Trần Văn Lâm tổng hợp và chọn lọc các bài toán trong hai box "High School Pre-Olympiad (Ages 14+)" "High School Intermediate Topics (Ages 13+)" sáng ngày 14/7/2013

Chú ý: Đa số các bài toán thuộc box "High School Pre-Olympiad (Ages 14+)"

☐ Let x,y be a real number amd x,y different from 0 satisfying that $(x+y)xy = x^2+y^2-xy$.find max of $\frac{1}{x^3}+\frac{1}{y^3}$

Solution

$x^2+y^2=r^2$ Then exists $\alpha \in [\frac{-\pi}{2},\frac{\pi}{2}]$ such that $x = rcos(\alpha)$ and $y = rsin(\alpha)$

$(x+y)xy = x^2+y^2-xy \Rightarrow r^3(cos(\alpha)+sin(\alpha))cos(\alpha)sin(\alpha) = r^2(cos^2(\alpha)+sin^2(\alpha)-cos(\alpha)sin(\alpha))$

$r(cos(\alpha)+sin(\alpha))sin(2\alpha) = 2(1-cos(\alpha)sin(\alpha))$

$\frac{2}{r} = \frac{sin(2\alpha)(cos(\alpha)+sin(\alpha))}{(1-cos(\alpha)sin(\alpha))}$; we see $cos(\alpha)sin(\alpha) \neq 1$ for all α

Now: $\frac{1}{x^3}+\frac{1}{y^3} = \frac{(cos(\alpha)+sin(\alpha))(1-cos(\alpha)sin(\alpha))}{r^3cos^3(\alpha)sin^3(\alpha)}$

$= \frac{(cos(\alpha)+sin(\alpha))^2}{r^2cos^2(\alpha)sin^2(\alpha)}$

$= \left(\frac{2}{r}\right)^2\left(\frac{1+sin(2\alpha)}{sin^2(2\alpha)}\right)$

$= \left(\frac{sin(2\alpha)(cos(\alpha)+sin(\alpha))}{(1-cos(\alpha)sin(\alpha))}\right)^2\left(\frac{1+sin(2\alpha)}{sin^2(2\alpha)}\right)$

$= \frac{(1+sin(2\alpha))^2}{1-sin(2\alpha)+cos^2(\alpha)sin^2(\alpha)}$

$= \left(\frac{1+sin(2\alpha)}{1-\frac{sin(2\alpha)}{2}}\right)^2$

we see that max of $\frac{1}{x^3}+\frac{1}{y^3}$ in $\alpha = \frac{\pi}{4}$

$\frac{1}{x^3}+\frac{1}{y^3} = 16$

$x = \frac{1}{2}$ $y = \frac{1}{2}$

☐ let $(u_n) = 1,2,2,3,3,3,4,4,4,4,\ldots\ldots..$ What is the value of u_k

Solution

Easy to see that $u_n = k$ when $1+2+\ldots+(k-1)+1 \leq n \leq 1+2+\ldots+k \Leftrightarrow \frac{(k-1)k}{2}+1 \leq n \leq \frac{k(k+1)}{2}$ $\Leftrightarrow 4k^2-4k+8 \leq 8n \leq 4k^2+4k \Leftrightarrow (2k-1)^2 < 8n < (2k+1)^2 \Leftrightarrow k < \sqrt{2n}+\frac{1}{2} < k+1 \Leftrightarrow k = [\sqrt{2n}+\frac{1}{2}]$
Result $u_n = [\sqrt{2n}+\frac{1}{2}]$

☐ The function $f : N \to N$ is such that $f(1) = 1, f(2n) = 2f(n)$, and $nf(2n+1) = (2n+1)(f(n)+n)$ for all $n \geq 1$.

Prove that $f(n)$ is always an integer, and for how many positive integers less than 2007 is $f(n) = 2n$?

Solution

Call $g(n) = \frac{f(n)}{n}$ then $g(1) = 1, g(2n) = g(n), g(2n+1) = g(n)+1$. Easy to check that $g(n) \in Z^+\forall n$ by induction! (Assume that $g(n) \in Z^+\forall n \leq k$: if $k+1 = 2m$ then $g(k+1) = g(m) \in Z^+$; if $k+1 = 2m+1$ then $g(k+1) = g(m)+1 \in Z^+$ (Because in all 2 case we have $m \leq k$)) We must find how many positive integers n less than 2007 so that $g(n) = 2$! With $n = 2^k.l : k \geq 0, l$ odd we have $g(n) = g(l)$ then if $g(n) = 1$ it mean $g(l) = 1 \Leftrightarrow l = 1$ (If $l = 2m+1 \geq 3$ then $g(l) = g(m)+1 \geq 2$) Then $g(n) = 1 \Leftrightarrow n = 2^k$! Now with $n = 2^k.(2m+1), k \geq 0, m \geq 1$ we have if $g(n) = 2$ it mean $g(2m+1) = 2 \Leftrightarrow g(m) = 1 \Leftrightarrow m = 2^h, h \geq 0 \Leftrightarrow n = 2^{k+h+1}+2^k, k, h \geq 0 \Leftrightarrow n = 2^p+2^q, p > q \geq 0$ Finally, $n \leq 2007 \Leftrightarrow 0 \leq q < p \leq 10$, it have $C_{11}^2 = 55$ numbers n so that $f(n) = 2n$!

☐ For a given $a \in \mathbb{R}$: $\forall x,y \in \mathbb{R}$ $f(x+y) = f(x)\cdot f(a-y)+f(y)\cdot f(a-x)$ assuming f is a real-valued function and $f(0) = \frac{1}{2}$. $f(2008) =?$

Solution

-) $x = y = 0 : f(0) = 2f(0)f(a) \Leftrightarrow f(a) = \frac{1}{2}$ -) $y = 0 : f(x) = f(x)f(a)+f(0)f(a-x) \Leftrightarrow f(x) = f(a-x) \Rightarrow f(x+y) = 2f(x)f(y)$ -) $y = a : f(x+a) = 2f(x)f(a) = f(x) \Rightarrow f(a-x) = f(x) = f(x-a)$

$\Rightarrow f(x)=f(-x)$ Then $f(x+y)=2f(x)f(y)=2f(x)f(-y)=f(x-y)$ for all x,y, it mean $f(x)=c=\frac{1}{2}\ \forall x$

$\square$ Prove that the ortocentre of a triangle whose vertices are $(a\cos\alpha, a\sin\alpha)$, $(a\cos\beta, a\sin\beta)$, $(a\cos\gamma, a\sin\gamma)$ is $(a\cos\alpha+a\cos\beta+a\cos\gamma, a\sin\alpha+a\sin\beta+a\sin\gamma)$

Solution

Call $A(a\cos\alpha, a\sin\alpha), B(a\cos\beta, a\sin\beta), C(a\cos\gamma, a\sin\gamma)$ and $H(a\cos\alpha+a\cos\beta+a\cos\gamma, a\sin\alpha+a\sin\beta+a\sin\gamma)$ we have $\overrightarrow{AH}(a\cos\beta+a\cos\gamma, a\sin\beta+a\sin\gamma)$ and $\overrightarrow{BC}(-a\cos\beta+a\cos\gamma, -a\sin\beta+a\sin\gamma)$ then $\overrightarrow{AH}.\overrightarrow{BC}=(a\cos\beta+a\cos\gamma)(-a\cos\beta+a\cos\gamma)$ $+(a\sin\beta+a\sin\gamma)(-a\sin\beta+a\sin\gamma)$ $=-a^2\cos^2\beta+a^2\cos^2\gamma-a^2\sin^2\beta+a^2\sin^2\gamma=0$ Similarly then OK!

$\square$ The number of factors of p in n! (p prime, n positive int) is $(n-s_n)/(p-1)$ where s_n is [i]the sum of the digits of n when expressed in base p

Solution

This is due to Legendre. Let $n=n_0+n_1p+\cdots+n_kp^k$ be the expansion of n in base p.

It is well-known that the valuation of $n!$ modulo p, say $\nu_p(n!)$ (that is the exponent of p in the prime decomposition of $n!$) is : $\nu_p(n!)=\sum_{i=1}^{+\infty}[\frac{n}{p^i}]$, where [.] denotes the integer part.

It easily follows that $\nu_p(n!)=(n_1+n_2p\cdots+n_kp^{k-1})+(n_2\cdots+n_kp^{k-2})+\cdots+(n_k)=n_1+n_2(1+p)+n_3(1+p+p^2)+\cdots+n_k(1+p+\cdots p^{k-1})=\sum_{i=1}^{k}n_i\frac{p^i-1}{p-1}=\sum_{i=0}^{k}n_i\frac{p^i-1}{p-1}=\frac{n-s_p(n)}{p-1}$, as desired.

$\square$ Prove that you can color positive rational numbers with two colors such that for each positive rational number q is color of q same as color of $\frac{1}{q}$ and different to color of $q+1$.

Solution

Let a,b be two positive integers. Now, use the euclidean algorithm to define two sequences of integers, namely (q_i) and (r_i) such that $a=r_0$ $b=q_1a+r_1$, (1) $a=r_0=q_2r_1+r_2$, $\ldots r_{i-1}=q_{i+1}r_i+r_{i+1}$ and so on.

Since $0\le r_i<r_{i-1}$ the algorithm will stop, and since the only reason to stop is to reach $r_i=0$ for some i, we may consider the integer n such that $r_n=0$. Now let $f(\frac{a}{b})=q_1+q_2+\cdots+q_n$.

Now assume that $b>a$. Then $a=0\cdot b+a$ and then the algorithm associated to $\frac{a}{b}$ follows the one associated to $\frac{b}{a}$. Therefore $f(\frac{a}{b})=f(\frac{b}{a})$. (2) In another hand $\frac{b}{a}+1=\frac{a+b}{a}$, and $(b+a)=(q_1+1)a+r_1$ where q_1,r_1 are the same as in (1). Thus, from that step, the algorithm associated to $\frac{a+b}{a}$ is the same as the one associated to $\frac{b}{a}$. Therefore $f(\frac{b}{a}+1)=1+f(\frac{b}{a})$. (3)

Thus, it suffices to define the coloring as follows : For each positive rational q let $q=\frac{b}{a}$ where $\gcd(a,b)=1$. Then color q in red if $f(\frac{b}{a})$ is even, and in blue otherwise. From (2) and (3), we deduce that this coloring as the desired property.

Remark. We may slightly simplify the above solution by defining $f(\frac{b}{a})$ to be the first non-zero term of the sequence (q_i)

$\square$ Determine all $f:\mathbb{R}\to\mathbb{R}$ for which

$$2\cdot f(x)-g(x)=f(y)-y \text{ and } f(x)\cdot g(x)\ge x+1.$$

Solution

The LHS of the first condition is independant of y, so $f(y)=y+c$, where c is a constant. Clearly $c=f(0)$. Thus $g(x)=2x+c$. The second condition may be rewritten as $(x+c)(x+2c)\ge x+1$ for all x. Expanding, we get a quadratic expression in x which has to be non-negative, and whose disriminant is $\Delta=(c-3)^2$. But, if $\Delta>0$ then the quadratic takes positive and negative values... then $c=3$, and $f(x)=x+3$, which is a solution. $\square$ Determine all polynomials satisfying $xP(x-1)=(x-23)P(x)$

$\forall x$.

Solution

Lemma : If $P(x)$ is a polynomial such that $P(x) = P(x-1)$ for all real number x, then P is constant.

Proof. Assume that such a polynomial P is nonconstant. Let α be one of its (complex) root. It is easy to verify that $\alpha+1$ is also a root of P. Thus, considering the sequence $U_0 = \alpha$ and $U_{n+1} = U_n+1$, we deduce that P has an infinite number of root, that is $P = 0$, a contradiction.

Now, let $P_0 = P$ where P is a polynomial satisfying the statement of the problem. Clearly, $P(0) = P(22) = 0$, which gives that $P(x) = x(x-22)P_1(x)$ where P_1 satisfies $(x-1)P_1(x-1) = (x-22)P_1(x)$ for all $x \neq 0$ and $x \neq 22$ which means that this equality is in fact satisfied for all x.

As above $P_1(1) = P_1(21) = 0$ so that $P_1(x) = (x-1)(x-21)P_2(x)$ and $(x-2)P_2(x-1) = (x-21)P_2(x)$.

And so on until we reach $(x-11)P_{11}(x-1) = (x-12)P_{11}(x)$ for all x, from which we deduce that $P_{11}(x) = (x-11)P_{12}(x)$ and $P_{12}(x-1) = P_{12}(x)$ for all x. From the lemma, we deduce that P_{12} is constant.

It follows that $P(x) = kx(x-1)(x-2)\dots(x-22)$, for some sonctant k.

Conversely, it is easy to verify that these polynomials are solutions of the problem.

□ Find all positive integer solutions x and n of the equation: $x^2 + 615 = 2^n$

Solution

Note that 615 is divisible by 3, so since 2^n is not, x is not divisible by 3; It follows that $x^2 = 1 \mod [3]$, so that $2^n = 1 \mod [3]$. Therefore $n = 2k$ is even. The equation becomes $615 = (2^k - x)(2^k + x)$. with $2^k + x > 2^k - x$. Since $2^k + x > 0$, we then have $2^k - x$. Then, $2^k + x$ and $2^k - x$ are two positive divisors of 615 whose product is 615. The positive divisors of $615 = 3 \cdot 5 \cdot 41$ are $1, 3, 5, 5, 41, 123, 205, 615$. Thus $(2^k + x, 2^k - x)$ is one of the couples $(615, 1), (205, 3), (123, 5), 41, 15)$. Direct checking shows that the only solution is $k = 6$ and $x = 59$, which leads to the solution $x = 59$ and $n = 12$.

□ Prove that for every positive integer n , the difference $s_n = (\sum_{k=1}^{n} [\frac{n}{k}]) - [\sqrt{n}]$ is an even integer , where [x] denotes the integer part of x

Solution

Let $A = \{1, 2, \cdots, n\}$ and $S = \sum_{k=1}^{n} [\frac{n}{k}]$.

For each k, the number $[\frac{n}{k}]$ denotes the number of multiples of k in A. Therefore, each element i from A contributes for 1 in S exactly as many times as its number of positive divisors, say $d(i)$. It follows that $S = \sum_{k=1}^{n} d(k)$. But, it is well known that $d(k)$ is odd if and only if k is a perfect square. Thus S has the same parity than the number of squares in A, which is $[\sqrt{n}]$, and the desired result now follows easily.

□ Denote by $u(k)$ the greatest odd divisor of $k \in \mathbb{N}$. Prove that $\forall n \in \mathbb{N}$ we have:

$$\frac{1}{2^n} \cdot \sum_{k=1}^{2^n} \frac{u(k)}{k} > \frac{2}{3}.$$

Solution

Let $S_n = \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{u(k)}{k}$. Then, $2^n \cdot S_n = \sum_{k=1}^{2^n} \frac{u(k)}{k}$.

Now observe that $\frac{u(2k)}{2k} = \frac{1}{2} \cdot \frac{u(k)}{k}$. Then : $2^n \cdot S_n = \sum_{k odd} \frac{u(k)}{k} + \sum_{k=1}^{2^{n-1}} \frac{u(2k)}{2k} = 2^{n-1} + 2^{n-2} S_{n-1}$ Thus : $S_n = \frac{1}{2} + \frac{S_{n-1}}{4}$. (1)

The desired result now follows easily by induction. But, we may go further : Let $S_n = U_n + \frac{2}{3}$. It is easy from (1) to see that the sequence (U_n) is geometrical with ratio $\frac{1}{4}$. Since $S_0 = 1$ and $U_0 = \frac{1}{3}$, it follws that $S_n = \frac{2}{3} + \frac{1}{3} \cdot (\frac{1}{4})^n$.

$\square$ A regular 1997-gon is decomposed into triangles using non-intersecting diagonals. How many of these triangles are acute?

Solution

Inscribe the polygon into a circle, with center O. That a triangle T is acute is equivalent to the fact that O is an interior point of T. Since we have a triangulation of the polygon, the interior of the triangles are pairwise disjoints. Therefore, there is at most one acute triangle. In another hand, if there is no acute triangle, it means that O belongs to the side of a triangle, so that there are two vertices which form a diameter. But this is impossible since 1997 is odd. Thus, there is at least one acute triangle.

Then, there is exactly one acute triangle.

This is clearly true for any odd n instead of 1997. The first part of the reasoning remains true for even n. If I've no mistake, if $n = 4, 6$ there is no acute triangle. If $n \geq 8$ there can be one acute triangle or not, according to chosen triangulation of the polygon.

$\square$ $a, b, c \in \mathbb{N}$, show that $a^2 + b^3 = b^c$ has no solution.

Solution

$b^c - b^3 = a^2\ so\ b^3(b^{c-3} - 1) = a^2$. We can easily show that $b^3 and b^{c-3} - 1$ share no factors because $b^{c-3} - 1$ has no factors of b in it. Therefore, this means both b^3 and $b^{c-3} - 1$ must be squares. Since b^3 must be a square, b must be in the form x^2. so that means that $x^{2^{c-3}} - 1$ is a square. Which is untrue, since that $+1$ is a square, and x isnt equal to 0 by the very nature of the problem. Therefore $a^2 + b^3 = b^c$ has no solution in natural numbers.

$\square$ Let A be a subset of $\mathbb{R}$ which staisfies the three following properties : 1) $1 \in A$ 2) If $x \in A$ then $x^2 \in A$ 3) If $(x-2)^2 \in A$ then $x \in A$.

Prove that $2004 + \sqrt{2005} \in A$.

Solution

We have that $x \in A \Rightarrow x^2 = (2 + x - 2)^2 \in A \Rightarrow 2 + x \in A$ Thus given that $1 \in A$ we know that all odd numbers are also in A. So $2005 \in A$.

Also we have that $x > 0;\ x \in A \Rightarrow (2 + \sqrt{x} - 2)^2 \in A \Rightarrow 2 + \sqrt{x} \in A$ But then $2 + \sqrt{2005} \in A$ and using the first result succesively proves $2004 + \sqrt{2005} \in A$

$\square$ Determine the least natural number n for which the following holds: No matter how the numbers 1 to n is divided into two disjoint sets, in at least one of the sets, there exist four (not necessarily distinct) elements w, x, y, z st $w + x + z = y$.

Solution

The minimal n is $n = 11$.

Assume that A, B form a partition of $\{1, \cdots, 11\}$ with no solution of the equation in one of the sets. Wlog, we may assume that $1 \in A$. Thus $3 \in B$. It follows that $9 \in A$ so that $11 \in B$. Now : - If $2 \in A$ then $4, 5, 6 \in B$. In that case $11 = 3 + 3 + 4$ gives a solution in B. A contradiction. - If $2 \in B$ then $6, 7, 8 \in A$, but $8 = 6 + 1 + 1$ gives a solution in A. A contradiction. It follows that the minimal n satisfies $n \leq 11$.

In another hand, for $n = 10$ and if $A = \{1, 2, 9, 10\}$ and $B = \{3, 4, 5, 6, 7, 8\}$, we have a partition into two parts with no solution of the equation. Hence result.

$\square$ Solve system:

$x+y^2=z^3$
$x^2+y^3=z^4$
$x^3+y^4=z^5$

Solution

We have $(x+y^2)(x^3+y^4)=z^8=(x^2+y^3)^2 \Leftrightarrow x^3y^2+xy^4=2x^2y^3 \Leftrightarrow xy^2(x-y)^2=0$ -) $x=0$ or $y=0$: Easy! -) $x=y$ then $x+x^2=z^3$ and $x^2+x^3=z^4$, it mean $(x+x^2)^4=(x^2+x^3)^3$. OK!

$\square$ Let f be a function from the set Q of the rational numbers onto itself such that $f(x+y)=f(x)+f(y)+2547$ for all rational numbers x,y. Moreover $f(2004)=2547$. Determine $f(2547)$.

Solution

$f(x+y)=f(x)+f(y)+2547$

Put $x=y=0$,we have $2547=-f(0)$, so we obtain $f(x+y)-f(x)=f(y)-f(0)$.

$\lim_{y\to 0}\frac{f(x+y)-f(x)}{y}=\lim_{y\to 0}\frac{f(y)-f(0)}{y}$

$f'(x)=f'(0)\iff f(x)=f'(0)x+C$.From $f(2004)=2547$,we have $C=2547-2004f'(0)$. Therefore by $f(2547)=2547f'(0)+C$, the answer is $f(2547)=543f'(0)+2547$.

$\square$ Let P be an internal point of triangle ABC. The line through P parallel to AB meets BC at L, the line through P parallel to BC meets CA at M, and the line through P parallel to CA meets AB at N.

Prove that $\frac{BL}{LC}\times\frac{CM}{MA}\times\frac{AN}{NB}\le\frac{1}{8}$

Solution

Denote $AP\cap BC=D, PN\cap BC=L'$ then $\frac{LC}{BL}\cdot\frac{MA}{CM}\cdot\frac{NB}{AN}=\frac{LC}{BL}\cdot\frac{PA}{PD}\cdot\frac{L'B}{CL'}=\frac{LC}{BL}\cdot\frac{L'C}{L'D}\cdot\frac{L'B}{CL'}=\frac{LC}{BL}\cdot\frac{L'B}{L'D}=\frac{(b+c+d)(a+b+c)}{ac}\left(BL=a, LD=b, DL'=c, L'C=d : ac=bd \text{ because } \frac{a}{b}=\frac{AP}{PD}=\frac{d}{c}\right)$
$\ge\frac{(d+2\sqrt{bc})(a+2\sqrt{bc})}{\sqrt{abcd}}\ge\frac{2\sqrt{2d\sqrt{bc}}\times 2\sqrt{2a\sqrt{bc}}}{\sqrt{abcd}}=8$ Equality when $a=2b=2c=d$, it mean $P\equiv G$!

$\square$ What is triangle ABC if $2\sin A+3\sin B+4\sin C=5\cos\frac{A}{2}+3\cos\frac{B}{2}+\cos\frac{C}{2}$

Solution

We have: $\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\le 2\cos\frac{C}{2}$ Similarly: $5\sin B+5\sin C\le 10\cos\frac{A}{2}$; $3\sin C+3\sin A\le 6\cos\frac{B}{2}$ Add 3 equalities we have: $4\sin A+6\sin B+8\sin C\le 10\cos\frac{A}{2}+6\cos\frac{B}{2}+2\cos\frac{C}{2}$ So triangle is an equilateral triangle .

$\square$ Let $X=\{1;2;3;\dots;15\}$. How many subset $A\subset X$ have 5 elements so that A have at least 2 consecutive numbers. Example: $A=\{1;2;4;5;7\}$ have 2 pair consecutive numbers. $B=\{1;3;5;7;9\}$ isn't satisfy!

Solution

A set $A=\{a;b;c;d;e\}$ with $a<b<c<d<e$ don't satisfy condition is five number so that $1\le a<b-1<c-2<d-3<e-4\le 11$, then have C_{11}^5 sets. Then we have $C_{15}^5-C_{11}^5$ sets satisfy our problem.

$\square$ Let ABC be a triangle and let E,F be the projections of the its orthocenter to the side-lines AC , AB respectively. Prove that $A=60^\circ$ if and only if the middlepoint of the segment $[EF]$ is the radical center for the circles $C(B,BF)$, $C(C,CE)$, $C\left(A,\frac{|b-c|}{2}\right)$.

Solution

Let the midpoint of segment EF be D. Let $BF=R_1$, $CE=R_2$, $EF=d$. Then the Power of point D to circle B is:

$DB^2-R_1^2=\frac{1}{2}R_1^2+\frac{1}{2}BE^2-\frac{1}{4}d^2-R_1^2=\frac{1}{2}\left(BC^2-R_2^2-R_1^2\right)-\frac{1}{4}d^2.$

Similarly we can get that the power of point D to circle C is also:

$DB^2-R_1^2=\frac{1}{2}R_1^2+\frac{1}{2}BE^2-\frac{1}{4}d^2-R_1^2=\frac{1}{2}\left(BC^2-R_1^2-R_2^2\right)-\frac{1}{4}d^2.$

Hence D lies on the radical axis of circle B and circle C.

The power of point D to circle A is:

$DA^2 - \left(\frac{|b-c|}{2}\right)^2 = \frac{1}{2}AE^2 + \frac{1}{2}AF^2 - \frac{1}{4}d^2 - \frac{b^2-2bc+c^2}{4} =$

Since $\angle A = 60^o$ so $a^2 = b^2 + c^2 - bc$ and there is also $AE = \frac{c}{2}$, $AF = \frac{b}{2}$, so the above can be written as:

$\frac{1}{2} \cdot \frac{1}{4}c^2 + \frac{1}{2} \cdot \frac{1}{4}b^2 - \frac{2a^2-b^2-c^2}{4} - \frac{1}{4}d^2 = \frac{3}{8}c^2 + \frac{3}{8}b^2 - \frac{1}{2}a^2 - \frac{d^2}{4}$

Since $\frac{c^2}{4} = AE^2 = c^2 - BE^2 = c^2 - a^2 + R_2^2$, so $\frac{3}{4}c^2 = a^2 - R_2^2$, similarly there is $\frac{3}{4}b^2 = a^2 - R_1^2$, Pluge them back, we get that the Power of point D to circle A is:

$\frac{1}{2}\left(BC^2 - R_2^2 - R_1^2\right) - \frac{1}{4}d^2$

Therefore D is the radical centre of the three circles.

□ Let H an interior point of a triangle ABC. Lines AH, BH, CH meet the sides of the triangle at points D, E, F respectively. If H is the incenter of the triangle DEF , prove that H is the orthocenter of triangle ABC.

Solution

Here is an easy and well-known property : " $AD \perp BC \iff \widehat{EDA} \equiv \widehat{FDA}$ ". **Proof.** I"ll with the orientate segments. Let d be the line for which $A \in d$ and $d \parallel BC$. Denote the intersections $M \in d \cap DE$ and $N \in d \cap DF$. $d \parallel BC \Longrightarrow \frac{EC}{EA} = \frac{DC}{MA}$, $\frac{FA}{FB} = \frac{DB}{NA}$. Apply the Ceva's theorem to the point H for the triangle ABC : $\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = -1 \Longrightarrow MA = AN$. Therefore, $AD \perp BC \iff DA \perp MN \iff \widehat{MDA} \equiv \widehat{NDA}$.

Remark. You prove immediately this problem with the harmonical division. Denote the intersections $X \in AD \cap EF$ and $Y \in EF \cap BC$. Thus, $(B, C; D, Y)$- h.d. $\Longrightarrow (E, F; X, Y)$- h.d. and $d \parallel BC \Longrightarrow NA = AM$. Therefore, $AD \perp BC \iff XD \perp BC \iff \widehat{EDX} \equiv \widehat{FDX} \iff \widehat{MDA} \equiv \widehat{NDA}$.

□ Find the n th term of the positive sequence $\{a_n\}$ such that $a_1 = 1,\ a_2 = 10,\ a_n^2 a_{n-2} = a_{n-1}^3\ (n = 3,\ 4,\ \cdots)$.

Solution

$\frac{a_n}{a_{n-1}} = \sqrt{\frac{a_{n-1}}{a_{n-2}}} \Longrightarrow \frac{a_n}{a_{n-1}} = \left(\frac{a_2}{a_1}\right)^{1/2^{n-2}} = 10^{1/2^{n-2}}, n \geqslant 2$

$$\begin{aligned}
\frac{a_n}{a_1} &= \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} \\
&= 10^{1/2^{n-2}+1/2^{n-3}+\cdots+1} \\
&= 10^{2-(1/2^{n-2})}
\end{aligned}$$

Since $a_1 = 1$, we get $a_n = 10^{2-(1/2^{n-2})}$

□ Find the n th term of the sequence $\{a_n\}$ such that $a_1 = a\ (\neq -1),\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right)\ (n \geq 1)$.

Solution

Put $a_n = \frac{p_n}{q_n}, p_1 = a, q_1 = 1$

Then $\frac{p_{n+1}}{q_{n+1}} = \frac{p_n^2+q_n^2}{2p_nq_n}$, hence we can put

$\begin{cases} p_{n+1} = p_n^2 + q_n^2 \\ q_{n+1} = 2p_nq_n \end{cases}$

From this we get

$p_{n+1} + q_{n+1} = (p_n + q_n)^2 \iff p_n + q_n = (p_1 + q_1)^{2^{n-1}} = (a+1)^{2^{n-1}}$ $p_{n+1} - q_{n+1} = (p_n - q_n)^2 \iff p_n - q_n = (p_1 - q_1)^{2^{n-1}} = (a - 1)^{2^{n-1}}$

That yields

$p_n = \frac{1}{2}\left((a+1)^{2^{n-1}} + (a-1)^{2^{n-1}}\right)$ $q_n = \frac{1}{2}\left((a+1)^{2^{n-1}} - (a-1)^{2^{n-1}}\right)$

giving

$a_n = \frac{(a+1)^{2^{n-1}}+(a-1)^{2^{n-1}}}{(a+1)^{2^{n-1}}-(a-1)^{2^{n-1}}}$

NOTE: The formula also works for both $a=1$ and $a=-1$

□ Define the sequence $\{a_n\}$ such that $a_1=-4,\ a_{n+1}=2a_n+2^{n+3}n-13\cdot 2^{n+1}\ (n=1,\ 2,\ 3,\ \cdots)$. Find the value of n for which a_n is minimized.

