TOPICS IN INEQUALITIES

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Introduction

Inequalities are useful in all fields of Mathematics. The purpose in this book is to present *standard techniques* in the theory of inequalities. The readers will meet classical theorems including Schur's inequality, Muirhead's theorem, the Cauchy-Schwartz inequality, AM-GM inequality, and Hölder's theorem, etc. There are many problems from Mathematical olympiads and competitions. The book is available at

http://my.netian.com/∼ideahitme/eng.html

I wish to express my appreciation to Stanley Rabinowitz who kindly sent me his paper *On The Computer Solution of Symmetric Homogeneous Triangle Inequalities*.

This is an **unfinished** manuscript. I would greatly appreciate hearing about any errors in the book, even minor ones. You can send all comments to the author at **hojoolee@korea.com**.

To Students

The given techniques in this book are just the tip of the inequalities iceberg. What young students read this book should be aware of is that they should find *their own creative methods* to attack problems. It's impossible to present *all* techniques in a small book. I don't even claim that the methods in this book are mathematically beautiful. For instance, although Muirhead's theorem and Schur's theorem which can be found at chapter 3 are extremely powerful to attack homogeneous symmetric polynomial inequalities, it's not a good idea for beginners to learn how to apply them to problems. (Why?) However, after mastering homogenization method using Muirhead's theorem and Schur's theorem, you can have a more broad mind in the theory of inequalities. That's why I include the methods in this book. **Have fun!**

Recommended Reading List

1. K. S. Kedlaya, $A < B$, http://www.unl.edu/amc/a-activities/a4-for-students/s-index.html
2. I. Niven, *Maxima and Minima Without Calculus*, MAA
3. T. Andreescu, Z. Feng, *103 Trigonometry Problems From the Training of the USA IMO Team*, Birkhauser
4. O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, P. M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen 1969

Contents

Chapter 1

100 **Problems**

Each problem that I solved became a rule, which served afterwards to solve other problems. Rene Descartes

I 1. (**Hungary 1996**) $(a+b=1,\ a,b>0)$

$$\frac{a^2}{a+1}+\frac{b^2}{b+1}\geq\frac{1}{3}$$

I 2. (**Columbia 2001**) $(x,y\in\mathbf{R})$

$$3(x+y+1)^2+1\geq 3xy$$

I 3. $(0<x,y<1)$

$$x^y+y^x>1$$

I 4. (**APMC 1993**) $(a,b\geq 0)$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2\leq\frac{a+\sqrt[3]{a^2b}+\sqrt[3]{ab^2}+b}{4}\leq\frac{a+\sqrt{ab}+b}{3}\leq\sqrt{\left(\frac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{2}\right)^3}$$

I 5. (**Czech and Slovakia 2000**) $(a,b>0)$

$$\sqrt[3]{2(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}\geq\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}}$$

I 6. (**Die $\sqrt{WURZEL}$, Heinz-Jürgen Seiffert**) $(xy>0, x,y\in\mathbf{R})$

$$\frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}}\geq\sqrt{xy}+\frac{x+y}{2}$$

I 7. (**Crux Mathematicorum, Problem 2645, Hojoo Lee**) $(a,b,c>0)$

$$\frac{2(a^3+b^3+c^3)}{abc}+\frac{9(a+b+c)^2}{(a^2+b^2+c^2)}\geq 33$$

I 8. $(x,y,z>0)$

$$\sqrt[3]{xyz}+\frac{|x-y|+|y-z|+|z-x|}{3}\geq\frac{x+y+z}{3}$$

I 9. $(a,b,c,x,y,z>0)$

$$\sqrt[3]{(a+x)(b+y)(c+z)}\geq\sqrt[3]{abc}+\sqrt[3]{xyz}$$

I 10. $(x, y, z > 0)$

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}} \leq 1$$

I 11. $(x+y+z=1,\ x, y, z > 0)$

$$\frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}+\frac{z}{\sqrt{1-z}} \geq \sqrt{\frac{3}{2}}$$

I 12. (**Iran 1998**) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2,\ x, y, z > 1\right)$

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$$

I 13. (**KMO Winter Program Test 2001**) $(a, b, c > 0)$

$$\sqrt{\left(a^2 b+b^2 c+c^2 a\right)\left(a b^2+b c^2+c a^2\right)} \geq a b c+\sqrt[3]{\left(a^3+a b c\right)\left(b^3+a b c\right)\left(c^3+a b c\right)}$$

I 14. (**KMO Summer Program Test 2001**) $(a, b, c > 0)$

$$\sqrt{a^4+b^4+c^4}+\sqrt{a^2 b^2+b^2 c^2+c^2 a^2} \geq \sqrt{a^3 b+b^3 c+c^3 a}+\sqrt{a b^3+b c^3+c a^3}$$

I 15. (**Gazeta Matematicã, Hojoo Lee**) $(a, b, c > 0)$

$$\sqrt{a^4+a^2 b^2+b^4}+\sqrt{b^4+b^2 c^2+c^4}+\sqrt{c^4+c^2 a^2+a^4} \geq a \sqrt{2 a^2+b c}+b \sqrt{2 b^2+c a}+c \sqrt{2 c^2+a b}$$

I 16. $(a, b, c \in \mathbf{R})$

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2} \geq \frac{3 \sqrt{2}}{2}$$

I 17. $(a, b, c > 0)$

$$\sqrt{a^2-a b+b^2}+\sqrt{b^2-b c+c^2} \geq \sqrt{a^2+a c+c^2}$$

I 18. (**Belarus 2002**) $(a, b, c, d > 0)$

$$\sqrt{(a+c)^2+(b+d)^2}+\frac{2|a d-b c|}{\sqrt{(a+c)^2+(b+d)^2}} \geq \sqrt{a^2+b^2}+\sqrt{c^2+d^2} \geq \sqrt{(a+c)^2+(b+d)^2}$$

I 19. (**Hong Kong 1998**) $(a, b, c \geq 1)$

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1} \leq \sqrt{c(a b+1)}$$

I 20. (**Carlson's inequality**) $(a, b, c > 0)$

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}} \geq \sqrt{\frac{a b+b c+c a}{3}}$$

I 21. (**Korea 1998**) $(x+y+z=x y z,\ x, y, z > 0)$

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}$$

I 22. (**IMO 2001**) $(a, b, c > 0)$

$$\frac{a}{\sqrt{a^2+8 b c}}+\frac{b}{\sqrt{b^2+8 c a}}+\frac{c}{\sqrt{c^2+8 a b}} \geq 1$$

I 23. (**IMO Short List 2004**) $(a b+b c+c a=1,\ a, b, c > 0)$

$$\sqrt[3]{\frac{1}{a}+6 b}+\sqrt[3]{\frac{1}{b}+6 c}+\sqrt[3]{\frac{1}{c}+6 a} \leq \frac{1}{a b c}$$

I 24. $(a, b, c > 0)$

$$\sqrt{ab(a+b)} + \sqrt{bc(b+c)} + \sqrt{ca(c+a)} \geq \sqrt{4abc + (a+b)(b+c)(c+a)}$$

I 25. (**Macedonia 1995**) $(a, b, c > 0)$

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \geq 2$$

I 26. (**Nesbitt's inequality**) $(a, b, c > 0)$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

I 27. (**IMO 2000**) $(abc = 1,\ a, b, c > 0)$

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

I 28. (**[ONI], Vasile Cirtoaje**) $(a, b, c > 0)$

$$\left(a + \frac{1}{b} - 1\right)\left(b + \frac{1}{c} - 1\right) + \left(b + \frac{1}{c} - 1\right)\left(c + \frac{1}{a} - 1\right) + \left(c + \frac{1}{a} - 1\right)\left(a + \frac{1}{b} - 1\right) \geq 3$$

I 29. (**IMO Short List 1998**) $(xyz = 1,\ x, y, z > 0)$

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

I 30. (**IMO Short List 1996**) $(abc = 1,\ a, b, c > 0)$

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1$$

I 31. (**IMO 1995**) $(abc = 1,\ a, b, c > 0)$

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

I 32. (**IMO Short List 1993**) $(a, b, c, d > 0)$

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}$$

I 33. (**IMO Short List 1990**) $(ab + bc + cd + da = 1,\ a, b, c, d > 0)$

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}$$

I 34. (**IMO 1968**) $(x_1, x_2 > 0, y_1, y_2, z_1, z_2 \in R, x_1y_1 > {z_1}^2, x_2y_2 > {z_2}^2)$

$$\frac{1}{x_1y_1 - {z_1}^2} + \frac{1}{x_2y_2 - {z_2}^2} \geq \frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2}$$

I 35. (**Romania 1997**) $(a, b, c > 0)$

$$\frac{a^2}{a^2 + 2bc} + \frac{b^2}{b^2 + 2ca} + \frac{c^2}{c^2 + 2ab} \geq 1 \geq \frac{bc}{a^2 + 2bc} + \frac{ca}{b^2 + 2ca} + \frac{ab}{c^2 + 2ab}$$

I 36. (**Canada 2002**) $(a, b, c > 0)$

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

I 37. (**USA 1997**) $(a, b, c > 0)$

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\le\frac{1}{abc}.$$

I 38. (**Japan 1997**) $(a, b, c > 0)$

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\ge\frac{3}{5}$$

I 39. (**USA 2003**) $(a, b, c > 0)$

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8$$

I 40. (**Crux Mathematicorum, Problem 2580, Hojoo Lee**) $(a, b, c > 0)$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{b+c}{a^2+bc}+\frac{c+a}{b^2+ca}+\frac{a+b}{c^2+ab}$$

I 41. (**Crux Mathematicorum, Problem 2581, Hojoo Lee**) $(a, b, c > 0)$

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\ge a+b+c$$

I 42. (**Crux Mathematicorum, Problem 2532, Hojoo Lee**) $(a^2+b^2+c^2=1,\ a, b, c > 0)$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge 3+\frac{2(a^3+b^3+c^3)}{abc}$$

I 43. (**Belarus 1999**) $(a^2+b^2+c^2=3,\ a, b, c > 0)$

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\ge\frac{3}{2}$$

I 44. (**Crux Mathematicorum, Problem 3032, Vasile Cirtoaje**) $(a^2+b^2+c^2=1,\ a, b, c > 0)$

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\le\frac{9}{2}$$

I 45. (**Moldova 2005**) $(a^4+b^4+c^4=3,\ a, b, c > 0)$

$$\frac{1}{4-ab}+\frac{1}{4-bc}+\frac{1}{4-ca}\le 1$$

I 46. (**Greece 2002**) $(a^2+b^2+c^2=1,\ a, b, c > 0)$

$$\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1}\ge\frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

I 47. (**Iran 1996**) $(a, b, c > 0)$

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\ge\frac{9}{4}$$

I 48. (**Albania 2002**) $(a, b, c > 0)$

$$\frac{1+\sqrt{3}}{3\sqrt{3}}(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge a+b+c+\sqrt{a^2+b^2+c^2}$$

I 49. (**Belarus 1997**) $(a, b, c > 0)$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{a+b}{c+a}+\frac{b+c}{a+b}+\frac{c+a}{b+c}$$

I 50. (**Belarus 1998**, I. Gorodnin) $(a, b, c > 0)$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{b+c} + \frac{b+c}{a+b} + 1$$

I 51. (**Poland 1996**) $\left(a + b + c = 1,\ a, b, c \geq -\frac{3}{4}\right)$

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq \frac{9}{10}$$

I 52. (**Bulgaria 1997**) $(abc = 1,\ a, b, c > 0)$

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}$$

I 53. (**Romania 1997**) $(xyz = 1,\ x, y, z > 0)$

$$\frac{x^9+y^9}{x^6+x^3y^3+y^6} + \frac{y^9+z^9}{y^6+y^3z^3+z^6} + \frac{z^9+x^9}{z^6+z^3x^3+x^6} \geq 2$$

I 54. (**Vietnam 1991**) $(x \geq y \geq z > 0)$

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2$$

I 55. (**Iran 1997**) $(x_1x_2x_3x_4 = 1, x_1, x_2, x_3, x_4 > 0)$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq max\left(x_1 + x_2 + x_3 + x_4, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right)$$

I 56. (**Hong Kong 2000**) $(abc = 1,\ a, b, c > 0)$

$$\frac{1+ab^2}{c^3} + \frac{1+bc^2}{a^3} + \frac{1+ca^2}{b^3} \geq \frac{18}{a^3+b^3+c^3}$$

I 57. (**Hong Kong 1997**) $(x, y, z > 0)$

$$\frac{3+\sqrt{3}}{9} \geq \frac{xyz(x+y+z+\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)(xy+yz+zx)}$$

I 58. (**Czech-Slovak Match 1999**) $(a, b, c > 0)$

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1$$

I 59. (**Moldova 1999**) $(a, b, c > 0)$

$$\frac{ab}{c(c+a)} + \frac{bc}{a(a+b)} + \frac{ca}{b(b+c)} \geq \frac{a}{c+a} + \frac{b}{b+a} + \frac{c}{c+b}$$

I 60. (**Baltic Way 1995**) $(a, b, c, d > 0)$

$$\frac{a+c}{a+b} + \frac{b+d}{b+c} + \frac{c+a}{c+d} + \frac{d+b}{d+a} \geq 4$$

I 61. (**[ONI], Vasile Cirtoaje**) $(a, b, c, d > 0)$

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geq 0$$

I 62. (**Poland 1993**) $(x, y, u, v > 0)$

$$\frac{xy+xv+uy+uv}{x+y+u+v} \geq \frac{xy}{x+y} + \frac{uv}{u+v}$$

I 63. (**Belarus 1997**) $(a, x, y, z > 0)$

$$\frac{a+y}{a+x}x + \frac{a+z}{a+x}y + \frac{a+x}{a+y}z \geq x+y+z \geq \frac{a+z}{a+z}x + \frac{a+x}{a+y}y + \frac{a+y}{a+z}z$$

I 64. (**Lithuania 1987**) $(x, y, z > 0)$

$$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2} \geq \frac{x+y+z}{3}$$

I 65. (**Klamkin's inequality**) $(-1 < x, y, z < 1)$

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \geq 2$$

I 66. $(xy + yz + zx = 1,\ x, y, z > 0)$

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \geq \frac{2x(1-x^2)}{(1+x^2)^2} + \frac{2y(1-y^2)}{(1+y^2)^2} + \frac{2z(1-z^2)}{(1+z^2)^2}$$

I 67. (**Russia 2002**) $(x + y + z = 3,\ x, y, z > 0)$

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx$$

I 68. (**APMO 1998**) $(a, b, c > 0)$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

I 69. (**Elemente der Mathematik, Problem 1207, Šefket Arslanagić**) $(x, y, z > 0)$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{x+y+z}{\sqrt[3]{xyz}}$$

I 70. (**Die $\sqrt{WURZEL}$, Walther Janous**) $(x + y + z = 1,\ x, y, z > 0)$

$$(1+x)(1+y)(1+z) \geq (1-x^2)^2 + (1-y^2)^2 + (1-z^2)^2$$

I 71. (**United Kingdom 1999**) $(p + q + r = 1,\ p, q, r > 0)$

$$7(pq + qr + rp) \leq 2 + 9pqr$$

I 72. (**USA 1979**) $(x + y + z = 1,\ x, y, z > 0)$

$$x^3 + y^3 + z^3 + 6xyz \geq \frac{1}{4}.$$

I 73. (**IMO 1984**) $(x + y + z = 1,\ x, y, z \geq 0)$

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$$

I 74. (**IMO Short List 1993**) $(a + b + c + d = 1,\ a, b, c, d > 0)$

$$abc + bcd + cda + dab \leq \frac{1}{27} + \frac{176}{27}abcd$$

I 75. (**Poland 1992**) $(a, b, c \in R)$

$$(a+b-c)^2(b+c-a)^2(c+a-b)^2 \geq (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

I 76. (**Canada 1999**) $(x+y+z=1,\ x,y,z\geq 0)$

$$x^2y+y^2z+z^2x\leq\frac{4}{27}$$

I 77. (**Hong Kong 1994**) $(xy+yz+zx=1,\ x,y,z>0)$

$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)\leq\frac{4\sqrt{3}}{9}$$

I 78. (**Vietnam 1996**) $(2(ab+ac+ad+bc+bd+cd)+abc+bcd+cda+dab=16, a,b,c,d\geq 0)$

$$a+b+c+d\geq\frac{2}{3}(ab+ac+ad+bc+bd+cd)$$

I 79. (**Poland 1998**) $\left(a+b+c+d+e+f=1,\ ace+bdf\geq\frac{1}{108}\ a,b,c,d,e,f>0\right)$

$$abc+bcd+cde+def+efa+fab\leq\frac{1}{36}$$

I 80. (**Italy 1993**) $(0\leq a,b,c\leq 1)$

$$a^2+b^2+c^2\leq a^2b+b^2c+c^2a+1$$

I 81. (**Czech Republic 2000**) $(m,n\in N,\ x\in[0,1])$

$$(1-x^n)^m+(1-(1-x)^m)^n\geq 1$$

I 82. (**Ireland 1997**) $(a+b+c\geq abc,\ a,b,c\geq 0)$

$$a^2+b^2+c^2\geq abc$$

I 83. (**BMO 2001**) $(a+b+c\geq abc,\ a,b,c\geq 0)$

$$a^2+b^2+c^2\geq\sqrt{3}abc$$

I 84. (**Bearus 1996**) $(x+y+z=\sqrt{xyz},\ x,y,z>0)$

$$xy+yz+zx\geq 9(x+y+z)$$

I 85. (**Poland 1991**) $(x^2+y^2+z^2=2,\ x,y,z\in\mathbf{R})$

$$x+y+z\leq 2+xyz$$

I 86. (**Mongolia 1991**) $(a^2+b^2+c^2=2,\ a,b,c\in\mathbf{R})$

$$|a^3+b^3+c^3-abc|\leq 2\sqrt{2}$$

I 87. (**Vietnam 2002, Dung Tran Nam**) $(a^2+b^2+c^2=9,\ a,b,c\in\mathbf{R})$

$$2(a+b+c)-abc\leq 10$$

I 88. (**Vietnam 1996**) $(a,b,c>0)$

$$(a+b)^4+(b+c)^4+(c+a)^4\geq\frac{4}{7}\left(a^4+b^4+c^4\right)$$

I 89. $(x,y,z\geq 0)$

$$xyz\geq(y+z-x)(z+x-y)(x+y-z)$$

I 90. (**Latvia 2002**) $\left(\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1,\ a,b,c,d>0\right)$

$$abcd\geq 3$$

I 91. (**Proposed for 1999 USAMO**, [AB, pp.25]) $(x, y, z > 1)$

$$x^{x^2+2yz} y^{y^2+2zx} z^{z^2+2xy} \geq (xyz)^{xy+yz+zx}$$

I 92. (**APMO 2004**) $(a, b, c > 0)$

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca)$$

I 93. (**USA 2004**) $(a, b, c > 0)$

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a+b+c)^3$$

I 94. (**USA 2001**) $(a^2+b^2+c^2+abc=4,\ a, b, c \geq 0)$

$$0 \leq ab+bc+ca-abc \leq 2$$

I 95. (**Turkey, 1999**) $(c \geq b \geq a \geq 0)$

$$(a+3b)(b+4c)(c+2a) \geq 60abc$$

I 96. (**Macedonia 1999**) $(a^2+b^2+c^2=1,\ a, b, c > 0)$

$$a+b+c+\frac{1}{abc} \geq 4\sqrt{3}$$

I 97. (**Poland 1999**) $(a+b+c=1,\ a, b, c > 0)$

$$a^2+b^2+c^2+2\sqrt{3abc} \leq 1$$

I 98. (**Macedonia 2000**) $(x, y, z > 0)$

$$x^2+y^2+z^2 \geq \sqrt{2}\,(xy+yz)$$

I 99. (**APMC 1995**) $(m, n \in \mathbf{N},\ x, y > 0)$

$$(n-1)(m-1)(x^{n+m}+y^{n+m})+(n+m-1)(x^n y^m+x^m y^n) \geq nm(x^{n+m-1}y+xy^{n+m-1})$$

I 100. (**[ONI], Gabriel Dospinescu, Mircea Lascu, Marian Tetiva**) $(a, b, c > 0)$

$$a^2+b^2+c^2+2abc+3 \geq (1+a)(1+b)(1+c)$$

Chapter 2

Substitutions

2.1 Euler's Theorem and the Ravi Substitution

Many inequalities are simplified by some suitable substitutions. We begin with a classical inequality in triangle geometry.

