S.O.S TECHNIQUE

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A. INTRODUCITON

Inequality is one of the most difficult and interesting aspect of Mathematics. We can't deny that there are many people being absorbed by its attraction. Nowadays, inequality problems are more and more complex, wider, and even "rougher". It acquires us to have more new methods, techniques to defeat these fascinating monsters. And we will introduce to you one of those new methods, S.O.S techniques. This is a very helpful tool which allows you to handle with the "roughest" ones in the rough inequality history.

Be thankful to my teacher, Mr. Tran Tuan Anh, who introduced me this method, to Mathlinks.ro's members, who help me to have a lot of ideas, to editorial staff who allow me to perform this article.

B. *S.O.S TECHNIQUE*

a)

- **METHOD**

Firstly, we will talk a little about some Old inequality, including Cauchy, Holder, Bernoulli … Take a notice of it, and all of these inequalities can be proved by Cauchy inequality, so it seems that Cauchy inequality is the origin of almost other old inequality. But is there any origin which is more standard than Cauchy inequality? The answer is yes, because we need a more standard one to prove Cauchy inequality, that is $x^2 \geq 0$, or more concrete $(a-b)^2 \geq 0$. So is that all inequalities, which have two variables and the equality occurs when two variables are equal , can be perform in form $(a-b)^2$, and then we just need to transform in this form then the problem can be solved ? The answer is yes, and this is also the main idea of the S.O.S technique as well.

We will first make acquaintance with the way to transform a two variables inequality to the form $(a-b)^2 g(a,b)$. When this step is done, the rest job is proving that $g(a,b) \geq 0$, and this is usually easy than the fist one, since the assessment is not strict anymore.

To bring out this idea, we will begin this principle now. But in spite of some examples, we will begin with a more interesting action which is creating inequality from some identities

We will begin with some familiar identities:

$$a^2+b^2-2ab=(a-b)^2$$
$$(a+b)^2-4ab=(a-b)^2$$
$$2(a^2+b^2)-(a+b)^2=(a-b)^2$$
$$a^3+b^3-ab(a+b)=(a+b)(a-b)^2$$
$$a^4+b^4-ab(a^2+b^2)=(a^2+ab+b^2)(a-b)^2$$

As you can see, the first three one are equal and quite strict, it is quite difficult to create anything from those. But the fourth and the fifth is interesting. For instance, check the fifth one, using the assessment: $a^2+ab+b^2\ge 3ab$, we find something exciting:

$$a^4+b^4-ab\left(a^2+b^2\right)\ge 3ab(a-b)^2$$
$$\Leftrightarrow\ a^4+b^4+6a^2b^2\ge 4ab\left(a^2+b^2\right)$$

The last result is stricter and harder. The inequality $a^4+b^4\ge ab\left(a^2+b^2\right)$, however, still can be solved by Cauchy inequality, but this one is not, and the only way is transforming to square form: $(a-b)^4\ge 0$.

Further more, we still can have the S.O.S form for fraction form or root form. Let have some example:

$$\sqrt{2(a^2+b^2)}-(a+b)=\frac{2(a^2+b^2)-(a+b)^2}{\sqrt{2(a^2+b^2)}+a+b}=\frac{(a-b)^2}{\sqrt{2(a^2+b^2)}+a+b}\ (1)$$
$$\frac{1}{a}+\frac{1}{b}-\frac{4}{a+b}=\frac{(a+b)^2-4ab}{ab(a+b)}=\frac{(a-b)^2}{ab(a+b)}$$

With some below identities, we again can create more difficult inequalities. Now let handle with (1), use the assessment: $2\sqrt{2\left(a^2+b^2\right)}\ge\sqrt{2\left(a^2+b^2\right)}+a+b\ge 2(a+b)$, and we obtain:

$$a+b+\frac{(a-b)^2}{2\sqrt{2(a^2+b^2)}}\le\sqrt{2(a^2+b^2)}\le a+b+\frac{(a-b)^2}{2(a+b)}$$

Well, now do you believe me S.O.S is the only one way to solve above Problem ☺

That is the story of creating, our problem is solving, but now I think everything becomes easier when you know the way people create it, have a look at this:

- **EXAMPLES**

Example one [Tran Tuan Anh]:

Let a, b be positive real number. Prove that:

$$\frac{a^2}{b}+\frac{b^2}{a}+7(a+b)\geq 8\sqrt{2(a^2+b^2)}$$

Solution:

The inequality is equal to:

$$(a-b)^2\left(\frac{1}{a}+\frac{1}{b}-\frac{8}{\sqrt{2(a^2+b^2)}+a+b}\right)\geq 0(*)$$

We can prove that the expression in brackets is non-negative. Indeed:

$$\frac{1}{a}+\frac{1}{b}\geq\frac{4}{a+b}\geq\frac{8}{\sqrt{2\left(a^2+b^2\right)}+a+b}\Rightarrow\frac{1}{a}+\frac{1}{b}-\frac{8}{\sqrt{2\left(a^2+b^2\right)}+a+b}\geq 0$$

And the proof is finished then.

The most different in creating and solving here is that we haven't known to have the Square form yet. Here we use some identities, which come from familiar inequalities:

$$\frac{a^2}{b}+\frac{b^2}{a}-a-b=\frac{a^3+b^3-a^2b-b^2a}{ab}=\frac{(a-b)^2(a+b)}{ab}$$

$$8(\sqrt{2\left(a^2+b^2\right)}-a-b)=8.\frac{2\left(a^2+b^2\right)-\left(a+b^2\right)}{\sqrt{2\left(a^2+b^2\right)}+a+b}=\frac{8(a-b)^2}{\sqrt{2\left(a^2+b^2\right)}+a+b}$$

Then subtract them and we have (*).

So, what is the trick here, we subtracted $\frac{a^2}{b}+\frac{b^2}{a}$ and $\sqrt{2(a^2+b^2)}$ with $a+b$, which is arithmetic mean value (of course we also use geometric mean value but it is less effective, since it is not in polynomial form at all). We will try to find the factor of $a+b$ such that the result obtained when $a=b$ is zero. Comment more that two expressions we obtain is assessment product of Cauchy inequality, so the difference square expression $(a-b)^2$ must appear.

