

A Collection of Limits

Contents

1	Short theoretical introduction	1
2	Problems	12
3	Solutions	23

Chapter 1

Short theoretical introduction

Consider a sequence of real numbers $(a_n)_{n \geq 1}$, and $l \in \overline{\mathbb{R}}$. We'll say that l represents the limit of $(a_n)_{n \geq 1}$ if any neighborhood of l contains all the terms of the sequence, starting from a certain index. We write this fact as $\lim_{n \rightarrow \infty} a_n = l$, or $a_n \rightarrow l$.

We can rewrite the above definition into the following equivalence:

$$\lim_{n \rightarrow \infty} a_n = l \Leftrightarrow (\forall) V \in \mathcal{V}(l), (\exists) n_V \in \mathbb{N}^* \text{ such that } (\forall) n \geq n_V \Rightarrow a_n \in V.$$

One can easily observe from this definition that if a sequence is constant then it's limit is equal with the constant term.

We'll say that a sequence of real numbers $(a_n)_{n \geq 1}$ is convergent if it has limit and $\lim_{n \rightarrow \infty} a_n \in \mathbb{R}$, or divergent if it doesn't have a limit or if it has the limit equal to $\pm\infty$.

Theorem: If a sequence has limit, then this limit is unique.

Proof: Consider a sequence $(a_n)_{n \geq 1} \subseteq \mathbb{R}$ which has two different limits $l', l'' \in \overline{\mathbb{R}}$. It follows that there exist two neighborhoods $V' \in \mathcal{V}(l')$ and $V'' \in \mathcal{V}(l'')$ such that $V' \cap V'' = \emptyset$. As $a_n \rightarrow l' \Rightarrow (\exists) n' \in \mathbb{N}^*$ such that $(\forall) n \geq n' \Rightarrow a_n \in V'$. Also, since $a_n \rightarrow l'' \Rightarrow (\exists) n'' \in \mathbb{N}^*$ such that $(\forall) n \geq n'' \Rightarrow a_n \in V''$. Hence $(\forall) n \geq \max\{n', n''\}$ we have $a_n \in V' \cap V'' = \emptyset$.

Theorem: Consider a sequence of real numbers $(a_n)_{n \geq 1}$. Then we have:

$$(i) \quad \lim_{n \rightarrow \infty} a_n = l \in \mathbb{R} \Leftrightarrow (\forall) \varepsilon > 0, (\exists) n_\varepsilon \in \mathbb{N}^* \text{ such that } (\forall) n \geq n_\varepsilon \Rightarrow |a_n - l| < \varepsilon.$$

(ii) $\lim_{n \rightarrow \infty} a_n = \infty \Leftrightarrow (\forall)\varepsilon > 0, (\exists)n_\varepsilon \in \mathbb{N}^*$ such that $(\forall)n \geq n_\varepsilon \Rightarrow a_n > \varepsilon$.

(iii) $\lim_{n \rightarrow \infty} a_n = -\infty \Leftrightarrow (\forall)\varepsilon > 0, (\exists)n_\varepsilon \in \mathbb{N}^*$ such that $(\forall)n \geq n_\varepsilon \Rightarrow a_n < -\varepsilon$

Theorem: Let $(a_n)_{n \geq 1}$ a sequence of real numbers.

1. If $\lim_{n \rightarrow \infty} a_n = l$, then any subsequence of $(a_n)_{n \geq 1}$ has the limit equal to l .
2. If there exist two subsequences of $(a_n)_{n \geq 1}$ with different limits, then the sequence $(a_n)_{n \geq 1}$ is divergent.
3. If there exist two subsequences of $(a_n)_{n \geq 1}$ which cover it and have a common limit, then $\lim_{n \rightarrow \infty} a_n = l$.

Definition: A sequence $(x_n)_{n \geq 1}$ is a Cauchy sequence if $(\forall)\varepsilon > 0, (\exists)n_\varepsilon \in \mathbb{N}$ such that $|x_{n+p} - x_n| < \varepsilon, (\forall)n \geq n_\varepsilon, (\forall)p \in \mathbb{N}$.

Theorem: A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Theorem: Any increasing and unbounded sequence has the limit ∞ .

Theorem: Any increasing and bounded sequence converge to the upper bound of the sequence.

Theorem: Any convergent sequence is bounded.

Theorem(Cesaro lemma): Any bounded sequence of real numbers contains at least one convergent subsequence.

Theorem(Weierstrass theorem): Any monotonic and bounded sequence is convergent.

Theorem: Any monotonic sequence of real numbers has limit.

Theorem: Consider two convergent sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ such that $a_n \leq b_n, (\forall)n \in \mathbb{N}^*$. Then we have $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$.

Theorem: Consider a convergent sequence $(a_n)_{n \geq 1}$ and a real number a such that $a_n \leq a, (\forall)n \in \mathbb{N}^*$. Then $\lim_{n \rightarrow \infty} a_n \leq a$.

Theorem: Consider a convergent sequence $(a_n)_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} a_n = a$. Then $\lim_{n \rightarrow \infty} |a_n| = |a|$.

Theorem: Consider two sequences of real numbers $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ such that $a_n \leq b_n$, $(\forall)n \in \mathbb{N}^*$. Then:

1. If $\lim_{n \rightarrow \infty} a_n = \infty$ it follows that $\lim_{n \rightarrow \infty} b_n = \infty$.
2. If $\lim_{n \rightarrow \infty} b_n = -\infty$ it follows that $\lim_{n \rightarrow \infty} a_n = -\infty$.

Limit operations:

Consider two sequences a_n and b_n which have limit. Then we have:

1. $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$ (except the case $(\infty, -\infty)$).
2. $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$ (except the cases $(0, \pm\infty)$).
3. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$ (except the cases $(0, 0)$, $(\pm\infty, \pm\infty)$).
4. $\lim_{n \rightarrow \infty} a_n^{b_n} = (\lim_{n \rightarrow \infty} a_n)^{\lim_{n \rightarrow \infty} b_n}$ (except the cases $(1, \pm\infty)$, $(\infty, 0)$, $(0, 0)$).
5. $\lim_{n \rightarrow \infty} (\log_{a_n} b_n) = \log \lim_{n \rightarrow \infty} a_n (\lim_{n \rightarrow \infty} b_n)$.

Trivial consequences:

1. $\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$;
2. $\lim_{n \rightarrow \infty} (\lambda a_n) = \lambda \lim_{n \rightarrow \infty} a_n$ ($\lambda \in \mathbb{R}$);
3. $\lim_{n \rightarrow \infty} \sqrt[k]{a_n} = \sqrt[k]{\lim_{n \rightarrow \infty} a_n}$ ($k \in \mathbb{N}$);

Theorem (Squeeze theorem): Let $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$, $(c_n)_{n \geq 1}$ be three sequences of real numbers such that $a_n \leq b_n \leq c_n$, $(\forall)n \in \mathbb{N}^*$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = l \in \overline{\mathbb{R}}$. Then $\lim_{n \rightarrow \infty} b_n = l$.

Theorem: Let $(x_n)_{n \geq 1}$ a sequence of real numbers such that $\lim_{n \rightarrow \infty} (x_{n+1} - x_n) = \alpha \in \overline{\mathbb{R}}$.

1. If $\alpha > 0$, then $\lim_{n \rightarrow \infty} x_n = \infty$.
2. If $\alpha < 0$, then $\lim_{n \rightarrow \infty} x_n = -\infty$.

Theorem (Ratio test): Consider a sequence of real positive numbers $(a_n)_{n \geq 1}$, for which $l = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \in \overline{\mathbb{R}}$.

1. If $l < 1$ then $\lim_{n \rightarrow \infty} a_n = 0$.
2. If $l > 1$ then $\lim_{n \rightarrow \infty} a_n = \infty$.

Proof: 1. Let $V = (\alpha, \beta) \in \mathcal{V}(l)$ with $l < \beta < 1$. Because $l = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$, there is some $n_0 \in \mathbb{N}^*$ such that $(\forall) n \geq n_0 \Rightarrow \frac{a_{n+1}}{a_n} \in V$, hence $(\forall) n \geq n_0 \Rightarrow \frac{a_{n+1}}{a_n} < 1$. That means starting from the index n_0 the sequence $(a_n)_{n \geq 1}$ is strictly decreasing. Since the sequence is strictly decreasing and it contains only positive terms, the sequence is bounded. Using Weierstrass Theorem, it follows that the sequence is convergent. We have:

$$a_{n+1} = \frac{a_{n+1}}{a_n} \cdot a_n \Rightarrow \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \cdot \lim_{n \rightarrow \infty} a_n$$

which is equivalent with:

$$\lim_{n \rightarrow \infty} a_n(1 - l) = 0$$

which implies that $\lim_{n \rightarrow \infty} a_n = 0$.

2. Denoting $b_n = \frac{1}{a_n}$ we have $\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = \frac{1}{l} < 1$, hence $\lim_{n \rightarrow \infty} b_n = 0$ which implies that $\lim_{n \rightarrow \infty} a_n = \infty$.

Theorem: Consider a convergent sequence of real non-zero numbers $(x_n)_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} n \left(\frac{x_n}{x_{n-1}} - 1 \right) \in \mathbb{R}^*$. Then $\lim_{n \rightarrow \infty} x_n = 0$.

Theorem (Cesaro-Stolz lemma): 1. Consider two sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ such that:

- (i) the sequence $(b_n)_{n \geq 1}$ is strictly increasing and unbounded;
- (ii) the limit $\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$ exists.

Then the sequence $\left(\frac{a_n}{b_n} \right)_{n \geq 1}$ is convergent and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l$.

Proof: Let's consider the case $l \in \mathbb{R}$ and assume $(b_n)_{n \geq 1}$ is a strictly increasing sequence, hence $\lim_{n \rightarrow \infty} b_n = \infty$. Now let $V \in \mathcal{V}(l)$, then there exists $\alpha > 0$ such

that $(l - \alpha, l + \alpha) \subseteq V$. Let $\beta \in \mathbb{R}$ such that $0 < \beta < \alpha$. As $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l$, there exists $k \in \mathbb{N}^*$ such that $(\forall)n \geq k \Rightarrow \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \in (l - \beta, l + \beta)$, which implies that:

$$(l - \beta)(b_{n+1} - b_n) < a_{n+1} - a_n < (l + \beta)(b_{n+1} - b_n), \quad (\forall)n \geq k$$

Now writing this inequality from k to $n - 1$ we have:

$$(l - \beta)(b_{k+1} - b_k) < a_{k+1} - a_k < (l + \beta)(b_{k+1} - b_k)$$

$$(l - \beta)(b_{k+2} - b_{k+1}) < a_{k+2} - a_{k+1} < (l + \beta)(b_{k+2} - b_{k+1})$$

...

$$(l - \beta)(b_n - b_{n-1}) < a_n - a_{n-1} < (l + \beta)(b_n - b_{n-1})$$

Summing all these inequalities we find that:

$$(l - \beta)(b_n - b_k) < a_n - a_k < (l + \beta)(b_n - b_k)$$

As $\lim_{n \rightarrow \infty} b_n = \infty$, starting from an index we have $b_n > 0$. The last inequality rewrites as:

$$\begin{aligned} (l - \beta) \left(1 - \frac{b_k}{b_n}\right) &< \frac{a_n}{b_n} - \frac{a_k}{b_n} < (l + \beta) \left(1 - \frac{b_k}{b_n}\right) \Leftrightarrow \\ \Leftrightarrow (l - \beta) + \frac{a_k + (\beta - l)b_k}{b_n} &< \frac{a_n}{b_n} < l + \beta + \frac{a_k - (\beta + l)b_k}{b_n} \end{aligned}$$

As

$$\lim_{n \rightarrow \infty} \frac{a_k + (\beta - l)b_k}{b_n} = \lim_{n \rightarrow \infty} \frac{a_k - (\beta + l)b_k}{b_n} = 0$$

there exists an index $p \in \mathbb{N}^*$ such that $(\forall)n \geq p$ we have:

$$\frac{a_k + (\beta - l)b_k}{b_n}, \frac{a_k - (\beta + l)b_k}{b_n} \in (\beta - \alpha, \alpha - \beta)$$

We shall look for the inequalities:

$$\frac{a_k + (\beta - l)b_k}{b_n} > \beta - \alpha$$

and

$$\frac{a_k - (\beta + l)b_k}{b_n} < \alpha - \beta$$

Choosing $m = \max\{k, p\}$, then $(\forall)n \geq m$ we have:

$$l - \alpha < \frac{a_n}{b_n} < l + \alpha$$

which means that $\frac{a_n}{b_n} \in V \Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l$. It remains to prove the theorem when $l = \pm\infty$, but these cases can be proven analogous choosing $V = (\alpha, \infty)$ and $V = (-\infty, \alpha)$, respectively.

2. Let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ such that:

(i) $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 0$, $y_n \neq 0$, $(\forall)n \in \mathbb{N}^*$;

(ii) the sequence $(y_n)_{n \geq 1}$ is strictly decreasing;

(iii) the limit $\lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = l \in \overline{\mathbb{R}}$.

Then the sequence $\left(\frac{x_n}{y_n}\right)_{n \geq 1}$ has a limit and $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = l$.

Remark: In problem's solutions we'll write directly $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$, and if the limit we arrive to belongs to $\overline{\mathbb{R}}$, then the application of Cesaro-Stolz lemma is valid.

Trivial consequences:

1. Consider a sequence $(a_n)_{n \geq 1}$ of strictly positive real numbers for which exists $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l$. Then we have:

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$$

Proof: Using Cesaro-Stolz theorem we have:

$$\lim_{n \rightarrow \infty} (\ln \sqrt[n]{a_n}) = \lim_{n \rightarrow \infty} \frac{\ln a_n}{n} = \lim_{n \rightarrow \infty} \frac{\ln a_{n+1} - \ln a_n}{(n+1) - n} = \lim_{n \rightarrow \infty} \ln \left(\frac{a_{n+1}}{a_n} \right) = \ln l$$

Then:

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} e^{\ln \sqrt[n]{a_n}} = e^{\lim_{n \rightarrow \infty} (\ln \sqrt[n]{a_n})} = e^{\ln l} = l$$

2. Let $(x_n)_{n \geq 1}$ a sequence of real numbers which has limit. Then:

$$\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = \lim_{n \rightarrow \infty} x_n$$

3. Let $(x_n)_{n \geq 1}$ a sequence of real positive numbers which has limit. Then:

$$\lim_{n \rightarrow \infty} \sqrt[n]{x_1 x_2 \dots x_n} = \lim_{n \rightarrow \infty} x_n$$

Theorem (Reciprocal Cesaro-Stolz): Let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ two sequences of real numbers such that:

(i) $(y_n)_{n \geq 1}$ is strictly increasing and unbounded;

(ii) the limit $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = l \in \overline{\mathbb{R}}$;

(iii) the limit $\lim_{n \rightarrow \infty} \frac{y_n}{y_{n+1}} \in \mathbb{R}_+ \setminus \{1\}$.

Then the limit $\lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$ exists and it is equal to l .

Theorem (exponential sequence): Let $a \in \mathbb{R}$. Consider the sequence $x_n = a^n$, $n \in \mathbb{N}^*$.

1. If $a \leq -1$, the sequence is divergent.

2. If $a \in (-1, 1)$, then $\lim_{n \rightarrow \infty} x_n = 0$.

3. If $a = 1$, then $\lim_{n \rightarrow \infty} x_n = 1$.

4. If $a > 1$, then $\lim_{n \rightarrow \infty} x_n = \infty$.

Theorem (power sequence): Let $a \in \mathbb{R}$. Consider the sequence $x_n = n^a$, $n \in \mathbb{N}^*$.

1. If $a < 0$, then $\lim_{n \rightarrow \infty} x_n = 0$.

2. If $a = 0$, then $\lim_{n \rightarrow \infty} x_n = 1$.

3. If $a > 0$, then $\lim_{n \rightarrow \infty} x_n = \infty$.

Theorem (polynomial sequence): Let $a_n = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0$, $(a_k \neq 0)$.

1. If $a_k > 0$, then $\lim_{n \rightarrow \infty} a_n = \infty$.

2. If $a_k < 0$, then $\lim_{n \rightarrow \infty} a_n = -\infty$.

Theorem: Let $b_n = \frac{a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0}{b_p n^p + b_{p-1} n^{p-1} + \dots + b_1 n + b_0}$, ($a_k \neq 0 \neq b_p$).

1. If $k < p$, then $\lim_{n \rightarrow \infty} b_n = 0$.
2. If $k = p$, then $\lim_{n \rightarrow \infty} b_n = \frac{a_k}{b_p}$.
3. If $k > p$, then $\lim_{n \rightarrow \infty} b_n = \frac{a_k}{b_p} \cdot \infty$.

Theorem: The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}^*$ is a strictly increasing and bounded sequence and $\lim_{n \rightarrow \infty} a_n = e$.

Theorem: Consider a sequence $(a_n)_{n \geq 1}$ of real non-zero numbers such that $\lim_{n \rightarrow \infty} a_n = 0$. Then $\lim_{n \rightarrow \infty} (1 + a_n)^{\frac{1}{a_n}} = e$.

Proof: If $(b_n)_{n \geq 1}$ is a sequence of non-zero positive integers such that $\lim_{n \rightarrow \infty} b_n = \infty$, we have $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{b_n}\right)^{b_n} = e$. Let $\varepsilon > 0$. From $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{b_n}\right)^{b_n} = e$, it follows that there exists $n'_\varepsilon \in \mathbb{N}^*$ such that $(\forall) n \geq n'_\varepsilon \Rightarrow \left| \left(1 + \frac{1}{b_n}\right)^{b_n} - e \right| < \varepsilon$. Also, since $\lim_{n \rightarrow \infty} b_n = \infty$, there exists $n''_\varepsilon \in \mathbb{N}^*$ such that $(\forall) n \geq n''_\varepsilon \Rightarrow b_n > n'_\varepsilon$. Therefore there exists $n_\varepsilon = \max\{n'_\varepsilon, n''_\varepsilon\} \in \mathbb{N}^*$ such that $(\forall) n \geq n_\varepsilon \Rightarrow \left| \left(1 + \frac{1}{b_n}\right)^{b_n} - e \right| < \varepsilon$. This means that: $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{b_n}\right)^{b_n} = e$. The same property is fulfilled if $\lim_{n \rightarrow \infty} b_n = -\infty$.

If $(c_n)_{n \geq 1}$ is a sequence of real numbers such that $\lim_{n \rightarrow \infty} c_n = \infty$, then $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{c_n}\right)^{c_n} = e$. We can assume that $c_n > 1$, $(\forall) n \in \mathbb{N}^*$. Let's denote $d_n = \lfloor c_n \rfloor \in \mathbb{N}^*$. In this way $(d_n)_{n \geq 1}$ is sequence of positive integers with $\lim_{n \rightarrow \infty} d_n = \infty$. We have:

$$d_n \leq c_n < d_n + 1 \Rightarrow \frac{1}{d_n + 1} < \frac{1}{c_n} \leq \frac{1}{d_n}$$

Hence it follows that:

$$\left(1 + \frac{1}{d_n + 1}\right)^{d_n} < \left(1 + \frac{1}{c_n}\right)^{d_n} \leq \left(1 + \frac{1}{c_n}\right)^{c_n} < \left(1 + \frac{1}{c_n}\right)^{d_n + 1} \leq \left(1 + \frac{1}{d_n}\right)^{d_n + 1}$$

Observe that:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{d_n + 1}\right)^{d_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{d_n + 1}\right)^{d_n + 1} \cdot \left(1 + \frac{1}{d_n + 1}\right)^{-1} = e$$

and

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{d_n}\right)^{d_n + 1} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{d_n}\right)^{d_n} \cdot \left(1 + \frac{1}{d_n}\right) = e$$

Using the Squeeze Theorem it follows that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{c_n}\right)^{c_n} = e$. The same property is fulfilled when $\lim_{n \rightarrow \infty} c_n = -\infty$.

