that there exist two sets of parallel lines K and L, each consisting of four lines, such that any line of $K \cup L$ contains exactly four points of M.

Solution: Let T be the midpoint of the arc AB of the circumcircle of ABC, I the incenter of ABC, and I_B, I_C the excenters of ABC opposite B and C, respectively. We first show $TI = TA = TB = TI_C$. Note that

$$\angle TAI = \angle TAB + \angle BAI = (\angle C + \angle A)/2 = \angle ICA + \angle IAC = \angle TAI,$$

so $TI = TA$, and similarly $TI = TB$. Moreover, in the right triangle AI_CI, $\angle AI_CT = \pi/2 - \angle AIT = \pi/2 - \angle TAI = \angle TAI_C$, so $TA = TI_C$ also.

We next show that the midpoint U of I_BI_C is also the midpoint of the arc BAC. Note that the line I_BI_C bisects the exterior angles of ABC at A, so the line I_BI_C passes through the midpoint V of the arc BAC. Considering the right triangles I_BBI_C and I_BCI_C, we note $BU = (I_B I_C)/2 = CU$, so U lies on the perpendicular bisector of BC, which suffices to show $U = V$. (Note that I_B and I_C lie on the same side of BC as A, so the same is true of U.)

Let E, F, G, H be the midpoints of the arcs AB, BC, CD, DA. Let I_A, I_B, I_C, I_D be the incenters of the triangles BCD, CDA, DAB, ABC, respectively. Let A_B, A_C, A_D be the excenters of BCD opposite B, C, D, respectively, and so on.

By the first observation, $I_C I_D C_D D_C$ is a rectangle with center E, and the diagonals, which contain the points C and D, have length $2EA = 2EB$. Similarly, we obtain rectangles centered at F, G, H.

Now consider the excenters of the form X_Y, where X and Y are opposite vertices in $ABCD$. We shall prove the claim with

$$K = \{B_C C_B, I_C I_B, I_D I_A, A_D D_A\}, \quad L = \{A_B B_A, I_A I_B, I_C I_D, C_D D_C\}.$$

Consider the rectangle $B_C I_D B_A P$, where P is an unknown point. From the second observation above, the midpoint K of diagonal $B_A B_C$ is the midpoint of arc CDA, so it lies on the internal bisector BK of triangle ABC. Again by the first observation, we conclude $M = D_A$, so D_A lies on the lines $B_C C_B$ and $B_A A_B$, and so on, proving the claim.

5. Given $a \in \mathbb{R}$ and $f_1, f_2, \dots, f_n : \mathbb{R} \to \mathbb{R}$ additive functions such that $f_1(x) f_2(x) \cdots f_n(x) = ax^n$ for all $x \in \mathbb{R}$. Prove that there exists $b \in \mathbb{R}$ and $i \in \{1, 2, \dots, n\}$ such that $f_i(x) = bx$ for all $x \in \mathbb{R}$.

   Solution: Let $c_i = f_i(1)$. Then for any integer x,

$$\prod_{i=1}^{n} f_i(1 + mx) = \prod_{i=1}^{n} [c_i + m f_i(x)] = a(1 + mx)^n.$$

   First suppose $a \neq 0$, in which case $c_i \neq 0$ for all i. Then we have an equality of polynomials in T:

$$\prod_{i=1}^{n} [c_i + f_i(x) T] = a(1 + xT)^n,$$

   and so by unique factorization, $c_i + f_i(x) T = b_i(1 + xT)$ for some real number b_i. Equating coefficients gives $b_i = c_i$ and $f_i(x) = b_i x = c_i x$ for all x.

Now suppose $a = 0$; we shall show that f_i is identically zero for some i. Assume on the contrary that there exist a_i for all i such that $f_i(a_i) \neq 0$. Let

$$x_m = a_1 + ma_2 + \cdots + m^{n-1}a_m$$

for any integer m. Then

$$0 = \prod_{i=1}^{n} f_i(x_m) = \prod_{i=1}^{n}[f_i(a_1) + f_i(a_2)m + \cdots + f_i(a_n)m^{n-1}].$$

Hence for some i, the polynomial $f_i(a_1)+f_i(a_2)m+\cdots+f_i(a_n)m^{n-1}$ is identically zero, contradicting the fact that $f_i(a_i) \neq 0$. Thus for some i, $f_i(x) = 0$ for all x, proving the claim with $b = 0$.

6. The sequence $\{a_n\}_{n\geq 2}$ is defined as follows: if $p_1, p_2, \ldots, p_k$ are the distinct prime divisors of n, then $a_n = p_1^{-1} + p_2^{-1} + \ldots + p_k^{-1}$. Show that for any positive integer $N \geq 2$,

$$\sum_{n=2}^{N} a_2 a_3 \cdots a_n < 1.$$

Solution: It is easily seen that

$$\sum_{k=2}^{n} a_k = \sum_{k=2}^{n}\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}\right) = \sum_{p\leq n} \frac{1}{p}\left\lfloor \frac{n}{p} \right\rfloor.$$

On the other hand, we have the inequalities

$$\begin{aligned}
\sum_{p\leq n} \frac{1}{p}\left\lfloor \frac{n}{p} \right\rfloor &\leq \sum_{p\leq n} \frac{n}{p^2} \\
&< n\left(\frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}\right) \\
&< \frac{n}{4}\left(\sum_{k=1}^{\infty} \frac{1}{k(k+1)}\right) = \frac{n}{2}.
\end{aligned}$$

Thus $\sum_{k=2}^{n} a_k < n/2$ for all $n \geq 2$. Now using the AM-GM inequality,

$$\begin{aligned} a_2 a_3 \cdots a_n &< \left(\frac{a_2 + a_3 + \cdots + a_n}{n-1}\right)^{n-1} \\ &< \frac{1}{2^{n-1}}\left(1 + \frac{1}{n-1}\right)^{n-1} < \frac{e}{2^{n-1}} < \frac{3}{2^{n-1}}. \end{aligned}$$

Adding these inequalities,

$$\begin{aligned} \sum_{n=2}^{\infty} a_2 \cdots a_n &< \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{60} + 3\left(\frac{1}{2^5} + \frac{1}{2^6} + \cdots\right) \\ &= \frac{46}{60} + \frac{3}{2^5}\left(1 + \frac{1}{2} + \cdots\right) = \frac{46}{60} + \frac{6}{32} < 1. \end{aligned}$$

7. Let $n \geq 3$ be an integer and $x_1, x_2, \ldots, x_{n-1}$ nonnegative integers such that

$$\begin{aligned} x_1 + x_2 + \ldots + x_{n-1} &= n \\ x_1 + 2x_2 + \ldots + (n-1)x_{n-1} &= 2n - 2. \end{aligned}$$

Find the minimum of the sum

$$F(x_1, \ldots, x_{n-1}) = \sum_{k=1}^{n-1} k x_k (2n - k).$$

Solution: The desired sum can be written as $2n(2n-2) - \sum_{k=1}^{n-1} k^2 x_k$. Now note

$$\sum_{k=1}^{n-1} k^2 x_k = \sum_{k=1}^{n-1} x_k + (k-1)(k+1)x_k \leq n + n \sum_{k=1} n-1(k-1)x_k = n + n(2n-2-n) = n^2 - n.$$

Hence the quantity in question is at most $2n(2n-2) - (n^2 - n) = 3n^2 - 3n$, with equality for $x_1 = n-1, x_2 = \cdots = x_{n-2} = 0, x_{n-1} = 1$.

8. Let n, r be positive integers and A a set of lattice points in the plane, such that any open disc of radius r contains a point of A. Show that for any coloring of the points of A using n colors, there exist four points of the same color which are the vertices of a rectangle.

Solution: Consider a square of side length $L = 4nr^2$ with sides parallel to the coordinate axes. One can draw $(2nr)^2 = 4n^2r^2$ disjoint disks of radius r inside the square, hence such a square contains at least $4n^2r^2$ points of A. The lattice points in A lie on $L-1 = 4nr^2-1$ vertical lines; by the pigeonhole principle, some vertical line contais $n+1$ points of A. Again by the pigeonhole principle, two of these points are colored in the same color.

Now consider an infinite horizontal strip made of ribbons of side length L; some two of them have two points in the same position in the same color, and these four points form the vertices of a rectangle.

9. Find all prime numbers p, q for which the congruence

$$\alpha^{3pq} \equiv \alpha \pmod{3pq}$$

holds for all integers α.

Solution: Without loss of generality assume $p \leq q$; the unique solution will be $(11, 17)$, for which one may check the congruence using the Chinese Remainder Theorem. We first have $2^{3pq} \equiv 2 \pmod 3$, which means p and q are odd. In addition, if α is a primitive root mod p, then $\alpha^{3pq-1} \equiv 1 \pmod p$ implies that $p-1$ divides $3pq-1$ as well as $3pq-1-3q(p-1) = 3q-1$, and conversely that $q-1$ divides $3p-1$. If $p = q$, we now deduce $p = q = 3$, but $4^{27} \equiv 1 \pmod{27}$, so this fails. Hence $p < q$.

Since p and q are odd primes, $q \geq p+2$, so $(3p-1)/(q-1) < 3$. Since this quantity is an integer, and it is clearly greater than 1, it must be 2. That is, $2q = 3p+1$. On the other hand, $p-1$ divides $3q-1 = (9p+1)/2$ as well as $(9p+1)-(9p-9) = 10$. Hence $p = 11, q = 17$.

10. Let $n \geq 3$ be an integer and $p \geq 2n-3$ a prime. Let M be a set of n points in the plane, no three collinear, and let $f : M \to \{0, 1, \ldots, p-1\}$ be a function such that:

   (i) only one point of M maps to 0, and

   (ii) if A, B, C are distinct points in M and k is the circumcircle of the triangle ABC, then

$$\sum_{P \in M \cap k} f(P) \equiv 0 \pmod{p}.$$

Show that all of the points of M lie on a circle.

Solution: Let X be the point mapping to 0. We first show that if every circle through X and two points of M contains a third point of M, then all of the points of M lie on a circle. Indeed, consider an inversion with center at X. Then the image of $M - \{X\}$ has the property that the line through any two of its points contains a third point; it is a standard result that this means the points are collinear. (Otherwise, find a triangle ABC minimizing the length of the altitude AH; there is another point N on BC, but then either ABN or ACN has a shorter altitude than AH, contradiction.)

Now suppose the points of M do not lie on a circle. By the above, there exists a circle passing through M and only two points A, B of M. Let $f(A) = i$, so that by the hypothesis, $f(B) = p - i$. Let a be the number of circles passing through X, A and at least one other point of M, let b be the number of circles passing through X, B and at least one other point of M, and let S be the sum of $f(P)$ over all P in M. By adding the relations obtained from the circles through X and A, we get $S + (a-1)i \equiv 0 \pmod{p}$, and similarly, $S + (b-1)(p-i) \equiv 0 \pmod{p}$. Therefore $a + b - 2 \equiv 0 \pmod{p}$; since $a + b \leq 2n + 4 < p$, we have $a + b = 2$ and so $a = b = 1$, contradicting the assumption that the points do not all lie on a circle.

11. Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be positive reals such that $x_1 + x_2 + \cdots + x_n = x_{n+1}$. Prove that

$$\sum_{i=1}^{n} \sqrt{x_i(x_{n+i} - x_i)} \leq \sqrt{\sum_{i=1}^{n} x_{n+1}(x_{n+1} - x_i)}.$$

Solution: First note that

$$\sum_{i=1}^{n} x_{n+1}(x_{n+1} - x_i) = n x_{n+1}^2 - x_{n+1} \sum_{i=1}^{n} x_i = (n-1)x_{n+1}.$$

Hence the given inequality may be rewritten as

$$\sum_{i=1}^{n} \sqrt{\frac{1}{n-1}\frac{x_i}{x_{n+1}}\left(1-\frac{x_i}{x_{n+1}}\right)} \leq 1$$

On the other hand, by the arithmetic-geometry mean inequality, the left side is at most

$$\sum_{i=1}^{n} \frac{x_i}{2x_{n+i}} + \frac{1-x_i/x_{n+1}}{2(n-1)} = \frac{1}{2} + \frac{(n-1)}{2(n-1)} = 1.$$

12. Let x, y, z be real numbers. Prove that the following conditions are equivalent.

    (i) $x, y, z > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 1$.

    (ii) For every quadrilateral with sides a, b, c, d, $a^2x + b^2y + c^2z > d^2$.

Solution: To show (i) implies (ii), note that

$$\begin{aligned} a^2x + b^2y + c^2z &\geq (a^2x + b^2y + c^2z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{x}\right) \\ &\geq (a+b+c)^2 > d^2, \end{aligned}$$

using Cauchy-Schwarz after the first inequality.

To show (i) implies (ii), first note that if $x \leq 0$, we may take a quadrilateral of sides $a = n, b = 1, c = 1, d = n$ and get $y + z > n^2(1-x)$, a contradiction for large n. Thus $x > 0$ and similarly $y > 0, z > 0$. Now use a quadrilateral of sides $1/x, 1/y, 1/z$ and $1/x + 1/y + 1/z - 1/n$, where n is large. We then get

$$\frac{x}{x^2} + \frac{y}{y^2} + \frac{z}{z^2} > \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{n}\right)^2.$$

Since this holds for all n, we may take the limit as $n \to \infty$ and get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{n}\right)^2,$$

and hence $1/x + 1/y + 1/z \leq 1$.

13. Let n be a positive integer and D a set of n concentric circles in the plane. Prove that if the function $f : D \to D$ satisfies $d(f(A), f(B)) \geq d(A, B)$ for all $A, B \in D$, then $d(f(A), f(B)) = d(A, B)$ for every $A, B \in D$.

    Solution: Label the circles $D_1, \dots, D_n$ in increasing order of radius, and let r_i denote the radius of D_i. Clearly the maximum of $d(A, B)$ occurs when A and B are antipodal points on D. Let $ABCD$ be the vertices of a square inscribed in D_n; then $f(A)$ and $f(C)$ are antipodal, as are $f(B)$ and $f(D)$. In addition, each of the minor arcs $f(A)f(B)$ and $f(B)f(C)$ must be at least a quarter arc, thus $f(B)$ bisects one of the semicircles bounded by $f(A)$ and $f(C)$, and $f(D)$ bisects the other. Now if P is any point on the minor arc AB, then the arcs $f(P)f(A)$ and $f(P)f(B)$, which are at least as long as the arcs PA and PB, add up to the quarter arc $f(P)f(B)$. We conclude f is isometric on D_n.

    Since f is clearly injective and is now bijective on D_n, f maps $D_1 \cup \ldots \cup D_{n-1}$ into itself. Thus we may repeat the argument to show that f is isometric on each D_i. To conclude, it suffices to show that distances between adjacent circles, say D_1 and D_2, are preserved. This is easy; choose a square $ABCD$ on D_1 and let A', B', C', D' be the points on D_2 closest to A, B, C, D, respectively. Then $A'B'C'D'$ also form a square, and the distance from A to C' is the maximum between any point on D_1 and any point on D_3. Hence the eight points maintain their relative position under f, which suffices to prove isometry.

14. Let $n \geq 3$ be an integer and $X \subseteq \{1, 2, \dots, n^3\}$ a set of $3n^2$ elements. Prove that one can find nine distinct numbers $a_1, \dots, a_9$ in X such that the system

$$\begin{aligned} a_1x + a_2y + a_3z &= 0 \\ a_4x + a_5y + a_6z &= 0 \\ a_7x + a_8y + a_9z &= 0 \end{aligned}$$

    has a solution (x_0, y_0, z_0) in nonzero integers.

    Solution: Label the elements of X in increasing order $x_1 < \cdots < x_{3n^2}$, and put

$$X_1 = \{x_1, \dots, x_{n^2}\}, \quad X_2 = \{x_{n^2+1}, \dots, x_{2n^2}\}, \quad X_3 = \{x_{2n^2+1}, \dots, x_{3n^2}\}.$$

Define the function $f : X_1 \times X_2 \times X_3 \to X \times X$ as follows:

$$f(a, b, c) = (b - a, c - b).$$

The domain of f contains n^6 elements. The range of f, on the other hand, is contained in the subset of $X \times X$ of pairs whose sum is at most n^3, a set of cardinality

$$\sum_{k=1}^{n^3-1} k = \frac{n^3(n^3-1)}{2} < \frac{n^6}{2}.$$

By the pigeonhole principle, some three triples (a_i, b_i, c_i) $(i = 1, 2, 3)$ map to the same pair, in which case $x = b_1 - c_1, y = c_1 - a_1, z = a_1 - b_1$ is a solution in nonzero integers. Note that a_i cannot equal b_j since X_1 and X_2 and so on, and that $a_1 = a_2$ implies that the triples (a_1, b_1, c_1) and (a_2, b_2, c_2) are identical, a contradiction. Hence the nine numbers chosen are indeed distinct.

1.14 Russia

1. Which are there more of among the natural numbers from 1 to 1000000, inclusive: numbers that can be represented as the sum of a perfect square and a (positive) perfect cube, or numbers that cannot be?

   Solution: There are more numbers not of this form. Let $n = k^2 + m^3$, where $k, m, n \in \mathbb{N}$ and $n \leq 1000000$. Clearly $k \leq 1000$ and $m \leq 100$. Therefore there cannot be more numbers in the desired form than the 100000 pairs (k, m).

2. The centers O_1, O_2, O_3 of three nonintersecting circles of equal radius are positioned at the vertices of a triangle. From each of the points O_1, O_2, O_3 one draws tangents to the other two given circles. It is known that the intersection of these tangents form a convex hexagon. The sides of the hexagon are alternately colored red and blue. Prove that the sum of the lengths of the red sides equals the sum of the lengths of the blue sides.

   Solution: Let A, B, C, D, E, F be the vertices of the hexagon in order, with A on the tangents to O_1, C on the tangents to O_2, and E on the tangents to O_3. Since the given circles have equal radius,

$$X_1O_2 = O_1Y_2,\ Y_1O_3 = O_2Z_2,\ Z_1O_1 = O_3X_2$$

   or

$$\begin{aligned} X_1A + AB + BO_2 &= O_1B + BC + CY_2, \\ Y_1C + CD + DO_3 &= O_2D + DE + EZ_2, \\ Z_1E + EF + FO_1 &= O_3F + FA + AX_2. \end{aligned}$$

   Adding these equations and noting that

$$X_1A = AX_2,\ Y_1C = CY_2,\ Z_1E = EZ_2,$$

   (as these segments are tangents to a circle from a single point) and

$$BO_2 = O_1B,\ DO_3 = O_2D,\ FO_1 = O_3F$$

(since the circles have equal radii), we get

$$AB + CD + EF = BC + DE + FA,$$

as desired.

Note: The analogous statement is also true in the case where the hexagon has reflex angles at B, D, F. In both cases, we also have the equality $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$, or equivalently, the lines AD, BE, CF concur. Moreover, the latter statement remains true even if the assumption of equal radii is removed, and this fact leads to a proof of Brianchon's Theorem.

3. Let x, y, p, n, k be natural numbers such that

$$x^n + y^n = p^k.$$

Prove that if $n > 1$ is odd, and p is an odd prime, then n is a power of p.

Solution: Let $m = \gcd(x, y)$. Then $x = mx_1, y = my_1$ and by virtue of the given equation, $m^n(x_1^n + y_1^n) = p^k$, and so $m = p^\alpha$ for some nonnegative integer α. It follows that

$$x_1^n + y_1^n = p^{k-n^\alpha}. \tag{1}$$

Since n is odd,

$$\frac{x_1^n + y_1^n}{x_1 + y_1} = x_1^{n-1} - x_1^{n-2}y_1 + x_1^{n-3}y_1^2 - \cdots - x_1y_1^{n-2} + y_1^{n-1}.$$

Let A denote the right side of the equation. By the condition $p > 2$, it follows that at least one of x_1, y_1 is greater than 1, so since $n > 1$, $A > 1$.

From (1) it follows that $A(x_1 + y_1) = p^{k-n^\alpha}$, so since $x_1 + y_1 > 1$, and $A > 1$, both of these numbers are divisible by p; moreover, $x_1 + y_1 = p^\beta$ for some natural number β. Thus

$$\begin{aligned} A &= x_1^{n-1} - x_1^{n-2}(p^\beta - x_1) + \cdots - x_1(p^\beta - x_1)^{n-2} + (p^\beta - x_1)^{n-1} \\ &= nx_1^{n-1} + Bp. \end{aligned}$$

Since A is divisible by p and x_1 is relatively prime to p, it follows that n is divisible by p.

Let $n = pq$. Then $x^{pq} + y^{pq} = p^k$ or $(x^p)^q + (y^p)^q = p^k$. If $q > 1$, then by the same argument, p divides q. If $q = 1$, then $n = p$. Repeating this argument, we deduce that $n = p^\ell$ for some natural number ℓ.

4. In the Duma there are 1600 delegates, who have formed 16000 committees of 80 persons each. Prove that one can find two committees having no fewer than four common members.

First Solution: Suppose any two committees have at most three common members. Have two deputies count the possible ways to choose a chairman for each of three sessions of the Duma. The first deputy assumes that any deputy can chair any session, and so gets 1600^3 possible choices. The second deputy makes the additional restriction that all of the chairmen belong to a single committee. Each of the 16000 committees yields 80^3 choices, but this is an overcount; each of the $16000(16000 - 1)/2$ pairs of committees give at most 3^3 overlapping choices. Since the first deputy counts no fewer possibilities than the second, we have the inequality

$$1600^3 \geq 16000 \cdot 80^3 - \frac{16000 \cdot 15999}{2} 3^3.$$

However,

$$\begin{aligned} 16000 \cdot 80^3 - \frac{16000 \cdot 15999}{2} 3^3 &> 16000 \cdot 80^3 - \frac{16000 \cdot 15999}{2} \frac{4^2}{2} \\ &= \frac{16000 \cdot 4^3}{4} + 2^{13} \cdot 10^6 - 2^{12} \cdot 10^6 \\ &> 2^{12} \cdot 10^6 = 1600^3. \end{aligned}$$

We have a contradiction.

Second Solution: Suppose we have N committees such that no two have more than three common members. For each deputy we write down all of the unordered pairs of committees she belongs to. If a person deputy to K committees, she gives rise to $K(K - 1)/2$ pairs.

Let $K_1, \ldots, K_{1600}$ be the number of committees that deputies $1, \ldots, 1600$ belong to (under some labeling of the deputies). The total number of pairs written down is

$$
\begin{aligned}
&\frac{K_1(K_1-1)}{2} + \ldots + \frac{K_{1600}(K_{1600}-1)}{2} \\
&= \frac{K_1^2 + \ldots + K_{1600}^2}{2} - \frac{K_1 + \ldots + K_{1600}}{2} \\
&\geq \frac{1}{2}\left(\frac{(K_1 + \ldots + K_{1600})^2}{1600} - (K_1 + \ldots + K_{1600})\right) \\
&= \frac{1}{2}\left(\frac{(80N)^2}{1600} - 80N\right) = \frac{1}{2}N(4N-80)
\end{aligned}
$$

since $K_1 + \ldots + K_{1600} = 80N$.

Since no two committees have more than three common members, the total number of pairs written cannot exceed $3N(N-1)/2$. Hence $N(4N-80)/2 \leq 3N(N-1)/2$, i.e. $N \leq 77$. In particular, if $N = 16000$, this cannot be the case.

5. Show that in the arithmetic progression with first term 1 and ratio 729, there are infinitely many powers of 10.

Solution: We will show that for all natural numbers n, $10^{81n} - 1$ is divisible by 729. In fact, $10^{81n} - 1 = (10^{81})^n - 1^n = (10^{81} - 1) \cdot A$, and

$$
\begin{aligned}
10^{81} - 1 &= \underbrace{9\ldots9}_{81} \\
&= \underbrace{9\ldots9}_{9} \cdots \underbrace{10\ldots01}_{8}\underbrace{10\ldots01}_{8} \ldots \underbrace{10\ldots01}_{8} \\
&= 9\underbrace{1\ldots1}_{9} \cdots \underbrace{10\ldots01}_{8}\underbrace{10\ldots01}_{8} \ldots \underbrace{10\ldots01}_{8}.
\end{aligned}
$$

The second and third factors are composed of 9 units, so the sum of their digits is divisible by 9, that is, each is a multiple of 9. Hence $10^{81} - 1$ is divisible by $9^3 = 729$, as is $10^{81n} - 1$ for any n.

6. In the isosceles triangle ABC ($AC = BC$) point O is the circumcenter, I the incenter, and D lies on BC so that lines OD and BI are perpendicular. Prove that ID and AC are parallel.

First Solution: If the given triangle is equilateral (i.e. $O = I$) the statement is obvious. Otherwise, suppose O lies between I and C. Draw the altitude CE and note that

$$\angle EIB = 90^\circ - \frac{1}{2}\angle ABC \quad \text{and} \quad \angle ODB = 90^\circ - \frac{1}{2}\angle ABC,$$

so $\angle OIB + \angle ODB = 180^\circ$, that is, the points B, I, O, D lie on a circle. Thus $\angle IDB = \angle IOB$ (both angles are inscribed in arc IB), but $\angle IOB = \frac{1}{2}\angle AOB = \angle ACB$. Therefore $\angle IDB = \angle ACB$, and so $ID||AC$. The argument is similar in the cases where I lies between O and C.

Second Solution: Extend the angle bisector BI to meet the circumcircle at E. Next extend the line ED to meet the circumcircle at F. Let G and K be the intersections of ED with AF and the altitude CH, respectively. The line OD contains the diameter perpendicular to EB, and so $DE = DB$, i.e. the triangle EDB is isosceles and $\angle DEB = \angle DBE$. But then $\angle DEB = \angle ABE$, hence $EF||AB$ and $EF \perp CI$. By inscribed angles,

$$\angle CEF = \angle IEF = \angle CFE = \angle IFE,$$

so $ECFI$ is a rhombus. Thus $CK = KI$, and (by the symmetry of G and D across CH) $GK = KD$. This means $GKDI$ is also a rhombus and $CG||DI$.

7. Two piles of coins lie on a table. It is known that the sum of the weights of the coins in the two piles are equal, and for any natural number k, not exceeding the number of coins in either pile, the sum of the weights of the k heaviest coins in the first pile is not more than that of the second pile. Show that for any natural number x, if each coin (in either pile) of weight not less than x is replaced by a coin of weight x, the first pile will not be lighter than the second.

Solution: Let the first pile have n coins of weights $x_1 \geq x_2 \geq \cdots \geq x_n$, and let the second pile have m coins of weights $y_1 \geq y_2 \geq \cdots \geq y_m$, where $x_1 \geq \cdots \geq x_s \geq x \geq x_{s+1} \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_t \geq x \geq y_{t+1} \geq \cdots \geq y_m$. (If there are no coins of weight greater than x, the result is clear.) We need to show that $xs + x_{s+1} + \cdots + x_n \geq xt + y_{t+1} + \cdots + y_m$. Since $x_1 + \cdots + x_n = y_1 + \ldots + y_m = A$, this inequality can be equivalently written

$$xs + (A - x_1 - \cdots - x_m) \geq xt + (A - y_1 - \ldots - y_t),$$

which in turn can be rewritten

$$x_1 + \ldots + x_s + x(t-s) \leq y_1 + \ldots + y_t,$$

which is what we will prove.

If $t \geq s$, then

$$\begin{aligned} x_1 + \ldots + x_s + x(t-s) &= (x_1 + \ldots + x_s) + \underbrace{(x + \cdots + x)}_{t-s} \\ &\leq (y_1 + \ldots + y_s) + (y_{s+1} + \ldots + y_t), \end{aligned}$$

since $x_1 + \ldots + x_s \leq y_1 + \ldots + y_s$ (from the given condition) and $y_{s+1} \geq \ldots \geq y_t \geq x$.