Example two [Nguyen Anh Cuong]:

Give $a,b\geq 0$. Prove that:

$$\frac{(a-b)^2}{4(a+b)}\leq\frac{a+b}{2}-\sqrt{ab}$$

Solution:

The idea to solve this problem is similar to example one. However, this one is the beginning for something more interesting we will discuss in some part below, so let check this problem first.

Firstly, take $a=x^2;b=y^2$ (the root is not so convenient anyway), and we have an equivalent problem:

Give $x, y \ge 0$. Prove that:

$$\frac{(x^2 - y^2)^2}{4(x^2 + y^2)} \le \frac{x^2 + y^2}{2} - xy$$

This is equal to:

$$\frac{(x + y)^2 (x - y)^2}{4(x^2 + y^2)} \le \frac{(x - y)^2}{2}$$

And we refer to the problem: $(x + y)^2 \le 2(x^2 + y^2) \Leftrightarrow (x - y)^2 \ge 0$. Therefore, the problem was done.

In another part, we will discuss the general problem,

> Give $a_1, a_2, ..., a_n \ge 0$. Find k such that:
>
> $$k \frac{\sum_{1 \le i < j \le n} (a_i - a_j)^2}{a_1 + a_2 + ... + a_n} \le \frac{a_1 + a_2 + .. + a_n}{n} - \sqrt[n]{a_1 a_2 ... a_n}$$
>
> The best result is $k = \frac{1}{2n}$, but this result is hard for $S.O.S$. Until now, I still can't solve it by S.O.S technique. We will discuss about the best k in ABC Theorem, which is a better method for this problem.

- **PROPOSED**

- **PROPOSED PROBLEMS**

Problem one: [VMO for secondary student-1995]

Prove that :

$$\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2 \ge 3\left(\frac{x}{y} + \frac{y}{x}\right) - 4, \forall x, y \ne 0.$$

Problem two: [Math & Youth magazine]

Prove that this inequality for all positive real numbers a, b

$$\frac{a + b}{2} \ge \sqrt{ab} + \frac{(a - b)^2 (a + 3b)(b + 3a)}{16(a + b)^3}$$

Problem three: [Given by Nickolas – www.Mathlinks.ro]

Give $a, b \ge \frac{1}{2}$. Prove that:

$$\left(\frac{a^2 - b^2}{2}\right)^2 \ge \sqrt{\frac{a^2 + b^2}{2}} - \frac{a + b}{2}$$

b)

- **METHOD**

Now, we will handle with the problems which have more variables. Give a function $f(a_1,a_2,...,a_n): R^n \to R^n$; the problem is given that proving $f(a_1,a_2,...,a_n) \geq 0$.
Follow the main idea in two variables problems, we will try to change $f(a_1,a_2,...,a_n)$ to the form $\sum g_{ij}(a)(a_i - a_j)^2$ and assess the value of $g_{ij}(a)$. The problem is solved clearly if we can prove that $g_{ij}(a)$ is non-negative. The idea may be simple but it is very helpful, let bring out it value through some examples:

- **EXAMPLE**

<u>*Example one [Nguyen Anh Cuong]:*</u>
Let a, b, c be real number satisfying $a+b+c=1$. Prove that:

$$4(a^3+b^3+c^3-3abc) \geq 3(a^2+b^2-2ab)$$

<u>*Solution:*</u>

The inequality is equal to:

$$2(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \geq 3(a-b)^2$$

$$\Leftrightarrow 2(b-c)^2+2(c-a)^2 \geq (a-b)^2$$

$$\Leftrightarrow (a+b-2c)^2 \geq 0$$

So, beside the equality point which is can be realized easily $a=b=c$, we discover a new equality point $a+b=2c$. And one more time, we used two identity expressions, which are assessment product of Cauchy inequality:

$$a^3+b^3+c^3-3abc = \frac{(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]}{2}$$

$$a^2+b^2-2ab = (a-b)^2$$

<u>*Example two: [Nguyen Anh Cuong]*</u>
Let x, y, z be real numbers satisfying some conditions:
$y^2 \geq xz; x \geq y \geq z \geq 0$. Prove that:

$$\frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y} \geq 2(x^2+y^2+z^2)-xy-xz-yz.$$

<u>*Solution:*</u>

$$\frac{y-z}{z}(x-z)^2+\frac{z-x}{x}(y-x)^2+\frac{x-y}{y}(y-z)^2 \geq 0(*)$$

Let $A = \frac{(y-z)(x-z)}{z} - \frac{(x-y)^2}{x}; B = \frac{(y-z)^2}{y} - \frac{(x-y)(x-z)}{x}$

Then $xzA + xyB = 2(x-z)(y^2-xz) \geq 0.$

So A or B is non-negative.
If $A \geq 0$, then

$$(*) \Leftrightarrow (x-z)A+(x-y)\frac{(y-z)^2}{y} \geq 0$$

If $B \geq 0$, then

$$(*) \Leftrightarrow (x-y)B+(y-z)\frac{(x-z)^2}{z} \geq 0$$

So the problem is done.

- **PROPOSED PROBLEMS**

*Problem one: [*L.Panaitopol, V. Bandila, M.Lascu – Inegalitati book*]*

Prove the following inequality:

$(x+y+z)(xy+yz+zx) \geq 9xyz+(y-x)(z-x)^2$, where $0 \leq x \leq y \leq z$

Problem two: [Nguyen Anh Cuong]

Let $a,b,c>0$ and $a+b+c=1$. Prove that:

$$\frac{a^2+3b}{b+c}+\frac{b^2+3c}{c+a}+\frac{c^2+3a}{a+b} \geq 5$$

Problem three: [Nam Dung]

Prove the inequality for $a,b,c>0$

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c \geq 2\sqrt{3(a^2+b^2+c^2)}$$

c)

- **METHOD**

So we can see some useful application of this technique. However, in above examples, we are lucky since the expression $g(a)$ is non-negative. Now, we will introduce to you a theorem which allow us to handle the inequality in unexpected cases

> **S.O.S Theorem:** Let real numbers a, b, c and A, B, C satisfying one of these below conditions:
>
> i) $A+B \geq 0, B+C \geq 0, C+A \geq 0$.
>
> If c is in the middle of a and b then $C \geq 0$
>
> ii) $A+B+C>0$ and $AB+AC+BC \geq 0$
>
> iii) If $C=\min\{A,B,C\}$ then $A+2C \geq 0$ and $B+2C \geq 0$
>
> Then we have $A(b-c)^2+B(c-a)^2+C(a-b)^2 \geq 0$.