What is the first[1] *nontrivial* geometric inequality ?

In 1765, Euler showed that

Theorem 1. *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Then, we have $R \geq 2r$ and the equality holds if and only if ABC is equilateral.*

Proof. Let $BC = a$, $CA = b$, $AB = c$, $s = \frac{a+b+c}{2}$ and $S = [ABC]$.[2] Recall the well-known identities : $S = \frac{abc}{4R}$, $S = rs$, $S^2 = s(s-a)(s-b)(s-c)$. Hence, $R \geq 2r$ is equivalent to $\frac{abc}{4S} \geq 2\frac{S}{s}$ or $abc \geq 8\frac{S^2}{s}$ or $abc \geq 8(s-a)(s-b)(s-c)$. We need to prove the following. ☐

Theorem 2. (**[AP], A. Padoa**) *Let a, b, c be the lengths of a triangle. Then, we have*

$$abc \geq 8(s-a)(s-b)(s-c) \;\; \text{or} \;\; abc \geq (b+c-a)(c+a-b)(a+b-c)$$

and the equality holds if and only if $a = b = c$.

First Proof. We use the *Ravi* Substitution : Since a, b, c are the lengths of a triangle, there are positive reals x, y, z such that $a = y+z$, $b = z+x$, $c = x+y$. (Why?) Then, the inequality is $(y+z)(z+x)(x+y) \geq 8xyz$ for x, y, $z > 0$. However, we get $(y+z)(z+x)(x+y) - 8xyz = x(y-z)^2 + y(z-x)^2 + z(x-y)^2 \geq 0$. ☐

Second Proof. (**[RI]**) We may assume that $a \geq b \geq c$. It's equivalent to

$$a^3 + b^3 + c^3 + 3abc \geq a^2(b+c) + b^2(c+a) + c^2(a+b).$$

Since $c(a+b-c) \geq b(c+a-b) \geq c(a+b-c)$[3], applying the Rearrangement inequality, we obtain

$$a \cdot a(b+c-a) + b \cdot b(c+a-b) + c \cdot c(a+b-c) \leq a \cdot a(b+c-a) + c \cdot b(c+a-b) + a \cdot c(a+b-c),$$

$$a \cdot a(b+c-a) + b \cdot b(c+a-b) + c \cdot c(a+b-c) \leq c \cdot a(b+c-a) + a \cdot b(c+a-b) + b \cdot c(a+b-c).$$

Adding these two inequalities, we get the result. ☐

Exercise 1. *Let ABC be a right triangle. Show that $R \geq (1+\sqrt{2})r$. When does the equality hold ?*

It's natural to ask that the inequality in the theorem 2 holds for arbitrary positive reals a, b, c? Yes ! It's possible to prove the inequality without the additional condition that a, b, c are the lengths of a triangle :

[1] The first geometric inequality is the Triangle Inequality : $AB + BC \geq AC$

[2] In this book, $[P]$ stands for the area of the polygon P.

[3] For example, we have $c(a+b-c) - b(c+a-b) = (b-c)(b+c-a) \geq 0$.

Theorem 3. *Let $x,\ y,\ z > 0$. Then, we have $xyz \geq (y+z-x)(z+x-y)(x+y-z)$. The equality holds if and only if $x = y = z$.*

Proof. Since the inequality is symmetric in the variables, without loss of generality, we may assume that $x \geq y \geq z$. Then, we have $x + y > z$ and $z + x > y$. If $y + z > x$, then $x,\ y,\ z$ are the lengths of the sides of a triangle. And by the theorem 2, we get the result. Now, we may assume that $y + z \leq x$. Then, $xyz > 0 \geq (y+z-x)(z+x-y)(x+y-z)$. □

The inequality in the theorem 2 holds when some of $x,\ y,\ z$ are zeros :

Theorem 4. *Let $x,\ y,\ z \geq 0$. Then, we have $xyz \geq (y+z-x)(z+x-y)(x+y-z)$.*

Proof. Since $x, y, z \geq 0$, we can find *positive* sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ for which

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} z_n = z.$$

(For example, take $x_n = x + \frac{1}{n}$ $(n = 1, 2, \cdots)$, etc.) Applying the theorem 2 yields

$$x_n y_n z_n \geq (y_n + z_n - x_n)(z_n + x_n - y_n)(x_n + y_n - z_n)$$

Now, taking the limits to both sides, we get the result.

□

Clearly, the equality holds when $x = y = z$. However, $xyz = (y+z-x)(z+x-y)(x+y-z)$ and $x,\ y,\ z \geq 0$ does not guarantee that $x = y = z$. In fact, for $x, y, z \geq 0$, the equality $xyz = (y+z-x)(z+x-y)(x+y-z)$ is equivalent to

$$x = y = z \ \ \textit{or} \ \ x = y, z = 0 \ \ \textit{or} \ \ y = z, x = 0 \ \ \textit{or} \ \ z = x, y = 0.$$

It's straightforward to verify the equality

$$xyz - (y+z-x)(z+x-y)(x+y-z) = x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y).$$

Hence, the theorem 4 is a particular case of Schur's inequality.[4]

Problem 1. (**IMO 2000/2**) *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

First Solution. Since $abc = 1$, we make the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for $x,\ y,\ z > 0$.[5] We rewrite the given inequality in the terms of $x,\ y,\ z$:

$$\left(\frac{x}{y} - 1 + \frac{z}{y}\right)\left(\frac{y}{z} - 1 + \frac{x}{z}\right)\left(\frac{z}{x} - 1 + \frac{y}{x}\right) \leq 1 \ \Leftrightarrow\ xyz \geq (y+z-x)(z+x-y)(x+y-z).$$

□

The *Ravi* Substitution is useful for inequalities for the lengths $a,\ b,\ c$ of a triangle. After the *Ravi* Substitution, we can remove the condition that they are the lengths of the sides of a triangle.

Problem 2. (**IMO 1983/6**) *Let $a,\ b,\ c$ be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Solution. After setting $a = y + z$, $b = z + x$, $c = x + y$ for $x, y, z > 0$, it becomes

$$x^3z + y^3x + z^3y \geq x^2yz + xy^2z + xyz^2 \ \text{ or } \ \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z,$$

which follows from the Cauchy-Schwartz inequality

$$(y+z+x)\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}\right) \geq (x+y+z)^2.$$

□

[4]See the theorem 10 in the chapter 3. Take $r = 1$.
[5]For example, take $x = 1$, $y = \frac{1}{a}$, $z = \frac{1}{ab}$.

Problem 3. (**IMO 1961/2, Weitzenböck's inequality**) *Let a, b, c be the lengths of a triangle with area S. Show that*

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Solution. Write $a = y + z$, $b = z + x$, $c = x + y$ for $x, y, z > 0$. It's equivalent to

$$((y + z)^2 + (z + x)^2 + (x + y)^2)^2 \geq 48(x + y + z)xyz,$$

which can be obtained as following :

$$((y + z)^2 + (z + x)^2 + (x + y)^2)^2 \geq 16(yz + zx + xy)^2 \geq 16 \cdot 3(xy \cdot yz + yz \cdot zx + xy \cdot yz).^{6}$$

□

Exercise 2. (**Hadwiger-Finsler inequality**) *Show that, for any triangle with sides a, b, c and area S, $2ab + 2bc + 2ca - (a^2 + b^2 + c^2) \geq 4\sqrt{3}S$.*

Exercise 3. (**Pedoe's inequality**) *Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Show that*

$${a_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) + {b_1}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {c_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) \geq 16F_1F_2.$$

[6] Here, we used the well-known inequalities $p^2 + q^2 \geq 2pq$ and $(p + q + r)^2 \geq 3(pq + qr + rp)$.

2.2 Trigonometric Substitutions

If you are faced with an integral that contains square root expressions such as

$$\int \sqrt{1-x^2}\, dx,\quad \int \sqrt{1+y^2}\, dy,\quad \int \sqrt{z^2-1}\, dz$$

then trigonometric substitutions such as $x = \sin t$, $y = \tan t$, $z = \sec t$ are very useful. When dealing with square root expressions, making a suitable *trigonometric* substitution simplifies the given inequality.

Problem 4. (**Latvia 2002**) *Let a, b, c, d be the positive real numbers such that*

$$\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1.$$

Prove that $abcd \geq 3$.

Solution. We can write $a^2 = \tan A$, $b^2 = \tan B$, $c^2 = \tan C$, $d^2 = \tan D$, where $A, B, C, D \in \left(0, \frac{\pi}{2}\right)$. Then, the algebraic identity becomes the following trigonometric identity :

$$\cos^2 A + \cos^2 B + \cos^2 C + \cos^2 D = 1.$$

Applying the AM-GM inequality, we obtain

$$\sin^2 A = 1 - \cos^2 A = \cos^2 B + \cos^2 C + \cos^2 D \geq 3\left(\cos B \cos C \cos D\right)^{\frac{2}{3}}.$$

Similarly, we obtain

$$\sin^2 B \geq 3\left(\cos C \cos D \cos A\right)^{\frac{2}{3}}, \sin^2 C \geq 3\left(\cos D \cos A \cos B\right)^{\frac{2}{3}},\ \text{and}\ \sin^2 D \geq 3\left(\cos A \cos B \cos C\right)^{\frac{2}{3}}.$$

Multiplying these inequalities, we get the result! ☐

Exercise 4. (**[ONI], Titu Andreescu, Gabriel Dosinescu**) *Let a, b, c, d be the real numbers such that*

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) = 16.$$

Prove that $-3 \leq ab + ac + ad + bc + bd + cd - abcd \leq 5$.

Problem 5. (**Korea 1998**) *Let x, y, z be the positive reals with $x + y + z = xyz$. Show that*

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}.$$

Since the function f is not concave down on R^+, we cannot apply Jensen's inequality to the function $f(t) = \frac{1}{\sqrt{1+t^2}}$. However, the function $f(\tan\theta)$ is concave down on $\left(0, \frac{\pi}{2}\right)$!

Solution. We can write $x = \tan A$, $y = \tan B$, $z = \tan C$, where $A, B, C \in \left(0, \frac{\pi}{2}\right)$. Using the fact that $1 + \tan^2\theta = \left(\frac{1}{\cos\theta}\right)^2$, where $\cos\theta \neq 0$, we rewrite it in the terms of A, B, C :

$$\cos A + \cos B + \cos C \leq \frac{3}{2}.$$

It follows from $\tan(\pi - C) = -z = \frac{x+y}{1-xy} = \tan(A+B)$ and from $\pi - C, A + B \in (0, \pi)$ that $\pi - C = A + B$ or $A + B + C = \pi$. Hence, it suffices to show the following. ☐

Theorem 5. *In any acute triangle ABC, we have* $\cos A + \cos B + \cos C \leq \frac{3}{2}$.

Proof. Since $\cos x$ is concave down on $\left(0, \frac{\pi}{2}\right)$, it's a direct consequence of Jensen's inequality. ☐

We note that the function $\cos x$ is not concave down on $(0, \pi)$. In fact, it's concave *up* on $\left(\frac{\pi}{2}, \pi\right)$. One may think that the inequality $\cos A + \cos B + \cos C \leq \frac{3}{2}$ doesn't hold for any triangles. However, it's known that it also holds for any triangles.

Theorem 6. *In any triangle ABC, we have* $\cos A + \cos B + \cos C \leq \frac{3}{2}$.

First Proof. It follows from $\pi - C = A + B$ that $\cos C = -\cos(A+B) = -\cos A \cos B + \sin A \sin B$ or

$$3 - 2(\cos A + \cos B + \cos C) = (\sin A - \sin B)^2 + (\cos A + \cos B - 1)^2 \geq 0.$$

□

Second Proof. Let $BC = a$, $CA = b$, $AB = c$. Use the Cosine Law to rewrite the given inequality in the terms of a, b, c :

$$\frac{b^2+c^2-a^2}{2bc} + \frac{c^2+a^2-b^2}{2ca} + \frac{a^2+b^2-c^2}{2ab} \leq \frac{3}{2}.$$

Clearing denominators, this becomes

$$3abc \geq a(b^2+c^2-a^2) + b(c^2+a^2-b^2) + c(a^2+b^2-c^2),$$

which is equivalent to $abc \geq (b+c-a)(c+a-b)(a+b-c)$ in the theorem 2. □

In case even when there is no condition such as $x+y+z = xyz$ or $xy+yz+zx = 1$, the trigonometric substitutions are useful.

Problem 6. (**APMO 2004/5**) *Prove that, for all positive real numbers* a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

Proof. Choose $A, B, C \in \left(0, \frac{\pi}{2}\right)$ with $a = \sqrt{2}\tan A$, $b = \sqrt{2}\tan B$, and $c = \sqrt{2}\tan C$. Using the well-known trigonometric identity $1 + \tan^2\theta = \frac{1}{cos^2\theta}$, one may rewrite it as

$$\frac{4}{9} \geq \cos A \cos B \cos C \left(\cos A \sin B \sin C + \sin A \cos B \sin C + \sin A \sin B \cos C\right).$$

One may easily check the following trigonometric identity

$$\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C.$$

Then, the above trigonometric inequality takes the form

$$\frac{4}{9} \geq \cos A \cos B \cos C \left(\cos A \cos B \cos C - \cos(A+B+C)\right).$$

Let $\theta = \frac{A+B+C}{3}$. Applying the AM-GM inequality and Jesen's inequality, we have

$$\cos A \cos B \cos C \leq \left(\frac{\cos A + \cos B + \cos C}{3}\right)^3 \leq \cos^3\theta.$$

We now need to show that

$$\frac{4}{9} \geq \cos^3\theta(\cos^3\theta - \cos 3\theta).$$

Using the trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \quad \text{or} \quad \cos 3\theta - \cos 3\theta = 3\cos\theta - 3\cos^3\theta,$$

it becomes

$$\frac{4}{27} \geq \cos^4\theta\left(1 - \cos^2\theta\right),$$

which follows from the AM-GM inequality

$$\left(\frac{\cos^2\theta}{2} \cdot \frac{\cos^2\theta}{2} \cdot \left(1-\cos^2\theta\right)\right)^{\frac{1}{3}} \leq \frac{1}{3}\left(\frac{\cos^2\theta}{2} + \frac{\cos^2\theta}{2} + \left(1-\cos^2\theta\right)\right) = \frac{1}{3}.$$

One find that the equality holds if and only if $\tan A = \tan B = \tan C = \frac{1}{\sqrt{2}}$ if and only if $a = b = c = 1$. □

Exercise 5. (**[TZ], pp.127**) *Let x, y, z be real numbers such that $0 < x, y, z < 1$ and $xy + yz + zx = 1$. Prove that*

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \geq \frac{3\sqrt{3}}{2}.$$

Exercise 6. (**[TZ], pp.127**) *Let x, y, z be positive real numbers such that $x + y + z = xyz$. Prove that*

$$\frac{x}{\sqrt{1+x^2}} + \frac{y}{\sqrt{1+y^2}} + \frac{z}{\sqrt{1+z^2}} \leq \frac{3\sqrt{3}}{2}.$$

Exercise 7. (**[ONI], Florina Carlan, Marian Tetiva**) *Prove that if x, y, $z > 0$ satisfy the condition $x + y + z = xyz$ then*

$$xy + yz + zx \geq 3 + \sqrt{1+x^2} + \sqrt{1+y^2} + \sqrt{1+z^2}.$$

Exercise 8. (**[ONI], Gabriel Dospinescu, Marian Tetiva**) *Let x, y, z be positive real numbers such that $x + y + z = xyz$. Prove that*

$$(x-1)(y-1)(z-1) \leq 6\sqrt{3} - 10.$$

Exercise 9. (**[TZ], pp.113**) *Let a, b, c be real numbers. Prove that*

$$(a^2+1)(b^2+1)(c^2+1) \geq (ab + bc + ca - 1)^2.$$

Exercise 10. (**[TZ], pp.149**) *Let a and b be positive real numbers. Prove that*

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} \geq \frac{2}{\sqrt{1+ab}}$$

if either (1) $0 < a, b \leq 1$ or (2) $ab \geq 3$.

In the theorem 1 and 2, we see that the *geometric* inequality $R \geq 2r$ is equivalent to the *algebraic* inequality $abc \geq (b+c-a)(c+a-b)(a+b-c)$. We now find that, in the proof of the theorem 6, $abc \geq (b+c-a)(c+a-b)(a+b-c)$ is equivalent to the *trigonometric* inequality $\cos A + \cos B + \cos C \leq \frac{3}{2}$. One may ask that

> In any triangles ABC, is there a *natural* relation between $\cos A + \cos B + \cos C$ and $\frac{R}{r}$, where R and r are the radii of the circumcircle and incircle of ABC ?