If $t < s$, then $x_1 + \ldots + x_s + x(t-s) \leq y_1 + \ldots + y_t$ is equivalent to

$$x_1 + \ldots + x_s \leq y_1 + \ldots + y_t + \underbrace{(x + \ldots + x)}_{t-s}.$$

The latter inequality follows from the fact that

$$x_1 + \ldots + x_s \leq y_1 + \ldots + y_s = (y_1 + \ldots + y_t) + (y_{t+1} + \ldots + y_s)$$

and $y_s \leq \ldots \leq y_{t+1} \leq x$.

8. Can a 5×7 checkerboard be covered by L's (figures formed from a 2×2 square by removing one of its four 1×1 corners), not crossing its borders, in several layers so that each square of the board is covered by the same number of L's?

First Solution: No such covering exists. Suppose we are given a covering of a 5×7 checkerboard with L's, such that every cell is covered by exactly k L's. Number the rows $1, \ldots, 5$ and the columns $1, \ldots, 7$, and consider the 12 squares lying at the intersections of odd-numbered rows with odd-numbered columns. Each of these cells is coverd by k L's, so at least $12k$ L's must be used in total. But these cover $3 \cdot 12k > 35k$ cells in total, a contradiction.

Second Solution: Color the cells of the checkerboard alternately black and white, so that the corners are all black. In each black square we write the number -2, and in each white square 1. Note that the sum of the numbers in the cells covered by each L is nonnegative, and consequently if we are given a covering of the board in k layers, the sum over each L of the numbers covered by that L is nonnegative. But if this number is S and s is the sum of the numbers on the board, then

$$S = ks = k(-2 \cdot 12 + 23 \cdot 1) = -k < 0.$$

We have a contradiction.

Note: It is proved analogously that a covering of the desired form does not exist if the checkerboard has dimensions $3 \times (2n+1)$ or 5×5. The 2×3 board can be covered by one layer of two L's, the 5×9 by one layer of 15 L's, and the 2×2 by three layers using four L's. Combining these three coverings, it is not hard to show that all remaining $m \times n$ boards $(m, n \geq 2)$ can be covered.

9. Points E and F are given on side BC of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.

Solution: By the equality of angles EAF and FDE, the quadrilateral $AEFD$ is cyclic. Therefore $\angle AEF + \angle FDA = 180°$. By the equality of angles BAE and CDF we have

$$\angle ADC + \angle ABC = \angle FDA + \angle CDF + \angle AEF - \angle BAE = 180°.$$

Hence the quadrilateral $ABCD$ is cyclic, so $\angle BAC = \angle BDC$. It follows that $\angle FAC = \angle EDB$.

10. On a coordinate plane are placed four counters, each of whose centers has integer coordinates. One can displace any counter by the vector joining the centers of two of the other counters. Prove that any two

preselected counters can be made to coincide by a finite sequence of moves.

Solution:

Lemma 1 *If three counters lie on a line and have integer coordinates, then we can make any two of them coincide.*

PROOF: Let A and B be the counters between which the smallest of the three pairwise distances occurs, and let C be the other one. By repeatedly moving C either by the vector AB or its reverse, we can put C on the segment AB, thus decreasing the minimum of the pairwise distances. Since the points have integer coordinates, repeating this process must eventually bring the minimum distance down to zero. If the desired counters coincide, we are done; otherwise, the one that coincides with the third counter can be moved to the location of the other one. □

Project the counters onto one of the axes. The projections behave like counters, in that if a counter is displaced by a vector, its projection is displaced by the projection of the vector. As described in the lemma, we can make the projections of our chosen counters coincide, using one of the remaining counters as the third counter. We can now make a third projection coincide with these by treating our chosen counters as one. (That is, each time we displace one, we displace the other by the same amount.) Now our two chosen counters and one more lie on a line perpendicular to the axis, and by the lemma we can make the desired counters coincide.

11. Find all natural numbers n, such that there exist relatively prime integers x and y and an integer $k > 1$ satisfying the equation $3^n = x^k + y^k$.

Solution: The only solution is $n = 2$.

Let $3^n = x^k + y^k$, where x, y are relatively prime integers with $x > y$, $k > 1$, and n a natural number. Clearly neither x nor y is a multiple of 3. Therefore, if k is even, x^k and y^k are congruent to 1 mod 3, so their sum is congruent to 2 mod 3, and so is not a power of 3.

If k is odd and $k > 1$, then $3^n = (x + y)(x^{k-1} - \ldots + y^{k-1})$. Thus $x + y = 3^m$ for some $m \geq 1$. We will show that $n \geq 2m$. Since $3|k$

(see the solution to Russia 3), by putting $x_1 = x^{k/3}$ and $y_1 = y^{k/3}$ we may assume $k = 3$. Then $x^3 + y^3 = 3^m$ and $x + y = 3^n$. To prove the inequality $n \geq 2m$, it suffices to show that $x^3 + y^3 \geq (x + y)^2$, or $x^2 - xy + y^2 \geq x + y$. Since $x \geq y + 1$, $x^2 - x = x(x - 1) \geq xy$, and $(x^2 - x + xy) + (y^2 - y) \geq y(y - 1) \geq 0$, and the inequality $n \geq 2m$ follows.

From the identity $(x + y)^3 - (x^3 + y^3) = 3xy(x + y)$ it follows that

$$3^{2m-1} - 3^{n-m-1} = xy.$$

But $2m - 1 \geq 1$, and $n - m - 1 \geq n - 2m \geq 0$. If strict inequality occurs in either place in the last inequality, then $3^{2m-1} - 3^{n-m-1}$ is divisible by 3 while xy is not. Hence $n - m - 1 = n - 2m = 0$, and so $m = 1, n = 2$ and $3^2 = 2^3 + 1^3$.

Note: The inequality $x^2 - xy + y^2 \geq x + y$ can alternatively be shown by noting that

$$x^2 - xy + y^2 - x - y = (x - y)^2 + (x - 1)(y - 1) - 1 \geq 0,$$

since $(x - y)^2 \geq 1$.

12. Show that if the integers $a_1, \ldots, a_m$ are nonzero and for each $k = 0, 1, \ldots, m$ $(n < m - 1)$,

$$a_1 + a_2 2^k + a_3 3^k + \ldots + a_m m^k = 0,$$

then the sequence $a_1, \ldots, a_m$ contains at least $n + 1$ pairs of consecutive terms having opposite signs.

Solution: We may assume $a_m > 0$, since otherwise we may multiply each of the numbers by -1. Consider the sequence $b_1, \ldots, b_m$, where $b_i = \sum_{j=0}^{n} c_j i^j$ for an arbitrary sequence of real numbers $c_0, \ldots, c_n$. From the given condition

$$\sum_{i=1}^{m} a_i b_i = \sum_{i=1}^{m} a_i \sum_{j=0}^{n} c_j i^j = \sum_{j=0}^{n} c_j \sum_{i=1}^{m} a_i i^j = 0.$$

Suppose now that the sequence $a_1, \ldots, a_m$ has k pairs of neighbors that differ in sign, where $k < n + 1$, and let $i_1, \ldots, i_k$ be the indices of the first members of these pairs. Let $b_i = f(i) = (i - x_1)(i - x_2)\dots(i - x_k)$, where $x_\ell = i_\ell + 1/2$ $(i = 1, 2, \dots, k)$. The function f changes sign only at the points $x_1, \dots, x_k$, and so b_i and b_{i+1} have different signs if and only one of the x_ℓ falls between them, which means $i = i_\ell$. We deduce that the sequences $a_1, \dots, a_m$ and $b_1, \dots, b_m$ have the same pairs of neighbors of opposite sign. Since a_m and b_m are positive, we have that a_i and b_i have the same sign for $i = 1, \dots, m$, and so $\sum_{i=1}^m a_i b_i > 0$, a contradiction.

13. At the vertices of a cube are written eight pairwise distinct natural numbers, and on each of its edges is written the greatest common divisor of the numbers at the endpoints of the edge. Can the sum of the numbers written at the vertices be the same as the sum of the numbers written at the edges?

    Solution: This is not possible. Note that if a and b are natural numbers with $a > b$, then $\gcd(a, b) \le b$ and $\gcd(a, b) \le a/2$. It follows that if $a \ne b$, then $\gcd(a, b) \le (a + b)/3$. Adding 12 such inequalities, corresponding to the 12 edges, we find that the desired condition is only possible if $\gcd(a, b) = (a+b)/3$ in each case. But in this case the larger of a and b is twice the smaller; suppose $a = 2b$. Consider the numbers c and d assigned to the vertices of the other endpoints of the other two edges coming out of the vertex labeled a. Each of these is either half of or twice a. If at least one is less than a, it equals b; otherwise, both are equal. Either option contradicts the assumption that the numbers are distinct.

14. Three sergeants and several solders serve in a platoon. The sergeants take turns on duty. The commander has given the following orders:

    (a) Each day, at least one task must be issued to a soldier.

    (b) No soldier may have more than two task or receive more than one tasks in a single day.

    (c) The lists of soldiers receiving tasks for two different days must not be the same.

    (d) The first sergeant violating any of these orders will be jailed.

    Can at least one of the sergeants, without conspiring with the others, give tasks according to these rules and avoid being jailed?

Solution: The sergeant who goes third can avoid going to jail. We call a sequence of duties by the first, second and third sergeants in succession a round. To avoid going to jail, the third sergeant on the last day of each round gives tasks to precisely those soldiers who received one task over the previous two days. (Such soldiers exist by the third condition.) With this strategy, at the end of each cycle each soldier will have received either two tasks or none, and the number of the latter will have decreased. It will end up, at some point, that all of the soldiers have received two tasks, and the first sergeant will go to jail.

15. A convex polygon is given, no two of whose sides are parallel. For each side we consider the angle the side subtends at the vertex farthest from the side. Show that the sum of these angles equals $180°$.

Solution: Denote by P_a the vertex of the polygon farthest from the line containing side a. Choose an arbitrary point O in the plane. We call the two vertical angles, consisting of all lines through O and parallel to the segment P_aQ for some Q on side a, the angles corresponding to side a.

We prove first that the angles corresponding to different sides do not overlap. Let a ray ℓ with vertex O lie inside one of the angles corresponding to a. The line parallel to this ray passing through P_a intersects side a at some interior point A. Draw through P_a the line b parallel to the line c containing side a. From the convexity of the polygon and the definition of P_a, it follows that the polygon lies in the strip bounded by b and c. Moreover, since the polygon has no parallel sides, P_a is the only vertex of the polygon lying on b. Therefore the segment P_aA is strictly longer than any other segment formed as the intersection of the polygon with a line parallel to ℓ. If ℓ lay inside the angle corresponding to another side b, then contrary to this conclusion, the longest such segment would be P_bB for some B, and hence this cannot occur. In other words, the angles corresponding to a and b do not overlap.

We now prove that the angles we have constructed cover the entire plane. Suppose this were not the case. Then there would exist some angle with vertex O not covered by any of the angles constructed.

Choose within this angle a ray m, not parallel to any side or diagonal of the polygon. Of all of the segments formed by intersecting the polygon with a line parallel to m, choose the one of maximum length. Clearly one of its vertices must be a vertex P of the polygon, while the other lies on some side a. Draw the line c through P parallel to the line b containing a. If one of the sides adjacent to P did not lie inside the strip bounded by b and c, then we could have found a line parallel to m intersecting the polygon in a segment longer than PA. Consequently, our polygon lies within the strip bounded by b and c, from which we deduce that P is the farthest vertex from the line b containing side a. This means m lies in the angle corresponding to a, contradicting our choice of m.

We thus conclude that our constructed angles cover the plane without overlap, and hence the sum of their measures is 360°. To finish the proof, simply note that the sum of the desired angles is half that of the constructed angles.

16. Goodnik writes 10 numbers on the board, then Nogoodnik writes 10 more numbers, all 20 of the numbers being positive and distinct. Can Goodnik choose his 10 numbers so that no matter what Nogoodnik writes, he can form 10 quadratic trinomials of the form $x^2 + px + q$, whose coefficients p and q run through all of the numbers written, such that the real roots of these trinomials comprise exactly 11 values?

Solution: We will prove that Goodnik can choose the numbers

$$1/4, 1/2, 1, 2, 5, 5^2, 5^4, 5^8, 5^{16}, 5^{32}.$$

Lemma 1 *(a) If $a > 4$ and $a > b$, then the trinomial $x^2 + ax + b$ has two distinct real roots.*

(b) If $a < 4$ and $b > 0$, then at least one of the trinomials $x^2 + ax + b, x^2 + bx + a$ does not have real roots.

Proof: The first part is obvious, since the discriminant $D = a^2 - 4b > 4a - 4b > 0$. For the second part, note that if $b \leq a$, then $b^2 - 4a < 0$, while if $b > a$, then $a^2 - 4b < 0$. $\square$

Lemma 2 *Suppose $0 < a < b < c < d$ and both of the trinomials $x^2 + dx + a$ and $x^2 + cx + b$ have two real roots. Then all four of these roots are distinct.*

PROOF: Suppose the contrary, that these trinomials have a common root x_0. Then $x_0^2 + dx_0 + a = 0 = x_0^2 + cx_0 + b$ and consequently $x_0 = (b - a)/(d - c) > 0$. But if $x_0 > 0$, then $x_0^2 + dx_0 + a > 0$, a contradiction. □

Suppose Goodnik has written the aforementioned numbers. Consider all of Nogoodnik's numbers which are greater than 4. If there are an odd number of them, add to them any of Nogoodnik's other numbers. Call these numbers *distinguished.*

Add to the distinguished numbers members of the set $\{5, 5^2, 5^4, 5^8, 5^{16}, 5^{32}\}$ so that the total number of distinguished numbers is 12; if the powers of 5 do not suffice, add any of Nogoodnik's remaining numbers to make a total of 12. From the unused powers of 5 make trinomials $x^2 + px + q$ with $p < q$, which have negative discriminant and hence no real roots.

Let $n_1, \ldots, n_{12}$ be the 12 distinguished numbers in increasing order. Now form from them the 6 trinomials $x^2+n_{12}x+n_1, \ldots, x^2+n_7x+n_6$. By the construction of the 12 distinguished numbers, at least 6 are greater than 4. Hence by Lemma 1, each of these trinomials has two distinct real roots. By Lemma 2, all of these roots are distinct. Hence we have 12 distinct real roots of the "distinguished" trinomials.

Consider the trinomial $x^2 + 2x + 1$, whose unique root is -1. If this number occurs among the roots of the distinguished trinomials, we declare the corresponding trinomial "bad". If not, declare an arbitrary distinguised trinomial to be bad. Remove the bad trinomial, and from its coefficients and the numbers 1/2 and 1/4 form (by Lemma 1) two trinomials without real roots. Now the number of distinct real roots of the trinomials constructed so far is 11.

There may be some of Nogoodnik's numbers left; all except possibly one must be less than 4 (one may equal 4). By Lemma 1, we form trinomials from these with no real roots.

17. Can the number obtained by writing the numbers from 1 to n in order ($n > 1$) be the same when read left-to-right and right-to-left?

Solution: This is not possible. Suppose $N = 123\cdots 321$ is an m-digit symmetric number, formed by writing the numbers from 1 to n in succession. Clearly $m > 18$. Also let A and B be the numbers formed from the first and last k digits, respectively, of N, where $k = \lfloor m/2 \rfloor$. Then if 10^p is the largest power of 10 dividing A, then $n < 2 \cdot 10^{p+1}$, that is, n has at most $p + 2$ digits. Moreover, A and B must contain the fragments

$$\underbrace{99\ldots9}_{p}1\underbrace{00\ldots0}_{p}1 \quad \text{and} \quad 1\underbrace{00\ldots0}_{p}1\underbrace{99\ldots9}_{p},$$

respectively, which is impossible.

18. Several hikers travel at fixed speeds along a straight road. It is known that over some period of time, the sum of their pairwise distances is monotonically decreasing. Show that there is a hiker, the sum of whose distances to the other hikers is monotonically decreasing over the same period.

Solution: Let n be the number of hikers, who we denote $P_1, \ldots, P_n$. Let V_{ij} be the rate of approach between P_i and P_j (this is negative if they are getting further apart). Note that V_{ij} never increases, and can only decrease once: it changes sign if P_i and P_j meet.

By the given condition, at the end of the period in question the sum of the pairwise speeds must be positive:

$$\sum_{1 \le i < j \le n} V_{ij} > 0.$$

Since $V_{ij} = V_{ji}$, we have (putting $V_{ii} = 0$)

$$\sum_{j=1}^{n} \sum_{i=1}^{n} V_{ij} = 2 \sum_{1 \le i < j \le n} V_{ij} > 0.$$

Hence for some j, $\sum_{i=1}^{n} V_{ij} > 0$. Since V_{ij} cannot increase over time, the sum of the distances from P_j to the other hikers is decreasing throughout the period.

19. Show that for $n \geq 5$, a cross-section of a pyramid whose base is a regular n-gon cannot be a regular $(n+1)$-gon.

Solution: Suppose the regular $(n+1)$-gon $B_1 \dots B_{n+1}$ is a cross-section of the pyramid $SA_1 \dots A_n$, whose base $A_1 \dots A_n$ is a regular n-gon. We consider three cases: $n = 5$, $n = 2k-1$ $(k > 3)$ and $n = 2k$ $(k > 2)$.

Since the pyramid has $n+1$ faces, one side of the section must lie on each face. Therefore without loss of generality, we may assume that the points $B_1, \dots, B_{n+1}$ lie on the edges of the pyramid.

(a) $n = 5$. Since in the regular hexagon $B_1 \dots B_6$ the lines B_2B_3, B_5B_6 and B_1B_4 are parallel, while the planes A_2SA_3 and A_1SA_5 pass through B_2B_3 and B_5B_6, respectively, the line ST ($T = A_1A_5 \cap A_2A_3$) along which these planes meet is parallel to these lines, i.e. $ST || B_1B_4$. Draw the plane containing ST and B_1B_4. This plane intersects the plane of the base of the pyramid in the line B_1A_4, which must pass through the intersection of the line ST with the plane of the base, that is, through T. Hence the lines A_1A_5, A_4B_1 and A_2A_3 pass through a single point. It is proved analogously that the lines A_1A_2, A_3B_6 and A_4A_5 also meet in a point. From this it follows that A_4B_1 and A_3B_6 are axes of symmetry of the regular pentagon $A_1 \dots A_5$, which means their intersection O is the center of this pentagon. Now note that if Q is the center of the regular hexagon $B_1 \dots B_6$, then the planes SA_3B_6, SA_4B_1 and SB_2B_5 intersect in the line SQ. Consequently, the lines A_3B_6, A_4B_1 and A_2A_5 must intersect in a point, namely the intersection of line SQ with the plane of the base. This means the diagonal A_2A_5 of the regular pentagon $A_1 \dots A_5$ must pass through its center O, which is impossible.

(b) $n = 2k-1$ $(k > 3)$. Analogously to the first case one shows that since in the regular $2k$-gon $B_1 \dots B_{2k}$ the lines $B_1B_2, B_{k+1}B_{k+2}$, and B_kB_{k+3} are parallel, then the lines $A_1A_2, A_{k+1}A_{k+2}$ and A_kA_{k+3} must intersect in a point, which is impossible, since in the regular $(2k-1)$-gon $A_1 \dots A_{2k-1}$, $A_{k+1}A_{k+2} || A_kA_{k+3}$, but the lines A_1A_2 and $A_{k+1}A_{k+2}$ are not parallel.

(c) $n = 2k$ $(k > 2)$. Analogously to the preceding cases, the lines

$A_1A_2, A_{k+1}A_{k+2}$ and A_kA_{k+3} are parallel, and hence the lines $B_1B_2, B_{k+1}B_{k+2}$ and B_kB_{k+3} must meet in a point, which is impossible, since $B_{k+1}B_{k+2}||B_kB_{k+3}$, while the lines B_1B_2 and $B_{k+1}B_{k+2}$ are not parallel.

Note: For $n = 3, 4$, the statement of the problem is not true. For examples, consider a regular tetrahedron having a square as a cross-section, and a square pyramid whose lateral faces are equilateral triangles, which has a regular pentagon as a cross-section.

Also, the presented solution may be more concisely expressed using central projection, and the property that under central projection, the images of lines passing through a single point (or parallel) are lines passing through a single point (or parallel). It suffices to project the cross-section of the pyramid onto the plane of the base with center the vertex of the pyramid.

20. Do there exist three natural numbers greater than 1, such that the square of each, minus one, is divisible by each of the others?

Solution: Such integers do not exist. Suppose $a \geq b \geq c$ satisfy the desired condition. Since $a^2 - 1$ is divisible by b, the numbers a and b are relatively prime. Hence the number $c^2 - 1$, which is divisible by a and b, must be a multiple of ab, so in particular $c^2 - 1 \geq ab$. But $a \geq c$ and $b \geq c$, so $ab \geq c^2$, a contradiction.

21. In isosceles triangle ABC ($AB = BC$) one draws the angle bisector CD. The perpendicular to CD through the center of the circumcircle of ABC intersects BC at E. The parallel to CD through E meets AB at F. Show that $BE = FD$.

Solution: We use directed angles modulo π. Let O be the circumcircle of ABC, and K the intersection of BO and CD. From the equality of the acute angles BOE and DCA having perpendicular sides, it follows that $\angle BOE = \angle KCE$ (CD being an angle bisector), which means the points K, O, E, C lie on a circle. From this it follows that $\angle OKE = \angle OCE$; but $\angle OCE = \angle OBE$, so $OB = OC$, and hence $\angle BKE = \angle KBE$, or in other words $BE = KE$. Moreover, $\angle BKE = \angle KBE = \angle KBA$, and so $KE||AB$. Consequently,

$FEKD$ is a parallelogram and $DF = KE$. Therefore, $DF = KE = BE$ as desired.

22. Does there exist a finite set M of nonzero real numbers, such that for any natural number n a polynomial of degree no less than n with coefficients in M, all of whose roots are real and belong to M?

Solution: Such a set does not exist. Suppose on the contrary that $M = \{a_1, a_2, \ldots, a_n\}$ satisfies the desired property. Let $m = \min\{|a_1|, \ldots, |a_n|\}$ and $M = \max\{|a_1|, \ldots, |a_n|\}$; the condition implies $M \geq m > 0$.

Consider the polynomial $P(x) = b_k x^k + \cdots + b_1 x + b_0$, all of whose coefficients $b_0, \ldots, b_k$ and roots $x_1, \ldots, x_k$ lie in M. By Vieta's theorem,

$$\begin{aligned} -\frac{b_{k-1}}{b_k} &= x_1 + \ldots + x_k \\ x_1x_2 + x_1x_3 + \ldots + x_{k-1}x_k &= \frac{b_{k-2}}{b_k} \end{aligned}$$

and so

$$x_1^2 + \ldots + x_k^2 = \frac{b_{k-1}^2}{b_k^2} - 2\frac{b_{k-2}}{b_k}.$$

It follows that

$$km^2 \leq x_1^2 + \ldots + x_k^2 = \frac{b_{k-1}^2}{b_k^2} - 2\frac{b_{k-2}}{b_k} \leq \frac{M^2}{m^2} + 2\frac{M}{m}.$$

Hence $k \leq M^2/m^4 + 2M/m^3$, contradicting the fact that P may have arbitrarily large degree.

23. The numbers from 1 to 100 are written in an unknown order. One may ask about any 50 numbers and find out their relative order. What is the fewest questions needed to find the order of all 100 numbers?

Solution: Five questions are needed. To determine the order of $a_1, \ldots, a_{100}$ in the sequence, it is necessary that each of the pairs (a_i, a_{i+1}) $(i = 1, ..., 99)$ occur together in at least one question, or else the two sequences $a_1, \ldots, a_i, a_{i+1}, \ldots, a_n$ and $a_1, \ldots, a_{i+1}, a_i, \ldots, a_n$ will give the same answers. We will show that for any two questions, there can arise a situation where including all pairs of consecutive numbers not already included requires at least three questions. Let $k_1, \ldots, k_{50}$ be the sequence (in order) of numbers about which the first question was asked, and $k'_1, \ldots, k'_{50}$ the corresponding sequence for the second question. We will construct a sequence $a_1, \ldots, a_{100}$ for which we cannot, given two more questions, uniquely determine the order of the terms. We consider a situation where all of the numbers named in the first two questions appear in the answers in the very same places.

For our desired sequence we shall choose a set with

$$k_i, k'_i \in \{a_{2i-1}, a_{2i}\}, i = 1, \ldots, 50$$

and moreover, for each quadruple $(a_{4m-3}, a_{4m-2}, a_{4m-1}, a_{4m})$ $(m = 1, \ldots, 25)$, in the first two questions there is no comparison of a consecutive pair from this quadruple. We will show that such a set exists. Let X be the set of numbers not named in the first two questions. For each of the four cases

$$\begin{aligned}
1: &\quad k_{2m-1} = k'_{2m-1},\ k_{2m} = k'_{2m} \\
2: &\quad k_{2m-1} = k'_{2m-1},\ k_{2m} \neq k'_{2m} \\
3: &\quad k_{2m-1} \neq k'_{2m-1},\ k_{2m} \neq k'_{2m} \\
4: &\quad k_{2m-1} \neq k'_{2m-1},\ k_{2m} = k'_{2m},
\end{aligned}$$

we construct the quadruple $(a_{4m-3}, a_{4m-2}, a_{4m-1}, a_{4m})$ in the following manner:

$$\begin{aligned}
1: &\quad (k_{2m-1}, *, *, k_{2m}), \\
2: &\quad (k_{2m-1}, *, k_{2m}, k'_{2m}) \\
3: &\quad (k_{2m-1}, k'_{2m-1}, k_{2m}, k'_{2m}) \\
4: &\quad (k_{2m-1}, k'_{2m-1}, *, k_{2m}),
\end{aligned}$$

where in place of a $*$ we may choose any number in X not occuring in the previously constructed quadruples.

Hence we have shown that after any two questions, a situation is possible where no pair (a_i, a_{i+1}) occurs for i not a multiple of 4.

Each of the 100 numbers occurs in at least one of the nonincluded pairs, and so must appear in one of the remaining questions.

Suppose that in the given situation, all remaining pairs can be included in two questions; then each of the 100 numbers must appear in exactly one of these questions. Considering the quadruples of the form $(a_{4i-3}, a_{4i-2}, a_{4i-1}, a_{4i})$ $(i = 1, \dots, 25)$, we notice that if one of the numbers in the quadruple appears in some question, then the remaining three numbers must also appear in the question (or else not all of the pairs of consecutive numbers in the quadruple would be included). But then the number of numbers in one question would have to be a multiple of 4, which 50 is not, giving a contradiction.

Hence 4 questions do not suffice in general. We now show that 5 questions suffice. We ask the first question about $M_1 = \{1, \dots, 50\}$, and the second about $M_2 = \{51, \dots, 100\}$. The set M_3 consists of the 25 leftmost numbers from each of M_1 and M_2, while M_4 consists of the 25 rightmost numbers from each of M_1 and M_2. Clearly the answer to the third question locates the first 25 numbers, and the answer to the fourth question locates the last 25. The fifth question, asked about the other 50 numbers, completely determines the order.

1.15 Spain

1. The natural numbers a and b are such that

$$\frac{a+1}{b} + \frac{b+1}{a}$$

is an integer. Show that the greatest common divisor of a and b is not greater than $\sqrt{a+b}$.

Solution: Let $d = \gcd(a, b)$ and put $a = md$ and $b = nd$. Then we have $(md+1)/nd + (nd+1)/md = (m^2d + m + n^2d + n)/mnd$ is an integer, so that in particular, d divides $m^2d + m + n^2d + n$ and also $m + n$. However, this means $d \le m + n$, and so $d \le \sqrt{d(m+n)} = \sqrt{a+b}$.

