Problems in this Issue (Tap chi 3T)

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Problem 1. Find all postive integers n such that $1009 < n < 2009$ and n has exactly twelve factors one of which is 17.

Problem 2. Let x, y be real numbers which satisfy

$$x^3 + y^3 - 6(x^2 + y^2) + 13(x + y) - 20 = 0.$$

Find the numerical value of $A = x^3 + y^3 + 12xy$.

Problem 3. Let x, y be non-negative real numbers that satisfy $x^2 - 2xy + x - 2y \geq 0$. Find the greatest value of $M = x^2 - 5y^2 + 3x$.

Problem 4. Let $ABCD$ be a parallelogram. M is a point on the side AB such that $AM = \frac{1}{3}AB$, N is the mid-point of CD, G is the centroid of $\triangle BMN$, I is the intersection of AG and BC. Compute GA/GI and IB/IC.

Problem 5. Suppose that d is a factor of n^4+2n^2+2 such that $d > n^2+1$, where n is some natural number $n > 1$. Prove that $d > n^2+1+\sqrt{n^2+1}$.

Problem 6. Solve the simultaneous equations

$$\frac{1}{xy}+\frac{1}{yz}+\frac{1}{z}=2, \quad \frac{2}{xy^2z}-\frac{1}{z^2}=4.$$

Problem 7. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Prove that

$$\frac{ab}{c+1}+\frac{bc}{a+1}+\frac{ca}{b+1} \leq \frac{1}{4}.$$

Problem 8. Given a triangle ABC, d is a variable line that intersects AB, AC at M, N respectively such that $AB/AM+AC/AN=2009$. Prove that d has a fixed point.

Problem 9. Find all three-digit natural numbers that possess the following property: sum of digits of each number is 9, the right-most digit is 2 units less than its tens digit, and if the left-most digit and the right-most digit in each number are swapped, then the resulting number is 198 units greater than the original number.

Problem 10. Find the least value of the expression $f(x) = 6|x-1| + |3x-2| + 2x$.

Problem 11. Let a, b be positive real numbers. Prove that

$$\left(1+\frac{1}{a}\right)^4 + \left(1+\frac{1}{b}\right)^4 + \left(1+\frac{1}{c}\right)^4 \geq 3\left(1+\frac{3}{2+abc}\right)^4.$$

Problem 12. Let $ABCD$ be a trapzium with parallel sides AB, CD. Suppose that M is a point on the side AD and N is interior to the trapezium such that $\angle NBC = \angle MBA$, $\angle NCB = \angle MCD$. Let P be the fourth vertex of the parallelogram $MANP$. Prove that P is on the side CD.

Problem 13. Find all right-angled triangles that each have integral side lengths and the area is equal to the perimeter.

Problem 14. Find the least value of $A = x^2 + y^2$, where x, y are positive integers such that A is divisible by 2010.

Problem 15. Let x, y be positive real numbers such that $x^3 + y^3 = x - y$. Prove that $x^2 + 4y^2 < 1$.

Problem 16. Pentagon $ABCDE$ is inscribed in a circle. Let a, b, c denote the perpendicular distance from E to the lines AB, BC and CD. Compute the distance from E to the line AD in terms of a, b, c.

Problem 17. Let $a = 123456789$ and $b = 987654321$.

1. Find the greatest common factor of a and b.

2. Find the remainder when the least common multiple of a, b is divided by 11.

Problem 18. Solve the simultaneous equations

$$\frac{xy}{2} + \frac{5}{2x + y - xy} = 5,\ 2x + y + \frac{10}{xy} = 4 + xy.$$

Problem 19. Let x, y be real numbers such that $x \geq 2$, $x + y \geq 3$. Find the least value of the expression

$$P = x^2 + y^2 + \frac{1}{x} + \frac{1}{x + y}.$$

Problem 20. Triangle ABC is right isosceles with $AB = AC$. M is a point on the side AC such that $MC = 2MA$. The line through M that is perpendicular to BC meets AB at D. Compute the distance from point B to the line CD in terms of $AB = a$.

Problem 21. Let n be a positive integer and $x_1, x_2, ..., x_{n-1}$ and x_n be integers such that $x_1 + x_2 + \cdots + x_n = 0$ and $x_1 x_2 \cdots x_n = n$. Prove that n is a multiple of 4.

Problem 22. Find all natural numbers a, b, n such that $a + b = 2^{2007}$ and $ab = 2^n - 1$, where a, b are odd numbers and $b > a > 1$.