Now if the sequence $(a_n)_{n \geq 1}$ contains a finite number of positive or negative terms we can remove them and assume that the sequence contains only positive terms. Denoting $x_n = \frac{1}{a_n}$ we have $\lim_{n \rightarrow \infty} x_n = \infty$. Then we have

$$\lim_{n \rightarrow \infty} (1 + a_n)^{\frac{1}{a_n}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{x_n}\right)^{x_n} = e$$

If the sequence contains an infinite number of positive or negative terms, the same fact happens for the sequence $(x_n)_{n \geq 1}$ with $x_n = \frac{1}{a_n}$, $(\forall) n \in \mathbb{N}^*$. Let's denote by $(a'_n)_{n \geq 1}$ the subsequence of positive terms, and by $(a''_n)_{n \geq 1}$ the subsequence of negative terms. Also let $c'_n = \frac{1}{a'_n}$, $(\forall) n \in \mathbb{N}^*$ and $c''_n = \frac{1}{a''_n}$, $(\forall) n \in \mathbb{N}^*$. Then it follows that $\lim_{n \rightarrow \infty} c'_n = \infty$ and $\lim_{n \rightarrow \infty} c''_n = -\infty$. Hence:

$$\lim_{n \rightarrow \infty} (1 + a'_n)^{\frac{1}{a'_n}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{c'_n}\right)^{c'_n} = e$$

and

$$\lim_{n \rightarrow \infty} (1 + a''_n)^{\frac{1}{a''_n}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{c''_n}\right)^{c''_n} = e$$

Then it follows that: $\lim_{n \rightarrow \infty} (1 + a_n)^{\frac{1}{a_n}} = e$.

Consequence: Let $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$ two sequences of real numbers such that $a_n \neq 1$, $(\forall) n \in \mathbb{N}^*$, $\lim_{n \rightarrow \infty} a_n = 1$ and $\lim_{n \rightarrow \infty} b_n = \infty$ or $\lim_{n \rightarrow \infty} b_n = -\infty$. If there exists $\lim_{n \rightarrow \infty} (a_n - 1)b_n \in \overline{\mathbb{R}}$, then we have $\lim_{n \rightarrow \infty} a_n^{b_n} = e^{\lim_{n \rightarrow \infty} (a_n - 1)b_n}$.

Theorem: Consider the sequence $(a_n)_{n \geq 0}$ defined by $a_n = \sum_{k=0}^n \frac{1}{k!}$. We have $\lim_{n \rightarrow \infty} a_n = e$.

Theorem: Let $(c_n)_{n \geq 1}$, a sequence defined by

$$c_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n, \quad n \geq 1$$

Then $(c_n)_{n \geq 1}$ is strictly decreasing and bounded, and $\lim_{n \rightarrow \infty} c_n = \gamma$, where γ is the Euler constant.

Recurrent sequences

A sequence $(x_n)_{n \geq 1}$ is a k -order recurrent sequence, if it is defined by a formula of the form

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}), \quad n \geq 1$$

with given $x_1, x_2, \dots, x_k$. The recurrence is linear if f is a linear function. Second order recurrence formulas which are homogeneous, with constant coefficients, have the form $x_{n+2} = \alpha x_{n+1} + \beta x_n$, $(\forall) n \geq 1$ with given x_1, x_2, α, β . To this recurrence formula we attach the equation $r^2 = \alpha r + \beta$, with r_1, r_2 as solutions.

If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then $x_n = Ar_1^n + Br_2^n$, where A, B are two real numbers, usually found from the terms x_1, x_2 . If $r_1 = r_2 = r \in \mathbb{R}$, then $x_n = r^n(A + nB)$ and if $r_1, r_2 \in \mathbb{R}$, we have $r_1, r_2 = \rho(\cos \theta + i \sin \theta)$ so $x_n = \rho^n(\cos n\theta + i \sin n\theta)$.

Limit functions

Definition: Let $f : D \rightarrow \mathbb{R}$ ($D \subseteq \mathbb{R}$) and $x_0 \in \overline{\mathbb{R}}$ and accumulation point of D . We'll say that $l \in \overline{\mathbb{R}}$ is the limit of the function f in x_0 , and we write $\lim_{x \rightarrow x_0} f(x) = l$, if for any neighborhood $\mathcal{V}$ of l , there is a neighborhood U of x_0 , such that for any $x \in D \cap U \setminus \{x_0\}$, we have $f(x) \in \mathcal{V}$.

Theorem: Let $f : D \rightarrow \mathbb{R}$ ($D \subset \mathbb{R}$) and x_0 an accumulation point of D . Then $\lim_{x \rightarrow x_0} f(x) = l$ ($l, x_0 \in \mathbb{R}$) if and only if $(\forall)\varepsilon > 0$, $(\exists)\delta_\varepsilon > 0$, $(\forall)x \in D \setminus \{x_0\}$ such that $|x - x_0| < \delta_\varepsilon \Rightarrow |f(x) - l| < \varepsilon$.

If $l = \pm\infty$, we have:

$\lim_{x \rightarrow x_0} f(x) = \pm\infty \Leftrightarrow (\forall)\varepsilon > 0$, $(\exists)\delta_\varepsilon > 0$, $(\forall)x \in D \setminus \{x_0\}$ such that $|x - x_0| < \delta_\varepsilon$, we have $f(x) > \varepsilon$ ($f(x) < \varepsilon$).

Theorem: Let $f : D \subset \mathbb{R} \Rightarrow \mathbb{R}$ and x_0 an accumulation point of D . Then $\lim_{x \rightarrow x_0} f(x) = l$ ($l \in \overline{\mathbb{R}}$, $x_0 \in \mathbb{R}$), if and only if $(\forall)(x_n)_{n \geq 1}$, $x_n \in D \setminus \{x_0\}$, $x_n \rightarrow x_0$, we have $\lim_{n \rightarrow \infty} f(x_n) = l$.

One-side limits

Definition: Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $x_0 \in \overline{\mathbb{R}}$ an accumulation point of D . We'll say that $l_s \in \mathbb{R}$ (or $l_d \in \mathbb{R}$) is the left-side limit (or right-side limit) of f in x_0 if for any neighborhood $\mathcal{V}$ of l_s (or l_d), there is a neighborhood U of x_0 , such that for any $x < x_0$, $x \in U \cap D \setminus \{x_0\}$ ($x > x_0$ respectively), $f(x) \in \mathcal{V}$.

We write $l_s = \lim_{\substack{x \rightarrow x_0 \\ x < x_0}} f(x) = f(x_0 - 0)$ and $l_d = \lim_{\substack{x \rightarrow x_0 \\ x > x_0}} f(x) = f(x_0 + 0)$.

Theorem: Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}$ an accumulation point of the sets $(-\infty, x_0) \cap D$ and $(x_0, \infty) \cap D$. Then f has the limit $l \in \mathbb{R}$ if and only if f has equal one-side limits in x_0 .

Remarkable limits

If $\lim_{x \rightarrow x_0} f(x) = 0$, then:

$$1. \lim_{x \rightarrow x_0} \frac{\sin f(x)}{f(x)} = 1;$$

$$2. \lim_{x \rightarrow x_0} \frac{\tan f(x)}{f(x)} = 1;$$

$$3. \lim_{x \rightarrow x_0} \frac{\arcsin f(x)}{f(x)} = 1;$$

$$4. \lim_{x \rightarrow x_0} \frac{\arctan f(x)}{f(x)} = 1;$$

$$5. \lim_{x \rightarrow x_0} (1 + f(x))^{\frac{1}{f(x)}} = e$$

$$6. \lim_{x \rightarrow x_0} \frac{\ln(1 + f(x))}{f(x)} = 1;$$

$$7. \lim_{x \rightarrow x_0} \frac{a^{f(x)} - 1}{f(x)} = \ln a \quad (a > 0);$$

$$8. \lim_{x \rightarrow x_0} \frac{(1 + f(x))^r - 1}{f(x)} = r \quad (r \in \mathbb{R});$$

If $\lim_{x \rightarrow x_0} f(x) = \infty$, then:

$$9. \lim_{x \rightarrow x_0} \left(1 + \frac{1}{f(x)}\right)^{f(x)} = e;$$

$$10. \lim_{x \rightarrow x_0} \frac{\ln f(x)}{f(x)} = 0;$$

Chapter 2

Problems

1. Evaluate:

$$\lim_{n \rightarrow \infty} \left(\sqrt[3]{n^3 + 2n^2 + 1} - \sqrt[3]{n^3 - 1} \right)$$

2. Evaluate:

$$\lim_{x \rightarrow -2} \frac{\sqrt[3]{5x+2} + 2}{\sqrt{3x+10} - 2}$$

3. Consider the sequence $(a_n)_{n \geq 1}$, such that $\sum_{k=1}^n a_k = \frac{3n^2 + 9n}{2}$, $(\forall) n \geq 1$.

Prove that this sequence is an arithmetical progression and evaluate:

$$\lim_{n \rightarrow \infty} \frac{1}{na_n} \sum_{k=1}^n a_k$$

4. Consider the sequence $(a_n)_{n \geq 1}$ such that $a_1 = a_2 = 0$ and $a_{n+1} = \frac{1}{3}(a_n + a_{n-1}^2 + b)$, where $0 \leq b \leq 1$. Prove that the sequence is convergent and evaluate $\lim_{n \rightarrow \infty} a_n$.

5. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ such that $x_1 = 1$ and $x_n = 2x_{n-1} + \frac{1}{n}$, $(\forall) n \geq 2$. Evaluate $\lim_{n \rightarrow \infty} x_n$.

6. Evaluate:

$$\lim_{n \rightarrow \infty} \left(n \left(\frac{4}{5} \right)^n + n^2 \sin^n \frac{\pi}{6} + \cos \left(2n\pi + \frac{\pi}{n} \right) \right)$$

7. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k! \cdot k}{(n+1)!}$$

8. Evaluate:

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right)$$

9. Evaluate:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{3^{3n}(n!)^3}{(3n)!}}$$

10. Consider a sequence of real positive numbers $(x_n)_{n \geq 1}$ such that $(n+1)x_{n+1} - nx_n < 0$, $(\forall) n \geq 1$. Prove that this sequence is convergent and evaluate its limit.

11. Find the real numbers a and b such that:

$$\lim_{n \rightarrow \infty} \left(\sqrt[3]{1 - n^3} - an - b \right) = 0$$

12. Let $p \in \mathbb{N}$ and $\alpha_1, \alpha_2, \dots, \alpha_p$ positive distinct real numbers. Evaluate:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\alpha_1^n + \alpha_2^n + \dots + \alpha_p^n}$$

13. If $a \in \mathbb{R}^*$, evaluate:

$$\lim_{x \rightarrow -a} \frac{\cos x - \cos a}{x^2 - a^2}$$

14. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{x \rightarrow 0} \frac{\ln(1 + x + x^2 + \dots + x^n)}{nx}$$

15. Evaluate:

$$\lim_{n \rightarrow \infty} \left(n^2 + n - \sum_{k=1}^n \frac{2k^3 + 8k^2 + 6k - 1}{k^2 + 4k + 3} \right)$$

16. Find $a \in \mathbb{R}^*$ such that:

$$\lim_{x \rightarrow 0} \frac{1 - \cos ax}{x^2} = \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x}$$

17. Evaluate:

$$\lim_{x \rightarrow 1} \frac{\sqrt[3]{x^2 + 7} - \sqrt{x + 3}}{x^2 - 3x + 2}$$

18. Evaluate:

$$\lim_{n \rightarrow \infty} \left(\sqrt{2n^2 + n} - \lambda \sqrt{2n^2 - n} \right)$$

where λ is a real number.

19. If $a, b, c \in \mathbb{R}$, evaluate:

$$\lim_{x \rightarrow \infty} (a\sqrt{x+1} + b\sqrt{x+2} + c\sqrt{x+3})$$

20. Find the set $A \subset \mathbb{R}$ such that $ax^2 + x + 3 \geq 0$, $(\forall)a \in A, (\forall)x \in \mathbb{R}$. Then for any $a \in A$, evaluate:

$$\lim_{x \rightarrow \infty} (x + 1 - \sqrt{ax^2 + x + 3})$$

21. If $k \in \mathbb{R}$, evaluate:

$$\lim_{n \rightarrow \infty} n^k \left(\sqrt{\frac{n}{n+1}} - \sqrt{\frac{n+2}{n+3}} \right)$$

22. If $k \in \mathbb{N}$ and $a \in \mathbb{R}_+ \setminus \{1\}$, evaluate:

$$\lim_{n \rightarrow \infty} n^k (a^{\frac{1}{n}} - 1) \left(\sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n+2}} \right)$$

23. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{n^2 + k}}$$

24. If $a > 0$, $p \geq 2$, evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt[p]{n^p + ka}}$$

25. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{n!}{(1+1^2)(1+2^2) \cdots (1+n^2)}$$

26. Evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{2n^2 - 3}{2n^2 - n + 1} \right)^{\frac{n^2 - 1}{n}}$$

27. Evaluate:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin^2 x} - \cos x}{1 - \sqrt{1 + \tan^2 x}}$$

28. Evaluate:

$$\lim_{x \rightarrow \infty} \left(\frac{x + \sqrt{x}}{x - \sqrt{x}} \right)^x$$

29. Evaluate:

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} (\cos x)^{\frac{1}{\sin x}}$$

30. Evaluate:

$$\lim_{x \rightarrow 0} (e^x + \sin x)^{\frac{1}{x}}$$

31. If $a, b \in \mathbb{R}_+^*$, evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{a - 1 + \sqrt[n]{b}}{a} \right)^n$$

32. Consider a sequence of real numbers $(a_n)_{n \geq 1}$ defined by:

$$a_n = \begin{cases} 1 & \text{if } n \leq k, \ k \in \mathbb{N}^* \\ \frac{(n+1)^k - n^k}{\binom{n}{k-1}} & \text{if } n > k \end{cases}$$

i) Evaluate $\lim_{n \rightarrow \infty} a_n$.

ii) If $b_n = 1 + \sum_{k=1}^n k \cdot \lim_{n \rightarrow \infty} a_n$, evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{b_n^2}{b_{n-1} b_{n+1}} \right)^n$$

33. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ such that $x_{n+2} = \frac{x_{n+1} + x_n}{2}$, $(\forall) n \in \mathbb{N}^*$. If $x_1 \leq x_2$,

i) Prove that the sequence $(x_{2n+1})_{n \geq 0}$ is increasing, while the sequence $(x_{2n})_{n \geq 0}$ is decreasing;

ii) Prove that:

$$|x_{n+2} - x_{n+1}| = \frac{|x_2 - x_1|}{2^n}, \quad (\forall) n \in \mathbb{N}^*$$

iii) Prove that:

$$2x_{n+2} + x_{n+1} = 2x_2 + x_1, \quad (\forall) n \in \mathbb{N}^*$$

iv) Prove that $(x_n)_{n \geq 1}$ is convergent and that its limit is $\frac{x_1 + 2x_2}{3}$.

34. Let $a_n, b_n \in \mathbb{Q}$ such that $(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}$, $(\forall)n \in \mathbb{N}^*$. Evaluate $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$.

35. If $a > 0$, evaluate:

$$\lim_{x \rightarrow 0} \frac{(a+x)^x - 1}{x}$$

36. Consider a sequence of real numbers $(a_n)_{n \geq 1}$ such that $a_1 = \frac{3}{2}$ and $a_{n+1} = \frac{a_n^2 - a_n + 1}{a_n}$. Prove that $(a_n)_{n \geq 1}$ is convergent and find its limit.

37. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ such that $x_0 \in (0, 1)$ and $x_{n+1} = x_n - x_n^2 + x_n^3 - x_n^4$, $(\forall)n \geq 0$. Prove that this sequence is convergent and evaluate $\lim_{n \rightarrow \infty} x_n$.

38. Let $a > 0$ and $b \in (a, 2a)$ and a sequence $x_0 = b$, $x_{n+1} = a + \sqrt{x_n(2a - x_n)}$, $(\forall)n \geq 0$. Study the convergence of the sequence $(x_n)_{n \geq 0}$.

39. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n+1} \arctan \frac{1}{2k^2}$$

40. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{4k^4 + 1}$$

41. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1 + 3 + 3^2 + \dots + 3^k}{5^{k+2}}$$

42. Evaluate:

$$\lim_{n \rightarrow \infty} \left(n + 1 - \sum_{i=2}^n \sum_{k=2}^i \frac{k-1}{k!} \right)$$

43. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1^1 + 2^2 + 3^3 + \dots + n^n}{n^n}$$

44. Consider the sequence $(a_n)_{n \geq 1}$ such that $a_0 = 2$ and $a_{n-1} - a_n = \frac{n}{(n+1)!}$. Evaluate $\lim_{n \rightarrow \infty} ((n+1)! \ln a_n)$.

45. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ with $x_1 = a > 0$ and $x_{n+1} = \frac{x_1 + 2x_2 + 3x_3 + \dots + nx_n}{n}$, $n \in \mathbb{N}^*$. Evaluate its limit.

46. Using $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6}$, evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{(2k-1)^2}$$

47. Consider the sequence $(x_n)_{n \geq 1}$ defined by $x_1 = a$, $x_2 = b$, $a < b$ and $x_n = \frac{x_{n-1} + \lambda x_{n-2}}{1 + \lambda}$, $n \geq 3$, $\lambda > 0$. Prove that this sequence is convergent and find its limit.

48. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}}$$

49. Consider the sequence $(x_n)_{n \geq 1}$ defined by $x_1 = 1$ and $x_n = \frac{1}{1 + x_{n-1}}$, $n \geq 2$. Prove that this sequence is convergent and evaluate $\lim_{n \rightarrow \infty} x_n$.

50. If $a, b \in \mathbb{R}^*$, evaluate:

$$\lim_{x \rightarrow 0} \frac{\ln(\cos ax)}{\ln(\cos bx)}$$

51. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \begin{cases} \{x\} & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Find all $\alpha \in \mathbb{R}$ for which $\lim_{x \rightarrow \alpha} f(x)$ exists.

52. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \begin{cases} [x] & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Find all $\alpha \in \mathbb{R}$ for which $\lim_{x \rightarrow \alpha} f(x)$ exists.

53. Let $(x_n)_{n \geq 1}$ be a sequence of positive real numbers such that $x_1 > 0$ and $3x_n = 2x_{n-1} + \frac{a}{x_{n-1}^2}$, where a is a real positive number. Prove that x_n is convergent and evaluate $\lim_{n \rightarrow \infty} x_n$.

54. Consider a sequence of real numbers $(a_n)_{n \geq 1}$ such that $a_1 = 12$ and $a_{n+1} = a_n \left(1 + \frac{3}{n+1}\right)$. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{a_k}$$

55. Evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{n}{\sqrt{n^2 + 1}} \right)^n$$

56. If $a \in \mathbb{R}$, evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\lfloor k^2 a \rfloor}{n^3}$$

57. Evaluate:

$$\lim_{n \rightarrow \infty} 2^n \left(\sum_{k=1}^n \frac{1}{k(k+2)} - \frac{1}{4} \right)^n$$

58. Consider the sequence $(a_n)_{n \geq 1}$, such that $a_n > 0$, $(\forall) n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} n(a_{n+1} - a_n) = 1$. Evaluate $\lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} \sqrt[n]{a_n}$.

59. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1 + 2\sqrt{2} + 3\sqrt{3} + \dots + n\sqrt{n}}{n^2\sqrt{n}}$$

60. Evaluate:

$$\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\frac{1}{2x-\pi}}$$

61. Evaluate:

$$\lim_{n \rightarrow \infty} n^2 \ln \left(\cos \frac{1}{n} \right)$$

62. Given $a, b \in \mathbb{R}_+^*$, evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} \right)^n$$

63. Let $\alpha > \beta > 0$ and the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

i) Prove that $(\exists)(x_n)_{n \geq 1}, (y_n)_{n \geq 1} \in \mathbb{R}$ such that:

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^n = x_n A + y_n B, \quad (\forall) n \geq 1$$

ii) Evaluate $\lim_{n \rightarrow \infty} \frac{x_n}{y_n}$.