2. Let G be the centroid of the triangle ABC. Prove that if $AB + GC = AC + GB$, then ABC is isosceles.

Solution: Let a, b, c be the lengths of sides BC, CA, AB, respectively. By Stewart's theorem and the fact that G trisects each median (on the side further from the vertex), we deduce

$$9GB^2 = 2a^2 + 2c^2 - b^2, \qquad 9GC^2 = 2a^2 + 2b^2 - c^2.$$

Now assume $b > c$. Assuming $AB + GC = AC + GB$, we have

$$\begin{aligned} 3(b-c) &= \sqrt{2a^2 + 2b^2 - c^2} - \sqrt{2a^2 + 2c^2 - b^2} \\ &= \frac{3(b^2 - c^2)}{\sqrt{2a^2 + 2b^2 - c^2} + \sqrt{2a^2 + 2c^2 - b^2}} \\ &< \frac{3(b^2 - c^2)}{\sqrt{2(b-c)^2 + 2b^2 - c^2} + \sqrt{2(b-c)^2 + 2c^2 - b^2}} \end{aligned}$$

since $a^2 > (b-c)^2$ by the triangle inequality. However, $2(b-c)^2 + 2b^2 - c^2 = (2b-c)^2$, so we have

$$3(b-c) < \frac{3(b^2 - c^2)}{2b - c + |2c - b|}.$$

If $b \le 2c$ then the two sides are equal, a contradiction. If $b > 2c$ we get $9(b-c)^2 < 3(b^2 - c^2)$; upon dividing off $3(b-c)$ and rearranging, we get $2b < 4c$, again a contradiction. Thus we cannot have $b > c$ or similarly $b < c$, so $b = c$.

3. Let a, b, c be real numbers. Consider the functions

$$f(x) = ax^2 + bx + c, \qquad g(x) = cx^2 + bx + a.$$

Given that

$$|f(-1)| \leq 1, \quad |f(0)| \leq 1, \quad |f(1)| \leq 1,$$

show that for $-1 \leq x \leq 1$,

$$|f(x)| \leq \frac{5}{4} \quad \text{and} \quad |g(x)| \leq 2.$$

Solution: We may assume $a > 0$, so that f is convex; then $f(-1), f(1) \leq 1$ implies $f(x) \leq 1$ for $-1 \leq x \leq 1$, so it suffices to look at the point t where f takes its minimum. If t is not in the interval, we have $f(x) \geq -1$, so assume it is; without loss of generality, we may assume $t \geq 0$.

We now consider two cases. First suppose $t \leq 1/2$. In this case $f(-1) \geq f(1) \geq f(0)$, so it suffices to impose the conditions $f(-1) \leq 1, f(0) \geq -1$. If we write $f(x) = a(x-t)^2 + k$, we have $2 \geq f(-1) - f(0) = a(2t+1)$, so $a \leq 2/(2t+1)$. Then $f(0) \geq -1$ means $at^2 + k \geq -1$, so

$$k \geq -1 - at^2 \geq -1 - \frac{2t^2}{2t+1} = -1 - \frac{2t}{2+1/t},$$

which is decreasing in t (the numerator of the fraction is increasing, the denominator is decreasing and there is a minus sign in front). Thus $k \geq 5/4$.

Now suppose $t \geq 1/2$. In this case $f(-1) \geq f(0) \geq f(1)$, so the relevant conditions are $f(-1) \leq 1, f(1) \geq -1$. If we write $f(x) = a(x-t)^2 + k$, we have $2 \geq f(-1) - f(1) = 2at$, so $a \leq 1/t$. Then $f(1) \geq -1$ means $a(1-t)^2 + k \geq -1$, so

$$k \geq -1 - a(1-t)^2 \geq -1 - \frac{(1-t)^2}{t} = -1 - \frac{(1-t)}{t/(1-t)}$$

which is increasing in t (similar reasoning). Thus $k \geq 5/4$.

We move on to g. We assume $c > 0$ and that the minimum of g occurs in $[0, 1]$. Assuming $g(-1), g(1) \leq 1$, we again need only determine the minimum of g. Writing $g(x) = c(x-t)^2 + k$, we have $c \leq 1$ and $c(1-t)^2 + k \geq -1$, so

$$k \geq -1 - c(1-t)^2 \geq -1 - (1-t)^2 \geq -2.$$

4. Find all real solutions of the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$$

for each real value of p.

Solution: Squaring both sides, we get

$$x^2 = 5x^2 - 4 - p + 4\sqrt{(x^2-p)(x^2-1)}.$$

Isolating the radical and squaring again, we get

$$16(x^2-p)(x^2-1) = (4x^2 - p - 4)^2,$$

which reduces to $(16 - 8p)x^2 = p^2 - 8p + 16$. Since $x \geq 0$ (it is the sum of two square roots), we have

$$x = \frac{|p-4|}{\sqrt{16-8p}}$$

if a solution exists. We need only determine when this value actually satisfies. Certainly we need $p \leq 2$. In that case plugging in our claimed value of x and multiplying through by $\sqrt{16-8p}$ gives

$$|3p - 4| + 2|p| = 4 - p.$$

If $p \geq 4/3$ this becomes $6p = 8$, or $p = 4/3$; if $0 \leq p \leq 4/3$ this holds identically; if $p \leq 0$ this becomes $4p = 0$, or $p = 0$. We conclude there exists a solution if and only if $0 \leq p \leq 4/3$, in which case it is the solution given above.

5. At Port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered.

Solution: We say two agents are partners if neither watches the other. First note that each agent watches at least 7 others; if an agent were watching 6 or fewer others, we could take away 6 agents and leave a group of 10 which could not be arranged ina circle. Similarly, each agent is watched by at least 7 others. Hence each agent is allied with at most one other.

Given a group of 11 agents, there must be one agent x who is not allied with any of the others in the group (since allies come in pairs). Remove that agen t and arrange the other 10 in a circle. The removed agent watches at least one of the other 10 and is watched by at least one. Thus there exists a pair u, v of agents with u watching v, u watching x and x watching v (move around the circle until the direction of the arrow to x changes); thus x can be spliced into the loop between u and v.

6. A regular pentagon is constructed externally on each side of a regular pentagon of side 1. This figure is then folded and the two edges meeting at each vertex of the original pentagon but not belonging to the original pentagon are glued together. Determine the volume of water that can be poured into the resulting container without spillage.

Solution: The figure formed by the water is a prismatoid of height equal to the vertical component of one of the glued edges. To determine this component, introducte a coordinate system centered at one of the base vertices, such that $(\cos 36, \sin 36, 0)$ and $(-\cos 36, \sin 36, 0)$ are two vertices. (All angles are measured in degrees.) The third vertex adjacent to this one has coordinates $(0, y, z)$ for some y, z with $z > 0$, $y^2 + z^2 = 1$ and $y \cos 36 = \cos 108$ (this being the dot product of the vectors of the two edges). Therefore

$$y = \frac{\cos 108}{\cos 36} = \frac{(1 - \sqrt{5})/4}{(1 + \sqrt{5})/4}$$

and $z = 2 \cdot 5^{1/4}/(1 + \sqrt{5})$.

Now we must determine the areas of the bases of the prismatoid. The area of the lower base is the area of a regular pentagon of side 1, which is $5/4 \cot 36$. The area of the upper base is the area of a regular pentagon in which the circumradius has been increased by y, namely $5/4 \cot 36(1 + y \sin 36)^2$. The volume is the height times the average of the bases, namely

$$\frac{5^{5/4}}{2(1+\sqrt{5})} \cot 36(1 + \cos 108 \tan 36)^2 \approx 0.956207.$$

1.16 Turkey

1. Let

$$\prod_{n=1}^{1996}\left(1+nx^{3^n}\right) = 1 + a_1x^{k_1} + a_2x^{k_2} + \ldots + a_mx^{k_m},$$

where $a_1, a_2, \ldots, a_m$ are nonzero and $k_1 < k_2 < \ldots < k_m$. Find a_{1996}.

Solution: Note that k_i is the number obtained by writing i in base 2 and reading the result as a number in base 3, and a_i is the sum of the exponents of the powers of 3 used. In particular, $1996 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^3 + 2^2$, so

$$a_{1996} = 10 + 9 + 8 + 7 + 6 + 3 + 2 = 45.$$

2. In a parallelogram $ABCD$ with $\angle A < 90°$, the circle with diameter AC meets the lines CB and CD again at E and F, respectively, and the tangent to this circle at A meets BD at P. Show that P, F, E are collinear.

Solution: Without loss of generality, suppose B, D, P occur in that order along BD. Let G and H be the second intersections of AD and AB with the circle. By Menelaos's theorem, it suffices to show that

$$\frac{CE \cdot BP \cdot DF}{EB \cdot PD \cdot FC} = 1.$$

First note that

$$\frac{BP}{AB}\frac{AD}{DP} = \frac{\sin\angle BAP}{\sin\angle APB}\frac{\sin\angle APD}{\sin\angle DAP} = \frac{\sin\angle BAP}{\sin\angle DAP}.$$

Since AP is tangent to the circle, $\angle BAP = \angle HAP = \pi - \angle HCA = \pi - \angle FAC$; similarly, $\angle DAP = \angle GCA = \angle EAC$. We conclude

$$\frac{BP}{AB}\frac{AD}{DP} = \frac{\sin\angle FAC}{\sin\angle EAC} = \frac{FC}{EC}.$$

Finally we note that $DF/BE = DA/AB$ because the right triangles AFD and AEB have the same angles at B and D and are thus similar. This proves the claim.

3. Given real numbers $0 = x_1 < x_2 < \ldots < x_{2n} < x_{2n+1} = 1$ with $x_{i+1} - x_i \leq h$ for $1 \leq i \leq 2n$, show that

$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) < \frac{1+h}{2}.$$

Solution: The difference between the middle quantity and $1/2$ is the difference between the sum of the areas of the rectangles bounded by the lines $x = x_{2i-1}$, $x = x_{2i+1}$, $y = 0$, $y = x_{2i}$ and the triangle bounded by the lines $y = 0, x = 1, x = y$. The area contained in the rectangles but not the triangle is a union of triangles of total base less than 1 and height at most h, as is the area contained in the triangle but not the rectangles. Hence the sum differs from $1/2$ by at most $h/2$, as desired.

4. In a convex quadrilateral $ABCD$, triangles ABC and ADC have the same area. Let E be the the intersection of AC and BD, and let the parallels through E to the lines AD, DC, CB, BA meet AB, BC, CD, DA at K, L, M, N, respectively. Compute the ratio of the areas of the quadrilaterals $KLMN$ and $ABCD$.

Solution: The triangles EKL and DAC are homothetic, so the ratio of their areas equals $(EK/AD)(EL/CD) = (BE/BD)^2 = 1/4$, since B and D are equidistant from the line AC. Similarly the ratio of the areas of EMN and BCA is $1/4$, so the union of the triangles EKL and EMN has area $1/4$ that of $ABCD$.

As for triangle EKN, its base KN is parallel to BD and half as long, so its area is one-fourth that of ABD. Similarly EML has area one-fourth that of BCD, and so the union of the two triangles EKN and EML has area one-fourth that of $ABCD$, and so the quadrilateral $KLMN$ has area one-half that of $ABCD$.

5. Find the maximum number of pairwise disjoint sets of the form $S_{a,b} = \{n^2 + an + b : n \in \mathbb{Z}\}$ with $a, b \in \mathbb{Z}$.

Solution: Only two such sets are possible, for example, with $(a, b) = (0, 0)$ and $(0, 2)$ (since 2 is not a difference of squares). There is no loss of generality in assuming $a \in \{0, 1\}$ by a suitable shift of

n, and the sets generated by $(0, a)$ and $(1, b)$ have the common value $(a-b)^2 + a = (a-b)^2 + (a-b) + b$. Thus we have $a = 0$ or $a = 1$ universally.

First suppose $a = 0$. If $b - c \not\equiv 2 \pmod 4$, then $(0, b)$ and $(0, c)$ give a common value because $b - c$ is a difference of squares; clearly this precludes having three disjoint sets. Now suppose $a = 1$. If $b - c$ is even, we can find x, y such that $b - c = (x + y + 1)(x - y)$, and so $x^2 + x + b = y^2 + y + c$; again, this precludes having three disjoint sets.

6. For which ordered pairs of positive real numbers (a, b) is the limit of every sequence $\{x_n\}$ satisfying the condition

$$\lim_{n \to \infty} (ax_{n+1} - bx_n) = 0$$

zero?

Solution: This holds if and only if $b < a$. If $b > a$, the sequence $x_n = (b/a)^n$ satisfies the condition but does not go to zero; if $b = a$, the sequence $x_n = 1 + 1/2 + \cdots + 1/n$ does likewise. Now suppose $b < a$. If L and M are the limit inferior and limit superior of the given sequence, the condition implies $M \leq (b/a)L$; since $L \leq M$, we have $M \leq (b/a)M$, and so $L, M \geq 0$. Similarly, the condition implies $L \geq (b/a)M$, and since $M \geq L$, we have $L \geq (b/a)L$, so $L, M \leq 0$; therefore $L = M = 0$ and the sequence converges to 0.

1.17 United Kingdom

1. Consider the pair of four-digit positive integers

$$(M, N) = (3600, 2500).$$

Notice that M and N are both perfect squares, with equal digits in two places, and differing digits in the remaining two places. Moreover, when the digits differ, the digit in M is exactly one greater than the corresponding digit in N. Find all pairs of four-digit positive integers (M, N) with these properties.

Solution: If $M = m^2$ and $N = n^2$, then

$$(m+n)(m-n) \in \{11, 101, 110, 1001, 1010, 1100\}.$$

Since M and N are four-digit numbers, we must have $32 \le n < m \le 99$, and so $65 \le m + n \le 197$. Moreover, $m + n$ and $m - n$ are both odd or both even, so 11, 110 and 1010 lead to no solutions. From this we get exactly five acceptable factorizations:

$$\begin{aligned}
101 &= (m+n)(m-n) = 101 \times 1 \\
1001 &= (m+n)(m-n) = 143 \times 7 \\
1001 &= (m+n)(m-n) = 91 \times 11 \\
1001 &= (m+n)(m-n) = 77 \times 13 \\
1100 &= (m+n)(m-n) = 110 \times 10
\end{aligned}$$

giving the solutions

$$(M, N) = (2601, 2500), (5625, 4624), (2601, 1600), (2025, 1024), (3600, 2500).$$

2. A function f defined on the positive integers satisfies $f(1) = 1996$ and

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n) \quad (n > 1).$$

Calculate $f(1996)$.

Solution: An easy induction will show that

$$f(n) = \frac{2 \times 1996}{n(n+1)}.$$

Namely,

$$\begin{aligned}
f(n) &= \frac{1}{n^2-1}\left(\frac{3992}{1\cdot 2}+\cdots+\frac{3992}{(n-1)n}\right) \\
&= \frac{3992}{n^2-1}\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n-1}-\frac{1}{n}\right) \\
&= \frac{3992}{(n+1)(n-1)}\left(1-\frac{1}{n}\right) \\
&= \frac{3992}{(n+1)(n-1)}\frac{n-1}{n} = \frac{3992}{n(n+1)}.
\end{aligned}$$

In particular, $f(1996) = 2/1997$.

3. Let ABC be an acute triangle and O its circumcenter. Let S denote the circle through A, B, O. The lines CA and CB meet S again at P and Q, respectively. Prove that the lines CO and PQ are perpendicular.

Solution: The angles $\angle PAB$ and $\angle BQP$ are supplementary, so $\angle BQP = \angle CAB$ (as directed angles mod π). In other words, the line PQ makes the same angle with the line CQ as the tangent to the circumcircle of ABC through C. Hence PQ is parallel to the tangent, so perpendicular to OC.

4. Define

$$q(n) = \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rfloor \qquad (n = 1, 2, \ldots).$$

Determine all positive integers n for which $q(n) > q(n+1)$.

Solution: We have $q(n) > q(n+1)$ if and only if $n+1$ is a perfect square. Indeed, if $n+1 = m^2$, then

$$q(n) = \left\lfloor \frac{m^2-1}{m-1} \right\rfloor = m+1, \qquad q(n+1) = \left\lfloor \frac{m^2}{m} \right\rfloor = m.$$

On the other hand, for $n = m^2 + d$ with $0 \le d \le 2m$,

$$q(n) = \left\lfloor \frac{m^2+d}{m} \right\rfloor = m + \left\lfloor \frac{d}{m} \right\rfloor$$

which is nondecreasing.

5. Let a, b, c be positive real numbers.

   (a) Prove that $4(a^3 + b^3) \geq (a + b)^3$.

   (b) Prove that $9(a^3 + b^3 + c^3) \geq (a + b + c)^3$.

Solution: Both parts follow from the Power Mean inequality: for $r > 1$ and $x_1 \ldots, x_n$ positive,

$$\left(\frac{x_1^r + \cdots + x_n^r}{n}\right)^{1/r} \geq \frac{x_1 + \cdots + x_n}{n},$$

which in turn follows from Jensen's inequality applied to the convex function x^r.

6. Find all solutions in nonnegative integers x, y, z of the equation

$$2^x + 3^y = z^2.$$

Solution: If $y = 0$, then $2^x = z^2 - 1 = (z+1)(z-1)$, so $z+1$ and $z-1$ are powers of 2. The only powers of 2 which differ by 2 are 4 and 2, so $(x, y, z) = (3, 0, 3)$.

If $y > 0$, then 2^x is a quadratic residue modulo 3, hence x is even. Now we have $3^y = z^2 - 2^x = (z + 2^{x/2})(z - 2^{x/2})$. The factors are powers of 3, say $z + 2^{x/2} = 3^m$ and $z - 2^{x/2} = 3^n$, but then $3^m - 3^n = 2^{x/2+1}$. Since the right side is not divisible by 3, we must have $n = 0$ and

$$3^m - 1 = 2^{x/2+1}.$$

If $x = 0$, we have $m = 1$, yielding $(x, y, z) = (0, 1, 2)$. Otherwise, $3^m - 1$ is divisible by 4, so m is even and $2^{x/2+1} = (3^{m/2}+1)(3^{m/2}-1)$. The two factors on the right are powers of 2 differing by 2, so they are 2 and 4, giving $x = 4$ and $(x, y, z) = (4, 2, 5)$.

7. The sides a, b, c and u, v, w of two triangles ABC and UVW are related by the equations

$$\begin{aligned} u(v + w - u) &= a^2, \\ v(w + u - v) &= b^2, \\ w(u + v - w) &= c^2. \end{aligned}$$

Prove that ABC is acute, and express the angles U, V, W in terms of A, B, C.

Solution: Note that

$$a^2 + b^2 - c^2 = w^2 - u^2 - v^2 + 2uv = (w + u - v)(w - u + v) > 0$$

by the triangle inequality, so $\cos C > 0$. By this reasoning, all of the angles of triangle ABC are acute. Moreover,

$$\begin{aligned} \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \sqrt{\frac{(w + u - v)(w - u + v)}{4uv}} \\ &= \sqrt{\frac{w^2 - u^2 - v^2 + 2uv}{4uv}} \\ &= \frac{1}{\sqrt{2}}\sqrt{1 - \cos U} \end{aligned}$$

from which we deduce $\cos U = 1 - 2\cos^2 A = \cos(\pi - 2A)$. Therefore $U = \pi - 2A$, and similarly $V = \pi - 2B$, $W = \pi - 2C$.

8. Two circles S_1 and S_2 touch each other externally at K; they also touch a circle S internally at A_1 and A_2, respectively. Let P be one point of intersection of S with the common tangent to S_1 and S_2 at K. The line PA_1 meets S_1 again at B_1, and PA_2 meets S_2 again at B_2. Prove that B_1B_2 is a common tangent to S_1 and S_2.

Solution: It suffices to show that $\angle B_2B_1O_1 = \angle B_1B_2O_2 = \pi/2$, where O_1 and O_2 are the centers of S_1 and S_2, respectively. By power-of-a-point, $PA_1 \cdot PB_1 = PK^2 = PA_2 \cdot PB_2$, so triangles PA_1A_2 and PB_2B_1 are similar. Therefore $\angle PB_1B_2 = \angle PA_2A_1 = \frac{1}{2}\angle POA_1$, where O is the center of S.

Now note that the homothety at A_1 carrying S_1 to S takes O_1 to O and B_1 to P, so $\angle POA_1 = \angle B_1O_1A_1$. From this we deduce $\angle PB_1B_2 = \angle B_1O_1N$, where N is the midpoint of A_1B_1. Finally,

$$\angle B_2B_1O_1 = \pi - \angle PB_1B_2 - \angle O_1B_1N = \pi/2,$$

as desired.

9. Find all solutions in positive real numbers a, b, c, d to the following system of equations:

$$\begin{aligned} a+b+c+d &= 12 \\ abcd &= 27+ab+ac+ad+bc+bd+cd. \end{aligned}$$

Solution: The first equation implies $abcd \leq 81$ by the arithmetic-geometric mean inequality, with equality holding for $a = b = c = d = 3$. Again by AM-GM,

$$abcd \geq 27 + 6(abcd)^{1/2}.$$

However, $x^2 - 6x - 27 \geq 0$ for $x \leq -3$ or $x \geq 9$, so $(abcd)^{1/2} \geq 9$, hence $abcd \geq 81$. We conclude $abcd = 81$, and hence $a = b = c = d = 3$.

1.18 United States of America

1. Prove that the average of the numbers $n \sin n°$ $(n = 2, 4, 6, \ldots, 180)$ is $\cot 1°$.

   Solution: All arguments of trigonometric functions will be in degrees. We need to prove

   $$2\sin 2 + 4\sin 4 + \cdots + 178\sin 178 = 90\cot 1, \tag{2}$$

   which is equivalent to

   $$2\sin 2\cdot\sin 1+2(2\sin 4\cdot\sin 1)+\cdots+89(2\sin 178\cdot\sin 1) = 90\cos 1. \tag{3}$$

   Using the identity $2\sin a \cdot \sin b = \cos(a-b) - \cos(a+b)$, we find

   $$\begin{aligned}
   &2\sin 2 \cdot \sin 1 + 2(2\sin 4 \cdot \sin 1) + \cdots + 89(2\sin 178 \cdot \sin 1)\\
   &\quad= (\cos 1 - \cos 3) + 2(\cos 3 - \cos 5) + \cdots + 89(\cos 177 - \cos 179)\\
   &\quad= \cos 1 + \cos 3 + \cos 5 + \cdots + \cos 175 + \cos 177 - 89\cos 179\\
   &\quad= \cos 1 + (\cos 3 + \cos 177) + \cdots + (\cos 89 + \cos 91) - 89\cos 179\\
   &\quad= \cos 1 + 89\cos 1 = 90\cos 1,
   \end{aligned}$$

   so (1) is true.

   Note: An alternate solution involves complex numbers. One expresses $\sin n$ as $(e^{\pi i n/180} - e^{-\pi i n/180})/(2i)$ and uses the fact that

   $$\begin{aligned}
   x + 2x^2 + \cdots + nx^n &= (x + \cdots + x^n) + (x^2 + \cdots + x^n) + \cdots + x^n\\
   &= \frac{1}{x-1}[(x^{n+1} - x) + (x^{n+1} - x^2) + \cdots\\
   &\qquad + (x^{n+1} - x^n)]\\
   &= \frac{nx^{n+1}}{x-1} - \frac{x^{n+1} - x}{(x-1)^2}.
   \end{aligned}$$

2. For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S. Given a set A of n positive integers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned

into n classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

Solution: Let $A = \{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$. For $i = 1, 2, \ldots, n$ let $s_i = a_1 + a_2 + \cdots + a_i$ and take $s_0 = 0$. All the sums in question are less than or equal to s_n, and if σ is one of them, we have

$$s_{i-1} < \sigma \leq s_i \tag{1}$$

for an appropriate i. Divide the sums into n classes by letting C_i denote the class of sums satisfying (1). We claim that these classes have the desired property. To establish this, it suffices to show that (1) implies

$$\frac{1}{2}\, s_i < \sigma \leq s_i. \tag{2}$$

Suppose (1) holds. The inequality $a_1 + a_2 + \cdots + a_{i-1} = s_{i-1} < \sigma$ shows that the sum σ contains at least one addend a_k with $k \geq i$. Then since then $a_k \geq a_i$, we have

$$s_i - \sigma < s_i - s_{i-1} = a_i \leq a_k \leq \sigma,$$

which together with $\sigma \leq s_i$ implies (2).

Note: The result does not hold if 2 is replaced by any smaller constant c. To see this, choose n such that $c < 2 - 2^{-(n-1)}$ and consider the set $\{1, \ldots, 2^{n-1}\}$. If this set is divided into n subsets, two of $1, \ldots, 2^{n-1}, 1 + \cdots + 2^{n-1}$ must lie in the same subset, and their ratio is at least $(1 + \cdots + 2^{n-1})/(2^{n-1}) = 2 - 2^{-(n-1)} > c$.

3. Let ABC be a triangle. Prove that there is a line ℓ (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in ℓ has area more than 2/3 the area of triangle ABC.

First Solution: In all of the solutions, a, b, c denote the lengths of the sides BC, CA, AB, respectively, and we assume without loss of generality that $a \leq b \leq c$.

Choose ℓ to be the angle bisector of $\angle A$. Let P be the intersection of ℓ with BC. Since $AC \leq AB$, the intersection of triangles ABC and

$A'B'C'$ is the disjoint union of two congruent triangles, APC and APC'. Considering BC as a base, triangles APC and ABC have equal altitudes, so their areas are in the same ratio as their bases:

$$\frac{\text{Area}(APC)}{\text{Area}(ABC)} = \frac{PC}{BC}.$$

Since AP is the angle bisector of $\angle A$, we have $BP/PC = c/b$, so

$$\frac{PC}{BC} = \frac{PC}{BP + PC} = \frac{1}{c/b + 1}.$$

Thus it suffices to prove

$$\frac{2}{c/b + 1} > \frac{2}{3}. \tag{1}$$

But $2b \geq a + b > c$ by the triangle inequality, so $c/b < 2$ and thus (1) holds.

Second Solution: Let the foot of the altitude from C meet AB at D. We will use the notation $[XYZ]$ to denote the area of triangle XYZ.

First suppose $[BDC] > (1/3)[ABC]$. In this case we reflect through CD. If B' is the image of B, then $BB'C$ lies in ABC and the area of the overlap is at least $2/3[ABC]$.

Now suppose $[BDC] \leq (1/3)[ABC]$. In this case we reflect through the bisector of $\angle A$. If C' is the image of C, then triangle ACC' is contained in the overlap, and $[ACC'] > [ADC] \geq 2/3[ABC]$.

Note: Let $\mathcal{F}$ denote the figure given by the intersection of the interior of triangle ABC and the interior of its reflection in ℓ. Yet another approach to the problem involves finding the maximum attained for $\text{Area}(\mathcal{F})/\text{Area}(ABC)$ by taking ℓ from the family of lines perpendicular to AB. By choosing the best alternative between the angle bisector at C and the optimal line perpendicular to AB, one can ensure

$$\frac{\text{Area}(\mathcal{F})}{\text{Area}(ABC)} > \frac{2}{1+\sqrt{2}} = 2(\sqrt{2}-1) = 0.828427,\ldots$$

and this constant is in fact the best possible.

4. An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a *binary sequence of length* n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

First Solution: We refer to the binary sequences counted by (a_n) and (b_n) as "type A" and "type B", respectively. For each binary sequence $(x_1, x_2, \ldots, x_n)$ there is a corresponding binary sequence $(y_0, y_1, \ldots, y_n)$ obtained by setting $y_0 = 0$ and

$$y_i = x_1 + x_2 + \cdots + x_i \mod 2, \quad i = 1, 2, \ldots, n. \tag{2}$$

(Addition mod 2 is defined as follows: $0 + 0 = 1 + 1 = 0$ and $0 + 1 = 1 + 0 = 1$.) Then

$$x_i = y_i + y_{i-1} \mod 2, \quad i = 1, 2, \ldots, n,$$

and it is easily seen that (1) provides a one-to-one correspondence between the set of all binary sequences of length n and the set of binary sequences of length $n+1$ in which the first term is 0. Moreover, the binary sequence $(x_1, x_2, \ldots, x_n)$ has three consecutive terms equal to 0, 1, 0 in that order if and only if the corresponding sequence $(y_0, y_1, \ldots, y_n)$ has four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order, so the first is of type A if and only if the second is of type B. The set of type B sequences of length $n+1$ in which the first term is 0 is exactly half the total number of such sequences, as can be seen by means of the mapping in which 0's and 1's are interchanged.