Theorem 7. *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Then, we have $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$.*

Proof. Use the identity $a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2) = 2abc+(b+c-a)(c+a-b)(a+b-c)$. We leave the details for the readers. □

Exercise 11. *Let R and r be the radii of the circumcircle and incircle of the triangle ABC with $BC = a$, $CA = b$, $AB = c$. Let s denote the semiperimeter of ABC. Verify the follwing identities* [7] *:*

(1) $ab + bc + ca = s^2 + 4Rr + r^2$,
(2) $abc = 4Rrs$,
(3) $\cos A \cos B + \cos B \cos C + \cos C \cos A = \frac{s^2-4R^2+r^2}{4R^2}$,
(4) $\cos A \cos B \cos C = \frac{s^2-(2R+r)^2}{4R^2}$

Exercise 12. *(a) Let p, q, r be the positive real numbers such that $p^2 + q^2 + r^2 + 2pqr = 1$. Show that there exists an acute triangle ABC such that $p = \cos A$, $q = \cos B$, $r = \cos C$.*
(b) Let $p, q, r \geq 0$ with $p^2 + q^2 + r^2 + 2pqr = 1$. Show that there are $A, B, C \in \left[0, \frac{\pi}{2}\right]$ with $p = \cos A$, $q = \cos B$, $r = \cos C$, and $A + B + C = \pi$.

[7] For more identities, see the exercise 10.

Exercise 13. (**[ONI], Marian Tetiva**) *Let x, y, z be positive real numbers satisfying the condition*

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

Prove that

(1) $xyz \leq \frac{1}{8}$,
(2) $xy + yz + zx \leq \frac{3}{4}$,
(3) $x^2 + y^2 + z^2 \geq \frac{3}{4}$, *and*
(4) $xy + yz + zx \leq 2xyz + \frac{1}{2}$.

Exercise 14. (**[ONI], Marian Tetiva**) *Let x, y, z be positive real numbers satisfying the condition*

$$x^2 + y^2 + z^2 = xyz.$$

Prove that

(1) $xyz \geq 27$,
(2) $xy + yz + zx \geq 27$,
(3) $x + y + z \geq 9$, *and*
(4) $xy + yz + zx \geq 2(x + y + z) + 9$.

Problem 7. (**USA 2001**) *Let $a, b,$ and c be nonnegative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that $0 \leq ab + bc + ca - abc \leq 2$.*

Solution. Notice that $a, b, c > 1$ implies that $a^2+b^2+c^2+abc > 4$. If $a \leq 1$, then we have $ab+bc+ca-abc \geq (1-a)bc \geq 0$. We now prove that $ab + bc + ca - abc \leq 2$. Letting $a = 2p$, $b = 2q$, $c = 2r$, we get $p^2 + q^2 + r^2 + 2pqr = 1$. By the exercise 12, we can write

$$a = 2\cos A,\ b = 2\cos B,\ c = 2\cos C \ \text{ for some } A, B, C \in \left[0, \frac{\pi}{2}\right] \text{ with } A + B + C = \pi.$$

We are required to prove

$$\cos A\cos B + \cos B\cos C + \cos C\cos A - 2\cos A\cos B\cos C \leq \frac{1}{2}.$$

One may assume that $A \geq \frac{\pi}{3}$ or $1 - 2\cos A \geq 0$. Note that

$$\cos A\cos B + \cos B\cos C + \cos C\cos A - 2\cos A\cos B\cos C = \cos A(\cos B + \cos C) + \cos B\cos C(1 - 2\cos A).$$

We apply Jensen's inequality to deduce $\cos B + \cos C \leq \frac{3}{2} - \cos A$. Note that $2\cos B\cos C = \cos(B - C) + \cos(B + C) \leq 1 - \cos A$. These imply that

$$\cos A(\cos B + \cos C) + \cos B\cos C(1 - 2\cos A) \leq \cos A\left(\frac{3}{2} - \cos A\right) + \left(\frac{1 - \cos A}{2}\right)(1 - 2\cos A).$$

However, it's easy to verify that $\cos A\left(\frac{3}{2} - \cos A\right) + \left(\frac{1-\cos A}{2}\right)(1 - 2\cos A) = \frac{1}{2}$. $\square$

In the above solution, we showed that

$$\cos A\cos B + \cos B\cos C + \cos C\cos A - 2\cos A\cos B\cos C \leq \frac{1}{2}$$

holds for all *acute* triangles. Using the results (c) and (d) in the exercise (4), we can rewrite it in the terms of R, r, s :

$$2R^2 + 8Rr + 3r^2 \leq s^2.$$

In 1965, W. J. Blundon found the best possible inequalities of the form $A(R, r) \leq s^2 \leq B(R, r)$, where $A(x, y)$ and $B(x, y)$ are real quadratic forms $\alpha x^2 + \beta xy + \gamma y^2$: [8]

Exercise 15. *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Let s be the semiperimeter of ABC. Show that*

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2.$$

[8] For a proof, see [WJB].

2.3 Algebraic Substitutions

We know that some inequalities in triangle geometry can be treated by the *Ravi* substitution and *trigonometric* substitutions. We can also transform the given inequalities into easier ones through some clever *algebraic* substitutions.

Problem 8. (**IMO 2001/2**) *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

First Solution. To remove the square roots, we make the following substitution :

$$x=\frac{a}{\sqrt{a^2+8bc}},\ \ y=\frac{b}{\sqrt{b^2+8ca}},\ \ z=\frac{c}{\sqrt{c^2+8ab}}.$$

Clearly, $x, y, z \in (0, 1)$. Our aim is to show that $x+y+z \geq 1$. We notice that

$$\frac{a^2}{8bc}=\frac{x^2}{1-x^2},\ \ \frac{b^2}{8ac}=\frac{y^2}{1-y^2},\ \ \frac{c^2}{8ab}=\frac{z^2}{1-z^2} \quad\Longrightarrow\quad \frac{1}{512}=\left(\frac{x^2}{1-x^2}\right)\left(\frac{y^2}{1-y^2}\right)\left(\frac{z^2}{1-z^2}\right).$$

Hence, we need to show that

$$x+y+z\geq 1,\ \text{where } 0<x,y,z<1 \text{ and } (1-x^2)(1-y^2)(1-z^2)=512(xyz)^2.$$

However, $1 > x+y+z$ implies that, by the AM-GM inequality,

$$(1-x^2)(1-y^2)(1-z^2) > ((x+y+z)^2-x^2)((x+y+z)^2-y^2)((x+y+z)^2-z^2) = (x+x+y+z)(y+z)$$

$$(x+y+y+z)(z+x)(x+y+z+z)(x+y) \geq 4(x^2yz)^{\frac{1}{4}}\cdot 2(yz)^{\frac{1}{2}}\ \cdot 4(y^2zx)^{\frac{1}{4}}\cdot 2(zx)^{\frac{1}{2}}\ \cdot 4(z^2xy)^{\frac{1}{4}}\cdot 2(xy)^{\frac{1}{2}}$$

$= 512(xyz)^2$. This is a contradiction ! □

Problem 9. (**IMO 1995/2**) *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

First Solution. After the substitution $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$, we get $xyz = 1$. The inequality takes the form

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} \geq \frac{3}{2}.$$

It follows from the Cauchy-schwartz inequality that

$$[(y+z)+(z+x)+(x+y)]\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right) \geq (x+y+z)^2$$

so that, by the AM-GM inequality,

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} \geq \frac{x+y+z}{2} \geq \frac{3(xyz)^{\frac{1}{3}}}{2} = \frac{3}{2}.$$

□

We offer an alternative solution of the problem 5 :
(**Korea 1998**) Let x, y, z be the positive reals with $x+y+z = xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}.$$

Second Solution. The starting point is letting $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$. We find that $a+b+c = abc$ is equivalent to $1 = xy + yz + zx$. The inequality becomes

$$\frac{x}{\sqrt{x^2+1}} + \frac{y}{\sqrt{y^2+1}} + \frac{z}{\sqrt{z^2+1}} \leq \frac{3}{2}$$

or

$$\frac{x}{\sqrt{x^2+xy+yz+zx}} + \frac{y}{\sqrt{y^2+xy+yz+zx}} + \frac{z}{\sqrt{z^2+xy+yz+zx}} \leq \frac{3}{2}$$

or

$$\frac{x}{\sqrt{(x+y)(x+z)}} + \frac{y}{\sqrt{(y+z)(y+x)}} + \frac{z}{\sqrt{(z+x)(z+y)}} \leq \frac{3}{2}.$$

By the AM-GM inequality, we have

$$\frac{x}{\sqrt{(x+y)(x+z)}} = \frac{x\sqrt{(x+y)(x+z)}}{(x+y)(x+z)} \leq \frac{1}{2}\frac{x[(x+y)+(x+z)]}{(x+y)(x+z)} = \frac{1}{2}\left(\frac{x}{x+z} + \frac{x}{x+z}\right).$$

In a like manner, we obtain

$$\frac{y}{\sqrt{(y+z)(y+x)}} \leq \frac{1}{2}\left(\frac{y}{y+z} + \frac{y}{y+x}\right) \quad \text{and} \quad \frac{z}{\sqrt{(z+x)(z+y)}} \leq \frac{1}{2}\left(\frac{z}{z+x} + \frac{z}{z+y}\right).$$

Adding these three yields the required result. □

We now prove a classical theorem in various ways.

Theorem 8. (**Nesbitt, 1903**) *For all positive real numbers a, b, c, we have*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 1. *After the substitution $x = b+c$, $y = c+a$, $z = a+b$, it becomes*

$$\sum_{\text{cyclic}} \frac{y+z-x}{2x} \geq \frac{3}{2} \quad \text{or} \quad \sum_{\text{cyclic}} \frac{y+z}{x} \geq 6,$$

which follows from the AM-GM inequality as following:

$$\sum_{\text{cyclic}} \frac{y+z}{x} = \frac{y}{x} + \frac{z}{x} + \frac{z}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{z} \geq 6\left(\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right)^{\frac{1}{6}} = 6.$$

Proof 2. *We make the substitution*

$$x = \frac{a}{b+c},\ y = \frac{b}{c+a},\ z = \frac{c}{a+b}.$$

It follows that

$$\sum_{\text{cyclic}} f(x) = \sum_{\text{cyclic}} \frac{a}{a+b+c} = 1, \quad \text{where} \quad f(t) = \frac{t}{1+t}.$$

Since f is concave down on $(0, \infty)$, Jensen's inequality shows that

$$f\left(\frac{1}{2}\right) = \frac{2}{3} = \frac{1}{3}\sum_{\text{cyclic}} f(x) \geq f\left(\frac{x+y+z}{3}\right) \quad \text{or} \quad f\left(\frac{1}{2}\right) \geq f\left(\frac{x+y+z}{3}\right).$$

Since f is monotone decreasing, we have

$$\frac{1}{2} \leq \frac{x+y+z}{3} \quad \text{or} \quad \sum_{\text{cyclic}} \frac{a}{b+c} = x+y+z \geq \frac{3}{2}.$$

Proof 3. *As in the previous proof, it suffices to show that*

$$T \geq \frac{1}{2} \quad \textit{where} \quad T = \frac{x+y+z}{3} \quad \textit{and} \quad \sum_{\text{cyclic}} \frac{x}{1+x} = 1.$$

One can easily check that the condition

$$\sum_{\text{cyclic}} \frac{x}{1+x} = 1$$

becomes $1 = 2xyz + xy + yz + zx$. *By the AM-GM inequality, we have*

$$1 = 2xyz + xy + yz + zx \leq 2T^3 + 3T^2 \quad \Leftrightarrow \quad 2T^3 + 3T^2 - 1 \geq 0 \quad \Leftrightarrow \quad (2T-1)(T+1)^2 \geq 0 \quad \Leftrightarrow \quad T \geq \frac{1}{2}.$$

Proof 4. *Since the inequality is symmetric in the three variables, we may assume that* $a \geq b \geq c$. *After the substitution* $x = \frac{a}{c}, y = \frac{b}{c}$, *we have* $x \geq y \geq 1$. *It becomes*

$$\frac{\frac{a}{c}}{\frac{b}{c}+1} + \frac{\frac{b}{c}}{\frac{a}{c}+1} + \frac{1}{\frac{a}{c}+\frac{b}{c}} \geq \frac{3}{2} \quad \textit{or} \quad \frac{x}{y+1} + \frac{y}{x+1} \geq \frac{3}{2} - \frac{1}{x+y}.$$

We apply the AM-GM inequality to obtain

$$\frac{x+1}{y+1} + \frac{y+1}{x+1} \geq 2 \quad \textit{or} \quad \frac{x}{y+1} + \frac{y}{x+1} \geq 2 - \frac{1}{y+1} + \frac{1}{x+1}.$$

It suffices to show that

$$2 - \frac{1}{y+1} + \frac{1}{x+1} \geq \frac{3}{2} - \frac{1}{x+y} \quad \Leftrightarrow \quad \frac{1}{2} - \frac{1}{y+1} \geq \frac{1}{x+1} - \frac{1}{x+y} \quad \Leftrightarrow \quad \frac{y-1}{2(1+y)} \geq \frac{y-1}{(x+1)(x+y)}.$$

However, the last inequality clearly holds for $x \geq y \geq 1$.

Proof 5. *As in the previous proof, we need to prove*

$$\frac{x}{y+1} + \frac{y}{x+1} + \frac{1}{x+y} \geq \frac{3}{2} \quad \textit{where} \quad x \geq y \geq 1.$$

Let $A = x + y$ *and* $B = xy$. *It becomes*

$$\frac{x^2+y^2+x+y}{(x+1)(y+1)} + \frac{1}{x+y} \geq \frac{3}{2} \quad \textit{or} \quad \frac{A^2-2B+A}{A+B+1} + \frac{1}{A} \geq \frac{3}{2} \quad \textit{or} \quad 2A^3 - A^2 - A + 2 \geq B(7A-2).$$

Since $7A - 2 > 2(x+y-1) > 0$ *and* $A^2 = (x+y)^2 \geq 4xy = 4B$, *it's enough to show that*

$$4(2A^3 - A^2 - A + 2) \geq A^2(7A-2) \quad \Leftrightarrow \quad A^3 - 2A^2 - 4A + 8 \geq 0.$$

However, it's easy to check that $A^3 - 2A^2 - 4A + 8 = (A-2)^2(A+2) \geq 0$.

We now present alternative solutions of problem 1.

(**IMO 2000/2**) Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Second Solution. (**[IV], Ilan Vardi**) Since $abc = 1$, we may assume that $a \geq 1 \geq b$. [9] It follows that

$$1 - \left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) = \left(c + \frac{1}{c} - 2\right)\left(a + \frac{1}{b} - 1\right) + \frac{(a-1)(1-b)}{a}. \text{ [10]}$$

□

[9] Why? Note that the inequality is not symmetric in the three variables. Check it!
[10] For a verification of the identity, see [IV].

Third Solution. As in the first solution, after the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for $x,\ y,\ z > 0$, we can rewrite it as $xyz \geq (y+z-x)(z+x-y)(x+y-z)$. Without loss of generality, we can assume that $z \geq y \geq x$. Set $y - x = p$ and $z - x = q$ with $p, q \geq 0$. It's straightforward to verify that

$$xyz \geq (y+z-x)(z+x-y)(x+y-z) = (p^2 - pq + q^2)x + (p^3 + q^3 - p^2q - pq^2).$$

Since $p^2 - pq + q^2 \geq (p-q)^2 \geq 0$ and $p^3 + q^3 - p^2q - pq^2 = (p-q)^2(p+q) \geq 0$, we get the result. $\square$

Fourth Solution. (based on work by an IMO 2000 contestant from Japan) Putting $c = \frac{1}{ab}$, it becomes

$$\left(a - 1 + \frac{1}{b}\right)(b - 1 + ab)\left(\frac{1}{ab} - 1 + \frac{1}{a}\right) \leq 1$$

or

$$a^3b^3 - a^2b^3 - ab^3 - a^2b^2 + 3ab^2 - ab + b^3 - b^2 - b + 1 \geq 0.$$

Setting $x = ab$, it becomes $f(x) \geq 0$, where

$$f_b(t) = t^3 + b^3 - b^2t - bt^2 + 3bt - t^2 - b^2 - t - b + 1.$$

Fix a positive number $b \geq 1$. We need to show that $F(t) := f_b(t) \geq 0$ for all $t \geq 0$. It's easy to check that the cubic polynomial $F^/(t) = 3t^2 - 2(b+1)t - (b^2 - 3b + 1)$ has two real roots

$$\frac{b + 1 - \sqrt{4b^2 - 7b + 4}}{3} \quad \text{and} \quad \lambda = \frac{b + 1 + \sqrt{4b^2 - 7b + 4}}{3}.$$

Since F has a local minimum at $t = \lambda$, we find that $F(t) \geq Min\ \{F(0),\ F(\lambda)\}$ for all $t \geq 0$. We have to prove that $F(0) \geq 0$ and $F(\lambda) \geq 0$. Since

$$F(0) = b^3 - b^2 - b + 1 = (b-1)^2(b+1) \geq 0,$$

it remains to show that $F(\lambda) \geq 0$. Notice that λ is a root of $F^/(t)$. After long division, we get

$$F(t) = F^/(t)\left(\frac{1}{3}t - \frac{b+1}{9}\right) + \frac{1}{9}\left((-8b^2 + 14b - 8)t + 8b^3 - 7b^2 - 7b + 8\right).$$

Putting $t = \lambda$, we have

$$F(\lambda) = \frac{1}{9}\left((-8b^2 + 14b - 8)\lambda + 8b^3 - 7b^2 - 7b + 8\right).$$

Thus, our job is now to establish that, for all $b \geq 0$,

$$(-8b^2 + 14b - 8)\left(\frac{b + 1 + \sqrt{4b^2 - 7b + 4}}{3}\right) + 8b^3 - 7b^2 - 7b + 8 \geq 0,$$

which is equivalent to

$$16b^3 - 15b^2 - 15b + 16 \geq (8b^2 - 14b + 8)\sqrt{4b^2 - 7b + 4}\,.$$

Since both $16b^3 - 15b^2 - 15b + 16$ and $8b^2 - 14b + 8$ are positive,[11] it's equivalent to

$$(16b^3 - 15b^2 - 15b + 16)^2 \geq (8b^2 - 14b + 8)^2(4b^2 - 7b + 4)$$

or

$$864b^5 - 3375b^4 + 5022b^3 - 3375b^2 + 864b \geq 0 \quad \text{or} \quad 864b^4 - 3375b^3 + 5022b^2 - 3375b + 864 \geq 0.$$

Let $G(x) = 864x^4 - 3375x^3 + 5022x^2 - 3375x + 864$. We prove that $G(x) \geq 0$ for all $x \in \mathbf{R}$. We find that

$$G^/(x) = 3456x^3 - 10125x^2 + 10044x - 3375 = (x-1)(3456x^2 - 6669x + 3375).$$

Since $3456x^2 - 6669x + 3375 > 0$ for all $x \in \mathbf{R}$, we find that $G(x)$ and $x - 1$ have the same sign. It follows that $G(x)$ is monotone decreasing on $(-\infty, 1]$ and monotone increasing on $[1, \infty)$. We conclude that $G(x)$ has the global minimum at $x = 1$. Hence, $G(x) \geq G(1) = 0$ for all $x \in \mathbf{R}$. $\square$

[11] It's easy to check that $16b^3 - 15b^2 - 15b + 16 = 16(b^3 - b^2 - b + 1) + b^2 + b > 16(b^2 - 1)(b - 1) \geq 0$ and $8b^2 - 14b + 8 = 8(b-1)^2 + 2b > 0$.