Problem 23. Solve the equation

$$x + 2 = 3\sqrt{1 - x^2} + \sqrt{1 + x}.$$

Problem 24. Let a, b, c be positive real numbers whose sum is 2. Find the greatest value of

$$\frac{a}{ab + 2c} + \frac{b}{bc + 2a} + \frac{c}{ca + 2b}.$$

Problem 25. Let ABC be a right-angled triangle with hypotenuse BC and altitude AH. I is the midpoint of BH, K is a point on the opposite ray of AB such that $AK = BI$. Draw a circle with center O circumscribing the triangle IKC. A tangent of O, touching O at I, intersects KC at P. Another tangent PM of the circle is drawn. Compute the ratio $\frac{MI}{MK}$.

Problem 26. Evaluate the sum

$$S = \frac{4+\sqrt{3}}{\sqrt{1}+\sqrt{3}} + \frac{6+\sqrt{8}}{\sqrt{3}+\sqrt{5}} + \cdots + \frac{2n+\sqrt{n^2-1}}{\sqrt{n-1}+\sqrt{n+1}} + \cdots + \frac{240+\sqrt{14399}}{\sqrt{119}+\sqrt{121}}.$$

Problem 27. Solve the equation

$$\sqrt{6x+10x} = x^2 - 13x + 12.$$

Problem 28. Let x, y, z be real numbers $(x+1)^2 + (y+2)^2 + (z+3)^2 \leq 2010$. Find the least value of

$$A = xy + y(z-1) + z(x-2).$$

Problem 29. A triangle ABC has $AC = 3AB$ and the size of $\angle A$ is $60°$. On the side BC, D is chosen such that $\angle ADB = 30°$. The line through D that is perpendicular to AD intersects AB at E. Prove that triangle ACE is equilateral.

Problem 30. Compare the algebraic value of

$$\frac{\sqrt{2}}{2\sqrt[3]{1}+\sqrt[3]{2^2.1^2}+1\sqrt[3]{2}}+\frac{\sqrt{2}}{3\sqrt[3]{2}+\sqrt[3]{3^2.2^2}+2\sqrt[3]{3}}+\cdots+\frac{\sqrt{2}}{1728\sqrt[3]{1727}+\sqrt[3]{1728^2.1727^2}+1727\sqrt[3]{1728}}$$

and $\frac{11}{7}$.

Problem 31. Find all possible values of m, n such that the simultaneous equations have a unique solution

$$\begin{aligned} xyz + z &= m, \\ xyz^2 + z &= n, \\ x^2 + y^2 + z^2 &= 4. \end{aligned}$$

Problem 32. Let x be a positive real number. Find the minimum value of

$$P = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)^2 + 1.$$

Problem 33. A quadrilateral $ABCD$ has $\angle BCD = \angle BDC = 50°$, $\angle ACD = \angle ADB = 30°$. Let AC intersect BD at I. Prove that ABI is an isosceles triangle.

Problem 34. Solve the equation in the set of integers

$$x^3 - (x+y+z)^2 = (y+z)^2 + 34.$$

Problem 35. Solve the equation

$$x^2 - 3x + 9 = 9\sqrt[3]{x-2}.$$

Problem 36. Solve the system of equations

$$\begin{aligned} \sqrt{2x+3} + \sqrt{2y+3} + \sqrt{2z+3} &= 9, \\ \sqrt{x-2} + \sqrt{y-2} + \sqrt{z-2} &= 3. \end{aligned}$$

Problem 37. Given that $a, b, c \geq 1$, prove that

$$abc + 6029 \geq 2010\left(\sqrt[2010]{a} + \sqrt[2010]{b} + \sqrt[2010]{c}\right).$$

Problem 38. ABC is an isosceles triangle with $AB = AC$. Let D, E be the midpoints of AB and AC. M is a variable point on the line DE. A circle with center O touches AB, AC at B and C respectively. A circle with diameter OM cuts (O) at N, K. Find the location of M such that the radius of the circumcircle of triangle ANK is a minimum.

Problem 39. A circle with center I is inscribed in triangle ABC, touching the sides BC, CA, and AB at A_1, B_1, and C_1 respectively. C_1K is the diameter of (I). A_1K cuts B_1C_1 at D, CD meets C_1A_1 at P. Prove that

a) $CD \parallel AB$

b) P, K, B_1 are collinear.