Solution

$$\begin{aligned}
& a_{n+1}=2a_n+2^{n+3}n-13\cdot 2^{n+1} \\
\iff\quad & a_{n+1}-2^{n+1}[2(n+1)^2-15(n+1)]=2[a_n-2^n(2n^2-15n)] \\
\iff\quad & a_n-2^n(2n^2-15n)=2^{n-1}(a_1-2(2-15))=11\cdot 2^n \\
\iff\quad & a_n=2^n(2n^2-15n+11)
\end{aligned}$$

a_n is negative for $1\leqslant n\leqslant 6$ and the minimal value is $a_5=a_6=-448$

□ Find the n th term of the seuence $\{a_n\}$ such that $a_1=\frac{1}{2},\ a_2=\frac{1}{3},\ a_{n+2}=\frac{a_na_{n+1}}{2a_n-a_{n+1}+2a_na_{n+1}}$.

Solution

$$\begin{aligned}
& \frac{1}{a_{n+2}}=\frac{2}{a_{n+1}}-\frac{1}{a_n}+2 \\
\iff\quad & \left(\frac{1}{a_{n+2}}-(n+2)^2\right)-\left(\frac{1}{a_{n+1}}-(n+1)^2\right)=\left(\frac{1}{a_{n+1}}-(n+1)^2\right)-\left(\frac{1}{a_n}-n^2\right) \\
\iff\quad & \frac{1}{a_n}-n^2=\frac{1}{a_1}-1^2+(n-1)\left(\frac{1}{a_2}-2^2-\frac{1}{a_1}+1^2\right)=-2n+3 \\
\iff\quad & a_n=\frac{1}{n^2-2n+3}
\end{aligned}$$

□ Solve for a_n:

$a_1=1, a_{n+1}=\frac{a_n}{1+a_n}+1, n\geqslant 1$

Solution

Let $\alpha,\ \beta\ (\alpha<\beta)$ be the roots of the quadratic equation $x^2-x-1=0$, we have $\frac{a_{n+1}-\beta}{a_{n+1}-\alpha}=\frac{2-\beta}{2-\alpha}\cdot\frac{a_n-\alpha}{a_n-\beta}$, thus we obtain $\frac{a_n-\beta}{a_n-\alpha}=\frac{\beta}{\alpha}\left(\frac{2-\beta}{2-\alpha}\right)^n$, yielding $a_n=\frac{(2-\beta)^n-(2-\alpha)^n}{\alpha(2-\alpha)^n-\beta(2-\beta)^n}$.

□ Find the n th term of the sequence $\{a_n\}$ such that $\sum_{k=1}^n a_k=3n^2+4n+2\ (n=1,\ 2,\ 3,\ \cdots)$ and calculate $\sum_{k=1}^n a_k^2$.

Solution

For $n\geqslant 2$:

$a_n=(3n^2+4n+2)-[3(n-1)^2+4(n-1)+2]=3(2n-1)+4=6n+1$

For $n=1$:

$a_1=\sum_{k=1}^1 a_k=3\cdot 1^2+4\cdot 1+2=9$

Hence $a_1=9, a_n=6n+1, n\geqslant 2$

Then for $n\geqslant 2$:

$$\begin{aligned}
\sum_{k=1}^{n} a_k^2 &= 81+\sum_{k=2}^{n}(36k^2+12k+1)\\
&= 81+36\left(\frac{n(n+1)(2n+1)}{6}-1\right)+12\left(\frac{n(n+1)}{2}-1\right)+(n-1)\\
&= 81+6n(n+1)(2n+1)-36+6n(n+1)-12+n-1\\
&= 6n(n+1)[(2n+1)+1]+n+32\\
&= 12n(n+1)^2+n+32\\
&= 12n^3+24n^2+13n+32
\end{aligned}$$

and that also works for $\sum_{k=1}^{1} a_k^2=81$.

Hence $\sum_{k=1}^{n} a_k^2=12n^3+24n^2+13n+32, n\geqslant 1$

□ Find the n th term of the sequence $\{a_n\}$ such that $a_1=0,\ a_2=1,\ (n-1)^2a_n=\sum_{k=1}^{n}a_k\ \ (n\geq 1)$.

Solution

For $n\geqslant 3$

$(n-1)^2a_n-(n-2)^2a_{n-1}=a_n \iff n(n-2)a_n-(n-2)^2a_{n-1}=0$

Since $n-2\neq 0$, this gives

$na_n=(n-2)a_{n-1}$

Then

$$\begin{aligned}
na_n &= (n-2)a_{n-1}\\
(n-1)a_{n-1} &= (n-3)a_{n-2}\\
&\vdots\\
3a_3 &= 1\cdot a_2
\end{aligned}$$

Multiply all the equations and denote $P=a_3a_4\ldots a_{n-1}$:

$\frac{n!}{2!}Pa_n=(n-2)!Pa_2 \iff a_n=\frac{2a_2}{n(n-1)}=\frac{2}{n(n-1)}$

which also works for $n=2$, hence

Hence $a_1=0, a_n=\frac{2}{n(n-1)}, n\geqslant 2$

□ Find the n th term of the sequence $\{x_n\}$ such that $x_{n+1}=x_n(2-x_n)$ $(n=1,\ 2,\ 3,\ \cdots)$ in terms of x_1.

Solution

$$\begin{aligned}
x_{n+1}=2x_n-x_n^2 &\iff 1-x_{n+1}=1-2x_n+x_n^2=(1-x_n)^2\\
&\iff 1-x_n=(1-x_1)^{2^{n-1}}\\
&\iff x_n=1-(1-x_1)^{2^{n-1}}
\end{aligned}$$

□ Find the n th term of the sequence $\{a_n\}$ such that $a_1=1, a_{n+1}=\frac{1}{2}a_n+\frac{n^2-2n-1}{n^2(n+1)^2}$ $(n=1,2,3,\ldots.)$.

Solution

$\frac{n^2-2n-1}{n^2(n+1)^2}=\frac{n^2-[(n+1)^2-n^2]}{n^2(n+1)^2}=\frac{2}{(n+1)^2}-\frac{1}{n^2}$

hence

$$\begin{aligned}
& a_{n+1} = \frac{1}{2}a_n + \frac{n^2-2n-1}{n^2(n+1)^2} \\
\iff \quad & a_{n+1} - \frac{2}{(n+1)^2} = \frac{1}{2}a_n - \frac{1}{n^2} \\
\iff \quad & a_{n+1} - \frac{2}{(n+1)^2} = \frac{1}{2}\left(a_n - \frac{2}{n^2}\right) \\
\iff \quad & a_n - \frac{2}{n^2} = \left(\frac{1}{2}\right)^{n-1}\left(a_1 - \frac{2}{1^2}\right) = -\frac{1}{2^{n-1}} \\
\iff \quad & a_n = \frac{2}{n^2} - \frac{1}{2^{n-1}}
\end{aligned}$$

□ Find the n th term of the sequence $\{a_n\}$ such that $a_1 = \frac{1}{2},\ (n-1)a_{n-1} = (n+1)a_n\ (n \geq 2)$.

Solution

$$\begin{aligned}
(n-1)a_{n-1} &= (n+1)a_n \\
(n-2)a_{n-2} &= (n)a_{n-1} \\
&\vdots \\
1 \cdot a_1 &= 3 \cdot a_2
\end{aligned}$$

Multiply all the equations and denote $P = a_1a_2\ldots a_n$:
$(n-1)!\frac{P}{a_n} = \frac{(n+1)!}{2!}\cdot\frac{P}{a_1} \iff a_n = \frac{2a_1}{n(n+1)} = \frac{1}{n(n+1)}$

□ Find all positive real solutions to the equation:
$x + \left\lfloor \frac{x}{6} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor$
where $\lfloor t \rfloor$ denotes the largest integer less than or equal to the real number t.

Solution

Put $x = 6n + \varrho$, where $n \in \mathbb{Z}$ and ϱ is a real number satisfying $0 \leqslant \varrho < 6$.

Then the equation becomes
$6n + \varrho + n = 3n + \left[\frac{\varrho}{2}\right] + 4n + \left[\frac{2\varrho}{3}\right]$
or
$\varrho = \left[\frac{\varrho}{2}\right] + \left[\frac{2\varrho}{3}\right]$

Since RHS is integer, LHS must be too, therefore $\varrho \in \{0,1,2,3,4,5\}$. By checking we see that all the values except 1 satisfy the equation. Therefore the initial equation's solution set is
$\mathcal{R} = \mathbb{Z} \setminus (6\mathbb{Z}+1)$

□ Find the n th term of the sequence $\{a_n\}$ which is defined by $a_1 = 0,\ a_n = \left(1 - \frac{1}{n}\right)^3 a_{n-1} + \frac{n-1}{n^2}\ (n = 2,\ 3,\ \cdots)$.

Solution

Multiply the expression by n^3 and then substitute $b_n = n^3a_n$: $b_n = b_{n-1} + 2\binom{n}{2} = b_1 + 2\sum_{i=2}^{n}\binom{i}{2} = 0 + 0 + 2\binom{n+1}{3}$ (using the hockey-stick pattern in Pascal's triangle). So $a_n = \frac{2(n+1)n(n-1)}{6n^3} = \frac{n^2-1}{3n^2}$ for $n > 1$. Another way Or to rearrange further thus:

$$\begin{aligned}
& n^3a_n = (n-1)^3a_{n-1} + n(n-1) \\
\iff \quad & n^3a_n - \frac{(n-1)n(n+1)}{3} = (n-1)^3a_{n-1} - \frac{(n-2)(n-1)n}{3}
\end{aligned}$$

hence

$n^3a_n - \frac{(n-1)n(n+1)}{3} = \text{const} = 1^3a_1 - \frac{0\cdot1\cdot2}{3} = 0 \iff a_n = \frac{n^2-1}{3n^2}$

$\square$ Find the n th term of the sequence $\{a_n\}$ such that $a_1 = 1,\ a_{n+1} = 2a_n - n^2 + 2n\ (n = 1,\ 2,\ 3,\ \cdots)$.

Solution

by rearranging:

$a_{n+1} = 2a_n - n^2 + 2n \iff a_{n+1} - (n+1)^2 - 1 = 2a_n - n^2 + 2n - (n+1)^2 - 1$

which gives

$a_{n+1} - (n+1)^2 - 1 = 2(a_n - n^2 - 1)$

hence $a_n - n^2 - 1 = 2^{n-1}(a_1 - 1^2 - 1) = -2^{n-1} \iff a_n = n^2 - 2^{n-1} + 1$

$\square$ Find the n th term of the sequence $\{a_n\}$ such that $a_1 = 1,\ a_{n+1}^2 = -\frac{1}{4}a_n^2 + 4\ \ (a_n > 0,\ n \geq 1)$.

Solution

$$\begin{aligned}
a_{n+1}^2 = -\frac{1}{4}a_n^2 + 4 \quad &\iff \quad a_{n+1}^2 - \frac{16}{5} = -\frac{1}{4}a_n^2 + 4 - \frac{16}{5}\\
&\iff \quad a_{n+1}^2 - \frac{16}{5} = -\frac{1}{4}a_n^2 + \frac{4}{5}\\
&\iff \quad a_{n+1}^2 - \frac{16}{5} = -\frac{1}{4}\left(a_n^2 - \frac{16}{5}\right)\\
&\iff \quad a_n^2 - \frac{16}{5} = \left(-\frac{1}{4}\right)^{n-1}\left(a_1^2 - \frac{16}{5}\right) = \frac{44}{5}\left(-\frac{1}{4}\right)^n\\
&\iff \quad a_n = \sqrt{\frac{16}{5} + \frac{44}{5}\left(-\frac{1}{4}\right)^n}
\end{aligned}$$

$\square$ Find the n th term of the sequence $\{a_n\}$ such that $a_1 = 1,\ a_2 = 3,\ a_{n+1} - 3a_n + 2a_{n-1} = 2^n\ \ (n \geq 2)$.

Solution

Remark that $a_n - a_{n-1} = 2^{n-1} + 2(a_{n-1} - a_{n-2})$, so $\frac{a_n - a_{n-1}}{2^n} = \frac{a_{n-1} - a_{n-2}}{2^{n-1}} + \frac{1}{2}$. Therefore, $\frac{a_n - a_{n-1}}{2^n} = \frac{a_2 - a_1}{4} + \frac{n-2}{2} = \frac{n-1}{2}$, i.e. $a_n - a_{n-1} = (n-1)2^{n-1}$. Also, $(a_n - 2a_{n-1}) = 2^{n-1} + (a_{n-1} - 2a_{n-2})$, so $a_n - 2a_{n-1} = 2^{n-1} + 2^{n-2} + \ldots + 4 + (a_2 - 2a_1) = 2^n - 3$. And there we are, $a_n = 2(a_n - a_{n-1}) - (a_n - 2a_{n-1}) = (n-1)2^n - (2^n - 3) = (n-2)2^n + 3$. Another approach, using characteristic equations:

Writing $n+1$ instead of n we get

$a_{n+2} - 3a_{n+1} + 2a_n = 2^{n+1} = 2\cdot 2^n \qquad (*)$

From the initial equation $2^n = a_{n+1} - 3a_n + 2a_{n-1}$. Plugging that into $(*)$ and rearranging, we get

$a_{n+2} - 5a_{n+1} + 8a_n - 4a_{n-1} = 0$

Hence the characteristic equation is $t^3 - 5t^2 + 8t - 4 = 0$. Factorize the LHS:

$$\begin{aligned}
t^3 - 5t^2 + 8t - 4 &= t^3 - t^2 - 4t^2 + 4t + 4t - 4\\
&= t^2(t-1) - 4t(t-1) + 4(t-1)\\
&= (t-2)^2(t-1)
\end{aligned}$$

So the roots are $t_1 = t_2 = 2, t_3 = 1$, hence the general solution is $a_n = (An+B)2^n + C\cdot 1^n$. Since $a_3 = 11$, we make the system

$2(A+B) + C = 1\ 4(2A+B) + C = 3\ 8(3A+B) + C = 11$

Solving it we get $A=1, B=-2, C=3$, hence $a_n=(n-2)2^n+3$.

□ Find the n th term of the sequence $\{a_n\}$ such that $a_1=1,\ a_{n+1}=\frac{a_n}{2a_n+3}\ (n\geq 1)$.

Solution

$$\begin{aligned}
a_{n+1}=\frac{a_n}{2a_n+3} &\iff \frac{1}{a_{n+1}}=2+\frac{3}{a_n}\\
&\iff 1+\frac{1}{a_{n+1}}=3+\frac{3}{a_n}\\
&\iff 1+\frac{1}{a_{n+1}}=3\left(1+\frac{1}{a_n}\right)\\
&\iff 1+\frac{1}{a_n}=3^{n-1}\left(1+\frac{1}{a_1}\right)=2\cdot 3^{n-1}\\
&\iff a_n=\frac{1}{2\cdot 3^{n-1}-1}
\end{aligned}$$

□ Find the n th term of the sequence $\{a_n\}$ such that $a_1=1,\ a_{n+1}=2a_n^2\ (n=1,\ 2,\ 3,\ \cdots)$.

Solution

$2a_{n+1}=(2a_n)^2 \implies 2a_n=(2a_1)^{2^{n-1}}=2^{2^{n-1}} \implies a_n=2^{2^{n-1}-1}$

□ For $\{m,n\}\subset N^*$ given are the polynomials $F=X^m-1$ and $G=X^n-1$. Denote $D=X^d-1$, where $d=(m,n)$. Then $D=(F,G)$.

Solution

The polynomials F,G have only simple roots (essentially !). There is $\{u,v\}\subset Z^*$ so that $d=um+vn$. Therefore, $F(\alpha)=G(\alpha)=0\implies\alpha^m=\alpha^n=1\implies\alpha^d=\alpha^{um+vn}=(\alpha^m)^u\cdot(\alpha^n)^v=1\implies D(\alpha)=0$, i.e. the polynomial D has the all common (simple) roots of the polynomials F and G . In conclusion, $D=(F,G)$.

□ Find the constant term of $\left(1+x+\frac{2}{x^3}\right)^8$.

Solution

Solution 1: Use the Multinomial Series. What you are looking for is $\sum Ca_1^{n_1}+\sum Ca_1^{n_1}a_2^{3n_3}a_3^{n_3}$ with $n_k\leq 8$. This gives

$$n_1=8 \implies \sum Ca_1^{n_1}=1$$

$$n_1+4n_3=8 \implies \sum Ca_1^{n_1}a_2^{3n_3}a_3^{n_3}=4\cdot\frac{8!}{6!2!}+2\cdot\frac{8!}{4!3!}=672$$

Therefore our answer is $672+1=\boxed{673}$. Solution 2: We'll use trinomial expansion

$(p+q+r)^n=\sum_{i+j+k=n}\frac{n!}{i!j!k!}p^iq^jr^k$

Hence we must find all pairs i,j such that

$\frac{8!}{i!j!(8-i-j)!}1^ix^j\left(\frac{2}{x^3}\right)^{8-i-j}$

doesn't depend on x

The exponent of x is $j-3(8-i-j)=j-24+3i+3j=3i+4j-24$. Equating it with zero we get $4j=24-3i \iff j=6-\frac{3i}{4}$. Since both i,j are non-negative integers not greater than 8, i is divisible by 4, hence can only be $0,4,8$. For those values we get $j=6,3,0$ respectively.

Therefore we have three constant terms:

$\frac{8!}{0!6!2!}1^0x^6\left(\frac{2}{x^3}\right)^2=\frac{8\cdot 7}{2}\cdot 2^2=112$

$\frac{8!}{4!3!1!}1^4x^3\left(\frac{2}{x^3}\right)^1=\frac{8\cdot 7\cdot 6\cdot 5}{6}\cdot 2^1=560$

$\frac{8!}{8!0!0!}1^8x^0\left(\frac{2}{x^3}\right)^0=1$

and they add up to $112+560+1=\boxed{673}$ Solution 3: There are three ways we can get a constant term.

Case 1. We take the 1 from each of the eight factors, resulting in $1^8 = 1$.

Case 2. We take the x from three factors and the $\frac{2}{x^3}$ from one factor. This case results in $\binom{8}{3}\binom{5}{1}x^3 \cdot \frac{2}{x^3} = 560$.

Case 3. We take the x from six factors and the $\frac{2}{x^3}$ from two factors. This case results in $\binom{8}{6}\binom{2}{2}x^6\left(\frac{2}{x^3}\right)^2 = 112$.

The constant term is $1 + 560 + 112 = 673$.

□

In triangle ABC, M is the midpoint of BC. A line passing through M divides the perimeter of triangle ABC into two equal parts. Show that this line is parallel to the internal bisector of $\measuredangle A$.

Solution

Standard markings a, b, c. WLOG assume $b < c$. Then the line in question will intersect AB and not AC. (Assume the opposite and denote the intersection by N. Then $CN = \frac{b+c}{2} > b$, hence N is outside AC.) If AD is the bisector of $\angle A$, where $D \in BC$, then $b < c$ implies that D is between C and M. (By Angle Bisector Theorem, $CD = \frac{ab}{b+c} < \frac{ab}{b+b} = \frac{a}{2}$.) By Angle Bisector Theorem $BD = \frac{ac}{b+c}$ and by construction $BM = \frac{a}{2}, BN = \frac{b+c}{2}$ (since $BM + BN = \frac{a+b+c}{2}$), hence

$BD : BA = \frac{ac}{b+c} : c = \frac{a}{b+c} = \frac{a}{2} : \frac{b+c}{2} = BM : BN$

which means that $\triangle BMN \sim \triangle BDA \implies MN \parallel AD$.

If $b = c$, then the two lines coincide and represent the symmetrial axis of the given isosceles triangle. Another way We assume $AB > AC$.

Construct a point F on the extension of BA such that $AF = AC$. Then triangle FAC is isoceles with $\angle AFC = \angle ACF = \theta$. Then $\angle BAC = 2\theta$ and the angle bisector of A makes an angle of θ with the line BF. Hence it is parallel to CF.

Let the line from M intersect AB at D. Then D is the midpoint of BF. But since M is the midpoint of BC, triangle DBM is similar to triangle FBC. Hence MD is parallel to CF.

So BF is parallel to MD.

□ Find the volume of the tetrahedon $ABCD$ such that $AB = 6,\ BC = \sqrt{13},\ AD = BD = CD = CA = 5$.

Solution

If we take $\triangle ABC$ as basis, then $DA = DB = DC$ means that D projects into the circumcenter of $\triangle ABC$.

By Heron's

$$\begin{aligned}
[ABC]^2 &= \frac{11+\sqrt{13}}{2} \cdot \frac{1+\sqrt{13}}{2} \cdot \frac{-1+\sqrt{13}}{2} \cdot \frac{11-\sqrt{13}}{2} \\
&= \frac{(121-13)(13-1)}{16} \\
&= \frac{108 \cdot 12}{16} \\
&= 81
\end{aligned}$$

hence $[ABC] = 9$. Then $R = \frac{5 \cdot 6 \cdot \sqrt{13}}{4 \cdot 9} = \frac{5\sqrt{13}}{6}$, giving $H^2 = 5^2 - R^2 = 25\left(1 - \frac{13}{36}\right) = \frac{25 \cdot 23}{36} \implies H = \frac{5\sqrt{23}}{6}$

Then $V = \frac{1}{3}[ABC]H = \frac{5\sqrt{23}}{2}$

□ Prove that $(a + b)/(c + d)$ is irreducible if $ad - bc = 1$.

Solution

Assume $a+b$ and $c+d$ have a common divisor $m > 1$. Then $a+b = mx, c+d = my$ for some integer x, y. Therefore $b = mx - a, d = my - c$, giving

$1 = ad - bc = a(my - c) - (mx - a)c = amy - ac - cmx + ac = m(ay - cx)$, which means that $m \mid 1$, and that's impossible.

Another way The quadrilateral formed by $(0,0), (a,c), (b,d), (a+b, c+d)$ has area 1, so it contains no interior lattice points by Pick's Theorem. Hence there exists no point (x, y) on the line between $(0,0), (a+b, c+d)$. QED.

□ Given two circles of radius r_1 and r_2, with both external tanges drawn, and one internal tangent drawn. The internal tangent intersects the external tangents at P and Q. What is the relationship between the length of PQ and the lengths of the external tangents?

Solution

Thiếu hình vẽ Here's a significantly simpler solution. See the attached diagram.

By the tangent property:

$PA = PR$ $QD = QR$

Adding those two, we get

$PA + QD = PQ \qquad (1)$

Also

$PB = PS$ $QC = QS$

Adding those two, we get

$PB + QC = PQ \qquad (2)$

Adding (1) and (2) we get

$(PA + PB) + (QC + QD) = 2PQ \implies AB + CD = 2PQ$

But because of the symmetry, we have $AB = CD$, hence

$2AB = 2PQ \implies PQ = AB$

Another way let $PR = x$, $RS = y$, and $SQ = z$ then $AP = x$, $PB = x + y$, $QC = z$, $QD = y + z$, then $AB = 2x + y$, and $CD = 2z + y$, but $AB = CD$, so $x = z$

then $AB = x + y + x = x + y + z = PQ$ as desired

□ Prove that for every nonzero number n may be uniquely represented in the form

$n = \sum_{j=0}^{s} c_j 3^j$

where Yes c_j is neither -1 or 0 or 1.

Solution

I'm just going to ignore the $c_j \neq 0$. For uniqueness suppose we have

$\sum c_j 3^j = \sum b_j 3^j$

where there exists some j such that $c_j \neq b_j$. Then

$\sum (c_j - b_j) 3^j = 0.$

Let s be the largest integer such that $c_j - b_j \neq 0$. Then $|(c_s - b_s)3^s| \geq 3^s$. However,

$\left|\sum_{j=0}^{s-1} (c_j - b_j) 3^j\right| \leq \sum_{j=0}^{s-1} |(c_j - b_j)3^j| \leq 2\sum_{j=0}^{s-1} 3^j = 3^s - 1.$

So

$|\sum (c_j - b_j) 3^j| \geq |(c_s - b_s)3^s| - \left|\sum_{j=0}^{s-1} (c_j - b_j) 3^j\right| \geq 3^s - (3^s - 1) = 1,$

which means it cannot be zero and our assumption was false.

□ Edit's grandmother's great grandmother's age was 1/31 of her own birth year when she died. (Count her age in full years.) How old was she in 1900?

Solution

Since she obviously lived before and after 1900, we can put $1900-x$ as the year of birth and $1900+y$ as the year of death, giving

$x+y=\frac{1}{31}(1900-x) \iff 32x+31y=1900$

By any of the known ways of solving linear diophantics in two variables, we get $(x,y)=(-31n+9, 32n+52), n\in\mathbb{Z}$. Since both x and y must be non-negative, we get $n\in\{-1,0\}$, yielding $(x,y)\in\{(9,52),(40,20)\}$. Therefore we have two possibilites:

1. She was born in 1891 and died in 1952, meaning that in 1900 she was 9 years old;
2. She was born in 1860 and died in 1920, meaning that in 1900 she was 40 years old.

(By an abundance of words "great" in the problem, I guess the creator opted for the second solution.)

☐ Show that $\tan n\theta = \frac{\binom{n}{1}\tan\theta-\binom{n}{3}\tan^3\theta\ldots}{\binom{n}{0}-\binom{n}{2}\tan^2\theta\ldots}$.

Solution

$$\begin{aligned}
\tan n\theta &= \frac{\sin n\theta}{\cos n\theta} \\
&= \frac{\Im\{e^{in\theta}\}}{\Re\{e^{in\theta}\}} \\
&= \frac{\Im\{(\cos\theta+i\sin\theta)^n\}}{\Re\{(\cos\theta+i\sin\theta)^n\}} \\
&= \frac{\binom{n}{1}\cos^{n-1}\theta\sin\theta-\binom{n}{3}\cos^{n-3}\theta\sin^3\theta+\ldots}{\binom{n}{0}\cos^n\theta-\binom{n}{2}\cos^{n-2}\sin^2\theta+\ldots} \\
&= \frac{\binom{n}{1}\tan\theta-\binom{n}{3}\tan^3\theta+\ldots}{\binom{n}{0}-\binom{n}{2}\tan^2\theta+\ldots}
\end{aligned}$$

☐ Find a_n if $a_1=4, a_2=9, a_{n+2}=5a_{n+1}-6a_n-2n^2+6n+1, n\geqslant 1$

Solution

We can rewrite the given recursion in two ways as follows.

$$a_{n+2}+(n+2)^2-2\{a_{n+1}+(n+1)^2\}=3\{a_{n+1}+(n+1)^2-2(a_n+n^2)\}$$

$$a_{n+2}+(n+2)^2-3\{a_{n+1}+(n+1)^2\}=2\{a_{n+1}+(n+1)^2-3(a_n+n^2)\}$$

Thus

$$a_{n+1}+(n+1)^2-2(a_n+n^2)=3^{n-1}\{a_2+(1+1)^2-2(a_1+1^2)\}=3^n$$

$$a_{n+1}+(n+1)^2-3(a_n+n^2)=2^{n-1}\{a_2+(1+1)^2-3(a_1+1^2)\}=-2^n$$

Subtracting both sides, yielding $a_n=3^n+2^n-n^2$ $(n\geq 1)$.

☐ Let $P(x)$ be a polynomail with integer coefficients that satisfies $P(17)=10$ and $P(24)=17$. Given that $P(n)=n+3$ has two distinct integer solutions n_1 and n_2, find the product n_1n_2.

Solution

A way to simplify the divisibilities by the Euclidean Algorithm...

$(m-24)|(m-14)\Rightarrow(m-24)|10$

so $m=14,19,22,23,25,26,29,34$ and

$(m-17)|(m-7)\Rightarrow(m-17)|10$

so $m=7,12,15,16,18,19,22,27$.

Just match to get $19\cdot 22=418$.

$\square$ Use AP (addition principle) and MP (multiplication principle) to slove the following problem: Let $x = 1, 2, 3, \ldots 100$ and let $S = (a, b, c)|a, b, c \in X, a < b, a < c$. (1) Find $|S|$.

Solution

For every chosen a, we can pick b and c among $100 - a$ numbers which are greater than a. Since the ordered triples are asked for, and there's no condition on $b < c, b = c$ or $b > c$, we assume the most general case where any of the $100 - a$ numbers can be put instead of b and c, repeating allowed. That gives $(100 - a)^2$ possibilities. Since a can't exceed 99, we get

$|S| = \sum_{a=1}^{99}(100 - a)^2 = \sum_{n=1}^{99} n^2 = \frac{99 \cdot 100 \cdot 199}{6} = 328350$

$\square$ Can you solve in constants k,j,n,m?