Fifth Solution. (**From the IMO 2000 Short List**) Using the condition $abc = 1$, it's straightforward to verify the equalities

$$2 = \frac{1}{a}\left(a - 1 + \frac{1}{b}\right) + c\left(b - 1 + \frac{1}{c}\right),$$

$$2 = \frac{1}{b}\left(b - 1 + \frac{1}{c}\right) + a\left(c - 1 + \frac{1}{a}\right),$$

$$2 = \frac{1}{c}\left(c - 1 + \frac{1}{a}\right) + b\left(a - 1 + \frac{1}{c}\right).$$

In particular, they show that at most one of the numbers $u = a - 1 + \frac{1}{b}$, $v = b - 1 + \frac{1}{c}$, $w = c - 1 + \frac{1}{a}$ is negative. If there is such a number, we have

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) = uvw < 0 < 1.$$

And if $u, v, w \geq 0$, the AM-GM inequality yields

$$2 = \frac{1}{a}u + cv \geq 2\sqrt{\frac{c}{a}uv},\ \ 2 = \frac{1}{b}v + aw \geq 2\sqrt{\frac{a}{b}vw},\ \ 2 = \frac{1}{c}w + aw \geq 2\sqrt{\frac{b}{c}wu}.$$

Thus, $uv \leq \frac{a}{c}$, $vw \leq \frac{b}{a}$, $wu \leq \frac{c}{b}$, so $(uvw)^2 \leq \frac{a}{c} \cdot \frac{b}{a} \cdot \frac{c}{b} = 1$. Since $u, v, w \geq 0$, this completes the proof. ☐

It turns out that the substitution $p = x + y + z$, $q = xy + yz + zx$, $r = xyz$ is powerful for the three variables inequalities. We need the following lemma.

Lemma 1. *Let x, y, z be non-negative real numbers numbers. Set $p = x + y + z$, $q = xy + yz + zx$, and $r = xyz$. Then, we have* [12]

(1) $p^3 - 4pq + 9r \geq 0$,
(2) $p^4 - 5p^2q + 4q^2 + 6pr \geq 0$,
(3) $pq - 9r \geq 0$.

Proof. They are equivalent to

(1′) $x(x - y)(x - z) + y(y - z)(y - x) + z(z - x)(z - y) \geq 0$,
(2′) $x^2(x - y)(x - z) + y^2(y - z)(y - x) + z^2(z - x)(z - y) \geq 0$,[13]
(3′) $x(y - z)^2 + y(z - x)^2 + z(x - y)^2 \geq 0$.

We leave the details for the readers. ☐

Problem 10. (**Iran 1996**) *Let x, y, z be positive real numbers. Prove that*

$$(xy + yz + zx)\left(\frac{1}{(x + y)^2} + \frac{1}{(y + z)^2} + \frac{1}{(z + x)^2}\right) \geq \frac{9}{4}.$$

First Solution. We make the *substitution* $p = x + y + z$, $q = xy + yz + zx$, $r = xyz$. Notice that $(x + y)(y + z)(z + x) = (x + y + z)(xy + yz + zx) - xyz = pq - r$. One may easily rewrite the given inequality in the terms of p, q, r :

$$q\left(\frac{(p^2 + q)^2 - 4p(pq - r)}{(pq - r)^2}\right) \geq \frac{9}{4}$$

or

$$4p^4q - 17p^2q^2 + 4q^3 + 34pqr - 9r^2 \geq 0$$

or

$$pq(p^3 - 4pq + 9r) + q(p^4 - 5p^2q + 4q^2 + 6pr) + r(pq - 9r) \geq 0.$$

We find that every term on the left hand side is nonnegative by the lemma. ☐

[12] When does equality hold in each inequality? For more p-q-r inequalities, visit the site [ESF].
[13] See the theorem 10.

Problem 11. *Let x, y, z be nonnegative real numbers with $xy + yz + zx = 1$. Prove that*

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{5}{2}.$$

First Solution. Rewrite the inequality in the terms of $p = x + y + z$, $q = xy + yz + zx$, $r = xyz$:

$$4p^4q + 4q^3 - 17p^2q^2 - 25r^2 + 50pqr \geq 0.$$

It can be rewritten as

$$3pq(p^3 - 4pq + 9r) + q(p^4 - 5p^2q + 4q^2 + 6pr) + 17r(pq - 9r) + 128r^2 \geq 0.$$

However, the every term on the left hand side is nonnegative by the lemma. $\square$

Exercise 16. (**Carlson's inequality**) *Prove that, for all positive real numbers a, b, c,*

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}} \geq \sqrt{\frac{ab+bc+ca}{3}}.$$

Exercise 17. (**Bulgaria 1997**) *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

We close this section by presenting a problem which can be solved by two algebraic substitutions and a trigonometric substitution.

Problem 12. (**Iran 1998**) *Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,*

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

First Solution. We begin with the algebraic substitution $a = \sqrt{x-1}$, $b = \sqrt{y-1}$, $c = \sqrt{z-1}$. Then, the condition becomes

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{1}{1+c^2} = 2 \quad \Leftrightarrow \quad a^2b^2 + b^2c^2 + c^2a^2 + 2a^2b^2c^2 = 1$$

and the inequality is equivalent to

$$\sqrt{a^2+b^2+c^2+3} \geq a + b + c \quad \Leftrightarrow \quad ab + bc + ca \leq \frac{3}{2}.$$

Let $p = bc$, $q = ca$, $r = ab$. Our job is to prove that $p + q + r \leq \frac{3}{2}$ where $p^2 + q^2 + r^2 + 2pqr = 1$. By the exercise 12, we can make the trigonometric substitution

$$p = \cos A,\ q = \cos B,\ r = \cos C \ \text{ for some } A, B, C \in \left[0, \frac{\pi}{2}\right) \text{ with } A + B + C = \pi.$$

What we need to show is now that $\cos A + \cos B + \cos C \leq \frac{3}{2}$. However, it follows from Jensen's inequality! $\square$

2.4 Supplementary Problems for Chapter 2

Exercise 18. *Let x, y, and z be positive numbers. Let $p = x + y + z$, $q = xy + yz + zx$, and $r = xyz$. Prove the following inequalities :*

(a) $p^2 \geq 3q$
(b) $p^3 \geq 27r$
(c) $q^2 \geq 3pr$
(d) $2p^3 + 9r \geq 7pq$
(e) $p^2q + 3pr \geq 4q^2$
(f) $p^2q \geq 3pr + 2q^2$
(g) $p^4 + 3q^2 \geq 4p^2q$
(h) $pq^2 \geq 2p^2r + 3qr$
(i) $2q^3 + 9r^3 \geq 7pqr$
(j) $q^3 + 9r^2 \geq 4pqr$
(k) $p^3r + q^3 \geq 6pqr$

Exercise 19. (**[ONI], Mircea Lascu, Marian Tetiva**) *Let x, y, z be positive real numbers satisfying the condition*

$$xy + yz + zx + 2xyz = 1.$$

Prove that

(1) $xyz \leq \frac{1}{8}$,
(2) $x + y + z \leq \frac{3}{2}$,
(3) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 4(x + y + z)$, *and*
(4) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 4(x + y + z) \geq \frac{(2z-1)^2}{z(2z+1)}$, *where $z \geq x, y$.*

Exercise 20. *Let $f(x, y)$ be a real polynomial such that, for all $\theta \in \mathbf{R}^3$,*

$$f(\cos\theta, \sin\theta) = 0.$$

Show that the polynomial $f(x, y)$ is divisible by $x^2 + y^2 - 1$.

Exercise 21. *Let $f(x, y, z)$ be a real polynomial. Suppose that*

$$f(\cos\alpha, \cos\beta, \cos\gamma) = 0,$$

for all $\alpha, \beta, \gamma \in \mathbf{R}^3$ with $\alpha + \beta + \gamma = \pi$. Show that $f(x, y, z)$ is divisible by $x^2 + y^2 + z^2 + 2xyz - 1$. [14]

Exercise 22. (**IMO Unused 1986**) *Let a, b, c be positive real numbers. Show that*

$$(a + b - c)^2(a - b + c)^2(-a + b + c)^2 \geq (a^2 + b^2 - c^2)(a^2 - b^2 + c^2)(-a^2 + b^2 + c^2).$$ [15]

Exercise 23. *With the usual notation for a triangle, verify the following identities :*

(1) $\sin A + \sin B + \sin C = \frac{s}{R}$
(2) $\sin A \sin B + \sin B \sin C + \sin C \sin A = \frac{s^2+4Rr+r^2}{4R^2}$
(3) $\sin A \sin B \sin C = \frac{sr}{2R^2}$
(4) $\sin^3 A + \sin^3 B + \sin^3 C = \frac{s(s^2-6Rr-3r^2)}{4R^3}$
(5) $\cos^3 A + \cos^3 B + \cos^3 C = \frac{(2R+r)^3-3rs^2-4R^3}{4R^3}$
(6) $\tan A + \tan B + \tan C = \tan A \tan B \tan C = \frac{2rs}{s^2-(2R+r)^2}$
(7) $\tan A \tan B + \tan B \tan C + \tan C \tan A = \frac{s^2-4Rr-r^2}{s^2-(2R+r)^2}$
(8) $\cot A + \cot B + \cot C = \frac{s^2-4Rr-r^2}{2sr}$
(9) $\sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} = \frac{r}{4R}$
(10) $\cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} = \frac{s}{4R}$

[14] For a proof, see [JmhMh].

[15] If we assume that there is a triangle ABC with $BC = a$, $CA = b$, $AB = c$, then it's equivalent to the inequality $s^2 \leq 4R^2 + 4Rr + 3r^2$ in the exercise 6.

Exercise 24. *Let a, b, c be the lengths of the sides of a triangle. Let s be the semi-perimeter of the triangle. Then, the following inequalities holds.*

(a) $3(ab+bc+ca) \leq (a+b+c)^2 < 4(ab+bc+ca)$
(b) [JfdWm] $a^2+b^2+c^2 \geq \frac{36}{35}\left(s^2+\frac{abc}{s}\right)$
(c) [AP] $8(s-a)(s-b)(s-c) \leq abc$
(d) [EC] $8abc \geq (a+b)(b+c)(c+a)$
(e) [AP] $3(a+b)(b+c)(c+a) \leq 8(a^3+b^3+c^3)$
(f) [MC] $2(a+b+c)(a^2+b^2+c^2) \geq 3(a^3+b^3+c^3+3abc)$
(g) $abc < a^2(s-a)+b^2(s-b)+c^2(s-c) \leq \frac{3}{2}abc$
(h) $bc(b+c)+ca(c+a)+ab(a+b) \geq 48(s-a)(s-b)(s-c)$
(i) $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} \geq \frac{9}{s}$
(j) [AMN], [MP] $\frac{3}{2} \leq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} < 2$
(k) $\frac{15}{4} \leq \frac{s+a}{b+c}+\frac{s+b}{c+a}+\frac{s+c}{a+b} < \frac{9}{2}$
(l) [SR2] $(a+b+c)^3 \leq 5[ab(a+b)+bc(b+c)+ca(c+a)]-3abc$

Exercise 25. (**[RS], R. Sondat**) *Let R, r, s be positive real numbers. Show that a necessary and sufficient condition for the existence of a triangle with circumradius R, inradius r, and semiperimeter s is*

$$s^4-2(2R^2+10Rr-r^2)s^2+r(4R+r)^2 \leq 0.$$

Exercise 26. *With the usual notation for a triangle, show that $4R+r \geq \sqrt{3}s$.* [16]

Exercise 27. (**[WJB2],[RAS], W. J. Blundon**) *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Let s be the semiperimeter of ABC. Show that*

$$s \geq 2R+(3\sqrt{3}-4)r.$$

Exercise 28. *Let G and I be the centroid and incenter of the triangle ABC with inradius r, semiperimeter s, circumradius R. Show that*

$$GI^2 = \frac{1}{9}\left(s^2+5r^2-16Rr\right).$$ [17]

Exercise 29. *Show that, for any triangle with sides a, b, c,*

$$2 > \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

[16] It's equivalent to the Hadwiger-Finsler inequality.
[17] See the exercise 6. For a solution, see [KWL].

Chapter 3

Homogenizations

3.1 Homogeneous Polynomial Inequalities

Many inequality problems come with constraints such as $ab = 1$, $xyz = 1$, $x+y+z = 1$. A non-homogeneous *symmetric* inequality can be transformed into a homogeneous one. Then we apply two powerful theorems : Shur's inequality and Muirhead's theorem. We begin with a simple example.

Problem 13. (**Hungary 1996**) *Let a and b be positive real numbers with $a+b=1$. Prove that*

$$\frac{a^2}{a+1} + \frac{b^2}{b+1} \geq \frac{1}{3}.$$

Solution. Using the condition $a+b=1$, we can reduce the given inequality to homogeneous one, i. e.,

$$\frac{1}{3} \leq \frac{a^2}{(a+b)(a+(a+b))} + \frac{b^2}{(a+b)(b+(a+b))} \quad \text{or} \quad a^2b+ab^2 \leq a^3+b^3,$$

which follows from $(a^3+b^3)-(a^2b+ab^2) = (a-b)^2(a+b) \geq 0$. The equality holds if and only if $a = b = \frac{1}{2}$. $\square$

The above inequality $a^2b+ab^2 \leq a^3+b^3$ can be generalized as following :

Theorem 9. *Let a_1, a_2, b_1, b_2 be positive real numbers such that $a_1 + a_2 = b_1 + b_2$ and $max(a_1, a_2) \geq max(b_1, b_2)$. Let x and y be nonnegative real numbers. Then, we have $x^{a_1}y^{a_2} + x^{a_2}y^{a_1} \geq x^{b_1}y^{b_2} + x^{b_2}y^{b_1}$.*

Proof. Without loss of generality, we can assume that $a_1 \geq a_2, b_1 \geq b_2, a_1 \geq b_1$. If x or y is zero, then it clearly holds. So, we also assume that both x and y are nonzero. It's easy to check

$$\begin{aligned} x^{a_1}y^{a_2} + x^{a_2}y^{a_1} - x^{b_1}y^{b_2} - x^{b_2}y^{b_1} &= x^{a_2}y^{a_2}\left(x^{a_1-a_2} + y^{a_1-a_2} - x^{b_1-a_2}y^{b_2-a_2} - x^{b_2-a_2}y^{b_1-a_2}\right) \\ &= x^{a_2}y^{a_2}\left(x^{b_1-a_2} - y^{b_1-a_2}\right)\left(x^{b_2-a_2} - y^{b_2-a_2}\right) \\ &= \frac{1}{x^{a_2}y^{a_2}}\left(x^{b_1} - y^{b_1}\right)\left(x^{b_2} - y^{b_2}\right) \geq 0. \end{aligned}$$

$\square$

Remark 1. *When does the equality hold in the theorem 8?*

We now introduce two summation notations $\sum_{\text{cyclic}}$ and $\sum_{\text{sym}}$. Let $P(x, y, z)$ be a three variables function of x, y, z. Let us define :

$$\sum_{\text{cyclic}} P(x,y,z) = P(x,y,z) + P(y,z,x) + P(z,x,y),$$

$$\sum_{\text{sym}} P(x,y,z) = P(x,y,z) + P(x,z,y) + P(y,x,z) + P(y,z,x) + P(z,x,y) + P(z,y,x)$$

For example, we know that

$$\sum_{\text{cyclic}} x^3y = x^3y + y^3z + z^3x, \quad \sum_{\text{sym}} x^3 = 2(x^3 + y^3 + z^3)$$

$$\sum_{\text{sym}} x^2y = x^2y + x^2z + y^2z + y^2x + z^2x + z^2y, \quad \sum_{\text{sym}} xyz = 6xyz.$$

Problem 14. (**IMO 1984/1**) *Let x, y, z be nonnegative real numbers such that $x + y + z = 1$. Prove that $0 \le xy + yz + zx - 2xyz \le \frac{7}{27}$.*

Solution. Using the condition $x + y + z = 1$, we reduce the given inequality to homogeneous one, i. e.,

$$0 \le (xy + yz + zx)(x + y + z) - 2xyz \le \frac{7}{27}(x + y + z)^3.$$

The left hand side inequality is trivial because it's equivalent to $0 \le xyz + \sum_{\text{sym}} x^2y$. The right hand side inequality simplifies to $7\sum_{\text{cyclic}} x^3 + 15xyz - 6\sum_{\text{sym}} x^2y \ge 0$. In the view of

$$7\sum_{\text{cyclic}} x^3 + 15xyz - 6\sum_{\text{sym}} x^2y = \left(2\sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y\right) + 5\left(3xyz + \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y\right),$$

it's enough to show that $2\sum_{\text{cyclic}} x^3 \ge \sum_{\text{sym}} x^2y$ and $3xyz + \sum_{\text{cyclic}} x^3 \ge \sum_{\text{sym}} x^2y$. Note that

$$2\sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y = \sum_{\text{cyclic}} (x^3 + y^3) - \sum_{\text{cyclic}} (x^2y + xy^2) = \sum_{\text{cyclic}} (x^3 + y^3 - x^2y - xy^2) \ge 0.$$

The second inequality can be rewritten as

$$\sum_{\text{cyclic}} x(x - y)(x - z) \ge 0,$$

which is a particular case of the theorem 10 in the next section. $\square$

3.2 Schur's Theorem

Theorem 10. (**Schur**) *Let x, y, z be nonnegative real numbers. For any $r > 0$, we have*

$$\sum_{\text{cyclic}} x^r(x-y)(x-z) \geq 0.$$

Proof. Since the inequality is symmetric in the three variables, we may assume without loss of generality that $x \geq y \geq z$. Then the given inequality may be rewritten as

$$(x-y)[x^r(x-z) - y^r(y-z)] + z^r(x-z)(y-z) \geq 0,$$

and every term on the left-hand side is clearly nonnegative. □

Remark 2. *When does the equality hold in Theorem 10?*

The following special case of Schur's inequality is useful :

$$\sum_{\text{cyclic}} x(x-y)(x-z) \geq 0 \Leftrightarrow 3xyz + \sum_{\text{cyclic}} x^3 \geq \sum_{\text{sym}} x^2y \Leftrightarrow \sum_{\text{sym}} xyz + \sum_{\text{sym}} x^3 \geq 2\sum_{\text{sym}} x^2y.$$

Exercise 30. (**[TZ], pp.142**) *Prove that for any acute triangle ABC,*

$$\cot^3 A + \cot^3 B + \cot^3 C + 6\cot A \cot B \cot C \geq \cot A + \cot B + \cot C.$$

Exercise 31. (**Korea 1998**) *Let I be the incenter of a triangle ABC. Prove that*

$$IA^2 + IB^2 + IC^2 \geq \frac{BC^2 + CA^2 + AB^2}{3}.$$

Exercise 32. (**[IN], pp.103**) *Let a, b, c be the lengths of a triangle. Prove that*

$$a^2b + a^2c + b^2c + b^2a + c^2a + c^2b > a^3 + b^3 + c^3 + 2abc.$$

We present another solution of the problem 1 :

(**IMO 2000/2**) Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Second Solution. It is equivalent to the following homogeneous inequality[1] :

$$\left(a - (abc)^{1/3} + \frac{(abc)^{2/3}}{b}\right)\left(b - (abc)^{1/3} + \frac{(abc)^{2/3}}{c}\right)\left(c - (abc)^{1/3} + \frac{(abc)^{2/3}}{a}\right) \leq abc.$$

After the substitution $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$, it becomes

$$\left(x^3 - xyz + \frac{(xyz)^2}{y^3}\right)\left(y^3 - xyz + \frac{(xyz)^2}{z^3}\right)\left(z^3 - xyz + \frac{(xyz)^2}{x^3}\right) \leq x^3y^3z^3,$$

which simplifies to

$$\left(x^2y - y^2z + z^2x\right)\left(y^2z - z^2x + x^2y\right)\left(z^2x - x^2y + y^2z\right) \leq x^3y^3z^3$$

or

$$3x^3y^3z^3 + \sum_{\text{cyclic}} x^6y^3 \geq \sum_{\text{cyclic}} x^4y^4z + \sum_{\text{cyclic}} x^5y^2z^2$$

or

$$3(x^2y)(y^2z)(z^2x) + \sum_{\text{cyclic}} (x^2y)^3 \geq \sum_{\text{sym}} (x^2y)^2(y^2z)$$

which is a special case of Schur's inequality. □

[1] For an alternative homogenization, see the problem 1 in the chapter 2.

