



A Note on Cosine Power Sums

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Abstract

Using the multisection series method, we establish formulas for various power sums of cosine functions. As corollaries we derive several combinatorial identities.

1 Introduction

In [2, 3], we presented two open problems concerning the asymptotic behaviour of cosine power sums,

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^p \left(\frac{k\pi}{n} \right), \quad (1)$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x . Under certain conditions, these cosine power sums can be determined exactly, without approximations [5].

If

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

is a finite or convergent infinite series, then for $0 \leq r < n$ the sum

$$a_rx^r + a_{r+n}x^{r+n} + a_{r+2n}x^{r+2n} + \cdots$$

is given by the multisection formula (see [1, Ch. 16], [4, Ch. 4, S. 4.3] and [6])

$$\sum_{k \geq 0} a_{r+kn} x^{r+kn} = \frac{1}{n} \sum_{k=0}^{n-1} z^{-kr} f(z^k x), \quad (2)$$

where $z = e^{\frac{2\pi i}{n}}$ is the n th root of 1.

We shall use the multisection formula to prove:

Theorem 1. *Let n and p be two positive integers. Then*

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^{2p} \left(\frac{k\pi}{n} \right) = -\frac{1}{2} + \frac{n}{2^{2p+1}} \sum_{k=-\lfloor \frac{p}{n} \rfloor}^{\lfloor \frac{p}{n} \rfloor} \binom{2p}{p+kn}.$$

Theorem 2. *Let n and p be two positive integers such that $n \equiv p \pmod{2}$. Then*

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^p \left(\frac{k\pi}{n} \right) = -\frac{1}{2} + \frac{n}{2^{p+1}} \sum_{k=-\lfloor \frac{p}{2n} \rfloor}^{\lfloor \frac{p}{2n} \rfloor} \binom{p}{\frac{p+n}{2} + kn},$$

where $[x]$ denotes the nearest integer to x , i.e., $[x] = \lfloor x + \frac{1}{2} \rfloor$.

Theorem 3. *Let n and p be two positive integers. Then*

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \cos^{2p} \left(\frac{2k-1}{n} \cdot \frac{\pi}{2} \right) = \frac{n}{2^{2p+1}} \sum_{k=-\lfloor \frac{p}{n} \rfloor}^{\lfloor \frac{p}{n} \rfloor} (-1)^k \binom{2p}{p+kn}.$$

We note that the formula proved in [5] is the case $p < n$ of Theorem 1. The conjecture published in [3] is solved by Theorem 2. In the final section of this paper we present some relations that can be obtained immediately by the above theorems.

2 Proof of Theorem 1

We apply the multisection formula (2) for

$$r = p - n \left\lfloor \frac{p}{n} \right\rfloor \quad \text{and} \quad f(x) = (1+x)^{2p}.$$

Thus, when $x = 1$, we get

$$\sum_{k=-\lfloor \frac{p}{n} \rfloor}^{\lfloor \frac{p}{n} \rfloor} \binom{2p}{p+kn} = \frac{1}{n} \sum_{k=0}^{n-1} z^{-k(p-n\lfloor \frac{p}{n} \rfloor)} (1+z^k)^{2p}, \quad (3)$$

where $z = e^{\frac{2\pi i}{n}}$. Having

$$\begin{aligned} e^{-ir\varphi} (1 + e^{i\varphi})^p &= e^{-ir\varphi} \left(e^{\frac{i\varphi}{2} - \frac{i\varphi}{2}} + e^{\frac{i\varphi}{2} + \frac{i\varphi}{2}} \right)^p \\ &= e^{-ir\varphi} e^{\frac{ip\varphi}{2}} \left(e^{-\frac{i\varphi}{2}} + e^{\frac{i\varphi}{2}} \right)^p \\ &= 2^p \cos^p \frac{\varphi}{2} \cdot e^{i(\frac{p}{2}-r)\varphi}, \end{aligned} \quad (4)$$

we can write

$$z^{-k(p-n\lfloor \frac{p}{n} \rfloor)} (1+z^k)^{2p} = 2^{2p} \cos^{2p} \left(\frac{k\pi}{n} \right) \cdot e^{i2k\lfloor \frac{p}{n} \rfloor \pi}.$$

So, we may rewrite (3) as

$$\sum_{k=-\lfloor \frac{p}{n} \rfloor}^{\lfloor \frac{p}{n} \rfloor} \binom{2p}{p+kn} = \frac{2^{2p}}{n} \sum_{k=0}^{n-1} \cos^{2p} \left(\frac{k\pi}{n} \right), \quad (5)$$

Taking into account that

$$\cos \left(\frac{(n-k)\pi}{n} \right) = -\cos \left(\frac{k\pi}{n} \right), \quad k = 0, 1, \dots, n,$$

Theorem 1 is proved.