Proof:

i) Assume that c is in the middle of a and b (the rest case can be considers similarly), take $x = b - c; y = c - a$. The inequality is equal to:

$$Ax^2 + By^2 + C(x + y)^2 \geq 0$$

$$\Leftrightarrow \quad (A + C)x^2 + (B + C)y^2 + 2Cxy \geq 0$$

ii) If $ABC = 0$, assume that C $\Rightarrow \ A \geq 0, B \geq 0 \Rightarrow \ dpcm.$

If $ABC \neq 0$, we will prove that there are at least two in three number A, B, C positive. Indeed, assume that:

$C < 0 \Rightarrow \ A + B > 0 \Rightarrow \ AB > 0 \Rightarrow \ A > 0, B > 0$.

And then, we have:

$$(\frac{1}{A} + \frac{1}{B})\left[A(b - c)^2 + B(c - a)^2\right] \geq (b - c + c - a)^2 = (a - b)^2$$

$$\Leftrightarrow \quad A(b - c)^2 + B(c - a)^2 \geq (\frac{AB}{A + B})(a - b)^2$$

$$\Leftrightarrow \quad A(b - c)^2 + B(c - a)^2 + C(a - b)^2 \geq (\frac{AB}{A + B} + C)(a - b)^2 = (\frac{AB + AC + BC}{A + B})(a - b)^2 \geq 0$$

iii) Assume that $C = \min\{A, B, C\}$. If $C \geq 0$, the inequality is trivial. If $C < 0$, we have: $\quad C(a - b)^2 = C(b - c + c - a)^2 \geq 2C(b - c)^2 + 2C(c - a)^2$

Hence:

$$A(b - c)^2 + B(c - a)^2 + C(a - b)^2 \geq (2C + A)(b - c)^2 + (2C + B)(c - a)^2 \geq 0$$

So the proof is finished.

Now, we will consider some concrete example to put the theorem in practice.

From now, the technique that changing the expression to sum of square form is skipped, we will have a summary about this in next paragraphs later.

<u>*Example one [Nguyen Anh Cuong]*</u>

Let a, b, c be non-negative real number. Prove that:

$$a^3 + b^3 + c^3 + 3abc \geq ab\sqrt{2(a^2 + b^2)} + bc\sqrt{2(b^2 + c^2)} + ca\sqrt{2(c^2 + a^2)}$$

- **EXAMPLES**

<u>Solution:</u>

In the first sight, you can see that this refers to Schur inequality which is:

$$a^3 + b^3 + c^3 + 3abc \geq ab(a + b) + bc(b + c) + ca(c + a)$$

WLOG, we can assume that $a \geq b \geq c$. The inequality is equal to:

$$\sum (a + b - c - \frac{2ab}{\sqrt{2(a^2 + b^2)} + a + b})(a - b)^2 \geq 0$$

We have:

$$a + c - b - \frac{2ac}{\sqrt{2(a^2 + c^2)} + a + c} \geq (a - b) + c(1 - \frac{2}{\sqrt{2} + 1}) \geq 0$$

Further more

$$B+C=2a-\frac{2ab}{\sqrt{2(a^2+b^2)}+a+b}-\frac{2ac}{\sqrt{2(a^2+c^2)}+a+c}\geq 2a(1-\frac{b}{2(a+b)}-\frac{c}{2(a+c)})$$

$$=a(\frac{a}{a+b}+\frac{a}{a+c})\geq 0$$

Hence the inequality is proved referring to the theorem. ☺

Example 2: [Nguyen Anh Cuong]

Let a, b, c be three positive real numbers. Prove that:

$$a^3+b^3+c^3+6(ab^2+bc^2+ca^2)\geq 3(a^2b+b^2c+c^2a)+12abc$$

Solution:

The inequality is equal to:

$$(2a+2b-c)(b-c)^2+(2b+2c-a)(c-a)^2+(2c+2a-b)(a-b)^2\geq 0$$

WLOG, assume that $2a+2b-c=\min\{(2a+2b-c),(2b+2c-a),(2c+2a-b)\}$ then:

$$2(2a+2b-c)+2b+2c-a=3a+6b>0;2(2a+2b-c)+2c+2a-b=6a+2b>0$$

So it is proved too. ☺

Example 3:[Nguyen Anh Cuong]

Let a, b, c be positive number satisfying that $\sqrt{ab},\sqrt{ac},\sqrt{bc}$ are the side lengths of a triangle.

Prove that:

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0$$

Solution:

The inequality is equal to:

$$(ab+ac-bc)(a-b)^2+(bc+ba-ac)(b-c)^2+(ac+bc-ab)(c-a)^2\geq 0$$

Clearly: $(ab+ac-bc)+(bc+ba-ac)+(ac+bc-ab)=ab+ac+bc>0$

Furthermore:

$$(ab+ac-bc)(bc+ba-ac)+(ab+ac-bc)(ac+bc-ab)+(bc+ba-ac)(ac+bc-ab)$$

$$=2abc(a+b+c)-a^2b^2-b^2c^2-a^2c^2$$

$$=(\sqrt{ab}+\sqrt{ac}-\sqrt{bc})(\sqrt{ab}-\sqrt{ac}+\sqrt{bc})(-\sqrt{ab}+\sqrt{ac}+\sqrt{bc})(\sqrt{ab}+\sqrt{ac}+\sqrt{bc})\geq 0$$

So, the proof is finished. ☺

Example 4: [Vasile Cirtoaje]

Let a, b, c be three side lengths of a triangle. Prove that:

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+3$$

Solution:

The inequality is equal to:

$$(5a-5b+3c)(a-b)^2+(5b-5c+3a)(b-c)^2+(5c-5a+3b)(c-a)^2 \ge 0(1)$$

Take $x=a+b; y=b+c; z=c+a.$

If $x \ge y \ge z$ then

$(a-b)(b-c)(c-a) \Leftrightarrow (a-b)^3+(b-c)^3+(c-a)^3 \Rightarrow$ (1) is trivial.