Here is another inequality problem with the constraint $abc = 1$.

Problem 15. (**Tournament of Towns 1997**) *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

Solution. We can rewrite the given inequality as following :

$$\frac{1}{a+b+(abc)^{1/3}} + \frac{1}{b+c+(abc)^{1/3}} + \frac{1}{c+a+(abc)^{1/3}} \leq \frac{1}{(abc)^{1/3}}.$$

We make the substitution $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$. Then, it becomes

$$\frac{1}{x^3+y^3+xyz} + \frac{1}{y^3+z^3+xyz} + \frac{1}{z^3+x^3+xyz} \leq \frac{1}{xyz}$$

which is equivalent to

$$xyz \sum_{\text{cyclic}} (x^3+y^3+xyz)(y^3+z^3+xyz) \leq (x^3+y^3+xyz)(y^3+z^3+xyz)(z^3+x^3+xyz)$$

and hence to $\sum_{\text{sym}} x^6y^3 \geq \sum_{\text{sym}} x^5y^2z^2$, which is a special case of theorem 11 in the next section. $\square$

3.3 Muirhead's Theorem

Theorem 11. (**Muirhead**) *Let $a_1, a_2, a_3, b_1, b_2, b_3$ be real numbers such that*

$$a_1 \geq a_2 \geq a_3 \geq 0, b_1 \geq b_2 \geq b_3 \geq 0, a_1 \geq b_1, a_1 + a_2 \geq b_1 + b_2, a_1 + a_2 + a_3 = b_1 + b_2 + b_3.^{2}$$

Let x, y, z be positive real numbers. Then, we have $\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} \geq \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}$.

Proof. Case 1. $b_1 \geq a_2$: It follows from $a_1 \geq a_1 + a_2 - b_1$ and from $a_1 \geq b_1$ that $a_1 \geq max(a_1 + a_2 - b_1, b_1)$ so that $max(a_1, a_2) = a_1 \geq max(a_1 + a_2 - b_1, b_1)$. From $a_1 + a_2 - b_1 \geq b_1 + a_3 - b_1 = a_3$ and $a_1 + a_2 - b_1 \geq b_2 \geq b_3$, we have $max(a_1 + a_2 - b_1, a_3) \geq max(b_2, b_3)$. Apply the theorem 8 twice to obtain

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} &= \sum_{\text{cyclic}} z^{a_3} (x^{a_1} y^{a_2} + x^{a_2} y^{a_1}) \\
&\geq \sum_{\text{cyclic}} z^{a_3} (x^{a_1 + a_2 - b_1} y^{b_1} + x^{b_1} y^{a_1 + a_2 - b_1}) \\
&= \sum_{\text{cyclic}} x^{b_1} (y^{a_1 + a_2 - b_1} z^{a_3} + y^{a_3} z^{a_1 + a_2 - b_1}) \\
&\geq \sum_{\text{cyclic}} x^{b_1} (y^{b_2} z^{b_3} + y^{b_3} z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.
\end{aligned}$$

Case 2. $b_1 \leq a_2$: It follows from $3b_1 \geq b_1 + b_2 + b_3 = a_1 + a_2 + a_3 \geq b_1 + a_2 + a_3$ that $b_1 \geq a_2 + a_3 - b_1$ and that $a_1 \geq a_2 \geq b_1 \geq a_2 + a_3 - b_1$. Therefore, we have $max(a_2, a_3) \geq max(b_1, a_2 + a_3 - b_1)$ and $max(a_1, a_2 + a_3 - b_1) \geq max(b_2, b_3)$. Apply the theorem 8 twice to obtain

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} &= \sum_{\text{cyclic}} x^{a_1} (y^{a_2} z^{a_3} + y^{a_3} z^{a_2}) \\
&\geq \sum_{\text{cyclic}} x^{a_1} (y^{b_1} z^{a_2 + a_3 - b_1} + y^{a_2 + a_3 - b_1} z^{b_1}) \\
&= \sum_{\text{cyclic}} y^{b_1} (x^{a_1} z^{a_2 + a_3 - b_1} + x^{a_2 + a_3 - b_1} z^{a_1}) \\
&\geq \sum_{\text{cyclic}} y^{b_1} (x^{b_2} z^{b_3} + x^{b_3} z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.
\end{aligned}$$

□

Remark 3. *The equality holds if and only if $x = y = z$. However, if we allow $x = 0$ or $y = 0$ or $z = 0$,* [3] *then one may easily check that the equality holds if and only if*

$$x = y = z \ \text{ or } \ x = y, \ z = 0 \ \text{ or } \ y = z, \ x = 0 \ \text{ or } \ z = x, \ y = 0.$$

We can use Muirhead's theorem to prove Nesbitt's inequality.

(**Nesbitt**) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

[2] Note the equality in the final equation.

[3] However, in this case, we assume that $0^0 = 1$ in the sense that $\lim_{x \to 0^+} x^0 = 1$. In general, 0^0 is not defined. Note also that $\lim_{x \to 0^+} 0^x = 0$.

Proof 6. *Clearing the denominators of the inequality, it becomes*

$$2\sum_{\text{cyclic}} a(a+b)(a+c) \geq 3(a+b)(b+c)(c+a) \quad \textit{or} \quad \sum_{\text{sym}} a^3 \geq \sum_{\text{sym}} a^2 b.$$

Problem 16. ((**IMO 1995**) *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution. It's equivalent to

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2(abc)^{4/3}}.$$

Set $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$. Then, it becomes $\sum_{\text{cyclic}} \frac{1}{x^9(y^3+z^3)} \geq \frac{3}{2x^4y^4z^4}$. Clearing denominators, this becomes

$$\sum_{\text{sym}} x^{12}y^{12} + 2\sum_{\text{sym}} x^{12}y^9z^3 + \sum_{\text{sym}} x^9y^9z^6 \geq 3\sum_{\text{sym}} x^{11}y^8z^5 + 6x^8y^8z^8$$

or

$$\left(\sum_{\text{sym}} x^{12}y^{12} - \sum_{\text{sym}} x^{11}y^8z^5\right) + 2\left(\sum_{\text{sym}} x^{12}y^9z^3 - \sum_{\text{sym}} x^{11}y^8z^5\right) + \left(\sum_{\text{sym}} x^9y^9z^6 - \sum_{\text{sym}} x^8y^8z^8\right) \geq 0,$$

and every term on the left hand side is nonnegative by Muirhead's theorem. $\square$

We can also attack problem 10 and problem 11 with Schur's inequality and Muirhead's theorem.

(**Iran 1996**) Let x, y, z be positive real numbers. Prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

Second Solution. It's equivalent to

$$4\sum_{\text{sym}} x^5y + 2\sum_{\text{cyclic}} x^4yz + 6x^2y^2z^2 - \sum_{\text{sym}} x^4y^2 - 6\sum_{\text{cyclic}} x^3y^3 - 2\sum_{\text{sym}} x^3y^2z \geq 0.$$

We rewrite this as following

$$\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^4y^2\right) + 3\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^3y^3\right) + 2xyz\left(\sum_{\text{cyclic}} x(x-y)(x-z)\right) \geq 0.$$

By Muirhead's theorem and Schur's inequality, it's a sum of three terms which are nonnegative. $\square$

Let x, y, z be nonnegative real numbers with $xy + yz + zx = 1$. Prove that

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{5}{2}.$$

Second Solution. Using $xy + yz + zx = 1$, we homogenize the given inequality as following :

$$(xy + yz + zx)\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right)^2 \geq \left(\frac{5}{2}\right)^2$$

or

$$4\sum_{\text{sym}} x^5y + \sum_{\text{sym}} x^4yz + 14\sum_{\text{sym}} x^3y^2z + 38x^2y^2z^2 \geq \sum_{\text{sym}} x^4y^2 + 3\sum_{\text{sym}} x^3y^3$$

or

$$\left(\sum_{\rm sym} x^5y - \sum_{\rm sym} x^4y^2\right) + 3\left(\sum_{\rm sym} x^5y - \sum_{\rm sym} x^3y^3\right) + xyz\left(\sum_{\rm sym} x^3 + 14\sum_{\rm sym} x^2y + 38xyz\right) \geq 0.$$

By Muirhead's theorem, we get the result. In the above inequality, without the condition $xy + yz + zx = 1$, the equality holds if and only if $x = y, z = 0$ *or* $y = z, x = 0$ *or* $z = x, y = 0$. Since $xy + yz + zx = 1$, the equality occurs when $(x, y, z) = (1, 1, 0), (1, 0, 1), (0, 1, 1)$. $\square$

Now, we apply Muirhead's theorem to obtain a geometric inequality [ZsJc] :

Problem 17. *If m_a,m_b,m_c are medians and r_a,r_b,r_c the exradii of a triangle, prove that*

$$\frac{r_a r_b}{m_a m_b} + \frac{r_b r_c}{m_b m_c} + \frac{r_c r_a}{m_c m_a} \geq 3.$$

An Impossible Verification. Let $2s = a + b + c$. Using the well-known identities

$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}},\ m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2},\ etc.$$

we have

$$\sum_{\rm cyclic} \frac{r_b r_c}{m_b m_c} = \sum_{\rm cyclic} \frac{4s(s-a)}{\sqrt{(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)}}.$$

Applying the AM-GM inequality, we obtain

$$\sum_{\rm cyclic} \frac{r_b r_c}{m_b m_c} \geq \sum_{\rm cyclic} \frac{8s(s-a)}{(2c^2 + 2a^2 - b^2) + (2a^2 + 2b^2 - c^2)} = \sum_{\rm cyclic} \frac{2(a+b+c)(b+c-a)}{4a^2 + b^2 + c^2}.$$

We now give a **moonshine** proof of the inequality

$$\sum_{\rm cyclic} \frac{2(a+b+c)(b+c-a)}{4a^2 + b^2 + c^2} \geq 3.$$

After expanding the above inequality, it becomes

$$2\sum_{\rm cyclic} a^6 + 4\sum_{\rm cyclic} a^4bc + 20\sum_{\rm sym} a^3b^2c + 68\sum_{\rm cyclic} a^3b^3 + 16\sum_{\rm cyclic} a^5b \geq 276a^2b^2c^2 + 27\sum_{\rm cyclic} a^4b^2.$$

We see that this cannot be directly proven by applying Muirhead's theorem. Since a, b, c are the sides of a triangle, we can make the *Ravi* Substitution $a = y + z$, $b = z + x$, $c = x + y$, where $x, y, z > 0$. After some brute-force algebra, we can rewrite the above inequality as

$$25\sum_{\rm sym} x^6 + 230\sum_{\rm sym} x^5y + 115\sum_{\rm sym} x^4y^2 + 10\sum_{\rm sym} x^3y^3 + 80\sum_{\rm sym} x^4yz$$

$$\geq 336\sum_{\rm sym} x^3y^2z + 124\sum_{\rm sym} x^2y^2z^2.$$

Now, by Muirhead's theorem, we get the result ! $\square$

3.4 Polynomial Inequalities with Degree 3

The solution of problem 13 shows us difficulties in applying Muirhead's theorem. Furthermore, there exist homogeneous symmetric polynomial inequalities which cannot be verified by just applying Muirhead's theorem. See the following inequality :

$$5\sum_{\text{cyclic}} x^6 + 15\sum_{\text{sym}} x^4y^2 + 2\sum_{\text{sym}} x^3y^2z + 3x^2y^2z^2 \geq 8\sum_{\text{sym}} x^5y + 8\sum_{\text{cyclic}} x^4yz + 16\sum_{\text{cyclic}} x^3y^3$$

This holds for all positive real numbers x, y, and z. However, it is not a direct consequence of Muirhead's theorem because the coefficients of $\sum_{\text{sym}} x^5y$ and $\sum_{\text{cyclic}} x^3y^3$ are too big. In fact, it is equivalent to

$$\frac{1}{6}\sum_{\text{cyclic}} (y-z)^4(x^2+15y^2+15z^2+8xy+4yz+8zx) \geq 0.$$ [4]

Another example is

$$\frac{1}{2}\sum_{\text{cyclic}} x^4 + \frac{3}{2}\sum_{\text{cyclic}} x^2y^2 \geq \sum_{\text{sym}} x^3y.$$

We realized that the above inequality is stronger than

$$\sum_{\text{cyclic}} x^2(x-y)(x-z) \geq 0 \ \text{ or } \sum_{\text{cyclic}} x^4 + \sum_{\text{cyclic}} x^2y^2 \geq \sum_{\text{sym}} x^3y.$$

It can be proved by the identities

$$4\left(\frac{1}{2}\sum_{\text{cyclic}} x^4 + \frac{3}{2}\sum_{\text{cyclic}} x^2y^2 - \sum_{\text{sym}} x^3y\right) = (x-y)^4+(y-z)^4+(z-x)^4$$

or

$$2\left(\frac{1}{2}\sum_{\text{cyclic}} x^4 + \frac{3}{2}\sum_{\text{cyclic}} x^2y^2 - \sum_{\text{sym}} x^3y\right) = (x^2+y^2+z^2-xy-yz-zx)^2.$$

As I know, there is no general criterion to attack the symmetric polynomial inequalities. However, there is a result for the homogeneous symmetric polynomial inequalities with degree 3. It's a direct consequence of Muirhead's theorem and Schur's inequality.

Theorem 12. *Let $P(u,v,w) \in \mathbf{R}[u,v,w]$ be a homogeneous symmetric polynomial with degree 3. Then the following two statements are equivalent.*

(a) $P(1,1,1), P(1,1,0), P(1,0,0) \geq 0$.
(b) $P(x,y,z) \geq 0$ for all $x,y,z \geq 0$.

Proof. (See [SR].) We only prove that (a) implies (b). Let

$$P(u,v,w) = A\sum_{\text{cyclic}} u^3 + B\sum_{\text{sym}} u^2v + Cuvw.$$

Let $p = P(1,1,1) = 3A+6B+C$, $q = P(1,1,0) = A+B$, and $r = P(1,0,0) = A$. We have $A = r$, $B = q-r$, $C = p-6q+3r$, and $p,q,r \geq 0$. Let $x,y,z \geq 0$. It follows that

$$P(x,y,z) = r\sum_{\text{cyclic}} x^3 + (q-r)\sum_{\text{sym}} x^2y + (p-6q+3r)xyz.$$

However, we see that

$$P(x,y,z) = r\left(\sum_{\text{cyclic}} x^3 + 3xyz - \sum_{\text{sym}} x^2y\right) + q\left(\sum_{\text{sym}} x^2y - 6xyz\right) + pxyz \geq 0.$$

$\square$

[4] See [JC].

Here is an alternative way to prove of the fact that $P(x,y,z) \geq 0$.

Case 1. $q \geq r$ *: We find that*

$$P(x,y,z) = \frac{r}{2}\left(\sum_{\text{sym}} x^3 - \sum_{\text{sym}} xyz\right) + (q-r)\left(\sum_{\text{sym}} x^2y - \sum_{\text{sym}} xyz\right) + pxyz.$$

and that the every term on the right hand side is nonnegative.

Case 2. $q \leq r$ *: We find that*

$$P(x,y,z) = \frac{q}{2}\left(\sum_{\text{sym}} x^3 - \sum_{\text{sym}} xyz\right) + (r-q)\left(\sum_{\text{cyclic}} x^3 + 3xyz - \sum_{\text{sym}} x^2y\right) + pxyz.$$

and that the every term on the right hand side is nonnegative.

For example, we can apply the theorem 11 to *check* the inequality in the problem 14.

> (**IMO 1984/1**) Let x, y, z be nonnegative real numbers such that $x+y+z=1$. Prove that $0 \leq xy+yz+zx-2xyz \leq \frac{7}{27}$.