$(x - \frac{j^2 - k^2}{4x})^2 + (y - \frac{n^2 - k^2}{4y})^2 = k^2$

$(x + \frac{j^2 - k^2}{4x})^2 + (y - \frac{n^2 - k^2}{4y})^2 = j^2$

$(x - \frac{j^2 - k^2}{4x})^2 + (y + \frac{n^2 - k^2}{4y})^2 = n^2$

$(x + \frac{j^2 - k^2}{4x})^2 + (y + \frac{n^2 - k^2}{4y})^2 = m^2$

Solution

After expanding the squares, we get

$$\begin{aligned}
x^2 - \frac{j^2 - k^2}{2} + \left(\frac{j^2 - k^2}{4x}\right)^2 + y^2 - \frac{n^2 - k^2}{2} + \left(\frac{n^2 - k^2}{4y}\right)^2 &= k^2 \\
x^2 + \frac{j^2 - k^2}{2} + \left(\frac{j^2 - k^2}{4x}\right)^2 + y^2 - \frac{n^2 - k^2}{2} + \left(\frac{n^2 - k^2}{4y}\right)^2 &= j^2 \\
x^2 - \frac{j^2 - k^2}{2} + \left(\frac{j^2 - k^2}{4x}\right)^2 + y^2 + \frac{n^2 - k^2}{2} + \left(\frac{n^2 - k^2}{4y}\right)^2 &= n^2 \\
x^2 + \frac{j^2 - k^2}{2} + \left(\frac{j^2 - k^2}{4x}\right)^2 + y^2 + \frac{n^2 - k^2}{2} + \left(\frac{n^2 - k^2}{4y}\right)^2 &= m^2
\end{aligned}$$

Subtracting the second, the third and the fourth equation from the first we get

$$\begin{aligned}
x^2 - \frac{j^2 - k^2}{2} + \left(\frac{j^2 - k^2}{4x}\right)^2 + y^2 - \frac{n^2 - k^2}{2} + \left(\frac{n^2 - k^2}{4y}\right)^2 &= k^2 \\
-(j^2 - k^2) &= k^2 - j^2 \\
-(n^2 - k^2) &= k^2 - n^2 \\
-(j^2 - k^2) - (n^2 - k^2) &= k^2 - m^2
\end{aligned}$$

Therefore, the second and the third equations are redundant, since they reduce to $0 = 0$, and the fourth equation can be valid if and only if $-j^2 + k^2 - n^2 + k^2 = k^2 - m^2 \iff k^2 + m^2 = j^2 + n^2$. In that case, the fourth equation also becomes redundant, and the system is reduced to the first equation:

$x^2 + \left(\frac{j^2 - k^2}{4x}\right)^2 + y^2 + \left(\frac{n^2 - k^2}{4y}\right)^2 = \frac{n^2 + j^2}{2}$

which obviously has infinitely many solutions.

$\square$ Let $\triangle ABC$ be a triangle with unequal sides. Let $D \in [AC]$ and $E \in [AB]$ such that $\widehat{EDB} = \widehat{BCD}$. If $|BC| = |AD| = 2$ and $|AE| = |DC| = 1$, then what is $|EB|$?

Solution

Standard markings α, β, γ. Denote $\phi = \angle CBD$. Then $\angle ADE = \phi$

Sine Law for $\triangle CBD$: $\frac{2}{\sin(\gamma+\phi)} = \frac{1}{\sin\phi}$

Sine Law for $\triangle ADE$: $\frac{2}{\sin(\alpha+\phi)} = \frac{1}{\sin\phi}$

Therefore $\sin(\alpha+\phi) = \sin(\gamma+\phi)$, which gives either $\alpha+\phi = \gamma+\phi$ or $(\alpha+\phi)+(\gamma+\phi) = \pi$

The first possibility yields $\alpha = \gamma \iff CB = AB$ and that's extraneous since the given triangle is scalene.

The second possibility yields $2\phi = \beta \iff CD : DA = CB : BA \iff BA = 4 \iff EB = 3$

□ solve for x:

$(5+2\sqrt{6})^{\sin x} + (5-2\sqrt{6})^{\sin x} = 2\sqrt{3}$

Solution

Put $u = (\sqrt{3}+\sqrt{2})^{2\sin x}, v = (\sqrt{3}-\sqrt{2})^{2\sin x}$. Then

$u + v = 2\sqrt{3}, uv = 1$

Hence u, v are the solutions of $t^2 - 2t\sqrt{3} + 1 = 0$, and those are $t_{1,2} = \sqrt{3} \pm \sqrt{2}$

For $(\sqrt{3}+\sqrt{2})^{2\sin x} = \sqrt{3}+\sqrt{2}$ we get $\sin x = \frac{1}{2}$

For $(\sqrt{3}+\sqrt{2})^{2\sin x} = \sqrt{3}-\sqrt{2}$ we get $\sin x = -\frac{1}{2}$

Therefore the solutions are $x = \pm\frac{\pi}{6} + 2k\pi \vee x = \pm\frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$

□ What are both primes $p > 0$ for which $\frac{1}{p}$ has a purely periodic decimal expansion with a period 5 digits long? [Note: $\frac{1}{37} = 0.\overline{027}$ starts to repeat immediately, so it's purely periodic. Its period is 3 digits long.]

Solution

Five-digit periodic numbers have the form $\frac{k}{99999} = \frac{k}{3^2 \cdot 41 \cdot 271}$, hence the desired numbers are 41 and 271: $\frac{1}{41} = 0.\overline{02439}, \frac{1}{271} = 0.\overline{00369}$

□ Let a, b, c, be random integers 1-9. What is the expected value of the zeros of the quadratic f(x) with coefficients a, b, and c?

Solution

$$
\begin{aligned}
E &= \sum_{a,b,c} (x_1(a,b,c)P(a,b,c) + x_2(a,b,c)P(a,b,c)) \\
&= \sum_{a,b,c} \left(-\frac{b}{a} \cdot \frac{1}{9^3}\right) \\
&= -\frac{1}{9^3} \sum_{a,b,c} \frac{b}{a} \\
&= -\frac{1}{9^3} \sum_{a,b} \left(9 \cdot \frac{b}{a}\right) \\
&= -\frac{1}{9^2} \sum_{a} \left(\sum_{b} \frac{b}{a}\right) \\
&= -\frac{1}{9^2} \sum_{a} \left(\frac{1}{a} \cdot \frac{9 \cdot 10}{2}\right) \\
&= -\frac{5}{9} \sum_{a} \frac{1}{a} \\
&= -\frac{5}{9} \left(1 + \frac{1}{2} + \cdots + \frac{1}{9}\right)
\end{aligned}
$$

The last expression can be reduced to a fraction, giving $E = -\frac{7129}{4536}$

□ If $a_n = \frac{2^n+2(-1)^n}{2^n-(-1)^n}$ for $n \geqslant 1$, find the recursive equation $a_{n+1} = f(a_n)$.

Solution

$a_n = \frac{p_n}{q_n}$ $p_n = 2^n + 2(-1)^n$ $q_n = 2^n - (-1)^n$

Solving the system to isolate the powers, $p_n - q_n = 3(-1)^n$ $p_n + 2q_n = 3(2)^n$

Identifying the recursion, $p_{n+1} - q_{n+1} = (-1)(p_n - q_n)$ $p_{n+1} + 2q_{n+1} = (2)(p_n + 2q_n)$

Solving the system, $p_{n+1} = 2q_n$ $q_{n+1} = p_n + q_n$

Finally, $a_1 = 0$ $a_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \frac{2q_n}{p_n+q_n} = \frac{2}{a_n+1}$.

□ Determine the shape of triangle ABC such that $\sin C = \frac{\sin A+\sin B}{\cos A+\cos B}$.

Solution

$\sin C = \sin(\pi - (A + B)) = \sin(A + B) = \sin A \cos B + \cos A \sin B$, which gives

$\sin A + \sin B = (\cos A + \cos B)(\sin A \cos B + \cos A \sin B)$

$\sin A + \sin B = \sin A \cos A \cos B + \sin B \cos^2 A + \sin A \cos^2 B + \cos A \cos B \sin B$

$\sin A - \sin A \cos^2 B + \sin B - \sin B \cos^2 A = \sin A \cos A \cos B + \cos A \cos B \sin B$

$\sin A \sin^2 B + \sin B \sin^2 A = \cos A \cos B(\sin A + \sin B)$

$(\sin A + \sin B)(\cos A \cos B - \sin A \sin B) = 0$

$(\sin A + \sin B) \cos(A + B) = 0$

Since $\sin A+\sin B \neq 0$ for the angles in a triangle ($B \neq -A, B \neq A+\pi$), it follows $A+B = \frac{\pi}{2} = C$, hence the triangle is right.

□ A hexagon is inscribed in a circle. Proceeding clockwise the lengths of its edges are $1, 1, 1, 2, 2, 2$. What is the area of this hexagon?

Solution

Let α be the central angle corresponding to the side of the length 1 and β the central angle corresponding to the side of the length 2. If we rearrange the sides thus: $1, 2, 1, 2, 1, 2$, we see that $\alpha + \beta = 120°$. Hence if the radius of the circle is r, then in the triangle $1, 2, r\sqrt{3}$ there's an angle $\frac{180°-\alpha}{2} + \frac{180°-\beta}{2} = 120°$ between 1 and 2. Applying Cosine Law we find

$3r^2 = 1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cos 120° = 7 \iff r = \sqrt{\frac{7}{3}}$

Now $S = 3 \cdot \frac{1}{2}\sqrt{\frac{7}{3} - \frac{1^2}{4}} + 3 \cdot \frac{2}{2}\sqrt{\frac{7}{3} - \frac{2^2}{4}} = \frac{3}{2} \cdot \frac{5}{2\sqrt{3}} + 3 \cdot \frac{2}{\sqrt{3}} = \frac{13}{4}\sqrt{3}$

□ The incircle of triangle ABC touched side BC at D. Let the midpoint of BC be M. Show that MI bisects AD where I is the incentre of triangle ABC.

Solution

Proof 1. Suppose w.l.o.g. that $b > c$. Denote the intersections $S \in BC \cap AI$, $N \in AD \cap MI$. Show easily that $MD = \frac{b-c}{2}$, $MS = \frac{a(b-c)}{2(b+c)}$, $SD = \frac{(b-c)(p-a)}{b+c}$ and $\frac{IA}{IS} = \frac{b+c}{a}$. Apply the Menelaus' theorem to the transversal $\overline{MIN}$ for the triangle ADS : $\frac{MS}{MD} \cdot \frac{ND}{NA} \cdot \frac{IA}{IS} = 1 \implies \frac{a(b-c)}{2(b+c)} \cdot \frac{2}{b-c} \cdot \frac{ND}{NA} \cdot \frac{b+c}{a} = 1$ $\implies \boxed{NA = ND}$. Apply the Menelaus' theorem to the transversal $\overline{AIS}$ for the triangle NDM : $\frac{AN}{AD} \cdot \frac{SD}{SM} \cdot \frac{IM}{IN} = 1 \implies \frac{1}{2} \cdot \frac{(b-c)(p-a)}{b+c} \cdot \frac{2(b+c)}{a(b-c)} \cdot \frac{IM}{IN} = 1 \implies \boxed{\frac{IN}{IM} = \frac{p-a}{a}}$.

Remark. Denote the projection P of the vertex A to the opposite sideline BC and the intersection $R \in AP \cap MI$. Prove easily that $NR = NI$ and $AR = ID$, i.e. $AR = r$. Example. The orthocenter $H \in MI \iff H \equiv R \iff AH = ID \iff 2R|\cos A| = r \iff |\cos A| = \frac{r}{2R}$ a.s.o.

Lemma (well-known). Given are two concurrent (in the point A) fixed lines d_1, d_2 and four fixed points $\{A, B\} \subset d_1$, $\{C, D\} \subset d_2$. Then the geometrical locus of the point L for which $[LAB] = [LCD]$ is a parallelogram. **Particular case.** Given is a quadrilateral $ABCD$ which is circumscribed to the circle $w = C(I, r)$ Denote the the middlepoints M, N of the diagonals AC, BD. Then $I \in MN$ (the Newton's line). Indeed, $[IAB] + [ICD] = [MAB] + [MCD] = [NAB] + [NCD] = \frac{1}{2} \cdot [ABCD]$.

Proof 2. Consider ABDC as a degenerate tangential quadrilateral with diagonals AD, BC and incircle (I). Newton line of any quadrilateral ABDC connecting midpoints M, N of its diagonals BC, AD is the locus of points P such that area sums $S_{\triangle PAB} + S_{\triangle PDC} = S_{\triangle PBD} + S_{\triangle PCA}$ are equal. If ABDC is tangential with incircle (I, r), then $S_{\triangle IAB} + S_{\triangle IDC} = \frac{r}{2}(AB + DC) = \frac{r}{2}(BD + CA) = S_{\triangle IBD} + S_{\triangle ICA}$, hence $I \in MN$.

Proof 3. Let $w = (I, r), w' = (I', r')$ be respectively the incircle and the A-excircle of $\triangle ABC$

Let DE be a diameter of w. The circle w' touches the side BC at E'. It's known that the point M is the midpoint of DE'

A is the homothety center of the circles w, w'. The directed segments $IE, I'E'$ have the same direction, so the points A, E, E' are collinear.

I is the midpoint of DE M is the midpoint of DE'

So $MI \parallel AE' \Rightarrow MI$ bisects AD

□ Triangle ABC has $BC = 1$ and $AC = 2$. What is the maximum possible value of $\angle A$?

Solution

Proof 1 (synthetical). Suppose that the points B, C are fixed, the values $CA = b$, $CB = a$ are constantly and w.l.o.g. $a < b$. Particularly, $A < 90$. Denote the (fixed) circle $w \equiv w(C, 2)$ and the second intersections A', B' of the circle w with the rays $[AC$, $[AB$ respectively. Therefore, A is maximum $\Longleftrightarrow$ A' is minimum $\Longleftrightarrow$ the length of the cord AB' is minimum $\Longleftrightarrow$ the distance of the center C to the cord AB' is maximum $\Longleftrightarrow AB \perp BC \Longleftrightarrow \sin A = \frac{a}{b}$. i.e. $B = 90 \Longleftrightarrow$ $\boxed{A = \arcsin\frac{a}{b}}$.

Proof 2 (metrical). A From the relation $4\cos^2\frac{A}{2} = \frac{4p(p-a)}{bc} = \frac{(b+c)^2 - a^2}{bc} = 2 + \frac{1}{b} \cdot \left(c + \frac{b^2 - a^2}{c}\right)$ obtain : A is maximum $\Longleftrightarrow \cos^2\frac{A}{2}$ is minimum $\Longleftrightarrow c + \frac{b^2 - a^2}{c}$ is minimum . But $\frac{b^2 - a^2}{c} \cdot c = b^2 - a^2$ (constant). Therefore, A is maximum $\Longleftrightarrow \frac{b^2 - a^2}{c} = c$, i.e. $b^2 = a^2 + c^2 \Longleftrightarrow B = 90 \Longleftrightarrow \sin A = \frac{a}{b}$ $\Longleftrightarrow \boxed{A = \arcsin\frac{a}{b}}$.

□ Find a_n, b_n if $a_1 = 3, b_1 = -3$ and
$$\begin{cases} a_{n+1} = a_n - b_n + n \\ b_{n+1} = b_n - a_n + n^2 \end{cases}$$
for $n \geqslant 1$

Solution

Rewrite the given system of recursion in two ways as follows.

$$a_{n+1} + b_{n+1} = n^2 + n$$

$$a_{n+1} - b_{n+1} - (n+1)^2 - (n+1) - 2 = 2(a_n - b_n - n^2 - n - 2)$$

Thus

$$a_n + b_n = n^2 - n$$

$$a_n - b_n - n^2 - n - 2 = 2^{n-1}(a_1 - b_1 - 1^2 - 1 - 2) = 2^n$$

Solve the system of recursion,yielding $a_n = 2^{n-1} + n^2 + 1,\ b_n = -2^{n-1} - n - 1\ (n \geq 1)$.

$\square$ $5\log_{\frac{x}{9}} x + \log_{\frac{9}{x}} x^3 + 8\log_{9x^2} x^2 = 2$

Solution

$$\begin{aligned} 5\log_{\frac{x}{9}} x + \log_{\frac{9}{x}} x^3 + 8\log_{9x^2} x^2 = &= \frac{5\log_9 x}{\log_9 \frac{x}{9}} + \frac{3\log_9 x}{\log_9 \frac{9}{x}} + \frac{16\log_9 x}{\log_9 9x^2} \\ &= \frac{5\log_9 x}{\log_9 x - 1} + \frac{3\log_9 x}{1 - \log_9 x} + \frac{16\log_9 x}{1 + 2\log_9 x} \\ &= \frac{2\log_9 x}{\log_9 x - 1} + \frac{16\log_9 x}{1 + 2\log_9 x} \end{aligned}$$

Putting $t = \log_9 x$, we get the equation

$$\begin{aligned} \frac{2t}{t-1} + \frac{16t}{1+2t} = 2 &\iff 2t + 4t^2 + 16t^2 - 16t = 2(t + 2t^2 - 1 - 2t) \\ &\iff 20t^2 - 14t = 4t^2 - 2t - 2 \\ &\iff 16t^2 - 12t + 2 = 0 \\ &\iff 8t^2 - 6t + 1 = 0 \\ &\iff t_{1,2} = \frac{6 \pm \sqrt{36 - 32}}{16} \\ &\iff t \in \{\frac{1}{2}, \frac{1}{4}\} \end{aligned}$$

Therefore $x_1 = 9^{t_1} = 3, x_2 = 9^{t_2} = \sqrt{3}$

$\square$ Solve the equation
$\lfloor 3x + \frac{1}{2} \rfloor + \{2x - \frac{1}{3}\} = 8x + 5$
in real numbers.

Solution

From the obvious $A - 1 < \lfloor A \rfloor \leq A$ and $0 \leq \{B\} < 1$ we have $3x - \frac{1}{2} < 8x + 5 < 3x + 3/2$ or $-\frac{11}{10} < x < -\frac{7}{10}$. So $-\frac{94}{15} < 6x + \frac{1}{3} < -\frac{58}{15}$. Since $6x + \frac{1}{3}$ is integer we have $6x + \frac{1}{3} = -6$ or $6x + \frac{1}{3} = -5$ or $6x + \frac{1}{3} = -4$ giving $x = -\frac{19}{18}$, $x = -\frac{8}{9}$ or $x = -\frac{13}{18}$. Checking only $x = -\frac{8}{9}$ works.

$\square$ ten cards 1-10 are arranged in a stack face down so that the first card is removed; the second card is put at the bottom of the stack; the third card is recoved; the fourth card is put at the bottom of the stack; and so on, until only one card remains. The removed cards, in order, are 1-9. The remaining card is 10. In the original stack, wat was the sum of the cards adjacent to card 10?

Solution

It's easiest to go backwards - adding a card at the time to the top and putting the bottom card over it. That gives

$$10 \to \begin{matrix}9\\10\end{matrix} \to \begin{matrix}10\\9\end{matrix} \to \begin{matrix}8\\10\\9\end{matrix} \to \begin{matrix}9\\8\\10\end{matrix} \to \begin{matrix}7\\9\\8\\10\end{matrix} \to \begin{matrix}10\\7\\9\\8\end{matrix} \to \begin{matrix}6\\10\\7\\9\\8\end{matrix} \to \begin{matrix}8\\6\\10\\7\\9\end{matrix} \to \begin{matrix}5\\8\\6\\10\\7\\9\end{matrix} \to \begin{matrix}9\\5\\8\\6\\10\\7\end{matrix} \to \begin{matrix}4\\9\\5\\8\\6\\10\\7\end{matrix} \to \begin{matrix}7\\4\\9\\5\\8\\6\\10\end{matrix} \to \begin{matrix}3\\7\\4\\9\\5\\8\\6\\10\end{matrix} \to \begin{matrix}10\\3\\7\\4\\9\\5\\8\\6\end{matrix} \to \begin{matrix}2\\10\\3\\7\\4\\9\\5\\8\\6\end{matrix} \to \begin{matrix}6\\2\\10\\3\\7\\4\\9\\5\\8\end{matrix} \to \begin{matrix}1\\6\\2\\10\\3\\7\\4\\9\\5\\8\end{matrix}$$

Hence $2+3=5$

$\square$ A, B, C and D are four positive whole numbers with the following properties:

(i) each is less than the sum of the other three, and (ii) each is a factor of the sum of the other three. Prove that at least two of the numbers must be equal. (An example of four such numbers: A=4, B=4, C=2, D=2.)

Solution

By the second condition, there must exist positive integers x, y, z, t such that

$$\begin{aligned} b+c+d &= xa \\ a+c+d &= yb \\ a+b+d &= zc \\ a+b+c &= td \end{aligned}$$

and also, all of them must be at least 2 (if, for example, $x=1$, then $b+c+d=a$ and we must have $b+c+d>a$ by the first condition).

Assume two of the numbers are equal - WLOG we'll take $a=b$. Then $a+c+d=b+c+d \implies xa=yb \implies \frac{x}{y}=\frac{b}{a}=1 \implies x=y$. Now assume that two of x,y,z,t are equal - WLOG we'll take $x=y$. Then $a+b+c+d=xa+a=a(x+1)$ and $a+b+c+d=yb+b=b(y+1)$, hence $a(x+1)=b(y+1) \implies \frac{a}{b}=\frac{y+1}{x+1}=1 \implies a=b$.

Therefore, we've proven that $a=b \iff x=y$, hence the problem is equivalent to proving that two of x,y,z,t must be equal.

Take the initial four equations and add a to the first, b to the second, c to the third and d to the fourth. If we denote $s=a+b+c+d$, then we get

$$\begin{aligned} s &= (x+1)a \\ s &= (y+1)b \\ s &= (z+1)c \\ s &= (t+1)d \end{aligned}$$

which gives

$a=\frac{s}{x+1}, b=\frac{s}{y+1}, c=\frac{s}{z+1}, d=\frac{s}{t+1}$

Therefore

$s=a+b+c+d=\frac{s}{x+1}+\frac{s}{y+1}+\frac{s}{z+1}+\frac{s}{t+1}$

which gives

$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}+\frac{1}{t+1}=1$

Now assume all x,y,z,t are different. Since they are all at least 2, the RHS is at most $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{19}{20}<1$, hence can never be equal to 1. Therefore, at least two of x,y,z,t must be equal. QED

□ The Word Problem: A sporting goods manufacturer makes a 5.00 profit on soccer balls and a 4.00 profit on volleyballs. Cutting requires 2 hours to make 75 soccer balls and 3 hours to make 60 volleyballs. Sewing needs 3 hours to make 75 soccer balls and 2 hours to make 60 volley balls. Cutting has 500 hours available, and Sewing has 450 hours available. How many soccer balls and volley balls should be made to maximize the profit?

Please explain this in detail as to how you get the answer. Thanks.

Solution

For easier calculation, we'll express time in minutes:

Cutting:

For a soccer ball $\frac{2}{75}\cdot 60$ min $=1.6$ min For a volleyball $\frac{3}{60}\cdot 60$ min $=3$ min

Sowing:

For a soccer ball $\frac{3}{75}\cdot 60$ min $=2.4$ min For a volleyball $\frac{2}{60}\cdot 60$ min $=2$ min

Therefore, if we have s soccer balls and v volleyballs, then

$1.6s+3v\leqslant 30000$ (500 hours converted into minutes) $2.4s+2v\leqslant 27000$ (450 hours converted into minutes)

For the profit, we know that $p=5s+4v$

From the first inequality we get $v\leqslant\frac{30000-1.6s}{3}=10000-\frac{8}{15}s$, hence $p\leqslant 5s+4\left(10000-\frac{8}{15}s\right)=\frac{43}{15}s+40000$

From the second inequality we get $v\leqslant\frac{27000-2.4s}{2}=13500-\frac{6}{5}s$, hence $p\leqslant 5s+4\left(13500-\frac{6}{5}s\right)=\frac{1}{5}s+54000$

Now, those conditions for p must be satisfied simultaneously, therefore it would be the best if those expressions have the same value (if possible), because otherwise we'd be limited by the smaller of the two (graphically, it means that we're looking for the intersection of the two lines):

$\frac{43}{15}s+40000=\frac{1}{5}s+54000\iff\frac{8}{3}s=14000\iff s=5250.$

Now from $v\leqslant 10000-\frac{8}{15}s$ we get $v\leqslant 7200$, and from $v\leqslant 13500-\frac{6}{5}s$ we get $v\leqslant 7200$. Therefore, we can produce $v=7200$ volleyballs.

For those values, maximal profit will be $p_{max}=\$5\cdot 5250+\$4\cdot 7200=\$55050$

□ Let $x=cy+bz, y=az+cx, z=bx+ay$. Find $\frac{(x-y)(y-x)(z-x)}{xyz}$ in terms of a,b, c .

Solution

$\frac{x-y}{z}=\frac{cy+bz-az-cx}{z}=-c\frac{x-y}{z}+b-a$, hence $(1+c)\frac{x-y}{z}=b-a\iff\frac{x-y}{z}=\frac{b-a}{1+c}$

Similarly for $\frac{y-z}{x}$ and $\frac{z-x}{y}$.

Thus $E=\frac{(b-a)(c-b)(a-c)}{(1+a)(1+b)(1+c)}$

□ Maximum value of $f(x)=a\sin^2 x+b\sin x\cos x+c\cos^2 x-\frac{1}{2}(a-c)$

Solution

Rewrite the function as

$f(x)=a\frac{1-\cos 2x}{2}+b\frac{\sin 2x}{2}+c\frac{1+\cos 2x}{2}-\frac{a-c}{2}$

$f(x)=\frac{a+c}{2}-\frac{a-c}{2}+\frac{c-a}{2}\cos 2x+\frac{b}{2}\sin 2x$

$f(x)=c+\frac{c-a}{2}\cos 2x+\frac{b}{2}\sin 2x$

The maximum value of $f(x)$ is obviously

$\max f(x) = c + \frac{1}{2}\sqrt{(a-c)^2 + b^2}$

attained for $x = \frac{1}{2}\arctan\frac{b}{c-a}$ (where it is understood that the arctangent takes the value $\pm\frac{\pi}{2}$ if $a = c$, depending on the sign of b. If $a = c$ and $b = 0$, the function is constant and equal to c, so maximum is attained for any value of x.)

□ Solve the equation: $max(x, y) + min(-x, y) = 0$.

Solution

Let alone your using completely incorrect formulas, as $\max(a, b) = \frac{a+b+|a-b|}{2}$ and $\min(a, b) = \frac{a+b-|a-b|}{2}$

Using the correct formulas we get $x + y + |x - y| - x + y - | - x - y| = 0$

$2y + |x - y| = |x + y| \quad (*)$

Squaring:

$4y^2 + 4y|x - y| + x^2 - 2xy + y^2 = x^2 + 2xy + y^2$

$4y|x - y| = 4xy - 4y^2 \quad (**)$

If $y = 0$, then $(*)$ becomes $|x| = |x|$, which is satisfied for all real x. Thus one solution is $x \in \mathbb{R}, y = 0$.

If $y \neq 0$, then $(**)$ becomes $|x - y| = x - y \iff x - y \geqslant 0$, turning $(*)$ into $2y + x - y = |x + y| \iff |x + y| = x + y \iff x + y \geqslant 0$.

Thus we get $-x \leqslant y \leqslant x$ with $x \geqslant 0$ as another solution.

□ Expanding $(1 + 0.2)^{1000}$ by the binomial theorem and doing no further manipulation gives

$$\begin{aligned}&\binom{1000}{0}(0.2)^0 + \binom{1000}{1}(0.2)^1 + \binom{1000}{2}(0.2)^2 + \cdots + \binom{1000}{1000}(0.2)^{1000}\\ &= A_0 + A_1 + A_2 + \cdots + A_{1000},\end{aligned}$$

where $A_k = \binom{1000}{k}(0.2)^k$ for $k = 0, 1, 2, \ldots, 1000$. For which k is A_k the largest?

Solution

For $k > 0$ we can write A_k as

$A_k = \binom{1000}{k}\frac{1}{5^k} = \frac{1000\cdot999\cdot998\ldots(1001-k)}{k!}\frac{1}{5^k} = \frac{1000}{5\cdot1}\cdot\frac{999}{5\cdot2}\cdot\frac{998}{5\cdot3}\cdots\frac{1001-k}{5k}$.

It follows that $A_k = A_{k-1}\frac{1001-k}{5k}$. Hence, A_k will increase as long as $\frac{1001-k}{5k} \geq 1$. Solving the inequality gives $k \leq 166\frac{5}{6}$. Therefore, the largest A_k is A_{166}.