Second Solution: The expression $2a_{n-1} - a_n$ counts the number of type A sequences of length $n-1$ that do not remain of type A when 0 is attached, or in other words the number of type A sequences of length $n-1$ ending in 0, 1. Since a type A sequence followed by 1 is still of type A, this number in turn equals the number of type A sequences of length $n-2$ ending in 0. However, this number can be viewed as the number of type A sequences of length $n-2$, minus the number of such sequences ending in 1. There are as many type A sequences of length $n-2$ ending in 1 as there are type A sequences of length $n-3$, again since every type A sequence remains type A when 1 is added. We conclude that $2a_{n-1} - a_n = a_{n-2} - a_{n-3}$, or equivalently

$$a_n = 2a_{n-1} - a_{n-2} + a_{n-3}.$$

We also note the initial values $a_0 = 1, a_1 = 2, a_2 = 4$.

Similarly, the expression $2b_{n-1} - b_n$ counts the number of type B sequences of length $n-1$ ending in 0,0,1 or 1,1,0. Since adding 1 to a type B sequence ending in 0, or 0 to a type B sequence ending in 1, yields another type B sequence, this number is the same as the number of type B sequences of length $n-2$ ending in 0,0 or 1,1, or equivalently the number not ending in 0,1 or 1,0. As in the type A case, we note that each type B sequence of length $n-3$ generates a unique type B sequence of length $n-2$ ending in 0, 1 or 1, 0, and so $2b_{n-1} - b_n = b_{n-2} - b_{n-3}$. Since $b_1 = 2, b_2 = 4, b_4 = 8$, we conclude by induction that $b_{n+1} = 2a_n$ for all n.

5. Triangle ABC has the following property: there is an interior point P such that $\angle PAB = 10°$, $\angle PBA = 20°$, $\angle PCA = 30°$, and $\angle PAC = 40°$. Prove that triangle ABC is isosceles.

First Solution: All angles will be in degrees. Let $x = \angle PCB$. Then $\angle PBC = 80 - x$. By the Law of Sines,

$$\begin{aligned} 1 &= \frac{PA}{PB}\frac{PB}{PC}\frac{PC}{PA} \\ &= \frac{\sin\angle PBA}{\sin\angle PAB}\frac{\sin\angle PCB}{\sin\angle PBC}\frac{\sin\angle PAC}{\sin\angle PCA} \\ &= \frac{\sin 20 \sin x \sin 40}{\sin 10 \sin(80-x)\sin 30} = \frac{4\sin x \sin 40 \cos 10}{\sin(80-x)}. \end{aligned}$$

The identity $2\sin a \cdot \cos b = \sin(a-b) + \sin(a+b)$ now yields

$$1 = \frac{2\sin x(\sin 30 + \sin 50)}{\sin(80-x)} = \frac{\sin x(1 + 2\cos 40)}{\sin(80-x)},$$

so

$$2\sin x \cos 40 = \sin(80-x) - \sin x = 2\sin(40-x)\cos 40.$$

This gives $x = 40 - x$ and thus $x = 20$. It follows that $\angle ACB = 50 = \angle BAC$, so triangle ABC is isosceles.

Second Solution: Let D be the reflection of A across the line BP. Then triangle APD is isosceles with vertex angle

$$\angle APD = 2(180 - \angle BPA) = 2(\angle PAB + \angle ABP) = 2(10 + 20) = 60,$$

and so is equilateral. Also, $\angle DBA = 2\angle PBA = 40$. Since $\angle BAC = 50$, we have $DB \perp AC$.

Let E be the intersection of DB with CP. Then

$$\angle PED = 180 - \angle CED = 180 - (90 - \angle ACE) = 90 + 30 = 120$$

and so $\angle PED + \angle DAP = 180$. We deduce that the quadrilateral $APED$ is cyclic, and therefore $\angle DEA = \angle DPA = 60$.

Finally, we note that $\angle DEA = 60 = \angle DEC$. Since $AC \perp DE$, we deduce that A and C are symmetric across the line DE, which implies that $BA = BC$, as desired.

6. Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

First Solution: Yes, there is such a subset. If the problem is restricted to the nonnegative integers, it is clear that the set of integers whose representations in base 4 contains only the digits 0 and 1 satisfies the desired property. To accommodate the negative integers as well, we switch to "base -4". That is, we represent every integer in the form $\sum_{i=0}^{k} c_i(-4)^i$, with $c_i \in \{0, 1, 2, 3\}$ for all i and $c_k \neq 0$, and let X be the set of numbers whose representations use only the digits 0 and 1. This X will again have the desired property, once we show that every integer has a unique representation in this fashion.

To show base -4 representations are unique, let $\{c_i\}$ and $\{d_i\}$ be two distinct finite sequences of elements of $\{0, 1, 2, 3\}$, and let j be the smallest integer such that $c_j \neq d_j$. Then

$$\sum_{i=0}^{k} c_i(-4)^i \not\equiv \sum_{i=0}^{k} d_i(-4)^i \pmod{4^j},$$

so in particular the two numbers represented by $\{c_i\}$ and $\{d_i\}$ are distinct. On the other hand, to show that n admits a base -4 representation, find an integer k such that $1 + 4^2 + \cdots + 4^{2k} \geq n$ and express $n + 4 + \cdots + 4^{2k-1}$ as $\sum_{i=0}^{2k} c_i 4^i$. Now set $d_{2i} = c_{2i}$ and $d_{2i-1} = 3 - c_{2i-1}$, and note that $n = \sum_{i=0}^{2k} d_i(-4)^i$.

Second Solution: For any $S \subset \mathbb{Z}$, let $S^* = \{a+2b|\, a, b \in S\}$. Call a finite set of integers $S = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{Z}$ *good* if $|S^*| = |S|^2$, i.e., if the values $a_i + 2a_j$ $(1 \leq i, j \leq m)$ are distinct. We first prove that given a good set and $n \in \mathbb{Z}$, we can always find a good superset T of S such that $n \in T^*$. If $n \in S^*$ already, take $T = S$. Otherwise take $T = S \cup \{k, n-2k\}$ where k is to be chosen. Then put $T^* = S^* \cup Q \cup R$, where

$$Q = \{3k, 3(n-2k), k + 2(n-2k), (n-2k) + 2k\}$$

and

$$R = \{k + 2a_i,\ (n-2k) + 2a_i,\ a_i + 2k,\ a_i + 2(n-2k)|\ 1 \leq i \leq m\}.$$

Note that for any choice of k, we have $n = (n-2k) + 2k \in Q \subset T^*$. Except for n, the new values are distinct nonconstant linear forms in k, so if k is sufficiently large, they will all be distinct from each other and from the elements of S^*. This proves that T^* is good.

Starting with the good set $X_0 = \{0\}$, we thus obtain a sequence of sets $X_1, X_2, X_3, \ldots$ such that for each positive integer j, X_j is a good superset of X_{j-1} and X_j^* contains the jth term of the sequence $1, -1, 2, -2, 3, -3, \ldots$. It follows that

$$X = \bigcup_{j=0}^{\infty} X_j$$

has the desired property.

1.19 Vietnam

1. Solve the system of equations:

$$\begin{aligned}\sqrt{3x}\left(1+\frac{1}{x+y}\right) &= 2\\ \sqrt{7y}\left(1-\frac{1}{x+y}\right) &= 4\sqrt{2}.\end{aligned}$$

Solution: Let $u = \sqrt{x}, v = \sqrt{y}$, so the system becomes

$$u + \frac{u}{u^2+v^2} = \frac{2}{\sqrt{3}}$$
$$v - \frac{v}{u^2+v^2} = \frac{4\sqrt{2}}{\sqrt{7}}.$$

Now let $z = u + vi$; the system then reduces to the single equation

$$z + \frac{1}{z} = 2\left(\frac{1}{\sqrt{3}} + \frac{2\sqrt{2}}{\sqrt{7}}i\right).$$

Let t denote the quantity inside the parentheses; then

$$\begin{aligned} z &= t \pm \sqrt{t^2-1}\\ &= \frac{1}{\sqrt{3}} + \frac{2\sqrt{2}}{\sqrt{7}}i \pm \left(\frac{2}{\sqrt{21}} + \sqrt{2}I\right)\end{aligned}$$

from which we deduce

$$u = \left(\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{21}}\right)^2, \qquad v = \left(\frac{2\sqrt{2}}{\sqrt{7}} \pm \sqrt{2}\right)^2.$$

2. Let $ABCD$ be a tetrahedron with $AB = AC = AD$ and circumcenter O. Let G be the centroid of triangle ACD, let E be the midpoint of BG, and let F be the midpoint of AE. Prove that OF is perpendicular to BG if and only if OD is perpendicular to AC.

Solution: We identify points with their vectors originating from the circumcenter, so that $A \cdot B = A \cdot C = A \cdot D$ and $A^2 = B^2 =$

$C^2 = D^2$. Now

$$\begin{aligned}
(O-F)\cdot(B-G) &= \frac{1}{2}(A+E)\cdot(B-G) \\
&= \frac{1}{4}[(2A+B+G)\cdot(B-G)] \\
&= \frac{1}{36}[18A\cdot B - 6A\cdot(A+C+D) + 9B^2 - (A+C+D)^2] \\
&= \frac{1}{36}[2A\cdot D - 2C\cdot D].
\end{aligned}$$

Therefore $OF \perp BF$ if and only if $OD \perp AC$.

3. Determine, as a function of n, the number of permutations of the set $\{1, 2, \ldots, n\}$ such that no three of $1, 2, 3, 4$ appear consecutively.

Solution: There are $n!$ permutations in all. Of those, we exclude $(n-2)!$ permutations for each arrangement of $1, 2, 3, 4$ into an ordered triple and one remaining element, or $24(n-2)!$ in all. However, we have twice excluded each of the $24(n-3)!$ permutations in which all four of $1, 2, 3, 4$ occur in a block. Thus the number of permutations of the desired form is $n! - 24(n-2)! + 24(n-3)!$.

4. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying (for all $n \in \mathbb{N}$)

$$f(n) + f(n+1) = f(n+2)f(n+3) - 1996.$$

Solution: From the given equation, we deduce

$$f(n) - f(n+2) = f(n+3)[f(n+2) - f(n+4)].$$

If $f(1) > f(3)$, then by induction, $f(2m-1) > f(2m+1)$ for all $m > 0$, giving an infinite decreasing sequence $f(1), f(3), \ldots$ of positive integers, a contradiction. Hence $f(1) \leq f(3)$, and similarly $f(n) \leq f(n+2)$ for all n.

Now note that

$$\begin{aligned}
0 &= 1996 + f(n) + f(n+1) - f(n+2)f(n+3) \\
&\leq 1996 + f(n+2) + f(n+3) - f(n+2)f(n+3) \\
&= 1997 - [f(n+2) - 1][f(n+3) - 1].
\end{aligned}$$

In particular, either $f(n+2) = 1$ or $f(n+3) \leq 1997$, and vice versa.

The numbers $f(2m+1) - f(2m-1)$ are either all zero or all positive, and similarly for the numbers $f(2m+2) - f(2m)$. If they are both positive, eventually $f(n+2)$ and $f(n+3)$ both exceed 1997, a contradiction.

We now split into three cases. If $f(2m)$ and $f(2m+1)$ are both constant, we have $[f(2m) - 1][f(2m+1) - 1] = 1997$ and so either $f(2m) = 1$ and $f(2m+1) = 1997$ or vice versa. If $f(2m+1)$ is constant but $f(2m)$ is not, then $f(2m+1) = 1$ for all m and $f(2m+2) = f(2m) + 1997$, so $f(2m) = 1997(m-1) + f(2)$. Similarly, if $f(2m)$ is not constant, then $f(2m) = 1$ and $f(2m+1) = 1997m + f(1)$.

5. Consider triangles ABC where $BC = 1$ and $\angle BAC$ has a fixed measure $\alpha > \pi/3$. Determine which such triangle minimizes the distance between the incenter and centroid of ABC, and compute this distance in terms of α.

Solution: If we fix B and C and force A to lie above the line BC, then A is constrained to an arc. The centroid of ABC is constrained to the image of that arc under a 1/3 homothety at the midpoint of BC. On the other hand, the incenter subtends an angle of $(\pi+\alpha)/2$ at BC, so it is also constrained to lie on an arc, but its arc passes through B and C. Since the top of the incenter arc lies above the top of the centroid arc, the arcs cannot intersect (or else their circles would intersect four times). Moreover, if we dilate the centroid arc about the midpoint of BC so that its image is tangent to the incenter arc at its highest point, the image lies between the incenter arc and BC.

In other words, the distance from the incenter to the centroid is always at least the corresponding distance for ABC isosceles. Hence we simply compute the distance in that case. The incenter makes an isosceles triangle of vertex angle $(\pi + \alpha)/2$, so its altitude is $1/2 \cot(\pi + \alpha)/4$. Meanwhile, the distance of the centroid to BC is 1/3 that of A to BC, or $1/6 \cot(\alpha/2)$. The desired distance is thus

$$\frac{1}{2} \cot \frac{\pi + \alpha}{4} - \frac{1}{6} \cot \frac{\alpha}{2}.$$

6. Let a, b, c, d be four nonnegative real numbers satisfying the condition

$$2(ab + ac + ad + bc + bd + cd) + abc + abd + acd + bcd = 16.$$

Prove that

$$a + b + c + d \geq \frac{2}{3}(ab + ac + ad + bc + bd + cd)$$

and determine when equality occurs.

Solution: For $i = 1, 2, 3$, define s_i as the average of the products of the i-element subsets of $\{a, b, c, d\}$. Then we must show

$$3s_2 + s_3 = 4 \Rightarrow s_1 \geq s_2.$$

It suffices to prove the (unconstrained) homogeneous inequality

$$3s_2^2 s_1^2 + s_3 s_1^3 \geq 4s_2^3,$$

as then $3s_2 + s_3 = 4$ will imply $(s_1 - s_2)^3 + 3(s_1^3 - s_2^3) \geq 0$.

We now recall two basic inequalities about symmetric means of nonnegative real numbers. The first is Schur's inequality:

$$3s_1^3 + s_3 \geq 4s_1 s_2,$$

while the second,

$$s_1^2 \geq s_2$$

is a case of Maclaurin's inequality $s_i^{i+1} \geq s_{i+1}^i$. These combine to prove the claim:

$$3s_2^2 s_1^2 + s_3 s_1^3 \geq 3s_2^2 s_1^2 + \frac{s_2^2 s_3}{s_1} \geq 4s_2^3.$$

Finally, for those who have only seen Schur's inequality in three variables, note that in general any inequality involving $s_1, \ldots, s_k$ which holds for $n \geq k$ variables also holds for $n + 1$ variables, by replacing the variables $x_1, \ldots, x_{n+1}$ by the roots of the derivative of the polynomial $(x - x_1) \cdots (x - x_{n+1})$.

2 1996 Regional Contests: Problems and Solutions

2.1 Asian Pacific Mathematics Olympiad

1. Let $ABCD$ be a quadrilateral with $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > BD/2$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of the hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

   Solution: The lengths of AM, MN, NC are all linear in the distance between the segments MN and AC; if this distance is h, extrapolating from the extremes $MN = AC$ and $M = N = D$ gives that
   $$AM + MN + NC = AC + \frac{2AB - AC}{BD/2}h.$$
   In particular, if the segments MN and PQ maintain constant total distance from AC, as they do if their distance remains constant, the total perimeter of the hexagon is constant.

2. Let m and n be positive integers such that $n \leq m$. Prove that
   $$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2 + m)^n.$$

   Solution: The quantity in the middle is $(m+n)(m+n-1)\cdots(m-n+1)$. If we pair off terms of the form $(m + x)$ and $(m + 1 - x)$, we get products which do not exceed $m(m + 1)$, since the function $f(x) = (m + x)(m + 1 - x)$ is a concave parabola with maximum at $x = 1/2$. From this the right inequality follows. For the left, we need only show $(m + x)(m + 1 - x) \geq 2x$ for $x \leq n$; this rearranges to $(m - x)(m + 1 + x) \geq 0$, which holds because $m \geq n \geq x$.

3. Let P_1, P_2, P_3, P_4 be four points on a circle, and let I_1 be the incenter of the triangle $P_2P_3P_4$, I_2 be the incenter of the triangle $P_1P_3P_4$, I_3 be the incentre of the triangle $P_1P_2P_4$, and I_4 be the incenter of the

triangle $P_1P_2P_3$. Prove that I_1, I_2, I_3 and I_4 are the vertices of a rectangle.

Solution: Without loss of generality, assume P_1, P_2, P_3, P_4 occur on the circle in that order. Let $M_{12}, M_{23}, M_{34}, M_{41}$ be the midpoints of arcs $P_1P_2, P_2P_3, P_3P_4, P_4P_1$, respectively. Then the line P_3M_1 is the angle bisector of $\angle P_2P_3P_1$ and so passes through I_4. Moreover, the triangle $M_{12}P_2I_4$ is isosceles because

$$\begin{aligned}\angle I_4M_{12}P_2 &= \angle P_3P_1P_2 \\ &= \pi - 2\angle P_1P_2I_4 - 2\angle M_{12}P_2P_1 \\ &= \pi - 2\angle M_{12}P_2I_4.\end{aligned}$$

Hence the circle centered at M passing through P_1 and P_2 also passes through I_4, and likewise through I_3.

From this we determine that the angle bisector of $\angle P_3M_{12}P_4$ is the perpendicular bisector of I_3I_4. On the other hand, this angle bisector passes through M_{34}, so it is simply the line $M_{12}M_{34}$; by symmetry, it is also the perpendicular bisector of I_1I_2. We conclude that $I_1I_2I_3I_4$ is a parallelogram.

To show that $I_1I_2I_3I_4$ is actually a rectangle, it now suffices to show that $M_{12}M_{34} \perp M_{23}M_{41}$. To see this, simply note that the angle between these lines is half the sum of the measures of the arcs $M_{12}M_{23}$ and $M_{34}M_{41}$, but these arcs clearly comprise half of the circle.

4. The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

   (a) All members of the group must be of the same sex, i.e. they are either all male or all female.

   (b) The difference in the size of any two groups is either 0 or 1.

   (c) All groups have at least one member.

   (d) Each person must belong to one and only one group.

   Find all values of n, $n \leq 1996$, for which this is possible. Justify your answer.

Solution: Clearly $n \geq 9$ since each of 17 groups must contain at least one member. Suppose there are k groups of men and $17 - k$

groups of women; without loss of generality, we assume $k \leq 8$. If m is the minimum number of members in a group, then the number of men in the groups is at most $k(m+1)$, while the number of women is at least $(k+1)m$. As there are the same number as men as women, we have $k(m+1) \geq (k+1)m$, so $m \leq k \leq 8$, and the maximum number of couples is $k(k+1) \leq 72$. In fact, any number of couples between 9 and 72 can be distributed: divide the men as evenly as possible into 8 groups, and divide the women as evenly as possible into 9 groups. Thus $9 \leq n \leq 72$ is the set of acceptable numbers of couples.

5. Let a, b and c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

and determine when equality occurs.

Solution: By the triangle inequality, $b+c-a$ and $c+a-b$ are positive. For any positive x, y, we have

$$2(x+y) \geq x+y+2\sqrt{xy} = (\sqrt{x}+\sqrt{y})^2$$

by the AM-GM inequality, with equality for $x = y$. Substituting $x = a+b-c, y = b+c-a$, we get

$$\sqrt{a+b-c} + \sqrt{b+c-a} \leq 2\sqrt{a},$$

which added to the two analogous inequalities yields the desired result. Inequality holds for $a+b-c = b+c-a = c+a-b$, i.e. $a = b = c$.

2.2 Austrian-Polish Mathematics Competition

1. Let $k \geq 1$ be an integer. Show that there are exactly 3^{k-1} positive integers n with the following properties:

   (a) The decimal representation of n consists of exactly k digits.

   (b) All digits of k are odd.

   (c) The number n is divisible by 5.

   (d) The number $m = n/5$ has k odd (decimal) digits.

   Solution: The multiplication in each place must produce an even number of carries, since these will be added to 5 in the next place and an odd digit must result. Hence all of the digits of m must be $1, 5$ or 9, and the first digit must be 1, since m and n have the same number of decimal digits. Hence there are 3^{k-1} choices for m and hence for n.

2. A convex hexagon $ABCDEF$ satisfies the following conditions:

   (a) Opposite sides are parallel (i.e. $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$).

   (b) The distances between opposite sides are equal (i.e. $d(AB, DE) = d(BC, EF) = d(CD, FA)$, where $d(g, h)$ denotes the distance between lines g and h).

   (c) The angles $\angle FAB$ and $\angle CDE$ are right.

   Show that diagonals BE and CF intersect at an angle of $45°$.

   Solution: The conditions imply that A and D are opposite vertices of a square $APDQ$ such that B, C, E, F lie on AP, PD, DQ, QA, respectively, and that all six sides of the hexagon are tangent to the inscribed circle of the square. The diagonals BE and CF meet at the center O of the square. Let T, U, V be the feet of perpendiculars from O to AB, BC, CD; then $\angle TOB = \angle BOU$ by reflection across OB, and similarly $\angle UOC = \angle COV$. Therefore $\pi/2 = 2\angle BOC$, proving the claim.

3. The polynomials $P_n(x)$ are defined by $P_0(x) = 0$, $P_1(x) = x$ and

$$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x) \qquad n \geq 2.$$

For every natural number $n \geq 1$, find all real numbers x satisfying the equation $P_n(x) = 0$.

Solution: One shows by induction that

$$P_n(x) = \frac{x}{x-2}[(x-1)^n - 1].$$

Hence $P_n(x) = 0$ if and only if $x = 0$ or $x = 1 + e^{2\pi i k/n}$ for some $k \in \{1, \ldots, n-1\}$.

4. The real numbers x, y, z, t satisfy the equalities $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that

$$-1 \leq xy + yz + zt + tx \leq 0.$$

Solution: The inner expression is $(x+z)(y+t) = -(x+z)^2$, so the second inequality is obvious. As for the first, note that

$$1 = (x^2 + z^2) + (y^2 + t^2) \geq \frac{1}{2}[(x+z)^2 + (y+t)^2] \geq |(x+z)(y+t)|$$

by two applications of the power mean inequality.

5. A convex polyhedron P and a sphere S are situated in space such that S intercepts on each edge AB of P a segment XY with $AX = XY = YB = \frac{1}{3}AB$. Prove that there exists a sphere T tangent to all edges of P.

Solution: Let AB and BC be two edges of the polyhedron, so that the sphere meets AB in a segment XY with $AX = XY = YB$ and meets BC in a segment ZW with $BZ = ZW = WC$. In the plane ABC, the points X, Y, Z, W lie on the cross-section of the sphere, which is a circle. Therefore $BY \cdot BX = BZ \cdot BW$ by power-of-a-point; this clearly implies $AB = BC$, and so the center of S is equidistant from AB and BC. We conclude that any two edges of P are equidistant from S, and so there is a sphere concentric with S tangent to all edges.

6. Natural numbers k, n are given such that $1 < k < n$. Solve the system of n equations

$$x_i^3(x_i^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \qquad 1 \le i \le n$$

in n real unknowns $x_1, \ldots, x_n$. (Note: $x_0 = x_n, x_1 = x_{n+1}$, etc.)

Solution: The only solution is $x_1 = \cdots x_n = k^{-1/3}$. Let L and M be the smallest and largest of the x_i, respectively. If $M = x_i$, then

$$kM^3L^2 \le x_i^3(x_i^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \le M^2$$

and so $M \le 1/(kL^2)$. Similarly, if $L = x_j$, then

$$kL^3M^2 \ge x_j^3(x_j^2 + \cdots + x_{j+k-1}^2) = x_{j-1}^2 \ge L^2$$

and so $L \ge 1/(kM^2)$. Putting this together, we get

$$L \ge \frac{1}{kM^2} \ge kL^4$$

and so $L \ge k^{-1/3}$; similarly, $M \le k^{-1/3}$. Obviously $L \le M$, so we have $L = M = k^{-1/3}$ and $x_1 = \cdots = x_n = k^{-1/3}$.

7. Show that there do not exist nonnegative integers k and m such that $k! + 48 = 48(k+1)^m$.

Solution: Suppose such k, m exist. We must have $48|k!$, so $k \ge 6$; one checks that $k = 6$ does not yield a solution, so $k \ge 7$. In that case $k!$ is divisible by 32 and by 9, so that $(k! + 48)/48$ is relatively prime to 6, as then is $k + 1$.

If $k + 1$ is not prime, it has a prime divisor greater than 3, but this prime divides $k!$ and not $k! + 48$. Hence $k + 1$ is prime, and by Wilson's theorem $k!+1$ is a multiple of $k+1$. Since $k!+48$ is as well, we find $k + 1 = 47$, and we need only check that $46!/48 + 1$ is not a power of 47. We check that $46!/48 + 1 \equiv 29 \pmod{53}$ (by cancelling as many terms as possible in 46! before multiplying), but that 47 has order 13 modulo 53 and that none of its powers is congruent to 29 modulo 53.

8. Show that there is no polynomial $P(x)$ of degree 998 with real coefficients satisfying the equation $P(x)^2 - 1 = P(x^2 + 1)$ for all real numbers x.

Solution: The equation implies $P(x) = \pm P(-x)$; since P has even degree, it must be an even polynomial, that is, $P(x) = Q(x^2)$ for some polynomial Q of degree 499. Then $Q(t)^2 - 1 = Q(t^2 + 2t + 1)$ for infinitely many values of t (namely $t \geq 0$), so this equation is also a polynomial identity. However, it implies that $Q(t) = \pm Q(-2 - t)$; if we put $R(t) = Q(t - 1)$, we have $R(t) = \pm R(-t)$, so that R is an odd polynomial. In particular, $R(0) = 0$, so $Q(-1) = 0$. But now we find $Q(1) = -1, Q(4) = 0, Q(25) = -1, \ldots$; this process produces infinitely many zeroes of Q, a contradiction.

9. We are given a collection of rectangular bricks, no one of which is a cube. The edge lengths are integers. For every triple of positive integers (a, b, c), not all equal, there is a sufficient supply of $a \times b \times c$ bricks. Suppose that the bricks completely tile a $10 \times 10 \times 10$ box.

   (a) Assume that at least 100 bricks have been used. Prove that there exist at least two parallel bricks, that is, if AB is an edge of one of the bricks, $A'B'$ is an edge of the other and $AB \parallel A'B'$, then $AB = A'B'$.

   (b) Prove the same statement with 100 replaced by a smaller number. The smaller the number, the better the solution.

Solution: We prove the claim with 97 bricks. For each integer up to 16, we tabulate the number of nonparallel bricks of that volume (disallowing cubical bricks and bricks with a dimension greater than 10) and their total volume:

Volume	2	3	4	5	6	7	8	9	10	12	14	15	16
Number	3	3	6	3	9	3	9	6	9	15	6	6	12
Total	6	9	24	15	54	21	72	54	90	180	74	90	192

Assuming no two bricks are parallel, the 90 smallest bricks have total volume 891. The 7 other bricks each have volume at least 18, giving a total volume of at least 1017, a contradiction.

We have not determined the optimal constant (one can improve the above bound to 96 easily), but we note that an arrangement with 73 nonparallel bricks is possible.

2.3 Balkan Mathematical Olympiad

1. Let O and G be the circumcenter and centroid, respectively, of triangle ABC. If R is the circumradius and r the inradius of ABC, show that
$$OG \leq \sqrt{R(R-2r)}.$$

Solution: Using vectors with origin at O, we note that
$$OG^2 = \frac{1}{9}(A+B+C)^2 = \frac{1}{3}R^2 + \frac{2}{9}R^2(\cos 2A + \cos 2B + \cos 2C).$$
Hence $R^2 - OG^2 = (a^2+b^2+c^2)/9$. On the other hand, by the standard area formula $K = rs = abc/4R$, we have $2rR = abc/(a+b+c)$. We now note that
$$(a^2+b^2+c^2)(a+b+c) \geq 9abc$$
by two applications of the AM-GM inequality, so $2rR \leq R^2 - OG^2$, proving the claim.