64. If $a \in \mathbb{R}$ such that $|a| < 1$ and $p \in \mathbb{N}^*$ is given, evaluate:

$$\lim_{n \rightarrow \infty} n^p \cdot a^n$$

65. If $p \in \mathbb{N}^*$, evaluate:

$$\lim_{n \rightarrow \infty} \frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}}$$

66. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{\sin(n \arccos x)}{\sqrt{1 - x^2}}$$

67. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{1 - \cos(n \arccos x)}{1 - x^2}$$

68. Study the convergence of the sequence:

$$x_{n+1} = \frac{x_n + a}{x_n + 1}, \quad n \geq 1, \quad x_1 \geq 0, \quad a > 0$$

69. Consider two sequences of real numbers $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ such that $x_0 = y_0 = 3$, $x_n = 2x_{n-1} + y_{n-1}$ and $y_n = 2x_{n-1} + 3y_{n-1}$, $(\forall) n \geq 1$. Evaluate $\lim_{n \rightarrow \infty} \frac{x_n}{y_n}$.

70. Evaluate:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^2}$$

71. Evaluate:

$$\lim_{x \rightarrow 0} \frac{\tan x - \arctan x}{x^2}$$

72. Let $a > 0$ and a sequence of real numbers $(x_n)_{n \geq 0}$ such that $x_n \in (0, a)$ and $x_{n+1}(a - x_n) > \frac{a^2}{4}$, $(\forall) n \in \mathbb{N}$. Prove that $(x_n)_{n \geq 1}$ is convergent and evaluate $\lim_{n \rightarrow \infty} x_n$.

73. Evaluate:

$$\lim_{n \rightarrow \infty} \cos(n\pi \sqrt[n]{e})$$

74. Evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{\tan \frac{(n-1)\pi}{2n}}$$

75. Evaluate:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\prod_{k=1}^n \binom{n}{k}}$$

76. If $a > 0$, evaluate:

$$\lim_{n \rightarrow \infty} \frac{a + \sqrt{a} + \sqrt[3]{a} + \dots + \sqrt[n]{a} - n}{\ln n}$$

77. Evaluate:

$$\lim_{n \rightarrow \infty} n \ln \tan \left(\frac{\pi}{4} + \frac{\pi}{n} \right)$$

78. Let $k \in \mathbb{N}$ and $a_0, a_1, a_2, \dots, a_k \in \mathbb{R}$ such that $a_0 + a_1 + a_2 + \dots + a_k = 0$. Evaluate:

$$\lim_{n \rightarrow \infty} \left(a_0 \sqrt[3]{n} + a_1 \sqrt[3]{n+1} + \dots + a_k \sqrt[3]{n+k} \right)$$

79. Evaluate:

$$\lim_{n \rightarrow \infty} \sin \left(n\pi \sqrt[3]{n^3 + 3n^2 + 4n - 5} \right)$$

80. Evaluate:

$$\lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{2 \arcsin x - \pi}{\sin \pi x}$$

81. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=2}^n \frac{1}{k \ln k}$$

82. Evaluate:

$$\lim_{n \rightarrow \infty} \left[\lim_{x \rightarrow 0} \left(1 + \sum_{k=1}^n \sin^2(kx) \right)^{\frac{1}{n^3 x^2}} \right]$$

83. If $p \in \mathbb{N}^*$, evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(k+1)(k+2) \cdot \dots \cdot (k+p)}{n^{p+1}}$$

84. If $\alpha_n \in \left(0, \frac{\pi}{4} \right)$ is a root of the equation $\tan \alpha + \cot \alpha = n$, $n \geq 2$, evaluate:

$$\lim_{n \rightarrow \infty} (\sin \alpha_n + \cos \alpha_n)^n$$

85. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\sqrt{\binom{n+k}{2}}}{n^2}$$

86. Evaluate:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\prod_{k=1}^n \left(1 + \frac{k}{n}\right)}$$

87. Evaluate:

$$\lim_{x \rightarrow 0} \frac{\arctan x - \arcsin x}{x^3}$$

88. If $\alpha > 0$, evaluate:

$$\lim_{n \rightarrow \infty} \frac{(n+1)^\alpha - n^\alpha}{n^{\alpha-1}}$$

89. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^2}{2^k}$$

90. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(k+1)(k+2)}{2^k}$$

91. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ such that $x_1 \in (0, 1)$ and $x_{n+1} = x_n^2 - x_n + 1$, $(\forall) n \in \mathbb{N}$. Evaluate:

$$\lim_{n \rightarrow \infty} (x_1 x_2 \cdots x_n)$$

92. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{x \rightarrow 0} \frac{1 - \cos x \cdot \cos 2x \cdots \cos nx}{x^2}$$

93. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ such that x_n is the real root of the equation $x^3 + nx - n = 0$, $n \in \mathbb{N}^*$. Prove that this sequence is convergent and find its limit.

94. Evaluate:

$$\lim_{x \rightarrow 2} \frac{\arctan x - \arctan 2}{\tan x - \tan 2}$$

95. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1 + \sqrt[2]{2!} + \sqrt[3]{3!} + \dots + \sqrt[n]{n!}}{n}$$

96. Let $(x_n)_{n \geq 1}$ such that $x_1 > 0$, $x_1 + x_1^2 < 1$ and $x_{n+1} = x_n + \frac{x_n^2}{n^2}$, $(\forall) n \geq 1$.
Prove that the sequences $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 2}$, $y_n = \frac{1}{x_n} - \frac{1}{n-1}$ are convergent.

97. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{2i}{n^2}$$

98. If $a > 0$, $a \neq 1$, evaluate:

$$\lim_{x \rightarrow a} \frac{x^x - a^x}{a^x - a^a}$$

99. Consider a sequence of positive real numbers $(a_n)_{n \geq 1}$ such that $a_{n+1} - \frac{1}{a_{n+1}} = a_n + \frac{1}{a_n}$, $(\forall) n \geq 1$. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right)$$

100. Evaluate:

$$\lim_{x \rightarrow 0} \frac{2^{\arctan x} - 2^{\arcsin x}}{2^{\tan x} - 2^{\sin x}}$$

Chapter 3

Solutions

1. Evaluate:

$$\lim_{n \rightarrow \infty} \left(\sqrt[3]{n^3 + 2n^2 + 1} - \sqrt[3]{n^3 - 1} \right)$$

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\sqrt[3]{n^3 + 2n^2 + 1} - \sqrt[3]{n^3 - 1} \right) &= \lim_{n \rightarrow \infty} \frac{n^3 + 2n^2 + 1 - n^3 + 1}{\sqrt[3]{(n^3 + 2n^2 + 1)^2} + \sqrt[3]{(n^3 - 1)(n^3 + 2n^2 + 1)} + \sqrt[3]{(n^3 - 1)^2}} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 \left(2 + \frac{2}{n} \right)}{n^2 \left[\sqrt[3]{\left(1 + \frac{2}{n} + \frac{1}{n^3} \right)^2} + \sqrt[3]{\left(1 - \frac{1}{n^3} \right) \left(1 + \frac{2}{n} + \frac{1}{n^3} \right)} + \sqrt[3]{\left(1 - \frac{1}{n^3} \right)^2} \right]} \\ &= \frac{2}{3} \end{aligned}$$

2. Evaluate:

$$\lim_{x \rightarrow -2} \frac{\sqrt[3]{5x+2} + 2}{\sqrt{3x+10} - 2}$$

Solution:

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{\sqrt[3]{5x+2} + 2}{\sqrt{3x+10} - 2} &= \lim_{x \rightarrow -2} \frac{\frac{5x+10}{\sqrt[3]{(5x+2)^2 - 2\sqrt[3]{5x+2}+4}}}{\frac{3x+6}{\sqrt{3x+10}+2}} \\ &= \frac{5}{3} \lim_{x \rightarrow -2} \frac{\sqrt{3x+10} + 2}{\sqrt[3]{(5x+2)^2} - 2\sqrt[3]{5x+2} + 4} \\ &= \frac{5}{9} \end{aligned}$$

3. Consider the sequence $(a_n)_{n \geq 1}$, such that $\sum_{k=1}^n a_k = \frac{3n^2 + 9n}{2}$, $(\forall) n \geq 1$.

Prove that this sequence is an arithmetical progression and evaluate:

$$\lim_{n \rightarrow \infty} \frac{1}{na_n} \sum_{k=1}^n a_k$$

Solution: For $n = 1$ we get $a_1 = 6$. Then $a_1 + a_2 = 15$, so $a_2 = 9$ and the ratio is $r = 3$. Therefore the general term is $a_n = 6 + 3(n - 1) = 3(n + 1)$. So:

$$\lim_{n \rightarrow \infty} \frac{1}{na_n} \sum_{k=1}^n a_k = \lim_{n \rightarrow \infty} \frac{n+3}{2n+2} = \frac{1}{2}$$

4. Consider the sequence $(a_n)_{n \geq 1}$ such that $a_1 = a_2 = 0$ and $a_{n+1} = \frac{1}{3}(a_n + a_{n-1}^2 + b)$, where $0 \leq b < 1$. Prove that the sequence is convergent and evaluate $\lim_{n \rightarrow \infty} a_n$.

Solution: We have $a_2 - a_1 = 0$ and $a_3 - a_2 = \frac{b}{3} \geq 0$, so assuming $a_{n-1} \geq a_{n-2}$ and $a_n \geq a_{n-1}$, we need to show that $a_{n+1} \geq a_n$. The recurrence equation gives us:

$$a_{n+1} - a_n = \frac{1}{3}(a_n - a_{n-1} + a_{n-1}^2 - a_{n-2}^2)$$

Therefore it follows that the sequence is monotonically increasing. Also, because $b \leq 1$, we have $a_3 = \frac{b}{3} < 1$, $a_4 = \frac{4b}{9} < 1$. Assuming that $a_{n-1}, a_n < 1$, it follows that:

$$a_{n+1} = \frac{1}{3}(b + a_n + a_{n-1}^2) < \frac{1}{3}(1 + 1 + 1) = 1$$

Hence $a_n \in [0, 1)$, $(\forall)n \in \mathbb{N}^*$, which means the sequence is bounded. From Weierstrass theorem it follows that the sequence is convergent. Let then $\lim_{n \rightarrow \infty} a_n = l$. By passing to limit in the recurrence relation, we have:

$$l^2 - 2l + b = 0 \Leftrightarrow (l - 1)^2 = 1 - b \Rightarrow l = 1 \pm \sqrt{1 - b}$$

Because $1 + \sqrt{1 - b} > 1$ and $a_n \in [0, 1)$, it follows that $\lim_{n \rightarrow \infty} a_n = 1 - \sqrt{1 - b}$.

5. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ such that $x_1 = 1$ and $x_n = 2x_{n-1} + \frac{1}{2}$, $(\forall)n \geq 2$. Evaluate $\lim_{n \rightarrow \infty} x_n$.

Solution: Let's evaluate a few terms:

$$x_2 = 2 + \frac{1}{2}$$

$$x_3 = 2^2 + 2 \cdot \frac{1}{2} + \frac{1}{2} = 2^2 + \frac{1}{2}(2^2 - 1)$$

$$x_4 = 2^3 + 2^2 - 1 + \frac{1}{2} = 2^3 + \frac{1}{2}(2^3 - 1)$$

$$x_5 = 2^4 + 2^3 - 1 + \frac{1}{2} = 2^4 + \frac{1}{2}(2^4 - 1)$$

and by induction we can show immediately that $x_n = 2^{n-1} + \frac{1}{2}(2^{n-1} - 1)$. Thus $\lim_{n \rightarrow \infty} x_n = \infty$.

6. Evaluate:

$$\lim_{n \rightarrow \infty} \left(n \left(\frac{4}{5} \right)^n + n^2 \sin^n \frac{\pi}{6} + \cos \left(2n\pi + \frac{\pi}{n} \right) \right)$$

Solution: We have:

$$\lim_{n \rightarrow \infty} \frac{\frac{4^{n+1} \cdot (n+1)}{5^{n+1}}}{\frac{4^n \cdot n}{5^n}} = \lim_{n \rightarrow \infty} \frac{4(n+1)}{5n} = \frac{4}{5} < 1$$

Thus using the ratio test it follows that $\lim_{n \rightarrow \infty} n \left(\frac{4}{5} \right)^n = 0$. Also

$$\lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{2n^2} = \frac{1}{2} < 1$$

From the ratio test it follows that $\lim_{n \rightarrow \infty} \frac{n^2}{2^n} = \lim_{n \rightarrow \infty} n^2 \sin^n \frac{\pi}{6} = 0$. Therefore the limit is equal to

$$\lim_{n \rightarrow \infty} \cos \left(2n\pi + \frac{\pi}{n} \right) = \lim_{n \rightarrow \infty} \cos \frac{\pi}{n} = \cos 0 = 1$$

7. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k! \cdot k}{(n+1)!}$$

Solution:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k! \cdot k}{(n+1)!} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(k+1)! - k!}{(n+1)!} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{(n+1)!} \right) = 1$$

8. Evaluate:

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^2} \right) \left(1 - \frac{1}{3^2} \right) \cdots \left(1 - \frac{1}{n^2} \right)$$

Solution:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) &= \lim_{n \rightarrow \infty} \prod_{r=2}^n \left(1 - \frac{1}{r^2}\right) \\
&= \lim_{n \rightarrow \infty} \prod_{r=2}^n \left(\frac{r^2 - 1}{r^2}\right) \\
&= \lim_{n \rightarrow \infty} \prod_{r=2}^n \left(\frac{(r-1)(r+1)}{r^2}\right) \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \\
&= \frac{1}{2}
\end{aligned}$$

9. Evaluate:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{3^{3n}(n!)^3}{(3n)!}}$$

Solution: Define $a_n = \frac{3^{3n}(n!)^3}{(3n)!}$. Then:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \sqrt[n]{a_n} &= \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n \rightarrow \infty} \frac{3^{3n+3}[(n+1)!]^3}{(3n+3)!} \cdot \frac{(3n)!}{3^{3n}(n!)^3} \\
&= \lim_{n \rightarrow \infty} \frac{27(n+1)^3}{(3n+1)(3n+2)(3n+3)} \\
&= 1
\end{aligned}$$

10. Consider a sequence of real positive numbers $(x_n)_{n \geq 1}$ such that $(n+1)x_{n+1} - nx_n < 0$, $(\forall)n \geq 1$. Prove that this sequence is convergent and evaluate its limit.

Solution: Because $nx_n > (n+1)x_{n+1}$, we deduce that $x_1 > 2x_2 > 3x_3 > \dots > nx_n$, whence $0 < x_n < \frac{x_1}{n}$. Using the Squeeze Theorem it follows that $\lim_{n \rightarrow \infty} x_n = 0$.

11. Find the real numbers a and b such that:

$$\lim_{n \rightarrow \infty} \left(\sqrt[3]{1 - n^3} - an - b\right) = 0$$

Solution: We have:

$$\begin{aligned}
b &= \lim_{n \rightarrow \infty} \left(\sqrt[3]{1 - n^3} - an \right) \\
&= \lim_{n \rightarrow \infty} \frac{1 - n^3 - a^3 n^3}{\sqrt[3]{(1 - n^3)^2} + \sqrt[3]{an(1 - n^3)} + \sqrt[3]{a^2 n^2}} \\
&= \lim_{n \rightarrow \infty} \frac{n \left(-1 - a^3 + \frac{1}{n^3} \right)}{\sqrt[3]{\left(\frac{1}{n^3} - 1 \right)^2} + \sqrt[3]{a \left(\frac{1}{n^3} - \frac{1}{n^2} \right)} + \sqrt[3]{\frac{a^2}{n^4}}}
\end{aligned}$$

If $-1 - a^3 \neq 0$, it follows that $b = \pm\infty$, which is false. Hence $a^3 = -1 \Rightarrow a = -1$ and so $b = 0$.

12. Let $p \in \mathbb{N}$ and $\alpha_1, \alpha_2, \dots, \alpha_p$ positive distinct real numbers. Evaluate:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\alpha_1^n + \alpha_2^n + \dots + \alpha_p^n}$$

Solution: WLOG let $\alpha_j = \max\{\alpha_1, \alpha_2, \dots, \alpha_p\}$, $1 \leq j \leq p$. Then:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \sqrt[n]{\alpha_1^n + \alpha_2^n + \dots + \alpha_p^n} &= \lim_{n \rightarrow \infty} \alpha_j \sqrt[n]{\left(\frac{\alpha_1}{\alpha_j}\right)^n + \left(\frac{\alpha_2}{\alpha_j}\right)^n + \dots + \left(\frac{\alpha_{j-1}}{\alpha_j}\right)^n + 1 + \left(\frac{\alpha_{j+1}}{\alpha_j}\right)^n + \dots + \left(\frac{\alpha_p}{\alpha_j}\right)^n} \\
&= \alpha_j \\
&= \max\{\alpha_1, \alpha_2, \dots, \alpha_p\}
\end{aligned}$$

13. If $a \in \mathbb{R}^*$, evaluate:

$$\lim_{x \rightarrow -a} \frac{\cos x - \cos a}{x^2 - a^2}$$

Solution:

$$\begin{aligned}
\lim_{x \rightarrow -a} \frac{\cos x - \cos a}{x^2 - a^2} &= \lim_{x \rightarrow -a} \frac{-2 \sin \frac{x+a}{2} \cdot \sin \frac{x-a}{2}}{(x-a)(x+a)} \\
&= \lim_{x \rightarrow -a} \frac{\sin \frac{x+a}{2}}{\frac{x+a}{2}} \cdot \lim_{x \rightarrow -a} \frac{\sin \frac{x-a}{2}}{a-x} \\
&= \lim_{x \rightarrow -a} \frac{\sin \frac{x-a}{2}}{a-x} \\
&= -\frac{\sin a}{2a}
\end{aligned}$$

14. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{x \rightarrow 0} \frac{\ln(1 + x + x^2 + \dots + x^n)}{nx}$$

Solution: Using $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$, we have:

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\ln(1 + x + x^2 + \dots + x^n)}{nx} &= \lim_{x \rightarrow 0} \frac{\ln(1 + x + x^2 + \dots + x^n)}{x + x^2 + \dots + x^n} \cdot \lim_{x \rightarrow 0} \frac{x + x^2 + \dots + x^n}{nx} \\
&= \lim_{x \rightarrow 0} \frac{x + x^2 + \dots + x^n}{nx} \\
&= \lim_{x \rightarrow 0} \frac{1 + x + \dots + x^{n-1}}{n} \\
&= \frac{1}{n}
\end{aligned}$$

15. Evaluate:

$$\lim_{n \rightarrow \infty} \left(n^2 + n - \sum_{k=1}^n \frac{2k^3 + 8k^2 + 6k - 1}{k^2 + 4k + 3} \right)$$

Solution: Telescoping, we have:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(n^2 + n - \sum_{k=1}^n \frac{2k^3 + 8k^2 + 6k - 1}{k^2 + 4k + 3} \right) &= \lim_{n \rightarrow \infty} \left(n^2 + n - 2 \sum_{k=1}^n k + \frac{1}{2} \sum_{k=1}^n \frac{1}{k+1} - \frac{1}{2} \sum_{k=1}^n \frac{1}{k+3} \right) \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k+1} - \sum_{k=1}^n \frac{1}{k+3} \right) \\
&= \frac{5}{12} - \frac{1}{2} \lim_{n \rightarrow \infty} \left(\frac{1}{n+2} + \frac{1}{n+3} \right) \\
&= \frac{5}{12}
\end{aligned}$$

16. Find $a \in \mathbb{R}^*$ such that:

$$\lim_{x \rightarrow 0} \frac{1 - \cos ax}{x^2} = \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x}$$

Solution: Observe that:

$$\lim_{x \rightarrow 0} \frac{1 - \cos ax}{x^2} = \frac{a^2}{4} \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{ax}{2}}{\frac{a^2 x^2}{4}} = \frac{a^2}{2}$$

and

$$\lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x} = \lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi - x} = 1$$

Therefore $\frac{a^2}{2} = 1$, which implies $a = \pm\sqrt{2}$.