Problem 40. For each positive integer n, let

$$S_n = \frac{1}{5} + \frac{3}{85} + \frac{5}{629} + \cdots + \frac{2n-1}{16n^4 - 32n^3 + 24n^2 - 8n + 5}.$$

Compute the value of S_{100}.

Problem 41. Find the value of

$$\frac{(xy + 2z^2)(yz + 2x^2)(zx + 2y^2)}{(2xy^2 + 2yz^2 + 2zx^2 + 3xyz)^2},$$

if x, y, z are real numbers satisfying $x + y + z = 0$.

Problem 42. Solve the equation

$$2x^2 + 3\sqrt[3]{x^3 - 9} = \frac{10}{x}.$$

Problem 43. Let m, n be constants and a, b be real numbers such that

$$m \leq n \leq 2m,\ 0 < a \leq b \leq m,\ a + b \leq n.$$

Find the greatest value of $S = a^2 + b^2$.

Problem 44. Let ABC be a right triangle with hypotenuse BC. A square $MNPQ$ is inscribed in the triangle such that M is on the side AB, N is on the side AC and P, Q are on the side BC. Let BN meet MQ at E, CM intersect NP at F. Prove that $AE = AF$ and $\angle EAB = \angle FAC$.

Problem 45. Let BC be a fixed chord of a circle with center O and radius R ($BC \neq 2R$). A is a variable point on the major arc BC. The bisector of $\angle BAC$ meets BC at D. Let r_1 and r_2 be the radius of the incircles of triangles ADB and DAC, respectively. Determine the location of A such that $r_1 + r_2$ is a maximum.

Problem 46. A natural number is said to be *intriguing* if it is a multiple of 11111 and all of its digits are distinct. Find the number of intriguing numbers that have ten digits each.

Problem 47. Find all the digits a, b, c such that $\sqrt{\overline{abc}} - \sqrt{\overline{acb}} = 1$.

Problem 48. Find the greatest and the least value of $y = \sqrt{x+1} + \sqrt{5-4x}$.

Problem 49. Let a, b, c be positive real numbers such that $a \neq c$ and $a + \sqrt{b + \sqrt{c}} = c + \sqrt{b + \sqrt{a}}$. Prove that $ac < \frac{1}{40}$.

Problem 50. ABC is an isosceles triangle with $AB = AC$. Let M, D be the midpoints of BC and AM. Let H be the perpendicular projection of M onto CD. AH meets BC at N, BH intersects AM at E. Prove that E is the orthocenter of triangle ABN.

Problem 51. Let $ABCDE$ be a convex pentagon. Triangles ABC, BCD, CDE and DEA each have area $\sqrt{2010}$. Find the area of the pentagon.

Problem 52. Without the aid of a calculator, compare the value of

$$A = \sqrt{2008} + \sqrt{2009} + \sqrt{2010}, \; B = \sqrt{2005} + \sqrt{2007} + \sqrt{2015}.$$

Problem 53. Solve the equation

$$x^3 - 2012x^2 + 1012037x - \sqrt{2x - 2011} - 1005 = 0.$$

Problem 54. Solve the system of equations

$$\begin{cases} \sqrt{335x - 2010} & = 12 - y^2, \\ xy & = x^2 + 3. \end{cases}$$

Problem 55. Let a, b, c be non-negative real numbers that adds up to 1. Find the minimum value of

$$P = a^2 + b^2 + c^2 + \frac{abc}{2}.$$

Problem 56. Triangle ABC is right at A and $AB = 3AC$. M is a point in the interior of the triangle such that $MA : MB : MC = 1 : 4 : \sqrt{2}$. Find the measure of angle BMC.

Problem 57. ABC is a triangle. Points K, N and M are the midpoints of AB, BC and AK. Prove that the perimeter of triangle AKC is greater than that of triangle CMN.

Problem 58. A natural number is said to be *intriguing* if it is a multiple of 11111 and all of its digits are distinct. Find the number of intriguing numbers that have ten digits each.

Problem 59. Find all the digits a, b, c such that $\sqrt{abc} - \sqrt{acb} = 1$.

Problem 60. Find the greatest and the least value of $y = \sqrt{x+1} + \sqrt{5-4x}$.

Problem 61. Let a, b, c be positive real numbers such that $a \neq c$ and $a + \sqrt{b+\sqrt{c}} = c + \sqrt{b+\sqrt{a}}$. Prove that $ac < \frac{1}{40}$.