2. Let $p > 5$ be a prime number and $X = \{p - n^2 | n \in \mathbb{N}, n^2 < p\}$. Prove that X contains two distinct elements x, y such that $x \neq 1$ and x divides y.

Solution: Write $p = m^2 + k$ with $k \leq 2m$. If $1 < k < 2m$ and k is either odd or a multiple of 4, we can write $k = a(2m-a) = m^2 - (m-a)^2$, and then $k | p^2 - (m-a)^2$. If k is even but not a multiple of 4, write $2k = a(2m-a)$ and proceed as above, which still works because $2k < m^2$ for $p > 5$.

We can't have $k = 2m$ since $m^2 + 2m = m(m+2)$ is composite, so the only case left is $p = m^2+1$. In this case let $t = 2m = p-(m-1)^2$ and write either t or $2t$ as a difference of squares $(m-1)^2-(m-a)^2$; this still works because $2t < (m-1)^2$ for $p \geq 7$.

3. Let $ABCDE$ be a convex pentagon, and let M, N, P, Q, R be the midpoints of sides AB, BC, CD, DE, EA, respectively. If the segments AP, BQ, CR, DM have a common point, show that this point also lies on EN.

Solution: Let T be the common point, which we take as the origin of a vector system. Then $A \times P = 0$, or equivalently $A \times (C+D) = 0$, which we may write $A \times C = D \times A$. Similarly, we have $B \times D = E \times B$, $C \times E = A \times C$, $D \times A = B \times D$. Putting these equalities together gives $E \times B = C \times E$, or $E \times (B + C) = 0$, which means the line EN also passes through the origin T.

4. Show that there exists a subset A of the set $\{1, 2, \ldots, 1996\}$ having the following properties:

   (a) $1, 2^{1996} - 1 \in A$;

   (b) every element of A, except 1, is the sum of two (not necessarily distinct) elements of A;

   (c) A contains at most 2012 elements.

Solution: We state the problem a bit differently: we want to write down at most 2012 numbers, starting with 1 and ending with $2^{1996} - 1$, such that every number written is the sum of two numbers previously written. If $2^n - 1$ has been written, then $2^n(2^n - 1)$ can be obtained by n doublings, and $2^{2n} - 1$ can be obtained in one more step. Hence we can obtain $2^2 - 1, 2^4 - 1, \ldots, 2^{256} - 1$ in $(1 + 1) + (2 + 1) + \cdots + (128 + 1) = 263$ steps. In 243 steps, we turn $2^{256} - 1$ into $2^{499} - 2^{243}$. Now notice that the numbers $2^{243} - 2^{115}, 2^{115} - 2^{51}, 2^{51} - 2^{19}, 2^{19} - 2^3, 2^3 - 2^1, 2^1 - 1$ have all been written down; in 6 steps, we now obtain $2^{499} - 1$. We make this into $2^{998} - 1$ in 500 steps, and make $2^{1996} - 1$ in 999 steps. Adding 1 for the initial 1, we count

$$1 + 263 + 243 + 6 + 500 + 999 = 2012$$

numbers written down, as desired.

2.4 Czech-Slovak Match

1. Let $\mathbb{Z}^*$ denote the set of nonzero integers. Show that an integer $p > 3$ is prime if and only if for any $a, b \in \mathbb{Z}^*$, exactly one of the numbers

$$N_1 = a + b - 6ab + \frac{p-1}{6}, \quad N_2 = a + b + 6ab + \frac{p+1}{6}$$

belongs to $\mathbb{Z}^*$.

Solution: If $N_1 = 0$, then $p = (6a-1)(6b-1)$ is composite; similarly, $N_2 = 0$ implies $p = -(6a+1)(6b+1)$ is composite. Conversely, suppose that p is composite. If $p \equiv 0, 2, 3$ or $4 \pmod 6$, then N_1 and N_2 are not integers. Otherwise, all divisors of p are congruent to $\pm 1 \pmod 6$, so there exist natural numbers c, d such that

$$p = (6c+1)(6d+1)\text{or}(6c-1)(6d-1)\text{or}(6c+1)(6d-1).$$

In the first case, N_2 is not an integer and $N_1 = 0$ for $a = -c, b = -d$. In the second case, N_2 is not an integer and $N_1 = 0$ for $a = c, b = d$. In the third case, N_1 is not an integer and $N_2 = 0$ for $a = c, b = -d$.

2. Let M be a nonempty set and $*$ a binary operation on M. That is, to each pair $(a, b) \in M \times M$ one assigns an element $a * b$. Suppose further that for any $a, b \in M$,

$$(a * b) * b = a \qquad \text{and} \qquad a * (a * b) = b.$$

   (a) Show that $a * b = b * a$ for all $a, b \in M$.

   (b) For which finite sets M does such a binary operation exist?

Solution:

(a) First note that $[a * (a * b)] * (a * b) = a$ by the first rule. By the second rule, we may rewrite the left side as $b * (a * b)$, so $b * (a * b) = a$ and so $b * a = b * [b * (a * b)]$. By the second rule, this equals $a * b$, so $a * b = b * a$.

(b) Such sets exist for all finite sets M. Identify M with $\{1, \ldots, n\}$ and define

$$a * b = c \Leftrightarrow a + b + c \equiv 0 \pmod n.$$

It is immediate that the axioms are satisfied.

3. A pyramid π is given whose base is a square of side $2a$ and whose lateral edges have length $a\sqrt{17}$. Let M be a point in the interior of the pyramid, and for each face of π, consider the pyramid similar to π whose vertex is M and whose base lies in the plane of the face. Show that the sum of the surface areas of these five pyramids is greater than or equal to one-fifth the surface area of π, and determine for which M equality holds.

Solution: All faces of π have the same area $S_1 = 4a^2$. The segments connecting the point M with the vertices of the pyramid partition π into a quadrilateral pyramid and four tetrahedra. Let $v_1, \ldots, v_5$ denote the heights of these five bodies from vertex M; then the volumes of the bodies sum to the volume of π, which means

$$\sum_{i=1}^{5} \frac{1}{3} S_1 v_i = \frac{1}{3} S_1 v.$$

In other words, $\sum v_i = v$. On the other hand, the v_i are also the heights of the small pyramids similar to π, so

$$\sum_{i=1}^{5} \frac{v_i}{v} = 1 = \sum_{i=1}^{5} k_i = \sum_{i=1}^{5} \sqrt{\frac{S_i}{S}},$$

where S denotes the surface area of π, S_i that of the i-th small pyramid, and k_i the coefficient of similarity between the i-th pyramid and π. We conclude

$$S = \left(\sum_{i=1}^{5} \sqrt{S_i} \right)^2 \leq 5 \sum_{i=1}^{5} S_i$$

by the power mean (or Cauchy-Schwarz) inequality. Equality holds only when all of the S_i are equal, as are the v_i, which occurs when M is the center of the inscribed sphere of the pyramid.

4. Determine whether there exists a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for each $k = 0, 1, \ldots, 1996$ and for each $m \in \mathbb{Z}$ the equation $f(x) + bx = m$ has at least one solution $x \in \mathbb{Z}$.

Solution: Each integer y can be written uniquely as $1997m + k$ with $m \in \mathbb{Z}$ and $k \in \{0, \ldots, 1996\}$. Define the function f by $f(y) =$

$m - ky$; then $f(x) + kx = m$ has the solution $x = 1997m + k$, so the condition is satisfied.

5. Two sets of intervals A, B on a line are given. The set A contains $2m - 1$ intervals, every two of which have a common interior point. Moreover, each interval in A contains at least two disjoint intervals of B. Show that there exists an interval in B which belongs to at least m intervals from A.

   Solution: Let $\alpha_i = [a_i, b_i]$ $(i = 1, \ldots, 2m - 1)$ be the intervals, indexed so that $a_1 \leq a_2 \leq \ldots \leq a_{2m-1}$. Choose $k \in \{m, \ldots, 2m-1\}$ to minimize b_k. By assumption, the interval α_k contains two disjoint intervals from B, say $\beta_1 = [c_1, d_1]$ and $\beta_2 = [c_2, d_2]$. Without loss of generality, assume

   $$a_k \leq c_1 < d_1 < c_2 < d_2 \leq b_k.$$

   If $d_1 \leq b_i$ for $i = 1, 2, \ldots, m$, then $\beta_1 \subset \alpha_i$ for $i = 1, 2, \ldots, m$, so β_1 satisfies the desired property. Otherwise, $d_1 > b_s$ for some $s \in \{1, 2, \ldots, m\}$. By assumption, $c_2 > d_1 > b_s$. Since no two of the α are disjoint, we have $b_s \geq a_i$ for all i, so $c_2 > a_i$. On the other hand, by the choice of k, $b_k \leq b_i$ for $i = m, \ldots, 2m_1$. Therefore $a_i < c_2 < d_2 \leq b_k \leq b_i$ for each $i \in \{m, m+1, \ldots, 2m - 1\}$, and so β_2 has the desired property.

6. The points E and D lie in the interior of sides AC and BC, respectively, of a triangle ABC. Let F be the intersection of the lines AD and BE. Show that the area of the triangles ABC and ABF satisfies

   $$\frac{S_{ABC}}{S_{ABF}} = \frac{|AC|}{|AE|} + \frac{|BC|}{|BD|} - 1.$$

   Solution: Let the line parallel to BC through F meet AB at K and AC at N; let the line parallel to CA through F meet BC at M and AB at P; let the line parallel to AB through F meet BC at L and CA at O. Let v_C and v_F be the distances of C and F, respectively, to the line AB. Then

   $$\frac{S_{ABC}}{S_{ABF}} = \frac{v_C}{v_F} = \frac{BC}{FK} = \frac{BL + LM + MC}{FK}.$$

Under the homothety through B carrying F to E, the segment PM maps to AC. Thus

$$\frac{LM}{FK} = \frac{FM}{FP} = \frac{EC}{AC} = \frac{AC}{AE} - 1$$

and similarly

$$\frac{CM}{FK} = \frac{NF}{FK} = \frac{CD}{BD} = \frac{BC}{BD} - 1.$$

The required assertion follows by putting this all together and noting $BL = FK$.

2.5 Iberoamerican Olympiad

1. Let n be a natural number. A cube of side length n can be divided into 1996 cubes whose side lengths are also natural numbers. Determine the smallest possible value of n.

   Solution: Since $1996 > 12^3$, we must have $n \geq 13$, and we now show $n = 13$ suffices. Inside a cube of edge 13, we place one cube of edge 5, one cube of length 4, and 2 of length 2, and fill the remainder with cubes of edge 1. The number of cubes used is

   $$13^3 - (5^3 - 1) - (4^3 - 1) - 2(2^3 - 1) = 2197 - 124 - 63 - 2(7) = 1996,$$

   as desired.

2. Let M be the midpoint of the median AD of triangle ABC. The line BM intersects side AC at the point N. Show that AB is tangent to the circumcircle of NB if and only if the following equality holds:

   $$\frac{BM}{BN} = \frac{BC^2}{BN^2}.$$

   Solution: First note that (by the Law of Sines in triangles ABM and AMN)

   $$\frac{BM}{MN} = \frac{\sin\angle MAB}{\sin\angle ABM}\frac{\sin\angle MNA}{\sin\angle NAM}.$$

   Then note that (by the Law of Sines in triangles ABD and ADC)

   $$\frac{\sin\angle MAB}{\sin\angle NAM} = \frac{BD}{DC}\frac{\sin\angle ABD}{\sin\angle DCA}.$$

   By the Law of Sines in triangle BNC,

   $$\frac{BC^2}{BN^2} = \frac{\sin^2\angle BNC}{\sin^2\angle BCN};$$

   therefore $BM/MN = BC^2/BN^2$ if and only if

   $$\frac{\sin\angle ABD}{\sin\angle ABM} = \frac{\sin\angle BNC}{\sin\angle BCN},$$

   which if we put $\alpha = \angle ABM, \beta = \angle BCN, \theta = \angle NBC$ becomes

   $$\sin(\alpha + \theta)\sin\beta = \sin(\beta + \theta)\sin\alpha.$$

Rewriting each side as a difference of cosines and cancelling, this becomes

$$\cos(\alpha + \theta - \beta) = \cos(\beta - \alpha + \theta).$$

Both angles in this equation are between $-\pi$ and π, so the angles are either equal or negatives of each other. The latter implies $\theta = 0$, which is untrue, so we deduce $\alpha = \beta$, and so $BM/MN = BC^2/BN^2$ if and only if $\angle ABM = \angle BCN$, that is, if AB is tangent to the circumcircle of BNC.

3. We have a square table of $k^2 - k + 1$ rows and $k^2 - k + 1$ columns, where $k = p + 1$ and p is a prime number. For each prime p, give a method of distributing the numbers 0 and 1, one number in each square of the table, such that in each row and column there are exactly k zeroes, and moreover no rectangle with sides parallel to the sides of the table has zeroes at all four corners.

Solution: The projective plane of order p is defined as the set of equivalence classes in the set $(\mathbb{Z} \bmod p)^3 - (0, 0, 0)$ where (a, b, c) is equivalent to (ma, mb) whenever m is coprime to p. Label the rows and columns by elements of the projective plane, and place a 1 in row (a, b, c) and column (d, e, f) if $ad + be + cf = 0$; then the desired condition is immediately verified.

Solution: This holds for $n = 2$, and we prove it in general by induction on n. Assume the result for $n - 1$. The fractions newly added are those with $b = n$ and a relatively prime to n. Those removed have $a + b = n$. Now note that the added fractions $1/an$ and $1/bn$ precisely cancel the removed fraction $1/ab$, so the sum remains unchanged.

4. Given a natural number $n \geq 2$, consider all of the fractions of the form $\frac{1}{ab}$, where a and b are relatively prime natural numbers such that $a < b \leq n$ and $a + b > n$. Show that the sum of these fractions is $1/2$.

5. Three counters A, B, C are placed at the corners of an equilateral triangle of side n. The triangle is divided into triangles of side length 1. Initially all lines of the figure are painted blue. The counters move

along the lines, painting their paths red, according to the following rules:

(i) First A moves, then B, then C, then A, and so on in succesion. On each turn, each counter moves the full length of a side of one of the short triangles.

(ii) No counter may retrace a segment already painted red, though it can stop on a red vertex, even if another counter is already there.

Show that for all integers $n > 0$ it is possible to paint all of the segments red in this fashion.

Solution: The cases $n = 1, 2$ are trivial; we use them as the base cases for an inductive proof. We describe the moves for A, understanding that the moves for B and C are the same moves rotated by $2\pi/3$ and $4\pi/3$, respectively. To fix directions, imagine the triangle is oriented with one side parallel to the horizontal and the third vertex above it, and suppose A starts at the bottom left. We first move A right for $n-1$ steps. We then alternate moving it up to the left and down to the left for a total of $2n-5$ steps. We then trace a path through the inner triangle of side $n-2$ using the induction hypothesis, ending at another corner. Finally, we follow the unused edges from that corner, ending three steps later.

6. In the plane are given n distinct points $A_1, \dots, A_n$, and to each point A_i is assigned a nonzero real number λ_i such that $(A_iA_j)^2 = \lambda_i + \lambda_j$ for all $i \neq j$. Show that

   (a) $n \leq 4$;

   (b) If $n = 4$, then $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} = 0$.

Solution: For any four points A_i, A_j, A_k, A_m, we have

$$A_iA_k^2 - A_kA_j^2 = A_iA_m^2 - A_mA_j^2 = \lambda_i - \lambda_j$$

and by an elementary lemma, this means A_iA_j is perpendicular to A_kA_m. Since this holds for all permutations of i, j, k, m, we conclude that A_m is the orthocenter of triangle $A_iA_jA_k$, so that in particular no other points can be given, hence $n \leq 4$.

Now suppose $n = 4$; without loss of generality, assume that $A_1A_2A_3$ form an acute triangle inside which A_4 lies. Note that

$$2\lambda_1 = A_1A_2^2 + A_1A_3^2 - A_1A_3^2 = 2A_1A_2 \cdot A_1A_3 \cos \angle A_3A_1A_2.$$

Using this and analogous formulae, and that $\angle A_1A_3A_4 = \angle A_1A_2A_4$ and $\angle A_3A_1A_2 = \pi - \angle A_3A_4A_2$, we get

$$\begin{aligned}
\frac{\lambda_1\lambda_2}{\lambda_3\lambda_4} &= \frac{(A_1A_2 \cdot A_1A_3 \cos \angle A_3A_1A_2)(A_1A_2 \cdot A_2A_4 \cos \angle A_1A_2A_4)}{(A_3A_1 \cdot A_3A_4 \cos \angle A_1A_3A_4)(A_2A_4 \cdot A_3A_4 \cos \angle A_2A_4A_3)} \\
&= -\frac{A_1A_2^2}{A_3A_4}^2 = -\frac{\lambda_1 + \lambda_2}{\lambda_3 + \lambda_4}.
\end{aligned}$$

Therefore $1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 + 1/\lambda_4 = 0$, as desired.

2.6 St. Petersburg City Mathematical Olympiad

1. Several one-digit numbers are written on a blackboard. One can replace any one of the numbers by the last digit of the sum of all of the numbers. Prove that the initial collection of numbers can be recovered by a sequence of such operations.

   Solution: At each step, erase the leftmost number and write the sum on the right, so that $(x_1, \dots, x_n)$ becomes $(x_2, \dots, x_{n-1}, x_1 + \cdots + x_n)$. Since there are finitely many n-tuples, some n-tuple must repeat, and the first such must be the original configuration, since each n-tuple $(y_1, \dots, y_n)$ comes from a unique n-tuple, namely $(y_n - y_1 - \cdots - y_{n-1}, y_1, \dots, y_{n-1})$.

2. Fifty numbers are chosen from the set $\{1, \dots, 99\}$, no two of which sum to 99 or 100. Prove that the chosen numbers must be $50, 51, \dots,$ 99.

   Solution: In the sequence

   $$99, 1, 98, 2, 97, 3, \dots, 51, 49, 50,$$

   any two adjacent numbers sum to 99 or 100, so both cannot occur. Grouping the numbers into 49 pairs plus one extra, we see at most 50 numbers can occur, and 50 must be one of them. Since we must step at least two terms along the list to make the next choice, the numbers must indeed be $50, 51, \dots, 99$. Clearly we maximize the number of chosen numbers by taking them two apart, and the list has odd length, so taking $99, 98, \dots, 50$ is the only Draw a graph with $\{1, \dots, 99\}$ as vertices, where two numbers are adjacent if they sum to 99 or 100.

3. Let M be the intersection of the diagonals of the trapezoid $ABCD$. A point P such that $\angle APM = \angle DPM$ is chosen on the base BC. Prove that the distance from C to the line AP is equal to the distance from B to the line DP.

   Solution: Since M lies on the internal angle bisector of angle $\angle APD$, it lies at the same distance from the lines AP and DP. The ratio of this distance to the distance from C to AP is AM/AC, while the ratio of this distance to the distance from B to DP is BM/MD. But $AM/MC = BM/MD$ by similar triangles, so the latter two distances are indeed the same.

4. In a group of several people, some are acquainted with each other and some are not. Every evening, one person invites all of his acquaintances to a party and introduces them to each other. Suppose that after each person has arranged at least one party, some two people are still unacquainted. Prove that they will not be introduced at the next party.

   Solution: We claim that two people unacquainted after each person has held at least one party lie in different connected components of the original (and final) graph of acquaintance. If two people are connected by a path of length n, they will be connected by a path of length $n-1$ after one person along the path (including either of the two people at the ends) holds a party, by a path of length $n-2$ after two of them hold a party, and so on. After each person holds a party, the two people on the ends will be acquainted.

5. Let M be the intersection of the diagonals of a cyclic quadrilateral, N the intersections of the lines joining the midpoints of opposite sides, and O the circumcenter. Prove that $OM \geq ON$.

   Solution: We use vectors. If A, B, C, D are the vertices of the quadrilateral in order, then $N = (A+B+C+D)/4$; in particular, if E and F are the midpoints of AC and BD, respectively, then N is the midpoint of EF. The circle with diameter OM passes through E and F, so $OM \geq OE$ and $OM \geq OF$; moreover, in any triangle, the median to a side is no longer than the average of the other two sides (rotate the triangle by π about the foot of the median, so twice the median becomes a diagonal of a parallelogram, and use the triangle inequality). Hence $OM \geq ON$.

6. Prove that for every polynomial $P(x)$ of degree 10 with integer coefficients, there is an infinite (in both directions) arithmetic progression which does not contain $P(k)$ for any integer k.

   Solution: Since P is not linear, there exists x such that $P(x+1) - P(x) = n > 1$. Since $P(t+k) \equiv P(t) \pmod{k}$, each value of

P is congruent to one of $P(x), P(x+1), \ldots, P(x+n-1)$ modulo n. However, since $P(x+1) - P(x) = n$, the values of P cover at most $n-1$ distinct residue classes, and so there is an arithmetic progression of difference n containing no values of P.

7. There are n parking spaces along a one-way road down which n drivers are traveling. Each driver goes to his favorite parking space and parks there if it is free; otherwise, he parks at the nearest free place down the road. If there is no free space after his favorite, he drives away. How many lists $a_1, \ldots, a_n$ of favorite parking spaces are there which permit all of the drivers to park?

   Solution: There are $(n+1)^{n-1}$ such lists. To each list of preferences $(a_1, \ldots, a_n)$ which allows all drivers to park, associate the list $(b_2, \ldots, b_n)$, where b_i is the difference mod $n+1$ between the numbers of the space driver i wants and the space the previous driver took. Clearly any two lists give rise to different sequences of b_i.

   We now argue that any list of b_i comes from a list of preferences. Imagine that the n parking spaces are arranged in a circle with an extra phantom space put in at the end. Put the first driver in any space, then for $i = 2, \ldots, n$, put driver i in the first available space after the space b_i away from the space taken by driver $i-1$; this gives a list of preferences if and only if the one space not taken at the end is the phantom space. However, by shifting the position of the first driver, we can always ensure that the phantom space is the space not taken.

   Thus the sequences of b_i are equal in number to the lists of preferences, so there are $(n+1)^{n-1}$ of each.

8. Find all positive integers n such that $3^{n-1} + 5^{n-1}$ divides $3^n + 5^n$.

   Solution: This only occurs for $n = 1$. Let $s_n = 3^n + 5^n$ and note that

   $$s_n = (3+5)s_{n-1} - 3 \cdot 5 \cdot s_{n-2}$$

   so s_{n-1} must also divide $3 \cdot 5 \cdot s_{n-2}$. If $n > 1$, then s_{n-1} is coprime to 3 and 5, so s_{n-1} must divide s_{n-2}, which is impossible since $s_{n-1} > s_{n-2}$.

9. Let M be the midpoint of side BC of triangle ABC, and let r_1 and r_2 be the radii of the incircles of triangles ABM and ACM. Prove that $r_1 < 2r_2$.

Solution: Recall that the area of a triangle equals its inradius times half its perimeter. Since ABM and ACM have equal area, we have

$$\frac{r_1}{r_2} = \frac{AC + AM + CM}{AB + AM + BM}$$

and it suffices to show $AC + AM + CM < 2AB + 2AM + 2BM$; since $BM = CM$, this simplifies to $AC < 2AB + AM + CM$. In fact, by the triangle inequality, $AC < AM + CM$, so we are done.

10. Several positive integers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

Solution: If a, b are erased and $c < d$ are written instead, we have $c \leq \min(a, b)$ and $d \geq \max(a, b)$; moreover, $ab = cd$. From this we may conclude $a+b \leq c+d$ by writing $ab+a^2 = cd+a^2 \leq ac+ad$ (the latter since $(d-a)(c-a) \leq 0$) and dividing both sides by a. Thus the sum of the numbers never decreases, and it is obviously bounded (e.g. by n times the product of the numbers, where n is the number of numbers on the board); hence it eventually stops changing, at which time the numbers never change.

11. No three diagonals of a convex 1996-gon meet in a point. Prove that the number of triangles lying in the interior of the 1996-gon and having sides on its diagonals is divisible by 11.

Solution: There is exactly one such triangle for each choice of six vertices of the 1996-gon: if A, B, C, D, E, F are the six vertices in order, the corresponding triangle is formed by the lines AD, BE, CF. Hence the number of triangles is $\binom{1996}{6}$; since 1991 is a multiple of 11, so is the number of triangles.

12. Prove that for every polynomial $x^2 + px + q$ with integer coefficients, there exists a polynomial $2x^2 + rx + s$ with integer coefficiets such

that the sets of values of the two polynomials on the integers are disjoint.

Solution: If p is odd, then $x^2 + px + q$ has the same parity as q for all integers x, and it suffices to choose r even and s of the opposite parity as q. If $p = 2m$ is even, then $x^2 + px + q = (x + m)^2 + (q - m^2)$ which is congruent to $q - m^2$ or $q - m^2 + 1$ modulo 4. Now it suffices to choose r even and s congruent to $q - m^2 + 2$ modulo 4.

13. In a convex pentagon $ABCDE$, $AB = BC$, $\angle ABE + \angle DBC = \angle EBD$, and $\angle AEB + \angle BDC = \pi$. Prove that the orthocenter of triangle BDE lies on AC.

    Solution: By the assumption $\angle AEB + \angle BDC = \pi$, there exists a point F on AC such that $\angle AFB = \angle AEB$ and $\angle BFC = \angle BDC$; this means F is the second intersection of the circumcircles of BCD and ABE. The triangle ABC is isosceles, so $\angle FCB = (\pi - \angle ABC)/2$. Hence $\angle FDB = \angle FCB = \pi/2 - \angle DBE$ by the assumption $\angle ABE + \angle DBC = \angle EBD$, and so $DF \perp BE$. Similarly $EF \perp BD$, and so F is the orthocenter of BDE.

14. In a federation consisting of two republics, each pair of cities is linked by a one-way road, and each city can be reached from each other city by these roads. The Hamilton travel agency provides n different tours of the cities of the first republic (visiting each city once and returning to the starting city without leaving the republic) and m tours of the second republic. Prove that Hamilton can offer mn such tours around the whole federation.

    Solution: From each pair of tours, we construct a tour of both republics from which the two original tours can be reconstructed (so in particular, all such pairs will be distinct). We first look for cities u, v in the first republic and w, x in the second, such that u immediately precedes v in the first tour, x immediately precedes w in the second, and there are roads from u to w and from x to v. In that case, we may start from v, tour the first republic ending at u, go to w, tour the second republic ending at x, and return to v. Call this a "direct" tour; the two original tours can be determined from a direct tour by reading off the cities of either republic in the order they appear.

Now suppose no direct tour exists; without loss of generality, assume the number of cities in the second republic does not exceed the number in the first republic. In this case, whenever u is some number of cities ahead of v in the first tour, and x is the same number ahead of w in the second tour, the "parallel" roads uw and vx must point in the same direction (by repeated application of the above observation). Since each city is reachable from each other city, given a vertex v, we can find x immediately preceding w in the second tour such that there are roads from w to v and from v to x. We now form an "alternating" tour starting at v by alternately taking roads parallel to vx and wv; once the second republic is exhausted, we tour the remaining cities of the first republic in order, returning to v. The two original tours can be determined from an alternating tour by reading off the cities of the first republic in the order they appear, and those of the second republic in the reverse of the order in which they appear.

15. Sergey found 11 different solutions to the equation $f(19x-96/x) = 0$. Prove that if he had tried harder, he could have found at least one more solution.

Solution: The equation $19x - 96/x = t$ can be rewritten $19x^2 - tx - 96 = 0$; since $t^2 + 19 \cdot 96 > 0$, it always has two real roots. Therefore the number of zeroes of f (if finite) is an even integer, so Sergey can find at least one more zero.