Another way Let n be the value of k such that A_k is the largest. Then $A_n > A_{n-1}$ and $A_n > A_{n+1}$. In other words,

$$\begin{aligned}\binom{1000}{n}(0.2)^n &> \binom{1000}{n-1}(0.2)^{n-1}\\ \binom{1000}{n}(0.2)^n &> \binom{1000}{n+1}(0.2)^{n+1}\end{aligned}$$

From (1), we get

$$\begin{aligned}
\binom{1000}{n}(0.2)^n &> \binom{1000}{n-1}(0.2)^{n-1} \\
\frac{1000!}{n!(1000-n)!}(0.2) &> \frac{1000!}{(n-1)!(1000-n+1)!} \\
\frac{1}{5n!(1000-n)!} &> \frac{1}{(n-1)!(1000-n+1)!} \\
(n-1)!(1000-n+1)! &> 5n!(1000-n)! \\
1000-n+1 &> 5n \\
1001 &> 6n,
\end{aligned}$$

so $n \leq \lfloor 1001/6 \rfloor = 166$.

To check, we can also solve (2), to get

$$\begin{aligned}
\binom{1000}{n}(0.2)^n &> \binom{1000}{n+1}(0.2)^{n+1} \\
\frac{1000!}{n!(1000-n)!} &> \frac{1000!}{(n+1)!(1000-n-1)!}(0.2) \\
\frac{1}{n!(1000-n)!} &> \frac{1}{5(n+1)!(1000-n-1)!} \\
5(n+1)!(1000-n-1)! &> n!(1000-n)! \\
5(n+1) &> 1000-n \\
6n &> 995,
\end{aligned}$$

so $n \geq \lceil 995/6 \rceil = 166$.

If $166 \leq n \leq 166$, then $n = \boxed{166}$.

□ The number of solution of the equation $\{\mathbf{x}\} + \{\frac{\mathbf{1}}{\mathbf{x}}\} = \mathbf{1}$

where $\{\mathbf{x}\}$ denote fractional part of $\mathbf{x}$

Solution

We aim to show that there are infinitely many solutions, and for that it will be sufficient to show that there are infinitely many positive solutions. Put $x = n + a$ where $n = [x], a = \{x\}$ with $n \geqslant 2$ to get

$a + \frac{1}{n+a} = 1$

$(n+a)(1-a) = 1$

$a^2 + (n-1)a + 1 - n = 0$

$a_{1,2} = \frac{1-n\pm\sqrt{(n-1)^2+4(n-1)}}{2}$

Since we need $0 \leqslant a < 1$, we take only the plus sign:

$a = \frac{1-n+\sqrt{n^2+2n-3}}{2} \iff x = n + a = \frac{n+1+\sqrt{n^2+2n-3}}{2}$ for $n \geqslant 2$, which can be rewritten as $x = \frac{n+\sqrt{n^2-4}}{2}$ for $n \geqslant 3$.

Thus there are indeed infinitely many solutions to the initial equation. (Not all of them are exhausted by the above formula, though.)

□ If a, b, c are rationals and $a\sqrt{2} + b\sqrt{3} + c\sqrt{5} = 0$ then show that $a = b = c = 0$ $a\sqrt{2} + b\sqrt{3} = -c\sqrt{5} \implies 2a^2 + 3b^2 + 2ab\sqrt{6} = 5c^2$

Solution

Thus $ab = 0$, since otherwise $\sqrt{6}$ would be rational.

(i) If $a=0$ then $b\sqrt{3}=-c\sqrt{5} \implies b\sqrt{15}=-5c$, hence $b=0$ since otherwise $\sqrt{15}$ would be rational. This in turn yields $c=0$.

(ii) If $b=0$, similar reasoning.

□ Given a sequence $\{a_i\}$, where i is a positive integer. Given that $a_1=a_2=1$ and $a_{n+2}=\frac{2}{a_{n+1}}+a_n$, for $n\geq 1$. a. Find an explicit formula for finding the value of a_k, k is a positive integer, if there is. b. Determine the value of a_{2011}.

Solution

$a_{n+2}a_{n+1}-a_{n+1}a_n=2$, thus $a_{n+1}a_n$ is an arithmetic sequence. With $a_2a_1=1$, this yields $a_{n+1}a_n=2n-1 \iff a_na_{n-1}=2n-3$

Thus $a_n=\frac{2n-3}{a_{n-1}}=\frac{2n-3}{\frac{2n-5}{a_{n-2}}}=\frac{2n-3}{2n-5}a_{n-2}=\frac{2n-3}{2n-5}\cdot\frac{2n-7}{2n-9}a_{n-4}=\dots$

The product continues while the fractions remain positive.

Therefore $a_k=\prod_{i=0}^{\left[\frac{k-3}{2}\right]}\frac{2k-3-4i}{2k-5-4i}, k\geqslant 3$

Hence $a_{2011}=\frac{4019}{4017}\cdot\frac{4015}{4013}\cdot\dots\cdot\frac{3}{1}$

□ Prove that if 13 divides $3a-2b$, then it also divides a^2+b^2

Solution

We have $3a=13k+2b$ for some integer k.

Then $9a^2=169k^2+52kb+4b^2 \iff 9a^2+9b^2=169k^2+52kb+13b^2$

Thus $13\mid 9(a^2+b^2)$, but as $\gcd(13,9)=1$, this implies $13\mid a^2+b^2$. QED

□ The angle bisectors of triangle ABC intersect its circumcircle at A', B', and C'. Prove that $[A'B'C']=\frac{Rs}{2}$, where R denotes the circumradius and s denotes the semiperimeter of ABC.

Solution

If O is the circumcentre, then $\angle A'OB'=\alpha+\beta$, since $\angle A'AC=\frac{\alpha}{2}\wedge\angle B'BC=\frac{\beta}{2}$. Similarly for the other two.

Therefore $[A'B'C']=\frac{R^2}{2}\left(\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)\right)$

$[A'B'C']=\frac{R^2}{2}(\sin\alpha+\sin\beta+\sin\gamma)$

$[A'B'C']=\frac{R^2}{2}\left(\frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}\right)$

$[A'B'C']=\frac{R^2}{2}\cdot\frac{s}{R}=\frac{Rs}{2}$. QED

□ Let x,y,z,w be different positive real numbers such that $x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{w}=w+\frac{1}{x}=t$. Find t.

Solution

$$x=t-\frac{1}{y}, y=t-\frac{1}{z}, z=t-\frac{1}{w}, w=t-\frac{1}{x}$$

Let $f(x)=w=t-\frac{1}{x}\Rightarrow f^{-1}(x)=\frac{1}{t-x}$

From the four equations above, we can get $ffff(x)=x$.

$$f^{-1}(x)=fff(x)$$

$$\frac{1}{t-x}=fff(x)$$

$$ff(x)=f^{-1}(\frac{1}{t-x})$$

$$\frac{xt^2-t-x}{xt-1}=\frac{t-x}{t^2-xt-1}$$

After expanding and factorizing,

$$2(x^2 - xt + 1) - t^2(x^2 - xt + 1) = 0 \Rightarrow t = \sqrt{2}$$

Another way
$x + \frac{1}{y} = t \iff y = \frac{1}{t-x}$
$y + \frac{1}{z} = t \iff \frac{1}{t-x} + \frac{1}{z} = t \iff t - x + z = tz(t-x) \quad (1)$
$z + \frac{1}{w} = t \iff w = \frac{1}{t-z}$
$w + \frac{1}{x} = t \iff \frac{1}{t-z} + \frac{1}{x} = t \iff t - z + x = tx(t-z) \quad (2)$
Subtracting (2) from (1) we get
$2(z-x) = t^2(z-x) \iff t = \sqrt{2}$ (since $z \neq x$)

$\square$ $ABCD$ is a trapeziod with $AB//DC$ and $AB > DC$. E is a point on AB such that $AE = DC$. AC meets DE and DB at F and G respectively. Find the value of $\frac{AB}{CD}$ for which $\frac{[\triangle DFG]}{[ABCD]}$ is maximum. ($[X_1X_2 \dots X_n]$) denotes the area of the polygon.)

Solution

Thiếu hình vẽ See the attached diagram. First we'll deduce an auxiliary result shown in the Figure 1.

To prove $[APD] = [BPC]$, it's enough to see that $[ACD] = [BCD]$, since they share the base b and the altitude h. And when we subtract $[PCD]$ from both of them, we obtain the result.

As for $S = \frac{abh}{2(a+b)}$, first we note that $\triangle PAB \sim \triangle PCD$, thus $\frac{PM}{PN} = \frac{a}{b}$ and $PM + PN = h$. These two equations yield $PM = \frac{ah}{a+b} \wedge PN = \frac{bh}{a+b}$. Now $S = [ACD] - [PCD] = \frac{bh}{2} - \frac{b \cdot PN}{2} = \frac{b \cdot PM}{2} = \frac{abh}{2(a+b)}$.

Let's now consider the given problem, shown on Figure 2. Since $AE \# CD$, quadrilateral $AECD$ is a paralellogram, hence F is the midpoint of both of its diagonals. Draw $FH \parallel AB$ such that $H \in BD$. Since F is the midpoint of ED, then FH is the midline in $\triangle EBD$, hence $FH = \frac{EB}{2} = \frac{a-b}{2}$. Also, FH being the midline means that the altitude of the trapezoid $FHCD$ is $\frac{h}{2}$.

Now we're ready. To get $S = [DFG]$, we apply the auxiliary result to the trapezoid $FHCD$ and obtain
$S = \frac{\frac{a-b}{2} \cdot b \cdot \frac{h}{2}}{2\left(\frac{a-b}{2} + b\right)} = \frac{(a-b)bh}{4(a+b)}$
Since the area of $[ABCD]$ is $S_0 = \frac{(a+b)h}{2}$, we get
$r = \frac{S}{S_0} = \frac{(a-b)b}{2(a+b)^2} = \frac{\frac{a}{b} - 1}{2\left(\frac{a}{b} + 1\right)^2}$
Put $x := \frac{a}{b}$. Then $r(x) = \frac{x-1}{2(x+1)^2}$ must be maximized, which means that $\frac{1}{r(x)} = \frac{2(x+1)^2}{x-1}$ must be minimized. Write it thus:
$\frac{1}{r(x)} = 2\left(x - 1 + \frac{4}{x-1} + 4\right)$
By AM-GM, this is minimized when $x - 1 = \frac{4}{x-1} \iff x = 3$, and then $r_{\max} = r(3) = \frac{1}{16}$
Thus the required ratio is $AB = 3CD$.

$\square$ Triangle ABC has sides $AB = 13, BC = 14$ and $AC = 15$. E and F are on AB and AC respectively. Triangle AEF is folded along crease EF such that A lies on BC and $EFCB$ is a cyclic quadrilateral after the fold. What is the length of EF?

Solution

Thiếu hình vẽ See the attached diagram for additional notation.

As $\angle BEF + \angle C = 180^\circ \implies \angle AEF = \angle C$ and similarly $\angle AFE = \angle B$, we have that $\triangle AFE \sim \triangle ABC$. Let the similarity factor be k. Then $AE = 13k, AF = 15k$.

Let AD be the altitude of $\triangle ABC$ and AM the altitude of $\triangle AEF$. By Heron's, $[ABC] = 84 \implies AD = 12$, hence $AM = 12k$. If N is the image of A on BC after the folding, then by the problem condition $MN = AM = 12k$. Also, $DC = \sqrt{13^2 - 12^2} = 5$

Draw the bisector AS of $\angle A$. By the Angle Bisector Theorem, we have $SC = 14 \cdot \frac{13}{13+15} = 6.5$, thus $SD = 1.5$.

Note that $\angle EAM = \angle DAC = 90° - \gamma$, hence AS bisects $\angle NAD$ as well. Apply the Angle Bisector Theorem to $\triangle NAD$:

$SD = ND \cdot \frac{AD}{AD+AN}$

$\frac{3}{2} = \sqrt{(24k)^2 - 12^2} \cdot \frac{12}{12+24k}$

$\frac{3}{2} = 12\sqrt{\frac{24k-12}{24k+12}}$

$\sqrt{\frac{2k-1}{2k+1}} = \frac{1}{8}$

$\frac{2k+1}{2k-1} = 64$

$2k + 1 = 128k - 64$

$126k = 65$

$k = \frac{65}{126}$

Now $EF = kBC = \frac{65}{126} \cdot 14 = \frac{65}{9}$

□ The sum of a number and its reciprocal is 1. Find the sum of the n-th power of the number and the n-th power of its reciprocal.

Solution

Using complex numbers, we see that $a = \cos\frac{\pi}{3} \pm i \sin\frac{\pi}{3}$. Hence by De Moivre,

$$a^n + \frac{1}{a^n} = 2\cos\frac{n\pi}{3} = \begin{cases} 1 & n \equiv \pm 1 \pmod 6 \\ -1 & n \equiv \pm 2 \pmod 6 \\ -2 & n \equiv 3 \pmod 6 \\ 2 & n \equiv 0 \pmod 6 \end{cases}$$

□ There are p arithmetic progressions and each of the progressions has n members. Initial term of each progression is 1,2,3,. . . ,p respectably and the common difference of each progression respectably is 1,3,5. . . ,2p-1 . Prove that the sum of all progressions is equal to np(np+1)/2.

Solution

The sum of the first terms in all the sequences is $1 + 2 + \cdots + p$.

The sum of all the second terms is greater than this by $1 + 3 + \cdots + (2p - 1) = p^2$, which is the same as if all the first terms were increased by p, since $\underbrace{p + p + \cdots + p}_{\text{p times}} = p^2$ - hence it is as if the second terms were $p + 1, p + 2, \ldots, 2p$

With the similar reason, we find that it is as if all the third terms were $2p + 1, 2p + 2, \ldots, 3p$.

etc.

Therefore, it is as if we have all the numbers from 1 to np, and their sum is $\frac{np(np+1)}{2}$.

□ Polygon $A_1A_2 \ldots A_n$ is a regular n-gon. For some integer $k < n$, quadrilateral $A_1A_2A_kA_{k+1}$ is a rectangle of area 6. If the area of $A_1A_2 \ldots A_n$ is 60, compute n.

Solution

Since any regular polygon can admit a circumscribed circle, we have that A_1A_k is a diameter, and so is A_2A_{k+1}. If the center of the circle is O, then $[OA_1A_2] = [OA_kA_{k+1}] = \frac{60}{n}$. But also $\triangle OA_2A_k$ has a same base as $\triangle OA_1A_2$ - namely $OA_1 = OA_k$, and they share the altitude from the vertex A_2. Thus $[OA_2A_k] = [OA_1A_{k+1}] = \frac{60}{n}$.

Therefore $\frac{240}{n} = 6 \implies n = 40$.

□ Solve in $\mathbb{R}^3$:

$$\begin{cases} \frac{4\sqrt{x^2+1}}{x} = \frac{4\sqrt{y^2+1}}{y} = \frac{4\sqrt{z^2+1}}{z} \\ x+y+z = xyz \end{cases}$$

Solution

$\frac{\sqrt{x^2+1}}{x} = \frac{\sqrt{y^2+1}}{y} \implies \frac{x^2+1}{x^2} = \frac{y^2+1}{y^2} \implies \frac{1}{x^2} = \frac{1}{y^2} \implies y = \pm x$

Plugging that into $\frac{\sqrt{x^2+1}}{x} = \frac{\sqrt{y^2+1}}{y}$, we get $y = x$. Similarly $z = y$, hence $x = y = z$. Thus $3x = x^3 \implies x(x^2-3)$. As $x \neq 0$, the solutions are $x = y = z = \sqrt{3}$ and $x = y = z = -\sqrt{3}$

☐ For $n \geq 1$, let a_n denote the number of n-digit strings consisting of the digits 0,1, and 2 respectively, such that no three consecutive terms in the sequence are all different. Find a_n in closed form.

Solution

Let 00_n denote the number of such strings ending in 00.

Let 01_n denote the number of such strings ending in 01.

etc.

Let 22_n denote the number of such strings ending in 22.

Then we have following equations:

$a_n = 00_n + 01_n + 02_n + 10_n + 11_n + 12_n + 20_n + 21_n + 22_n \quad (*)$ $00_n = 00_{n-1} + 10_{n-1} + 20_{n-1} \quad (1)$ $01_n = 00_{n-1} + 10_{n-1}$ $02_n = 00_{n-1} + 20_{n-1}$ $10_n = 01_{n-1} + 11_{n-1}$ $11_n = 01_{n-1} + 11_{n-1} + 21_{n-1} \quad (2)$ $12_n = 11_{n-1} + 21_{n-1}$ $20_n = 02_{n-1} + 22_{n-1}$ $21_n = 12_{n-1} + 22_{n-1}$ $22_n = 02_{n-1} + 12_{n-1} + 22_{n-1} \quad (3)$

Summing up the last nine equations and using $(*)$, we get

$a_n = 2a_{n-1} + 00_{n-1} + 11_{n-1} + 22_{n-1} \quad (**)$

Summing up $(1), (2), (3)$ and using $(*)$, we get

$00_n + 11_n + 22_n = a_{n-1}$

Thus $(**)$ becomes

$a_n = 2a_{n-1} + a_{n-2}$.

The characteristic equation is $t^2 - 2t - 1 = 0$ and the roots are $t_{1,2} = 1 \pm \sqrt{2}$, hence $a_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$

Since $a_1 = 3$ and $a_2 = 9$, we get

$A(1+\sqrt{2}) + B(1-\sqrt{2}) = 3$ $A(3+2\sqrt{2}) + B(3-2\sqrt{2}) = 9$

The solution is $A = B = \frac{3}{2}$.

So finally $a_n = \frac{3}{2}\left((1+\sqrt{2})^n + (1-\sqrt{2})^n\right)$

☐ Find value of x in $4x^2 - 40[x] + 51 = 0$

Solution

If $n = [x], a = \{x\}$, then $4(n+a)^2 - 40n + 51 = 0 \iff a = \frac{\sqrt{40n-51}}{2} - n$

Since we must have $0 \leqslant \frac{\sqrt{40n-51}}{2} - n < 1$, by solving the inequalities we get $n \in \{2, 6, 7, 8\}$.

Hence $x = n + a = \frac{\sqrt{40n-51}}{2} \in \left\{\frac{\sqrt{29}}{2}, \frac{\sqrt{189}}{2}, \frac{\sqrt{229}}{2}, \frac{\sqrt{269}}{2}\right\}$

☐ An analog clock is manufactured with an hour hand and a minute hand that are indistinguishable from one another. (There is no second hand on the clock.) At some point in time between noon and midnight, a photograph of the clock face is to be taken. At how many such times will it be impossible to discern the time the photograph was taken from the image of the clock face? (Assume that the position of the clock's hands can be determined with complete accuracy.)

Solution

Let the radius corresponding to the number 12 on the dial be our reference point and let all the angles be measured clockwise, in the interval $(0, 2\pi)$

If at some point in time the hour leg takes an angle α, then the minute leg has traveled the angle 12α, since it moves 12 times faster. Therefore the actual angle it takes (against the reference radius) is $\beta = 12\alpha - 2\pi\left[\frac{12\alpha}{2\pi}\right] = 12\alpha - 2\pi\left[\frac{6\alpha}{\pi}\right]$ (we're cutting all the full circles it may have traveled in the meantime).

But if the positions of the legs are to be legitimately interchangeable, then the angles must satisfy the above relation the other way round, i.e. $\alpha = 12\beta - 2\pi\left[\frac{6\beta}{\pi}\right]$.

So we got ourselves a system:

$\beta = 12\alpha - 2\pi\left[\frac{6\alpha}{\pi}\right]$

$\alpha = 12\beta - 2\pi\left[\frac{6\beta}{\pi}\right]$

Plugging the first equation into the second, we get

$\alpha = 12\left(12\alpha - 2\pi\left[\frac{6\alpha}{\pi}\right]\right) - 2\pi\left[\frac{6\left(12\alpha-2\pi\left[\frac{6\alpha}{\pi}\right]\right)}{\pi}\right]$

$\alpha = 144\alpha - 24\pi\left[\frac{6\alpha}{\pi}\right] - 2\pi\left[\frac{72\alpha}{\pi} - 12\left[\frac{6\alpha}{\pi}\right]\right]$

$\alpha = 144\alpha - 24\pi\left[\frac{6\alpha}{\pi}\right] - 2\pi\left[\frac{72\alpha}{\pi}\right] + 24\pi\left[\frac{6\alpha}{\pi}\right]$

$\alpha = 144\alpha - 2\pi\left[\frac{72\alpha}{\pi}\right]$

$143\alpha = 2\pi\left[\frac{72\alpha}{\pi}\right]$

Let's put $x = \frac{\alpha}{2\pi}$. Then the equation becomes

$143x = [144x]$

Since $[144x]$ is an integer, it follows that $x = \frac{n}{143}$ for some integer n. Then

$n = \left[\frac{144n}{143}\right] = n + \left[\frac{n}{143}\right] \iff \left[\frac{n}{143}\right] = 0$

Since we're not counting either midnight or midday, we have $0 < \frac{n}{143} < 1 \iff 1 \leqslant n \leqslant 142$. Therefore there are 142 moments in half a day when the positions of the legs are legitimately interchangeable. The actual times are easily calculated: $\alpha_n = 2\pi x_n = 2\pi\frac{n}{143}$, and since the hour leg travels 2π in 12 hours, we have $t_n = \frac{12n}{143}$ o'clock where $1 \leqslant n \leqslant 142$

□ ind value of x in the equation $x^2 + [\frac{x}{2}] + [\frac{x}{3}] = 10$

Solution

Since $x^2 = 10 - \left[\frac{x}{2}\right] - \left[\frac{x}{3}\right]$, it follows that x^2 is integer, hence $x = \sqrt{n}$ or $x = -\sqrt{n}$ for some natural n.

Let's try $x = \sqrt{n}$. Then $f(n) = n + \left[\frac{\sqrt{n}}{2}\right] + \left[\frac{\sqrt{n}}{3}\right]$. Plugging $n = 8$ we get $f(8) = 9$ and plugging $n = 9$ we get $f(9) = 11$, hence the initial equation has no solution in this case.

If $x = -\sqrt{n}$, then $f(n) = n + \left[-\frac{\sqrt{n}}{2}\right] + \left[-\frac{\sqrt{n}}{3}\right]$. Plugging $n = 13, 14, 15$, we get $f(13) = 9, f(14) = 10, f(15) = 11$, hence the only solution is $x = -\sqrt{14}$.

□ Find all pairs of polynomials $P(x)$ and $Q(x)$ such that for all x that are not roots of $Q(x)$, $\frac{P(x)}{Q(x)} - \frac{P(x+1)}{Q(x+1)} = \frac{1}{x(x+2)}$.

Solution

Let $f(x) = \frac{P(x)}{Q(x)}$. Then $f(x) - f(x+1) = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right) = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1} - \frac{1}{x+1} - \frac{1}{x+2}\right)$

Hence $\frac{P(x)}{Q(x)} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{2x+1}{2x(x+1)}$, so $P(x) = (2x+1)R(x) \wedge Q(x) = 2x(x+1)R(x)$ where $R(x)$ is an arbitrary polynomial.

□ find value of x that satisfy $xx = [x]$

Solution

Let $x = n + a$ where $n = [x], a = \{x\}$. By the given equation we have $a \cdot |x| = |n|$. If $a = 0$ then $n = 0 \implies x = 0$. If $0 < a < 1$ then $|n| < |x| \iff x > 0 \iff n \geqslant 0$.

Now $(n+a)a = n \implies a^2 + na - n = 0 \implies a = \frac{-n+\sqrt{n^2+4n}}{2}$ (the negative solution is discarded).

If $n = 0$ then $a = 0$. If $n \geqslant 1$ then $n^2 + 4n > n^2 \implies a > 0$ and $n^2 + 4n < n^2 + 4n + 4 \implies a < \frac{-n+n+2}{2} = 1$, hence for all $n \geqslant 0$ we have $0 \leqslant a < 1$.

Therefore there are infinitely many solutions: $x_n = n + a = \frac{n+\sqrt{n^2+4n}}{2}$ for integer $n \geqslant 0$.

$\square$ Triangle ABC is drawn. Three parallels are drawn through each of the vertices. The line through A meets BC (extended if necessary) at X. The lines through B, and C meet AC, and BC, at Y and Z, respectively, all extended if necessary. Prove that the area of XYZ is twice the area of triangle ABC.

Solution

Thiếu hình vẽ See the attached diagram.

Since $XA \parallel CZ$, points X and A are equidistant from the line CZ, hence $[XCZ] = [ACZ]$.

Similarly, $YB \parallel CZ \implies [YCZ] = [BCZ]$

Therefore $[XCZ] + [YCZ] = [ACZ] + [BCZ] = [ABC]$

Also, $XA \parallel YB \implies \triangle XCA \sim \triangle BCY \implies \frac{CX}{CA} = \frac{CB}{CY} \iff CX \cdot CY = CA \cdot CB$, and since the vertical angles ACB and XCY are equal, we get $[XCY] = [ABC]$.

Therefore $[XYZ] = [XCY] + [XCZ] + [YCZ] = 2[ABC]$

$\square$ For a triangle ABC, let $\tan A$, $\tan B$, $\tan C$ be natural numbers. Find $\tan A$, $\tan B$, $\tan C$.

Solution

For a triangle it holds $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Hence we must find natural m, n, p such that $m + n + p = mnp$. WLOG take $m \leqslant n \leqslant p$. Then $mnp = m + n + p \leqslant 3p \implies mn \leqslant 3$. Therefore $(m, n) \in \{(1, 1), (1, 2), (1, 3)\}$. Solving these cases for p, we find that the only solution satisfying $m \leqslant n \leqslant p$ is $(m, n, p) = (1, 2, 3)$.

Thus $\{\tan A, \tan B, \tan C\} = \{1, 2, 3\}$.

$\square$ The first 44 positive integers are appended in order to to form the largest number $N = 123456789101112\ldots\ldots424344$. What is the remainder when N is divided by 45?

Solution

The number is obviously $\equiv 4 \pmod 5$. Let's see about the sum of its digits modulo 9.

$1 + 2 + \cdots + 9 = 45$

The sum of the digits of the numbers from 10 to 19 is $10 \cdot 1 + 45 = 55$

The sum of the digits of the numbers from 20 to 29 is $10 \cdot 2 + 45 = 65$

The sum of the digits of the numbers from 30 to 39 is $10 \cdot 3 + 45 = 75$

The sum of the digits of the numbers from 40 to 44 is $5 \cdot 4 + 10 = 30$

So the total sum of the digits is $45 + 55 + 65 + 75 + 30 = 270 \equiv 0 \pmod 9$

Thus the number is $\equiv 4 \pmod 5$ and $\equiv 0 \pmod 9$, hence it's $\equiv 9 \pmod{45}$.

$\square$ If $\sin x + \sin y = a$, $\cos x + \cos y = b$, prove that $\sin(x + y) = \dfrac{2ab}{a^2 + b^2}$.

Solution

If $u = \cos x + i \sin x, v = \cos y + i \sin y$, then $u + v = b + ai$. Also $|u| = 1 \implies u\bar{u} = 1 \implies \bar{u} = \frac{1}{u}$ and similarly $\bar{v} = \frac{1}{v}$.

Now $\sin(x + y) = \Im\{\cos(x + y) + i \sin(x + y)\} = \Im\{uv\}$.

But $uv = \frac{u+v}{\frac{u+v}{uv}} = \frac{u+v}{\frac{1}{u}+\frac{1}{v}} = \frac{u+v}{\bar{u}+\bar{v}} = \frac{b+ai}{b-ai}$

$uv = \frac{(b+ai)^2}{a^2+b^2} = \frac{b^2-a^2}{a^2+b^2} + i\frac{2ab}{a^2+b^2}$

Hence $\sin(x + y) = \Im\{uv\} = \frac{2ab}{a^2+b^2}$

$\square$ Let points D, E, and F be on sides BC, AC, and AB, respectively. Let point D' be on BC such that D' is on the line formed by reflecting line AD through the angle bisector of $\angle A$, and similarly define BE' and CF'. Prove that if AD, BE, and CF are concurrent, then so are the lines AD', BE', and CF'.

Solution

Observe that AD and AD' are isogonal. So by **Steiner**'s theorem we have: $\frac{BD}{DC} \cdot \frac{BD'}{D'C} = \left(\frac{AB}{AC}\right)^2$ (1) .

Analogous we obtain $\frac{CE}{EA} \cdot \frac{CE'}{E'A} = \left(\frac{BC}{BA}\right)^2$ (2), $\frac{AF}{FB} \cdot \frac{AF'}{F'B} = \left(\frac{CA}{CB}\right)^2$ (3)

So if AD, BE, CF are concurrent it means that $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$ (4) (**Ceva**'s theorem) .

Multiplying relations (1), (2), (3) $\Longrightarrow \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = 1 \overset{(4)}{\Longleftrightarrow} \frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = 1$ and the conclusion follows.

□ Find the area of a triangle ABC with altitudes of lengths 10 , 15 and 20 .