17. Evaluate:

$$\lim_{x \rightarrow 1} \frac{\sqrt[3]{x^2+7} - \sqrt{x+3}}{x^2 - 3x + 2}$$

Solution:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt[3]{x^2+7} - \sqrt{x+3}}{x^2 - 3x + 2} &= \lim_{x \rightarrow 1} \frac{\sqrt[3]{x^2+7} - 3}{x^2 - 3x + 2} + \lim_{x \rightarrow 1} \frac{2 - \sqrt{x+3}}{x^2 - 3x + 2} \\ &= \lim_{x \rightarrow 1} \frac{x+1}{(x-2) \left(\sqrt[3]{(x^2+7)^2} + 2\sqrt[3]{x^2+7} + 4 \right)} + \lim_{x \rightarrow 1} \frac{1}{(2-x)(2+\sqrt{x+3})} \\ &= -\frac{2}{12} + \frac{1}{4} \\ &= \frac{1}{12} \end{aligned}$$

18. Evaluate:

$$\lim_{n \rightarrow \infty} \left(\sqrt{2n^2 + n} - \lambda \sqrt{2n^2 - n} \right)$$

where λ is a real number.

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\sqrt{2n^2 + n} - \lambda \sqrt{2n^2 - n} \right) &= \lim_{n \rightarrow \infty} \frac{2n^2 + n - \lambda^2 (2n^2 - n)}{\sqrt{2n^2 + n} + \lambda \sqrt{2n^2 - n}} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2 (1 - \lambda^2) + n (1 + \lambda^2)}{n \left(\sqrt{2 + \frac{1}{n}} + \lambda \sqrt{2 - \frac{1}{n}} \right)} \\ &= \lim_{n \rightarrow \infty} \frac{2n (1 - \lambda^2) + (1 + \lambda^2)}{\sqrt{2 + \frac{1}{n}} + \lambda \sqrt{2 - \frac{1}{n}}} \\ &= \begin{cases} +\infty & \text{if } \lambda \in (-\infty, 1) \\ \frac{\sqrt{2}}{2} & \text{if } \lambda = 1 \\ -\infty & \text{if } \lambda \in (1, +\infty) \end{cases} \end{aligned}$$

19. If $a, b, c \in \mathbb{R}$, evaluate:

$$\lim_{x \rightarrow \infty} (a\sqrt{x+1} + b\sqrt{x+2} + c\sqrt{x+3})$$

Solution: If $a + b + c \neq 0$, we have:

$$\begin{aligned}
\lim_{x \rightarrow \infty} (a\sqrt{x+1} + b\sqrt{x+2} + c\sqrt{x+3}) &= \lim_{x \rightarrow \infty} \sqrt{x} \left(a\sqrt{1 + \frac{1}{x}} + b\sqrt{1 + \frac{2}{x}} + c\sqrt{1 + \frac{3}{x}} \right) \\
&= \lim_{x \rightarrow \infty} \sqrt{x} (a + b + c) \\
&= \begin{cases} -\infty & \text{if } a + b + c < 0 \\ \infty & \text{if } a + b + c > 0 \end{cases}
\end{aligned}$$

If $a + b + c = 0$, then:

$$\begin{aligned}
\lim_{x \rightarrow \infty} (a\sqrt{x+1} + b\sqrt{x+2} + c\sqrt{x+3}) &= \lim_{x \rightarrow \infty} (a\sqrt{x+1} - a + b\sqrt{x+2} - b + c\sqrt{x+3} - c) \\
&= \lim_{x \rightarrow \infty} \left(\frac{a}{\sqrt{\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x}}} + \frac{b + \frac{b}{x}}{\sqrt{\frac{1}{x} + \frac{2}{x^2} + \frac{1}{x}}} + \frac{c + \frac{2c}{x}}{\sqrt{\frac{1}{x} + \frac{3}{x^2} + \frac{1}{x}}} \right) \\
&= 0
\end{aligned}$$

20. Find the set $A \subset \mathbb{R}$ such that $ax^2 + x + 3 \geq 0$, $(\forall)a \in A, (\forall)x \in \mathbb{R}$. Then for any $a \in A$, evaluate:

$$\lim_{x \rightarrow \infty} (x + 1 - \sqrt{ax^2 + x + 3})$$

Solution: We have $ax^2 + x + 3 \geq 0$, $(\forall)x \in \mathbb{R}$ if $a > 0$ and $\Delta_x \leq 0$, whence $a \in \left[\frac{1}{12}, \infty \right)$. Then:

$$\begin{aligned}
\lim_{x \rightarrow \infty} (x + 1 - \sqrt{ax^2 + x + 3}) &= \lim_{x \rightarrow \infty} \frac{(1-a)x^2 + x - 2}{x + 1 + \sqrt{ax^2 + x + 3}} \\
&= \lim_{x \rightarrow \infty} \frac{(1-a)x + 1 - \frac{2}{x}}{1 + \frac{1}{x} + \sqrt{a + \frac{1}{x} + \frac{3}{x^2}}} \\
&= \begin{cases} \infty & \text{if } a \in \left[\frac{1}{12}, 1 \right) \\ \frac{1}{2} & \text{if } a = 1 \\ -\infty & \text{if } a \in (1, \infty) \end{cases}
\end{aligned}$$

21. If $k \in \mathbb{R}$, evaluate:

$$\lim_{n \rightarrow \infty} n^k \left(\sqrt{\frac{n}{n+1}} - \sqrt{\frac{n+2}{n+3}} \right)$$

Solution:

$$\begin{aligned}
\lim_{n \rightarrow \infty} n^k \left(\sqrt{\frac{n}{n+1}} - \sqrt{\frac{n+2}{n+3}} \right) &= \lim_{n \rightarrow \infty} \frac{n^k}{(n+1)(n+2)} \cdot \lim_{n \rightarrow \infty} \frac{-2}{\sqrt{\frac{n}{n+1}} + \sqrt{\frac{n+2}{n+3}}} \\
&= \lim_{n \rightarrow \infty} \frac{-n^k}{(n+1)(n+2)} \\
&= \begin{cases} 0 & \text{if } k < 2 \\ -1 & \text{if } k = 2 \\ -\infty & \text{if } k > 2 \end{cases}
\end{aligned}$$

22. If $k \in \mathbb{N}$ and $a \in \mathbb{R}_+ \setminus \{1\}$, evaluate:

$$\lim_{n \rightarrow \infty} n^k (a^{\frac{1}{n}} - 1) \left(\sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n+2}} \right)$$

Solution:

$$\begin{aligned}
\lim_{n \rightarrow \infty} n^k (a^{\frac{1}{n}} - 1) \left(\sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n+2}} \right) &= \lim_{n \rightarrow \infty} \frac{-n^k (a^{\frac{1}{n}} - 1)}{n(n+2)} \cdot \lim_{n \rightarrow \infty} \frac{2}{\sqrt{\frac{n-1}{n}} + \sqrt{\frac{n+1}{n+2}}} \\
&= \lim_{n \rightarrow \infty} \frac{-n^{k-1}}{n(n+2)} \cdot \lim_{n \rightarrow \infty} \frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}} \\
&= \ln a \cdot \lim_{n \rightarrow \infty} \frac{-n^{k-2}}{n+2} \\
&= \begin{cases} 0 & \text{if } k \in \{0, 1, 2\} \\ -\ln a & \text{if } k = 3 \\ \infty & \text{if } k \geq 4 \text{ and } a \in (0, 1) \\ -\infty & \text{if } k \geq 4 \text{ and } a > 1 \end{cases}
\end{aligned}$$

23. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{n^2 + k}}$$

Solution: Clearly

$$\frac{1}{\sqrt{n^2 + n}} \leq \frac{1}{\sqrt{n^2 + k}} \leq \frac{1}{\sqrt{n^2 + 1}}, \quad (\forall) 1 \leq k \leq n$$

Thus summing for $k = \overline{1, n}$, we get:

$$\frac{n}{\sqrt{n^2 + n}} \leq \sum_{k=1}^n \frac{1}{\sqrt{n^2 + k}} \leq \frac{n}{\sqrt{n^2 + 1}}$$

Because $\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n}}} = 1$ and $\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + 1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n^2}}} = 1$, using the squeeze theorem it follows that:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{n^2 + k}} = 1$$

24. If $a > 0$, $p \geq 2$, evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt[p]{n^p + ka}}$$

Solution: Obviously

$$\frac{1}{\sqrt[p]{n^p + na}} \leq \frac{1}{\sqrt[p]{n^p + ka}} \leq \frac{1}{\sqrt[p]{n^p + a}}, \quad (\forall) 1 \leq k \leq n$$

Thus summing for $k = \overline{1, n}$, we get:

$$\frac{n}{\sqrt[p]{n^p + na}} \leq \sum_{k=1}^n \frac{1}{\sqrt[p]{n^p + ka}} \leq \frac{n}{\sqrt[p]{n^p + a}}$$

Because $\lim_{n \rightarrow \infty} \frac{n}{\sqrt[p]{n^p + a}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[p]{1 + \frac{a}{n^p}}} = 1$ and $\lim_{n \rightarrow \infty} \frac{n}{\sqrt[p]{n^p + na}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[p]{1 + \frac{a}{n^{p-1}}}} = 1$, using the squeeze theorem it follows that:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt[p]{n^p + ka}} = 1$$

25. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{n!}{(1 + 1^2)(1 + 2^2) \cdots (1 + n^2)}$$

Solution: We have

$$\begin{aligned} 0 &\leq \frac{n!}{(1 + 1^2)(1 + 2^2) \cdots (1 + n^2)} \\ &< \frac{n!}{1^2 \cdot 2^2 \cdots n^2} \\ &= \frac{n!}{(1 \cdot 2 \cdots n) \cdot (1 \cdot 2 \cdots n)} \\ &= \frac{n!}{(n!)^2} \\ &= \frac{1}{n!} \end{aligned}$$

Thus using squeeze theorem it follows that:

$$\lim_{n \rightarrow \infty} \frac{n!}{(1+1^2)(1+2^2) \cdots (1+n^2)} = 0$$

26. Evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{2n^2 - 3}{2n^2 - n + 1} \right)^{\frac{n^2 - 1}{n}}$$

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{2n^2 - 3}{2n^2 - n + 1} \right)^{\frac{n^2 - 1}{n}} &= \lim_{n \rightarrow \infty} \left(1 + \frac{n - 4}{2n^2 - n + 1} \right)^{\frac{n^2 - 1}{n}} \\ &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{n - 4}{2n^2 - n + 1} \right)^{\frac{2n^2 - n + 1}{n - 4}} \right]^{\frac{(n-4)(n^2-1)}{2n^3 - 2n^2 + n}} \\ &= \lim_{n \rightarrow \infty} \frac{n^3 - 4n^2 - n + 4}{2n^3 - 2n^2 + n} \\ &= e^{\frac{1}{2}} \\ &= \sqrt{e} \end{aligned}$$

27. Evaluate:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin^2 x} - \cos x}{1 - \sqrt{1 + \tan^2 x}}$$

Solution:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin^2 x} - \cos x}{1 - \sqrt{1 + \tan^2 x}} &= \lim_{x \rightarrow 0} \frac{(1 + \sin^2 x - \cos^2 x)(1 + \sqrt{1 + \tan^2 x})}{(1 - 1 - \tan^2 x)(\sqrt{1 + \sin^2 x} + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin^2 x (1 + \sqrt{1 + \tan^2 x})}{-\tan^2 x (\sqrt{1 + \sin^2 x} + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{-2 \cos^2 x (1 + \sqrt{1 + \tan^2 x})}{\sqrt{1 + \sin^2 x} + \cos x} \\ &= -2 \end{aligned}$$

28. Evaluate:

$$\lim_{x \rightarrow \infty} \left(\frac{x + \sqrt{x}}{x - \sqrt{x}} \right)^x$$

Solution:

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \left(\frac{x + \sqrt{x}}{x - \sqrt{x}} \right)^x &= \lim_{x \rightarrow \infty} \left(1 + \frac{2\sqrt{x}}{x - \sqrt{x}} \right)^x \\
 &= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{2\sqrt{x}}{x - \sqrt{x}} \right)^{\frac{x - \sqrt{x}}{2\sqrt{x}}} \right]^{\frac{2x\sqrt{x}}{x - \sqrt{x}}} \\
 &= e^{\lim_{x \rightarrow \infty} \frac{2x\sqrt{x}}{x - \sqrt{x}}} \\
 &= e^{\lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{1 - \frac{1}{\sqrt{x}}}} \\
 &= e^\infty \\
 &= \infty
 \end{aligned}$$

29. Evaluate:

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} (\cos x)^{\frac{1}{\sin x}}$$

Solution:

$$\begin{aligned}
 \lim_{\substack{x \rightarrow 0 \\ x > 0}} (\cos x)^{\frac{1}{\sin x}} &= \lim_{\substack{x \rightarrow 0 \\ x > 0}} \left[(1 + (\cos x - 1))^{\frac{1}{\cos x - 1}} \right]^{\frac{\cos x - 1}{\sin x}} \\
 &= e^{\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{-2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}}} \\
 &= e^{\lim_{\substack{x \rightarrow 0 \\ x > 0}} -\tan \frac{x}{2}} \\
 &= e^0 \\
 &= 1
 \end{aligned}$$

30. Evaluate:

$$\lim_{x \rightarrow 0} (e^x + \sin x)^{\frac{1}{x}}$$

Solution:

$$\begin{aligned}
\lim_{x \rightarrow 0} (e^x + \sin x)^{\frac{1}{x}} &= \lim_{x \rightarrow 0} \left[e^x \left(1 + \frac{\sin x}{e^x} \right) \right]^{\frac{1}{x}} \\
&= \lim_{x \rightarrow 0} (e^x)^{\frac{1}{x}} \cdot \lim_{x \rightarrow 0} \left[\left(1 + \frac{\sin x}{e^x} \right)^{\frac{e^x}{\sin x}} \right]^{\frac{\sin x}{x e^x}} \\
&= e \cdot \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \frac{1}{e^x} \\
&= e^2
\end{aligned}$$

31. If $a, b \in \mathbb{R}_+^*$, evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{a - 1 + \sqrt[n]{b}}{a} \right)^n$$

Solution:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\frac{a - 1 + \sqrt[n]{b}}{a} \right)^n &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{\sqrt[n]{b} - 1}{a} \right)^{\frac{a}{\sqrt[n]{b} - 1}} \right]^{\frac{n(\sqrt[n]{b} - 1)}{a}} \\
&= e^{\frac{1}{a} \lim_{n \rightarrow \infty} \frac{b^{\frac{1}{n}} - 1}{\frac{1}{n}}} \\
&= e^{\frac{\ln b}{a}} \\
&= b^{\frac{1}{a}}
\end{aligned}$$

32. Consider a sequence of real numbers $(a_n)_{n \geq 1}$ defined by:

$$a_n = \begin{cases} 1 & \text{if } n \leq k, \ k \in \mathbb{N}^* \\ \frac{(n+1)^k - n^k}{\binom{n}{k-1}} & \text{if } n > k \end{cases}$$

i) Evaluate $\lim_{n \rightarrow \infty} a_n$.

ii) If $b_n = 1 + \sum_{k=1}^n k \cdot \lim_{n \rightarrow \infty} a_n$, evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{b_n^2}{b_{n-1} b_{n+1}} \right)^n$$

Solution: i) We have

$$\begin{aligned}
\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{(n+1)^k - n^k}{\binom{n}{k-1}} \\
&= \lim_{n \rightarrow \infty} \frac{(k-1)! \cdot k \cdot n^{k-1} + \dots + (k-1)!}{(n-k+2)(n-k+3) \cdot \dots \cdot n} \\
&= \frac{k! \cdot n^{k-1} + \dots}{n^{k-1} + \dots} \\
&= k!
\end{aligned}$$

ii) Then:

$$b_n = 1 + \sum_{k=1}^n k \cdot k! = 1 + \sum_{k=1}^n (k+1)! - \sum_{k=1}^n k! = (n+1)!$$

so

$$\lim_{n \rightarrow \infty} \left(\frac{b_n^2}{b_{n-1}b_{n+1}} \right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right)^n = e^{-1}$$

33. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ such that $x_{n+2} = \frac{x_{n+1} + x_n}{2}$, $(\forall) n \in \mathbb{N}^*$. If $x_1 \leq x_2$,

i) Prove that the sequence $(x_{2n+1})_{n \geq 0}$ is increasing, while the sequence $(x_{2n})_{n \geq 0}$ is decreasing;

ii) Prove that:

$$|x_{n+2} - x_{n+1}| = \frac{|x_2 - x_1|}{2^n}, \quad (\forall) n \in \mathbb{N}^*$$

iii) Prove that:

$$2x_{n+2} + x_{n+1} = 2x_2 + x_1, \quad (\forall) n \in \mathbb{N}^*$$

iv) Prove that $(x_n)_{n \geq 1}$ is convergent and that its limit is $\frac{x_1 + 2x_2}{3}$.

Solution: i) Using induction we can show that $x_{2n-1} \leq x_{2n}$. Then the sequence $(x_{2n+1})_{n \geq 0}$ will be increasing, because

$$x_{2n+1} = \frac{x_{2n} + x_{2n-1}}{2} \geq \frac{x_{2n-1} + x_{2n-1}}{2} = x_{2n-1}$$

Similarly, we can show that $(x_{2n})_{n \geq 1}$ is decreasing.

ii) For $n = 1$, we get $|x_3 - x_2| = \frac{|x_2 - x_1|}{2}$, so assuming it's true for some k , we have:

$$|x_{k+3} - x_{k+2}| = \left| \frac{x_{k+2} + x_{k+1}}{2} - x_{k+2} \right| = \frac{|x_{k+2} - x_{k+1}|}{2} = \frac{|x_2 - x_1|}{2^{n+1}}$$

Thus, by induction the equality is proven.

iii) Observe that:

$$2x_{n+2} + x_{n+1} = 2 \cdot \frac{x_{n+1} + x_n}{2} + x_{n+1} = 2x_{n+1} + x_n$$

and repeating the process, the demanded identity is showed.

iv) From i) it follows that the sequences $(x_{2n})_{n \geq 1}$ and $(x_{2n-1})_{n \geq 1}$ are convergent and have the same limit. Let $l = \lim_{n \rightarrow \infty} x_n = l$. Then from iii), we get

$$3l = x_1 + 2x_2 \Rightarrow l = \frac{x_1 + 2x_2}{3}$$

34. Let $a_n, b_n \in \mathbb{Q}$ such that $(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}$, $(\forall) n \in \mathbb{N}^*$. Evaluate $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$.

Solution: Because $(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}$, it follows that $(1 - \sqrt{2})^n = a_n - b_n\sqrt{2}$. Solving this system we find:

$$a_n = \frac{1}{2} \left[(1 + \sqrt{2})^n + (1 - \sqrt{2})^n \right]$$

and

$$b_n = \frac{1}{2\sqrt{2}} \left[(1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right]$$

and therefore $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \sqrt{2}$.

35. If $a > 0$, evaluate:

$$\lim_{x \rightarrow 0} \frac{(a+x)^x - 1}{x}$$

Solution:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{(a+x)^x - 1}{x} &= \lim_{x \rightarrow 0} \frac{e^{x \ln(a+x)} - 1}{x} \\ &= \lim_{x \rightarrow 0} \frac{e^{x \ln(a+x)} - 1}{x \ln(a+x)} \cdot \lim_{x \rightarrow 0} \ln(a+x) \\ &= \ln a \end{aligned}$$

36. Consider a sequence of real numbers $(a_n)_{n \geq 1}$ such that $a_1 = \frac{3}{2}$ and $a_{n+1} = \frac{a_n^2 - a_n + 1}{a_n}$. Prove that $(a_n)_{n \geq 1}$ is convergent and find its limit.