If $z \ge y \ge x$

$(1) \Leftrightarrow (4x-z)(x-z)^2+(4z-y)(y-z)^2+(4y-x)(x-y)^2(*) \ge 0.$

We only need to consider the case $4x < z$. We have:

$(x-z)^2 \le 2(y-z)^2+2(x-y)^2$ .Hence

$(*) \ge (8x+2z-y)(y-z)^2+(7x+4y-2z)(x-y)^2(**) \ge 0$

If $7x+4y-2z \ge 0$ then the inequality is proved. On the otherwise, we have:

$z \ge 2y \Rightarrow z-y \ge y-x \ge 0$. So:

$(**) \ge (x-y)^2(15x+3y) \ge 0.$

In conclusion, the inequality is proved. ☺

Example 5 [Iran Math Olympiad - 1996]:

Let a,b,c be three positive number .Prove that:

$$(ab+ac+bc)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right] \ge \frac{9}{4}$$

Solution:

The inequality is equal to:

$$\left[\frac{2}{(a+c)(b+c)}-\frac{1}{(a+b)^2}\right](a-b)^2+\left[\frac{2}{(a+b)(a+c)}-\frac{1}{(b+c)^2}\right](b-c)^2+$$
$$\left[\frac{2}{(a+b)(b+c)}-\frac{1}{(a+c)^2}\right](c-a)^2 \ge 0$$

$$\Leftrightarrow \left[\frac{2}{c^2(a+c)(b+c)}-\frac{1}{c^2(a+b)^2}\right](ac-bc)^2+\left[\frac{2}{a^2(a+b)(a+c)}-\frac{1}{a^2(b+c)^2}\right](ab-ac)^2+$$
$$\left[\frac{2}{b^2(a+b)(b+c)}-\frac{1}{b^2(a+c)^2}\right](cb-ab)^2 \ge 0$$

Take $S_A = \dfrac{2}{a^2(a+b)(a+c)} - \dfrac{1}{a^2(b+c)^2}$, $S_B = \dfrac{2}{b^2(a+b)(c+b)} - \dfrac{1}{b^2(a+c)^2}$,

$S_C = \dfrac{2}{c^2(a+b)(a+c)} - \dfrac{1}{c^2(a+b)^2}$

WLOG, assume that $a \ge b \ge c$ then it is not too difficult to prove that $S_B \ge 0$. So according to the first condition of the theorem, we just need to prove that $S_A + S_B, S_B + S_C, S_C + S_A$ are non-negative. Here we will prove that $S_A + S_B \ge 0$, since the others are the same.

$$S_A + S_B = \frac{2}{a^2(a+b)(b+c)} + \frac{2}{b^2(b+a)(b+c)} - \frac{1}{a^2(b+c)^2} - \frac{1}{b^2(c+a)^2}$$

$$> \frac{2}{a+b}\left[\frac{1}{a^2(a+c)} + \frac{1}{b^2(b+c)}\right] - \frac{1}{a^2b(b+c)} - \frac{1}{b^2a(a+c)}$$

$$= \frac{1}{a(a+c)}\left[\frac{2}{a(a+b)} - \frac{1}{b^2}\right] + \frac{1}{b(b+c)}\left[\frac{2}{b(a+b)} - \frac{1}{a^2}\right]$$

$$= \frac{(b-a)(2b+a)}{a^2b^2(a+c)(a+b)} + \frac{(a-b)(2a+b)}{a^2b^2(b+c)(a+b)} = \frac{(2a+2b+c)(a-b)^2}{a^2b^2(a+b)(b+c)(c+a)} \ge 0$$

So it is done here.

This problem gives us a idea about S.O.S form. For the standard form:

$A(b-c)^2 + B(c-a)^2 + C(a-b)^2 \ge 0$

We can transform it to:

$\dfrac{A}{a^2}(ab-ac)^2 + \dfrac{B}{b^2}(bc-ba)^2 + \dfrac{C}{c^2}(ca-cb)^2$ and apply $S.O.S$ theorem for this form (consider ab, bc, ca as a, b, c in the theorem). Then we will have more condition, such as:

i) $Ab^2 + Ba^2 \ge 0, Bc^2 + Cb^2 \ge 0, Ca^2 + Ac^2 \ge 0$.

If c is in the middle of a and b then $C \ge 0$

ii) $Ab^2c^2 + Ba^2c^2 + Ca^2b^2 > 0$ and $ABc^2 + ACb^2 + BCa^2 \ge 0$

iii) If $\dfrac{C}{c^2} = \min\left\{\dfrac{A}{a^2}, \dfrac{B}{b^2}, \dfrac{C}{c^2}\right\}$ then $Ac^2 + 2Ca^2 \ge 0$ and $Bc^2 + 2Cb^2 \ge 0$

Another transformation available is that:

$$Ab^2c^2\left(\frac{1}{b} - \frac{1}{c}\right)^2 + Ba^2c^2\left(\frac{1}{c} - \frac{1}{a}\right)^2 + Ca^2b^2\left(\frac{1}{a} - \frac{1}{b}\right)^2$$

- **PROPOSED PROBLEMS**

Problem one: [Nguyen Anh Cuong]

Prove following inequality for $a,b,c > 0.$ Prove that:

$$\frac{abc}{a^3+b^3+c^3}+\frac{2}{3}\geq\frac{ab+bc+ca}{a^2+b^2+c^2}$$

Problem two: [Vasile Cirtoaje]

If a,b,c are positive numbers such that $a+b+c=3$, then:

$$\frac{a}{b^2+2}+\frac{b}{c^2+2}+\frac{c}{a^2+2}\geq 1$$

Problem three: [Nguyen Anh Cuong]

Let a,b,c be three side length of a triangle. Prove that:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+3\frac{(a-b)(b-c)(c-a)}{abc}\geq 9+\frac{(a-b)^2}{ac}+\frac{(b-c)^2}{ab}+\frac{(c-a)^2}{cb}$$

Problem four: [Nguyen Anh Cuong]

Change the expression to S.O.S form and prove it for $x,y,z>0$

$$\sqrt{\frac{x^4+y^4+z^4}{x^2y^2+y^2z^2+z^2x^2}}+\sqrt{\frac{2(xy+yz+zx)}{x^2+y^2+z^2}}\geq 1+\sqrt{2}$$

Problem five [Nguyen Anh Cuong]:

Let a,b,c be three positive number satisfying: $\min\{a,b,c\}\geq\frac{1}{4}\max\{a,b,c\}$. Prove that:

$$(ab+ac+bc)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right]\geq\frac{9}{4}+\frac{1}{16}\left[\left(\frac{a-b}{a+b}\right)^2+\left(\frac{b-c}{b+c}\right)^2+\left(\frac{c-a}{c+a}\right)^2\right]$$

d)

- **METHOD**

The above theorem is really useful then; however, the inequality which is in the form S.O.S is too rare .So the first trouble we must handle is transfer the expression to the S.O.S form. This is really a big question too, and we will fine the way to deal with it now.