Solution

Let h_a, h_b, h_c be the altitudes to $\triangle ABC$. Then we have

$$2\triangle = 2|ABC| = ah_a = bh_b = ch_c,$$

and from Heron's formula,

$$4\triangle = \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

Solving the first set of equations for a, b, c and substituting the result into the second, we obtain

$$16\triangle^2 = (2\triangle)^4 \left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\left(-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}-\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}-\frac{1}{h_c}\right),$$

and assuming $\triangle > 0$, we easily find

$$\triangle^{-2} = (h_a^{-1}+h_b^{-1}+h_c^{-1})(-h_a^{-1}+h_b^{-1}+h_c^{-1})(h_a^{-1}-h_b^{-1}+h_c^{-1})(h_a^{-1}+h_b^{-1}-h_c^{-1}).$$

This of course requires that the altitudes in fact form a constructible triangle–which is possible if and only if the right-hand side product is nonnegative.

Substituting the given values for the altitudes in no particular order yields the result $|ABC| = \frac{60^2}{\sqrt{455}}$.

□ Prove, **<u>without the use of a calculator</u>**, that : $\boxed{\sin\ 40^\circ\ <\ \sqrt{\frac{5}{12}}}$. **<u>Proof.</u>** Note first that : $\sin\ 40^\circ\ <\ \sqrt{\frac{5}{12}}$ is equivalent to : $\frac{1-\cos 80^\circ}{2}\ <\ \frac{5}{12}$, or $\cos\ 80^\circ\ >\ \frac{1}{6}$ which is the same

as : $\sin\ 10^\circ\ >\ \frac{1}{6}$. Let $c = \sin\ 10^\circ$. Then $0 < c < 1$. From $\frac{1}{2} = \sin\ 30^\circ = 3\sin\ 10^\circ - 4\sin^3\ 10^\circ = 3c - 4c^3$,

we obtain : $8c^3 - 6c + 1 = 0$. Since $8c^3 > 0$, we must have : $-6c + 1 < 0$. Hence, $c > \frac{1}{6}$, and we are done . Another solution It ist knows $\sin x > \frac{3}{\pi}x$ in this case $(x = \frac{\pi}{18})$ $\sin\frac{\pi}{18} > \frac{3}{\pi}\frac{\pi}{18}$ $\sin\frac{\pi}{18} > \frac{1}{6}$ Q.E.D

□ Compute the coefficient of x^9 in the expansion of $(x^3 + x^2 + 1)^8$

Solution

The idea is to observe that the coefficient of $x_1^{k_1}x_2^{k_2}\cdot\ldots\cdot x_m^{k_m}$ in the multinomial expansion of $(x_1 + x_2 + \cdots + x_m)^n$ is given by

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!k_2!\ldots k_m!},$$

where $k_1 + k_2 + \cdots + k_m = n$. For the case $m = 3$ and $n = 8$ (and for simplicity of notation, we let $x_1 = a$, $x_2 = b$, $x_3 = c$, $k_1 = p$, $k_2 = q$, $k_3 = r$), we then find the coefficient of $a^pb^qc^r$ is $\binom{8}{p,q,r}$. Now since $a = x^3$, $b = x^2$, $c = 1$, we need to consider all integers $0 \le p, q, r \le 8$ such that $3p + 2q = 9$ and

$r = 8 - p - q$. By inspection, the only such solutions are $(p, q, r) \in \{(3, 0, 5), (1, 3, 4)\}$. Therefore, the coefficient of x^9 is
$$\binom{8}{3,0,5} + \binom{8}{1,3,4} = \frac{8!}{3!0!5!} + \frac{8!}{1!3!4!} = 336.$$

☐ Let $M = \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}\right\}$. Prove that any subset of M containing 6 elements has 4 distinct numbers so that the sum of two of them is equal with the sum of the other two .

Solution

Presumably $M = \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}\right\}$. Then $2M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and the statement stays the same. Let $1 \le x_1 < x_2 < x_3 < x_4 < x_5 < x_6 \le 9$ be 6 elements. Then, by contradiction, assume $x_2 - x_1$, $x_4 - x_3$ and $x_6 - x_5$ distinct (hence their sum at least $1 + 2 + 3 = 6$), and $x_3 - x_2$, $x_5 - x_4$ distinct (hence their sum at least $1 + 2 = 3$). It follows $x_6 - x_1 \ge 6 + 3 = 9$, absurd. Notice that this remains true no more for 5 elements - a model is $\{1, 2, 3, 5, 7\}$; and neither for 6 elements out of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ - a model is $\{1, 2, 3, 5, 7, 10\}$.

☐ How many number of possible pair(s) (x, y) of positive integers are there that satisfy this equation:

$x^{2}\check{}\,y! = 2001$

Solution

If $y \geqslant 7$, then $y! + 2001 \equiv 6 \pmod 7$, but 6 is not a quadratic residue modulo 7, hence there's no solution.

Then we simply check $y \in \{1, 2, 3, 4, 5, 6\}$ and find that the only solution is $(x, y) = (45, 4)$

☐

let : $\begin{array}{l} x_1 = y_1 = \sqrt{3} \\ x_{n+1} = x_n + \sqrt{x_n^2 + 1} \\ \& \\ y_{n+1} = \frac{y_n}{1+\sqrt{1+y_n^2}} \end{array}$

then prove that : $2 < (x_{2010^{2010}}).(y_{2010^{2010}}) < 3$

Solution

We'll use following substitutions: $x_n = \cot \alpha_n, y_n = \tan \beta_n$ where α_n, β_n are acute angles.

Then $\cot \alpha_{n+1} = \frac{\cos \alpha_n}{\sin \alpha_n} + \frac{1}{\sin \alpha_n} = \frac{2\cos^2 \frac{\alpha_n}{2}}{2 \sin \frac{\alpha_n}{2} \cos \frac{\alpha_n}{2}} = \cot \frac{\alpha_n}{2}$

Therefore $x_n = \cot \frac{\alpha_1}{2^{n-1}}$. Since $\cot \alpha_1 = \sqrt{3} \implies \alpha_1 = \frac{\pi}{6}$, we get $x_n = \cot \frac{\pi}{6 \cdot 2^{n-1}}$

Similarly, $\tan \beta_{n+1} = \tan \frac{\beta_n}{2}$, and with $\tan \beta_1 = \sqrt{3} \implies \beta_1 = \frac{\pi}{3}$, we have $y_n = \tan \frac{\pi}{3 \cdot 2^{n-1}}$

If we now put $\theta = \frac{\pi}{6 \cdot 2^{n-1}}$, then $\frac{1}{x_n} = \tan \theta$ and $y_n = \tan 2\theta$. Using double-angle formulas, we have $y_n = \frac{2\frac{1}{x_n}}{1 - \frac{1}{x_n^2}} = \frac{2x_n}{x_n^2 - 1}$

Therefore $x_n y_n = \frac{2x_n^2}{x_n^2 - 1} = 2 + \frac{2}{x_n^2 - 1}$

Now we note that cotangent is a decreasing function, hence x_n is an increasing sequence (because the argument of the corresponding cotangent is decreasing). Therefore, for $n > 1$, we have $x_n > x_1 = \sqrt{3}$. That yields $x_n^2 - 1 > 2 \implies 0 < \frac{2}{x_n^2 - 1} < 1 \implies 2 < 2 + \frac{2}{x_n^2 - 1} < 3$.

Therefore, for every $n > 1$, we have $2 < x_n y_n < 3$, hence for $n = 2010^{2010}$ as well.

☐ a_n is the integer nearest to $\sqrt{n}$ Find the value of $\sum_{n=1}^{1980} \frac{1}{a_n}$

Solution

If k is the integer closest to $\sqrt{n}$, then $\left(k - \frac{1}{2}\right)^2 < n < \left(k + \frac{1}{2}\right)^2 \implies k^2 - k + 1 \leqslant n \leqslant k^2 + k$

Thus we have $(k^2 + k) - (k^2 - k + 1) + 1 = 2k$ numbers with the above property.

Since $1980 = 44^2 + 44$, the desired sum can be split thus:

$S = \sum_{k=1}^{44} \sum_{n=k^2-k+1}^{k^2+k} \frac{1}{a_n} = \sum_{k=1}^{44} \sum_{n=k^2-k+1}^{k^2+k} \frac{1}{k} = \sum_{k=1}^{44} \frac{2k}{k} = 88$

$\square$ Solve the following.
$\frac{1}{[x]} + \frac{1}{[2x]} = x - [x] + \frac{1}{3}$

Solution

Let $x = n + a$ where $n = [x], a = \{x\}$.

Case 1. $0 \leqslant a < \frac{1}{2}$.

Then $[2x] = 2n$, hence $\frac{3}{2n} = a + \frac{1}{3} \iff a = \frac{9-2n}{6n}$

We must have $0 \leqslant \frac{9-2n}{6n} < \frac{1}{2}$

The LHS yields $n \in \{1, 2, 3, 4\}$, and the RHS yields $n \leqslant -1 \vee n \geqslant 2$.

Hence the combined solution is $n \in \{2, 3, 4\}$

$n = 2 \implies a = \frac{9-2n}{6n} = \frac{5}{12} \implies x = n + a = \frac{29}{12}$

$n = 3 \implies x = \frac{19}{6}$

$n = 4 \implies x = \frac{97}{24}$

Case 2. $\frac{1}{2} \leqslant a < 1$.

Then $[2x] = 2n + 1$, hence $\frac{1}{n} + \frac{1}{2n+1} = a + \frac{1}{3}$.

By the constraint we have $\frac{5}{6} \leqslant \frac{1}{n} + \frac{1}{2n+1} < \frac{4}{3}$, thus obviously $n > 0$.

For $n = 1$ the value of the expression is $\frac{4}{3}$, which doesn't satisfy, and for $n = 2$ the value of the expression is $\frac{7}{10} < \frac{5}{6}$. For $n > 2$ the value is decreasing, hence we have no solution in this case.

Conclusion. The solutions are $x \in \left\{\frac{29}{12}, \frac{19}{6}, \frac{97}{24}\right\}$

$\square$ How many n-digit base-4 numbers are there that start with the digit 3 and in which each digit is exactly one more or one less than the previous digit? (For example, 321010121 is such a 9-digit number.)

Solution

Let a_n be the quantity of such numbers, and let $0_n, 1_n, 2_n, 3_n$ denote the respective quantities of such numbers ending in $0, 1, 2, 3$.

Then

$$\begin{aligned}
a_n &= 0_n + 1_n + 2_n + 3_n \\
0_{n+1} &= 1_n \\
1_{n+1} &= 0_n + 2_n \\
2_{n+1} &= 1_n + 3_n \\
3_{n+1} &= 2_n
\end{aligned}$$

Summing up (2) to (5) we get $a_{n+1} = a_n + 1_n + 2_n$. Summing up (3) and (4) we get $1_{n+1} + 2_{n+1} = a_n \iff 1_n + 2_n = a_{n-1}$, thus $a_{n+1} = a_n + a_{n-1}$. Since $a_1 = 1$ (number 3) and $a_2 = 1$ (number 32), we get simple Fibonacci sequence, hence $a_n = F_n$.

$\square$ Given a triangle ABC which inscribed in a circle with radius $\sqrt{\frac{5}{2}}$ and let the area of the triangle is 1. If $2\sin(A + B)\sin C = 1$, then find the side lengths of the triangle.

Solution

$\sin(A + B) = \sin C \implies \sin C = \frac{1}{\sqrt{2}} \implies C = \frac{\pi}{4}$.

Therefore the central angle corresponding to c is $\frac{\pi}{2}$, which yields $c = R\sqrt{2} = \sqrt{5}$

Now $ab = \frac{4[ABC]R}{c} = 2\sqrt{2}$

By Heron's, $2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = 16[ABC]^2$, hence

$16 + 10a^2 + 10b^2 - a^4 - b^4 - 25 = 16$

$a^4 + b^4 - 10(a^2 + b^2) + 25 = 0$

$a^4+b^4=(a^2+b^2)^2-2a^2b^2=(a^2+b^2)^2-16$, hence

$(a^2+b^2)^2-10(a^2+b^2)+9=0 \implies a^2+b^2\in\{1,9\}$

Since $a^2+b^2\geqslant 2ab$, we take $a^2+b^2=9$

Now $a^2+b^2=9\wedge a^2b^2=8 \implies \{a^2,b^2\}=\{1,8\} \iff \{a,b\}=\{1,2\sqrt{2}\}$.

□ Let ABC be a right triangle with hypotenuse BC. Suppose that M is the midpoint of BC and H is the feet of the perpendicular dropped from A onto BC. A point P, distinct from A, is chosen on the opposite ray of ray AM. Let the line through H perpendicular to AB intersect PB at Q; and let the line through H perpendicular to AC meet PC at R. Prove that A is the orthocenter of triangle PQR.

Solution

Let us use barycentric coordinates with respect to $\triangle ABC$. Thus, the coordinates of P lying on the A-median of $\triangle ABC$ can be written as $P\equiv(1:k:k)$ for $k\in\mathbb{R}$. then:

$\implies CP\equiv kx-y=0\ ,\ BP\equiv kx-z=0$.

Therefore, the infinity point of the line BP is $B_\infty\equiv(1:-1-k:k)$

Since $H\equiv(0:S_C:S_B)$, the parallel ℓ from H to AB has equation $S_Bx+S_By-S_Cz=0$. Hence the coordinates of $R\equiv CP\cap\ell$ are $R\equiv(S_C:kS_C1+k)S_B)\implies AR\equiv S_B(1+k)y-kS_Cz=0$.

Keeping in mind that $S_A=0\iff 90^\circ$, infinite point T_∞ of the orthogonal gradient to AR is

$T_\infty\equiv(-kS_BS_C+S_CS_B(1+k):-S_BS_C(1+k):S_BS_Ck)\equiv(1:-1-k:k)$

$T_\infty\equiv B_\infty\implies RA\perp PQ$. Similarly, we'll get $QA\perp PR$ and the conclusion follows.

Another approach.

Consider the homothety ι with center H,ratio AB/AC and rotation angle 90°. Hence $\iota(A)=C$ and let $\iota(P)=S$. Let R' be the intersection of PC with SA. We now show that $R\equiv R'$, which implies $\angle QAB=\angle SCA=\angle RCA$, hence $QA\perp PR$ and $RA\perp PQ$, hence A is the orthocenter of $\triangle PQR$.

Let T be the intersection of AP and SC. Since $PA\perp SC$ and $\angle CAT=\angle HCA$, $HT\parallel AC$. Moreover, $\overline{AP}/\overline{CS}=\overline{AB}/\overline{AC}$. Therefore, we can reformulate the problem as follows: [i]Let $\triangle ABC$ be a triangle with $\angle BCA=90^\circ$. Consider points P and Q on the rays CA and CB, such that $\overline{AQ}/\overline{BP}=\overline{BC}/\overline{CA}$. Let S be the intersection of AP and BQ. Then the foot T of the perpendicular from S onto AB is the isometric conjugate of the foot D from C onto AB.[/i]

To prove this, it's enough to show that

$$\frac{\overline{BT}}{\overline{TA}}=\frac{\overline{AC}^2}{\overline{BC}^2}=\frac{\overline{AD}}{\overline{DB}}.$$

Let M and N be the feet of the perpendiculars from P and Q onto AB, respectively. We get $\overline{NA}=\overline{MB}$ and hence $\overline{BN}=\overline{AM}$ as well. Since $\triangle BTS\sim\triangle BNQ$ and $\triangle ATS\sim\triangle AMP$, we get

$$\frac{\overline{BT}}{\overline{TA}}=\frac{\overline{BS}\cdot\overline{AP}}{\overline{AS}\cdot\overline{QB}}=\frac{[APB]}{[AQB]}=\frac{\overline{BP}\cdot\overline{AC}}{\overline{AQ}\cdot\overline{BC}}=\frac{\overline{AC}^2}{\overline{BC}^2},$$

which is what we had to prove. □

□ Solve the equation $tan^{-1}\sqrt{x^2+x}+sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$

Solution

Since $|\sin\theta|\leq 1$, we must have $0\leq\sqrt{x^2+x+1}\leq 1$, or $-1\leq x\leq 0$. But then $x^2+x\leq 0$ on this interval, so the only permissible values of x for which the left-hand side is defined are $x=-1$ and $x=0$, yielding $x^2+x=0$, and hence

$$\tan^{-1}\sqrt{x^2+x}+\sin^{-1}\sqrt{x^2+x+1}=\tan^{-1}0+\sin^{-1}1=\frac{\pi}{2}.$$

Thus the only real-valued solutions are $x = -1, 0$.

□ Given $S_n = \binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \dots\dots\dots + \binom{n}{n-1}\binom{n}{n}$, $\frac{S_{n+1}}{S_n} = \frac{15}{4}$,find the sum of two possible values of n.

Solution

The sum S_n is a special case of Vandermonde's identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k},$$

for nonnegative integers m, n, r, for which there are a variety of proofs.* With the choice $m = n$ and $r = n - 1$, we immediately obtain

$$S_n = \binom{2n}{n-1}.$$

Consequently,

$$\begin{aligned}\frac{15}{4} = \frac{S_{n+1}}{S_n} &= \frac{\binom{2n+2}{n}}{\binom{2n}{n-1}} \\ &= \frac{(2n+2)(2n+1)}{n(n+2)} \\ &= 4 + \frac{1}{n} - \frac{3}{n+2}.\end{aligned}$$

Simplifying and solving easily gives $n = 2$, $n = 4$ as solutions.

* For a combinatorial proof, count the number of ways to choose r objects from $m + n$ distinct objects which are grouped into two sets of m and n objects each. Clearly this is $\binom{m+n}{r}$, but it is also the sum of the number of ways to select k objects from the group of m objects and $r - k$ objects from the group of n objects, for each $k = 0, 1, 2, \dots, r$, or $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$.

□ In triangle ABC, $AC = 13$, $AB = 14$ and $BC = 15$. E is the foot of the angle bisector of angle A on segment BC and F is the foot of the angle bisector of angle B on segment AC. If P is the intersection of segments EA and FB, what is $\sin \angle EPF$?

Solution

The point P, being the intersection of the angle bisectors of $\triangle ABC$, is the incenter. Consider $\triangle ABP$, whose altitude from P to $\overline{AB}$ we call $PH = r$, where r is the inradius of $\triangle ABC$. We calculate $s = (13+14+15)/2 = 21$, and by writing the area in two ways, $|\triangle ABC| = rs = \sqrt{s(s-a)(s-b)(s-c)}$, or

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{8 \cdot 7 \cdot 6}{21}} = 4.$$

Thus

$$\begin{aligned}AP &= \sqrt{(s-b)^2 + r^2} = \sqrt{8^2 + 4^2} = 4\sqrt{5}, \\ BP &= \sqrt{(s-a)^2 + r^2} = \sqrt{6^2 + 4^2} = 2\sqrt{13}.\end{aligned}$$

Again considering the area in two ways,

$$|\triangle ABP| = \frac{1}{2}(AP)(BP)\sin \angle APB = \frac{1}{2}(AB)(PH),$$

or equivalently,

$$\sin \angle APB = \frac{14 \cdot 4}{4\sqrt{5} \cdot 2\sqrt{13}} = \frac{7}{\sqrt{65}}.$$

But $\angle APB = \angle EPF$, so $\sin \angle EPF = 7/\sqrt{65}$.

$\square$ Calculate : $\sum_{i=1}^{n} \frac{1}{\cos(i.\alpha).\cos((i+1).\alpha)}$ for $\alpha \in (-\frac{\pi}{2n}, \frac{\pi}{2n})$.

Solution

We observe that

$$\begin{aligned} \frac{\sin x}{\cos(k+1)x \cos kx} &= \frac{\sin((k+1)x - kx)}{\cos(k+1)x \cos kx} \\ &= \frac{\sin(k+1)x \cos kx - \sin kx \cos(k+1)x}{\cos(k+1)x \cos kx} \\ &= \tan(k+1)x - \tan kx. \end{aligned}$$

Hence,

$$\begin{aligned} S &= \sum_{k=1}^{n} \frac{1}{\cos kx \cos(k+1)x} \\ &= \frac{\tan(n+1)x - \tan x}{\sin x}. \end{aligned}$$

$\square$ find the solutions to $1 + [x] = [nx]$ where n is a natural number and x is a real number.

Solution

First, we see that when $n = 1$, there is no solution, since the condition would imply $1 = 0$. So suppose $n > 1$. We then observe that for all x, $x < 1 + \lfloor x \rfloor \leq x + 1$ and $nx - 1 < \lfloor nx \rfloor \leq nx$. Therefore, $nx - 1 < x + 1$ and $x < nx$ from which it follows that $0 < x < \frac{2}{n-1}$. For $n > 2$, the right-hand side of this inequality is less than 1, so we then have $1 + \lfloor x \rfloor = 1$, and therefore we require $\lfloor nx \rfloor = 1$, or

$$1/n \leq x < 2/n, \quad n > 2.$$

If $n = 2$, then there are two sub-cases. When $0 < x < 1$, we have $1 = \lfloor 2x \rfloor$, or $1/2 \leq x < 1$; and when $1 \leq x < 2$, we have $2 = \lfloor 2x \rfloor$, or $1 \leq x < 3/2$. So we can summarize the solution as follows:

$n = 1$: No solution.

$n = 2$: $x \in [1/2, 3/2)$.

$n > 2$: $x \in [1/n, 2/n)$.

$\square$ For how many integers n is $n^2 + n + 1$ a divisor of $n^{2010} + 20$?

Solution

We first claim that $n^{2010} \equiv 1 \pmod{n^2 + n + 1}$ for all positive integers n. To see why, note that

$$\begin{aligned} n^{2010} - 1 &= (n^3)^{670} - 1 \\ &= (n^3 - 1) \sum_{k=0}^{669} n^{3k} \\ &= (n-1)(n^2 + n + 1) \sum_{k=0}^{669} n^{3k}. \end{aligned}$$

Therefore, $n^2 + n + 1 | n^{2010} - 1$, and the claim immediately follows.

Therefore, $n^{2010} + 20 \equiv 21 \pmod{n^2 + n + 1}$, and it is now easy to see that we need to check only those integers for which $n^2 + n - 20 \leq 0$; i.e., $-5 \leq n \leq 4$. Substitution gives the solutions $n \in \{-5, -3, -2, -1, 0, 1, 2, 4\}$.

$\square$

find value of x that satisfy $[x]x = 1991x$

Solution

Let $[x] = n, \{x\} = a$. Then $na = 1991(n+a) \implies a = \frac{1991n}{n-1991} = 1991 + \frac{1991^2}{n-1991}$.

By definition we must have $0 \leqslant a < 1$, hence

$-1991 \leqslant \frac{1991^2}{n-1991} < -1990$

$1990 < \frac{1991^2}{1991-n} \leqslant 1991$

$\frac{1}{1991} \leqslant \frac{1991-n}{1991^2} < \frac{1}{1990}$

$1991 \leqslant 1991 - n < \frac{1991^2}{1990}$

$0 \leqslant -n < \frac{1991^2}{1990} - 1991 = \frac{1991}{1990}$

$-\frac{1991}{1990} < n \leqslant 0$

Therefore $n \in \{-1, 0\}$.

$n = -1 \implies a = \frac{1991\cdot(-1)}{-1-1991} = \frac{1991}{1992} \implies x = n + a = -\frac{1}{1992}$

$n = 0 \implies a = 0 \implies x = 0$

Hence $x \in \left\{-\frac{1}{1992}, 0\right\}$

□ Two circles T_1 and T_2 are internally tangent at A and T_1 is bigger than T_2. A variable tangent of T_2 cuts T_1 at B, C. Then the locus of the incenter of $\triangle ABC$ is another circle tangent to T_1, T_2 through A.

Solution

Let V be the tangency point of T_2 with BC. It is known that AV bisects $\angle BAC$. Then

$\frac{AI}{IV} = \frac{CA+AB}{BC}$

If ray AV cuts T_1 at P, by Ptolemy's theorem for $ABPC$ we have

$BC \cdot AP = CA \cdot PB + AB \cdot PC$

Since $PB = PC \implies \frac{AP}{PB} = \frac{CA+AB}{BC} \implies \frac{AI}{IV} = \frac{AP}{PB}$

Note that $\triangle PAB \sim \triangle PVB$ are similar because of $\angle VBP = \angle BAP$, thus we have $PB^2 = AP \cdot PV$. Combining this one with the previous expression yields

$\frac{IV^2}{AI^2} = \frac{PV}{AP}$. But $PV = AP - AV$ and $IV = AV - AI$

$\implies \left(\frac{AW}{AI} - 1\right)^2 = 1 - \frac{AV}{AP}$

Ratio $\frac{AV}{AP} =$ const is the coefficient k of the direct homothecy taking T_1 into T_2. Therefore, locus of the incenter I is the homothetic circumference of T_2 under the homothety with center A and coefficient $\frac{1}{\sqrt{1-k}+1}$.

□ Prove (or explain) why there are no polyhedra having exactly seven edges.

Solution

Denote by F, V, E the number of faces, vertices and edges of the polyhedron. Since each face contains at least three edges, then the number of edges will be $\geq \frac{3}{2}F$, since each edge lies on two faces, in other words, $3F \leq 2E$. Analogously, at each vertex, at least three edges come together and each of them connects two vertices, then it follows that $3V \leq 2E$. By combining these two latter inequalities with Euler's formula $F + V - E = 2$, we obtain $3F + 3V = 3E + 6 \leq 4E \implies E \geq 6$.

Furthermore, $3E + 6 = 3F + 3V \leq 2E + 3V$, from which $E + 6 \leq 3V \leq 2E$ and analogously, we'll have $E + 6 \leq 3F \leq 2E$. Combining both inequalities with $E \geq 6$, we get $F \geq 4$ and $V \geq 4$. These inequalities show the imposibility of a seven-edged Eulerian polyhedron, since between $E + 6 = 13$ and $2E = 14$ there is no integer number. Obviously, equalities hold when each face is a triangle and each vertex is a concurrency point of three edges, i.e. the tetrahedron.

□ Let triangle ABC, AB=AC. $P \in$ triangle ABC prove that: $PA^2 + PB.PC \leq AB^2$

Solution

Let M be the midpoint of BC and WLOG assume that P lies inside $\triangle ABM$. Let $Q \in \overrightarrow{CP}$ such that $PB = PQ$. The circumcenter O' of $\triangle QBC$ lies on AM such that $\angle BO'C = 2\angle BQC = \angle BPC$. Thus the power of P WRT (O') is $PC \cdot PQ = PC \cdot PB = O'C^2 - O'P^2$.

$S \equiv PC \cap AM$ and N is the midpoint of CP. For each P lying on AB, the midpoint of the cevian CP lies on the C-midline. Therefore, for each P inside $\triangle ABM$, the midpoint N of CP lies inside $\triangle AMC \Longrightarrow S$ is between P, N.

On the other hand, since O' is the midpoint of the arc BPC, the ray $\overrightarrow{CP}$ is internal to $\angle O'CB$ $\Longrightarrow O'$ is between S and A. Then it follows that the orthogonal projection X of O' onto CP lies between C and the orthogonal projection Y of A onto CP. Thus

$AC^2 - AP^2 = YC^2 - YP^2 = 2PC \cdot NY$

$O'C^2 - O'P^2 = XC^2 - XP^2 = 2PC \cdot NX$

$NY \geq NX \Longrightarrow AC^2 - AP^2 \geq O'C^2 - O'P^2 \Longrightarrow AB^2 - PA^2 \geq PB \cdot PC.$

□ In a parallelogram ABCD with $\angle A < 90$, the circle with diameter AC meets the lines CB and CD again at E and F, respectively, and the tangent to this circle at A meets BD at P. Show that P, F, and E are collinear.

Solution

Let O be the center of the parallelogram and Q the orthogonal projection of A on the diagonal DB. Since $AF \perp DC$ and $AE \perp CB$, the quadrilaterals $ABEQ$ and $ADFQ$ are both cyclic $\Longrightarrow$ $\angle DQF = \angle DAF = \angle BAE = \angle BQE$. Therefore, AQ and DB are the internal and external bisector of $\angle EQF$, Thus, the perpendicular bisector of EF meets DB at the midpoint of the arc EFQ of $\odot(EFQ) \Longrightarrow OEFQ$ is cyclic. EF is the radical axis of $\odot(OEFQ)$ and $\odot(O)$, DB is the radical axis of $\odot(OEFQ)$ and $\odot(AQO)$, the tangent to $\odot(O)$ at A is the radical axis of (O) and $\odot(AQO)$ $\Longrightarrow AP, DB, EF$ concur at the radical center P of (O), $\odot(AQO)$, $\odot(OEFQ)$. Hence, P, F, E are collinear.