Solution: By *AM - GM* we have $a_{n+1} = a_n + \frac{1}{a_n} - 1 \geq 1$, $(\forall)n \geq 2$, so the sequence is lower bounded. Also $a_{n+1} - a_n = \frac{1}{a_n} - 1 \leq 0$, hence the sequence is decreasing. Therefore $(a_n)_{n \geq 1}$ is bounded by 1 and $a_1 = \frac{3}{2}$. Then, because $(a_n)_{n \geq 1}$ is convergent, denote $\lim_{n \rightarrow \infty} a_n = l$, to obtain $l = \frac{l^2 - l + 1}{l} \Rightarrow l = 1$

37. Consider sequence $(x_n)_{n \geq 1}$ of real numbers such that $x_0 \in (0, 1)$ and $x_{n+1} = x_n - x_n^2 + x_n^3 - x_n^4$, $(\forall)n \geq 0$. Prove that this sequence is convergent and evaluate $\lim_{n \rightarrow \infty} x_n$.

Solution: It's easy to see that the recurrence formula can be written as: $x_{n+1} = x_n(1 - x_n)(1 + x_n^2)$, $n \in \mathbb{N}$, then because $1 - x_0 > 0$, it's easy to show by induction that $x_n \in (0, 1)$. Now rewrite the recurrence formula as $x_{n+1} - x_n = -x_n^2(x_n^2 - x_n + 1) < 0$. It follows that the sequence is strictly decreasing, thus convergent. Let $\lim_{n \rightarrow \infty} x_n = l$. Then

$$l = l - l^2 + l^3 - l^4 \Rightarrow l^2(l^2 - l + 1) = 0 \Rightarrow l = 0$$

38. Let $a > 0$ and $b \in (a, 2a)$ and a sequence $x_0 = b$, $x_{n+1} = a + \sqrt{x_n(2a - x_n)}$, $(\forall)n \geq 0$. Study the convergence of the sequence $(x_n)_{n \geq 0}$.

Solution: Let's see a few terms: $x_1 = a + \sqrt{2ab - b^2}$ and also

$$x_2 = a + \sqrt{(a + \sqrt{2ab - b^2})(a - \sqrt{2ab - b^2})} = a + \sqrt{a^2 - 2ab + b^2} = a + |a - b| = b$$

Thus the sequence is periodic, with $x_{2k} = b$ and $x_{2k+1} = a + \sqrt{2ab - b^2}$, $(\forall)k \in \mathbb{N}$. Then $\lim_{k \rightarrow \infty} x_{2k} = b$ and $\lim_{n \rightarrow \infty} x_{2k+1} = a + \sqrt{2ab - b^2}$. The sequence is convergent if and only if $b = a + \sqrt{2ab - b^2}$, which implies that $b = \left(1 + \frac{1}{\sqrt{2}}\right)a$, which is also the limit in this case.

39. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n+1} \arctan \frac{1}{2k^2}$$

Solution: We can check easily that $\arctan \frac{1}{2k^2} = \arctan \frac{k}{k+1} - \arctan \frac{k-1}{k}$. Then:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n+1} \arctan \frac{1}{2k^2} = \lim_{n \rightarrow \infty} \arctan \frac{n}{n+1} = \frac{\pi}{4}$$

40. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{4k^4 + 1}$$

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{4k^4 + 1} &= \lim_{n \rightarrow \infty} \left(\frac{1}{2} \sum_{k=1}^n \frac{1}{2k^2 - 2k + 1} - \frac{1}{2} \sum_{k=1}^n \frac{1}{2k^2 + 2k + 1} \right) \\ &= \frac{1}{4} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n^2 + 2n + 1} \right) \\ &= \frac{1}{4} \end{aligned}$$

41. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1 + 3 + 3^2 + \dots + 3^k}{5^{k+2}}$$

Solution: In the numerator we have a geometrical progression, so:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1 + 3 + 3^2 + \dots + 3^k}{5^{k+2}} &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{3^{k+1} - 1}{2 \cdot 5^{k+2}} \\ &= \frac{1}{10} \lim_{n \rightarrow \infty} \sum_{k=2}^n \left(\frac{3^k}{5^k} - \frac{1}{5^k} \right) \\ &= \frac{1}{10} \left(\frac{9}{10} - \frac{1}{20} \right) \\ &= \frac{17}{200} \end{aligned}$$

42. Evaluate:

$$\lim_{n \rightarrow \infty} \left(n + 1 - \sum_{i=2}^n \sum_{k=2}^i \frac{k-1}{k!} \right)$$

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(n + 1 - \sum_{i=2}^n \sum_{k=2}^i \frac{k-1}{k!} \right) &= \lim_{n \rightarrow \infty} \left(n + 1 - \sum_{i=2}^n \sum_{k=2}^i \left(\frac{1}{(k-1)!} - \frac{1}{k!} \right) \right) \\ &= \lim_{n \rightarrow \infty} \left(n + 1 - \sum_{i=2}^n \left(1 - \frac{1}{i!} \right) \right) \\ &= \lim_{n \rightarrow \infty} \left(1 + \sum_{i=1}^n \frac{1}{i!} \right)^n \\ &= e \end{aligned}$$

43. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1^1 + 2^2 + 3^3 + \dots + n^n}{n^n}$$

Solution: Using Cesaro-Stolz theorem we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1^1 + 2^2 + 3^3 + \dots + n^n}{n^n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1)^{n+1} - n^n} \\ &= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^{n+1} - \frac{1}{n}} \\ &= \frac{e}{e - 0} \\ &= 1 \end{aligned}$$

44. Consider the sequence $(a_n)_{n \geq 1}$ such that $a_0 = 2$ and $a_{n-1} - a_n = \frac{n}{(n+1)!}$. Evaluate $\lim_{n \rightarrow \infty} ((n+1)! \ln a_n)$.

Solution: Observe that

$$a_k - a_{k-1} = \frac{-k}{(k+1)!} = \frac{1}{(k+1)!} - \frac{1}{k!}, \quad (\forall) 1 \leq k \leq n$$

Letting $k = 1, 2, 3 \dots n$ and summing, we get $a_n - a_0 = \frac{1}{(n+1)!} - 1$. Since $a_0 = 2$ we get $a_n = 1 + \frac{1}{(n+1)!}$. Using the result $\lim_{f(x) \rightarrow 0} \frac{\ln(1+f(x))}{f(x)} = 1$, we conclude that

$$\lim_{n \rightarrow \infty} (n+1)! \ln a_n = \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{(n+1)!}\right)}{\frac{1}{(n+1)!}} = 1$$

45. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ with $x_1 = a > 0$ and $x_{n+1} = \frac{x_1 + 2x_2 + 3x_3 + \dots + nx_n}{n}$, $n \in \mathbb{N}^*$. Evaluate it's limit.

Solution: The sequence is strictly increasing because:

$$x_{n+1} - x_n = \frac{x_1 + 2x_2 + 3x_3 + \dots + nx_n}{n} - x_n = \frac{x_1 + 2x_2 + 3x_3 + \dots + (n-1)x_{n-1}}{n} > 0$$

Then

$$x_{n+1} > \frac{a + 2a + \dots + na}{n} = \frac{(n+1)a}{2}$$

It follows that $\lim_{n \rightarrow \infty} x_n = \infty$.

46. Using $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6}$, evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{(2k-1)^2}$$

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{(2k-1)^2} &= \lim_{n \rightarrow \infty} \sum_{k=1}^{2n} \frac{1}{k^2} - \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{(2k)^2} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^{2n} \frac{1}{k^2} - \frac{1}{4} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} \\ &= \frac{\pi}{6} - \frac{\pi}{24} \\ &= \frac{\pi}{8} \end{aligned}$$

47. Consider the sequence $(x_n)_{n \geq 1}$ defined by $x_1 = a$, $x_2 = b$, $a < b$ and $x_n = \frac{x_{n-1} + \lambda x_{n-2}}{1 + \lambda}$, $n \geq 3$, $\lambda > 0$. Prove that this sequence is convergent and find its limit.

Solution: The sequence isn't monotonic because $x_3 = \frac{b + \lambda a}{1 + \lambda} \in [a, b]$. We can prove by induction that $x_n \in [a, b]$. The sequences $(x_{2n})_{n \geq 1}$ and $(x_{2n-1})_{n \geq 1}$ are monotonically increasing. Also, we can show by induction, that:

$$x_{2k} - x_{2k-1} = \left(\frac{\lambda}{1 + \lambda} \right)^{2k} (b - a)$$

It follows that the sequences $(x_{2n})_{n \geq 1}$ and $(x_{2n-1})_{n \geq 1}$ have the same limit, so $(x_n)_{n \geq 1}$ is convergent. The recurrence formulas can be written as

$$x_k - x_{k-1} = \lambda(x_{k-2} - x_k), \quad (\forall) k \geq 3$$

Summing for $k = 3, 4, 5, \dots, n$, we have:

$$x_n - b = \lambda(a + b - x_{n-1} - x_n) \Leftrightarrow (1 + \lambda)x_n + \lambda x_{n-1} = (1 + \lambda)b + \lambda a$$

By passing to limit, it follows that:

$$\lim_{n \rightarrow \infty} x_n = \frac{b + \lambda(a + b)}{1 + 2\lambda}$$

48. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}}$$

Solution: Using the consequence of Cesaro-Stolz lemma, we have:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{n!}} \\
 &= \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} \\
 &= \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{n^n \cdot (n+1)} \\
 &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \\
 &= e
 \end{aligned}$$

49. Consider the sequence $(x_n)_{n \geq 1}$ defined by $x_1 = 1$ and $x_n = \frac{1}{1 + x_{n-1}}$, $n \geq 2$. Prove that this sequence is convergent and evaluate $\lim_{n \rightarrow \infty} x_n$.

Solution: We can show easily by induction that $x_n \in (0, 1)$ and that the sequence $(x_{2n})_{n \geq 1}$ is increasing, while the sequence $(x_{2n-1})_{n \geq 1}$ is decreasing. Observe that:

$$x_{2n+2} = \frac{1}{1 + x_{2n+1}} = \frac{1}{1 + \frac{1}{1 + x_{2n}}} = \frac{1 + x_{2n}}{2 + x_{2n}}$$

The sequence $(x_{2n})_{n \geq 1}$ is convergent, so it has the limit $\frac{\sqrt{5}-1}{2}$. Similarly

$\lim_{n \rightarrow \infty} x_{2n-1} = \frac{\sqrt{5}-1}{2}$. Therefore $(x_n)_{n \geq 1}$ is convergent and has the limit equal to $\frac{\sqrt{5}-1}{2}$.

50. If $a, b \in \mathbb{R}^*$, evaluate:

$$\lim_{x \rightarrow 0} \frac{\ln(\cos ax)}{\ln(\cos bx)}$$

Solution:

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\ln(\cos ax)}{\ln(\cos bx)} &= \lim_{x \rightarrow 0} \frac{(\cos ax - 1) \cdot \ln(1 + \cos ax - 1) \frac{1}{\cos ax - 1}}{(\cos bx - 1) \cdot \ln(1 + \cos bx - 1) \frac{1}{\cos bx - 1}} \\
 &= \lim_{x \rightarrow 0} \frac{-2 \sin^2 \frac{ax}{2}}{-2 \sin^2 \frac{bx}{2}} \\
 &= \frac{a^2}{b^2}
 \end{aligned}$$

51. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \begin{cases} \{x\} & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Find all $\alpha \in \mathbb{R}$ for which $\lim_{x \rightarrow \alpha} f(x)$ exists.

Solution: Let $f = g - h$, where $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = x$, $(\forall)x \in \mathbb{R}$ and $h : \mathbb{R} \rightarrow \mathbb{R}$, $h(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. If $\alpha \in \mathbb{R} \setminus [0, 1)$, we can find two sequences $x_n \in \mathbb{Q}$ and $y_n \in \mathbb{R} \setminus \mathbb{Q}$ going to α , such that the sequences $(f(x_n))$ and $(f(y_n))$ have different limits. If $\alpha \in [0, 1)$, $h(x) = 0$ and $f(x) = x$, thus $(\forall)\alpha \in [0, 1)$, we have $\lim_{x \rightarrow \alpha} f(x) = \alpha$.

52. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Find all $\alpha \in \mathbb{R}$ for which $\lim_{x \rightarrow \alpha} f(x)$ exists.

Solution: Divide the problem in two cases:

Case I: $\alpha = k \in \mathbb{Z}$. Consider a sequence (x_n) , $x_n \in (k-1, k) \cap \mathbb{Q}$ and (y_n) , $y_n \in (k-1, k) \cap (\mathbb{R} \setminus \mathbb{Q})$, both tending to k . Then:

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} \lfloor x_n \rfloor = \lim_{n \rightarrow \infty} (k-1) = k-1$$

and $\lim_{n \rightarrow \infty} f(y_n) = \lim_{n \rightarrow \infty} y_n = k$. Therefore $\lim_{x \rightarrow \alpha} f(x)$ doesn't exist.

Case II: $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Let $\lfloor \alpha \rfloor = k$. Consider a sequence (x_n) , $x_n \in (k, k+1) \cap \mathbb{Q}$ and (y_n) , $y_n \in (k, k+1) \cap (\mathbb{R} \setminus \mathbb{Q})$, which tend both to α . Then:

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} \lfloor x_n \rfloor = \lim_{n \rightarrow \infty} k = k$$

and $\lim_{n \rightarrow \infty} f(y_n) = \lim_{n \rightarrow \infty} y_n = \alpha$. Again, in this case, $\lim_{x \rightarrow \alpha} f(x)$ doesn't exist.

53. Let $(x_n)_{n \geq 1}$ be a sequence of positive real numbers such that $x_1 > 0$ and $3x_n = 2x_{n-1} + \frac{a}{x_{n-1}^2}$, where a is a real positive number. Prove that x_n is convergent and evaluate $\lim_{n \rightarrow \infty} x_n$.

Solution: By AM-GM

$$x_{n+1} = \frac{x_n + x_n + \frac{a}{x_n^2}}{3} \geq \sqrt[3]{x_n \cdot x_n \cdot \frac{a}{x_n^2}} = \sqrt[3]{a} \Rightarrow x_n \geq \sqrt[3]{a}$$

Also

$$3(x_{n+1} - x_n) = \frac{a}{x_n^2} - x_n = \frac{a - x_n^3}{x_n^3} \leq 0 \Rightarrow x_{n+1} - x_n \leq 0, \forall n \in \mathbb{N}, n \geq 2$$

Therefore, the sequence $(x_n)_{n \geq 1}$ is decreasing and lower bounded, so it's convergent. By passing to limit in the recurrence formula we obtain $\lim_{n \rightarrow \infty} x_n = \sqrt[3]{a}$.

54. Consider a sequence of real numbers $(a_n)_{n \geq 1}$ such that $a_1 = 12$ and $a_{n+1} = a_n \left(1 + \frac{3}{n+1}\right)$. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{a_k}$$

Solution: Rewrite the recurrence formula as

$$a_{n+1} = a_n \cdot \frac{n+4}{n+1}$$

Writing it for $n = 1, 2, \dots, n-1$ and multiplying the obtained equalities, we find that:

$$a_n = \frac{(n+1)(n+2)(n+3)}{2}, \quad (\forall) n \in \mathbb{N}^*$$

Then:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{a_k} &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2}{(k+1)(k+2)(k+3)} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{k+1} - \frac{2}{k+2} + \frac{1}{k+3} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{6} - \frac{1}{n+2} + \frac{1}{n+3} \right) \\ &= \frac{1}{6} \end{aligned}$$

55. Evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{n}{\sqrt{n^2 + 1}} \right)^n$$

Solution:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\frac{n}{\sqrt{n^2 + 1}} \right)^n &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{n - \sqrt{n^2 + 1}}{\sqrt{n^2 + 1}} \right)^{\frac{\sqrt{n^2 + 1}}{n - \sqrt{n^2 + 1}}} \right]^{\frac{n(n - \sqrt{n^2 + 1})}{\sqrt{n^2 + 1}}} \\
&= e^{\lim_{n \rightarrow \infty} \frac{n(n - \sqrt{n^2 + 1})}{\sqrt{n^2 + 1}}} \\
&= e^{\lim_{n \rightarrow \infty} \frac{-n}{\sqrt{n^2 + 1} \cdot (\sqrt{n^2 + 1} + n)}} \\
&= e^{\lim_{n \rightarrow \infty} \frac{-n}{n^2 + 1 + n\sqrt{n^2 + 1}}} \\
&= e^{\lim_{n \rightarrow \infty} \frac{-1}{n + \frac{1}{n} + \sqrt{n^2 + 1}}} \\
&= e \\
&= e^0 \\
&= 1
\end{aligned}$$

56. If $a \in \mathbb{R}$, evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\lfloor k^2 a \rfloor}{n^3}$$

Solution: We have $x - 1 < \lfloor x \rfloor \leq x$, $(\forall) x \in \mathbb{R}$. Choosing $x = k^2 a$, letting k to take values from 1 to n and summing we have:

$$\sum_{k=1}^n (k^2 a - 1) < \sum_{k=1}^n \lfloor k^2 a \rfloor \leq \sum_{k=1}^n k^2 a \Leftrightarrow \frac{\sum_{k=1}^n (k^2 a - 1)}{n^3} < \frac{\sum_{k=1}^n \lfloor k^2 a \rfloor}{n^3} \leq \frac{\sum_{k=1}^n k^2 a}{n^3}$$

Now observe that:

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n (k^2 a - 1)}{n^3} = \lim_{n \rightarrow \infty} \frac{a \cdot \frac{n(n+1)(2n+1)}{6} - n}{n^3} = \frac{a}{3}$$

and

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n k^2 a}{n^3} = \lim_{n \rightarrow \infty} \frac{an(n+1)(2n+1)}{6n^3} = \frac{a}{3}$$

So using the Squeeze Theorem it follows that:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\lfloor k^2 a \rfloor}{n^3} = \frac{a}{3}$$

57. Evaluate:

$$\lim_{n \rightarrow \infty} 2^n \left(\sum_{k=1}^n \frac{1}{k(k+2)} - \frac{1}{4} \right)^n$$

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} 2^n \left(\sum_{k=1}^n \frac{1}{k(k+2)} - \frac{1}{4} \right)^n &= \lim_{n \rightarrow \infty} 2^n \left(\frac{1}{2} \sum_{k=1}^n \frac{1}{k} - \frac{1}{2} \sum_{k=1}^n \frac{1}{k+2} - \frac{1}{4} \right)^n \\ &= \lim_{n \rightarrow \infty} 2^n \left(\frac{3}{4} - \frac{1}{2} \left(\frac{1}{n+1} + \frac{1}{n+2} \right) - \frac{1}{4} \right)^n \\ &= \lim_{n \rightarrow \infty} \left(1 - \frac{2n+3}{(n+1)(n+2)} \right)^n \\ &= \lim_{n \rightarrow \infty} \left[\left(1 - \frac{2n+3}{(n+1)(n+2)} \right)^{\frac{-(n+1)(n+2)}{2n+3}} \right]^{\frac{-n(2n+3)}{(n+1)(n+2)}} \\ &= \lim_{n \rightarrow \infty} \frac{-2n^2 - 3n}{n^2 + 3n + 2} \\ &= e^{-2} \end{aligned}$$

58. Consider the sequence $(a_n)_{n \geq 1}$, such that $a_n > 0$, $(\forall) n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} n(a_{n+1} - a_n) = 1$. Evaluate $\lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} \sqrt[n]{a_n}$.

