PROOF OF THE LEGENDRE'S CONJECTURE

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Abstract. Legendre's conjecture states that there exists a prime between n^2 and $(n+1)^2$, for every positive integer n. Here I prove that for sufficiently large n, there is a prime number between n^2 and $(n+1)^2$. The proof relies on the idea of counting the maximum power, $o_p(n)$ of a prime $p \leq n$ such that $p^{o_p(n)}||n$.

1. Notations

We fix some notations and conventions that will be used throughout this paper. n will always denote a sufficiently large positive integer, whose meaning would be understood from the context.

- $[x]$ denotes the greatest integer less than or equal to x.
- p will always denote a prime number.

Theorem 1.1. *(Legendre's Conjecture)*
There is a prime number between n^2 and $(n+1)^2$ for every positive integer n

To prove (1.1), we need the following results.

2. Some established results

We state below some important results without proof.

Lemma 2.1. *Given a positive natural number $n > 1$. If p denotes any prime less than n. Then the maximum power of p (denoted by $o_p(n!)$) that occurs in $n!$ is,*

$$\sum_{k=1}^{\left[\frac{\log n}{\log p}\right]} \left[\frac{n}{p^k}\right] \tag{2.1}$$

Theorem 2.2. *Let n and k be positive integers with $n \leq k$. Suppose $p \leq n$ be any prime such that $p^m|\binom{n}{k}$, for any natural number m. Then*

$$p^m \leq n$$

Theorem 2.3. *For any positive real number x, denote*

$$\nu(x) = \sum_{p \leq x} \log p \tag{2.2}$$

Then $\nu(x) < 1.01624x$

$\nu(x)$ is also known as the *first Chebychev's function.*

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Theorem 2.4. *The number of primes less than or equal to n, denoted by $\pi(n)$ satisfies,*

$$\frac{1}{6}\frac{n}{\log n} < \pi(n) < 6\frac{n}{\log n}$$

for any $n > 1$.

3. Proof of 1.1

Proof. Let there does not exist any prime between n^2 and $(n+1)^2$. Hence all the numbers between n^2 and $(n+1)^2$ are composite and their prime factors p satisfies

$$p \leq \frac{(n+1)^2}{2}$$

The theorem will be proved if we can show that

$$\prod_{p\leq\frac{(n+1)^2}{2}} p^{o_p((n+1)^2!)-o_p(n^2!)} < \frac{(n+1)^2!}{n^2!}. \tag{3.1}$$

We divide the proof into the following subsections.

3.1. Estimation of the product $\prod_{p\leq\frac{(n+1)^2}{2}} p^{o_p((n+1)^2!)-o_p(n^2)!}$.

Case A: If $p \geq 2n+1$, then it is clear that $o_p((n+1)^2!) - o_p(n^2!) \leq 1$.
Case B: If $p < 2n+1$, we estimate the following.
We know that

$$\frac{(n+1)^2!}{n^2!} = (2n+1)!\binom{(n+1)^2}{2n+1}$$

Thus

$$o_p((n+1)^2!) - o_p(n^2!) = o_p((2n+1)!) + o_p\left(\binom{(n+1)^2}{2n+1}\right)$$

Hence

$$\begin{aligned}\prod_{p<2n+1} p^{(o_p((n+1)^2!)-o_p(n^2!))} &= \prod_{p<2n+1} p^{\left(o_p((2n+1)!)+o_p\left(\binom{(n+1)^2}{2n+1}\right)\right)}\\ &= \underbrace{\prod_{p<2n+1} p^{o_p((2n+1)!)}}_{P_1}\underbrace{\prod_{p<2n+1} p^{o_p\left(\binom{(n+1)^2}{2n+1}\right)}}_{P_2} \qquad (3.2)\end{aligned}$$

3.1.1. *Estimation of P_1.*

From Lemma 2.1, we obtain

$$\begin{aligned}o_p((2n+1)!) &= \sum_{k=1}^{\left[\frac{\log(2n+1)}{\log p}\right]}\left[\frac{2n+1}{p^k}\right]\\ &\leq \sum_{k=1}^{\left[\frac{\log(2n+1)}{\log p}\right]}\frac{2n+1}{p^k}\\ &\leq \frac{2n}{p-1} \qquad (3.3)\end{aligned}$$

Thus

$$p^{o_p((2n+1)!)} \leq p^{\frac{2n}{p-1}} \tag{3.4}$$

3.1.2. *Estimation of P_2.*
To estimate P_2, we appeal to Theorem 2.2 to obtain

$$\prod_{p<2n+1} p^{o_p\left(\binom{(n+1)^2}{2n+1}\right)} \leq (n+1)^{2\pi(2n)} \tag{3.5}$$

Thus putting the estimates of (3.4) and (3.5) in (3.3), we obtain

$$\prod_{p<2n+1} p^{(o_p((n+1)^2!)-o_p(n^2!))} \leq \left(\prod_{p<2n+1} p^{\frac{2n}{p-1}}\right)(n+1)^{2\pi(2n)}.$$