□ ABC triangle has sides a,b,c and P is a point in ABC triangle and $m(\widehat{APB} = m(\widehat{APC} = m(\widehat{BPC} = 120°$. If A(ABC)=S $|AP| + |BP| + |CP| = \sqrt{\frac{a^2+b^2+c^2}{2} + 2S\sqrt{3}}$

Solution

Construct outwardly on the sides BC, CA, AB of $\triangle ABC$ the equilateral triangles BCA', CAB', ABC' whose centers are X, Y, Z. Then it's well-known that $P \equiv (X) \cap (Y) \cap (Z)$ and $P \equiv AA' \cap BB' \cap CC'$. The lines PA, PB, PC are pairwise radical axes of $(X), (Y), (Z)$. Thus, the sidelines of $\triangle XYZ$ are perpendicular to PA, PB, PC, respectively $\Longrightarrow \triangle XYZ$ is equilateral and A, B, C are the reflections of P across YZ, ZX, XY. If $X', Y'Z'$ denote the projections of P onto YZ, ZX, XY, then $PA + PB + PC = 2(PX' + PY' + PZ')$.

Let L be the side-lenght of $\triangle XYZ$. By Viviani's theorem, the sum $(PX' + PY' + PZ')$ equals the altitude of $\triangle XYZ \Longrightarrow PA + PB + PC = \sqrt{3}L$. Thus, it remains to find the side-lenght L of $\triangle XYZ$ in terms of AB, AC, BC

By cosine law in $\triangle AYZ$, kepping in mind that AY, AZ are circumradii of the equilateral $\triangle CAB', \triangle ABC'$, we have

$L^2 = AY^2 + AZ^2 - AY \cdot AZ \cdot 2\cos(A + 60°)$

$L^2 = \frac{AB^2+AC^2}{3} + \frac{AB \cdot AC \cdot (\sqrt{3}\sin A - \cos A)}{3}$

Using the identities

$AB \cdot AC \cdot \sin A = 2S$, $BC^2 = AB^2 + AC^2 - AB \cdot AC \cdot 2\cos A$

$\Longrightarrow L^2 = \frac{AB^2+AC^2+BC^2}{6} + \frac{2\sqrt{3}}{3}S$

$L = \sqrt{\frac{AB^2+AC^2+BC^2}{6} + \frac{2\sqrt{3}}{3}S}$

$PA + PB + PC = \sqrt{3}L = \sqrt{\frac{AB^2+AC^2+BC^2}{2} + 2\sqrt{3}S}$

□ In $\triangle ABC$, points H, I, and J lie on lines AB, BC, and CA respectively. BJ and CH intersect at P, CH and AI intersect at Q, AI and BJ intersect at R, $CH \perp AB$, and $\frac{3BH}{BA} = \frac{3AJ}{AC} = \frac{3CI}{CB} = \frac{AR}{AQ} = \frac{mHP}{HC} = 1$. Compute m.

Solution

Thiếu hình vẽ Here's the solution using area ratios. See the attached diagram.

If $[PHB] = x$, then $[PHA] = 2x$, since $HA = 2HB$. Similarly, $[RAJ] = y \implies [RCJ] = 2y$ and $[QCI] = z \implies [QBI] = 2z$. Since $AR = QR$, we have $[CQR] = [CAR] = 3y$ and also $[PAR] = [PQR] = u$. Since $BH = \frac{1}{2}AH$, we have $[BPQ] = \frac{1}{2}[APQ] = u$.

Now

$$[BHC] = \frac{1}{3}[ABC] \iff 2[BHC] = [AHC] \iff 2(x + u + 3z) = 2x + 2u + 6y \iff \boxed{z = y}$$

Similarly, $2[CAI] = [BAI] \iff 2(6y + z) = 3x + 3u + 2y \iff 3u = 10y + 2z - 3x$, but as $z = y$, we get $3u = 12y - 3x \iff \boxed{u = 4y - x}$

Also, $2[ABJ] = [CBJ] \iff 2(3x + u + y) = 2u + 5y + 3z \iff 6x = 3y + 3z$, but as $z = y$, we get $\boxed{x = y}$, which in turn yields $\boxed{u = 3x}$ and $\boxed{z = x}$

Now $m = \frac{CH}{PH} = \frac{[CHB]}{[PHB]} = \frac{3z+u+x}{x} = \frac{3x+3x+x}{x} = 7$

□ $\triangle ABC$ is a triangle with side lengths a, b, c. D, E, F denote the midpoints of BC, CA, AB. $EF = \frac{1}{2}a = x$, $FD = \frac{1}{2}b = y$ and $DE = \frac{1}{2}c = z$. Triangles $\triangle AFE$, $\triangle BFD$, $\triangle CDE$ are rotated about EF, FD, DE in such a way that A, B, C coincide at P producing the tetrahedron $PDEF$ with edges $PD = EF = x$, $FD = PE = y$, $DE = PF = z$. Then prove that the volume V of $PDEF$ is given by

$$V^2 = \frac{1}{72}(x^2 + y^2 - z^2)(x^2 + z^2 - y^2)(y^2 + z^2 - x^2)$$

Solution

It is clear that the projection of P on the face DEF is the orthocenter H of $\triangle ABC$. Hence, if the A-altitude AH_a meets EF at D', the length of the altitude h on the face ABC is given by

$h^2 = (AD')^2 - (AD' - HH_a)^2 = (AH + HH_a)HH_a - (HH_a)^2 = AH \cdot HH_a$

But $AH \cdot HH_a$ is the power k^2 of the negative inversion that takes the circumcircle of $\triangle ABC$ into its nine-point circle. Hence, $h^2 = k^2 = \frac{1}{2}(a^2 + b^2 + c^2) - 4R^2$.

$V^2 = \frac{1}{9}[\triangle DEF]^2 \left(\frac{1}{2}(a^2 + b^2 + c^2) - 4R^2\right)$

But, keeping in mind that $4R^2 = \frac{a^2b^2c^2}{64[\triangle DEF]}$, we have

$V^2 = \frac{16(x^2+y^2+z^2)[\triangle DEF]^2 - 8x^2y^2z^2}{72}$

Using Heron's formula $[\triangle DEF]^2 = \frac{1}{16}(x + y + z)(x + y - z)(x + z - y)(y + z - x)$

$\implies V^2 = \frac{1}{72}(x^2 + y^2 - z^2)(x^2 + z^2 - y^2)(y^2 + z^2 - x^2)$

□ Let be given triangle ABC with $AB = AC$. E is the midpoint of AB, and G is the centroid of triangle ACE. If O is the circumcenter of triangle ABC, prove that $GO \perp CE$.

Solution

AO and CE are medians of $\triangle ABC$ intersecting at its centroid M. $F \equiv CG \cap AB$ is the midpoint of $\overline{AE}$. Since $\overline{GC} : \overline{GF} = \overline{EB} : \overline{EF} = -2$ and $\overline{GC} : \overline{GF} = \overline{MC} : \overline{ME} = -2$, it follows that $GE \parallel BC$ and $GM \parallel AB \implies OM$ and OE are perpendicular to $BC \parallel GE$ and $AB \parallel GM$, respectively $\implies O$ becomes orthocenter of $\triangle GEM \implies GO \perp ME \equiv CE$, as desired.

□ Let $\triangle ABC$ be an isosceles triangle with $AB = AC = L$. D is a point on BC, such that the radii of the incircle of $\triangle ABD$ and the A-exincircle of $\triangle ADC$ are equal to r. Show that the altitude

h on the leg L is four times r.

Solution

Drop perpendiculars DP and DQ from D to AB and AC, respectively. Then, using the well-known formulae of the inradii and exinradii in terms of altitudes, we get

$DP = \frac{r(L+AD+BD)}{L}$, $DQ = \frac{r(AD+L-DC)}{L}$

$DP + DQ = \frac{r(2L+2AD+BD-DC)}{L}$

On the other hand, $DP + DQ = h \Longrightarrow h = \frac{r(2L+2AD+BD-DC)}{L}$ $(\star)$

Since these two circles are congruent, the tangent segments from D to both are equal.

$L + DC - AD = AD + BD - L \implies 2L = 2AD + BD - DC$

Combining with $(\star)$ yields $h = \frac{r(2L+2L)}{L} = 4r$.

☐ Let M be a point inside the equilateral triangle ABC with side lenght a.

Prove that $MA + MB + MC < 2a$.

Solution

We prove a more general result:

Lemma: M is a point inside $\triangle ABC$ whose shortest side is BC. Then we have that $b + c > MA + MB + MC$.

Draw the parallel to BC passing through M that cuts AC and AB at X, Y, respectively. Draw the altitude AH and WLOG assume that M lies inside $\triangle AHB$. We have $YA > MA$ (1) and since $AC > CB \Longrightarrow AX > XY$, due to the similarity $\triangle ABC \sim \triangle AXY$. Thus, $AX + XC = b > XY + XC$ (2)

By triangle inequality $MX + XC > MC$, $MY + YB > MB$. Adding these two inequalities gives

$CX + XY + YB > MB + MC$ (3)

Adding (1), (2), (3) yields

$b + YA + YB + XC + XY > MA + MB + MC + XY + XC$

$\Longrightarrow b + c > MA + MB + MC.$

☐ Prove that $\sin \frac{\pi}{14}$ is a root of the polynomial equation

$$8x^3 - 4x^2 - 4x + 1 = 0$$

Solution

The proposed problem is equilavent to show that: If a is the side-lenght of a regular 14-gon, then its circumradius R is a real positive solution of $R^3 + a^3 - a^2R - 2aR^2 = 0$.

Let O be the center of the 14-gon and B, C two consecutive vertices. Thus $\angle BOC = \frac{180°}{7}$. There exists two points P, Q on OC, OB such that $BP = PQ = QO = a$. Draw parallels $QT = x$ and $PS = y$ to BC. Then $\triangle CBP$ and $\triangle QOT$ are congruent $\Longrightarrow PC = QT = x$, but $\triangle BCP$ and $\triangle OBC$ are similar

$\Longrightarrow \frac{PC}{BC} = \frac{BC}{R} \Longrightarrow x = \frac{a^2}{R}$ (1)

$QTPS$ is a trapezoid with $PS = QS = y$ and since $\triangle OSP \sim \triangle OBC$, we get:

$\frac{SP}{BC} = \frac{OS}{OB} \Longrightarrow \frac{y}{a} = \frac{y+a}{R} \Longrightarrow y = \frac{a^2}{R-a}$ (2)

$QS = TP = y \Longrightarrow TP + PC = OC - OT \Longrightarrow \ y + x = R - a$ (3)

Combining (1), (2) and (3) yields:

$\frac{a^2}{R-a} + \frac{a^2}{R} = R - a \Longrightarrow R^3 + a^3 = a^2R + 2aR^2.$

☐ In a triangle ABC prove that there is a point D on side AB such that CD is the geometric mean of AD and DB if and only if $\sin A \sin B \leq \sin^2 \frac{C}{2}$

Solution

Let D' be the second intersection of the ray CD with the circumcircle (O) of $\triangle ABC$. From the power of D WRT (O), we have $CD \cdot DD' = AD \cdot BD = CD^2 \implies CD = DD'$. Hence, D' lies on the homologous line ℓ of AB under the homothety with center C and coefficient 2. Thus, there exist such a D on BC if and only if ℓ cuts (O).

Let M, N be the midpoints of AB and the arc AB and H, H' the orthogonal projections of C, D' on AB. Since $\triangle CHD \cong \triangle D'H'D \implies CH = D'H'$. Thereby, there exists at most two points D if and only if $CH < MN$, there exists one point D if and only if $D' \equiv N$, i.e. ℓ is tangent to (O) and there is no such a point D if $CH > MN$. Therefore, the necessary condition for the existence of at least one solution is that $CH \le MN$.

Since $\angle NAM = \angle NCB = \frac{1}{2}\angle C$, we have $MN = \frac{1}{2}AB \cdot \tan\frac{C}{2} \implies$
$CH \le \frac{1}{2}AB \cdot \tan\frac{C}{2} \implies \frac{2CH}{AB} \cdot \cos\frac{C}{2} \cdot \sin\frac{C}{2} \le \sin^2\frac{C}{2} \implies$
$\frac{CH}{AB} \cdot \sin C \le \sin^2\frac{C}{2} \implies \sin A \cdot \sin B \le \sin^2\frac{C}{2}$

$\square$ Prove that for any complex number z,
$|z+1| \ge \frac{1}{\sqrt{2}}$ or $|z^2+1| \ge 1$

Solution

Suppose there exists a complex number z such that $|z+1| \le \frac{1}{\sqrt{2}}$ and $|z^2+1| \le 1$. Write $A := |z|^2 - 1$ and $B := \mathrm{Re}(z)$. We see that the two inequalities are equivalent to

$$A + 2B + \frac{3}{2} \le 0 \text{ and } A^2 + 4B^2 \le 1\,.$$

Thus, $|A| \le 1$, and therefore, $2B \le -\frac{3}{2} - A \le -\frac{1}{2} < 0$. This means

$$\left(\frac{3}{2} + A\right)^2 \le (2B)^2 = 4B^2 \le 1 - A^2\,.$$

Consequently,

$$\left(A + \frac{3}{4}\right)^2 + \frac{1}{16} = A^2 + \frac{3}{2}A + \frac{5}{8} \le 0\,,$$

which is absurd. Hence, for all $z \in \mathbb{C}$, we must have $|z+1| > \frac{1}{\sqrt{2}}$ or $|z^2+1| > 1$.

We can improve the original problem yet another way: for all z, either $|z+1| \ge \sqrt{2-\sqrt{2}}$ or $|z^2+1| \ge 1$.

$\square$ Prove that the area of a right angled triangle which has integral lengths is even.

Solution

Let a, b, c be the sides of right-angled $\triangle ABC$ (with hypotenuse $\overline{AB}$), then
$a^2 + b^2 = c^2$.

Since the quadratic residues $\mod 4$ are 0 and 1, we have two cases (all the terms are divisible by 4, or one is divisible by 4 on the LHS and the rest are not):

Case 1: $a^2 \equiv b^2 \equiv 0 \mod 4$
$a \equiv b \equiv 0 \mod 2 \implies a = 2a_0$ and $b = 2b_0$
$\implies [ABC] = \frac{ab}{2} = 2a_0b_0,$
which concludes this case.

Case 2: WLOG, $a^2 \equiv c^2 \equiv 1 \mod 4$ and $b^2 \equiv 0 \mod 4$
$\implies a \equiv c \equiv 1 \mod 2$ and $b \equiv 0 \mod 2 \implies a = 2a_0 + 1$, $c = 2c_0 + 1$, and $b = 2b_0$
$\implies (2a_0+1)^2 + (2b_0)^2 = (2c_0+1)^2$
$\implies a_0^2 + a_0 + b_0^2 = c_0^2 + c_0$

We already know that $a_0^2 \equiv c_0^2 \equiv 1 \mod 4$, so a_0 and c_0, may be 1 or $-1 \mod 4$. Also, b_0^2 may be congruent to 0 or 1.

Suppose that $b_0^2 \equiv 1 \mod 4$, then
$a_0^2 + a_0 + b_0^2 = c_0^2 + c_0 \implies 1 + a_0 + 1 \equiv 1 + c_0 \mod 4$
$\implies 1 + a_0 \equiv c_0$; contradiction.
So, $b_0 \equiv 0 \mod 2 \implies b_0 = 2k \implies b = 4k$, and therefore,
$[ABC] = \frac{ab}{2} = \frac{(2a_0+1)(4k)}{2} = 2k(2a_0+1)$,
which concludes our last case.
QED Another approach All the non-primitive Pythagorean triplets are given by
$(\{a,b\},c) = (\{2mnk, k(m^2-n^2)\}, k(m^2+n^2))$ for positive integers k, m, n with $m > n$
Thus the required area is $k^2mn(m^2-n^2)$
If k, m, n are all odd, then $m^2 - n^2$ is even. QED

□ For rational numbers $a,\ b$ with $0 < a \leq b \leq 1$, let $f(n) = an^3 + bn$. Find all pairs of $a,\ b$ such that for all integers $f(n)$ is integer and if n is even, then $f(n)$ is even as well.

Solution

Let $n = 1$, then $f(1) = a + b \in \mathbb{Z} \implies a + b \in \{1, 2\}$, since $0 \leq a \leq b \leq 1$.

Let's take the easy case first: if $a + b = 2$, then $a = b = 1$. Hence, $f(n) = n^3 + n$ is always an integer for integer n, and if $2|n$ i.e. n is even, obviously $2|n^3 + n = f(n)$, so $f(n)$ is even (actually, $f(n)$ is even for all integer n, in this case).

Now the other case: $a + b = 1$, we may substitute this into $f(n)$ to get
$f(n) = an^3 + (1-a)n = n + a(n^3 - n)$.

Now, since a is rational, we may express it as $\frac{p}{q}$, where p and q are relatively prime positive integers, and $p < q$; so
$f(n) = n + \frac{p(n^3-n)}{q}$.

But since $f(n)$ is an integer for all integer n, we must have that $\frac{p(n^3-n)}{q} \in \mathbb{Z} \implies q \mid n^3 - n = n(n-1)(n+1)$, for all integer n.

From here, it's easy to see that for that to happen, we must have that $q \in \{2, 3, 6\}$, so we now have three cases:

Case 1: $q = 2 \implies p = 1 \implies f(n) = \frac{1}{2}(n^3 + n)$. But if n is even, then $f(n)$ is not necessarily even, so we dismiss this case.

Case 2: $q = 3 \implies p \in \{1, 2\} \implies f(n) \in \{\frac{1}{3}n^3 + \frac{2}{3}n, \frac{2}{3}n^3 + \frac{1}{3}n\}$. It's easy to see that $2 \mid n \implies 2 \mid f(n)$, by plugging in $n = 2k$.

Case 3: $q = 6 \implies p \in \{1, 5\} \implies f(n) \in \{\frac{1}{6}n^3 + \frac{5}{6}n, \frac{5}{6}n^3 + \frac{1}{6}n\}$, but as in case 1, if n is even, then $f(n)$ is not necessarily even (a simple counterexample does the job, or just by plugging in $n = 2k$), so we dismiss this case.

The only solutions from these cases are from case 2, where we have that $a \in \{\frac{1}{3}, \frac{2}{3}\}$, but since $b = 1 - a$, we have the solutions $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$.

Therefore, the only solutions for (a, b) are $\boxed{(1,1), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3})}$. Another solution By a given condition, $8an^3 + 2bn$ is an even number for all n, hence $4an^3 + bn$ is an integer. Therefore $(4an^3 + bn) - (an^3 + bn) = 3an^3$ is an integer for all n, yielding $a \in \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$.

If $a = \frac{1}{3}$, then $\frac{n^3}{3} + bn = \frac{n^3-n}{3} + n\left(b + \frac{1}{3}\right)$ must be an integer for all n. Since the first term is always an integer, it follows that $b + \frac{1}{3}$ must be an integer (since the complete second term must be an integer for all n), hence $b = \frac{2}{3}$.

If $a = \frac{2}{3}$, then $\frac{2n^3}{3} + bn = \frac{2(n^3-n)}{3} + n\left(b + \frac{2}{3}\right)$ must be an integer for all n, and similarly as above we get $b = \frac{1}{3}$

If $a = 1$, then $n^3 + bn$ must be an integer, hence bn must be an integer (for all n), hence $b = 1$.

□ In a triangle ABC it is $\angle B = 75$ and $BC = 2AD$, where AD is the altitude from A. Prove tht $\angle C = 30$

Solution

Remark. It is well-known that $bc = 2Rh_a$ and $\frac{1}{\tan B} + \frac{1}{\tan C} = \frac{a}{h_a}$. Therefore, $a = \lambda \cdot h_a \iff \frac{1}{\tan B} + \frac{1}{\tan C} = \lambda$.

]**A metrical proof.** It is well-known that $\tan 75^\circ = 2 + \sqrt{3}$. For $\lambda = 2$ obtain : $\frac{1}{\tan B} + \frac{1}{\tan C} = 2 \iff 2 - \sqrt{3} + \frac{1}{\tan C} = 2 \iff \tan C = \frac{1}{\sqrt{3}} \iff C = 30^\circ$.

A synthetical proof. Denote the middlepoint M of the side $[BC]$ and the interior point E for which the triangle ABE is equilateral. $AB = BE$, $AD = BM$, $\widehat{BAD} \equiv \widehat{EBM} \Longrightarrow$ (s.a.s.) $\triangle ABD \equiv \triangle BEM \Longrightarrow EM \perp BC \Longrightarrow$ The point E is the circumcenter of the triangle $ABC \Longrightarrow AB = R$- the circumradius $\Longrightarrow C = 30^\circ$.

□ Find all real values of x, y and z such that $x - \sqrt{yz} = 42$ $y - \sqrt{xz} = 6$ $z - \sqrt{xy} = -30$

Solution

Since xy, yz, zx must be non-negative, x, y, z are all of the same sign. From $x = 42 + \sqrt{yz}$ it follows $x \geqslant 0$, hence all of them are non-negative.

Put $a = \sqrt{x}, b = \sqrt{y}, c = \sqrt{z}$ to get

$$\begin{aligned} a^2 - bc &= 42 \\ b^2 - ac &= 6 \\ c^2 - ab &= -30 \end{aligned}$$

Subtracting (2) from (1) and (3) from (2) we get

$a^2 - b^2 + ac - bc = 36 \iff (a-b)(a+b+c) = 36$ $b^2 - c^2 + ab - ac = 36 \iff (b-c)(a+b+c) = 36$

Hence $a - b = b - c \iff a = 2b - c$. Plugging that into (2) and (3) we get

$$\begin{aligned} b^2 - 2bc + c^2 &= 6 \\ -2b^2 + bc + c^2 &= -30 \end{aligned}$$

Multiplying (4) by 2 and adding to (5) we get

$$-3bc + 3c^2 = -18 \iff c(b-c) = 6$$

Since a, b, c are non-negative, that gives $b - c > 0$, hence from (4) we get $b - c = \sqrt{6}$, and then from (6) we get $c = \sqrt{6}$. Now $b = c + \sqrt{6} = 2\sqrt{6}$ and $a = 2b - c = 3\sqrt{6}$, which gives $(x, y, z) = (54, 24, 6)$

□ $a + b + c = 1$ and a,b,$c \in [0, 1]$ Find the maximum of $(a-b)(b-c)(c-a)$

Solution

Let $c = \max\{a, b, c\}$. For a maximum, we need $P = (a-b)(b-c)(c-a) = (c-a)(c-b)(b-a)$ to be positive, so we take $c \geq b \geq a$. Substituting $c = 1 - b - a$ gives $P = (1 - b - 2a)(1 - 2b - a)(b - a)$ and so clearly for a maximum, $a = 0$ giving $P = b(1-b)(1-2b)$ This is easy to maximise using calculus. Otherwise using AM-GM

$$P = 4\left[\frac{b}{\sqrt{3}-1} \cdot \frac{1-b}{\sqrt{3}+1} \cdot \frac{1-2b}{2}\right] \leq \frac{4}{27}\left[\frac{b}{\sqrt{3}-1} + \frac{1-b}{\sqrt{3}+1} + \frac{1-2b}{2}\right]^3 = \frac{1}{6\sqrt{3}}$$

with equality at $\left(0, \frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{2} + \frac{1}{2\sqrt{3}}\right)$ and cyclic permutations.

$\square$ Prove that $\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin 9x}{\cos 27x}=\frac{1}{2}(\tan 27x-\tan x)$.

Solution

Observe that $\tan 3\alpha-\tan\alpha=\frac{\sin 3\alpha}{\cos 3\alpha}-\frac{\sin\alpha}{\cos\alpha}=\frac{\sin(3\alpha-\alpha)}{\cos 3\alpha\cos\alpha}=\frac{2\sin\alpha\cos\alpha}{\cos 3\alpha\cos\alpha}=$

$\frac{2\sin\alpha}{\cos 3\alpha}\implies\boxed{\tan 3\alpha-\tan\alpha=\frac{2\sin\alpha}{\cos 3\alpha}}\ (*)$. Apply the relation $(*)$ for $\alpha\in\{x,3x,9x\}$:

$$\left\{\begin{array}{c}\tan 27x-\tan 9x=\frac{2\sin 9x}{\cos 27x}\\ \\ \tan 9x-\tan 3x=\frac{2\sin 3x}{\cos 9x}\\ \\ \tan 3x-\tan x=\frac{2\sin x}{\cos 3x}\end{array}\right\| \bigoplus \implies \frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin 9x}{\cos 27x}=\frac{1}{2}(\tan 27x-\tan x)\ .$$

Remark. Prove similarly that $\frac{\cos x}{\sin 3x}+\frac{\cos 3x}{\sin 9x}+\frac{\cos 9x}{\sin 27x}=\frac{1}{2}(\cot x-\cot 27x)$ or by the substitution $x:=\frac{\pi}{2}-x$.

$\square$ Find the integer solutions for $x^2(y-1)+y^2(x-1)=1$.

Solution

The method is classical; although a number theory problem, the answer comes from algebraic inequalities.

We cannot simultaneously have $x,y<1$, so assume $y\geq 1$. Write $(y-1)x^2+y^2x-(y^2+1)=0$ as quadratic in x, of discriminant $\Delta=y^4+4(y-1)(y^2+1)=y^4+4y^3-4y^2+4y-4$ needing to be a perfect square. But $\Delta-(y^2+2y-4)^2=20(y-1)\geq 0$. On the other hand $\Delta-(y^2+2y-3)^2=-2y^2+16y-13<0$ for $y>7$, so we only have to check by hand $y\in\{1,2,3,4,5,6,7\}$. The only ones that works are $y=1$, with $x=2$, and $y=2$, with $x=1$ or $x=-5$. So the complete set of solutions is $(x,y)\in\{(1,2),(2,1),(2,-5),(-5,2)\}$. Another way

Alternatively, we can write the equation as $xy(x+y)-(x^2+y^2)=1\iff xy(x+y)-(x+y)^2+2xy=1$

Putting $a:=x+y,b=xy$ we get $ab-a^2+2b=1\iff b=\frac{a^2+1}{a+2}=\frac{a^2-4+5}{a+2}=a-2+\frac{5}{a+2}$

Hence $a+2\in\{\pm1,\pm5\}\iff a\in\{-7,-3,-1,3\}$, yielding the pairs $(a,b)\in\{(-7,-10),(-3,-10),(-1,2$

Now solving the system in each case (using Vieta and a quadratic) yields the solutions already posted.

$\square$ Let a, b, and c be three real numbers such that $\frac{a(b-c)}{b(c-a)}=\frac{b(c-a)}{c(b-a)}=k>0$ for some constant k. Find the greatest integer less than or equal to k.

Solution

Let $x:=ab,y:=bc,z:=ca$. Then the equations rewrite as

$(k+1)x-ky-z=0$

$-x+(1-k)y+kz=0$

Since $k>0$, we can multiply the first equation by k and add it to the second:

$(k^2+k-1)(x-y)=0$

Assume $x=y\iff ab=bc\iff b(a-c)=0$, which is impossible due to the form of the first fraction given.

Hence $k^2+k-1=0\overset{k>0}{\iff}k=\frac{\sqrt{5}-1}{2}$, and $\lfloor k\rfloor=0$

$\square$

Prove that if $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ and $\frac{a}{c}+\frac{b}{a}+\frac{c}{b}$ are integers, then $|a|=|b|=|c|$.

Solution

Consider the cubic equation whose roots are $\frac{a}{b},\frac{b}{c},\frac{c}{a}$. Then

$K:=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ is integer and $L:=\frac{a}{b}\cdot\frac{b}{c}+\frac{b}{c}\cdot\frac{c}{a}+\frac{c}{a}\cdot\frac{a}{b}$ is also integer.

Therefore by Vieta the equation is $t^3 - Kt^2 + Lt - 1 = 0$. Since all the coefficients are integer, and the roots are rational by the initial assumption, it follows directly (by the Rational Root Theorem) that only possible roots are ± 1. The conclusion follows.

$\square$ Solve the system: $yz = 3y + 2z - 8, zx = 4z + 3x - 8, xy = 2x + y - 1$.

Solution

Rewrite the equations as

$$\begin{cases} (y-2)(z-3) = -2 \\ (x-4)(z-3) = 4 \\ (x-1)(y-2) = 1 \end{cases}$$

Now $\frac{(2)\cdot(3)}{(1)} \iff (x-1)(x-4) = -2 \iff x^2 - 5x + 6 = 0 \iff x \in \{2, 3\}$

So $x = 2 \implies y = 3 \wedge z = 1$ and $x = 3 \implies y = \frac{5}{2} \wedge z = -1$

Hence the solutions are $(x, y, z) \in \left\{(2, 3, 1), \left(3, \frac{5}{2}, -1\right)\right\}$

$\square$

If p is prime and $p = 1 \pmod 4$ such that p can be written as the sum of 4 numbers greater than zero a, b, c, d,prove that $a.d$ cannot equal bc.