Solution: Start with the ε criterion

$$\lim_{n \rightarrow \infty} n(a_{n+1} - a_n) = 1 \Leftrightarrow (\forall) \varepsilon > 0, (\exists) n_\varepsilon \in \mathbb{N}, (\forall) n \geq n_\varepsilon \Rightarrow |n(a_{n+1} - a_n) - 1| < \varepsilon$$

Let $\varepsilon \in (0, 1)$. Then for $n \geq n_\varepsilon$, we have:

$$-\varepsilon < n(a_{n+1} - a_n) - 1 < \varepsilon \Rightarrow \frac{1 - \varepsilon}{n} < a_{n+1} - a_n < \frac{1 + \varepsilon}{n}$$

Summing for $n = n_\varepsilon, n_\varepsilon + 1, \dots, n$, we get:

$$(1 - \varepsilon) \left(\frac{1}{n_\varepsilon} + \frac{1}{n_\varepsilon + 1} + \dots + \frac{1}{n} \right) < a_{n+1} - a_{n_\varepsilon} < (1 + \varepsilon) \left(\frac{1}{n_\varepsilon} + \frac{1}{n_\varepsilon + 1} + \dots + \frac{1}{n} \right)$$

By passing to limit, it follows that $\lim_{n \rightarrow \infty} a_n = \infty$. To evaluate $\lim_{n \rightarrow \infty} \sqrt[n]{a_n}$, recall that in the above conditions we have:

$$\frac{1-\varepsilon}{n} < a_{n+1} - a_n < \frac{1+\varepsilon}{n} \Rightarrow \frac{1-\varepsilon}{na_n} < \frac{a_{n+1}}{a_n} - 1 < \frac{1+\varepsilon}{na_n}$$

Thus $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$, and the root test implies that $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$

59. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1 + 2\sqrt{2} + 3\sqrt{3} + \dots + n\sqrt{n}}{n^2\sqrt{n}}$$

Solution: Using Cesaro-Stolz lemma, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1 + 2\sqrt{2} + 3\sqrt{3} + \dots + n\sqrt{n}}{n^2\sqrt{n}} &= \lim_{n \rightarrow \infty} \frac{(n+1)\sqrt{n+1}}{(n+1)^2\sqrt{n+1} - n^2\sqrt{n}} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{(n+1)^3}}{\sqrt{(n+1)^5} - \sqrt{n^5}} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{(n+1)^3} \left(\sqrt{(n+1)^5} + \sqrt{n^5} \right)}{(n+1)^5 - n^5} \\ &= \lim_{n \rightarrow \infty} \frac{n^4 + 4n^3 + 6n^2 + 4n + 1 + \sqrt{n^8 + 3n^7 + 3n^6 + n^5}}{5n^4 + 10n^3 + 10n^2 + 5n + 1} \\ &= \lim_{n \rightarrow \infty} \frac{1 + \frac{4}{n} + \frac{6}{n^2} + \frac{4}{n^3} + \frac{1}{n^4} + \sqrt{1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}}}{5 + \frac{10}{n} + \frac{10}{n^2} + \frac{5}{n^3} + \frac{1}{n^4}} \\ &= \frac{2}{5} \end{aligned}$$

60. Evaluate:

$$\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\frac{1}{2x-\pi}}$$

Solution:

$$\begin{aligned}
\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\frac{1}{2x-\pi}} &= \lim_{x \rightarrow \frac{\pi}{2}} \left[(1 + \sin x - 1)^{\frac{1}{\sin x - 1}} \right]^{\frac{\sin x - 1}{2x - \pi}} \\
&= e^{\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{2x - \pi}} \\
&= e^{\lim_{y \rightarrow 0} \frac{\cos y - 1}{2y}} \\
&= e^{\lim_{y \rightarrow 0} \frac{-\sin^2 \frac{y}{2}}{y}} \\
&= e^{\lim_{y \rightarrow 0} \left(\frac{\sin \frac{y}{2}}{\frac{y}{2}} \right)^2 \cdot \left(-\frac{y}{4} \right)} \\
&= e^0 \\
&= 1
\end{aligned}$$

61. Evaluate:

$$\lim_{n \rightarrow \infty} n^2 \ln \left(\cos \frac{1}{n} \right)$$

Solution: We'll use the well-known limit $\lim_{x_n \rightarrow 0} \frac{\ln(1 + x_n)}{x_n} = 1$. We have:

$$\begin{aligned}
\lim_{n \rightarrow \infty} n^2 \ln \left(\cos \frac{1}{n} \right) &= \lim_{n \rightarrow \infty} \left[n^2 \left(\cos \frac{1}{n} - 1 \right) \right] \cdot \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{n} - 1 \right)}{\cos \frac{1}{n} - 1} \\
&= \lim_{n \rightarrow \infty} -2n^2 \cdot \sin^2 \frac{1}{2n} \\
&= \lim_{n \rightarrow \infty} -\frac{1}{2} \cdot \left(\frac{\sin \frac{1}{2n}}{\frac{1}{2n}} \right)^2 \\
&= -\frac{1}{2}
\end{aligned}$$

62. Given $a, b \in \mathbb{R}_+^*$, evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} \right)^n$$

Solution: Using the limits $\lim_{x_n \rightarrow \infty} \frac{1}{(1 + x_n)^{x_n}} = e$ and $\lim_{n \rightarrow \infty} n(\sqrt[n]{a} - 1) = \ln a$, we have:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} \right)^n &= \lim_{n \rightarrow \infty} \left(1 + \frac{\sqrt[n]{a} - 1 + \sqrt[n]{b} - 1}{2} \right)^n \\
&= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{\sqrt[n]{a} - 1 + \sqrt[n]{b} - 1}{2} \right)^{\frac{2}{\sqrt[n]{a} - 1 + \sqrt[n]{b} - 1}} \right]^{\frac{n(\sqrt[n]{a} - 1) + n(\sqrt[n]{b} - 1)}{2}} \\
&= e^{\lim_{n \rightarrow \infty} \frac{n(\sqrt[n]{a} - 1) + n(\sqrt[n]{b} - 1)}{2}} \\
&= e^{\frac{\ln a + \ln b}{2}} \\
&= e^{\ln \sqrt{ab}} \\
&= \sqrt{ab}
\end{aligned}$$

63. Let $\alpha > \beta > 0$ and the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

i) Prove that $(\exists)(x_n)_{n \geq 1}, (y_n)_{n \geq 1} \in \mathbb{R}$ such that:

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^n = x_n A + y_n B, \quad (\forall) n \geq 1$$

ii) Evaluate $\lim_{n \rightarrow \infty} \frac{x_n}{y_n}$.

Solution: i) We proceed by induction. For $n = 1$, we have

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} = \alpha A + \beta B$$

Hence $x_1 = \alpha$ and $y_1 = \beta$. Let

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^k = x_k A + y_k B$$

Then

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^{k+1} = (\alpha A + \beta B)(x_k A + y_k B)$$

Using $B^2 = A$, we have:

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^{k+1} = (\alpha x_k + \beta y_k) A + (\beta x_k + \alpha y_k) B$$

Thus $x_{k+1} = \alpha x_k + \beta y_k$ and $y_{k+1} = \beta x_k + \alpha y_k$.

ii) An easy induction shows that $x_n, y_n > 0$, $(\forall)n \in \mathbb{N}^*$. Let $X \in \mathcal{M}_2(\mathbb{R})$ such that $X^n = \begin{pmatrix} x_n & y_n \\ y_n & x_n \end{pmatrix}$. Because $\det(X^n) = (\det X)^n$, it follows that $(\alpha^2 - \beta^2)^n = x_n^2 - y_n^2$, and because $\alpha > \beta$, we have $x_n > y_n$, $(\forall)n \in \mathbb{N}^*$. Let $z_n = \frac{x_n}{y_n}$. Then:

$$z_{n+1} = \frac{x_{n+1}}{y_{n+1}} = \frac{\alpha x_n + \beta y_n}{\beta x_n + \alpha y_n} = \frac{\alpha z_n + \beta}{\beta z_n + \alpha}$$

It's easy to see that the sequence is bounded by 1 and $\frac{\alpha}{\beta}$. Also the sequence is strictly decreasing, because

$$z_{n+1} - z_n = \frac{\alpha z_n + \beta}{\beta z_n + \alpha} - z_n = \frac{\beta(1 - z_n^2)}{\beta z_n + \alpha} < 0$$

Therefore the sequence is convergent. Let $\lim_{n \rightarrow \infty} z_n = l$, then

$$l = \frac{\alpha l + \beta}{\beta l + \alpha} \Rightarrow l^2 = 1$$

l can't be -1 , because $z_n \in \left(1, \frac{\alpha}{\beta}\right)$, hence $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = 1$.

64. If $a \in \mathbb{R}$ such that $|a| < 1$ and $p \in \mathbb{N}^*$ is given, evaluate:

$$\lim_{n \rightarrow \infty} n^p \cdot a^n$$

Solution: If $a = 0$, we get $n^p \cdot a^n = 0$, $(\forall)n \in \mathbb{N}$. If $a \neq 0$, since $|a| < 1$, there is a $\alpha > 0$ such that $|a| = \frac{1}{1 + \alpha}$. Let now $n > p$, then from binomial expansion we get:

$$(1 + \alpha)^n > C_n^{p+1} \cdot \alpha^{p+1} \Leftrightarrow \frac{1}{(1 + \alpha)^n} < \frac{(p+1)!}{n(n-1)(n-2) \cdots (n-p) \cdot \alpha^{p+1}}$$

Then:

$$\begin{aligned} 0 &< |n^p \cdot a^n| \\ &= n^p \cdot |a|^n \\ &< \frac{n^p \cdot (p+1)!}{n(n-1)(n-2) \cdots (n-p) \cdot \alpha^{p+1}} \\ &= \frac{n^{p-1} \cdot (p+1)!}{(n-1)(n-2) \cdots (n-p) \cdot \alpha^{p+1}} \end{aligned}$$

Keeping in mind that

$$\lim_{n \rightarrow \infty} \frac{n^{p-1} \cdot (p+1)!}{(n-1)(n-2) \cdots (n-p) \cdot \alpha^{p+1}} = 0$$

and using the Squeeze Theorem, it follows that

$$\lim_{n \rightarrow \infty} n^p \cdot a^n = 0$$

65. If $p \in \mathbb{N}^*$, evaluate:

$$\lim_{n \rightarrow \infty} \frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}}$$

Solution: Using Cesaro-Stolz lemma we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}} &= \lim_{n \rightarrow \infty} \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}} \\ &= \lim_{n \rightarrow \infty} \frac{n^p + \binom{p}{1} n^{p-1} + \dots}{\binom{p+1}{1} n^p + \binom{p+1}{2} n^{p-1} + \dots} \\ &= \frac{1}{\binom{p+1}{1}} \\ &= \frac{1}{p+1} \end{aligned}$$

66. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{\sin(n \arccos x)}{\sqrt{1-x^2}}$$

First solution: Recall the identity:

$$\cos nt + i \sin nt = \binom{n}{0} \cos^n t + i \binom{n}{1} \cos^{n-1} t \cdot \sin t + \dots + i^n \binom{n}{n} \sin^n t$$

For $t = \arccos x$, we have:

$$\sin(n \arccos x) = \binom{n}{1} x^{n-1} \cdot \sqrt{1-x^2} - \binom{n}{3} x^{n-3} (\sqrt{1-x^2})^3 + \binom{n}{5} x^{n-5} (\sqrt{1-x^2})^5 - \dots$$

Then:

$$\lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{\sin(n \arccos x)}{\sqrt{1-x^2}} = \lim_{\substack{x \rightarrow 1 \\ x < 1}} \left(\binom{n}{1} x^{n-1} - \binom{n}{3} x^{n-3} (1-x^2) + \binom{n}{5} x^{n-5} (1-x^2)^2 - \dots \right) = n$$

Second solution:

$$\begin{aligned}
 \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{\sin(n \arccos x)}{\sqrt{1-x^2}} &= \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{\sin(n \arccos x)}{n \arccos x} \cdot \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{n \arccos x}{\sqrt{1-x^2}} \\
 &= \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{n \arccos x}{\sqrt{1-x^2}} \\
 &= \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{ny}{\sqrt{1-\cos y}} \\
 &= \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{ny}{\sin y} \\
 &= n
 \end{aligned}$$

67. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{1 - \cos(n \arccos x)}{1 - x^2}$$

Solution:

$$\begin{aligned}
 \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{1 - \cos(n \arccos x)}{1 - x^2} &= \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{2 \sin^2 \left(\frac{n \arccos x}{2} \right)}{1 - x^2} \\
 &= \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{2 \sin^2 \left(\frac{n \arccos x}{2} \right)}{\left(\frac{n \arccos x}{2} \right)^2} \cdot \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{n^2 \arccos^2 x}{4(1-x^2)} \\
 &= \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{n^2 \arccos^2 x}{4(1-x^2)} \\
 &= \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{n^2 y^2}{2 \sin^2 y} \\
 &= \frac{n^2}{2}
 \end{aligned}$$

68. Study the convergence of the sequence:

$$x_{n+1} = \frac{x_n + a}{x_n + 1}, \quad n \geq 1, \quad x_1 \geq 0, \quad a > 0$$

Solution: Consider a sequence $(y_n)_{n \geq 1}$ such that $x_n = \frac{y_{n+1}}{y_n} - 1$. Thus, our recurrence formula reduces to : $y_{n+2} - 2y_{n+1} + (1-a)y_n = 0$, whence $y_n = \alpha \cdot (1 + \sqrt{a})^n + \beta \cdot (1 - \sqrt{a})^n$. Finally:

$$\begin{aligned}
\lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \frac{\alpha \cdot (1 + \sqrt{a})^{n+1} + \beta \cdot (1 - \sqrt{a})^{n+1}}{\alpha \cdot (1 + \sqrt{a})^n + \beta \cdot (1 - \sqrt{a})^n} - 1 \\
&= \lim_{n \rightarrow \infty} \frac{\alpha \cdot (1 + \sqrt{a}) + \beta \cdot \left(\frac{1 - \sqrt{a}}{1 + \sqrt{a}}\right)^n \cdot (1 - \sqrt{a})}{\alpha + \beta \cdot \left(\frac{1 - \sqrt{a}}{1 + \sqrt{a}}\right)^n} - 1 \\
&= \frac{\alpha \cdot (1 + \sqrt{a})}{\alpha} - 1 \\
&= \sqrt{a}
\end{aligned}$$

69. Consider two sequences of real numbers $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ such that $x_0 = y_0 = 3$, $x_n = 2x_{n-1} + y_{n-1}$ and $y_n = 2x_{n-1} + 3y_{n-1}$, $(\forall) n \geq 1$. Evaluate $\lim_{n \rightarrow \infty} \frac{x_n}{y_n}$.

First solution: Summing the hypothesis equalities, we have

$$x_n + y_n = 4(x_{n-1} + y_{n-1}), \quad n \geq 1$$

Then $x_n + y_n = 4(x_0 + y_0) = 6 \cdot 4^n$. Subtracting the hypothesis equalities, we get

$$y_n - x_n = 2y_{n-1}, \quad n \geq 1$$

Summing with the previous equality we have $2y_n = 2y_{n-1} + 6 \cdot 4^n \Rightarrow y_n - y_{n-1} = 3 \cdot 4^n$. Then

$$y_1 - y_0 = 3 \cdot 4$$

$$y_2 - y_1 = 3 \cdot 4^2$$

$$y_3 - y_2 = 3 \cdot 4^3$$

...

$$y_n - y_{n-1} = 3 \cdot 4^n$$

Summing, it follows that:

$$y_n = y_0 + 3(4 + 4^2 + \dots + 4^n) = 3(1 + 4 + 4^2 + \dots + 4^n) = 3 \cdot \frac{4^{n+1} - 1}{4 - 1} = 4^{n+1} - 1$$

Then $x_n = 2 \cdot 4^n + 1$, and therefore:

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{2 \cdot 4^n + 1}{4 \cdot 4^n - 1} = \frac{1}{2}$$

Second solution: Define $a_n = \frac{x_n}{y_n}$ so that $a_n = \frac{2a_{n-1} + 1}{2a_{n-1} + 3}$. Now let $a_n = \frac{b_{n+1}}{b_n} - \frac{3}{2}$ to obtain $2b_{n+1} - 5b_n + b_{n-1} = 0$. Then $b_n = \alpha \cdot 2^n + \beta \cdot 2^{-n}$ for some $\alpha, \beta \in \mathbb{R}$. We finally come to:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{\alpha \cdot 2^{n+1} + \beta \cdot 2^{-n-1}}{\alpha \cdot 2^n + \beta \cdot 2^{-n}} - \frac{3}{2} \right) = 2 - \frac{3}{2} = \frac{1}{2}$$

70. Evaluate:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^2}$$

Solution: If $x \in \left(0, \frac{\pi}{2}\right)$, we have:

$$0 < \frac{\tan x - x}{x^2} < \frac{\tan x - \sin x}{x^2} = \frac{\tan x(1 - \cos x)}{x^2} = \frac{2 \tan x \cdot \sin^2 \frac{x}{2}}{x^2}$$

and because $\lim_{x \rightarrow 0} \frac{2 \tan x \cdot \sin^2 \frac{x}{2}}{x^2} = \lim_{x \rightarrow 0} \frac{\tan x}{2} \cdot \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = 0$, using the Squeeze Theorem it follows that:

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{\tan x - x}{x^2} = 0$$

Also

$$\lim_{\substack{x \rightarrow 0 \\ x < 0}} \frac{\tan x - x}{x^2} = \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{-\tan y + y}{y^2} = - \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{\tan y - y}{y^2} = 0$$

71. Evaluate:

$$\lim_{x \rightarrow 0} \frac{\tan x - \arctan x}{x^2}$$

Solution: Using the result from the previous problem, we have:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x - \arctan x}{x^2} &= \lim_{x \rightarrow 0} \frac{\tan x - x}{x^2} + \lim_{x \rightarrow 0} \frac{x - \arctan x}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{x - \arctan x}{x^2} \\ &= \lim_{y \rightarrow 0} \frac{\tan y - y}{\tan^2 y} \\ &= \lim_{y \rightarrow 0} \frac{\tan y - y}{y^2} \cdot \lim_{y \rightarrow 0} \frac{y^2}{\tan^2 y} \\ &= 0 \end{aligned}$$

72. Let $a > 0$ and a sequence of real numbers $(x_n)_{n \geq 0}$ such that $x_n \in (0, a)$ and $x_{n+1}(a - x_n) > \frac{a^2}{4}$, $(\forall) n \in \mathbb{N}$. Prove that $(x_n)_{n \geq 1}$ is convergent and evaluate $\lim_{n \rightarrow \infty} x_n$.

Solution: Rewrite the condition as $\frac{x_{n+1}}{a} \left(1 - \frac{x_n}{a}\right) > \frac{1}{4}$. With the substitution $y_n = \frac{x_n}{a}$, we have $y_{n+1}(1 - y_n) > \frac{1}{4}$, with $y_n \in (0, 1)$. Then:

$$4y_{n+1} - 4y_n y_{n+1} - 1 > 0 \Leftrightarrow 4y_n y_{n+1} - 4y_{n+1}^2 + 4y_{n+1}^2 - 4y_{n+1} + 1 < 0 \Leftrightarrow 4y_n(y_{n+1} - y_n) > (2y_{n+1} - 1)^2$$

So $y_{n+1} - y_n > 0$, whence the sequence is strictly increasing. Let $\lim_{n \rightarrow \infty} y_n = l$.

Then $l(1 - l) \geq \frac{1}{4} \Leftrightarrow \left(l - \frac{1}{2}\right)^2 \leq 0$. Hence $l = \frac{1}{2} \Rightarrow \lim_{n \rightarrow \infty} x_n = \frac{a}{2}$.

73. Evaluate:

$$\lim_{n \rightarrow \infty} \cos(n\pi \sqrt[n]{e})$$

Solution: Using $\lim_{f(x) \rightarrow 0} \frac{a^{f(x)} - 1}{f(x)} = \ln a$, with $a \in \mathbb{R}$, we have:

$$\begin{aligned} \left| \lim_{n \rightarrow \infty} \cos(n\pi \sqrt[n]{e}) \right| &= \lim_{n \rightarrow \infty} |(-1)^n \cdot \cos(n\pi \sqrt[n]{e} - n\pi)| \\ &= \lim_{n \rightarrow \infty} \left| \cos\left(\frac{\pi}{2} \cdot \frac{e^{\frac{1}{2n}} - 1}{\frac{1}{2n}}\right) \right| \\ &= \left| \cos\left(\frac{\pi}{2} \cdot \lim_{n \rightarrow \infty} \frac{e^{\frac{1}{2n}} - 1}{\frac{1}{2n}}\right) \right| \\ &= \left| \cos\left(\frac{\pi}{2}\right) \right| \\ &= 0 \end{aligned}$$

It follows that $\lim_{n \rightarrow \infty} \cos(n\pi \sqrt[n]{e}) = 0$.