Solution. Using $x+y+z=1$, we homogenize the given inequality as following :

$$0 \leq (xy+yz+zx)(x+y+z) - 2xyz \leq \frac{7}{27}(x+y+z)^3$$

Let us define $L(u,v,w), R(u,v,w) \in \mathbf{R}[u,v,w]$ by

$$L(u,v,w) = (uv+vw+wu)(u+v+w) - 2uvw,$$

$$R(u,v,w) = \frac{7}{27}(u+v+w)^3 - (uv+vw+wu)(u+v+w) + 2uvw.$$

However, one may easily check that $L(1,1,1)=7$, $L(1,1,0)=2$, $L(1,0,0)=0$, $R(1,1,1)=0$, $R(1,1,0)=\frac{2}{27}$, and $R(1,0,0)=\frac{7}{27}$. ☐

Exercise 33. (**M. S. Klamkin [MEK2]**) *Determine the maximum and minimum values of*

$$x^2+y^2+z^2+\lambda xyz$$

where $x+y+z=1$, $x,y,z \geq 0$, and λ is a given constant.

Exercise 34. (**Walter Janous [MC]**) *let $x,y,z \geq 0$ with $x+y+z=1$. For fixed real numbers $a \geq 0$ and b, determine the maximum $c = c(a,b)$ such that*

$$a + bxyz \geq c(xy+yz+zx).$$

Here is the criterion for homogeneous symmetric polynomial inequalities for the triangles :

Theorem 13. (**K. B. Stolarsky**) *Let $P(u,v,w)$ be a real symmetric form of degree 3.*[5] *If*

$$P(1,1,1), P(1,1,0), P(2,1,1) \geq 0,$$

then we have $P(a,b,c) \geq 0$, where a,b,c are the lengths of the sides of a triangle.

Proof. Make the *Ravi* substitution $a=y+z$, $b=z+x$, $c=x+y$ and apply the above theorem. We leave the details for the readers. For an alternative proof, see [KBS]. ☐

As noted in [KBS], we can apply Stolarsky's theorem to prove cubic inequalities in triangle geometry. We recall the exercise 11.

[5] $P(x,y,z) = \sum_{\text{sym}} (px^3 + qx^2y + rxyz) \quad (p,q,r \in \mathbf{R}.)$

Let a, b, c be the lengths of the sides of a triangle. Let s be the semi-perimeter of the triangle. Then, the following inequalities holds.

(a) $3(ab+bc+ca) \leq (a+b+c)^2 < 4(ab+bc+ca)$
(b) [JfdWm] $a^2+b^2+c^2 \geq \frac{36}{35}\left(s^2+\frac{abc}{s}\right)$
(c) [AP] $8(s-a)(s-b)(s-c) \leq abc$
(d) [EC] $8abc \geq (a+b)(b+c)(c+a)$
(e) [AP] $3(a+b)(b+c)(c+a) \leq 8(a^3+b^3+c^3)$
(f) [MC] $2(a+b+c)(a^2+b^2+c^2) \geq 3(a^3+b^3+c^3+3abc)$
(g) $abc < a^2(s-a)+b^2(s-b)+c^2(s-c) \leq \frac{3}{2}abc$
(h) $bc(b+c)+ca(c+a)+ab(a+b) \geq 48(s-a)(s-b)(s-c)$
(i) $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} \geq \frac{9}{s}$
(j) [AMN], [MP] $\frac{3}{2} \leq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} < 2$
(k) $\frac{15}{4} \leq \frac{s+a}{b+c}+\frac{s+b}{c+a}+\frac{s+c}{a+b} < \frac{9}{2}$
(l) [SR] $(a+b+c)^3 \leq 5[ab(a+b)+bc(b+c)+ca(c+a)]-3abc$

Proof. For example, we show the right hand side inequality in (j). It's equivalent to the cubic inequality $T(a,b,c) \geq 0$, where

$$T(a,b,c) = 2(a+b)(b+c)(c+a)-(a+b)(b+c)(c+a)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

Since $T(1,1,1)=4$, $T(1,1,0)=0$, and $T(2,1,1)=6$, the result follows from Stolarsky's theorem. For alternative proofs of the above 12 inequalities, see [GI]. $\square$

3.5 Supplementary Problems for Chapter 3

Exercise 35. *Let x, y, z be positive real numbers. Prove that*

$$(x+y-z)(x-y)^2 + (y+z-x)(y-z)^2 + (z+x-y)(z-x)^2 \geq 0.$$

Exercise 36. *Let x, y, z be positive real numbers. Prove that*

$$(x^2+y^2-z^2)(x-y)^2 + (y^2+z^2-x^2)(y-z)^2 + (z^2+x^2-y^2)(z-x)^2 \geq 0.$$

Exercise 37. *(APMO 1998) Let a, b, c be positive real numbers. Prove that*

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right).$$

Exercise 38. *(Ireland 2000) Let $x, y \geq 0$ with $x+y=2$. Prove that*

$$x^2y^2(x^2+y^2) \leq 2.$$

Exercise 39. *(IMO Short-listed 1998) Let x, y, z be positive real numbers such that $xyz = 1$. Prove that*

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

Exercise 40. *(United Kingdom 1999) Some three nonnegative real numbers p, q, r satisfy $p+q+r=1$. Prove that $7(pq+qr+rp) \leq 2 + 9pqr$.*

Chapter 4

Normalizations

4.1 Normalizations

In the previous chapter, we transformed non-homogeneous inequalities into homogeneous ones. On the other hand, homogeneous inequalities also can be normalized in *various* ways. We offer two alternative solutions of the problem 8 by normalizations :

(**IMO 2001/2**) Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Second Solution. We make the substitution $x=\frac{a}{a+b+c}$, $y=\frac{b}{a+b+c}$, $z=\frac{c}{a+b+c}$.[1] The problem is

$$xf(x^2+8yz)+yf(y^2+8zx)+zf(z^2+8xy)\geq 1,$$

where $f(t)=\frac{1}{\sqrt{t}}$. Since the function f is convex down on R^+ and $x+y+z=1$, we apply (the weighted) Jensen's inequality to have

$$xf(x^2+8yz)+yf(y^2+8zx)+zf(z^2+8xy)\geq f(x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy)).$$

Note that $f(1)=1$. Since the function f is strictly decreasing, it suffices to show that

$$1\geq x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy).$$

Using $x+y+z=1$, we homogenize it as $(x+y+z)^3\geq x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy)$. However, this is easily seen from

$$(x+y+z)^3-x(x^2+8yz)-y(y^2+8zx)-z(z^2+8xy)=3[x(y-z)^2+y(z-x)^2+z(x-y)^2]\geq 0.$$

□

In the above solution, we normalized to $x+y+z=1$. We now prove it by normalizing to $xyz=1$.

Third Solution. We make the *substitution* $x=\frac{bc}{a^2}$, $y=\frac{ca}{b^2}$, $z=\frac{ab}{c^2}$. Then, we get $xyz=1$ and the inequality becomes

$$\frac{1}{\sqrt{1+8x}}+\frac{1}{\sqrt{1+8y}}+\frac{1}{\sqrt{1+8z}}\geq 1$$

which is equivalent to

$$\sum_{\text{cyclic}}\sqrt{(1+8x)(1+8y)}\geq\sqrt{(1+8x)(1+8y)(1+8z)}$$

[1] Dividing by $a+b+c$ gives the equivalent inequality $\sum_{\text{cyclic}}\frac{\frac{a}{a+b+c}}{\sqrt{\frac{a^2}{(a+b+c)^2}+\frac{8bc}{(a+b+c)^2}}}\geq 1$.

hence, after squaring both sides, equivalent to

$$8(x+y+z)+2\sqrt{(1+8x)(1+8y)(1+8z)}\sum_{\text{cyclic}}\sqrt{1+8x}\geq 510.$$

Recall that $xyz=1$. The AM-GM inequality gives us $x+y+z\geq 3$,

$$(1+8x)(1+8y)(1+8z)\geq 9x^{\frac{8}{9}}\cdot 9y^{\frac{8}{9}}\cdot 9z^{\frac{8}{9}}=729 \text{ and } \sum_{\text{cyclic}}\sqrt{1+8x}\geq\sum_{\text{cyclic}}\sqrt{9x^{\frac{8}{9}}}\geq 9(xyz)^{\frac{4}{27}}=9.$$

Using these three inequalities, we get the result. □

We now present another proofs of Nesbitt's inequality.

(**Nesbitt**) For all positive real numbers a,b,c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 7. *We may normalize to $a+b+c=1$. Note that $0<a,b,c<1$. The problem is now to prove*

$$\sum_{\text{cyclic}}\frac{a}{b+c}=\sum_{\text{cyclic}}f(a)\geq\frac{3}{2},\ \textit{where}\ f(x)=\frac{x}{1-x}.$$

Since f is concave down on $(0,1)$, Jensen's inequality shows that

$$\frac{1}{3}\sum_{\text{cyclic}}f(a)\geq f\left(\frac{a+b+c}{3}\right)=f\left(\frac{1}{3}\right)=\frac{1}{2}\quad\textit{or}\quad\sum_{\text{cyclic}}f(a)\geq\frac{3}{2}.$$

Proof 8. *As in the previous proof, we need to prove*

$$\sum_{\text{cyclic}}\frac{a}{1-a}\geq\frac{3}{2},\ \textit{where}\ a+b+c=1.$$

It follows from $4x-(1-x)(9x-1)=(3x-1)^2$ or $4x\geq(1-x)(9x-1)$ that

$$\sum_{\text{cyclic}}\frac{a}{1-a}\geq\sum_{\text{cyclic}}\frac{9a-1}{4}=\frac{9}{4}\sum_{\text{cyclic}}a-\frac{3}{4}=\frac{3}{2}.$$

4.2 Classical Theorems : Cauchy-Schwartz, (Weighted) AM-GM, and Hölder

We now illustrate the normalization technique to establish classical theorems.

Theorem 14. *(The Cauchy-Schwartz inequality) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers. Then, we have*

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

Proof. Let $A = \sqrt{{a_1}^2 + \cdots + {a_n}^2}$ and $B = \sqrt{{b_1}^2 + \cdots + {b_n}^2}$. In the case when $A = 0$, we get $a_1 = \cdots = a_n = 0$. Thus, the given inequality clearly holds. So, we now may assume that $A, B > 0$. Now, we make the substitution $x_i = \frac{a_i}{A}$ $(i = 1, \cdots, n)$. Then, it's equivalent to

$$({x_1}^2 + \cdots + {x_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (x_1 b_1 + \cdots + x_n b_n)^2.$$

However, we have ${x_1}^2 + \cdots + {x_n}^2 = 1$. (Why?). Hence, it's equivalent to

$${b_1}^2 + \cdots + {b_n}^2 \geq (x_1 b_1 + \cdots + x_n b_n)^2.$$

Next, we make the substitution $y_i = \frac{b_i}{B}$ $(i = 1, \cdots, n)$. Then, it's equivalent to

$$1 = {y_1}^2 + \cdots + {y_n}^2 \geq (x_1 y_1 + \cdots + x_n y_n)^2 \text{ or } 1 \geq |x_1 y_1 + \cdots + x_n y_n|.$$

Hence, we need to to show that

$$|x_1 y_1 + \cdots + x_n y_n| \leq 1, \text{ where } {x_1}^2 + \cdots + {x_n}^2 = {y_1}^2 + \cdots + {y_n}^2 = 1.$$

However, it's very easy. We apply the AM-GM inequality to deduce

$$|x_1 y_1 + \cdots + x_n y_n| \leq |x_1 y_1| + \cdots + |x_n y_n| \leq \frac{{x_1}^2 + {y_1}^2}{2} + \cdots + \frac{{x_n}^2 + {y_n}^2}{2} = \frac{A + B}{2} = 1.$$

□

Exercise 41. *Prove the Lagrange's identity :*

$$\sum_{i=1}^{n} {a_i}^2 \sum_{i=1}^{n} {b_i}^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)^2.$$

Exercise 42. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\sqrt{(a_1 + \cdots + a_n)(b_1 + \cdots + b_n)} \geq \sqrt{a_1 b_1} + \cdots + \sqrt{a_n b_n}.$$

Exercise 43. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\frac{{a_1}^2}{b_1} + \cdots + \frac{{a_n}^2}{b_n} \geq \frac{(a_1 + \cdots + a_n)^2}{b_1 + \cdots + b_n}.$$

Exercise 44. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\frac{a_1}{{b_1}^2} + \cdots + \frac{a_n}{{b_n}^2} \geq \frac{1}{a_1 + \cdots + a_n} \left(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}\right)^2.$$

Exercise 45. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} \geq \frac{(a_1 + \cdots + a_n)^2}{a_1 b_1 + \cdots + a_n b_n}.$$

We now apply the Cauchy-Schwartz inequality to prove Nesbitt's inequality.

(**Nesbitt**) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 9. *Applying the Cauchy-Schwartz inequality, we have*

$$((b+c)+(c+a)+(a+b))\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq 3^2.$$

It follows that

$$\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}\geq\frac{9}{2}\quad or\quad 3+\sum_{\text{cyclic}}\frac{a}{b+c}\geq\frac{9}{2}.$$

Proof 10. *The Cauchy-Schwartz inequality yields*

$$\sum_{\text{cyclic}}\frac{a}{b+c}\sum_{\text{cyclic}}a(b+c)\geq\left(\sum_{\text{cyclic}}a\right)^2\quad or\quad \sum_{\text{cyclic}}\frac{a}{b+c}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}\geq\frac{3}{2}.$$

Here is an extremely short proof of the problem 12 :

(**Iran 1998**) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$,

$$\sqrt{x+y+z}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

Second Solution. We notice that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2\quad\Leftrightarrow\quad\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=1.$$

We now apply the Cauchy-Schwartz inequality to deduce

$$\sqrt{x+y+z}=\sqrt{(x+y+z)\left(\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}\right)}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

□

Problem 18. (**Gazeta Matematicã, Hojoo Lee**) *Prove that, for all $a, b, c > 0$,*

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4}\geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

Solution. We obtain the chain of equalities and inequalities

$$\begin{aligned}
\sum_{\text{cyclic}}\sqrt{a^4+a^2b^2+b^4} &= \sum_{\text{cyclic}}\sqrt{\left(a^4+\frac{a^2b^2}{2}\right)+\left(b^4+\frac{a^2b^2}{2}\right)}\\
&\geq \frac{1}{\sqrt{2}}\sum_{\text{cyclic}}\left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{b^4+\frac{a^2b^2}{2}}\right)\qquad(\text{Cauchy}-\text{Schwartz})\\
&= \frac{1}{\sqrt{2}}\sum_{\text{cyclic}}\left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{a^4+\frac{a^2c^2}{2}}\right)\\
&\geq \sqrt{2}\sum_{\text{cyclic}}\sqrt[4]{\left(a^4+\frac{a^2b^2}{2}\right)\left(a^4+\frac{a^2c^2}{2}\right)}\qquad(\text{AM}-\text{GM})\\
&\geq \sqrt{2}\sum_{\text{cyclic}}\sqrt{a^4+\frac{a^2bc}{2}}\qquad(\text{Cauchy}-\text{Schwartz})\\
&= \sum_{\text{cyclic}}\sqrt{2a^4+a^2bc}\,.
\end{aligned}$$

□

Using the same idea in the proof of the Cauchy-Schwartz inequality, we find a natural generalization :

Theorem 15. *Let $a_{ij}(i,j=1,\cdots,n)$ be positive real numbers. Then, we have*

$$({a_{11}}^n+\cdots+{a_{1n}}^n)\cdots({a_{n1}}^n+\cdots+{a_{nn}}^n)\ge(a_{11}a_{21}\cdots a_{n1}+\cdots+a_{1n}a_{2n}\cdots a_{nn})^n.$$

Proof. Since the inequality is homogeneous, as in the proof of the theorem 11, we can normalize to

$$({a_{i1}}^n+\cdots+{a_{in}}^n)^{\frac{1}{n}}=1 \text{ or } {a_{i1}}^n+\cdots+{a_{in}}^n=1 \ (i=1,\cdots,n).$$

Then, the inequality takes the form $a_{11}a_{21}\cdots a_{n1}+\cdots+a_{1n}a_{2n}\cdots a_{nn}\le 1$ or $\sum_{i=1}^n a_{i1}\cdots a_{in}\le 1$. Hence, it suffices to show that, for all $i=1,\cdots,n$,

$$a_{i1}\cdots a_{in}\le\frac{1}{n}, \text{ where } a_{i1}+\cdots+a_{in}=1.$$

To finish the proof, it remains to show the following *homogeneous* inequality : ☐

Theorem 16. (**AM-GM inequality**) *Let $a_1,\cdots,a_n$ be positive real numbers. Then, we have*

$$\frac{a_1+\cdots+a_n}{n}\ge(a_1\cdots a_n)^{\frac{1}{n}}.$$

Proof. Since it's homogeneous, we may rescale $a_1,\cdots,a_n$ so that $a_1\cdots a_n=1$. [2] Hence, we want to show that

$$a_1\cdots a_n=1 \implies a_1+\cdots+a_n\ge n.$$

The proof is by induction on n. If $n=1$, it's trivial. If $n=2$, then we get $a_1+a_2-2=a_1+a_2-2\sqrt{a_1a_2}=(\sqrt{a_1}-\sqrt{a_2})^2\ge 0$. Now, we assume that it holds for some positive integer $n\ge 2$. And let $a_1,\cdots,a_{n+1}$ be positive numbers such that $a_1\cdots a_na_{n+1}$=1. We may assume that $a_1\ge 1\ge a_2$. (Why?) Since $(a_1a_2)a_3\cdots a_n=1$, by the induction hypothesis, we have $a_1a_2+a_3+\cdots+a_{n+1}\ge n$. Thus, it suffices to show that $a_1a_2+1\le a_1+a_2$. However, we have $a_1a_2+1-a_1-a_2=(a_1-1)(a_2-1)\le 0$. ☐

The following simple observation is not tricky :

> Let $a,b>0$ and $m,n\in\mathbf{N}$. Take $x_1=\cdots=x_m=a$ and $x_{m+1}=\cdots=x_{x_{m+n}}=b$. Applying the AM-GM inequality to $x_1,\cdots,x_{m+n}>0$, we obtain
>
> $$\frac{ma+nb}{m+n}\ge(a^mb^n)^{\frac{1}{m+n}} \quad\text{or}\quad \frac{m}{m+n}a+\frac{n}{m+n}b\ge a^{\frac{m}{m+n}}b^{\frac{n}{m+n}}.$$
>
> Hence, for all positive *rationals* ω_1 and ω_2 with $\omega_1+\omega_2=1$, we get
>
> $$\omega_1\,a+\omega_2\,b\ge a^{\,\omega_1}b^{\,\omega_2}.$$

We immediately have

Theorem 17. *Let ω_1, $\omega_2>0$ with $\omega_1+\omega_2=1$. Then, for all x, $y>0$, we have*

$$\omega_1\,x+\omega_2\,y\ge x^{\,\omega_1}y^{\,\omega_2}.$$

Proof. We can choose a positive *rational* sequence $a_1,a_2,a_3,\cdots$ such that

$$\lim_{n\to\infty}a_n=\omega_1.$$

And letting $b_i=1-a_i$, we get

$$\lim_{n\to\infty}b_n=\omega_2.$$

From the previous observation, we have

$$a_n\,x+b_n\,y\ge x^{a_n}y^{b_n}$$

Now, taking the limits to both sides, we get the result. ☐

[2]Set $x_i=\frac{a_i}{(a_1\cdots a_n)^{\frac{1}{n}}}$ $(i=1,\cdots,n)$. Then, we get $x_1\cdots x_n=1$ and it becomes $x_1+\cdots+x_n\ge n$.