Solution

Let $p > 2$ be a prime, $p = a + b + c + d$, with a, b, c, d positive integers. Assume $ad = bc$. We have $a + d \equiv -(b + c) \pmod p$, so $a^2 + 2ad + d^2 \equiv b^2 + 2bc + c^2 \pmod p$, hence $a^2 + d^2 \equiv b^2 + c^2 \pmod p$. Then $(a + b)(a - b) \equiv (c + d)(c - d) \pmod p$. But $a + b \equiv -(c + d) \pmod p$ and $a + c \equiv -(b + d) \pmod p$. Thus $(a + b)(a - b + c - d) \equiv 0 \pmod p$, or $2(a + b)(a + c) \equiv 0 \pmod p$, absurd. We do not need $p \equiv 1 \pmod 4$.

$\square$ Let $a, b, c \in \mathbb{Q}$ such that $a\sqrt[3]{3} + b\sqrt[3]{4} + c\sqrt[3]{5} = 0$. Prove that $a = b = c = 0$.

Solution

There is an advanced theory (where this means that the three cubic roots are linearly independent over $\mathbb{Q}$) that gives immediate answer to such questions. Otherwise just separate into $a\sqrt[3]{3} + b\sqrt[3]{4} = -c\sqrt[3]{5}$, cube it, group again with all rationals on one side, cube it, ..., until you get some linear system of equations with rational coefficients in $\sqrt[3]{6}$ and $\sqrt[3]{6^2}$. Solve this, and use the fact that these two cubic roots are irrational to get conditions on the (rational) coefficients, ..., until you finally should reach $a = b = c = 0$. Boooooring.

$\square$ If x , y, z and are positive integers such that $6xyz + 30xy + 21xz + 2yz + 105x + 10y + 7z = 812$, find $x + y + z$.

Solution

Factor x; this yields $3x(2y+7)(z+5)$. For the remaining terms, $2yz+10y+7z = (2y+7)(z+5)-35$.

So $(3x + 1)(2y + 7)(z + 5) = 812 + 35 = 847 = 7 \cdot 11^2$. Since x, y, z are positive integers, each of the three factors in the LHS is larger than one, so one of them equals 7, and the other two equal 11. Since $2y + 7 > 7$, it follows $2y + 7 = 11$, so $y = 2$. Since $3x + 1 \neq 11$, it follows $3x + 1 = 7$, so $x = 2$. It is left $z + 5 = 11$, so $z = 6$. Therefore $x + y + z = 10$.

$\square$ Find all primes $p \leq q \leq r$ such that the numbers

$$pq + r,\ pq + r^2,\ qr + p,\ qr + p^2,\ rp + q,\ rp + q^2$$

Are all primes.

If p is odd, than can we see that all the numbers are even and can't be prime. So $p = 2$. $pr + q$ has also to be odd, so $q > 2$. If $q > 3$, $qr + 2$ or $qr + 4$ is a mulitple of 3, so we can conclude that

$q = 3$. So, we have to find r such that $3r + 2, 2r + 3, 2r + 9, 3r + 4, 6 + r$ and $6 + r^2$ are prime, it is simple to find $r = 5$ as an example (3 is false, because 9 isn't prime). If $r > 5 : r$ can be $\equiv 1, 2, 3, 4 \pmod 5$ if $r \equiv 1 \pmod 5$ is $5|3r + 2$; if $r \equiv 2 \pmod 5$ is $5|3r + 4$; if $r \equiv 3 \pmod 5$ is $5|2r + 9$; if $r \equiv 4 \pmod 5$ is $5|6 + r$;

So $(2, 3, 5)$ is the only solution.

Another approach All three cannot be odd, so $p = 2$. Then $q > 2$, otherwise $qr + p$ is even. Now we must have $q = 3$, otherwise one of $qr + 2$, $qr + 4$ would be divisible by 3. Then $r > 3$, otherwise $pq+r$ is divisible by 3. Finally, $r = 5$, otherwise one of the expressions is divisible by 5 (easy to check modulo 5).

□ The sequence $1, 3, 4, 9, 10, 12, 13, \ldots$ is increasing and consists of all positive integers which are either powers of 3 or sums of at most 3 distinct powers of 3. Find the 100th term.

Solution

Within the positive integers whose representation in ternary basis has at most n digits there are $\binom{n}{1}$ having one digit 1, $\binom{n}{2}$ having two digits 1, and $\binom{n}{3}$ having three digits 1, the rest of the digits being zero. In order not to exceed 100 we therefore need $n + n(n-1)/2 + n(n-1)(n-2)/6 \leq 100$, and the largest such n is 8, for which there are 92 such numbers. The 93$^{\text{rd}}$ one is therefore $\overline{100000000}_{(3)} = 3^8 = 6561$. Now, small such numbers in ternary writing, having at most two digits 1 are succesively $1, \overline{10} = 3, \overline{11} = 4, \overline{100} = 9, \overline{101} = 10, \overline{110} = 12$, and finally $\overline{1000} = 27$, so the 100$^{\text{th}}$ number is $6561 + 27 = 6588$

□ Let $n \in \mathbb{N}$ such as $n \geq 5$ Prove that $2^n \nmid 3^n - 1$

Solution

If n is odd, then $3^n - 1 = (4-1)^n - 1 \equiv (-1)^n - 1 \equiv 2 \pmod 4$, so $2 \mid 3^n - 1$ but $4 \nmid 3^n - 1$. If n is even, let $n = 2^a b$, with integers $a \geq 1$ and b odd. Then $3^n - 1 = (3^b)^{2^a} - 1 = (3^b - 1)(3^b + 1)\prod_{k=1}^{a-1}(3^{2^k b} + 1)$. But $3^b - 1 \equiv 2 \pmod 4$, while $3^2 = 9 \equiv 1 \pmod 8$, so $3^{2c} + 1 \equiv 2 \pmod 8$ and $3^{2c+1} + 1 \equiv 4 \pmod 8$. Summing up, the exponent α such that $2^\alpha \mid 3^n - 1$ but $2^{\alpha+1} \nmid 3^n - 1$ is $\alpha = 1 + 2 + (a-1) = a + 2$. But $n = 2^a b \geq 2^a > a + 2$ for $a \geq 3$.

□ Find the maximum value and minimum value of function:

$$f(x) = \sum_{k=0}^{27} \left[\left(\begin{array}{c} 27 \\ k \end{array} \right) \left(\frac{x}{100}\right)^k \left(\frac{100-x}{100}\right)^{27-k} .(80k - 23x) \right]$$

in [0;100]

Solution

let us pursue further. $\sum_{k=0}^{27}\left[k\binom{27}{k}\left(\frac{x}{100}\right)^k\left(\frac{100-x}{100}\right)^{27-k}\right] = \frac{27x}{100}\sum_{j=0}^{26}\left[\binom{26}{j}\left(\frac{x}{100}\right)^j\left(\frac{100-x}{100}\right)^{26-j}\right] = \frac{27x}{100}$. Thus, $f(x) = \frac{80 \cdot 27x}{100} - 23x$. Thus $-140 = f(100) \leq f(x) \leq f(0) = 0$ for $x \in [0, 100]$. □

Can someone please explain this principle to me and also how it can be used for counting how many integers are relatively prime to another integer. For example, if I asked you: How many integers are relatively prime to 800? How would you go about counting them using PIE?

Solution

Let A be the set of positive integers between 1 and 800 inclusive, each divisible by 2; then its cardinality is $|A| = \lfloor 800/2 \rfloor = 400$. Let B be the set of positive integers between 1 and 800 inclusive, each divisible by 5; then its cardinality is $|B| = \lfloor 800/5 \rfloor = 160$. The number of positive integers between 1 and 800 inclusive, relatively prime with $800 = 2^5 \cdot 5$ is then $800 - |A \cup B|$.

By PIE we have $|A \cup B| = |A| + |B| - |A \cap B|$, and since $|A \cap B| = \lfloor 800/10 \rfloor = 80$, we have $|A \cup B| = 400 + 160 - 80 = 480$, so the number we seek is $800 - 480 = 320$. This checks with

$\varphi(800) = \varphi(2^5 \cdot 5) = (2-1)\cdot 2^4 \cdot (5-1)\cdot 5 = 2^6 \cdot 5 = 320$, the value of Euler's totient, which yields precisely what we seek. In fact, one proof for Euler's totient formula goes precisely by PIE – Find the nonzero digits a, b, c such that: $\sqrt{a} + \sqrt{ab} + \sqrt{abc} + \sqrt{a+b+c} = cc - bb - aa$

$\square$ Let $f(x) = 2x+1$ Solve the equation: $f(x)+f(f(x))+f(f(f(x)))+f(f(f(f(x)))) = N$, where $N \in \mathbb{R}$ is given.

Solution

$f(x) = 2x+1$ $f(f(x)) = 2(2x+1)+1 = 4x+3$ $f(f(f(x))) = 4(2x+1)+3 = 8x+7$ $f(f(f(f(x)))) = 8(2x+1)+7 = 16x+15$

So,

$$\begin{aligned} f(x)+f(f(x))+f(f(f(x)))+f(f(f(f(x)))) &= (2x+1)+(4x+3)+(8x+7)+(16x+15) \\ &= 30x+26 \\ &= N \end{aligned}$$

Thus, $x = \boxed{\dfrac{N-26}{30}}$

Tìm mavropnevma 13+ và 14+

$\square$

Let$a, b, c > 0, a+b+c = 3$. Prove that: $\sum \frac{1}{\sqrt{a^2-3a+3}} \le 3$

Solution

Put $x = a-1, y = b-1, z = c-1$ so that $x+y+z = 0$ and we need

$$\sum \frac{1}{\sqrt{x^2-x+1}} \le 3$$

Not all x, y, z are positive. Case 1: Suppose $x, y \ge 0$. Since $c \ge 0, z \ge -1, x+y \le 1$, so that at most one of $x, y \ge 1/2$ If $x \ge 1/2$, since $f(x) = \dfrac{1}{\sqrt{x^2-x+1}}$ is symmetrical about $x = 1/2$, we can replace x with its reflection in $x = 1/2$ and use the surplus to increase z (still negative), so increasing the overall sum since $f(z)$ is strictly increasing for negative z. Hence we can assume that $x, y \le 1/2$

$f(x) = \dfrac{1}{\sqrt{x^2-x+1}} \Longrightarrow f'(x) = -\dfrac{2x-1}{2(x^2-x+1)^{3/2}}$ and $f''(x) = \dfrac{8x^2-8x-1}{4(x^2-x+1)^{5/2}}$

$f''(x) = 0$ when $x = 1/4(2 \pm \sqrt{6})$ and $f''(0) = -\frac{1}{4}$, so $f(x)$ is concave throughout $0 \le x \le 1/2$.

By Jensen's inequality, $f(x)+f(y) \le 2f(\dfrac{x+y}{2})$. Put $m = \frac{x+y}{2}$ so that $z = -2m$, since $x+y+z = 0$.

We now have $f(x)+f(y)+f(z) \le 2f(m)+f(-2m) = P(m)$, say, and $0 \le m \le 1/2$

For turning points, $P'(m) = 0$.

$P'(m) = 2f'(m) - 2f'(-2m) = 0 \Longrightarrow f'(m) = f'(-2m)$

But for $0 \le m \le 1/2, f'(m) < 1/2$ and $f'(-2m) \ge 1/2$ so this is only possible when $m = 0$. $P''(m) = 2f''(m)+4f''(-2m)$ and so $P''(0) = 6f''(0) < 0$, giving a maximum value for $P(m)$ in this range of $3f(0) = 3$

Case 2: Suppose $x \ge 0$ and $y, z \le 0$. As before, we may suppose $x \ge 1/2$ Let $g(x) = 1/2x+1-\dfrac{1}{\sqrt{x^2-x+1}}$. Solving $g(x) = 0$,

$$(1/2x+1)^2(x^2-x+1) = 1 \Longrightarrow \frac{1}{4}x^2(x^2+3x+1) = 0$$

This gives solutions $x = 0$, (repeated, tangent), $x = 1/2(-3+\sqrt{5}) = -\theta$, say, and $x = 1/2(-3-\sqrt{5})$, (spurious, from squaring).

We have $g(x) \geq 0$ for $x \geq -\theta$ and strictly increasing for $x \geq 0$. $g(x)$ is also strictly increasing for $x \leq -\theta$ and so for $-1/2 \leq x \leq -\theta, g(x) \geq g(-1/2) = \frac{3}{4} - \frac{2}{\sqrt{7}}$

Suppose $z = min\{y, z\}$

If $z \geq -\theta$,both z and y are $\geq -\theta$ and $g(x) + g(y) + g(z) \geq 0$

If $z \leq -\theta$, then $-1/2 + \theta \leq y \leq 0$, $x \geq \theta$ and $g(x) + g(y) + g(z) \geq g(-1/2) + g(\theta) \geq 0$, since $g(\theta) > 0.04 > g(-1/2)$

In either case, $g(x) + g(y) + g(z) \geq 0$, i.e.

$$\frac{1}{2}x + 1 - \frac{1}{\sqrt{x^2 - x + 1}} + \frac{1}{2}y + 1 - \frac{1}{\sqrt{y^2 - y + 1}} + \frac{1}{2}z + 1 - \frac{1}{\sqrt{z^2 - z + 1}} \geq 0$$

and summing with $x + y + z = 0$ gives us the result we needed.

Alternatively, to prove the second case without using numbers, we can use the fact that $f'(x)$ is concave to show that $h(x) = x - f(x) + f(-x) = g(x) - g(-x)$ is non-decreasing which is equivalent to the result above.

□

How many four digits numbers less than 5000 are possible with: a. no repeatation of digits in the four digits b. digit 1 compulsorily appearing as one of the four digits c. the four digit number being divisible by 11 d. All ten digits between 0 to 9 (both including) are qualified to appear

Solution

Remember from primary school that if a number is divisible by 11 then the alternating subtraction and addition of the digits is also divisible by 11. In this case, if a four digit number in the form $abcd$ is divisible by 11 then $a - b + c - d$ is also divisible by 11, or $a + c \equiv b + d \mod 11$

Now one digit has to be 1 so for now we let $a = 1$ and $a + c = 1, 2 \dots 10$ So we have to find the partions of $\{1, 2 \dots 10\}$ in the form $b + d$ so that $a - b + c - d$ is divisible by 11

To do this consider a function, $f(x) = x^0 + x^1 + x^2 \dots x^9$, the expansion of $f(x)^2$ will show us all the possible sums of two digits, but instead of actually expanding this we just realise that the exponents will be $0, 1, 2, 3 \dots 18$ and the respective coefficients are $1, 2, 3 \dots 9, 10, 9 \dots 3, 2, 1$ Now since we are only concerned with numbers that sum to $1, 2 \dots 10$ we find that there are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 9$ such pair. (now it gets a little messy) From the first condition there can be no repeating digits so we have to subtract 1 from every even sum to get $2, 2, 4, 4, 6, 6, 8, 8, 10, 8$ Also, since we have counted the each pair twice (e.g. 8=5+3 and 3+5) we divide by 2 and get $1, 1, 2, 2, 3, 3, 4, 4, 5, 4$ And again; one of each solution will be the form (1+c) so we must subtract because (a+c) is already in that form$\dots 0, 0, 1, 1, 2, 2, 3, 3, 4, 3$

So the sum of these is $1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 4 = 29$ These groups of four can be organised 8 ways and still be divisible by 11 but 8 will begin with 0

So the answer is $29 \times 8 - 8 = 224$

□ Consider the following expression: $4f^3 + 9f^2 - 4f$ where f is a reduced fraction. for which value of f, the above expression is an integer ?

Solution

From the rational root theorem, if $f = \frac{p}{q} \longrightarrow q = \{1, 2, 4\}$ For now we let $q = 4$ because that covers everything.

$\Rightarrow \frac{p^3}{16} + \frac{p^2}{16} - p = k$ for some intiger $k \Rightarrow p^3 + 9p^2 - 16p \equiv 0 \mod 16$

Since f is a reduced fraction $\gcd(p, q) = (p, 4) = (p, 16) = 1$, so $p^2 + 9p - 16 \equiv 0 \mod 16$ $p + 9 \equiv 0 \mod 16$

$p = 7 + 16k$

Checking we see that $f\left(\frac{7}{4}\right) = 42$ which is an intiger so

$f = \dfrac{7+16k}{4}$ where k is a positive integer

Given any n + 2 integers, show that, for some pair of them, either their sum or their difference is divisible by 2n.

Solution

Every term in a set $S = a_0, a_1 \ldots a_{n+2}$ that is not divisible by $2n$ can be expressed as $b_n \mod 2n$ where $0 < b_n < 2n$ When adding or subracting two terms from the set, there equivalent values mod $2n$ cannot equal 0 or 2n. Therefore no two terms have equivalent values $\mod 2n$ therefore the largest possible set S contains terms with consecutive interget equivalents $b \mod 2n$ where $0 < b < n$ or $n < b < 2n-1$ Therefore the largest set contains n initgers

Due to the pigeonhole principal and set with $n+1$ intigers or indeed $n+2$ initgers must contain a pair of terms whose sum or difference is divisible by $2n$

$a+b+c=1$ and a,b,$c \in [0,1]$ Find the maximum of $(a-b)(b-c)(c-a)$

Solution

Let $c = \max\{a,b,c\}$. For a maximum, we need $P = (a-b)(b-c)(c-a) = (c-a)(c-b)(b-a)$ to be positive, so we take $c \geq b \geq a$. Substituting $c = 1-b-a$ gives $P = (1-b-2a)(1-2b-a)(b-a)$ and so clearly for a maximum, $a = 0$ giving $P = b(1-b)(1-2b)$ This is easy to maximise using calculus. Otherwise using AM-GM

$$P = 4\left[\frac{b}{\sqrt{3}-1}\cdot\frac{1-b}{\sqrt{3}+1}\cdot\frac{1-2b}{2}\right] \leq \frac{4}{27}\left[\frac{b}{\sqrt{3}-1}+\frac{1-b}{\sqrt{3}+1}+\frac{1-2b}{2}\right]^3 = \frac{1}{6\sqrt{3}}$$

with equality at $\left(0, \frac{1}{2}-\frac{1}{2\sqrt{3}}, \frac{1}{2}+\frac{1}{2\sqrt{3}}\right)$ and cyclic permutations.

Let a and b be real numbers such that $\left(a^2+b^2\right)^2 + 4a^2b^2 = 2a+2b$ a) Find max $(a+b)$; b) Find max $(a-b)$; c) when these maximumes occur.

Solution

Multiplying out and collecting, we have

$$(a^2+b^2)^2 + 4a^2b^2 = \frac{1}{2}[(a+b)^4 + (a-b)^4] = 2(a+b)$$

This gives (a)

$$(a+b)^4 = 4(a+b) - (a-b)^4 \leq 4(a+b) \Rightarrow a+b \leq \sqrt[3]{4}$$

since for max, $a+b \geq 0$. Equality when $a = b = \frac{1}{\sqrt[3]{2}}$

and (b)

$$(a-b)^4 \leq 4(a+b) - (a+b)^4 = 4u - u^4$$

where $u = a+b$. From

$$4u - u^4 \leq 3 \iff (u-1)^2(u^2+2u+3) \geq 0 \Longrightarrow a-b \leq \sqrt[4]{3}$$

Equality when $a+b = u = 1$ and $a-b = \sqrt[4]{3}$, i.e. when $a = \frac{1+\sqrt[4]{3}}{2}, b = \frac{1-\sqrt[4]{3}}{2}$

If reals $a,b,c \in [0,1]$, show that

$$\frac{1}{5-ab} + \frac{1}{5-bc} + \frac{1}{5-ca} \geq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{4}$$

Solution

Since $a,b,c\in[0,1]$, we have $(1-a)(1-b)\ge 0\Rightarrow 1+ab\ge a+b\Rightarrow \frac{1}{5-ab}\ge\frac{1}{6-(a+b)}$ and similars. After this and then C-S

$$LHS\ge\frac{1}{6-(a+b)}+\frac{1}{6-(b+c)}+\frac{1}{6-(c+a)}\ge\frac{9}{18-2(a+b+c)}$$

and we need

$$\frac{9}{18-2(a+b+c)}\ge\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{4}$$

i.e.

$$36+2(a+b+c)(\sqrt{a}+\sqrt{b}+\sqrt{c})\ge 18(\sqrt{a}+\sqrt{b}+\sqrt{c})$$

Putting $p=\sqrt{a}+\sqrt{b}+\sqrt{c}$ and noting that $a+b+c\ge\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}{3}$ we need

$$36+2\cdot\frac{p^3}{3}\ge 18p\iff (p-3)^2(p+6)\ge 0$$

which is obviously true.

$\square$

Prove that:

$$\frac{2011}{2}-\frac{2010}{3}+\frac{2009}{4}-\ldots+\frac{1}{2012}=\frac{1}{1007}+\frac{3}{1008}+\frac{5}{1009}+\ldots+\frac{2011}{2012}$$

Solution

Proof by induction. Let

$$S(n)=\frac{2n-1}{2}-\frac{2n-2}{3}+\ldots-\frac{2}{2n-1}+\frac{1}{2n}-\left[\frac{1}{n+1}+\frac{3}{n+2}+\ldots+\frac{2n-1}{2n}\right]$$

Then

$$\begin{aligned}S(n)-S(n-1)&=2\theta(n)+2\phi(n)-\frac{2}{2n-1}+\frac{1}{2n}-\frac{2n-3}{2n-1}-\frac{2n-1}{2n}+\frac{1}{n}\\&=2\theta(n)+2\phi(n)+\frac{2}{n}-2\end{aligned}$$

where

$$\theta(n)=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\ldots-\frac{1}{2n-3}+\frac{1}{2n-2},\qquad \phi(n)=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n-2}$$

Similarly

$$S(n-1)-S(n-2)=2\theta(n-1)+2\phi(n-1)+\frac{2}{n-1}-2$$

Then

$$\begin{aligned}S(n)+S(n-2)-2S(n-1)&=2[\theta(n)-\theta(n-1)]+2[\phi(n)-\phi(n-1)]+\frac{2}{n}-2-\frac{2}{n-1}+2\\&=2\left[-\frac{1}{2n-3}+\frac{1}{2n-2}\right]+2\left[-\frac{1}{n}+\frac{1}{2n-3}+\frac{1}{2n-2}\right]+\frac{2}{n}-\frac{2}{n-1}\\&=0\end{aligned}$$

We claim that $S(n)=0\quad\forall n\in\mathbb{N}$ We can easily show that this is true for $n=1$ and $n=2$ and from the previous, if $S(n-2)=S(n-1)=0$ then $S(n)=0$. Hence proved by induction for all positive integers and in particular, $n=1006$.

$\square$ Solve the system of equation $\begin{cases} x^5 + y^5 = 1 \\ x^6 + y^6 = 1 \end{cases}$

Solution

If we consider only real solutions then from (2) we know that $0 \leq x, y \leq 1$ Also if we subtract (1) from (2) then $x^5(x-1) = -y^5(y-1)$

Let $f(x) = x^5(x-1)$, we need to find x, y such that $f(x) = -f(y)$ however $f(x)$ has only two real roots, $x = 0, 1$ and $f(x) < 0, x \in (0,1)$ therefore there can be no solutions

Except, of coarse when $f(x) = 0$ so $(x, y) = (1, 0), (0, 1)$

$\square$ Let $a_0 = 1/2$ and let $a_{n+1} = 1 - (a_n)^2$. Find $lim_{n\to\infty} a_{2n+1}$.

Solution

$a_0 = \frac{1}{2}$ also if $0 < a_n < 1$ then $0 < 1 - a_n^2 < 1 \Longrightarrow 0 < a_{n+1} < 1$ So all terms $a_i \in (0,1)$

we define a sequence $b_0 = a_1 = \frac{3}{4}, b_{n+1} = b_n^2(2 - b_n^2)$ We show that $b_{n+1} > b_n$

$b_n^2(2 - b_n^2) > b_n \Longrightarrow b_n(2 - b_n^2) > 1$ $(1 - b_n)(b_n^2 + b_n - 1) > 0$ Which is true for $b_n > \frac{-1+\sqrt{5}}{2}$ $\left(\frac{3}{4} > \frac{-1+\sqrt{5}}{2}\right)$

So we have a strictly increasing set of positive real numbers $\{b_i\}$, bounded at 1

As $n \to \infty, a_n = a_{n+1}$ therefore $a_n(1 - a_n)(a_n^2 + a_n - 1) = 0$

This gives roots $a_n = 0, \frac{-1\pm\sqrt{5}}{2}, 1$ but all $\{b_i\}$ are greater that the first three roots so the least upper bound is 1.

$\square$ If an integer comes from four digits 0,6,8,9 , we call it Holi Numbers. The first 16 Holi Numbers list in the ascending order as the following:

6 8 9 60 66 68 69 80 86 88 89 90 96 98 99 600

How about the 2008th Holi Number ?

Solution

Convert 2008 from base 10 to base 4 $4^n = 1, 4, 16, 64, 256, 1024\ldots$ So $2008 = 1.(4^5) + 3.(4^4) + 3.(4^3) + 1.(4^2) + 2.(4^1) + 0.(4^0)$ We have 2008 base 10 = 133120 base 4

now substistuting $(0, 1, 2, 3)$ for $(0, 6, 8, 9)$

Holy number: 699680

$\square$ Three numbers a, b and c are selected from the interval $[0, 1]$, with $a \geq b \geq c$. Find the probability that $4a + 3b + 2c \geq 1$.

Solution

Feasible region for a, b, c is a tetrahedron bounded by planes $c = 0, a-b = 0, b-c = 0$ and $4a+3b+2c = 1$ This has vertices $(0, 0, 0), (\frac{1}{4}, 0, 0), (\frac{1}{7}, \frac{1}{7}, 0), (\frac{1}{9}, \frac{1}{9}, \frac{1}{9})$ Area of base triangle is $\frac{1}{2} \times \frac{1}{4} \times \frac{1}{7} = \frac{1}{56}$ Volume is $\frac{1}{3} \times \frac{1}{56} \times \frac{1}{9} = \frac{1}{1512} = V$ $P(4a + 3b + 2a \leq 1) = 3! \times V = \frac{1}{252}$

In general, with the same condition, $P(\alpha a + \beta b + \gamma c \leq 1) = \frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)}$

$\square$ Find all polynomials $p(x)$ with real coefficients such that

$p(x)p(x + 1) = p(x^2 + x + 1)$

Solution

First we show that $P(x)$ has no real roots Let a be a real root of $P(x)$ $0 = P(a)P(a+1) = P(a^2+a+1)$ and $0 = P(a-1)P(a) = P(a^2 - a + 1)$

So if $f(x) = x^2 + x + 1$ and a is a root of $P(x)$ then so is $f(a), f^2(a) \ldots f^n(a)$

Since $a^2 + a + 1 > 0 \Longrightarrow f^{n+1}(a) > f^n(a)$ and there are infinitely many roots of $P(x)$ - contradiction.

Therefore $P(x)$ has no real roots

Let $x = 0 \Longrightarrow P(0)P(1) = P(1) \Longrightarrow P(0) = 1(\because P(1) \neq 0)$

$P(0)=1$ implies that the product of all roots (complex of course) is 1 So if b is a complex root then $\|b\|=1, \|b^2+b+1\|=1, \|b^2-b+1\|=1$

However of one of $b^2\pm b+1$ we will have $1=\|b^2\pm b+1\|=\|b\|+\|b^2+1\|$

Therefore $\|b^2+1\|=0 \Longrightarrow b^2+1=0$

So we find the only polynomial $P(x)=(x^2+1)^n$

Also $P(x^2+x+1)=((x^2+x+1)^2+1)^n=((x^2+1)(x^2+2x+2))^n=(x^2+1)^n((x+1)^2+1)^n=P(x)P(x+1)$ Another way

If $P(x)=c$ is a constant polynomial, then $c\cdot c=c \Rightarrow c=0,1$. Thus, $P(x)=0$ and $P(x)=1$ are the only constant solutions.

Now assume that $P(x)$ is non-constant. Then $P(x)$ has a complex zero: z.

$x=z$ yields: $P(z^2+z+1)=P(z)P(z+1)=0$, so z^2+z+1 is also a zero.

$x=z-1$ yields: $P(z^2-z+1)=P(z-1)P(z)=0$, so z^2-z+1 is also a zero.

Lemma: If $Re(z)>0$, then $|z^2+z+1|>|z|$. If $Re(z)<0$, then $|z^2-z+1|>|z|$. If $Re(z)=0$, then $|z^2+z+1|, |z^2-z+1|\geq |z|$ with equality iff $z=\pm i$. Proof for Lemma Let $z=a+bi$.

$z^2+z+1=(a^2+a+1-b^2)+(2ab+b)i$ $|z^2+z+1|>|z| \Longleftrightarrow (a^2+a+1-b^2)^2+(2ab+b)^2>a^2+b^2$ $\Longleftrightarrow (a^4+2a^3+2a^2+2a)+(2a^2+2a)b^2+(b^2-1)^2>0$ which is true for all $Re(z)=a>0$.