3 Proof of Theorem 2

To prove the theorem we proceed as in Theorem 1. By the multisection formula (2), with

$$x = 1, \quad f(x) = (1+x)^p \quad \text{and} \quad r = \frac{p+n}{2} - n \left\lfloor \frac{p+n}{2n} \right\rfloor,$$

we get

$$\sum_{k=-\lfloor \frac{p+n}{2n} \rfloor}^{\lfloor \frac{p+n}{2n} \rfloor} \binom{p}{\frac{p+n}{2} + nk} = \frac{1}{n} \sum_{k=0}^{n-1} z^{-k(\frac{p+n}{2} - n\lfloor \frac{p+n}{2n} \rfloor)} (1+z^k)^p,$$

where $z = e^{\frac{2\pi i}{n}}$. By (4), it immediately follows that

$$z^{-k(\frac{p+n}{2} - n\lfloor \frac{p+n}{2n} \rfloor)} (1+z^k)^p = 2^p \cos^p \left(\frac{k\pi}{n} \right) \cdot e^{ik\pi(2\lfloor \frac{p+n}{2n} \rfloor - 1)}.$$

Thus we can write

$$\sum_{k=-\lfloor \frac{p+n}{2n} \rfloor}^{\lfloor \frac{p+n}{2n} \rfloor} \binom{p}{\frac{p+n}{2} + kn} = \frac{2^p}{n} \sum_{k=0}^{n-1} (-1)^k \cos^p \left(\frac{k\pi}{n} \right). \quad (6)$$

On the other hand, we have

$$\begin{aligned} & \sum_{k=0}^{n-1} (-1)^k \cos^p \left(\frac{k\pi}{n} \right) \\ &= 1 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^p \left(\frac{k\pi}{n} \right) + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-k} \cos^p \left(\frac{(n-k)\pi}{n} \right) \\ &= 1 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^p \left(\frac{k\pi}{n} \right) + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n+p-k} \cos^p \left(\frac{k\pi}{n} \right) \\ &= 1 + 2 \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^p \left(\frac{k\pi}{n} \right). \end{aligned} \quad (7)$$

According to (3) and (7), Theorem 2 is proved.

4 Proof of Theorem 3

To prove the theorem, we use the formula

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} f(2k-1) = \frac{1}{2} \left(\sum_{k=1}^{n-1} f(k) - \sum_{k=1}^{n-1} (-1)^k f(k) \right) \quad (8)$$

for $f(k) = \cos^{2p} \left(\frac{k\pi}{2n} \right)$. By Theorem 1 (with n replaced by $2n$) and Theorem 2 (with n and p replaced by $2n$ and $2p$, respectively), we have

$$\begin{aligned} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} f(2k-1) &= \frac{1}{2} \left(-\frac{1}{2} + \frac{2n}{2^{2p+1}} \sum_{k=-\lfloor \frac{p}{2n} \rfloor}^{\lfloor \frac{p}{2n} \rfloor} \binom{2p}{p+2kn} \right. \\ &\quad \left. + \frac{1}{2} - \frac{2n}{2^{2p+1}} \sum_{k=-\lfloor \frac{p}{2n} \rfloor}^{\lfloor \frac{p}{2n} \rfloor} \binom{2p}{p+(2k+1)n} \right) \\ &= \frac{n}{2^{2p+1}} \sum_{k=-\lfloor \frac{p}{2n} \rfloor}^{\lfloor \frac{p}{2n} \rfloor} \left(\binom{2p}{p+2kn} - \binom{2p}{p+(2k+1)n} \right) \\ &= \frac{n}{2^{2p+1}} \sum_{k=-\lfloor \frac{p}{2n} \rfloor}^{\lfloor \frac{p}{2n} \rfloor} (-1)^k \binom{2p}{p+kn}. \end{aligned}$$

Theorem 3 is proved.

5 Special cases

We begin this section with two immediate consequences of Theorems 1, 2 and 3.