Problem 62. ABC is an isosceles triangle with $AB = AC$. Let M, D be the midpoints of BC and AM. Let H be the perpendicular projection of M onto CD. AH meets BC at N, BH intersects AM at E. Prove that E is the orthocenter of triangle ABN.

Problem 63. Let $ABCDE$ be a convex pentagon. Triangles ABC, BCD, CDE and DEA each have area $\sqrt{2010}$. Find the area of the pentagon.

Problem 64. Solve for integers x, y

$$2008x^2 - 199y^2 = 2008.2009.2010.$$

Problem 65. For each real number x, we denote by $[x]$ the greatest integer not exceeding x. Prove that

$$\left[\sqrt{n} + \frac{1}{2}\right] = \left[\sqrt{n - \frac{3}{4}} + \frac{1}{2}\right].$$

Problem 66. Solve the system of equations

$$\begin{cases} 8x^2 + \frac{1}{\sqrt{y}} & = \frac{5}{2}, \\ 8y^2 + \frac{1}{\sqrt{x}} & = \frac{5}{2}. \end{cases}$$

Problem 67. Positive real numbers satisfy the relation $\sqrt{a^2+b^2} + \sqrt{b^2+c^2} + \sqrt{a^2+c^2} = 3\sqrt{2}$. Find the minimum value of the expression

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}.$$

Problem 68. ABC is an equilateral triangle. M is a point inside the triangle such that $MA^2 = MB^2 + MC^2$. Compute the area of triangle ABC in terms of the length of MB and MC.

Problem 69. ABC is a triangle. The angle bisectors BE and CF meet each other at I. AI meets EF at M. A line through M, parallel to BC, intersects AB and AC at N, P. Prove that $3NP > MB + MC$.

Problem 70. For a positive integer k, denote by $k! = 1 \times 2 \times \cdots \times k$. Given an integer $n > 3$, prove that

$$A_n = 1! + 2! + \cdots + n!$$

can not be written in the form a^b, where a, b are integers and $b > 1$.

Problem 71. Solve the integer equation

$$\frac{x+y}{x^2 - xy + y^2} = \frac{3}{7}.$$

Problem 72. Solve the equation

$$x^4 + 4x^3 + 5x^2 + 2x - 10 = 12\sqrt{x^2 + 2x + 5}.$$

Problem 73. Let a, b, c be positive real numbers such that $a \geq b \geq c$ and $3a - 4b + c = 0$. Find the minimum value of

$$M = \frac{a^2 - b^2}{c} - \frac{b^2 - c^2}{a} - \frac{c^2 - a^2}{b}.$$

Problem 74. Triangle ABC is isosceles at A and $\angle BAC = 40°$. Point M is inside the triangle such that $\angle MBC = 40°$, $\angle MCB = 20°$. Find the measure of $\angle MAB$.

Problem 75. Let O be a center with two mutually perpendicular diameters AB and CD. E is a point on the minor arc BD, E is distinct from B and D). AE meets CD at M, CE meets AB at N. Prove that

$$\frac{MD}{MO} + \frac{NB}{NO} \geq 2\sqrt{2}.$$

Problem 76. Solve the integer equation

$$(|x-y|+|x+y|)^3 = x^3 + |y|^3 + 6.$$

Problem 77. Let a, b, c be real numbers distinct from 0. Find all real numbers x, y, z such that

$$\frac{xy}{ay+bx} = \frac{yz}{bz+cy} = \frac{zx}{cx+az} = \frac{x^2+y^2+z^2}{a^2+b^2+c^2}.$$

Problem 78. Solve the system of equations

$$\frac{2x^2}{x^2+1} = y, \ \frac{3y^2}{y^4+y^2+1} = z, \ \frac{4z^2}{z^6+z^4+z^2+1} = x.$$

Problem 79. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2+b^2}{(a+b)^2} + \frac{b^2+c^2}{(b+c)^2} + \frac{c^2+a^2}{(c+a)^2} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq \frac{5}{2}.$$

Problem 80. Given two equilateral triangles ABC, $A'B'C'$ overlapping each other in such a way that the intersections of the sides form a regular hexagon, find the minimum value of the perimeter of the hexagon if the side-lengths of the two triangles are x, y.

Problem 81. Let BC be a fixed chord of a circle with center O; BC is not a diameter. M is the midpoint of the chord BC, A is a point that varies on the major arc BC, D is the intersection of AM and the minor arc BC, N is the intersection of AB and CD. Prove that N is on a fixed line when A moves on the major arc BC.