The first question is when can we transform an expression to the S.O.S form?

S.O.S Presentation Theorem:

We will call A and B have family relative if A-B can be performed as the S.O.S form.

Two cyclic homogeneous polynomials which have same degree have family relative. That means we can perform the difference:

$$\sum_{cyclic} a_1^{\alpha_1}a_2^{\alpha_2}...a_n^{\alpha_n}-\sum_{cyclic} a_1^{\beta_1}a_2^{\beta_2}...a_n^{\beta_n}$$

Where $\alpha_1+...+\alpha_n=\beta_1+...+\beta_n=m$ in the form $\sum P_{ij}(a)(a_i-a_j)^2$ where $a=\{a_i\}_{i=1}^n$

Proof:

Firstly, I will prove the lemma:

The difference $\sum\limits_{cyclic} a_1^m - \sum\limits_{cyclic} a_1^{\alpha_1} a_2^{\alpha_2} ... a_n^{\alpha_n}$ where $\alpha_1 + \alpha_2 + ... + \alpha_n = m$ can be performed as the S.O.S form $\sum P_{ij}(a)(a_i - a_j)^2$.

Then our theory will be proved correctly by using the equation as follow:

$$\sum_{cyclic} a_1^{\alpha_1} a_2^{\alpha_2} ... a_n^{\alpha_n} - \sum_{cyclic} a_1^{\beta_1} a_2^{\beta_2} ... a_n^{\beta_n}$$

$$= \left(\sum_{cyclic} a_1^{\alpha_1} a_2^{\alpha_2} ... a_n^{\alpha_n} - \sum_{cyclic} a_1^m \right) + \left(\sum_{cyclic} a_1^m - \sum_{cyclic} a_1^{\beta_1} a_2^{\beta_2} ... a_n^{\beta_n} \right)$$

Now we will prove the lemma by induction with k which is the number of non-zero value in the set $\{\alpha_i\}_{i=1}^n$ (n, m is fixed).

When $k = 1$, it is evident.

When $k = 2$, we need to prove that:

$\sum\limits_{cyclic} a_1^m - \sum\limits_{cyclic} a_1^t a_2^{m-t}$ can be perform as $\sum P_{ij}(a)(a_i - a_j)^2$.

This can be proved with the remark that:

$$ta^m + (m - t)b^m - ma^t b^{m-t} = P(a,b)(a - b)^2$$

Indeed, firstly note that the equation $f(x) = tx^n + (m - t) - mx^t = 0$ has a double root which is 1, since $f(1) = f'(1) = 0$. So $f(x)$ can be performed as $Q(x)(x - 1)^2, \deg(Q) = m - 2$

Now, take $x = \dfrac{a}{b}$, then we have:

$b^m f\left(\dfrac{a}{b}\right) = ta^m + (m - t)b^m - ma^t b^{m-t} = b^{m-2} Q(\dfrac{a}{b})(a - b)^2$. However $b^{m-2} Q\left(\dfrac{a}{b}\right)$ should be a polynomial with variables a, b since Q is a polynomial whose degree is $n - 2$.

Now, we assume that the statement is true for k, for $k + 1$, we can refer to k as follow:

$$a_1^{\alpha_1} a_2^{\alpha_2} ... a_{k+1}^{\alpha_{k+1}} = - \frac{\alpha_1 a_1^{\alpha_1 + \alpha_2} + \alpha_2 a_2^{\alpha_1 + \alpha_2} - (\alpha_1 + \alpha_2) a_1^{\alpha_1} a_2^{\alpha_2}}{\alpha_1 + \alpha_2} a_3^{\alpha_3} ... a_{k+1}^{\alpha_{K=1}}$$

$$\frac{\alpha_1}{\alpha_1 + \alpha_2} a_1^{\alpha_1 + \alpha_2} a_3^{\alpha_3} ... a_{k+1}^{\alpha_{k+1}} + \frac{\alpha_2}{\alpha_1 + \alpha_2} a_2^{\alpha_1 + \alpha_2} a_3^{\alpha_3} ... a_{k+1}^{\alpha_{K+1}}$$

Use the remark for $k = 2$: $\dfrac{\alpha_1 a_1^{\alpha_1 + \alpha_2} + \alpha_2 a_2^{\alpha_1 + \alpha_2} - (\alpha_1 + \alpha_2) a_1^{\alpha_1} a_2^{\alpha_2}}{\alpha_1 + \alpha_2} = H_{12}(a)(a_1 - a_2)^2$ then we have:

$$a_1^{\alpha_1} a_2^{\alpha_2} ... a_{k+1}^{\alpha_{k+1}} = Q_{12}(a)(a_1 - a_2)^2 + \frac{\alpha_1}{\alpha_1 + \alpha_2} a_1^{\alpha_1 + \alpha_2} a_3^{\alpha_3} ... a_{k+1}^{\alpha_{k+1}} + \frac{\alpha_2}{\alpha_1 + \alpha_2} a_2^{\alpha_1 + \alpha_2} a_3^{\alpha_3} ... a_{k+1}^{\alpha_{k+1}}$$