And hence from **Case A** and **Case B**, we get

$$\prod_{p\leq\frac{(n+1)^2}{2}} p \leq \left(\prod_{p<2n+1} p^{\frac{2n}{p-1}}\right)(n+1)^{2\pi(2n)}\left(\prod_{2n+1\leq p\leq\frac{(n+1)^2}{2}} p\right) \tag{3.6}$$

Taking logarithm on both sides of (3.6) we obtain

$$\begin{aligned}\log\left(\prod_{p\leq\frac{(n+1)^2}{2}} p^{o_p((n+1)^2)-o_p(n^2)}\right) &\leq \underbrace{2n\sum_{p<2n+1}\frac{\log p}{p-1}}_{S_1}+\underbrace{2\pi(2n)\log(n+1)}_{S_2}\\ &\quad+\underbrace{\sum_{2n+1\leq p\leq\frac{(n+1)^2}{2}}\log p}_{S_3}\end{aligned} \tag{3.7}$$

We evaluate each of the three terms on the right of (3.7) separately.

3.1.3. *Evaluation of S_1.*
We observe that $p-1$ is even for every prime $p>2$. Thus for sufficiently large n we obtain the following using Theorem 2.3

$$\begin{aligned}S_1 &\leq \frac{1}{2}\times 2n\sum_{p<2n+1}\log p\\ &\leq 1.01624\, n\log(2n+1)\end{aligned}$$

3.1.4. *Evaluation of S_2.*
From Theorem 2.4, we obtain

$$S_2 < 24n$$

3.1.5. *Evaluation of S_3.*
In **Case A**, we noticed that

$$o_p((n+1)^2)-o_p(n^2)\leq 1,\ \text{ if } 2n+1\leq p\leq\tfrac{(n+1)^2}{2}.$$

Thus for $2n+1\leq p\leq\frac{(n+1)^2}{2}$, only multiples of p of the form kp, for some integer k and $(k,p)=1$ may occur among numbers from n^2+1 to $(n+1)^2$. Said differently, if such a multiple of p occurs that is, if

$$n^2 < kp \leq (n+1)^2 \tag{3.8}$$

then there should exist a k satisfying

$$\frac{n^2}{p} < k \leq \frac{(n+1)^2}{p} \tag{3.9}$$

Thus the number, N of positive integers satisfying (3.9) satisfies

$$N \leq \frac{2n+1}{p}$$

Hence if $p > (2n+1)\log n$, then since for sufficiently large n, $N \to 0$, and since the left hand side of (3.9) is a strict inequality, we will not be able to find any k for which (3.9) holds and hence (3.8) does not hold as well. Thus we need to worry about primes $2n+1 < p \leq (2n+1)\log n$ in S_3, for sufficiently large n. Thus for sufficiently large n we have

$$\begin{aligned} S_3 &= \sum_{2n+1<p\leq(2n+1)\log n} \log p \\ &\leq \sum_{p\leq(2n+1)\log n} \log p \\ &\leq 1.01624\,(2n+1)\log n \qquad (3.10) \end{aligned}$$

Putting the estimates for S_1, S_2, and S_3 in (3.7) we obtain

$$\log\left(\prod_{p\leq\frac{(n+1)^2}{2}} p^{o_p((n+1)^2)-o_p(n^2)}\right) \leq 1.01624\,n\log(2n+1) + 24n + 1.01624\,(2n+1)\log n \tag{3.11}$$

3.2. **Estimation of $\frac{(n+1)^2!}{n^2!}$.**
We have

$$\begin{aligned} \frac{(n+1)^2!}{n^2!} &= (n^2+1)(n^2+2)....(n^2+2n+1) \qquad (3.12) \\ &> n^{2(2n+1)} \end{aligned}$$

Taking logarithm on both sides of (3.10), we obtain

$$\log\frac{(n+1)^2!}{n^2!} > 2(2n+1)\log n \tag{3.13}$$

3.3. **Verification of 3.1.**
From (3.9) and (3.11) we obtain

$$\begin{aligned} &\log\frac{(n+1)^2!}{n^2!} - \log\left(\prod_{p\leq\frac{(n+1)^2}{2}} p^{o_p((n+1)^2)-o_p(n^2)}\right) \\ &> 2(2n+1)\log n - [1.01624\,n\log(2n+1) + 24\,n + 1.01624\,(2n+1)\log n] \\ &> 0.98376(2n+1)\log n - 1.01624\,n\log(2n+1) - 24\,n \\ &> 0.98376(2n+1)\log n - 1.01624\,n\log n - 1.01624\,n\log(2+\frac{1}{n}) - 24\,n \\ &> 0.95128\,n\log n + 0.98376\log n - 1.01624\,n\log(2+\frac{1}{n}) - 24\,n \\ &> 0 \end{aligned}$$

for sufficiently large n. This verifies (3.1) and proves Theorem 1.1 for sufficiently large n. $\square$

References

[1] Rosser, J. Barkley; Schoenfeld, Lowell (1962): *Illinois J. Math. 6: 6494*, Approximate formulas for some functions of prime numbers.

[2] Apostol, Tom M.: *Springer International Student Edition*, Introduction to Analytic Number Theory.

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