Modifying slightly the above arguments, we obtain

Theorem 18. (**Weighted AM-GM inequality**) *Let $\omega_1, \cdots, \omega_n$ be positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, for all $x_1, \cdots, x_n > 0$, we have*

$$\omega_1\, x_1 + \cdots + \omega_n\, x_n \geq {x_1}^{\omega_1} \cdots {x_n}^{\omega_n}.$$

Recall that the AM-GM inequality is used to deduce the theorem 12, which is a generalization of the Cauchy-Schwartz inequality. Since we now get the *weighted* version of the AM-GM inequality, we establish *weighted* version of the Cauchy-Schwartz inequality. It's called Hölder's Inequality :

Theorem 19. (**Hölder**) *Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have*

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j}.$$

Proof. Since the inequality is homogeneous, as in the proof of the theorem 12, we may rescale $x_{1j}, \cdots, x_{mj}$ so that $x_{1j} + \cdots + x_{mj} = 1$ for each $j \in \{1, \cdots, n\}$. Then, we need to show that

$$\prod_{j=1}^{n} 1^{\omega_j} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j} \quad \text{or} \quad 1 \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j}.$$

The weighted AM-GM inequality provides that

$$\sum_{j=1}^{n} \omega_j x_{ij} \geq \prod_{j=1}^{n} {x_{ij}}^{\omega_j} \ \ (i \in \{1, \cdots, m\}) \implies \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j x_{ij} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j}.$$

However, we immediately have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j x_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} \omega_j x_{ij} = \sum_{j=1}^{n} \omega_j \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} \omega_j = 1.$$

□

4.3 Homogenizations and Normalizations

Here, we present an inequality problem which is solved by the techniques we studied : *normalization* and *homogenization*.

Problem 19. (**IMO 1999/2**) *Let n be an integer with $n \geq 2$.*

(a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.
(b) For this constant C, determine when equality holds.

Solution. (**Marcin E. Kuczma**[3]) For $x_1 = \cdots = x_n = 0$, it holds for any $C \geq 0$. Hence, we consider the case when $x_1 + \cdots + x_n > 0$. Since the inequality is homogeneous, we may normalize to $x_1 + \cdots + x_n = 1$. We denote

$$F(x_1, \cdots, x_n) = \sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2).$$

From the assumption $x_1 + \cdots + x_n = 1$, we have

$$F(x_1, \cdots, x_n) = \sum_{1 \leq i < j \leq n} {x_i}^3 x_j + \sum_{1 \leq i < j \leq n} x_i {x_j}^3 = \sum_{1 \leq i \leq n} {x_i}^3 \sum_{j \neq i} x_i$$
$$= \sum_{1 \leq i \leq n} {x_i}^3 (1 - x_i) = \sum_{1 \leq i \leq n} x_i ({x_i}^2 - {x_i}^3).$$

We claim that $C = \frac{1}{8}$. It suffices to show that

$$F(x_1, \cdots, x_n) \leq \frac{1}{8} = F\left(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0\right).$$

Lemma 2. $0 \leq x \leq y \leq \frac{1}{2}$ *implies* $x^2 - x^3 \leq y^2 - y^3$.

Proof. Since $x + y \leq 1$, we get $x + y \geq (x+y)^2 \geq x^2 + xy + y^2$. Since $y - x \geq 0$, this implies that $y^2 - x^2 \geq y^3 - x^3$ or $y^2 - y^3 \geq x^2 - x^3$, as desired. □

Case 1. $\frac{1}{2} \geq x_1 \geq x_2 \geq \cdots \geq x_n$

$$\sum_{1 \leq i \leq n} x_i ({x_i}^2 - {x_i}^3) \leq \sum_{1 \leq i \leq n} x_i \left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 \right) = \frac{1}{8} \sum_{1 \leq i \leq n} x_i = \frac{1}{8}.$$

Case 2. $x_1 \geq \frac{1}{2} \geq x_2 \geq \cdots \geq x_n$ *Let $x_1 = x$ and $y = 1 - x = x_2 + \cdots + x_n$.*

$$F(x_1, \cdots, x_n) = x^3 y + \sum_{2 \leq i \leq n} x_i ({x_i}^2 - {x_i}^3) \leq x^3 y + \sum_{2 \leq i \leq n} x_i (y^2 - y^3) = x^3 y + y(y^2 - y^3).$$

Since $x^3 y + y(y^2 - y^3) = x^3 y + y^3(1 - y) = xy(x^2 + y^2)$, it remains to show that

$$xy(x^2 + y^2) \leq \frac{1}{8}.$$

Using $x + y = 1$, we homogenize the above inequality as following.

$$xy(x^2 + y^2) \leq \frac{1}{8}(x+y)^4.$$

However, we immediately find that $(x+y)^4 - 8xy(x^2 + y^2) = (x - y)^4 \geq 0$.

□

[3] I slightly modified his solution in [Au99].

4.4 Supplementary Problems for Chapter 4

Exercise 46. (**IMO unused 1991**) *Let n be a given integer with $n \geq 2$. Find the maximum value of*

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i + x_j),$$

where $x_1, \cdots, x_n \geq 0$ and $x_1 + \cdots + x_n = 1$.

Exercise 47. (**[PF], S. S. Wagner**) *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Suppose that $x \in [0, 1]$. Show that*

$$\left(\sum_{i=1}^{n} {a_i}^2 + 2x \sum_{i \neq j} a_i a_j\right) \left(\sum_{i=1}^{n} {b_i}^2 + 2x \sum_{i \neq j} b_i b_j\right) \geq \left(\sum_{i=1}^{n} a_i b_i + x \sum_{i \neq j} a_i b_j\right)^2.$$

Exercise 48. *Prove the Cauchy-Schwartz inequality for complex numbers [4]:*

$$\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \geq \left|\sum_{k=1}^{n} a_k b_k\right|^2.$$

Exercise 49. *Prove the complex version of the Lagrange's identity [5]:*

$$\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \left|\sum_{k=1}^{n} a_k b_k\right|^2 = \sum_{1 \leq s < t \leq n} |\ \overline{a_s} b_t - a_t \overline{b_s}\ |^2.$$

[4] $|a + bi| = \sqrt{a^2 + b^2}$ $(a, b \in \mathbf{R})$

[5] $\overline{a + bi} = a - bi$ $(a, b \in \mathbf{R})$

Chapter 5

Multivariable Inequalities

M 1. (**IMO short-listed 2003**) *Let $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n)$ be two sequences of positive real numbers. Suppose that $(z_1, z_2, \cdots, z_n)$ is a sequence of positive real numbers such that*

$$z_{i+j}{}^2 \geq x_i y_j$$

for all $1 \leq i, j \leq n$. Let $M = max\{z_2, \cdots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \cdots + z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

M 2. (**Bosnia and Herzegovina 2002**) *Let $a_1, \cdots, a_n, b_1, \cdots, b_n, c_1, \cdots, c_n$ be positive real numbers. Prove the following inequality :*

$$\left(\sum_{i=1}^{n} a_i{}^3\right)\left(\sum_{i=1}^{n} b_i{}^3\right)\left(\sum_{i=1}^{n} c_i{}^3\right) \geq \left(\sum_{i=1}^{n} a_i b_i c_i\right)^3.$$

M 3. (**C2113, Marcin E. Kuczma**) *Prove that inequality*

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \geq \sum_{i=1}^{n} (a_i + b_i) \sum_{i=1}^{n} \frac{a_i b_i}{a_i + b_i}$$

for any positive real numbers $a_1, \cdots, a_n, b_1, \cdots, b_n$

M 4. (**Yogoslavia 1998**) *Let $n > 1$ be a positive integer and $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Prove the following inequality.*

$$\left(\sum_{i \neq j} a_i b_j\right)^2 \geq \sum_{i \neq j} a_i a_j \sum_{i \neq j} b_i b_j.$$

M 5. (**C2176, Sefket Arslanagic**) *Prove that*

$$((a_1 + b_1) \cdots (a_n + b_n))^{\frac{1}{n}} \geq (a_1 \cdots a_n)^{\frac{1}{n}} + (b_1 \cdots b_n)^{\frac{1}{n}}$$

where $a_1, \cdots, a_n, b_1, \cdots, b_n > 0$

M 6. (**Korea 2001**) *Let $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ be real numbers satisfying*

$$x_1{}^2 + \cdots + x_n{}^2 = y_1{}^2 + \cdots + y_n{}^2 = 1$$

Show that

$$2\left|1 - \sum_{i=1}^{n} x_i y_i\right| \geq (x_1 y_2 - x_2 y_1)^2$$

and determine when equality holds.

M 7. (**Singapore 2001**) *Let* $a_1, \cdots, a_n, b_1, \cdots, b_n$ *be real numbers between* 1001 *and* 2002 *inclusive. Suppose that*

$$\sum_{i=1}^{n} {a_i}^2 = \sum_{i=1}^{n} {b_i}^2.$$

Prove that

$$\sum_{i=1}^{n} \frac{{a_i}^3}{b_i} \leq \frac{17}{10} \sum_{i=1}^{n} {a_i}^2.$$

Determine when equality holds.

M 8. (**[EWW-AI], Abel's inequality**) *Let* $a_1, \cdots, a_N, x_1, \cdots, x_N$ *be real numbers with* $x_n \geq x_{n+1} > 0$ *for all* n*. Show that*

$$|a_1 x_1 + \cdots + a_N x_N| \leq A x_1$$

where

$$A = max\{|a_1|, |a_1 + a_2|, \cdots, |a_1 + \cdots + a_N|\}.$$

M 9. (**China 1992**) *For every integer* $n \geq 2$ *find the smallest positive number* $\lambda = \lambda(n)$ *such that if*

$$0 \leq a_1, \cdots, a_n \leq \frac{1}{2}, \;\; b_1, \cdots, b_n > 0, \;\; a_1 + \cdots + a_n = b_1 + \cdots + b_n = 1$$

then

$$b_1 \cdots b_n \leq \lambda(a_1 b_1 + \cdots + a_n b_n).$$

M 10. (**C2551, Panos E. Tsaoussoglou**) *Suppose that* $a_1, \cdots, a_n$ *are positive real numbers. Let* $e_{j,k} = n-1$ *if* $j = k$ *and* $e_{j,k} = n-2$ *otherwise. Let* $d_{j,k} = 0$ *if* $j = k$ *and* $d_{j,k} = 1$ *otherwise. Prove that*

$$\sum_{j=1}^{n} \prod_{k=1}^{n} e_{j,k} {a_k}^2 \geq \prod_{j=1}^{n} \left(\sum_{k=1}^{n} d_{j,k} a_k \right)^2$$

M 11. (**C2627, Walther Janous**) *Let* $x_1, \cdots, x_n (n \geq 2)$ *be positive real numbers and let* $x_1 + \cdots + x_n$*. Let* $a_1, \cdots, a_n$ *be non-negative real numbers. Determine the optimum constant C(n) such that*

$$\sum_{j=1}^{n} \frac{a_j (s_n - x_j)}{x_j} \geq C(n) \left(\prod_{j=1}^{n} a_j \right)^{\frac{1}{n}}.$$

M 12. (**Hungary-Israel Binational Mathematical Competition 2000**) *Suppose that* k *and* l *are two given positive integers and* $a_{ij} (1 \leq i \leq k, 1 \leq j \leq l)$ *are given positive numbers. Prove that if* $q \geq p > 0$*, then*

$$\left(\sum_{j=1}^{l} \left(\sum_{i=1}^{k} {a_{ij}}^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{k} \left(\sum_{j=1}^{l} {a_{ij}}^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}.$$

M 13. (**[EWW-KI] Kantorovich inequality**) *Suppose* $x_1 < \cdots < x_n$ *are given positive numbers. Let* $\lambda_1, \cdots, \lambda_n \geq 0$ *and* $\lambda_1 + \cdots + \lambda_n = 1$*. Prove that*

$$\left(\sum_{i=1}^{n} \lambda_i x_i \right) \left(\sum_{i=1}^{n} \frac{\lambda_i}{x_i} \right) \leq \frac{A^2}{G^2},$$

where $A = \frac{x_1 + x_n}{2}$ *and* $G = \sqrt{x_1 x_n}$*.*

M 14. (**Czech-Slovak-Polish Match 2001**) *Let* $n \geq 2$ *be an integer. Show that*

$$({a_1}^3 + 1)({a_2}^3 + 1) \cdots ({a_n}^3 + 1) \geq ({a_1}^2 a_2 + 1)({a_2}^2 a_3 + 1) \cdots ({a_n}^2 a_1 + 1)$$

for all nonnegative reals $a_1, \cdots, a_n$*.*

M 15. (**C1868, De-jun Zhao**) *Let $n \geq 3$, $a_1 > a_2 > \cdots > a_n > 0$, and $p > q > 0$. Show that*

$$a_1{}^p a_2{}^q + a_2{}^p a_3{}^q + \cdots + a_{n-1}{}^p a_n{}^q + a_n{}^p a_1{}^q \geq a_1{}^q a_2{}^p + a_2{}^q a_3{}^p + \cdots + a_{n-1}{}^q a_n{}^p + a_n{}^q a_1{}^p$$

M 16. (**Baltic Way 1996**) *For which positive real numbers a, b does the inequality*

$$x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1 \geq x_1{}^a x_2{}^b x_3{}^a + x_2{}^a x_3{}^b x_4{}^a + \cdots + x_n{}^a x_1{}^b x_2{}^a$$

holds for all integers $n > 2$ and positive real numbers $x_1, \cdots, x_n$.

M 17. (**IMO short List 2000**) *Let $x_1, x_2, \cdots, x_n$ be arbitrary real numbers. Prove the inequality*

$$\frac{x_1}{1 + x_1{}^2} + \frac{x_2}{1 + x_1{}^2 + x_2{}^2} + \cdots + \frac{x_n}{1 + x_1{}^2 + \cdots + x_n{}^2} < \sqrt{n}.$$

M 18. (**MM1479, Donald E. Knuth**) *Let M_n be the maximum value of the quantity*

$$\frac{x_n}{(1 + x_1 + \cdots + x_n)^2} + \frac{x_2}{(1 + x_2 + \cdots + x_n)^2} + \cdots + \frac{x_1}{(1 + x_n)^2}$$

over all nonnegative real numbers $(x_1, \cdots, x_n)$. At what point(s) does the maximum occur ? Express M_n in terms of M_{n-1}, and find $\lim_{n\to\infty} M_n$.

M 19. (**IMO 1971**) *Prove the following assertion is true for $n = 3$ and $n = 5$ and false for every other natural number $n > 2$: if $a_1, \cdots, a_n$ are arbitrary real numbers, then*

$$\sum_{i=1}^{n} \prod_{i \neq j} (a_i - a_j) \geq 0.$$

M 20. (**IMO 2003**) *Let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers.*
(a) Prove that

$$\left(\sum_{1 \leq i,j \leq n} |x_i - x_j| \right)^2 \leq \frac{2(n^2 - 1)}{3} \sum_{1 \leq i,j \leq n} (x_i - x_j)^2.$$

(b) Show that the equality holds if and only if $x_1, x_2, \cdots, x_n$ is an arithmetic sequence.

M 21. (**Bulgaria 1995**) *Let $n \geq 2$ and $0 \leq x_1, \cdots, x_n \leq 1$. Show that*

$$(x_1 + x_2 + \cdots + x_n) - (x_1 x_2 + x_2 x_3 + \cdots + x_n x_1) \leq \left[\frac{n}{2}\right],$$

and determine when there is equality.

M 22. (**MM1407, Murry S. Klamkin**) *Determine the maximum value of the sum*

$$x_1{}^p + x_2{}^p + \cdots + x_n{}^p - x_1{}^q x_2{}^r - x_2{}^q x_3{}^r - \cdots x_n{}^q x_1{}^r,$$

where p, q, r are given numbers with $p \geq q \geq r \geq 0$ and $0 \leq x_i \leq 1$ for all i.

M 23. (**IMO Short List 1998**) *Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n < 1.$$

Prove that

$$\frac{a_1 a_2 \cdots a_n (1 - (a_1 + a_2 + \cdots + a_n))}{(a_1 + a_2 + \cdots + a_n)(1 - a_1)(1 - a_2) \cdots (1 - a_n)} \leq \frac{1}{n^{n+1}}.$$

M 24. (**IMO Short List 1998**) *Let $r_1, r_2, \cdots, r_n$ be real numbers greater than or equal to 1. Prove that*

$$\frac{1}{r_1 + 1} + \cdots + \frac{1}{r_n + 1} \geq \frac{n}{(r_1 \cdots r_n)^{\frac{1}{n}} + 1}.$$

M 25. (**Baltic Way 1991**) *Prove that, for any real numbers* $a_1, \cdots, a_n$,

$$\sum_{1 \le i,j \le n} \frac{a_i a_j}{i+j-1} \ge 0.$$

M 26. (**India 1995**) *Let* $x_1, x_2, \cdots, x_n$ *be positive real numbers whose sum is* 1. *Prove that*

$$\frac{x_1}{1-x_1} + \cdots + \frac{x_n}{1-x_n} \ge \sqrt{\frac{n}{n-1}}.$$

M 27. (**Turkey 1997**) *Given an integer* $n \ge 2$, *Find the minimal value of*

$$\frac{{x_1}^5}{x_2+x_3+\cdots+x_n} + \frac{{x_2}^5}{x_3+\cdots+x_n+x_1} + \cdots \frac{{x_n}^5}{x_1+x_3+\cdots+x_{n-1}}$$

for positive real numbers $x_1, \cdots, x_n$ *subject to the condition* ${x_1}^2 + \cdots + {x_n}^2 = 1$.