Therefore:

$$\sum_{cyclic} a_1^m - \sum_{cyclic} a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_{k+1}^{\alpha_{k+1}} = -\sum_{cyclic} Q_{12}(a)(a_1 - a_2)^2 + \sum_{cyclic} a_1^m - \frac{\alpha_1}{\alpha_1 + \alpha_2} \sum_{cyclic} a_1^{\alpha_1+\alpha_2} a_3^{\alpha_3} \ldots a_{k+1}^{k+1} + \frac{\alpha_2}{\alpha_1 + \alpha_2} \sum_{cyclic} a_2^{\alpha_1+\alpha_2} a_3^{\alpha_3} \ldots a_{k+1}^{k+1}$$

$$\sum_{cyclic} a_1^m - \sum_{cyclic} a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_{k+1}^{\alpha_{k+1}} = -\sum_{cyclic} Q_{12}(a)(a_1 - a_2)^2 + \frac{\alpha_1}{\alpha_1 + \alpha_2}\left(\sum_{cyclic} a_1^m - \sum_{cyclic} a_1^{\alpha_1+\alpha_2} a_3^{\alpha_3} \ldots a_{k+1}^{k+1}\right) + \frac{\alpha_2}{\alpha_1 + \alpha_2}\left(\sum_{cyclic} a_1^m - \sum_{cyclic} a_2^{\alpha_1+\alpha_2} a_3^{\alpha_3} \ldots a_{k+1}^{k+1}\right)$$

In addition, two expression $\sum_{cyclic} a_1^m - \sum_{cyclic} a_1^{\alpha_1+\alpha_2} a_3^{\alpha_3} \ldots a_{k+1}^{k+1}$ and $\sum_{cyclic} a_1^m - \sum_{cyclic} a_2^{\alpha_1+\alpha_2} a_3^{\alpha_3} \ldots a_{k+1}^{k+1}$ and be written as S.O.S form, so we conclude this for $\sum_{cyclic} a_1^m - \sum_{cyclic} a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_{k+1}^{\alpha_{k+1}}$ too.

By induction principle, the proof should end here.

Since almost the inequality can be transform to homogeneous polynomial form, so it can be performed in S.O.S as well.

The second question is how can we transform an expression to the S.O.S form?

Firstly for polynomial, the above proof is also an algorithm to transform a polynomial to a S.O.S form if possible. For another form, we will consider below.

We will call B brother of A if we can perform A-B in S.O.S form easily. Here is some of the form we usually meet:

i) P (a, b, c) A (a, b, c)-Q (a, b, c) B (a, b, c). Where A and B, P and Q are brothers .So to get the S.O.S form, we will subtract and add Q (a, b, c) A (a, b, c)

$$P(a,b,c)A(a,b,c) - Q(a,b,c)A(a,b,c) + Q(a,b,c)A(a,b,c) - Q(a,b,c)B(a,b,c)$$
$$= A(a,b,c)\left[P(a,b,c) - Q(a,b,c)\right] + Q(a,b,c)\left[A(a,b,c) - Q(a,b,c)\right]$$

ii) Or in the fractional form

$$\frac{A(a,b,c)}{P(a,b,c)} - \frac{B(a,b,c)}{Q(a,b,c)} = \frac{A(a,b,c) - B(a,b,c)}{P(a,b,c)} + \frac{B(a,b,c)\left[Q(a,b,c) - P(a,b,c)\right]}{P(a,b,c)Q(a,b,c)}$$

iii) In root form:

$$\sqrt{A(a,b,c)P(a,b,c)} - \sqrt{B(a,b,c)Q(a,b,c)} = \frac{A(a,b,c)P(a,b,c) - B(a,b,c)Q(a,b,c)}{\sqrt{A(a,b,c)P(a,b,c)} + \sqrt{B(a,b,c)Q(a,b,c)}}$$

Here is the list of some form in 2, 3, 4 degree polynomial that you can reference:

$$a^2+b^2+c^2-ab-ac-bc=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}$$

$$a^3+b^3+c^3-3abc=\frac{(a+b+c)}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

$$a^2b+b^2c+c^2a-ab^2-bc^2-ca^2=\frac{(a-b)^3+(b-c)^3+(c-a)^3}{3}$$

$$a^3+b^3+c^3-a^2b-b^2c-c^2a=\frac{(2a+b)(a-b)^2+(2b+c)(b-c)^2+(2c+a)(c-a)^2}{3}$$

$$a^4+b^4+c^4-a^3b-b^3c-c^3a=$$
$$\frac{(3a^2+2ab+b^2)(a-b)^2+(3b^2+2bc+c^2)(b-c)^2+(3c^2+2ca+a^2)(a-c)^2}{4}$$

$$a^3b+b^3c+c^3a-ab^3-bc^3-ca^3=\frac{a+b+c}{3}\left[(b-a)^3+(c-b)^3+(a-c)^3\right]$$

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2=\frac{(a-b)^2(a+b)^2+(b-c)^2(b+c)^2+(c-a)^2(c+a)^2}{2}$$

- **EXAMPLES**

Now we will return to some problems in section c to find out the way to transform them in S.O.S form:

Changing the expression in S.O.S

$i) a^3+b^3+c^3+4(ab^2+bc^2+ca^2)-a^2b-b^2c-c^2a-12abc$

$ii) 3(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})-2(\frac{b}{a}+\frac{c}{b}+\frac{a}{c})-3$

$iii) a^3+b^3+c^3+3abc-ab\sqrt{2(a^2+b^2)}-bc\sqrt{2(b^2+c^2)}-ca\sqrt{2(c^2+a^2)}$

$iv) a^2b(a-b)+b^2c(b-c)+c^2a(c-a)$

$v) a_1^n+a_2^n+...+a_n^n-na_1a_2...a_n$

First problem

The first problem is very easy to change in S.O.S form, since all the expressions are in the table. We just need to apply it now:

$$a^3+b^3+c^3+6(ab^2+bc^2+ca^2)-3(a^2b+b^2c+c^2a)-12abc$$
$$=4\left(a^3+b^3+c^3-3abc\right)+6\left(ab^2+bc^2+ca^2-a^3-b^3-c^3\right)+3\left(a^3+b^3+c^3-a^2b-b^2c-c^2a\right)$$
$$=2(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]-(4b+2a)(a-b)^2-(4c+2b)(b-c)^2-(4a+2c)(c-a)^2$$
$$+(2a+b)(a-b)^2+(2b+c)(b-c)^2+(2c+a)(c-a)^2$$
$$=(2c+2a-b)(a-b)^2+(2a+2b-c)(b-c)^2+(2b+2c-a)(c-a)^2$$