74. Evaluate:

$$\lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^{\tan \frac{(n-1)\pi}{2n}}$$

Solution:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{\tan \frac{(n-1)\pi}{2n}} &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right)^n \right]^{\frac{1}{n} \tan \frac{(n-1)\pi}{2n}} \\
&= \lim_{n \rightarrow \infty} \frac{\tan \frac{(n-1)\pi}{2n}}{n} \\
&= \lim_{n \rightarrow \infty} \frac{\tan \left(\frac{\pi}{2} - \frac{\pi}{2n} \right)}{n} \\
&= \lim_{n \rightarrow \infty} \frac{\cot \frac{\pi}{2n}}{n} \\
&= \lim_{n \rightarrow \infty} \frac{1}{n \tan \frac{\pi}{2n}} \\
&= \lim_{n \rightarrow \infty} \frac{2}{\pi} \cdot \frac{\frac{\pi}{2n}}{\tan \frac{\pi}{2n}} \\
&= \frac{2}{\pi}
\end{aligned}$$

75. Evaluate:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\prod_{k=1}^n \binom{n}{k}}$$

Solution: Using AM-GM, we have:

$$\sqrt[n]{n!} = \sqrt[n]{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} < \frac{1 + 2 + \dots + n}{n} = \frac{n+1}{2}$$

Therefore $\frac{(n+1)^n}{n!} > 2^n \Rightarrow \lim_{n \rightarrow \infty} \frac{(n+1)^n}{n!} = \infty$. So:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\prod_{k=1}^n \binom{n}{k}} = \lim_{n \rightarrow \infty} \frac{\prod_{k=1}^{n+1} \binom{n+1}{k}}{\prod_{k=1}^n \binom{n}{k}} = \lim_{n \rightarrow \infty} \frac{(n+1)^n}{n!} = \infty$$

76. If $a > 0$, evaluate:

$$\lim_{n \rightarrow \infty} \frac{a + \sqrt{a} + \sqrt[3]{a} + \dots + \sqrt[n]{a} - n}{\ln n}$$

Solution:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{a + \sqrt{a} + \sqrt[3]{a} + \dots + \sqrt[n]{a} - n}{\ln n} &= \lim_{n \rightarrow \infty} \frac{{}^{n+1}\sqrt{a} - 1}{\ln(n+1) - \ln n} \\
&= \lim_{n \rightarrow \infty} \frac{n \cdot ({}^{n+1}\sqrt{a} - 1)}{\ln \left(1 + \frac{1}{n}\right)^n} \\
&= \lim_{n \rightarrow \infty} \left[\frac{a^{\frac{1}{n+1}} - 1}{\frac{1}{n+1}} \cdot \frac{n}{n+1} \right] \\
&= \ln a
\end{aligned}$$

77. Evaluate:

$$\lim_{n \rightarrow \infty} n \ln \tan \left(\frac{\pi}{4} + \frac{\pi}{n} \right)$$

Solution:

$$\begin{aligned}
\lim_{n \rightarrow \infty} n \ln \tan \left(\frac{\pi}{4} + \frac{\pi}{n} \right) &= \lim_{n \rightarrow \infty} \ln \tan \left(\frac{\pi}{4} + \frac{\pi}{n} \right)^n \\
&= \ln \lim_{n \rightarrow \infty} \left[\left(1 + \tan \left(\frac{\pi}{4} + \frac{\pi}{n} \right) - 1 \right)^{\frac{1}{\tan \left(\frac{\pi}{4} + \frac{\pi}{n} \right) - 1}} \right]^n \left(\tan \left(\frac{\pi}{4} + \frac{\pi}{n} \right) - 1 \right) \\
&= \ln e^{\lim_{n \rightarrow \infty} \left(\tan \left(\frac{\pi}{4} + \frac{\pi}{n} \right) - 1 \right)} \\
&= \lim_{n \rightarrow \infty} n \left(\tan \left(\frac{\pi}{4} + \frac{\pi}{n} \right) - 1 \right) \\
&= \lim_{n \rightarrow \infty} n \left(\frac{1 + \tan \frac{\pi}{n}}{1 - \tan \frac{\pi}{n}} - 1 \right) \\
&= \lim_{n \rightarrow \infty} \frac{2n \tan \frac{\pi}{n}}{1 - \tan \frac{\pi}{n}} \\
&= 2 \lim_{n \rightarrow \infty} n \tan \frac{\pi}{n} \\
&= 2\pi \lim_{n \rightarrow \infty} \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}} \\
&= 2\pi
\end{aligned}$$

78. Let $k \in \mathbb{N}$ and $a_0, a_1, a_2, \dots, a_k \in \mathbb{R}$ such that $a_0 + a_1 + a_2 + \dots + a_k = 0$. Evaluate:

$$\lim_{n \rightarrow \infty} \left(a_0 \sqrt[3]{n} + a_1 \sqrt[3]{n+1} + \dots + a_k \sqrt[3]{n+k} \right)$$

Solution:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left(a_0 \sqrt[3]{n} + a_1 \sqrt[3]{n+1} + \dots + a_k \sqrt[3]{n+k} \right) &= \lim_{n \rightarrow \infty} \left(a_0 \sqrt[3]{n} + \sum_{i=1}^k a_i \sqrt[3]{n+i} \right) \\
 &= \lim_{n \rightarrow \infty} \left(-\sqrt[3]{n} \cdot \sum_{i=1}^k a_i + \sum_{i=1}^k a_i \sqrt[3]{n+i} \right) \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^k a_i \left(\sqrt[3]{n+i} - \sqrt[3]{n} \right) \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^k \frac{ia_i}{\sqrt[3]{(n+i)^2} + \sqrt[3]{n(n+i)} + \sqrt[3]{n^2}} \\
 &= 0
 \end{aligned}$$

79. Evaluate:

$$\lim_{n \rightarrow \infty} \sin \left(n\pi \sqrt[3]{n^3 + 3n^2 + 4n - 5} \right)$$

Solution:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sin \left(n\pi \sqrt[3]{n^3 + 3n^2 + 4n - 5} \right) &= \lim_{n \rightarrow \infty} \sin \left(n\pi \sqrt[3]{n^3 + 3n^2 + 4n - 5} - n(n+1)\pi \right) \\
 &= \lim_{n \rightarrow \infty} \sin \left(n\pi \left(\sqrt[3]{n^3 + 3n^2 + 4n - 5} - n - 1 \right) \right) \\
 &= \lim_{n \rightarrow \infty} \sin \left(\frac{n(n-6)\pi}{\sqrt[3]{(n^3-5)^2} + (n+1)\sqrt[3]{n^3-5} + (n+1)^2} \right) \\
 &= \sin \left(\pi \lim_{n \rightarrow \infty} \frac{1 - \frac{6}{n}}{\sqrt[3]{\left(1 - \frac{5}{n^3}\right)^2} + \left(1 + \frac{1}{n}\right)\sqrt[3]{1 - \frac{5}{n^3}} + \left(1 + \frac{1}{n}\right)^2} \right) \\
 &= \sin \frac{\pi}{3} \\
 &= \frac{\sqrt{3}}{2}
 \end{aligned}$$

80. Evaluate:

$$\lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{2 \arcsin x - \pi}{\sin \pi x}$$

Solution:

$$\begin{aligned}
\lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{2 \arcsin x - \pi}{\sin \pi x} &= 2 \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{\arcsin x - \frac{\pi}{2}}{\sin \left(\arcsin x - \frac{\pi}{2} \right)} \cdot \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{\sin \left(\arcsin x - \frac{\pi}{2} \right)}{\sin \pi x} \\
&= 2 \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{-\sqrt{1-y^2}}{\sin \pi x} \\
&= -2 \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{\sqrt{y(2-y)}}{\sin \pi(1-y)} \\
&= -2 \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{\sqrt{2(1-y)}}{\sin \pi y} \\
&= -2 \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{\pi y}{\sin \pi y} \cdot \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{\sqrt{2-y}}{\pi \sqrt{y}} \\
&= -\infty
\end{aligned}$$

81. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=2}^n \frac{1}{k \ln k}$$

Solution: Using Lagrange formula we can deduce that

$$\frac{1}{k \ln k} > \ln(\ln(k+1)) - \ln(\ln k)$$

Summing from $k = 2$ to n it follows that

$$\sum_{k=2}^n \frac{1}{k \ln k} > \ln(\ln(n+1)) - \ln(\ln 2)$$

Then it is obvious that:

$$\lim_{n \rightarrow \infty} \sum_{k=2}^n \frac{1}{k \ln k} = \infty$$

82. Evaluate:

$$\lim_{n \rightarrow \infty} \left[\lim_{x \rightarrow 0} \left(1 + \sum_{k=1}^n \sin^2(kx) \right)^{\frac{1}{n^3 x^2}} \right]$$

Solution:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left[\lim_{x \rightarrow 0} \left(1 + \sum_{k=1}^n \sin^2(kx) \right)^{\frac{1}{n^3 x^2}} \right] &= \lim_{n \rightarrow \infty} \left[\lim_{x \rightarrow 0} \left(1 + \sum_{k=1}^n \sin^2(kx) \right)^{\frac{1}{\sum_{k=1}^n \sin^2(kx)}} \right]^{\frac{\sum_{k=1}^n \sin^2(kx)}{n^3 x^2}} \\
&= \lim_{n \rightarrow \infty} \left[\frac{1}{e n^3} \lim_{x \rightarrow 0} \frac{\sum_{k=1}^n \sin^2(kx)}{x^2} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3} \\
&= \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \sqrt[3]{e}
\end{aligned}$$

83. If $p \in \mathbb{N}^*$, evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(k+1)(k+2) \cdots (k+p)}{n^{p+1}}$$

Solution: Using Cesaro-Stolz, we have:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(k+1)(k+2) \cdots (k+p)}{n^{p+1}} &= \sum_{k=0}^n \frac{(k+p)!}{k! n^{p+1}} \\
&= \lim_{n \rightarrow \infty} \frac{\frac{(n+p+1)!}{(n+1)!}}{(n+1)^{p+1} - n^{p+1}} \\
&= \lim_{n \rightarrow \infty} \frac{(n+2)(n+3) \cdots (n+p+1)}{n^{p+1} + \binom{p+1}{1} n^p + \dots + 1 - n^{p+1}} \\
&= \lim_{n \rightarrow \infty} \frac{n^p + \dots}{(p+1)n^p + \dots} \\
&= \frac{1}{p+1}
\end{aligned}$$

84. If $\alpha_n \in \left(0, \frac{\pi}{4}\right)$ is a root of the equation $\tan \alpha + \cot \alpha = n$, $n \geq 2$, evaluate:

$$\lim_{n \rightarrow \infty} (\sin \alpha_n + \cos \alpha_n)^n$$

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} (\sin \alpha_n + \cos \alpha_n)^n &= \lim_{n \rightarrow \infty} [(\sin \alpha_n + \cos \alpha_n)^2]^{\frac{n}{2}} \\ &= \lim_{n \rightarrow \infty} (1 + 2 \cos \alpha_n \cdot \sin \alpha_n)^{\frac{n}{2}} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{2}{\frac{\sin^2 \alpha_n + \cos^2 \alpha_n}{\cos \alpha_n \cdot \sin \alpha_n}}\right)^{\frac{n}{2}} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{2}{\tan \alpha_n + \cot \alpha_n}\right)^{\frac{n}{2}} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^{\frac{n}{2}} \\ &= e \end{aligned}$$

85. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\sqrt{\binom{n+k}{2}}}{n^2}$$

First solution: Cesaro-Stolz gives:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\sqrt{\binom{n+k}{2}}}{n^2} &= \lim_{n \rightarrow \infty} \frac{\sqrt{\binom{2n+1}{2}} + \sqrt{\binom{2n+2}{2}} - \sqrt{\binom{n+1}{2}}}{2n+1} \\ &= \frac{1}{\sqrt{2}} \lim_{n \rightarrow \infty} \frac{\sqrt{2n(2n+1)} + \sqrt{(2n+1)(2n+2)} - \sqrt{n(n+1)}}{2n+1} \\ &= \frac{1}{\sqrt{2}} \lim_{n \rightarrow \infty} \frac{\sqrt{4 + \frac{2}{n}} + \sqrt{4 + \frac{6}{n} + \frac{2}{n^2}} - \sqrt{1 + \frac{1}{n}}}{2 + \frac{1}{n}} \\ &= \frac{3}{2\sqrt{2}} \end{aligned}$$

Second solution: Observe that:

$$\binom{n+k}{2} = \frac{(n+k-1)(n+k)}{2} = \frac{n^2}{2} \left(1 + \frac{k}{n}\right) \left(1 + \frac{k-1}{n}\right)$$

for which we have

$$\frac{n^2}{2} \left(1 + \frac{k-1}{n}\right)^2 \leq \frac{n^2}{2} \left(1 + \frac{k}{n}\right) \left(1 + \frac{k-1}{n}\right) \leq \frac{n^2}{2} \left(1 + \frac{k}{n}\right)^2$$

therefore

$$\frac{n}{\sqrt{2}} \left(1 + \frac{k-1}{n}\right) \leq \sqrt{\binom{n+k}{2}} \leq \frac{n}{\sqrt{2}} \left(1 + \frac{k}{n}\right)$$

Summing from $k = 1$ to n , we get:

$$\frac{1}{n\sqrt{2}} \sum_{k=1}^n \left(1 + \frac{k-1}{n}\right) \leq \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\sqrt{\binom{n+k}{2}}}{n^2} \leq \frac{1}{n\sqrt{2}} \sum_{k=1}^n \left(1 + \frac{k}{n}\right)$$

We can apply the Squeeze theorem because

$$\lim_{n \rightarrow \infty} \frac{1}{n\sqrt{2}} \sum_{k=1}^n \left(1 + \frac{k-1}{n}\right) = \lim_{n \rightarrow \infty} \frac{1}{n\sqrt{2}} \left(n + \frac{n-1}{2}\right) = \lim_{n \rightarrow \infty} \frac{3n-1}{2n\sqrt{2}} = \frac{3}{2\sqrt{2}}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n\sqrt{2}} \sum_{k=1}^n \left(1 + \frac{k}{n}\right) = \lim_{n \rightarrow \infty} \frac{1}{n\sqrt{2}} \left(n + \frac{n+1}{2}\right) = \lim_{n \rightarrow \infty} \frac{3n+1}{2n\sqrt{2}} = \frac{3}{2\sqrt{2}}$$

Thus

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\sqrt{\binom{n+k}{2}}}{n^2} = \frac{3}{2\sqrt{2}}$$

86. Evaluate:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\prod_{k=1}^n \left(1 + \frac{k}{n}\right)}$$

Solution: Using Cesaro-Stolz we'll evaluate:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \ln \sqrt[n]{\prod_{k=1}^n \left(1 + \frac{k}{n}\right)} &= \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \ln \left(1 + \frac{k}{n}\right)}{n} \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^{n+1} \ln \left(1 + \frac{k}{n+1}\right) - \sum_{k=1}^n \ln \left(1 + \frac{k}{n}\right) \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n \ln \frac{1 + \frac{k}{n+1}}{1 + \frac{k}{n}} + \ln 2 \\
&= \lim_{n \rightarrow \infty} \ln \left(\frac{4n+2}{n+1} \cdot \left(\frac{n}{n+1}\right)^n \right) \\
&= \ln 4 - 1
\end{aligned}$$

It follows that:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\prod_{k=1}^n \left(1 + \frac{k}{n}\right)} = 4e^{-1}$$

87. Evaluate:

$$\lim_{x \rightarrow 0} \frac{\arctan x - \arcsin x}{x^3}$$

Solution:

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\arctan x - \arcsin x}{x^3} &= \lim_{x \rightarrow 0} \frac{\arctan x - \arcsin x}{\tan(\arctan x - \arcsin x)} \cdot \lim_{x \rightarrow 0} \frac{\tan(\arctan x - \arcsin x)}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{\tan(\arctan x - \arcsin x)}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{1}{x^3} \cdot \frac{x - \frac{x}{\sqrt{1-x^2}}}{1 + \frac{x}{\sqrt{1-x^2}}} \\
&= \lim_{x \rightarrow 0} \frac{1}{x^2} \cdot \frac{\sqrt{1-x^2} - 1}{\sqrt{1-x^2} + x^2} \\
&= \lim_{x \rightarrow 0} \frac{-1}{(\sqrt{1-x^2} + x^2)(\sqrt{1-x^2} + 1)} \\
&= -\frac{1}{2}
\end{aligned}$$

88. If $\alpha > 0$, evaluate:

$$\lim_{n \rightarrow \infty} \frac{(n+1)^\alpha - n^\alpha}{n^{\alpha-1}}$$

Solution: Let

$$x_n = \frac{(n+1)^\alpha - n^\alpha}{n^{\alpha-1}} = \frac{n^\alpha \left[\left(1 + \frac{1}{n}\right)^\alpha - 1 \right]}{n^{\alpha-1}} = n \left[\left(1 + \frac{1}{n}\right)^\alpha - 1 \right]$$

Then $\lim_{n \rightarrow \infty} \frac{x_n}{n} = 0$. Observe that:

$$1 + \frac{x_n}{n} = \left(1 + \frac{1}{n}\right)^\alpha \Leftrightarrow \left[\left(1 + \frac{x_n}{n}\right) \frac{n}{x_n} \right]^{x_n} = \left[\left(1 + \frac{1}{n}\right)^n \right]^\alpha$$

By passing to limit, we have $\lim_{n \rightarrow \infty} e^{x_n} = e^\alpha$. Hence $\lim_{n \rightarrow \infty} x_n = \alpha$.

89. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^2}{2^k}$$

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^2}{2^k} &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{k(k+1)}{2^k} - \frac{k}{2^k} \right) \\ &= \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \left(\frac{k^2}{2^{k-1}} - \frac{(k+1)^2}{2^k} + \frac{3k+1}{2^k} \right) - \sum_{k=1}^n \frac{k}{2^k} \right] \\ &= \lim_{n \rightarrow \infty} \left[\left(1 - \frac{(n+1)^2}{2^n} \right) + \sum_{k=1}^n \frac{2k+1}{2^k} \right] \\ &= \lim_{n \rightarrow \infty} \left[\left(1 - \frac{(n+1)^2}{2^n} \right) + 2 \sum_{k=1}^n \frac{k}{2^k} + \sum_{k=1}^n \frac{1}{2^k} \right] \\ &= \lim_{n \rightarrow \infty} \left[\left(1 - \frac{(n+1)^2}{2^n} \right) + 2 \sum_{k=1}^n \left(\frac{k}{2^{k-1}} - \frac{k+1}{2^k} + \frac{1}{2^k} \right) + \sum_{k=1}^n \frac{1}{2^k} \right] \\ &= \lim_{n \rightarrow \infty} \left[\left(1 - \frac{(n+1)^2}{2^n} \right) + 2 \left(1 - \frac{n+1}{2^n} \right) + 3 \sum_{k=1}^n \frac{1}{2^k} \right] \\ &= \lim_{n \rightarrow \infty} \left[3 - \frac{n^2 + 4n + 3}{2^n} + 3 \left(1 - \frac{1}{2^n} \right) \right] \\ &= \lim_{n \rightarrow \infty} \left(6 - \frac{n^2 + 4n + 6}{2^n} \right) \\ &= 6 - \lim_{n \rightarrow \infty} \frac{n^2 + 4n + 6}{2^n} \end{aligned}$$

Because:

$$\lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2 + 4(n+1) + 6}{2^{n+1}}}{\frac{n^2 + 4n + 6}{2^n}} = \lim_{n \rightarrow \infty} \frac{n^2 + 6n + 11}{2n^2 + 8n + 12} = \frac{1}{2}$$

it follows that $\lim_{n \rightarrow \infty} \frac{n^2 + 4n + 6}{2^n} = 0$, therefore our limit is 6.

90. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(k+1)(k+2)}{2^k}$$

Solution: Using the previous limit, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(k+1)(k+2)}{2^k} &= \lim_{n \rightarrow \infty} \left(\sum_{k=0}^n \frac{k^2}{2^k} + 3 \cdot \sum_{k=0}^n \frac{k}{2^k} + \sum_{k=0}^n \frac{1}{2^{k-1}} \right) \\ &= 6 + 3 \lim_{n \rightarrow \infty} \left(2 - \frac{n+2}{2^n} \right) + \lim_{n \rightarrow \infty} \left(2 + \frac{1 - \frac{1}{2^n}}{\frac{1}{2}} \right) \\ &= 16 \end{aligned}$$

91. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ such that $x_1 \in (0, 1)$ and $x_{n+1} = x_n^2 - x_n + 1$, $(\forall) n \in \mathbb{N}$. Evaluate:

$$\lim_{n \rightarrow \infty} (x_1 x_2 \cdots x_n)$$

Solution: Subtracting x_n from both sides of the recurrence formula gives $x_{n+1} - x_n = x_n^2 - 2x_n + 1 = (x_n - 1)^2 \geq 0$ so $(x_n)_{n \geq 1}$ is an increasing sequence.

$x_1 \in (0, 1)$ is given as hypothesis. Now if there exists $k \in \mathbb{N}$ such that $x_k \in (0, 1)$, then $(x_k - 1) \in (-1, 0)$, so $x_k(x_k - 1) \in (-1, 0)$. Then $x_{k+1} = 1 + x_k(x_k - 1) \in (0, 1)$ as well, so by induction we see that the sequence is contained in $(0, 1)$.

$(x_n)_{n \geq 1}$ is increasing and bounded from above, so it converges. If $\lim_{n \rightarrow \infty} x_n = l$ then from the recurrence, $l = l^2 - l + 1$ which gives $l = 1$. Thus, $\lim_{n \rightarrow \infty} x_n = 1$.

Now rewrite the recurrence formula as $1 - x_{n+1} = x_n(1 - x_n)$. For $n = 1, 2, \dots, n$, we have:

$$1 - x_2 = x_1(1 - x_1)$$

$$1 - x_3 = x_2(1 - x_2)$$

...

$$1 - x_n = x_{n-1}(1 - x_{n-1})$$

$$1 - x_{n+1} = x_n(1 - x_n)$$

Multiplying them we have:

$$1 - x_{n+1} = x_1 x_2 \cdot \dots \cdot x_n (1 - x_1)$$

Thus:

$$\lim_{n \rightarrow \infty} (x_1 x_2 \cdot \dots \cdot x_n) = \lim_{n \rightarrow \infty} \frac{1 - x_{n+1}}{1 - x_1} = 0$$

92. If $n \in \mathbb{N}^*$, evaluate:

$$\lim_{x \rightarrow 0} \frac{1 - \cos x \cdot \cos 2x \cdot \dots \cdot \cos nx}{x^2}$$

Solution: Let

$$a_n = \lim_{x \rightarrow 0} \frac{1 - \cos x \cdot \cos 2x \cdot \dots \cdot \cos nx}{x^2}$$

Then

$$\begin{aligned} a_{n+1} &= \lim_{x \rightarrow 0} \frac{1 - \cos x \cdot \cos 2x \cdot \dots \cdot \cos nx \cdot \cos(n+1)x}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos x \cdot \cos 2x \cdot \dots \cdot \cos nx}{x^2} + \lim_{x \rightarrow 0} \frac{\cos x \cdot \cos 2x \cdot \dots \cdot \cos nx (1 - \cos(n+1)x)}{x^2} \\ &= a_n + \lim_{x \rightarrow 0} \frac{1 - \cos(n+1)x}{x^2} \\ &= a_n + \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{(n+1)x}{2}}{x^2} \\ &= a_n + \frac{(n+1)^2}{2} \lim_{x \rightarrow 0} \left(\frac{\sin \frac{(n+1)x}{2}}{\frac{n+1}{2}} \right)^2 \\ &= a_n + \frac{(n+1)^2}{2} \end{aligned}$$

Now let $n = 1, 2, 3, \dots, n-1$:

$$a_0 = 0$$

$$a_1 = a_0 + \frac{1}{2}$$

$$a_2 = a_1 + \frac{2^2}{2}$$

$$a_3 = a_2 + \frac{3^2}{2}$$

...

$$a_n = a_{n-1} + \frac{n^2}{2}$$

Summing gives:

$$a_n = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{n^2}{2} = \frac{1}{2}(1^2 + 2^2 + \dots + n^2) = \frac{1}{2} \cdot \frac{n(n+1)(2n+1)}{6}$$

Finally, the answer is

$$\lim_{x \rightarrow 0} \frac{1 - \cos x \cdot \cos 2x \cdot \dots \cdot \cos nx}{x^2} = \frac{n(n+1)(2n+1)}{12}$$

93. Consider a sequence of real numbers $(x_n)_{n \geq 1}$ such that x_n is the real root of the equation $x^3 + nx - n = 0$, $n \in \mathbb{N}^*$. Prove that this sequence is convergent and find its limit.

Solution: Let $f(x) = x^3 + nx - n$. Then $f'(x) = 3x^2 + n > 0$, so f has only one real root which is contained in the interval $(0, 1)$ (because $f(0) = -n$ and $f(1) = 1$, so $x_n \in (0, 1)$).

The sequence $(x_n)_{n \geq 1}$ is strictly increasing, because

$$x_{n+1} - x_n = \frac{1 - x_n}{x_{n+1}^2 + x_{n+1}x_n + x_n^2 + n} > 0$$

Therefore the sequence is convergent. From the equation, we have $x_n = 1 - \frac{x_n^3}{n}$.

By passing to limit, we find that $\lim_{n \rightarrow \infty} x_n = 1$.

94. Evaluate:

$$\lim_{x \rightarrow 2} \frac{\arctan x - \arctan 2}{\tan x - \tan 2}$$

Solution: Using $\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \cdot \tan b}$, we have:

$$\begin{aligned}
\lim_{x \rightarrow 2} \frac{\arctan x - \arctan 2}{\tan x - \tan 2} &= \lim_{x \rightarrow 2} \frac{\arctan x - \arctan 2}{\tan(\arctan x - \arctan 2)} \cdot \lim_{x \rightarrow 2} \frac{\tan(\arctan x - \arctan 2)}{\tan x - \tan 2} \\
&= \lim_{x \rightarrow 2} \frac{\frac{x-2}{1+2x}}{\frac{\sin(x-2)}{\cos x \cdot \cos 2}} \\
&= \lim_{x \rightarrow 2} \frac{x-2}{\sin(x-2)} \cdot \lim_{x \rightarrow 2} \frac{\cos x \cdot \cos 2}{1+2x} \\
&= \lim_{x \rightarrow 2} \frac{\cos x \cdot \cos 2}{1+2x} \\
&= \frac{\cos^2 2}{5}
\end{aligned}$$

95. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1 + \sqrt[2]{2!} + \sqrt[3]{3!} + \dots + \sqrt[n]{n!}}{n}$$

Solution: Using Cesaro-Stolz:

$$\lim_{n \rightarrow \infty} \frac{1 + \sqrt[2]{2!} + \sqrt[3]{3!} + \dots + \sqrt[n]{n!}}{n} = \lim_{n \rightarrow \infty} \frac{(n+1)^2 \sqrt{(n+1)!}}{(n+1) - n} = \lim_{n \rightarrow \infty} (n+1)^2 \sqrt{(n+1)!}$$

Also, an application of AM-GM gives:

$$\begin{aligned}
1 &\leq \frac{(n+1)^2 \sqrt{(n+1)!}}{(n+1)^2 \sqrt{(n+1)!}} \\
&= \frac{(n+1)^2 \sqrt{(n+1)!}}{(n+1)^2 \sqrt{1 \cdot 2 \cdot 3 \cdot \dots \cdot n \cdot (n+1)}} \\
&< \frac{(n+1)^2 \sqrt{1+2+3+\dots+n+n+1}}{(n+1)^2 \sqrt{n+1}} \\
&= \frac{(n+1)^2 \sqrt{n+2}}{(n+1)^2 \sqrt{2}}
\end{aligned}$$

Thus

$$1 \leq \lim_{n \rightarrow \infty} \frac{(n+1)^2 \sqrt{(n+1)!}}{(n+1)^2 \sqrt{(n+1)!}} \leq \lim_{n \rightarrow \infty} \frac{(n+1)^2 \sqrt{n+2}}{(n+1)^2 \sqrt{2}} = 1$$

From the Squeeze Theorem it follows that:

$$\lim_{n \rightarrow \infty} \frac{1 + \sqrt[2]{2!} + \sqrt[3]{3!} + \dots + \sqrt[n]{n!}}{n} = 1$$

96. Let $(x_n)_{n \geq 1}$ such that $x_1 > 0$, $x_1 + x_1^2 < 1$ and $x_{n+1} = x_n + \frac{x_n^2}{n^2}$, $(\forall) n \geq 1$.

Prove that the sequences $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 2}$, $y_n = \frac{1}{x_n} - \frac{1}{n-1}$ are convergent.

Solution: $x_{n+1} - x_n = \frac{x_n^2}{n^2}$, so the $(x_n)_{n \geq 1}$ is strictly increasing.

$$x_2 = x_1 + x_1^2 < 1 \Rightarrow \frac{1}{x_2} > 1 \Rightarrow y_2 = \frac{1}{x_2} - 1 > 0$$

Also

$$\begin{aligned} y_{n+1} - y_n &= \frac{1}{x_{n+1}} - \frac{1}{n} - \frac{1}{x_n} + \frac{1}{n-1} \\ &= \frac{1}{n(n-1)} - \frac{x_{n+1} - x_n}{x_n x_{n+1}} \\ &= \frac{1}{n(n-1)} - \frac{x_n}{n^2 x_{n+1}} \\ &> \frac{1}{n(n-1)} - \frac{1}{n^2} \\ &= \frac{1}{n^2(n-1)} \\ &> 0 \end{aligned}$$

Hence $(y_n)_{n \geq 2}$ is strictly increasing. Observe that $x_n = \frac{1}{y_n + \frac{1}{n-1}}$. So

$\lim_{n \rightarrow \infty} x_n = \frac{1}{\lim_{n \rightarrow \infty} y_n}$. Assuming that $\lim_{n \rightarrow \infty} y_n = \infty$, we have $\lim_{n \rightarrow \infty} x_n = 0$, which is a contradiction, because $x_1 > 0$ and the sequence $(x_n)_{n \geq 1}$ is strictly increasing. Hence $(y_n)_{n \geq 2}$ is convergent. It follows that $(x_n)_{n \geq 2}$ is also convergent.

97. Evaluate:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{2i}{n^2}$$

First solution: Let's start from

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \Leftrightarrow (\forall) \varepsilon > 0, (\exists) \delta > 0, (\forall) x \in (-\delta, \delta) \setminus \{0\} \Rightarrow \left| \frac{\sin x}{x} - 1 \right| < \varepsilon$$

Let some arbitrary $\varepsilon > 0$. For such ε , $(\exists) \delta > 0$ such that $(\forall) x \in (-\delta, \delta) \setminus \{0\}$, we have $1 - \varepsilon < \frac{\sin x}{x} < 1 + \varepsilon$. For $\delta > 0$, $(\exists) n_\varepsilon \in \mathbb{N}^*$ such that $\frac{2}{n} < \delta$, $(\forall) n \geq n_\varepsilon$.

Because $0 < \frac{2i}{n^2} \leq \frac{2}{n}$, $(\forall) 1 \leq i \leq n$, $n \geq n_\varepsilon$, we have:

$$1 - \varepsilon < \frac{\sin \frac{2i}{n^2}}{\frac{2i}{n^2}} < 1 + \varepsilon$$

Summing, we get:

$$(1 - \varepsilon) \sum_{i=1}^n \frac{2i}{n^2} < \sum_{i=1}^n \sin \frac{2i}{n^2} < (1 + \varepsilon) \sum_{i=1}^n \frac{2i}{n^2}$$

Or equivalently:

$$\frac{(1 - \varepsilon)(n + 1)}{n} < \sum_{i=1}^n \sin \frac{2i}{n^2} < \frac{(1 + \varepsilon)(n + 1)}{n}$$

By passing to limit:

$$1 - \varepsilon \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{2i}{n^2} \leq 1 + \varepsilon$$

Or

$$\left| \lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{2i}{n^2} - 1 \right| \leq \varepsilon, \quad (\forall) \varepsilon > 0$$

which implies that:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{2i}{n^2} = 1$$

Second solution: Start with the formula

$$\sum_{i=1}^n \sin(x + yi) = \frac{\sin \frac{(n+1)y}{2} \cdot \sin \left(x + \frac{ny}{2} \right)}{\sin \frac{y}{2}}$$

Setting $x = 0$, $y = \frac{2}{n^2}$, it rewrites as

$$\sum_{i=1}^n \sin \frac{2i}{n^2} = \frac{\sin \frac{n+1}{n^2} \sin \frac{1}{n}}{\sin \frac{1}{n^2}}$$

whence

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{2i}{n^2} = \lim_{n \rightarrow \infty} \frac{\frac{\sin \frac{n+1}{n^2}}{\frac{n+1}{n^2}} \cdot \frac{\sin \frac{1}{n}}{\frac{1}{n}}}{\frac{\sin \frac{1}{n^2}}{\frac{1}{n^2}}} \cdot \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$$

98. If $a > 0$, $a \neq 1$, evaluate:

$$\lim_{x \rightarrow a} \frac{x^x - a^x}{a^x - a^a}$$

Solution: As $\lim_{x \rightarrow a} x \ln \frac{x}{a} = 0$, we have:

$$\begin{aligned} \lim_{x \rightarrow a} \frac{x^x - a^x}{a^x - a^a} &= \lim_{x \rightarrow a} \frac{e^{x \ln x} - e^{x \ln a}}{a^x - a^a} \\ &= \lim_{x \rightarrow a} \frac{e^{x \ln a} (e^{x \ln \frac{x}{a}} - 1)}{a^a (a^{x-a} - 1)} \\ &= \left(\lim_{x \rightarrow a} \frac{e^{x \ln a}}{a^a} \right) \cdot \left(\lim_{x \rightarrow a} \frac{e^{x \ln \frac{x}{a}} - 1}{x \ln \frac{x}{a}} \right) \cdot \left(\lim_{x \rightarrow a} \frac{x - a}{a^{x-a} - 1} \right) \cdot \left(\lim_{x \rightarrow a} \frac{x \ln \frac{x}{a}}{x - a} \right) \\ &= \frac{1}{\ln a} \cdot \lim_{x \rightarrow a} \left[x \ln \left(\frac{x}{a} \right)^{\frac{1}{x-a}} \right] \\ &= \frac{a}{\ln a} \cdot \lim_{x \rightarrow a} \left[\left(1 + \frac{x-a}{a} \right)^{\frac{a}{x-a}} \right]^{\frac{1}{a}} \\ &= \frac{a}{\ln a} \cdot \ln e^{\frac{1}{a}} \\ &= \frac{1}{\ln a} \end{aligned}$$

99. Consider a sequence of positive real numbers $(a_n)_{n \geq 1}$ such that $a_{n+1} - \frac{1}{a_{n+1}} = a_n + \frac{1}{a_n}$, $(\forall) n \geq 1$. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right)$$

Solution: $(a_n)_{n \geq 1}$ is clearly an increasing sequence. If it has a finite limit, say l , then

$$l - \frac{1}{l} = l + \frac{1}{l} \Rightarrow \frac{2}{l} = 0$$

contradiction. Therefore a_n approaches infinity. Let $y_n = \frac{1}{a_n^2} + a_n^2$. Then $y_{n+1} = y_n + 4$. So

$$y_2 = y_1 + 4$$

$$y_3 = y_2 + 4$$

...

$$y_{n+1} = y_n + 4$$

Summing, it results that $y_{n+1} = y_1 + 4n$, which rewrites as

$$a_{n+1}^2 + \frac{1}{a_{n+1}^2} = y_1 + 4n \Leftrightarrow \left(a_{n+1} + \frac{1}{a_{n+1}}\right)^2 = y_1 + 2 + 4n \Leftrightarrow$$

$$a_{n+1} + \frac{1}{a_{n+1}} = \sqrt{4n + y_1 + 2} \Rightarrow a_{n+1}^2 - \sqrt{4n + y_1 + 2} \cdot a_{n+1} + 1 = 0$$

from which $a_{n+1} = \frac{\sqrt{4n + y_1 + 2} \pm \sqrt{4n + y_1 - 2}}{2}$. If we accept that $a_{n+1} = \frac{\sqrt{4n + y_1 + 2} - \sqrt{4n + y_1 - 2}}{2}$, then:

$$\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{\sqrt{4n + y_1 + 2} - \sqrt{4n + y_1 - 2}}{2} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{4n + y_1 + 2} + \sqrt{4n + y_1 - 2}} = 0$$

which is false, therefore $a_{n+1} = \frac{\sqrt{4n + y_1 + 2} + \sqrt{4n + y_1 - 2}}{2}$.

By Cesaro-Stolz, we obtain:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) &= \lim_{n \rightarrow \infty} \frac{\frac{1}{a_n}}{\sqrt{n+1} - \sqrt{n}} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{n} + \sqrt{n+1}}{a_{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n} + \sqrt{n+1})}{\sqrt{4n + y_1 + 2} + \sqrt{4n + y_1 - 2}} \\ &= \lim_{n \rightarrow \infty} \frac{2(1 + \sqrt{1 + \frac{1}{n}})}{\sqrt{4 + \frac{y_1}{n} + \frac{2}{n}} + \sqrt{4 + \frac{y_1}{n} - \frac{2}{n}}} \\ &= 1 \end{aligned}$$

100. Evaluate:

$$\lim_{x \rightarrow 0} \frac{2^{\arctan x} - 2^{\arcsin x}}{2^{\tan x} - 2^{\sin x}}$$

Solution:

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{2^{\arctan x} - 2^{\arcsin x}}{2^{\tan x} - 2^{\sin x}} &= \lim_{x \rightarrow 0} \frac{2^{\arcsin x} (2^{\arctan x - \arcsin x} - 1)}{2^{\sin x} (2^{\tan x - \sin x} - 1)} \\
&= \lim_{x \rightarrow 0} \frac{2^{\arctan x - \arcsin x} - 1}{2^{\tan x - \sin x} - 1} \\
&= \lim_{x \rightarrow 0} \frac{2^{\arctan x - \arcsin x} - 1}{\arctan x - \arcsin x} \cdot \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{2^{\tan x - \sin x} - 1} \cdot \lim_{x \rightarrow 0} \frac{\arctan x - \arcsin x}{\tan x - \sin x} \\
&= \ln 2 \cdot \frac{1}{\ln 2} \cdot \lim_{x \rightarrow 0} \frac{\arctan x - \arcsin x}{\tan x - \sin x} \\
&= \lim_{x \rightarrow 0} \frac{\arctan x - \arcsin x}{x^3} \cdot \lim_{x \rightarrow 0} \frac{x^3}{\tan x - \sin x} \\
&= \lim_{x \rightarrow 0} \frac{\arctan x - \arcsin x}{\tan(\arctan x - \arcsin x)} \cdot \lim_{x \rightarrow 0} \frac{\tan(\arctan x - \arcsin x)}{x^3} \cdot \lim_{x \rightarrow 0} \frac{x^3}{\tan x (1 - \cos x)} \\
&= \lim_{x \rightarrow 0} \frac{\frac{x - \frac{x}{\sqrt{1-x^2}}}{1 + \frac{x^2}{\sqrt{1-x^2}}}}{x^3} \cdot \lim_{x \rightarrow 0} \frac{x^3}{2 \tan x \cdot \sin^2 \frac{x}{2}} \\
&= \lim_{x \rightarrow 0} \frac{\sqrt{1-x^2} - 1}{x^2(\sqrt{1-x^2} + x^2)} \cdot \lim_{x \rightarrow 0} \frac{x}{\tan x} \cdot 2 \lim_{x \rightarrow 0} \left(\frac{\frac{x}{2}}{\sin \frac{x}{2}} \right)^2 \\
&= 2 \lim_{x \rightarrow 0} \frac{-x^2}{x^2(\sqrt{1-x^2} + x^2)(\sqrt{1-x^2} + 1)} \\
&= -1
\end{aligned}$$