M 28. (**China 1996**) *Suppose* $n \in \mathbf{N}$, $x_0 = 0$, $x_1, \cdots, x_n > 0$, *and* $x_1 + \cdots + x_n = 1$. *Prove that*

$$1 \le \sum_{i=1}^{n} \frac{x_i}{\sqrt{1+x_0+\cdots+x_{i-1}}\sqrt{x_i+\cdots+x_n}} < \frac{\pi}{2}$$

M 29. (**Vietnam 1998**) *Let* $x_1, \cdots, x_n$ *be positive real numbers satisfying*

$$\frac{1}{x_1+1998} + \cdots + \frac{1}{x_n+1998} = \frac{1}{1998}.$$

Prove that

$$\frac{(x_1 \cdots x_n)^{\frac{1}{n}}}{n-1} \ge 1998$$

M 30. (**C2768 Mohammed Aassila**) *Let* $x_1, \cdots, x_n$ *be* n *positive real numbers. Prove that*

$$\frac{x_1}{\sqrt{x_1x_2+{x_2}^2}} + \frac{x_2}{\sqrt{x_2x_3+{x_3}^2}} + \cdots + \frac{x_n}{\sqrt{x_nx_1+{x_1}^2}} \ge \frac{n}{\sqrt{2}}$$

M 31. (**C2842, George Tsintsifas**) *Let* $x_1, \cdots, x_n$ *be positive real numbers. Prove that*

$$(a)\ \frac{{x_1}^n + \cdots + {x_n}^n}{nx_1 \cdots x_n} + \frac{n(x_1 \cdots x_n)^{\frac{1}{n}}}{x_1+\cdots+x_n} \ge 2,$$

$$(b)\ \frac{{x_1}^n + \cdots + {x_n}^n}{x_1 \cdots x_n} + \frac{(x_1 \cdots x_n)^{\frac{1}{n}}}{x_1+\cdots+x_n} \ge 1.$$

M 32. (**C2423, Walther Janous**) *Let* $x_1, \cdots, x_n (n \ge 2)$ *be positive real numbers such that* $x_1 + \cdots + x_n = 1$. *Prove that*

$$\left(1+\frac{1}{x_1}\right) \cdots \left(1+\frac{1}{x_n}\right) \ge \left(\frac{n-x_1}{1-x_1}\right) \cdots \left(\frac{n-x_n}{1-x_n}\right)$$

Determine the cases of equality.

M 33. (**C1851, Walther Janous**) *Let* $x_1, \cdots, x_n (n \ge 2)$ *be positive real numbers such that*

$${x_1}^2 + \cdots + {x_n}^2 = 1.$$

Prove that

$$\frac{2\sqrt{n}-1}{5\sqrt{n}-1} \le \sum_{i=1}^{n} \frac{2+x_i}{5+x_i} \le \frac{2\sqrt{n}+1}{5\sqrt{n}+1}.$$

M 34. (**C1429, D. S. Mitirinovic, J. E. Pecaric**) *Show that*

$$\sum_{i=1}^{n} \frac{x_i}{{x_i}^2 + x_{i+1}x_{i+2}} \le n-1$$

where $x_1, \cdots, x_n$ *are* $n \ge 3$ *positive real numbers. Of course,* $x_{n+1} = x_1, x_{n+2} = x_2$. [1]

M 35. (**Belarus 1998 S. Sobolevski**) *Let* $a_1 \le a_2 \le \cdots \le a_n$ *be positive real numbers. Prove the inequalities*

$$(a) \quad \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \ge \frac{a_1}{a_n} \cdot \frac{a_1 + \cdots + a_n}{n},$$

$$(b) \quad \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \ge \frac{2k}{1+k^2} \cdot \frac{a_1 + \cdots + a_n}{n},$$

where $k = \frac{a_n}{a_1}$.

M 36. (**Hong Kong 2000**) *Let* $a_1 \le a_2 \le \cdots \le a_n$ *be* n *real numbers such that*

$$a_1 + a_2 + \cdots + a_n = 0.$$

Show that

$${a_1}^2 + {a_2}^2 + \cdots + {a_n}^2 + na_1a_n \le 0.$$

M 37. (**Poland 2001**) *Let* $n \ge 2$ *be an integer. Show that*

$$\sum_{i=1}^{n} {x_i}^i + \binom{n}{2} \ge \sum_{i=1}^{n} ix_i$$

for all nonnegative reals $x_1, \cdots, x_n$.

M 38. (**Korea 1997**) *Let* $a_1, \cdots, a_n$ *be positive numbers, and define*

$$A = \frac{a_1 + \cdots + a_n}{n}, G = (a_1 \cdots {}_n)^{\frac{1}{n}}, H = \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}$$

(a) If n *is even, show that*

$$\frac{A}{H} \le -1 + 2\left(\frac{A}{G}\right)^n.$$

(b) If n *is odd, show that*

$$\frac{A}{H} \le -\frac{n-2}{n} + \frac{2(n-1)}{n}\left(\frac{A}{G}\right)^n.$$

M 39. (**Romania 1996**) *Let* $x_1, \cdots, x_n, x_{n+1}$ *be positive reals such that*

$$x_{n+1} = x_1 + \cdots + x_n.$$

Prove that

$$\sum_{i=1}^{n} \sqrt{x_i(x_{n+1} - x_i)} \le \sqrt{x_{n+1}(x_{n+1} - x_i)}$$

M 40. (**C2730, Peter Y. Woo**) *Let* $AM(x_1, \cdots, x_n)$ *and* $GM(x_1, \cdots, x_n)$ *denote the arithmetic mean and the geometric mean of the positive real numbers* $x_1, \cdots, x_n$ *respectively. Given positive real numbers* $a_1, \cdots, a_n, b_1, \cdots, b_n$, *(a) prove that*

$$GM(a_1 + b_1, \cdots, a_n + b_n) \ge GM(a_1, \cdots, a_n) + GM(b_1, \cdots, b_n).$$

For each real number $t \ge 0$, *define*

$$f(t) = GM(t + b_1, t + b_2, \cdots, t + b_n) - t$$

(b) Prove that f *is a monotonic increasing function, and that*

$$\lim_{t \to \infty} f(t) = AM(b_1, \cdots, b_n)$$

[1]Original version is to show that $sup \sum_{i=1}^{n} \frac{x_i}{{x_i}^2 + x_{i+1}x_{i+2}} = n-1$.

M 41. (**C1578, O. Johnson, C. S. Goodlad**) *For each fixed positive real number a_n, maximize*

$$\frac{a_1a_2\cdots a_n}{(1+a_1)(a_1+a_2)(a_2+a_3)\cdots(a_{n-1}+a_n)}$$

over all positive real numbers $a_1,\cdots,a_{n-1}$.

M 42. (**C1630, Isao Ashiba**) *Maximize*

$$a_1a_2+a_3a_4+\cdots+a_{2n-1}a_{2n}$$

over all permutations $a_1,\cdots,a_{2n}$ of the set $\{1,2,\cdots,2n\}$

M 43. (**C1662, Murray S. Klamkin**) *Prove that*

$$\frac{{x_1}^{2r+1}}{s-x_1}+\frac{{x_2}^{2r+1}}{s-x_2}+\cdots\frac{{x_n}^{2r+1}}{s-x_n}\geq\frac{4^r}{(n-1)n^{2r-1}}\left(x_1x_2+x_2x_3+\cdots+x_nx_1\right)^r$$

where $n>3$, $r\geq\frac{1}{2}$, $x_i\geq 0$ for all i, and $s=x_1+\cdots+x_n$. Also, Find some values of n and r such that the inequality is sharp.

M 44. (**C1674, Murray S. Klamkin**) *Given positive real numbers r,s and an integer $n>\frac{r}{s}$, find positive real numbers $x_1,\cdots,x_n$ so as to minimize*

$$\left(\frac{1}{{x_1}^r}+\frac{1}{{x_2}^r}+\cdots+\frac{1}{{x_n}^r}\right)(1+x_1)^s(1+x_2)^s\cdots(1+x_n)^s.$$

M 45. (**C1691, Walther Janous**) *Let $n\geq 2$. Determine the best upper bound of*

$$\frac{x_1}{x_2x_3\cdots x_n+1}+\frac{x_2}{x_1x_3\cdots x_n+1}+\cdots+\frac{x_n}{x_1x_2\cdots x_{n-1}+1}$$

over all $x_1,\cdots,x_n\in[0,1]$.

M 46. (**C1892, Marcin E. Kuczma**) *Let $n\geq 4$ be an integer. Find the exact upper and lower bounds for the cyclic sum*

$$\sum_{i=1}^{n}\frac{x_i}{x_{i-1}+x_i+x_{i+1}}$$

over all n-tuples of nonnegative numbers $x_1,\cdots,x_n$ such that $x_{i-1}+x_i+x_{i+1}>0$ for all i. Of course, $x_{n+1}=x_1$, $x_0=x_n$. Characterize all cases in which either one of these bounds is attained.

M 47. (**C1953, Murray S. Klamkin**) *Determine a necessary and sucient condition on real constants $r_1,\cdots,r_n$ such that*

$${x_1}^2+{x_2}^2+\cdot+{x_n}^2\geq(r_1x_1+r_2x_2+\cdots+r_nx_n)^2$$

holds for all real numbers $x_1,\cdots,x_n$.

M 48. (**C2018, Marcin E. Kuczma**) *How many permutations $(x_1,\cdots,x_n)$ of $\{1,2,\cdots,n\}$ are there such that the cyclic sum*

$$|x_1-x_2|+|x_2-x_3|+\cdots+|x_{n-1}-x_n|+|x_n-x_1|$$

is (a) a minimum, (b) a maximum ?

M 49. (**C2214, Walther Janous**) *Let $n\geq 2$ be a natural number. Show that there exists a constant $C=C(n)$ such that for all $x_1,\cdots,x_n\geq 0$ we have*

$$\sum_{i=1}^{n}\sqrt{x_i}\leq\sqrt{\prod_{i=1}^{n}(x_i+C)}$$

Determine the minimum $C(n)$ for some values of n. (For example, $C(2)=1$.)

M 50. (**C2615, Murray S. Klamkin**) *Suppose that* $x_1, \cdots, x_n$ *are non-negative numbers such that*

$$\sum {x_i}^2 \sum (x_i x_{i+1})^2 = \frac{n(n+1)}{2}$$

where e the sums here and subsequently are symmetric over the subscripts $\{1, \cdots, n\}$. *(a) Determine the maximum of* $\sum x_i$. *(b) Prove or disprove that the minimum of* $\sum x_i$ *is* $\sqrt{\frac{n(n+1)}{2}}$.

M 51. (**Turkey 1996**) *Given real numbers* $0 = x_1 < x_2 < \cdots < x_{2n}, x_{2n+1} = 1$ *with* $x_{i+1} - x_i \leq h$ *for* $1 \leq i \leq n$, *show that*

$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) < \frac{1+h}{2}.$$

M 52. (**Poland 2002**) *Prove that for every integer* $n \geq 3$ *and every sequence of positive numbers* $x_1, \cdots, x_n$ *at least one of the two inequalities is satsified :*

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}} \geq \frac{n}{2}, \quad \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + x_{i-2}} \geq \frac{n}{2}.$$

Here, $x_{n+1} = x_1, x_{n+2} = x_2, x_0 = x_n, x_{-1} = x_{n-1}$.

M 53. (**China 1997**) *Let* $x_1, \cdots, x_{1997}$ *be real numbers satisfying the following conditions:*

$$-\frac{1}{\sqrt{3}} \leq x_1, \cdots, x_{1997} \leq \sqrt{3}, x_1 + \cdots + x_{1997} = -318\sqrt{3}$$

Determine the maximum value of ${x_1}^{12} + \cdots + {x_{1997}}^{12}$.

M 54. (**C2673, George Baloglou**) *Let* $n > 1$ *be an integer. (a) Show that*

$$(1 + a_1 \cdots a_n)^n \geq a_1 \cdots a_n (1 + {a_1}^{n-2}) \cdots (1 + {a_1}^{n-2})$$

for all $a_1, \cdots, a_n \in [1, \infty)$ *if and only if* $n \geq 4$.

(b) Show that

$$\frac{1}{a_1(1 + {a_2}^{n-2})} + \frac{1}{a_2(1 + {a_3}^{n-2})} + \cdots + \frac{1}{a_n(1 + {a_1}^{n-2})} \geq \frac{n}{1 + a_1 \cdots a_n}$$

for all $a_1, \cdots, a_n > 0$ *if and only if* $n \leq 3$.

(c) Show that

$$\frac{1}{a_1(1 + {a_1}^{n-2})} + \frac{1}{a_2(1 + {a_2}^{n-2})} + \cdots + \frac{1}{a_n(1 + {a_n}^{n-2})} \geq \frac{n}{1 + a_1 \cdots a_n}$$

for all $a_1, \cdots, a_n > 0$ *if and only if* $n \leq 8$.

M 55. (**C2557, Gord Sinnamon,Hans Heinig**) *(a) Show that for all positive sequences* $\{x_i\}$

$$\sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{i=1}^{j} x_i \leq 2 \sum_{k=1}^{n} \left(\sum_{j=1}^{k} x_j \right)^2 \frac{1}{x_k}.$$

(b) Does the above inequality remain true without the factor 2? (c) What is the minimum constant c that can replace the factor 2 in the above inequality?

M 56. (**C1472, Walther Janous**) *For each integer* $n \geq 2$, *Find the largest constant* C_n *such that*

$$C_n \sum_{i=1}^{n} |a_i| \leq \sum_{1 \leq i < j \leq n} |a_i - a_j|$$

for all real numbers $a_1, \cdots, a_n$ *satisfying* $\sum_{i=1}^{n} a_i = 0$.

M 57. (**China 2002**) *Given* $c \in \left(\frac{1}{2}, 1\right)$. *Find the smallest constant* M *such that, for any integer* $n \geq 2$ *and real numbers* $1 < a_1 \leq a_2 \leq \cdots \leq a_n$, *if*

$$\frac{1}{n}\sum_{k=1}^{n} ka_k \leq c\sum_{k=1}^{n} a_k,$$

then

$$\sum_{k=1}^{n} a_k \leq M\sum_{k=1}^{m} ka_k,$$

where m *is the largest integer not greater than* cn.

M 58. (**Serbia 1998**) *Let* $x_1, x_2, \cdots, x_n$ *be positive numbers such that*

$$x_1 + x_2 + \cdots + x_n = 1.$$

Prove the inequality

$$\frac{a^{x_1-x_2}}{x_1+x_2} + \frac{a^{x_2-x_3}}{x_2+x_3} + \cdots \frac{a^{x_n-x_1}}{x_n+x_1} \geq \frac{n^2}{2},$$

holds true for every positive real number a. *Determine also when the equality holds.*

M 59. (**MM1488, Heinz-Jurgen Seiffert**) *Let* n *be a positive integer. Show that if* $0 < x_1 \leq x_2 \leq x_n$, *then*

$$\prod_{i=1}^{n}(1+x_i)\left(\sum_{j=0}^{n}\prod_{k=1}^{j}\frac{1}{x_k}\right) \geq 2^n(n+1)$$

with equality if and only if $x_1 = \cdots = x_n = 1$.

Chapter 6

References

AB K. S. Kedlaya, *$A < B$*, http://www.unl.edu/amc/a-activities/a4-for-students/s-index.html

AI D. S. Mitinovi'c (in cooperation with P. M. Vasić), *Analytic Inequalities*, Springer

AMN A. M. Nesbitt, Problem 15114, Educational Times (2) 3(1903), 37-38

AP A. Padoa, Period. Mat. (4)5 (1925), 80-85

Au99 A. Storozhev, *AMOC Mathematics Contests 1999*, Australian Mathematics Trust

EC E. Cesáro, Nouvelle Correspondence Math. 6(1880), 140

ESF Elementare Symmetrische Funktionen,
http://hydra.nat.uni-magdeburg.de/math4u/var/PU4.html

EWW-KI Eric W. Weisstein. "Kantorovich Inequality." From MathWorld–A Wolfram Web Resource.
http://mathworld.wolfram.com/KantorovichInequality.html

EWW-AI Eric W. Weisstein. "Abel's Inequality." From MathWorld–A Wolfram Web Resource.
http://mathworld.wolfram.com/AbelsInequality.html

GI O. Bottema, R. Ž̃. Djordjević, R. R. Janić, D. S. Mitrinović, P. M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen 1969

HFS H. F. Sandham, Problem E819, Amer. Math. Monthly 55(1948), 317

IN I. Niven, *Maxima and Minima Without Calculus*, MAA

IV Ilan Vardi, *Solutions to the year 2000 International Mathematical Olympiad*
http://www.lix.polytechnique.fr/Labo/Ilan.Vardi/publications.html

JC Ji Chen, Problem 1663, Crux Mathematicorum 18(1992), 188-189

JfdWm J. F. Darling, W. Moser, Problem E1456, Amer. Math. Monthly 68(1961) 294, 230

JmhMh J. M. Habeb, M. Hajja, *A Note on Trigonometric Identities*, Expositiones Mathematicae 21(2003), 285-290

KBS K. B. Stolarsky, *Cubic Triangle Inequalities*, Amer. Math. Monthly (1971), 879-881

KWL Kee-Wai Liu, Problem 2186, Crux Mathematicorum with Mathematical Mayhem, 23(1997), 71-72

MC M. Cipu, Problem 2172, Crux Mathematicorum with Mathematical Mayhem, 23(1997), 439-440

MCo M. Colind, Educational Times 13(1870), 30-31

MEK Marcin E. Kuczma, Problem 1940, Crux Mathematicorum with Mathematical Mayhem, 23(1997), 170-171

MP M. Petrović, Računanje sa brojnim razmacima, Beograd 1932, 79

MEK2 Marcin E. Kuczma, Problem 1703, Crux Mathematicorum 18(1992), 313-314

NC *A note on convexity*, Crux Mathematicorum with Mathematical Mayhem, 23(1997), 482-483

ONI T. Andreescu, V. Cirtoaje, G. Dospinescu, M. Lascu, *Old and New Inequalities*

PF P. Flor, *Über eine Ungleichung von S. S. Wagner*, Elem. Math. 20, 136(1965)

RAS R. A. Satnoianu, *A General Method for Establishing Geometric Inequalities in a Triangle*, Amer. Math. Monthly 108(2001), 360-364

RI K. Wu, Andy Liu, *The Rearrangement Inequality*, ??

RS R. Sondat, Nouv. Ann. Math. (3) 10(1891), 43-47

SR S. Rabinowitz, *On The Computer Solution of Symmetric Homogeneous Triangle Inequalities*, Proceedings of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation (ISAAC 89), 272-286

SR2 S. Reich, Problem E1930, Amer. Math. Monthly 73(1966), 1017-1018

TZ T. Andreescu, Z. Feng, *103 Trigonometry Problems From the Training of the USA IMO Team*, Birkhauser

WJB W. J. Blundon, Canad. Math. Bull. 8(1965), 615-626

WJB2 W. J. Blundon, Problem E1935, Amer. Math. Monthly 73(1966), 1122

ZsJc Zun Shan, Ji Chen, Problem 1680, Crux Mathematicorum 18(1992), 251