Second problem

$ii) 3(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})-2(\frac{b}{a}+\frac{c}{b}+\frac{a}{c})-3$

This one is in fractional form; however we can change to the polynomial form by reducing to the same denominator

$$3(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})-2(\frac{b}{a}+\frac{c}{b}+\frac{a}{c})-3=\frac{3\left(a^2c+b^2a+c^2b\right)-2\left(b^2c+c^2a+a^2b\right)-3abc}{abc}$$

So we just need to transform the numerator in S.O.S form:

$$3\left(a^2c+b^2a+c^2b\right)-2\left(b^2c+c^2a+a^2b\right)-3abc$$

$$=-3\left(a^3+b^3+c^3-a^2c-b^2a-c^2a\right)+2\left(a^3+b^3+c^3-b^2c-c^2a-a^2b\right)+\left(a^3+b^3+c^3-3abc\right)$$

$$=-(a+2b)(a-b)^2-(b+2c)(b-c)^2-(c+2a)(c-a)^2+$$

$$\frac{(4a+2b)(a-b)^2+(4b+2c)(b-c)^2+(4c+2a)(c-a)^2}{3}+\frac{(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]}{2}$$

$$=\frac{(5a-5b+3c)(a-b)^2+(5b-5c+3a)(b-c)^2+(5c-5a+3b)(c-a)^2}{6}$$

Third problem

$$iii)a^3+b^3+c^3+3abc-ab\sqrt{2(a^2+b^2)}-bc\sqrt{2(b^2+c^2)}-ca\sqrt{2(c^2+a^2)}$$

Take a first look at it, we can see it is in root form, and transforming to the S.O.S form is not easy at all. But we can see the familiar expression $\sqrt{2\left(a^2+b^2\right)}$, which is handling by subtracting for $a+b$. From this idea, we have:

$$a^3+b^3+c^3+3abc-ab\sqrt{2(a^2+b^2)}-bc\sqrt{2(b^2+c^2)}-ca\sqrt{2(c^2+a^2)}$$

$$=a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)-ab\left(\sqrt{2\left(a^2+b^2\right)}-a-b\right)$$

$$-bc\left(\sqrt{2\left(b^2+c^2\right)}-b-c\right)-ca\left(\sqrt{2\left(c^2+a^2\right)}-a-c\right)$$

$$=\frac{(a+b-c)(a-b)^2+(b+c-a)(b-c)^2+(c+a-b)(c-a)^2}{2}-\frac{ab(a-b)^2}{a+b+\sqrt{2\left(a^2+b^2\right)}}$$

$$-\frac{bc(b-c)^2}{b+c+\sqrt{2\left(b^2+c^2\right)}}-\frac{ca(c-a)^2}{c+a+\sqrt{2\left(c^2+a^2\right)}}$$

$$=\frac{1}{2}\left[\left(a+b-c-\frac{2ab}{a+b+\sqrt{2\left(a^2+b^2\right)}}\right)(a-b)^2+\left(b+c-a-\frac{2bc}{b+c+\sqrt{2\left(b^2+c^2\right)}}\right)(b-c)^2\right.$$

$$\left.+\left(c+a-b-\frac{2ca}{c+a+\sqrt{2\left(a^2+c^2\right)}}\right)(c-a)^2\right]$$

Fourth problem:

This one is in polynomial form, and transforming to S.O.S is very easy, so what should we study here. Let take a look when we apply the table such as:

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)$$

$$=\left(a^3b+b^3c+c^3a-a^4-b^4-c^4\right)+\left(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2\right)$$

But here we will show you another way:

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)$$
$$=abc\left[\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}-\frac{ab}{c}-\frac{bc}{a}-\frac{ca}{b}\right]$$

Then we need to transform the expression in the bracket to S.O.S form

$$2\left(\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}-\frac{ab}{c}-\frac{bc}{a}-\frac{ca}{b}\right)$$
$$=2\left(\frac{a^2}{c}+c-2a\right)+2\left(\frac{b^2}{a}+a-2b\right)+2\left(\frac{c^2}{b}+b-2c\right)-\left(\frac{ab}{c}+\frac{bc}{a}-2b\right)-\left(\frac{bc}{a}+\frac{ca}{b}-2c\right)-\left(\frac{ab}{c}+\frac{ac}{b}-2a\right)$$
$$=\frac{2(a-c)^2}{c}+\frac{2(b-a)^2}{a}+\frac{2(c-b)^2}{b}-\frac{b(a-c)^2}{ac}-\frac{c(a-b)^2}{ab}-\frac{a(b-c)^2}{bc}$$

So

$$2\left[a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\right]$$
$$=\left(2bc-c^2\right)(a-b)^2+\left(2ca-a^2\right)(b-c)^2+\left(2ab-b^2\right)(c-a)^2$$

Opp, this S.O.S form is not good because this form is not concerned with $\sqrt{ab},\sqrt{bc},\sqrt{ca}$ at all. The form we got in the solution is much better:

$$(ab+ac-bc)(a-b)^2+(ab+bc-ca)(b-c)^2+(ac+bc-ab)(c-a)^2\,(*)$$

In this case, we have add the expression

$$(a-b)^2(b-c)(c-a)+(a-b)(b-c)^2(c-a)+(a-b)(b-c)(c-a)^2\,(**)$$

which is equal to 0

And we obtain exactly the expression $(*)$.

This transformation shows that there are many S.O.S form for an expression, and we can change it adding the 0 expressions, such as $(**)$. In many case, finding a suitable expression is very important to show the problem.

<u>Fifth problem:</u>

$v)\ a_1^n+a_2^n+\ldots+a_n^n-na_1a_2\ldots a_n$

This is a good problem to prove that you are a master in S.O.S transformation. This problem is a practice for you then ☺.

e)

- **METHOD**

Math Rivers never stop run by question cascade. We can see the use of S.O.S principle in solving three variables inequality, but what about many variables ones? In fact, it may be less helpful, but we still can apply it in many cases, and it usually accompany with the induction method and mixing variables technique. Let consider some example to bring out this problem.