16. The numbers $1, 2, \ldots, 2n$ are divided into two groups of n numbers. Prove that the pairwise sums of numbers in each group (the sum of each number with itself included) have the same remainders upon division by $2n$. (Note: each pair of distinct numbers should be added twice, and each remainder must occur the same number of times in the two groups.)

Solution: Let S and T be the groups, and let $P(x) = \sum_{i \in S} x^i$ and $Q(x) = \sum_{j \in T} x^j$; the claim amounts to showing

$$P(x)^2 \equiv Q(x)^2 \pmod{x^{2n} - 1}.$$

This follows by noting that $x-1|P(x)-Q(x)$ since both groups have the same number of elements, and that $x^{2n-1}+\cdots+1|P(x)+Q(x)$ since each of the numbers $1,\ldots,2n$ occurs exactly once.

17. The points A' and C' are chosen on the diagonal BD of a parallelogram $ABCD$ so that $AA' \parallel CC'$. The point K lies on the segment $A'C$, and the line AK meets CC' at L. A line parallel to BC is drawn through K, and a line parallel to BD is drawn through C; these meet at M. Prove that D, M, L are collinear.

    Solution: Let M' and M'' be the intersections of DM with MK and MC, respectively. Since $M'K \parallel DA$, we have $LM'/LD = LK/LA$. Since $CK \parallel C'A$, we have $LK/LA = LC/LC'$. Finally, since $CM \parallel C'D$, we have $LM''/LD = LC/LC'$. Therefore $LM'/LD = LM''/LD$ and so $M' = M'' = M$.

18. Find all quadruples of polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ with real coefficients such that for each quadruple of integers x, y, z, t such that $xy - zt = 1$, one has

    $$P_1(x)P_2(y) - P_3(z)P_4(t) = 1.$$

    Solution: If $P_1(1) = 0$, then $P_3(z)P_4(t) = -1$ for each pair of integers z, t, and so P_3 and P_4 are constant functions; moreover, $P_1(x)P_2(y) = 0$, so one of P_1 and P_2 is identically zero. Ignoring such cases, which are easily enumerated, we assume $P_i(1) \neq 0$ for all i.

    We first note that $P_1(x)P_2(1) = P_1(1)P_2(x)$ for all nonzero integers x, so that P_1 and P_2 are equal up to a scalar factor; similarly, P_3 and P_4 are equal up to a scalar factor. Now note that $P_1(x)P_2(ay) = P_1(ax)P_2(y)$ for all nonzero a, x, y, so that the difference between the two sides is identically zero as a polynomial in a. In particular, that means no term in $P_1(x)P_2(y)$ has unequal exponent in x and y, and the same is true of $P_1(x)P_1(y)$. On the other hand, if $P_1(x)$ has terms of more than one degree, then $P_1(x)P_1(y)$ contains a term with different degrees in x and y. Hence $P_1(x) = cx^k$ for some integer k and some constant c, and similarly $P_2(x) = dx^k$, $P_3(x) = ex^m, P_4(x) = fx^m$.

Thus we must determine when $cdx^ky^k - efz^mt^m = 1$ whenever $xy - zt = 1$ in integers. Clearly $k = m$ since otherwise one of the two terms on the left dominates the other, and $cd = 1$ by setting $x = y = 1$ and $z = t = 0$, and similarly $ef = 1$. Now note that $(xy)^k - (zt)^k = 1$ can only happen in general for $k = 1$, since for $k > 1$, there are no consecutive perfect k-th powers. We conclude $P_1(x) = cx, P_2(x) = x/c, P_3(x) = ex, P_4(x) = x/e$ for some nonzero real numbers c, e.

19. Two players play the following game on a 100×100 board. The first player marks a free square, then the second player puts a 1×2 domino down covering two free squares, one of which is marked. This continues until one player is unable to move. The first player wins if the entire board is covered, otherwise the second player wins. Which player has a winning strategy?

    Solution: The first player has a winning strategy. Let us say a position is *stable* if every square below or to the right of a free square is free. Then we claim the first player can always ensure that on his turn, either the position is stable or there is a free square with exactly one free neighbor (or both).

    Let us label the square in the i-th row and j-th column as (i, j), with $(1, 1)$ in the top left. We call a free square a *corner* if is not below or to the right of another free square. Let (a_1, b_1), (a_2, b_2), $\ldots$, (a_k, b_k) be the corners from top to bottom.

    First notice that if (a, b) is a corner such that both $(a+1, b-1)$ and $(a-1, b+1)$ are nonfree (or off the board), then the first player may mark (a, b), and however the second player moves, the result will be a stable position. More generally, if $(a, b), (a+1, b-1), \cdots, (a+k, b-k)$ are corners and $(a-1, b+1)$ and $(a+k+1, b-k-1)$ are both nonfree or off the board, the first player can be sure to return to a stable position.

    To show this, first note that we cannot have both $a = 1$ and $b - k = 1$, or else the number of nonfree squares would be odd, which is impossible. Without loss of generality, assume that $b - k \neq 1$ is not the final corner. The first player now marks (a, b). If the second player covers (a, b) and $(a, b+1)$, the position is again stable. Otherwise, the first player marks $(a+1, b-1)$ and the second player is forced to cover it and $(a+2, b-1)$. Then the first player marks

$(a+2, b-2)$ and the second player is forced to cover it and $(a+3, b-2)$, and so on. After $(a+k, b-k)$ is marked, the result is a stable position. (Note that the assumption $b-k \neq 1$ ensures that the moves described do not cross the edge of the board.)

To finish the proof, we need to show that such a chain of corners must exist. Write the labels $(a_1, b_1), \ldots, (a_k, b_k)$ in a row, and join two adjacent labels by a segment if they are of the form $(a, b), (a+1, b-1)$. If two adjacent labels $(a, b), (a+i, b-j)$ are not joined by a segment, then either $i = 1$ or $j = 1$ but not both. If $i = 1$, draw an arrow between the labels pointing towards $(a + i, b - j)$; otherwise draw the arrow the other way. Also draw arrows pointing to (a_1, b_1) and (a_k, b_k). There is now one more chain of corners (joined by segments) than arrows, so some chain has two arrows pointing to it. That chain satisfies the condition above, so the first player can use it to create another stable position. Consequently, the first player can ensure victory.

20. Let BD be the bisector of angle B in triangle ABC. The circumcircle of triangle BDC meets AB at E, while the circumcircle of triangle ABD meets BC at F. Prove that $AE = CF$.

Solution: By power-of-a-point, $AE \cdot AB = AD \cdot AC$ and $CF \cdot CB = CD \cdot CA$, so $AE/CF = (AD/CD)(BC/AB)$. However, $AB/CB = AD/CD$ by the angle bisector theorem, so $AE = CF$.

21. A 10×10 table consists of positive integers such that for every five rows and five columns, the sum of the numbers at their intersections is even. Prove that all of the integers in the table are even.

Solution: We denote the first five entries in a row as the "head" of that row. We first show that the sum of each head is even. We are given that the sum of any five heads is even; by subtracting two such sums overlapping in four heads, we deduce that the sum of any two heads is even. Now subtracting two such relations from a sum of five heads, we determine that the sum of any head is even.

By a similar argument, the sum of any five entries in a row is even. By the same argument as above, we deduce that each entry is even.

22. Prove that there are no positive integers a and b such that for each pair p, q of distinct primes greater than 1000, the number $ap + bq$ is also prime.

Solution: Suppose a, b are so chosen, and let m be a prime greater than $a+b$. By Dirichlet's theorem, there exist infinitely many primes in any nonzero residue class modulo m; in particular, there exists a pair p, q such that $p \equiv b \,(\text{mod } m), q \equiv -a \,(\text{mod } m)$, giving $ap + bq$ is divisible by m, a contradiction.

23. In triangle ABC, the angle A is $60°$. A point O is taken inside the triangle such that $\angle AOB = \angle BOC = 120°$. The points D and E are the midpoints of sides AB and AC. Prove that the quadrilateral $ADOE$ is cyclic.

Solution: Since $\angle OBA = 60° - \angle OAB = \angle OAC$, the triangles OAB and OCA are similar, so there is a spiral similarity about O carrying OAB to OCA. This similarity preserves midpoints, so it carries D to E, and therefore $\angle AOD = \angle COE = 120° - \angle AOE$. We conclude $\angle DOE = 120°$ and so $ADOE$ is cyclic.

24. There are 2000 towns in a country, each pair of which is linked by a road. The Ministry of Reconstruction proposed all of the possible assignments of one-way traffic to each road. The Ministry of Transportation rejected each assignment that did not allow travel from any town to any other town. Prove that more of half of the assignments remained.

Solution: We will prove the same statement for $n \geq 6$ towns. First suppose $n = 6$. In this case there are 2^{15} assignments, and an assignment is rejected only if either one town has road to all of the others in the same direction, or if there are two sets of three towns, such that within each town the roads point in a circle, but all of the roads from one set to the other point in the same direction. There are $5 \cdot 2^{11}$ bad assignments of the first kind and $20 \cdot 8$ of the second kind, so the fraction of good assignments is at least $5/8$.

For $n \geq 6$, we claim that the fraction of good assignments is at least

$$\frac{5}{8}\prod_{i=6}^{n-1}\left(1-\frac{1}{2^{i-1}}\right).$$

We show this by induction on n: a good assignment on $n-1$ vertices can be extended to a good assignment on n vertices simply by avoiding having all edges from the last vertex pointing in the same direction, which occurs in 2 cases out of 2^{n-1}.

Now it suffices to show that the above expression is more than 1/2. In fact,

$$\begin{aligned}
\prod_{i=5}^{\infty}\left(1-\frac{1}{2^i}\right)^{-1} &\leq 1+\sum_{i=5}^{\infty}\frac{i-4}{2^i} \\
&= 1+\frac{1}{2^5}\sum_{i=0}^{\infty}\frac{i+1}{2^i} \\
&= 1+\frac{1}{2^5}\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{1}{2^i} \\
&= 1+\frac{1}{2^5}\sum_{i=0}^{\infty}\frac{1}{2^{i-1}} \\
&= 1+\frac{4}{2^5}=\frac{9}{8}
\end{aligned}$$

Thus the fraction of good assignments is at least $(5/8)(8/9) = 5/9 > 1/2$.

25. The positive integers m, n, m, n are written on a blackboard. A generalized Euclidean algorithm is applied to this quadruple as follows: if the numbers x, y, u, v appear on the board and $x > y$, then $x-y, y, u+v, v$ are written instead; otherwise $x, y-x, u, v+u$ are written instead. The algorithm stops when the numbers in the first pair become equal (they will equal the greatest common divisor of m and n). Prove that the arithmetic mean of the numbers in the second pair at that moment equals the least common multiple of m and n.

Solution: Note that $xv + yu$ does not change under the operation, so it remains equal to $2mn$ throughout. Thus when the first two numbers both equal $\gcd(m, n)$, the sum of the latter two is $2mn/\gcd(m, n) = 2\text{lcm}(m, n)$.

26. A set of geometric figures consists of red equilateral triangles and blue quadrilaterals with all angles greater than 80° and less than 100°. A convex polygon with all of its angles greater than 60° is assembled from the figures in the set. Prove that the number of (entirely) red sides of the polygon is a multiple of 3.

Solution: We first enumerate the ways to decompose various angles α into sums of 60° angles (T) and angles between 80° and 100° (Q):

$$\begin{array}{rl} 60^\circ < \alpha < 180^\circ & \alpha = T, 2T, T+Q, 2Q \\ \alpha = 180^\circ & \alpha = 3T, 2Q \\ \alpha = 360^\circ & \alpha = 6T, 3T+2Q, 4Q \end{array}$$

(The range for Q cannot be increased, since $3Q$ ranges from 240° to 300°; even including the endpoints would allow for additional combinations above.)

The set of all of the vertices of all of the polygons can be divided into three categories, namely those which lie in the interior, on an edge, or at a vertex of the large polygon. The above computation shows that the number of T angles at interior or edge vertices is a multiple of 3; since the total number is three times the number of triangles, we deduce that the number of T angles at vertices of the large polygon is also a multiple of 3.

Next note that every edge is entirely of one color, since we cannot have both a T and a Q at a 180° angle. Additionally, no vertex of the large polygon consists of more than two angles, and a T cannot occur by itself. All this means that the number of red sides is half the number of T angles at the vertices, which is a multiple of 3.

27. The positive integers $1, 2, \ldots, n^2$ are placed in some fashion in the squares of an $n \times n$ table. As each number is placed in a square, the sum of the numbers already placed in the row and column containing that square is written on a blackboard. Give an arrangement of the numbers that minimizes the sum of the numbers written on the blackboard.

Solution: Rather than describe the arrangement, we demonstrate it for $n = 4$:

$$\begin{array}{cccc} 1 & 5 & 9 & 13 \\ 14 & 2 & 6 & 10 \\ 11 & 15 & 3 & 7 \\ 8 & 12 & 16 & 4 \end{array}$$

Note that the sum of the numbers written may also be computed as the product of each number with the number of empty spaces in its row and column at the time it was placed.

We now simply note that the contribution from rows is at least that of the minimal arrangement, and analogously for columns. This is because we end up multiplying n numbers by each of $0, 1, \ldots, n-1$. By the rearrangement inequality, the total is minimizing by multiplying $1, \ldots, n$ by $n-1$, $n+1, \ldots, 2n$ by $n-2$, and so on.

3 1997 National Contests: Problems

3.1 Austria

1. Solve the system

$$\begin{aligned}(x-1)(y^2+6) &= y(x^2+1)\\(y-1)(x^2+6) &= x(y^2+1).\end{aligned}$$

2. Consider the sequence of positive integers which satisfies $a_n = a_{n-1}^2 + a_{n-2}^2 + a_{n-3}^2$ for all $n \geq 3$. Prove that if $a_k = 1997$ then $k \leq 3$.

3. Let k be a positive integer. The sequence a_n is defined by $a_1 = 1$, and a_n is the n-th positive integer greater than a_{n-1} which is congruent to n modulo k. Find a_n in closed form. What happens if $k = 2$?

4. Given a parallelogram $ABCD$, inscribe in the angle $\angle BAD$ a circle that lies entirely inside the parallelogram. Similarly, inscribe a circle in the angle $\angle BCD$ that lies entirely inside the parallelogram and such that the two circles are tangent. Find the locus of the tangency point of the circles, as the two circles vary.

3.2 Bulgaria

1. Find all real numbers m such that the equation

$$(x^2 - 2mx - 4(m^2+1))(x^2 - 4x - 2m(m^2+1)) = 0$$

has exactly three different roots.

2. Let ABC be an equilateral triangle with area 7 and let M, N be points on sides AB, AC, respectively, such that $AN = BM$. Denote by O the intersection of BN and CM. Assume that triangle BOC has area 2.

   (a) Prove that MB/AB equals either $1/3$ or $2/3$.

   (b) Find $\angle AOB$.

3. Let $f(x) = x^2 - 2ax - a^2 - 3/4$. Find all values of a such that $|f(x)| \le 1$ for all $x \in [0,1]$.

4. Let I and G be the incenter and centroid, respectively, of a triangle ABC with sides $AB = c$, $BC = a$, $CA = b$.

   (a) Prove that the area of triangle CIG equals $|a-b|r/6$, where r is the inradius of ABC.

   (b) If $a = c+1$ and $b = c-1$, prove that the lines IG and AB are parallel, and find the length of the segment IG.

5. Let $n \ge 4$ be an even integer and A a subset of $\{1, 2, \dots, n\}$. Consider the sums $e_1x_1 + e_2x_2 + e_3x_3$ such that:

   - $x_1, x_2, x_3 \in A$;
   - $e_1, e_2, e_3 \in \{-1, 0, 1\}$;
   - at least one of e_1, e_2, e_3 is nonzero;
   - if $x_i = x_j$, then $e_ie_j \ne -1$.

   The set A is *free* if all such sums are not divisible by n.

   (a) Find a free set of cardinality $\lfloor n/4 \rfloor$.

   (b) Prove that any set of cardinality $\lfloor n/4 \rfloor + 1$ is not free.

6. Find the least natural number a for which the equation

$$\cos^2 \pi(a-x) - 2\cos \pi(a-x) + \cos \frac{3\pi x}{2a} \cos\left(\frac{\pi x}{2a} + \frac{\pi}{3}\right) + 2 = 0$$

has a real root.

7. Let $ABCD$ be a trapezoid ($AB||CD$) and choose F on the segment AB such that $DF = CF$. Let E be the intersection of AC and BD, and let O_1, O_2 be the circumcenters of ADF, BCF. Prove that the lines EF and O_1O_2 are perpendicular.

8. Find all natural numbers n for which a convex n-gon can be divided into triangles by diagonals with disjoint interiors, such that each vertex of the n-gon is the endpoint of an even number of the diagonals.

9. For any real number b, let $f(b)$ denote the maximum of the function

$$\left|\sin x + \frac{2}{3+\sin x} + b\right|$$

over all $x \in \mathbb{R}$. Find the minimum of $f(b)$ over all $b \in \mathbb{R}$.

10. Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O denote the orthocenter and circumcenter of the triangle ABC. Prove that H, O, D are collinear.

11. For any natural number $n \geq 3$, let $m(n)$ denote the maximum number of points lying within or on the boundary of a regular n-gon of side length 1 such that the distance between any two of the points is greater than 1. Find all n such that $m(n) = n - 1$.

12. Find all natural numbers a, b, c such that the roots of the equations

$$\begin{aligned} x^2 - 2ax + b &= 0 \\ x^2 - 2bx + c &= 0 \\ x^2 - 2cx + a &= 0 \end{aligned}$$

are natural numbers.

13. Given a cyclic convex quadrilateral $ABCD$, let F be the intersection of AC and BD, and E the intersection of AD and BC. Let M, N be the midpoints of AB, CD. Prove that

$$\frac{MN}{EF} = \frac{1}{2}\left|\frac{AB}{CD} - \frac{CD}{AB}\right|.$$

14. Prove that the equation

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0$$

has no solutions in rational numbers.

15. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$f(x) = f\left(x^2 + \frac{1}{4}\right).$$

16. Two unit squares K_1, K_2 with centers M, N are situated in the plane so that $MN = 4$. Two sides of K_1 are parallel to the line MN, and one of the diagonals of K_2 lies on MN. Find the locus of the midpoint of XY as X, Y vary over the interior of K_1, K_2, respectively.

17. Find the number of nonempty subsets of $\{1, 2, \ldots, n\}$ which do not contain two consecutive numbers.

18. For any natural number $n \geq 2$, consider the polynomial

$$P_n(x) = \binom{n}{2} + \binom{n}{5}x + \binom{n}{8}x^2 + \cdots + \binom{n}{3k+2}x^k,$$

where $k = \lfloor \frac{n-2}{3} \rfloor$.

(a) Prove that $P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x)$.

(b) Find all integers a such that $3^{\lfloor (n-1)/2 \rfloor}$ divides $P_n(a^3)$ for all $n \geq 3$.

19. Let M be the centroid of triangle ABC.

(a) Prove that if the line AB is tangent to the circumcircle of the triangle AMC, then

$$\sin\angle CAM + \sin\angle CBM \leq \frac{2}{\sqrt{3}}.$$

(b) Prove the same inequality for an arbitrary triangle ABC.

20. Let m, n be natural numbers and $m + i = a_i b_i^2$ for $i = 1, 2, \ldots, n$, where a_i and b_i are natural numbers and a_i is squarefree. Find all values of n for which there exists m such that $a_1 + a_2 + \cdots + a_n = 12$.

21. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \le \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

22. Let ABC be a triangle and M, N the feet of the angle bisectors of B, C, respectively. Let D be the intersection of the ray MN with the circumcircle of ABC. Prove that

$$\frac{1}{BD} = \frac{1}{AD} + \frac{1}{CD}.$$

23. Let X be a set of cardinality $n + 1$ $(n \ge 2)$. The ordered n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ of distinct elements of X are called *separated* if there exist indices $i \ne j$ such that $a_i = b_j$. Find the maximal number of n-tuples such that any two of them are separated.

3.3 Canada

1. How many pairs (x, y) of positive integers with $x \leq y$ satisfy $\gcd(x, y) = 5!$ and $\operatorname{lcm}(x, y) = 50!$?

2. Given a finite number of closed intervals of length 1, whose union is the closed interval $[0, 50]$, prove that there exists a subset of the intervals, any two of whose members are disjoint, whose union has total length at least 25. (Two intervals with a common endpoint are not disjoint.)

3. Prove that
$$\frac{1}{1999} < \frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{1997}{1998} < \frac{1}{44}.$$

4. Let O be a point inside a parallelogram $ABCD$ such that $\angle AOB + \angle COD = \pi$. Prove that $\angle OBC = \angle ODC$.

5. Express the sum
$$\sum_{k=0}^{n} \frac{(-1)^k}{k^3 + 9k^2 + 26k + 24} \binom{n}{k}$$
in the form $p(n)/q(n)$, where p, q are polynomials with integer coefficients.

3.4 China

1. Let $x_1, x_2, \ldots, x_{1997}$ be real numbers satisfying the following conditions:

   (a) $-\frac{1}{\sqrt{3}} \le x_i \le \sqrt{3}$ for $i = 1, 2, \ldots, 1997$;

   (b) $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$.

   Determine the maximum value of $x_1^{12} + x_2^{12} + \cdots + x_{1997}^{1}2$.

2. Let $A_1B_1C_1D_1$ be a convex quadrilateral and P a point in its interior. Assume that the angles PA_1B_1 and PA_1D_1 are acute, and similarly for the other three vertices. Define A_k, B_k, C_k, D_k as the reflections of P across the lines $A_{k-1}B_{k-1}, B_{k-1}C_{k-1}, C_{k-1}D_{k-1}, D_{k-1}A_{k-1}$.

   (a) Of the quadrilaterals $A_kB_kC_kD_k$ for $k = 1, \ldots, 12$, which ones are necessarily similar to the 1997th quadrilateral?

   (b) Assume that the 1997th quadrilateral is cyclic. Which of the first 12 quadrilaterals must then be cyclic?

3. Show that there exist infinitely many positive integers n such that the numbers $1, 2, \ldots, 3n$ can be labeled

   $$a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n$$

   in some order so that the following conditions hold:

   (a) $a_1 + b_1 + c_1 = \cdots = a_n + b_n + c_n$ is a multiple of 6;

   (b) $a_1 + \cdots + a_n = b_1 + \cdots + b_n = c_1 + \cdots + c_n$ is also a multiple of 6.

4. Let $ABCD$ be a cyclic quadrilateral. The lines AB and CD meet at P, and the lines AD and BC meet at Q. Let E and F be the points where the tangents from Q meet the circumcircle of $ABCD$. Prove that points P, E, F are collinear.

5. Let $A = \{1, 2, \ldots, 17\}$ and for a function $f\colon A \to A$, denote $f^{[1]}(x) = f(x)$ and $f^{[k+1]}(x) = f(f^{[k]}(x))$ for $k \in \mathbb{N}$. Find the largest natural number M such that there exists a bijection $f : A \to A$ satisfying the following conditions:

(a) If $m < M$ and $1 \leq i \leq 16$, then

$$f^{[m]}(i+1) - f^{[m]}(i) \not\equiv \pm 1 \pmod{17}.$$

(b) For $1 \leq i \leq 16$,

$$f^{[M]}(i+1) - f^{[M]}(i) \equiv \pm 1 \pmod{17}.$$

(Here $f^{[k]}(18)$ is defined to equal $f^{[k]}(1)$.)

6. Let $a_1, a_2, \ldots,$ be nonnegative integers satisfying

$$a_{n+m} \leq a_n + a_m \quad (m, n \in \mathbb{N}).$$

Prove that

$$a_n \leq m a_1 + \left(\frac{n}{m} - 1\right) a_m$$

for all $n \geq m$.

3.5 Colombia

1. We are given an $m \times n$ grid and three colors. We wish to color each segment of the grid with one of the three colors so that each unit square has two sides of one color and two sides of a second color. How many such colorings are possible?

2. We play the following game with an equilateral triangle of $n(n+1)/2$ pennies (with n pennies on each side). Initially, all of the pennies are turned heads up. On each turn, we may turn over three pennies which are mutually adjacent; the goal is to make all of the pennies show tails. For which values of n can this be achieved?

3. Let $ABCD$ be a fixed square, and consider all squares $PQRS$ such that P and R lie on different sides of $ABCD$ and Q lies on a diagonal of $ABCD$. Determine all possible positions of the point S.

4. Prove that the set of positive integers can be partitioned into an infinite number of (disjoint) infinite sets $A_1, A_2, \ldots$ so that if x, y, z, w belong to A_k for some k, then $x - y$ and $z - w$ belong to the same set A_i (where i need not equal k) if and only if $x/y = z/w$.

3.6 Czech and Slovak Republics

1. Let ABC be a triangle with sides a, b, c and corresponding angles α, β, γ. Prove that the equality $\alpha = 3\beta$ implies the inequality $(a^2 - b^2)(a - b) = bc^2$, and determine whether the converse also holds.

2. Each side and diagonal of a regular n-gon ($n \geq 3$) is colored red or blue. One may choose a vertex and change the color of all of the segments emanating from that vertex, from red to blue and vice versa. Prove that no matter how the edges were colored initially, it is possible to make the number of blue segments at each vertex even. Prove also that the resulting coloring is uniquely determined by the initial coloring.

3. The tetrahedron $ABCD$ is divided into five convex polyhedra so that each face of $ABCD$ is a face of one of the polyhedra (no faces are divided), and the intersection of any two of the five polyhedra is either a common vertex, a common edge, or a common face. What is the smallest possible sum of the number of faces of the five polyhedra?

4. Show that there exists an increasing sequence $\{a_n\}_{n=1}^{\infty}$ of natural numbers such that for any $k \geq 0$, the sequence $\{k + a_n\}$ contains only finitely many primes.

5. For each natural number $n \geq 2$, determine the largest possible value of the expression

$$V_n = \sin x_1 \cos x_2 + \sin x_2 \cos x_3 + \cdots + \sin x_n \cos x_1,$$

   where $x_1, x_2, \ldots, x_n$ are arbitrary real numbers.

6. A parallelogram $ABCD$ is given such that triangle ABD is acute and $\angle BAD = \pi/4$. In the interior of the sides of the parallelogram, points K on AB, L on BC, M on CD, N on DA can be chosen in various ways so that $KLMN$ is a cyclic quadrilateral whose circumradius equals those of the triangles ANK and CLM. Find the locus of the intersection of the diagonals of all such quadrilaterals $KLMN$.

3.7 France

1. Each vertex of a regular 1997-gon is labeled with an integer, such that the sum of the integers is 1. Starting at some vertex, we write down the labels of the vertices reading counterclockwise around the polygon. Can we always choose the starting vertex so that the sum of the first k integers written down is positive for $k = 1, \dots, 1997$?

2. Find the maximum volume of a cylinder contained in the intersection of a sphere with center O radius R and a cone with vertex O meeting the sphere in a circle of radius r, having the same axis as the cone?

3. Find the maximum area of the orthogonal projection of a unit cube onto a plane.

4. Given a triangle ABC, let a, b, c denote the lengths of its sides and m, n, p the lengths of its medians. For every positive real α, let $\lambda(\alpha)$ be the real number satisfying

$$a^\alpha + b^\alpha + c^\alpha = \lambda(\alpha)^\alpha (m^\alpha + n^\alpha + p^\alpha).$$

   (a) Compute $\lambda(2)$.

   (b) Determine the limit of $\lambda(\alpha)$ as α tends to 0.

   (c) For which triangles ABC is $\lambda(\alpha)$ independent of α?

3.8 Germany

1. Determine all primes p for which the system

$$\begin{aligned} p+1 &= 2x^2 \\ p^2+1 &= 2y^2 \end{aligned}$$

has a solution in integers x, y.

2. A square S_a is inscribed in an acute triangle ABC by placing two vertices on side BC and one on each of AB and AC. Squares S_b and S_c are inscribed similarly. For which triangles ABC will S_a, S_b, S_c all be congruent?