Corollary 4. *Let n be a positive integer. Then*

1. $\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^{2n} \left(\frac{k\pi}{n} \right) = -\frac{1}{2} + \frac{n}{2^{2n}} + \frac{n}{2^{2n+1}} \binom{2n}{n};$
2. $\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^n \left(\frac{k\pi}{n} \right) = -\frac{1}{2} + \frac{n}{2^n};$
3. $\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \cos^{2n} \left(\frac{2k-1}{n} \cdot \frac{\pi}{2} \right) = -\frac{n}{2^{2n}} + \frac{n}{2^{2n+1}} \binom{2n}{n}.$

Corollary 5. *Let n and p be two positive integers such that $p < n$. Then*

$$\begin{aligned}
1. \quad & \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^{2p} \left(\frac{k\pi}{n} \right) = -\frac{1}{2} + \frac{n}{2^{2p+1}} \binom{2p}{p}; \\
2. \quad & \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^p \left(\frac{k\pi}{n} \right) = -\frac{1}{2}, \quad n \equiv p \pmod{2}; \\
3. \quad & \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \cos^{2p} \left(\frac{2k-1}{n} \cdot \frac{\pi}{2} \right) = \frac{n}{2^{2p+1}} \binom{2p}{p}.
\end{aligned}$$

By Theorems 1, 2 and 3, for fixed values of n , we can obtain some combinatorial identities.

Corollary 6. *Let p be a positive integer. Then*

$$\begin{aligned}
1. \quad & \sum_{k=0}^p \binom{2p}{p-k} = \frac{1}{2} \binom{2p}{p} + 2^{2p-1}; \\
2. \quad & \sum_{k=0}^{\lfloor p/2 \rfloor} \binom{2p}{p-2k} = \frac{1}{2} \binom{2p}{p} + 2^{2p-2}; \\
3. \quad & \sum_{k=0}^{\lfloor p/3 \rfloor} \binom{2p}{p-3k} = \frac{1}{2} \binom{2p}{p} + \frac{1+2^{2p-1}}{3}; \\
4. \quad & \sum_{k=0}^{\lfloor p/4 \rfloor} \binom{2p}{p-4k} = \frac{1}{2} \binom{2p}{p} + 2^{2p-3} + 2^{p-2}; \\
5. \quad & \sum_{k=0}^{\lfloor p/5 \rfloor} \binom{2p}{p-5k} = \frac{1}{2} \binom{2p}{p} + \frac{(3+\sqrt{5})^p + (3-\sqrt{5})^p + 2^{3p-1}}{5 \cdot 2^p}; \\
6. \quad & \sum_{k=0}^{\lfloor p/6 \rfloor} \binom{2p}{p-6k} = \frac{1}{2} \binom{2p}{p} + \frac{3^p + 2^{2p-1} + 1}{6}.
\end{aligned}$$

Corollary 7. *Let p be a positive integer. Then*

$$\begin{aligned}
1. \quad & \sum_{k=1}^p \binom{2p-1}{p-k} = 4^{p-1}; \\
2. \quad & \sum_{k=1}^{\lfloor p/3 \rfloor} \binom{2p-3}{p-3k} = \frac{4^{p-2} - 1}{3}, \quad p > 1;
\end{aligned}$$

$$3. \sum_{k=1}^{\lfloor p/5 \rfloor} \binom{2p-5}{p-5k} = \frac{4^{p-3}}{5} - \frac{(\sqrt{5}+1)^{2p-5} - (\sqrt{5}-1)^{2p-5}}{5 \cdot 2^{2p-5}}, \quad p > 2.$$

Corollary 8. *Let p be a positive integer. Then*

$$1. \sum_{k=0}^p (-1)^k \binom{2p}{p-k} = \frac{1}{2} \binom{2p}{p};$$

$$2. \sum_{k=0}^{\lfloor p/2 \rfloor} (-1)^k \binom{2p}{p-2k} = \frac{1}{2} \binom{2p}{p} + 2^{p-1};$$

$$3. \sum_{k=0}^{\lfloor p/3 \rfloor} (-1)^k \binom{2p}{p-3k} = \frac{1}{2} \binom{2p}{p} + 3^{p-1};$$

$$4. \sum_{k=0}^{\lfloor p/4 \rfloor} (-1)^k \binom{2p}{p-4k} = \frac{1}{2} \binom{2p}{p} + \frac{(2+\sqrt{2})^p + (2-\sqrt{2})^p}{4};$$

$$5. \sum_{k=0}^{\lfloor p/5 \rfloor} (-1)^k \binom{2p}{p-5k} = \frac{1}{2} \binom{2p}{p} + \frac{(5+\sqrt{5})^p + (5-\sqrt{5})^p}{5 \cdot 2^p}.$$

Note that Corollary 6 is related in [7] with the sequences [A032443](#), [A114121](#), [A007583](#), [A007582](#), [A078789](#), [A085282](#), Corollary 7 with [A000302](#), [A002450](#), [A095931](#), Corollary 8 with [A088218](#), [A005317](#), [A191993](#), [A007052](#), [A081567](#), respectively.

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