- **EXAMPLES**

Example one [Vasile Cirtoaje]:

Give n positive real numbers $a_1, a_2, ..., a_n$ satisfying $a_1 + a_2 + ... + a_n = n$. Prove that:

$$n^2\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \geq 4(n-1)(a_1{}^2 + a_2{}^2 + ... + a_n{}^2) + n(n-2)^2$$

Solution:

We will prove the inequality by induction. And here we just consider the most complex part, that is prove the inequality for n+1 numbers when we knew that it is true for n or less than n numbers.

Firstly, notice that the inequality is equal to:

$$\left[n^2(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}) - n(n-2)^2\right](a_1 + ... + a_n)^2 \geq 4(n-1)n^2(a_1{}^2 + ... + a_n{}^2)$$

Hence if the inequality is true for $a_1 + ... + a_n = n$ then it will be true for $a_1 + ... + a_n \leq n$.

Take $f(a_1;...;a_n) = LHS - RHS$.WLOG, assume that:

$a_1 = \max\{a_1, a_2, ..., a_n\}$. We have:

$$f(a_1, ..., a_n) - f(a_1, \frac{a_2 + ... + a_n}{n-1}, ..., \frac{a_2 + ... + a_n}{n-1})$$

$$= n^2(\frac{1}{a_2} + ... + \frac{1}{a_n} - \frac{n^2}{a_2 + ... + a_n}) - 4(n-1)(a_2{}^2 + ... + a_n{}^2 - \frac{(a_2 + ... + a_n)^2}{n-1})$$

$$= \frac{n^2}{(a_2 + ... + a_n)}\sum \frac{(a_i - a_j)^2}{a_i a} - 4\sum (a_i - a_j)^2 \geq \frac{n^2}{n-1}\sum \frac{(a_i - a_j)^2}{a_i a_j} - 4\sum (a_i - a_j)^2 (*)$$

However, we have the inequality for n numbers:

$$(n-1)^2(\frac{1}{a_2} + ... + \frac{1}{a_n}) \geq 4(n-2)(a_2{}^2 + ... + a_n{}^2) + (n-1)(n-3)^2$$

$$\Leftrightarrow (n-1)\sum \frac{(a_i - a_j)^2}{a_i a_j} \geq \frac{4(n-2)}{n-1}\sum (a_i - a_j)^2$$

$$\Leftrightarrow \frac{(n-1)^2}{n-2}\sum \frac{(a_i - a_j)^2}{a_i a} \geq 4\sum (a_i - a_j)^2$$

And clearly $\frac{n^2}{n-1} > \frac{(n-1)^2}{n-2}$ then $(*) \geq 0$.

So we just need to prove that $f(a, d, ..., d) \geq 0, a + (n-1)d = n$. This is not too difficult, you will find the way easily.

Example two [Vasile Cirtoaje]:

Give n positive real numbers $a_1,...,a_n$. Prove this inequality:

$$a_1^n + ... + a_n^n + n(n-1)a_1...a_n \geq a_1...a_n(a_1 + ... + a_n)(\frac{1}{a_1} + ... + \frac{1}{a_n})$$

The solution below uses Suranyi inequality:

$(n-1)(a_1^n + ... + a_n^n) - \sum a_1(a_2^{n-1} + ... + a_n^{n-1}) \geq a_1^n + ... + a_n^n - na_1...a_n$

We will again prove the inequality by induction. And here we just consider the most complex part, that is prove the inequality for $n+1$ numbers when we knew that it is true for n or less than n numbers too.

The inequality is equal to:

$$\left(a_1^n + ... + a_n^n - na_1a_2...a_n\right) \geq a_1...a_n\left[(a_1 + ... + a_n)(\frac{1}{a_1} + ... + \frac{1}{a_n}) - n^2\right]$$

$$\Leftrightarrow \ (a_1^n + ... + a_n^n - na_1a_2...a_n) \geq a_i..a_j\left[\frac{(a_i - a_j)^2}{a_ia_j}\right]$$

Now, we will try to change the expression $a_1^n + ... + a_n^n - na_1...a_n$ to the form S.O.S. This can be done when we take $\sum a_1(a_2^{n-1} + ... + a_n^{n-1})$ as brother expression. We have:

$$(n-1)(a_1^n + ... + a_n^n - na_1a_2..a_n)$$

$$= (n-1)(a_1^n + ... + a_n^n) - \sum a_1(a_2^{n-1} + ... + a_n^{n-1}) + \sum a_1\left[a_2^n + ... + a_n^n - (n-1)a_2...a_n\right]$$

$$\geq a_1^n + ... + a_n^n - na_1...a_n + (n-2)\sum a_1...a_n\frac{(a_i - a_j)^2}{a_ia_j}$$

$$\Rightarrow \ a_1^n + ... + a_n^n - na_1a_2...a_n \geq \sum a_1...a_n\frac{(a_i - a_j)^2}{a_ia_j} = a_1a_2...a_n\left[(a_1 + ... + a_n)(\frac{1}{a_1} + ... + \frac{1}{a_n}) - n^2\right]$$

And so the proof is done.

So we saw the power of S.O.S technique even for n variables, however, we must combine S.O.S with induction flexible to obtain the best result. Indeed, we also can use S.O.S technique directly.

Now let discuss the problem I mentions in part a)

Example three: [Nguyen Anh Cuong]

Give $a_1, a_2,..., a_n \geq 0$. Find k such that:

$$k\frac{\sum\limits_{1\le i<j\le n}\left(a_i-a_j\right)^2}{a_1+a_2+...+a_n}\le\frac{a_1+a_2+..+a_n}{n}-\sqrt[n]{a_1a_2...a_n}$$

Firstly, to solve this problem directly by S.O.S technique, we must think about transform $x_1^n+x_2^n+...+x_n^n-nx_1x_2...x_n$ to S.O.S form.

C. __CONCLUSION__

Many thorny exercises are still unsolved even there are more and more methods are created to handle it. However, challenges shoot us forward and that is one of the reason for the birth of S.O.S technique, even this is only a grain of sand in the vast desert. There are still more and more ideas hidden inside every problem, maybe small but still valuable, and our job is finding and generalizing it. Wish for the success of your Math way ☺ .