3. In a park, 10000 trees have been placed in a square lattice. Determine the maximum number of trees that can be cut down so that from any stump, you cannot see any other stump. (Assume the trees have negligible radius compared to the distance between adjacent trees.)

4. In the circular segment AMB, the central angle $\angle AMB$ is less than 90°. FRom an arbitrary point on the arc AB one constructs the perpendiculars PC and PD onto MA and MB ($C \in MA$, $D \in MB$). Prove that the length of the segemnt CD does not depend on the position of P on the arc AB.

5. In a square $ABCD$ one constructs the four quarter circles having their respective centers at A, B, C and D and containing the two adjacent vertices. Inside $ABCD$ lie the four intersection points E, F, G and H, of these quarter circles, which form a smaller square $\mathcal{S}$. Let $\mathcal{C}$ be the circle tangent to all four quarter circles. Compare the areas of $\mathcal{S}$ and $\mathcal{C}$.

6. Denote by $u(k)$ the largest odd number that divides the natural number k. Prove that

$$\frac{1}{2^n} \cdot \sum_{k=1}^{2^n} \frac{u(k)}{k} \geq \frac{2}{3}.$$

7. Find all real solutions of the system of equations

$$x^3 = 2y - 1$$

$$\begin{aligned} y^3 &= 2z - 1 \\ z^3 &= 2x - 1 \end{aligned}$$

8. Define the functions

$$\begin{aligned} f(x) &= x^5 + 5x^4 + 5x^3 + 5x^2 + 1 \\ g(x) &= x^5 + 5x^4 + 3x^3 - 5x^2 - 1. \end{aligned}$$

Find all prime numbers p and for which there exists a natural number $0 \leq x < p$, such that both $f(x)$ and $g(x)$ are divisible by p, and for each such p, find all such x.

3.9 Greece

1. Let P be a point inside or on the sides of a square $ABCD$. Determine the minimum and maximum possible values of

$$f(P) = \angle ABP + \angle BCP + \angle CDP + \angle DAP.$$

2. Let $f : (0, \infty) \to \mathbb{R}$ be a function such that

   (a) f is strictly increasing;

   (b) $f(x) > -1/x$ for all $x > 0$;

   (c) $f(x)f(f(x) + 1/x) = 1$ for all $x > 0$.

   Find $f(1)$.

3. Find all integer solutions of

$$\frac{13}{x^2} + \frac{1996}{y^2} = \frac{z}{1997}.$$

4. Let P be a polynomial with integer coefficients having at least 13 distinct integer roots. Show that if $n \in \mathbb{Z}$ is not a root of P, then $|P(n)| \geq 7(6!)^2$, and give an example where equality is achieved.

3.10 Hungary

1. Each member of a committee ranks applicants A, B, C in some order. It is given that the majority of the committee ranks A higher than B, and also that the majority of the commitee ranks B higher than C. Does it follow that the majority of the committee ranks A higher than C?

2. Let a, b, c be the sides, m_a, m_b, m_c the lengths of the altitudes, and d_a, d_b, d_c the distances from the vertices to the orthocenter in an acute triangle. Prove that
$$m_a d_a + m_b d_b + m_c d_c = \frac{a^2 + b^2 + c^2}{2}.$$

3. Let R be the circumradius of triangle ABC, and let G and H be its centroid and orthocenter, respectively. Let F be the midpoint of GH. Show that $AF^2 + BF^2 + CF^2 = 3R^2$.

4. A box contains 4 white balls and 4 red balls, which we draw from the box in some order without replacement. Before each draw, we guess the color of the ball being drawn, always guessing the color more likely to occur (if one is more likely than the other). What is the expected number of correct guesses?

5. Find all solutions in integers of the equation
$$x^3 + (x + 1)^3 + (x + 2)^3 + \cdots + (x + 7)^3 = y^3.$$

6. We are given 1997 distinct positive integers, any 10 of which have the same least common multiple. Find the maximum possible number of pairwise coprime numbers among them.

7. Let AB and CD be nonintersecting chords of a circle, and let K be a point on CD. Construct (with straightedge and compass) a point P on the circle such that K is the midpoint of the intersection of the part of the segment CD lying inside triangle ABP.

8. We are given 111 unit vectors in the plane whose sum is zero. Show that there exist 55 of the vectors whose sum has length less than 1.

3.11 Iran

1. Suppose $w_1, \dots, w_k$ are distinct real numbers with nonzero sum. Prove that there exist integers $n_1, \dots, n_k$ such that $n_1 w_1 + \cdots + n_k w_k > 0$ and that for any permutation π of $\{1, \dots, k\}$ not equal to the identity, we have $n_1 w_{\pi(1)} + \cdots + n_k w_{\pi(k)} < 0$.

2. Suppose the point P varies along the arc BC of the circumcircle of triangle ABC, and let I_1, I_2 be the respective incenters of the triangles PAB, PAC. Prove that

   (a) the circumcircle of PI_1I_2 passes through a fixed point;

   (b) the circle with diameter I_1I_2 passes through a fixed point;

   (c) the midpoint of I_1I_2 lies on a fixed circle.

3. Suppose $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing continuous function such that for all $x, y \in \mathbb{R}^+$,

$$f(x+y) + f(f(x)+f(y)) = f(f(x+f(y))) + f(y+f(x)).$$

   Prove that $f(f(x)) = x$.

4. Let A be a matrix of zeroes and ones which is symmetric ($A_{ij} = A_{ji}$ for all i, j) such that $A_{ii} = 1$ for all i. Show that there exists a subset of the rows whose sum is a vector all of whose components are odd.

3.12 Ireland

1. Find all pairs (x, y) of integers such that $1 + 1996x + 1998y = xy$.

2. Let ABC be an equilateral triangle. For M inside the triangle, let D, E, F be the feet of the perpendiculars from M to BC, CA, AB, respectively. Find the locus of points M such that $\angle FDE = \pi/2$.

3. Find all polynomials $p(x)$ such that for all x,
$$(x - 16)p(2x) = 16(x - 1)p(x).$$

4. Let a, b, c be nonnegative real numbers such that $a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq abc$.

5. Let $S = \{3, 5, 7, \ldots\}$. For $x \in S$, let $\delta(x)$ be the unique integer such that $2^{\delta(x)} < x < 2^{\delta(x)+1}$. For $a, b \in S$, define
$$a * b = 2^{\delta(a)-1}(b - 3) + a.$$
   (a) Prove that if $a, b \in S$, then $a * b \in S$.
   (b) Prove that if $a, b, c \in S$, then $(a * b) * c = a * (b * c)$.

6. Let $ABCD$ be a convex quadrilateral with an inscribed circle. If $\angle A = \angle B = 2\pi/3$, $\angle D = \pi/2$ and $BC = 1$, find the length of AD.

7. Let A be a subset of $\{0, 1, \ldots, 1997\}$ containing more than 1000 elements. Prove that A contains either a power of 2, or two distinct integers whose sum is a power of 2.

8. Determine the number of natural numbers n satisfying the following conditions:
   (a) The decimal expansion of n contains 1000 digits.
   (b) All of the digits of n are odd.
   (c) The absolute value of the difference between any two adjacent digits of n is 2.

3.13 Italy

1. A rectangular strip of paper 3 centimeters wide is folded exactly once. What is the least possible area of the region where the paper covers itself?

2. Let f be a real-valued function such that for any real x,

   (a) $f(10 + x) = f(10 - x)$;

   (b) $f(20 + x) = -f(20 - x)$.

   Prove that f is odd ($f(-x) = -f(x)$) and periodic (there exists $T > 0$ such that $f(x + T) = f(x)$).

3. The positive quadrant of a coordinate plane is divided into unit squares by lattice lines. Is it possible to color some of the unit squares so as to satisfy the following conditions:

   (a) each square with one vertex at the origin and sides parallel to the axes contains more colored than uncolored squares;

   (b) each line parallel to the angle bisector of the quadrant at the origin passes through only finitely many colored squares?

4. Let $ABCD$ be a tetrahedron. Let a be the length of AB and let S be the area of the projection of the tetrahedron onto a plane perpendicular to AB. Determine the volume of the tetrahedron in terms of a and S.

5. Let X be the set of natural numbers whose decimal representations have no repeated digits. For $n \in X$, let A_n be the set of numbers whose digits are a permutation of the digits of n, and let d_n be the greatest common divisor of the numbers in A_n. Find the largest possible value of d_n.

3.14 Japan

1. Prove that among any ten points located in a circle of diameter 5, there exist two at distance less than 2 from each other.

2. Let a, b, c be positive integers. Prove the inequality

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5},$$

   and determine when equality holds.

3. Let G be a graph with 9 vertices. Suppose given any five points of G, there exist at least 2 edges with both endpoints among the five points. What is the minimum possible number of edges in G?

4. Let A, B, C, D be four points in space not lying in a plane. Suppose $AX + BX + CX + DX$ is minimized at a point $X = X_0$ distinct from A, B, C, D. Prove that $\angle AX_0B = \angle CX_0D$.

5. Let n be a positive integer. Show that one can assign to each vertex of a 2^n-gon one of the letters A or B such that the sequences of n letters obtained by starting at a vertex and reading counterclockwise are all distinct.

3.15 Korea

1. Show that among any four points contained in a unit circle, there exist two whose distance is at most $\sqrt{2}$.

2. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function satisfying

   (a) For every $n \in \mathbb{N}$, $f(n + f(n)) = f(n)$.

   (b) For some $n_0 \in \mathbb{N}$, $f(n_0) = 1$.

   Show that $f(n) = 1$ for all $n \in \mathbb{N}$.

3. Express $\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor$ in terms of n and $a = \lfloor \sqrt{n} \rfloor$.

4. Let C be a circle touching the edges of an angle $\angle XOY$, and let C_1 be the circle touching the same edges and passing through the center of C. Let A be the second endpoint of the diameter of C_1 passing through the center of C, and let B be the intersection of this diameter with C. Prove that the circle centered at A passing through B touches the edges of $\angle XOY$.

5. Find all integers x, y, z satisfying $x^2 + y^2 + z^2 - 2xyz = 0$.

6. Find the smallest integer k such that there exist two sequences $\{a_i\}$, $\{b_i\}$ $(i = 1, \ldots, k)$ such that

   (a) For $i = 1, \ldots, k$, $a_i, b_i \in \{1, 1996, 1996^2, \ldots\}$.

   (b) For $i = 1, \ldots, k$, $a_i \neq b_i$.

   (c) For $i = 1, \ldots, k - 1$, $a_i \leq a_{i+1}$ and $b_i \leq b_{i+1}$.

   (d) $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i$.

7. Let A_n be the set of all real numbers of the form $1 + \frac{\alpha_1}{\sqrt{2}} + \frac{\alpha_2}{(\sqrt{2})^2} + \cdots + \frac{\alpha_n}{(\sqrt{2})^n}$, where $\alpha_j \in \{-1, 1\}$ for each j. Find the number of elements of A_n, and find the sum of all products of two distinct elements of A_n.

8. In an acute triangle ABC with $AB \neq AC$, let V be the intersection of the angle bisector of A with BC, and let D be the foot of the perpendicular from A to BC. If E and F are the intersections of the circumcircle of AVD with CA and AB, respectively, show that the lines AD, BE, CF concur.

9. A word is a sequence of 8 digits, each equal to 0 or 1. Let x and y be two words differing in exactly three places. Show that the number of words differing from each of x and y in at least five places is 188.

10. Find all pairs of functions $f, g\colon \mathbb{R} \to \mathbb{R}$ such that

    (a) if $x < y$, then $f(x) < f(y)$;

    (b) for all $x, y \in \mathbb{R}$, $f(xy) = g(y)f(x) + f(y)$.

11. Let $a_1, \ldots, a_n$ be positive numbers, and define

$$\begin{aligned} A &= \frac{a_1 + \cdots + a_n}{n} \\ G &= (a_1 \cdots a_n)^{1/n} \\ H &= \frac{n}{a_1^{-1} + \cdots + a_n^{-1}}. \end{aligned}$$

    (a) If n is even, show that $\frac{A}{H} \le -1 + \left(\frac{A}{G}\right)^n$.

    (b) If n is odd, show that $\frac{A}{H} \le -\frac{n-2}{n} + \frac{2(n-1)}{n}\left(\frac{A}{G}\right)^n$.

12. Let $p_1, \ldots, p_r$ be distinct primes, and let $n_1, \ldots, n_r$ be arbitrary natural numbers. Prove that the number of pairs (x, y) of integers satisfying $x^3 + y^3 = p_1^{n_1} \cdots p_r^{n_r}$ is at most 2^{r-1}.

3.16 Poland

1. The positive integers $x_1, \ldots, x_7$ satisfy the conditions

$$x_6 = 144, \qquad x_{n+3} = x_{n+2}(x_{n+1} + x_n) \qquad n = 1, 2, 3, 4.$$

Compute x_7.

2. Solve the following system of equations in real numbers x, y, z:

$$\begin{aligned} 3(x^2 + y^2 + z^2) &= 1 \\ x^2y^2 + y^2z^2 + z^2x^2 &= xyz(x + y + z)^3. \end{aligned}$$

3. In a tetrahedron $ABCD$, the medians of the faces ABD, ACD, BCD from D make equal angles with the corresponding edges AB, AC, BC. Prove that each of these faces has area less than the sum of the areas of the other two faces.

4. The sequence $a_1, a_2, \ldots$ is defined by

$$a_1 = 0, \qquad a_n = a_{\lfloor n/2 \rfloor} + (-1)^{n(n+1)/2} \qquad n > 1.$$

For every integer $k \geq 0$, find the number of n such that

$$2^k \leq n < 2^{k+1} \quad \text{and} \quad a_n = 0.$$

5. Given a convex pentagon $ABCDE$ with $DC = DE$ and $\angle BCD = \angle DEA = \pi/2$, let F be the point on segment AB such that $AF/BF = AE/BC$. Show that

$$\angle FCE = \angle FDE \quad \text{and} \quad \angle FEC = \angle BDC.$$

6. Consider n points ($n \geq 2$) on a unit circle. Show that at most $n^2/3$ of the segments with endpoints among the n chosen points have length greater than $\sqrt{2}$.

3.17 Romania

1. In the plane are given a line Δ and three circles tangent to Δ and externally tangent to each other. Prove that the triangle formed by the centers of the circles is obtuse, and find all possible measures of the obtuse angle.

2. Determine all sets A of nine positive integers such that for any $n \geq 500$, there exists a subset B of A, the sum of whose elements is n.

3. Let $n \geq 4$ be an integer and M a set of n points in the plane, no three collinear and not all lying on a circle. Find all functions $f\colon M \to \mathbb{R}$ such that for any circle C containing at least three points of M,
$$\sum_{P \in M \cap C} f(P) = 0.$$

4. Let ABC be a triangle, D a point on side BC and ω the circumcircle of ABC. Show that the circles tangent to ω, AD, BD and to ω, AD, DC, respectively, are tangent to each other if and only if $\angle BAD = \angle CAD$.

5. Let $VA_1 \cdots A_n$ be a pyramid with $n \geq 4$. A plane Π intersects the edges $VA_1, \ldots, VA_n$ at $B_1, \ldots, B_n$, respectively. Suppose that the polygons $A_1 \cdots A_n$ and $B_1 \cdots B_n$ are similar. Prove that Π is parallel to the base of the pyramid.

6. Let A be the set of positive integers representable in the form a^2+2b^2 for integers a, b with $b \neq 0$. Show that if $p^2 \in A$ for a prime p, then $p \in A$.

7. Let $p \geq 5$ be a prime and choose $k \in \{0, \ldots, p-1\}$. Find the maximum length of an arithmetic progression, none of whose elements contain the digit k when written in base p.

8. Let p, q, r be distinct prime numbers and let A be the set
$$A = \{p^a q^b r^c : 0 \leq a, b, c \leq 5\}.$$
Find the smallest integer n such that any n-element subset of A contains two distinct elements x, y such that x divides y.

9. Let $ABCDEF$ be a convex hexagon. Let P, Q, R be the intersections of the lines AB and EF, EF and CD, CD and AB, respectively. Let S, T, U be the intersections of the lines BC and DE, DE and FA, FA and BC, respectively. Show that if $AB/PR = CD/RQ = EF/QP$, then $BC/US = DE/ST = FA/TU$.

10. Let P be the set of points in the plane and D the set of lines ithe plane. Determine whether there exists a bijective function $f : P \to D$ such that for any three collinear points A, B, C, the lines $f(A), f(B), f(C)$ are either parallel or concurrent.

11. Find all functions $f : \mathbb{R} \to [0, \infty)$ such that for all $x, y \in \mathbb{R}$,
$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy).$$

12. Let $n \geq 2$ be an integer and $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$ be a polynomial with positive integer coefficients. Suppose that $a_k = a_{n-k}$ for $k = 1, 2, \ldots, n-1$. Prove that there exist infinitely many pairs x, y of positive integers such that $x|P(y)$ and $y|P(x)$.

13. Let $P(x), Q(x)$ be monic irreducible polynomials over the rational numbers. Suppose P and Q have respective roots α and β such that $\alpha + \beta$ is rational. Prove that the polynomial $P(x)^2 - Q(x)^2$ has a rational root.

14. Let $a > 1$ be an integer. Show that the set
$$\{a^2 + a - 1, a^3 + a^2 - 1, \ldots\}$$
contains an infinite subset, any two members of which are relatively prime.

15. Find the number of ways to color the vertices of a regular dodecagon in two colors so that no set of vertices of a single color form a regular polygon.

16. Let Γ be a circle and AB a line not meeting Γ. For any point $P \in \Gamma$, let P' be the second intersection of the line AP with Γ and let $f(P)$ be the second intersection of the line BP' with Γ. Given a point P_0, define the sequence $P_{n+1} = f(P_n)$ for $n \geq 0$. Show that if a positive integer k satisfies $P_0 = P_k$ for a single choice of P_0, then $P_0 = P_k$ for all choices of P_0.

3.18 Russia

1. Show that the numbers from 1 to 16 can be written in a line, but not in a circle, so that the sum of any two adjacent numbers is a perfect square.

2. On equal sides AB and BC of an equilateral triangle ABC are chosen points D and K, and on side AC are chosen points E and M, so that $DA + AE = KC + CM = AB$. Show that the angle between the lines DM and KE equals $\pi/3$.

3. A company has 50000 employees. For each employee, the sum of the numbers of his immediate superiors and of his immediate inferiors is 7. On Monday, each worker issues an order and gives copies of it to each of his immediate inferiors (if he has any). Each day thereafter, each worker takes all of the orders he received on the previous day and either gives copies of them to all of his immediate inferiors if he has any, or otherwise carries them out himself. It turns out that on Friday, no orders are given. Show that there are at least 97 employees who have no immediate superiors.

4. The sides of the acute triangle ABC are diagonals of the squares K_1, K_2, K_3. Prove that the area of ABC is covered by the three squares.

5. The numbers from 1 to 37 are written in a line so that each number divides the sum of the previous numbers. If the first number is 37 and the second number is 1, what is the third number?

6. Find all paris of prime numbers p, q such that $p^3 - q^5 = (p+q)^2$.

7. (a) In Mexico City, to restrict traffic flow, for each private car are designated two days of the week on which that car cannot be driven on the streets of the city. A family needs to have use of at least 10 cars each day. What is the smallest number of cars they must possess, if they may choose the restricted days for each car?

   (b) The law is changed to restrict each car only one day per week, but the police get to choose the days. The family bribes the police so that for each car, they will restrict one of two days chosen by the family. Now what is the smallest number of cars the family needs to have access to 10 cars each day?

8. A regular 1997-gon is divided by nonintersecting diagonals into triangles. Prove that at least one of the triangles is acute.

9. On a chalkboard are written the numbers from 1 to 1000. Two players take turns erasing a number from the board. The game ends when two numbers remain: the first player wins if the sum of these numbers is divisible by 3, the second player wins otherwise. Which player has a winning strategy?

10. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 11/2 times any other group.

11. In Robotland, a finite number of (finite) sequences of digits are forbidden. It is known that there exists an infinite decimal fraction, not containing any forbidden sequences. Show that there exists an infinite periodic decimal fraction, not containing any forbidden sequences.

12. (a) A collection of 1997 numbers has the property that if each number is subtracted from the sum of the remaining numbers, the same collection of numbers is obtained. Prove that the product of the numbers is 0.

    (b) A collection of 100 numbers has the same property. Prove that the product of the numbers is positive.

13. Given triangle ABC, let A_1, B_1, C_1 be the midpoints of the broken lines CAB, ABC, BCA, respectively. Let l_A, l_B, l_C be the respective lines through A_1, B_1, C_1 parallel to the angle bisectors of A, B, C. Show that l_A, l_B, l_C are concurrent.

14. The MK-97 calculator can perform the following three operations on numbers in its memory:

    (a) Determine whether two chosen numbers are equal.

    (b) Add two chosen numbers together.

    (c) For chosen numbers a and b, find the real roots of $x^2 + ax + b$, or announce that no real roots exist.

    The results of each operation are accumulated in memory. Initially the memory contains a single number x. How can one determine, using the MK-97, whether x is equal to 1?

15. The circles S_1 and S_2 intersect at M and N. Show that if vertices A and C of a rectangle $ABCD$ lie on S_1 while vertices B and D lie on S_2, then the intersection of the diagonals of the rectangle lies on the line MN.

16. For natural numbers m, n, show that $2^n - 1$ divides $(2^m - 1)^2$ if and only if n divides $m(2^m - 1)$.

17. Can three faces of a cube of side length 4 be covered with 16 1×3 rectangles?

18. The vertices of triangle ABC lie inside a square K. Show that if the triangle is rotated $180°$ about its centroids, at least one vertex remains inside the square.

19. Let $S(N)$ denote the sum of the digits of the natural number N. Show that there exist infinitely many natural numbers n such that $S(3^n) \geq S(3^{n+1})$.

20. The members of Congress form various overlapping factions such that given any two (not necessarily distinct) factions A and B, the complement of $A \cup B$ is also a faction. Show that for any two factions A and B, $A \cup B$ is also a faction.

21. Show that if $1 < a < b < c$, then

$$\log_a(\log_a b) + \log_b(\log_b c) + \log_c(\log_c a) > 0.$$

22. Do there exist pyramids, one with a triangular base and one with a convex n-sided base ($n \geq 4$), such that the solid angles of the triangular pyramid are congruent to four of the solid angles of the n-sided pyramid?

23. For which α does there exist a nonconstant function $f \colon \mathbb{R} \to \mathbb{R}$ such that

$$f(\alpha(x + y)) = f(x) + f(y)?$$

24. Let $P(x)$ be a quadratic polynomial with nonnegative coefficients. Show that for any real numbers x and y, we have the inequality

$$P(xy)^2 \leq P(x^2)P(y^2).$$

25. Given a convex polygon M invariant under a $90°$ rotation, show that there exist two circles, the ratio of whose radii is $\sqrt{2}$, one containing M and the other contained in M.

26. (a) The Judgment of the Council of Sages proceeds as follows: the king arranges the sages in a line and places either a white hat or a black hat on each sage's head. Each sage can see the color of the hats of the sages in front of him, but not of his own hat or of the hats of the sages behind him. Then one by one (in an order of their choosing), each sage guesses a color. Afterward, the king executes those sages who did not correctly guess the color of their own hat.

    The day before, the Council meets and decides to minimize the number of executions. What is the smallest number of sages guaranteed to survive in this case?

    (b) The king decides to use three colors of hats: white, black and red. Now what is the smallest number of sages guaranteed to survive?

27. The lateral sides of a box with base $a \times b$ and height c (where a, b, c are natural numbers) are completely covered without overlap by rectangles whose edges are parallel to the edges of the box, each containing an odd number of unit squares. Prove that if c is odd, then the number of rectangles covering lateral edges of the box is even.

28. Do there exist real numbers b and c such that each of the equations $x^2+bx+c=0$ and $2x^2+(b+1)x+c+1=0$ have two integer roots?

29. A class consists of 33 students. Each student is asked how many other students in the class have his first name, and how many have his last name. It turns out that each number from 0 to 10 occurs among the answers. Show that there are two students in the class with the same first and last name.

30. The incircle of triangle ABC touches sides AB, BC, CA at M, N, K, respectively. The line through A parallel to NK meets MN at D. The line through A parallel to MN meets NK at E. Show that the line DE bisects sides AB and AC of triangle ABC.

31. The numbers from 1 to 100 are arranged in a 10×10 table so that no two adjacent numbers sum to S. Find the smallest value of S for which this is possible.

32. Find all integer solutions of the equation

$$(x^2 - y^2)^2 = 1 + 16y.$$

33. An $n \times n$ square grid ($n \geq 3$) is rolled into a cylinder. Some of the cells are then colored black. Show that there exist two parallel lines (horizontal, vertical or diagonal) of cells containing the same number of black cells.

34. Two circles intersect at A and B. A line through A meets the first circle again at C and the second circle again at D. Let M and N be the midpoints of the arcs BC and BD not containing A, and let K be the midpoint of the segment CD. Show that $\angle MKN = \pi/2$. (You may assume that C and D lie on opposite sides of A.)

35. A polygon can be divided into 100 rectangles, but not into 99. Prove that it cannot be divided into 100 triangles.

36. Do there exist two quadratic trinomials $ax^2 + bx + c$ and $(a+1)x^2 + (b+1)x + (c+1)$ with integer coefficients, both of which have two integer roots?

37. A circle centered at O and inscribed in triangle ABC meets sides AC, AB, BC at K, M, N, respectively. The median BB_1 of the triangle meets MN at D. Show that O, D, K are collinear.

38. Find all triples m, n, l of natural numbers such that

$$m + n = \gcd(m, n)^2, m + l = \gcd(m, l)^2, n + l = \gcd(n, l)^2.$$

39. On an infinite (in both directions) strip of squares, indexed by the natural numbers, are placed several stones (more than one may be placed on a single square). We perform a sequence of moves of one of the following types:

    (a) Remove one stone from each of the squares $n - 1$ and n and place one stone on square $n + 1$.

    (b) Remove two stones from square n and place one stone on each of the squares $n + 1$, $n - 2$.

    Prove that any sequence of such moves will lead to a position in which no further moves can be made, and moreover that this position is independent of the sequence of moves.

40. An $n \times n \times n$ cube is divided into unit cubes. We are given a closed non-self-intersecting polygon (in space), each of whose sides joins the centers of two unit cubes sharing a common face. The faces of unit cubes which intersect the polygon are said to be distinguished. Prove that the edges of the unit cubes may be colored in two colors so that each distinguished face has an odd number of edges of each color, while each nondistinguished face has an even number of edges of each color.

41. Of the quadratic trinomials $x^2 + px + q$ where p, q are integers and $1 \leq p, q \leq 1997$, which are there more of: those having integer roots or those not having real roots?

42. We are given a polygon, a line l and a point P on l in general position: all lines containing a side of the polygon meet l at distinct points differing from P. We mark each vertex of the polygon whose sides both meet the line l at points differing from P. Show that P lies inside the polygon if and only if for each choice of l there are an odd number of marked vertices.

43. A sphere inscribed in a tetrahedron touches one face at the intersection of its angle bisectors, a second face at the intersection of its altitudes, and a third face at the intersection of its medians. Show that the tetrahedron is regular.

44. In an $m \times n$ rectangular grid, where m and n are odd integers, 1×2 dominoes are initially placed so as to exactly cover all but one of the 1×1 squares at one corner of the grid. It is permitted to slide a domino towards the empty square, thus exposing another square. Show that by a sequence of such moves, we can move the empty square to any corner of the rectangle.

3.19 South Africa

1. From an initial triangle $A_0B_0C_0$ a sequence $A_1B_1C_1, A_2B_2C_2, \dots$ is formed such that at each stage, $A_{k+1}, B_{k+1}, C_{k+1}$ are the points where the incircle of $A_kB_kC_k$ touches the sides B_kC_k, C_kA_k, A_kB_k, respectively.

   (a) Express $\angle A_{k+1}B_{k+1}C_{k+1}$ in terms of $\angle A_kB_kC_k$.

   (b) Deduce that as $k \to \infty$, $\angle A_kB_kC_k \to 60°$.

2. Find all natural numbers with the property that, when the first digit is moved to the end, the resulting number is $3\frac{1}{2}$ times the original one.

3. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ which satisfy $f(m + f(n)) = f(m) + n$ for all $m, n \in \mathbb{Z}$.

4. A circle and a point P above the circle lie in a vertical plane. A particle moves along a straight line from P to a point Q on the circle under the influence of gravity. That is, the distance traveled from P in time t equals $\frac{1}{2}gt^2 \sin\alpha$, where g is a constant and α is the angle between PQ and the horizontal. Describe (geometrically) the point Q for which the time taken to move from P to Q is minimized.

5. Six points are joined pairwise by red or blue segments. Must there exist a closed path consisting of four of the segments, all of the same color?

3.20 Spain

1. Calculate the sum of the squares of the first 100 terms of an arithmetic progression, given that the sum of the first 100 terms is -1 and that the sum of the second, fourth, ..., and the hundredth terms is 1.

2. Let A be a set of 16 lattice points forming a square with 4 points on a side. Find the maximum number of points of A no three of which form an isosceles right triangle.

3. For each parabola $y = x^2 + px + q$ meeting the coordinate axes in three distinct points, a circle through these points is drawn. Show that all of the circles pass through a single point.

4. Let p be a prime number. Find all $k \in \mathbb{Z}$ such that $\sqrt{k^2 - pk}$ is a positive integer.

5. Show that in any convex quadrilateral of area 1, the sum of the lengths of the sides and diagonals is at least $2(2 + \sqrt{2})$.

6. The exact quantity of gas needed for a car to complete a single loop around a track is distributed among n containers placed along the track. Prove that there exists a position starting at which the car, beginning with an empty tank of gas, can complete a single loop around the track without running out of gas. (Assume the car can hold unlimited quantities of gas.)

3.21 Taiwan

1. Let a be a rational number, b, c, d be real numbers, and $f : \mathbb{R} \to [-1, 1]$ a function satisfying

$$f(x + a + b) - f(x + b) = c\lfloor x + 2a + \lfloor x \rfloor - 2\lfloor x + a \rfloor - \lfloor b \rfloor \rfloor + d$$

for each $x \in \mathbb{R}$. Show that f is periodic, that is, there exists $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$.

2. Let AB be a given line segment. Find all possible points C in the plane such that in the triangle ABC, the altitude from A and the median from B have the same length.

3. Let $n \geq 3$ be an integer, and suppose that the sequence $a_1, a_2, \ldots, a_n$ satisfies $a_{i-1} + a_{i+1} = k_i a_i$ for some sequence $k_1, k_2, \ldots, k_n$ of positive integers. (Here $a_0 = a_n$ and $a_{n+1} = a_1$.) Show that

$$2n \leq k_1 + k_2 + \cdots + k_n \leq 3n.$$

4. Let $k = 2^{2^n} + 1$ for some positive integer n. Show that k is a prime if and only if k is a factor of $3^{(k-1)/2} + 1$.

5. Let $ABCD$ be a tetrahedron. Show that

   (a) If $AB = CD$, $AD = BC$, $AC = BD$, then the triangles ABC, ACD, ABD, BCD are acute;

   (b) If ABC, ACD, ABD, BCD have the same area, then $AB = CD, AD = BC, AC = BD$.

6. Let X be the set of integers of the form

$$a_{2k}10^{2k} + a_{2k-2}10^{2k-2} + \cdots + a_2 10^2 + a_0,$$

where k is a nonnegative integer and $a_{2i} \in \{1, 2, \ldots, 9\}$ for $i = 0, 2, \ldots, 2k$. Show that every integer of the form $2^p 3^q$, for p and q nonnegative integers, divides some element of X.

7. Determine all positive integers k for which there exists a function $f : \mathbb{N} \to \mathbb{Z}$ such that

   (a) $f(1997) = 1998$;

(b) for all $a, b \in \mathbb{N}$, $f(ab) = f(a) + f(b) + kf(\gcd(a, b))$.

8. Let ABC be an acute triangle with circumcenter O and circumradius R. Let AO meet the circumcircle of OBC again at D, BO meet the circumcircle of OCA again at E, and CO meet the circumcircle of OAB again at F. Show that $OD \cdot OE \cdot OF \geq 8R^3$.

9. For $n \geq k \geq 3$, let $X = \{1, 2, \ldots, n\}$ and let F_k be a family of k-element subsets of X such that any two subsets in F_k have at most $k - 2$ common elements. Show that there exists a subset M_k of X with at least $\lfloor \log_2 n \rfloor + 1$ elements containing no subset in F_k.

3.22 Turkey

1. In a triangle ABC with a right angle at A, let H denote the foot of the altitude from A. Show that the sum of the inradii of the triangles ABC, ABH, ACH equals AH.

2. The sequences $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are defined as follows: $a_1 = \alpha$, $b_1 = \beta$, $a_{n+1} = \alpha a_n - \beta b_n$, $b_{n+1} = \beta a_n + \alpha b_n$ for all $n \geq 1$. How many pairs (α, β) of real numbers are there such that $a_{1997} = b_1$ and $b_{1997} = a_1$?

3. In a soccer league, when a player moves from a team X with x players to a team Y with y players, the federation receives $y - x$ million dollars from Y if $y \geq x$, but pays $x - y$ million dollars to X if $x > y$. A player may move as often as he wishes during a season. The league consists of 18 teams, each of which begins a certain season with 20 players. At the end of the season, 12 teams end up with 20 players, while the other 6 end up with $16, 16, 21, 22, 22, 23$ players. What is the maximum amount the federation could have earned during the season?

4. The edge AE of a convex pentagon $ABCDE$ with vertices on a unit circle passes through the center of the circle. If $AB = a$, $BC = b$, $CD = c$, $DE = d$ and $ab = cd = 1/4$, compute $AC + CE$ in terms of a, b, c, d.

5. Prove that for each prime $p \geq 7$, there exists a positive integer n and integers $x_1, \dots, x_n, y_1, \dots, y_n$ not divisible by p such that
$$\begin{aligned} x_1^2 + y_1^2 &\equiv x_2^2 \pmod{p} \\ x_2^2 + y_2^2 &\equiv x_3^2 \pmod{p} \\ &\vdots \\ x_n^2 + y_n^2 &\equiv x_1^2 \pmod{p}. \end{aligned}$$

6. Given an integer $n \geq 2$, find the minimal value of
$$\frac{x_1^5}{x_2 + x_3 + \cdots + x_n} + \frac{x_2^5}{x_3 + \cdots + x_n + x_1} + \cdots + \frac{x_n^5}{x_1 + \cdots + x_{n-1}}$$
for positive real numbers $x_1, \dots, x_n$ subject to the condition $x_1^2 + \cdots + x_n^2 = 1$.

3.23 Ukraine

1. A rectangular grid is colored in checkerboard fashion, and each cell contains an integer. It is given that the sum of the numbers in each row and the sum of the numbers in each column is even. Prove that the sum of all numbers in black cells is even.

2. Find all solutions in real numbers to the following system of equations:

$$\begin{aligned} x_1 + x_2 + \cdots + x_{1997} &= 1997 \\ x_1^4 + x_2^4 + \cdots + x_4^{1997} &= x_1^3 + x_2^3 + \cdots + x_{1997}^3. \end{aligned}$$

3. Let $d(n)$ denote the greatest odd divisor of the natural number n. We define the function $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(2n-1) = 2^n$ and $f(2n) = n + \frac{2n}{d(n)}$ for all $n \in \mathbb{N}$. Find all k such that $f(f(\cdots f(1)\cdots)) = 1997$, where f is iterated k times.

4. Two regular pentagons $ABCDE$ and $AEKPL$ are situated in space so that $\angle DAK = 60^\circ$. Prove that the planes ACK and BAL are perpendicular.

5. The equation $ax^3 + bx^2 + cx + d = 0$ is known to have three distinct real roots. How many real roots are there of the equation

$$4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2?$$

6. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Find all functions $f\colon \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$:

   (a) $f(x+1) = f(x) + 1$;

   (b) $f(x^2) = f(x)^2$.

7. Find the smallest integer n such that among any n integers, there exist 18 integers whose sum is divisible by 18.

8. Points K, L, M, N lie on the edges AB, BC, CD, DA of a (not necessarily right) parallelepiped $ABCDA_1B_1C_1D_1$. Prove that the centers of the circumscribed spheres of the tetrahedra A_1AKN, B_1BKL, C_1CLM, D_1DMN are the vertices of a parallelogram.

3.24 United Kingdom

1. (a) Let M and N be two 9-digit positive integers with the property that if any one digit of M is replaced by the digit of N in the corresponding place, the resulting integer is a multiple of 7. Prove that any number obtained by replacing a digit of N with the corresponding digit of M is also a multiple of 7.

   (b) Find an integer $d > 9$ such that the above result remains true when M and N are two d-digit positive integers.

2. In acute triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

3. Find the number of polynomials of degree 5 with distinct coefficients from the set $\{1, 2, \ldots, 9\}$ that are divisible by $x^2 - x + 1$.

4. The set $S = \{1/r : r = 1, 2, 3, \ldots\}$ of reciprocals of the positive integers contains arithmetic progressions of various lengths. For instance, $1/20, 1/8, 1/5$ is such a progression, of length 3 and common difference $3/40$. Moreover, this is a *maximal progression* in S of length 3 since it cannot be extended to the left or right within S ($-1/40$ and $11/40$ not being members of S).

   (a) Find a maximal progression in S of length 1996.

   (b) Is there a maximal progression in S of length 1997?

3.25 United States of America

1. Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integer k, define

$$x_k = 0 \quad \text{if} \quad x_{k-1} = 0, \qquad \left\{ \frac{p_k}{x_{k-1}} \right\} \quad \text{if} \quad x_{k-1} \neq 0,$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

2. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines EF, FD, DE, respectively, are concurrent.

3. Prove that for any integer n, there exists a unique polynomial Q with coefficients in $\{0, 1, \ldots, 9\}$ such that $Q(-2) = Q(-5) = n$.

4. To *clip* a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_8$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is greater than $1/3$, for all $n \geq 6$.

5. Prove that, for all positive real numbers a, b, c,

$$(a^3 + b^3 + abc)^{-1} + (b^3 + c^3 + abc)^{-1} + (c^3 + a^3 + abc)^{-1} \leq (abc)^{-1}.$$

6. Suppose the sequence of nonnegative integers $a_1, a_2, \ldots, a_{1997}$ satisfies

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$$

for all $i, j \geq 1$ with $i + j \leq 1997$. Show that there exists a real number x such that $a_n = \lfloor nx \rfloor$ for all $1 \leq n \leq 1997$.

3.26 Vietnam

1. Determine the smallest integer k for which there exists a graph on 25 vertices such that every vertex is adjacent to exactly k others, and any two nonadjacent vertices are both adjacent to some third vertex.

2. Find the largest real number α for which there exists an infinite sequence $a_1, a_2, \dots$ of positive integers satisfying the following properties.

   (a) For each $n \in \mathbb{N}$, $a_n > 1997^n$.

   (b) For every $n \geq 2$, a_n^α does not exceed the greatest common divisor of the set $\{a_i + a_j : i + j = n\}$.

3. Let $f\colon \mathbb{N} \to \mathbb{Z}$ be the function defined by

   $$f(0) = 2, f(1) = 503, f(n+2) = 503f(n+1) - 1996f(n).$$

   For $k \in \mathbb{N}$, let $s_1, \dots, s_k$ be integers not less than k, and let p_i be a prime divisor of $f(2^{s_i})$ for $i = 1, \dots, k$. Prove that for $t = 1, \dots, k$,

   $$\sum_{i=1}^{k} p_i \mid 2^t \qquad \text{if and only if} \qquad k \mid 2^t.$$

4. Find all pairs (a, b) of positive reals such that for every $n \in \mathbb{N}$ and every real number x satisfying

   $$4n^2 x = \log_2(2n^2 x + 1),$$

   we have $a^x + b^x \geq 2 + 3x$.

5. Let n, k, p be positive integers such that $k \geq 2$ and $k(p+1) \leq n$. Determine the number of ways to color n labeled points on a circle in blue or red, so that exactly k points are colored blue, and any arc whose endpoints are blue but contains no blue points in its interior contains exactly p red points.

4 1997 Regional Contests: Problems

4.1 Asian Pacific Mathematics Olympiad

1. Let

$$S = 1 + \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}+\frac{1}{6}} + \cdots + \frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}},$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers. Prove that $S > 1001$.

2. Find an integer n with $100 \leq n \leq 1997$ such that n divides $2^n + 2$.

3. Let ABC be a triangle and let

$$l_a = \frac{m_a}{M_a}, l_b = \frac{m_b}{M_b}, l_c = \frac{m_c}{M_c},$$

where m_a, m_b, m_c are the lengths of the internal angle bisectors and M_a, M_b, M_c are the lengths of the extensions of the internal angle bisectors to the circumcircle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3,$$

with equality if and only if ABC is equilateral.

4. The triangle $A_1A_2A_3$ has a right angle at A_3. For $n \geq 3$, let A_{n+1} be the foot of the perpendicular from A_n to $A_{n-1}A_{n-2}$.

   (a) Show that there is a unique point P in the plane interior to the triangles $A_{n-2}A_{n-1}A_n$ for all $n \geq 3$.

   (b) For fixed A_1 and A_3, determine the locus of P as A_2 varies.

5. Persons $A_1, \ldots, A_n$ ($n \geq 3$) are seated in a circle in that order, and each person A_i holds a number a_i of objects, such that $(a_1 + \cdots + a_n)/n$ is an integer. It is desired to redistribute the objects so that each person holds the same number; objects may only be passed from one person to either of her two neighbors. How should the redistribution take place so as to minimize the number of passes?

4.2 Austrian-Polish Mathematical Competition

1. Let P be the intersection of lines l_1 and l_2. Let S_1 and S_2 be two circles externally tangent at P and both tangent to l_1, and let T_1 and T_2 be two circles externally tangent at P and both tangent to l_2. Let A be the second intersection of S_1 and T_1, B that of S_1 and T_2, C that of S_2 and T_1, and D that of S_2 and T_2. Show that the points A, B, C, D are concyclic if and only if l_1 and l_2 are perpendicular.

2. Let m, n, p, q be positive integers, and consider an $m \times n$ checkerboard with a checker on each of its mn squares. A piece can be moved from (x, y) to (x', y') if and only if $|x - x'| = p$ and $|y - y'| = q$. How many ways can all of the pieces be moved simultaneously so that one piece ends up on each square?

3. On a blackbroad are written the numbers $48/k$ with $k = 1, 2, \dots, 97$. At each step, two numbers a, b are erased and $2ab - a - b + 1$ is written in their place. After 96 steps, a single number remains on the blackboard. Determine all possible such numbers.

4. In a convex quadrilateral $ABCD$, the sides AB and CD are parallel, the diagonals AC and BD intersect at E, and the triangles EBC and EAD have respective orthocenters F and G. Prove that the midpoint of GF lies on the line through E perpendicular to AB.

5. Let p_1, p_2, p_3, p_4 be distinct primes. Prove there does not exist a cubic polynomial $Q(x)$ with integer coefficients such that
$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3.$$

6. Prove there does not exist $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x + f(y)) = f(x) - y$ for all integers x, y.

7. (a) Prove that for all $p, q \in \mathbb{R}$, $p^2 + q^2 + 1 > p(q + 1)$.
   (b) Determine the largest real number x such that $p^2 + q^2 + 1 > bp(q + 1)$ for all $p, q \in \mathbb{R}$.
   (c) Determine the largest real number x such that $p^2 + q^2 + 1 > bp(q + 1)$ for all $p, q \in \mathbb{Z}$.

8. Let n be a natural number and M a set with n elements. Find the biggest integer k such that there exists a k-element family of three-element subsets of M, no two of which are disjoint.

9. Let P be a parallelepiped with volume V and surface area S, and let L be the sum of the lengths of the edges of P. For $t \geq 0$, let P_t be the set of points which lie at distance at most t from some point of P. Prove that the volume of P_t is

$$V + St + \frac{\pi}{4}Lt^2 + \frac{4\pi}{3}t^3.$$

4.3 Czech-Slovak Match

1. An equilateral triangle ABC is given. Points K and L are chosen on its sides AB and AC, respectively, such that $|BK| = |AL|$. Let P be the intersection of the segmentns BL and CK. Determine the ratio $|AK| : |KB|$ if it is known that the segments AP and CK are perpendicular.

2. In a community of more than six people, each member exchanges letters with precisely three other members of the community. Prove that the community can be divided into two nonempty groups so that each member exchanges letters with at least two members of the group he belongs to.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality
$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$
holds for all pairs of real numbers x, y.

4. Is it possible to place 100 solid balls in space so that no two of them have a common interior point, and each of them touches at least one-third of the others?

5. Several integers are given (some of them may be equal) whose sum is equal to 1492. Decide whether the sum of their seventh powers can equal
   (a) 1996;
   (b) 1998.

6. In a certain language there are only two letters, A and B. The words of this language obey the following rules:
   (a) The only word of length 1 is A.
   (b) A sequence of letters $X_1X_2 \cdots X_nX_{n+1}$, where $X_i \in \{A, B\}$ for each i, is a word if and only if it contains at least one A but is not of the form $X_1X_2 \cdots X_nA$ where $X_1X_2 \cdots X_n$ is a word.

   Show that there are precisely $\binom{3995}{1997} - 1$ words which do not begin with AA and which are composed of 1998 A's and 1998 B's.

4.4 Hungary-Israel Mathematics Competition

1. Is there an integer N such that
$$(\sqrt{1997}-\sqrt{1996})^{1998}=\sqrt{N}-\sqrt{N-1}?$$

2. Find all real numbers α with the following property: for any positive integer n, there exists an integer m such that
$$\left|\alpha-\frac{m}{n}\right|<\frac{1}{3n}.$$

3. The acute triangle ABC has circumcenter O. Let A_1, B_1, C_1 be the points where the diameters of the circumcircle through A, B, C meet the sides BC, CA, AB, respectively. Suppose the circumradius of ABC is $2p$ for some prime number p, and the lengths OA_1, OB_1, OC_1 are integers. What are the lengths of the sides of the triangle?

4. How many distinct sequences of length 1997 can be formed using each of the letters A, B, C an odd number of times (and no others)?

5. The three squares ACC_1A'', $ABB_1'A'$, $BCDE$ are constructed externally on the sides of a triangle ABC. Let P be the center of $BCDE$. Prove that the lines $A'C, A''B, PA$ are concurrent.

4.5 Iberoamerican Mathematical Olympiad

1. Let $r \geq 1$ be a real number such that for all m, n such that m divides n, $\lfloor mr \rfloor$ divides $\lfloor nr \rfloor$. Prove that r is an integer.

2. Let ABC be a triangle with incenter I. A circle centered at I meets the segment BC at D and P (with D closer to B), CA at E and Q (with E closer to C), and AB at F and R (with F closer to A). Let S, T, U be the intersections of the diagonals of the quadrilaterals $EQFR, FRDP, DPEQ$, respectively. Show that the circumcircles of the triangles FRT, DPU, EQS pass through a common point.

3. Let $n \geq 2$ be an integer and D_n the set of points (x, y) in the plane such that x, y are integers with $|x|, |y| \leq n$.

   (a) Prove that if each of the points in D_n is colored in one of three colors, there exist two points of D_n in the same color such that the line through them passes through no other point of D_n.

   (b) Show that the points of D_n can be colored in four colors so that if a line contains exactly two points of D_n, those two points have different colors.

4. Let n be a positive integer. Let O_n be the number of $2n$-tuples $(x_1, \dots, x_n, y_1, \dots, y_n)$ with values in 0 or 1 for which the sum $x_1 y_1 + \cdots + x_n y_n$ is odd, and let E_n be the number of $2n$-tuples for which the sum is even. Prove that

$$\frac{O_n}{E_n} = \frac{2^n - 1}{2^n + 1}.$$

5. Let AE and BF be altitudes, and H the orthocenter, of acute triangle ABC. The reflection of AE across the interior angle bisector of A meets the reflection of BF across the interior angle bisector of B meet in a point O. The lines AE and AO meet the circumcircle of ABC again at M and N, respectively. Let P, Q, R be the intersection of BC with HN, BC with OM, HR with OP, respectively. Show that $AHSO$ is a parallelogram.

6. Let $P = \{P_1, P_2, \dots, P_{1997}\}$ be a set of 1997 points in the interior of a circle of radius 1, with P_1 the center of the circle. For $k =$

$1, \ldots, 1997$, let x_k be the distance from P_k to the point of P closest to P_k. Prove that

$$x_1^2 + x_2^2 + \cdots + x_{1997}^2 \leq 9.$$

4.6 Nordic Mathematical Contest

1. For any set A of positive integers, let n_A denote the number of triples (x, y, z) of elements of A such that $x < y$ and $x + y = z$. Find the maximum value of n_A given that A contains seven distinct elements.

2. Let $ABCD$ be a convex quadrilateral. Assume that there exists an internal point P of $ABCD$ such that the areas of the triangles ABP, BCP, CDP, DAP are all equal. Prove that at least one of the diagonals of the quadrilateral bisects the other.

3. Assume that A, B, C, D are four distinct points in the plane. Three of the segments AB, AC, AD, BC, BD, CD have length a. The other three have length $b > a$. Find all possible values of the ratio b/a.

4. Let f be a function defined on $\{0, 1, 2, \ldots\}$ such that

   $$f(2x) = 2f(x), f(4x + 1) = 4f(x) + 3, f(4x - 1) = 2f(2x - 1) - 1.$$

   Prove that f is injective (if $f(x) = f(y)$, then $x = y$).

4.7 Rio Plata Mathematical Olympiad

1. Around a circle are written 1996 zeroes and and one 1. The only permitted operation is to choose a number and change its two neighbors, from 0 to 1 and vice versa. Is it possible to change all of the numbers to 1? And what if we started with 1997 zeroes?

2. Show that one cannot draw two triangles of area 1 inside a circle of radius 1 so that the triangles have no common point.

3. A benefit concert is attended by 1997 people from Peru, Bolivia, Paraguay and Venezuela. Each person paid for his ticket an integer number of dollars between 1 and 499, inclusive.

   (a) Prove that at least two people of the same nationality paid the same price.

   (b) It is known that each possible price was paid at least once, that the maximum number of times a price was repeated was 10, and that subject to these conditions, the smallest amount of money was collected. How many tickets were sold at each price?

4. A 4×4 square is divided into 1×1 squares. A secret number is written into each small square. All that is known is that the sum of the numbers in each row, each column, and each of the diagonals equals 1. Is it possible to determine from this information the sum of the numbers in the four corners, and the sum of the numbers in the four central squares? And if so, what are these sums?

5. What is the smallest multiple of 99 whose digits sum to 99 and which begins and ends with 97?

6. A tourist takes a trip through a city in stages. Each stage consist of three segments of length 100 meters separated by right turns of 60°. Between the last segment of one stage and the first segment of the next stage, the tourist makes a left turn of 60°. At what distance will the tourist be from his initial position after 1997 stages?

4.8 St. Petersburg City Mathematical Olympiad (Russia)

1. The incircle of a triangle is projected onto each of the sides. Prove that the six endpoints of the projections are concyclic.

2. Let a and b be integers. Prove that

$$\left|\frac{a+b}{a-b}\right|^{ab} \geq 1.$$

3. Prove that every positive integer has at least as many (positive) divisors whose last decimal digit is 1 or 9 as divisors whose last digit is 3 or 7.

4. Prove that opposite vertices of a 142×857 rectangle with vertices at lattice points cannot be joined by a five-edge broken line with vertices at lattice points such that the ratio of the lengths of the edges is $2 : 3 : 4 : 5 : 6$.

5. Do there exist 100 positive integers such that the sum of the fourth powers of every four of the integers is divisible by the product of the four numbers?

6. Let B' be the antipode of B on the circumcircle of triangle ABC, let I be the incenter of ABC, and let M be the point where the incircle touches AC. The points K and L are chosen on the sides AB and BC, respectively, so that $KB = MC$, $LB = AM$. Prove that the lines $B'I$ and KL are perpendicular.

7. Can a 1997×1997 square be dissected into squares whose side lengths are integers greater than 30?

8. At each vertex of a regular 1997-gon is written a positive integer. One may add 2 to any of the numbers and subtract 1 from the numbers k away from it in either direction, for some $k \in \{1, 2, \ldots, 1998\}$; the number k is then written on a blackboard. After some number of operations, the original numbers reappear at all of the vertices. Prove that at this time, the sum of the squares of the numbers written on the blackboard is divisible by 1997.

9. The positive integers x, y, z satisfy the equation $2x^x + y^y = z^z$. Prove that $x = y = z$.

10. The number N is the product of k diferent primes ($k \geq 3$). Two players play the following game. In turn, they write composite divisors of N on a blackboard. One may not write N. Also, there may never appear two coprime numbers or two numbers, one of which divides the other. The first player unable to move loses. Does the first player or the second player have a winning strategy?

11. Let K, L, M, N be the midpoints of sides AB, BC, CD, DA, respectively, of a cyclic quadrilateral $ABCD$. Prove that the orthocentres of triangles AKN, BKL, CLM, DMN are the vertices of a parallelogram.

12. A 100×100 square grid is folded several times along grid lines. Two straight cuts are also made along grid lines. What is the maximum number of pieces the square can be cut into?

13. The sides of a convex polyhedron are all triangles. At least 5 edges meet at each vertex, and no two vertices of degree 5 are connected by an edge. Prove that this polyhedron has a side whose vertices have degrees 5, 6, 6, respectively.

14. Given $2n + 1$ lines in the plane, prove that there are at most $n(n + 1)(2n + 1)/6$ acute triangles with sides on the lines.

15. Prove that the set of all 12-digit numbers cannot be divided into groups of 4 numbers so that the numbers in each group have the same digits in 11 places and four consecutive digits in the remaining place.

16. A circle is divided into equal arcs by 360 points. The points are joined by 180 nonintersecting chords. Consider also the 180 chords obtained from these by a rotation of $38°$ about the center of the circle. Prove that the union of these 360 chords cannot be a closed (self-intersecting) polygon.

17. Can a 75×75 table be partitioned into dominoes (1×2 rectangles) and crosses (five-square figures consisting of a square and its four neighbors)?

18. Prove that for $x, y, z \geq 2$, $(y^3 + x)(z^3 + y)(x^3 + z) \geq 125xyz$.

19. The circles S_1, S_2 intersect at A and B. Let Q be a point on S_1. The lines QA and QB meet S_2 at C and D, respectively, while the tangents to S_1 at A and B meet at P. Assume that Q lies outside S_2, and that C and D lie outside S_1. Prove that the line QP goes through the midpoint of CD.

20. Given a convex 50-gon with vertices at lattice points, what is the maximum number of diagonals which can lie on grid lines?

21. The number $99 \cdots 99$ (with 1997 nines) is written on a blackboard. Each minute, one number written on the blackboard is factored into two factors and erased, each factor is (independently) increased or diminished by 2, and the resulting two numbers are written. Is it possible that at some point all of the numbers on the blackboard equal 9?

22. A device consists of $4n$ elements, any two of which are joined by either a red or a blue wire. The numbers of red and blue wires are the same. The device is disabled by removing two wires of the same color connecting four different elements. Prove that the number of ways to disable the device by removing two blue wires is the same as the number of ways by removing two red wires.

23. An Aztec diamond of rank n is a figure consisting of those squares of a gridded coordinate plane lying entirely inside the diamond $\{(x, y) : |x|+|y| \leq n+1\}$. For any covering of an Aztec diamond by dominoes (1×2 rectangles), we may rotate by $90°$ any 2×2 square covered by exactly two dominoes. Prove that at most $n(n + 1)(2n + 1)/6$ rotations are needed to transform an arbitrary covering into the covering consisting only of horizontal dominoes.