$z^2-z+1=(a^2-a+1-b^2)+(2ab-b)i$ $|z^2-z+1|>|z| \Longleftrightarrow (a^2-a+1-b^2)^2+(2ab-b)^2>a^2+b^2$ $\Longleftrightarrow (a^4-2a^3+2a^2-2a)+(2a^2-2a)b^2+(b^2-1)^2>0$ which is true for all $Re(z)=a<0$.

If $Re(z)=a=0$ then $z=bi$, $|z|=b$ and: $z^2\pm z+1=(1-b^2)\pm(b)i$. It is easy to see that $|z^2\pm z+1|\geq |z|$ with equality when $b=\pm 1 \Rightarrow z=\pm i$.

Thus, if we can find a zero z_0, then we can construct an infinite sequence of roots: $\{z_n\}$ such that $z_{n+1}=z_n^2\pm z_n+1$ and $|z_n|\geq |z_{n-1}|$ for all $n\in\mathbb{N}$.

If $z_n\neq \pm i$ for all $n\in\mathbb{N}_0$, then $\{z_n\}$ is an infinite sequence whose magnitude is strictly increasing, and thus, $P(x)$ will have an infinite number of distinct roots, a contradiction.

So, there exists $n\in\mathbb{N}_0$ such that $z_n=\pm i$.

Suppose that $z_{n-1}\neq \pm i$. Then we must have $z_{n-1}^2\pm z_{n-1}+1=\pm i$ Solving yields $z_{n-1}=\pm(1+i)$ (Since $z\neq \pm i$).

However, then $|z_{n-1}|>|z_n|$ which is a contradiction.

Thus, the only possible sequence of roots with a finite number of distinct values is $z_n=\pm i$ for all $n\in\mathbb{N}_0$.

Therefore, the only possible roots are $z=\pm i$.

So, $P(x)=K(x-i)^{n_1}(x+i)^{n_2}$ for some $n_1,n_2\in\mathbb{N}_0$ and $K\in\mathbb{R}$ is non-zero.

Since $P(x)$ must be a real polynomial, $n_1=n_2=n$.

Therefore, $P(x)=K(x^2+1)^n$. Plugging this yields $K=1$ as the only non-zero solution.

Thus, the solution is $P(x)=(x^2+1)^n$.

$\square$ Find all triples (a,b,c) such that ,

$$\begin{cases} a^2-2b^2=1 \\ 2b^2-3c^2=1 \\ ab+bc+ca=1 \end{cases}$$

Solution

$ab+bc+ca=1 \Longrightarrow (b+c)(b+a)=1+b^2 \qquad (1)$

$a^2-2b^2=1 \Longrightarrow (a-b)(a+b)=1+b^2 \qquad (2)$

$a+b \neq 0$ otherwise $c(a+b)+ab=1 \Rightarrow ab=1$ which is impossible

So from (1) and (2), $a=2b+c(*)$

Similarly $ab+bc+ca=1 \Longrightarrow (c+b)(c+a)=1+c^2 \qquad (3)$

$2b^2-3c^2=1 \Longrightarrow 2(b-c)(b+c)=1+c^2 \qquad (4)$

Therefore $a+c=2(b-c) \Rightarrow a=2b-3c(**)$

From $(*)$ and $(**)$ clearly $c=0$

Hence $a=2b \Longrightarrow (2b)b+0(b+c)=1 \Longrightarrow b=\pm\frac{1}{\sqrt{2}}$

And $a=\pm\sqrt{2}$

This gives $(a,b,c)=(\sqrt{2},\frac{1}{\sqrt{2}},0)$ or $(-\sqrt{2},-\frac{1}{\sqrt{2}},0)$ as the only solutions

☐ Prove that :

$\left\lfloor \sum_{k=1}^{n} \sqrt[2k+1]{\frac{2k}{2k-1}} \right\rfloor = n$, $(\forall) n \in \mathbb{N}^*$.

Solution

$n < \sum_{k=1}^{n} \sqrt[2k+1]{\frac{2k}{2k-1}}$ is true since each term is greater than 1

now by Bernoulli,

$\left(1+\frac{1}{2k-1}\right)^{\frac{1}{2k+1}} < 1+\frac{1}{(2k-1)(2k+1)}$

So it is left to show that $\sum \frac{1}{(2k-1)(2k+1)} < 1$

However this series telescopes;

$\sum \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\sum\left(\frac{1}{2k-1}-\frac{1}{(2k+1)}\right) < \frac{1}{2}$

So $n < \sum_{k=1}^{n} \sqrt[2k+1]{\frac{2k}{2k-1}} < n+1$

☐ Find the positive numbers x, y, z such that

$x+y+z=1$ and $\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}=\dfrac{x+y}{y+z}+\dfrac{y+z}{x+y}+1$

Solution

Notice that the fractions on the RHS are the mediants of the fractions on the LHS, we write

$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=\frac{x+y}{y+z}+\frac{y+z}{x+y}+1$

$\Longrightarrow \left(\frac{x}{y}+\frac{y}{z}-\frac{x+y}{y+z}\right)+\left(\frac{z}{x}+\frac{y}{y}-\frac{y+z}{x+y}\right)=2$

$\Longrightarrow \frac{z}{y+z}\cdot\left(\frac{x}{y}\right)+\frac{y}{y+z}\cdot\left(\frac{y}{z}\right)+\frac{y}{y+x}\cdot\left(\frac{z}{x}\right)+\frac{x}{y+x}=2$

Now we substitute $a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}$ and we have $abc=1$

$\Longrightarrow \frac{a}{b+1}+\frac{b^2}{b+1}+\frac{c}{a+1}+\frac{a}{a+1}=2$

Sub in $c=\frac{1}{ab}$ and then multiply though $(a+1)(b+1)$

$(a+b^2)(a+1)+\frac{(a^2b+1)(b+1)}{ab}=2(a+1)(b+1)$

$a^2+b^2+ab^2+a+ab+a+\frac{1}{a}+\frac{1}{ab}=2(ab+a+b+1)$

$\Longrightarrow (a-b)^2+(ab-\frac{1}{ab})^2+\frac{1}{a}(ab-1)^2=0$

Equality occurs when $a=b$ and $ab=1$ therefore $a=b=1 \longrightarrow c=1$

Therefore we have $x=y=z$, and from our condition $x+y+z=1$ we get $(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ as the only solution.

☐ Let $P(x)$ be a polynomial with integer coefficients. It is known that $P(a)=P(b)=P(c)=-1$, where a, b and c are different integers. Prove that $P(x)$ does not have integer roots.

Solution

$P(x)=q(x)(x-a)(x-b)(x-c)-1$ with $a,b,c \in \mathbb{Z}$ and $q(x)$ has integer coefficients

Suppose there is an integer root $x=x_0$

$q(x_0)(x_0-a)(x_0-b)(x_0-c)=1$

we have $q(x_0),(x_0-a),(x_0-b),(x_0-c) \in \mathbb{Z}$ and the values are distinct because a,b,c are distinct.

But there are no three distinct integers whose product is 1.

□ Solve the following system of equations $\sqrt{x}-\frac{1}{y}=\sqrt{y}-\frac{1}{z}=\sqrt{z}-\frac{1}{x}=\frac{7}{4}$

Solution

First note that $x,y,z>0$ otherwise we have complex numbers etc.

Assume w.l.o.g. that $x\geq y\geq z\Longrightarrow \frac{1}{x}\leq\frac{1}{y}\leq\frac{1}{z}$

Then we have $\sqrt{x}-\frac{1}{y}=\sqrt{y}-\frac{1}{z}\Longrightarrow \sqrt{x}-\sqrt{y}=\frac{1}{y}-\frac{1}{z}$

However $\sqrt{x}-\sqrt{y}\geq 0$ and $\frac{1}{y}-\frac{1}{z}\leq 0$

Therefore $x=y=z$

$\Longrightarrow \sqrt{x}-\frac{1}{x}=\frac{7}{4}\Longrightarrow 4x^3-7x^2-14x+1=0\Longrightarrow (x-4)(16x^2+15x+4)=0$

Our only answer is $x=y=z=\boxed{4}$

Another way:

Let $f(x)=\left(\dfrac{7}{4}+\dfrac{1}{x}\right)^2$,

So that $f(y)=x, f(z)=y, f(x)=z$.

This comes from,

$\sqrt{x}-\frac{1}{y}=\frac{7}{4}\implies x=\left(\dfrac{7}{4}+\dfrac{1}{y}\right)^2=f(y)$

The same proccess for y and z.

Now we can use the fact that $f(f(f(x)))=x$, and since $f(x)$ is an increasing function, $f(x)=x$ so $\left(\dfrac{7}{4}+\dfrac{1}{x}\right)^2=x$ which from the previous post we know that $x=4$. Now we substitute 4 for x into the original problem and the only solution is $x=y=z=4$.

□ Let be $a>0$ and $b,c\in[1,2)$ such that $\frac{a+b}{b(1+c)}+\frac{a+c}{c(1+b)}=2$. Prove that a,b,c can be the side lenghts of a triangle .

Solution

$\frac{a+b}{b(1+c)}+\frac{a+c}{c(1+b)}=2$

$\frac{a-bc}{b(1+c)}+\frac{a-bc}{c(1+b)}=0$

$\implies a-bc=0$

$\implies 4>b+c\geq 2\sqrt{bc}=2\sqrt{a}>a$ Where the last inequality comes from $4>2\sqrt{a}$

Therefore $b+c>a$

Also $a=bc$ and $b,c\geq 1$ implies that $a+b>bc\geq c$. The same applies for $a+c$

So we have $b+c>a, a+b>c$ and $a+c>b$. Therefore they are sides of a trianlge .

□ Solve in $\mathbb{R}_+^*$ the following equation :

$\sqrt{x+\lfloor x\rfloor}+\sqrt{x+\{x\}}=\sqrt{x+\lfloor x\rfloor\cdot\{x\}}+\sqrt{x+1}$

Solution

By inspection notice that $\lfloor x\rfloor=0$ yields no solutions. For $\lfloor x\rfloor=1$ the equality holds.

So, to show there are no other solutions assume $\lfloor x\rfloor>1$

For simplicity let $\lfloor x\rfloor=a, \{x\}=b$

squaring both sides gives

$2x+a+b+2\sqrt{(x+a)(x+b)}=2x+ab+1+2\sqrt{(x+ab)(x+1)}$

$2\sqrt{(x+a)(x+b)}=(a-1)(b-1)+2\sqrt{(x+ab)(x+1)}$

From our conditions we have $(a-1)(b-1)<0$

$\implies 2\sqrt{(x+a)(x+b)}<2\sqrt{(x+ab)(x+1)}$

$\implies x^2+(a+b)x+ab<x^2+(ab+1)x+ab$

$\implies 0<(a-1)(b-1)$ contradiction.

Hence the only solutions are $\lfloor x\rfloor=1$, with $0\leq\{x\}<1$

$\square$ Given a, b, c and $\frac{ab+bc+ac}{\sqrt{abc}}$ are all positive integers, does that imply that $\sqrt{\frac{ac}{b}}, \sqrt{\frac{ab}{c}}, \sqrt{\frac{bc}{a}}$ must all be integers?

Solution

Clearly $\sqrt{abc} \in \mathbb{N}$ so $abc = k^2, k \in \mathbb{N}$

Write $M = (a, b, c) = (\alpha^2 xy, \beta^2 yz, \gamma^2 zx)$

With $\gcd(\alpha, \beta) = \gcd(\beta, \gamma) = \gcd(\gamma, \alpha) = 1$

Constructive proof Take $M = (a, b, c)$ and let $\gcd(a, b) = y \implies M = (a'y, b'y, c)$

Let $\gcd(a', c) = x \implies M = (a''yx, b'y, c'x)$

Let $\gcd(b', c') = z \implies M = (a''xy, b''yz, c''zx)$

Since $\gcd(a'', b'') = \gcd(b'', c'') = \gcd(c'', a'') = 1$ it follows that a'', b'', c'' are perfect squares.

$\implies M = (\alpha^2 xy, \beta^2 yz, \gamma^2 zx)$

This gives

$\frac{ab+bc+ca}{\sqrt{abc}} = \frac{\sum \alpha^2\beta^2 y}{\alpha\beta\gamma}$

Hence $\alpha|z, \beta|x$ and $\gamma|y$

Therefore

$\sqrt{\frac{ab}{c}} = \sqrt{\frac{\alpha^2 xy\beta^2 yz}{\gamma^2 zx}} = \frac{\alpha\beta y}{\gamma} \in \mathbb{N}$ because $\gamma|y$

$\square$ Find all pairs of integers (m, n) such that the numbers $A = n^2 + 2mn + 3m^2 + 2,\ B = 2n^2 + 3mn + m^2 + 2,\ C = 3n^2 + mn + 2m^2 + 1$ have a common divisor greater than 1.

Solution

Suppose p is prime and $p|A, B, C$.

$A - B = 2m^2 - mn - n^2 = (m - n)(2m + n) \qquad (1)$

$C - B = m^2 - 2mn + n^2 - 1 = (m - n)^2 - 1 \qquad (2)$

From (1), $p|(m - n)$ or $p|(2m + n)$ but clearly $p \not|(m - n)$ because of (2)

replacing $n \equiv -2m \mod p$ in A and C gives $3m^2 + 2 \equiv 12m^2 + 1 \mod p$

but $\gcd(3m^2 + 2, 12m^2 + 1) = \gcd(3m^2 + 2, 7)$ so the greatest common denominator is at most 7

so $3m^2 + 1 \equiv 0 \mod 7 \implies m \equiv 2, 5 \mod 7 \implies n \equiv 3, 4 \mod 7$

hence $(m, n) = (7k_1 + 2, 7k_2 + 3) or (7k_1 + 5, 7k_2 + 4)$

$\square$ Prove that there is no natural n that satisfy $2^n + 3^n = a^3$, where a is natural number.

Solution

we can consider integers modulo 3.

Let $x \equiv 0$ modulo 3. Obviously, $x^3 \equiv 0$ modulo 27, so $x^3 \equiv 9$.

Let $x \equiv 1$ modulo 3, so $x = 3k + 1$, so $x^3 = 27k^3 + 27k^2 + 9k + 1$, which is equivalent to 1 modulo 9.

Let $x \equiv 2$ modulo 3, so $x = 3k + 2$, so $x^3 = 27k^3 + 54k^2 + 18k + 8$, which is equivalent to 8 modulo 9.

$\square$ Five different four-digit integers all have the same initial digit, and their sum is divisible by four of them. Find all possible such sets of integers.

Solution

$M = \{k \cdot 10^4 + x_1, k \cdot 10^4 + x_2, \ldots, k \cdot 10^4 + x_5\}$ with $k \in \{1, 2, \ldots, 9\}$ and x_1, x_2, x_3, x_5, x_5 being 5 distinct three digit numbers. Assume that the four elements that divide the sum are x_1, x_2, x_3, x_4

and for brevity write $S = (k10^4 + x_1) + (k10^4 + x_2) + \cdots + (k10^4 + x_5)$

First we show that $k = 1$

We will show that $\frac{5k+1}{k+1} < \frac{S}{k10^4+x_i} < \frac{5k+4}{k} \qquad (*)$ proof: $\frac{5 \cdot k10^4+x_1+x_2+x_3+x_4+x_5}{k10^4+x_1} > \frac{5 \cdot k10^4+x_1}{k10^4+x_1}$
$> \frac{5 \cdot k10^4+1000}{k10^4+1000}$

$= \frac{5k+1}{k+1}$
Also
$\frac{5\cdot k10^4+x_1+x_2+x_3+x_4+x_5}{k10^4+x_1} < \frac{5\cdot k10^4+x_1+4000}{k10^4+x_1}$
$< \frac{5k+4}{k}$ $\square$
Therefore, since all elements in M are distinct, the four terms $\frac{S}{k10^4+x_i} \dots$ are distinct integers.
If $k=1$, From $(*)$, we have $3 < \frac{S}{k10^4+x_i} < 9$
implying that $\frac{S}{k10^4+x_i} \in \{4,5,6,7,8\}$
if $k>1$ however there aren't enough integers to have 4 distinct terms. For example, if $k=2$
$\frac{11}{2} < \frac{S}{k10^4+x_i} < 7$
Wich would imply that $\frac{S}{k10^4+x_i} = 6$ for $i=1,2,3,4$, which is impossible if x_1,x_2,x_3,x_4 are distinct.
Hence $k=1$
So we have that $k=1$ and as a result of the proof above we have that $\frac{S}{1000+x_i} \in \{4,5,6,7,8\}$
Clearly we cannot have two x_i, x_j such that $\frac{S}{1000+x_i} = 4$ and $\frac{S}{1000+x_j} = 8$
Otherwise $1000+2x_j = x_i$ which is impossible since $x_i < 1000$
This leaves two systems
system 1
$5000 + x_1 + \cdots + x_5 = 4(1000+x_1) = 5(1000+x_2) = 6(1000+x_3) = 7(1000+x_4)$
Letting $y_i = 1000 + x_i$ and we get simply
$y_1 + y_2 + \dots y_5 = 4y_1 = 5y_2 = 6y_3 = 7y_4$
Now $S = y_1 + \frac{4y_1}{5} + \frac{4y_1}{6} + \frac{4y_1}{7} + y_5 = 4y_4 \Rightarrow 101y_1 = 105y_5$
Similarly $101y_2 = 84y_5, 101y_3 = 70y_5, 101y_4 = 60y_5$
This gives $(y_1,y_2,y_3,y_4,y_5) = (105m, 84m, 70m, 60m, 101m)$ with $m \in \{17,18,19\}$
system 2
$y_1 + y_2 + \dots y_5 = 5y_1 = 6y_2 = 7y_3 = 8y_4$
And by a similar method as above we find
$(y_1,y_2,y_3,y_4,y_5) = (168m, 140m, 120m, 105m, 307m)$ but there is no m such that $1000 \le 120m \le 2000$ and $1000 \le 307m \le 2000$
So the only solutions are
$(y_1,y_2,y_3,y_4,y_5) = (105m, 84m, 70m, 60m, 101m)$ with $m \in \{17,18,19\}$
$\square$ $a,b,c \in Z^+$ and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \in Z^+$ .Prove that: abc is a perfect cube.

Solution

Let $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = k$
The LHS is homogeneous so assume $\gcd(a,b,c) = 1$
Assume there exists some prime p that divides a and b , but not c, $(\because (a,b,c)=1)$
Write $a = xp^n, b = yp^m, c = z$ and $(x,p) = (y,p) = (z,p) = 1$
Therefore $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \frac{x^2p^{2n}z + z^2yp^m + y^2p^{2m}xp^n}{xyzp^{m+n}}$
Since the expression is an integer and $p^{m+n} | y^2xp^{2m+n}$ we know that
$p^{m+n} | x^2zp^{2n} + z^2yp^m$
If $2n > m$ then $p^{n+m} | p^m(x^2zp^{2n-m} + z^2y)$, which is impossible as: $p^{n+m} \not| p^m$ because $p^{m+n} > p^m$ and $p \not| x^2zp^{2n-m} + z^2y$ because $(z,p) = (y,p) = 1$.
If $m > 2n$ then $p^{m+n} | p^{2n}(x^2z + y^2p^{m-2n})$ which again is impossible.
Hence $m = 2n$ and $p^{2n} | x^2z + z^2y$
$\implies m + n = 2n + n = 3n \equiv 0 \mod 3$

So if a prime, p divides any number a, b, c then it divides the product abc a number of times that is divisible by three. hence abc is a perfect cube.

$\square$ There are 35 objects that need to be carried away which in total weigh 18 pounds. A spaceship can carry away up to a total of three pounds per trip. Show that if the spaceship can carry away any combination of 34 of the objects in 7 trips, then it can carry away all 35 of the objects in 7 trips.

Solution

Label the objects a_i with $a_1 \geq a_2 \geq \ldots \geq a_{35}$

If 34 objects can be moved in 7 trips, and each trip can take at most 3 pounds, then there is a total of $3 \cdot 7 = 21$ pounds available among all 7 trips. Consider the first 34 objects, which we know can be taken in 7 trips. We have $a_1 + \ldots + a_{34} = 18 - a_{35}$

So by the box principle there is atleast one trip with $\frac{21-(18-a_{35})}{7}$ pounds of free space. If $\frac{21-(18-a_{35})}{7} \geq a_{35} \Leftrightarrow \frac{1}{2} \geq a_{35}$ then we are done.

So assume $a_{35} > \frac{1}{2}$

Since 34 objects (all more than $\frac{1}{2}$) can be taken in 7 trips, we must have each trip taking exactly 5 objects except one trip which takes 4 objects.

Consider the trip which takes 4 objects, the worst case is that the four objects are the heaviest.

$18 = \sum a_i \geq (a_1+a_2+a_3+a_4+a_{35})+30a_{35} > (a_1+a_2+a_3+a_4+a_{35})+\frac{30}{2} \Longrightarrow a_1+a_2+a_3+a_4+a_{35} < 3$

Since $a_1 + a_2 + a_3 + a_4 + a_{35} < 3$, we can send them together and we are done

$\square$ Let $f(\frac{x-1}{x+1}) + f(\frac{1}{x}) + f(\frac{1+x}{1-x}) = x$ Find $f(x)$

Solution

$f\left(\frac{x-1}{x+1}\right) + f\left(\frac{1}{x}\right) + f\left(\frac{1+x}{1-x}\right) = x \qquad (1)$

let $y = \frac{x-1}{x+1} \Longrightarrow x = \frac{y+1}{1-y}$

$\Longrightarrow f(y)+f\left(\frac{1-y}{y+1}\right)+f\left(\frac{-1}{y}\right) = \frac{y+1}{1-y}$ Then set $y = -x \Longrightarrow f(-x)+f\left(\frac{1+x}{1-x}\right)+f\left(\frac{1}{x}\right) = \frac{1-x}{1+x} \qquad (2)$

let $y = \frac{1+x}{1-x} \Longrightarrow x = \frac{y-1}{y+1}$

$\Longrightarrow f\left(\frac{-1}{y}\right)+f\left(\frac{1+y}{y-1}\right)+f(y) = \frac{y-1}{1+y}$ then set $y = -x \Longrightarrow f\left(\frac{1}{x}\right)+f\left(\frac{x-1}{x+1}\right)+f(-x) = \frac{x+1}{x-1} \qquad (3)$

Now $(2)-(1)$ gives $f(-x) - f\left(\frac{x-1}{x+1}\right) = \frac{x-1}{x+1} - x \Leftrightarrow f(-x) - (-x) = f\left(\frac{x-1}{x+1}\right) - \left(\frac{x-1}{x+1}\right) \qquad (*)$

Similarly $(3)-(2)$ gives $f\left(\frac{x-1}{x+1}\right) - \left(\frac{x-1}{x+1}\right) = f\left(\frac{x+1}{1-x}\right) - \left(\frac{x+1}{1-x}\right) \qquad (**)$

From (*) and (**) we find

$3\left(f\left(\frac{1}{x}\right) - \frac{1}{x}\right) = x - \left(\frac{x-1}{x+1} + \frac{1}{x} + \frac{1+x}{1-x}\right)$

$\Longrightarrow f\left(\frac{1}{x}\right) = \frac{x^4+5x^2-2}{3x(x^2-1)}$

$\Longrightarrow f(x) = \frac{2x^4-5x^2-1}{3x(x^2-1)}$

$\square$ $p > 3$ $p \in P$ $P \in 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots.$ $a, b, c \in Z^+$ $a+b+c = p+1$ $p|a^3+b^3+c^3-1 \Longrightarrow \frac{a^3+b^3+c^3-1}{p} \in \mathbb{Z}$

Prove that $a = 1$ v $b = 1$ v $c = 1$

Solution

$a+b+c = p+1$. Hence $1 \leq a < p, 1 \leq b < p, 1 \leq c < p$. $(a+b+c)^3 = a^3+b^3+c^3+3(a+b+c)(ab+ac+bc)-3abc$, $a+b+c \equiv 1(modp)$, $a^3+b^3+c^3 \equiv 1(modp)$. Hence, $1 \equiv 1+3\cdot1\cdot(ab+ac+bc)-3abc(modp)$. Hence, $abc-(ab+ac+bc) \equiv 0(modp)$. Hence, $abc-(ab+ac+bc)+a+b+c-1 \equiv 0(modp)$. Hence, $(a-1)(b-1)(c-1) \equiv 0(modp)$. Hence, $a-1 \equiv 0(modp) \vee b-1 \equiv 0(modp) \vee c-1 \equiv 0(modp)$. Hence, $a = 1 \vee b = 1 \vee c = 1$.

$\square$ Let a, b, c be real numbers satisifying $a+b+c = 2$ and $abc = 4$.

(1) Find the minimum of $\max\{a, b, c\}$.

(2) Find the minimum of $|a| + |b| + |c|$.

Solution

Let $a \leq b \leq c$. Then $a+b = 2-c$ and $ab = \frac{4}{c}$. Hence, a and b is roots of equation $z^2+(c-2)z+\frac{4}{c} = 0$. Hence, $(c-2)^2 - \frac{16}{c} \geq 0 \Leftrightarrow \frac{(c^2+4)(c-4)}{c} \geq 0$. Hence, $c \geq 4$ or $c < 0$. If $c < 0$ then $ab < 0$ and or $a > 0$ or $b > 0$. Contradiction($c = max\{a, b, c\}$). Thence, $c \geq 4$. Hence, $minmax\{a, b, c\} = 4$. Since $c \geq 4$ then $a + b < 0$ and $ab > 0$. Hence $a < 0, b < 0$. Thence, $|a| + |b| + |c| = -a - b + c = c - 2 + c = 2c - 2 \geq 2 \cdot 4 - 2 = 6$. Hence, $min(|a| + |b| + |c|) = 6$. $(a = -1, b = -1, c = 4)$

Problem: Solve the equation

$$x\sqrt{x^2+x+1} + \sqrt{x^2-x+1} = x + \sqrt{x^4+x^2+1}$$

Solution

We notice that $(x^2+x+1)(x^2-x+1) = x^4+x^2+1$.

Rearranging a bit, $x(\sqrt{x^2+x+1}-1) = \sqrt{x^2-x+1}(\sqrt{x^2+x+1}-1)$

Putting all the terms on one side gives us $(x - \sqrt{x^2-x+1})(\sqrt{x^2+x+1}-1) = 0$

Thus, either $x = \sqrt{x^2-x+1}$ or $\sqrt{x^2+x+1} = 1$.

$x^2 = x^2 - x + 1 \Longrightarrow x = 1$ Or $x^2 + x = 0 \Longrightarrow x = 0, -1$.

Thus, we have three solutions $x = -1, 0, 1$ –

$$\left(\left\lfloor x+\frac{7}{3}\right\rfloor\right)^2 - \left\lfloor x-\frac{9}{4}\right\rfloor = 16$$

–

□ Let array a_n is defined by $a_1 = \frac{27}{10}, a_{n+1}^3 - 3a_{n+1}(a_{n+1}-1) - a_n = 1 \forall n > 1$ Prove that array has a limit and find that limit

Solution

$a_{n+1}^3 - 3a_{n+1}(a_{n+1}-1) - a_n = 1 \Leftrightarrow (a_{n+1}-1)^3 = a_n \Leftrightarrow \Leftrightarrow (a_{n+1}-1)^3 - (A-1)^3 = a_n - (A-1)^3$. All this is $\forall A \in \mathbb{R}$. Let A is root of equation $(x-1)^3 = x$. $(x-1)^3 = x \Leftrightarrow x^3 - 3x^2 + 2x - 1 = 0$. Let $f(x) = x^3 - 3x^2 + 2x - 1$. Hence $f'(x) = 3x^2 - 6x + 2$. $f'(x) = 0 \Leftrightarrow x = x_1 = \frac{3+\sqrt{3}}{3}$ or $x = x_2 = \frac{3-\sqrt{3}}{3}$. Hence $x_{max} = x_2$ and $f(x_2) < 0$. Hence A is alone root of the equation and $A > 2.3$ since $f(2.3) < 0$. $a_2 = 1 + \sqrt[3]{2.7} = 2.39... > 2.3$. Hence $\forall n \in \mathbb{N}$ $a_{n+1} = 1 + \sqrt[3]{a_n} > 1 + \sqrt[3]{2.3} = 2.32... > 2.3$. Thence, $(x_{n+1}-1)^3 - (A-1)^3 = x_n - (A-1)^3 \Leftrightarrow\Leftrightarrow (x_{n+1}-A)((x_{n+1}-1)^2 + (x_{n+1}-1)(A-1) + (A-1)^2) = x_n - A$. Hence, $|x_{n+1} - A| = \frac{|x_n - A|}{(x_{n+1}-1)^2+(x_{n+1}-1)(A-1)+(A-1)^2} < < \frac{|x_n-A|}{1.3^2+1.3\cdot1.3+1.3^2} < \frac{|x_n-A|}{2}$. Id est, $\forall n \in \mathbb{N}$ $|a_{n+1} - A| < \frac{1}{2} \cdot |a_n - A|$. Hence, $|a_n - A| < \frac{1}{2} \cdot |a_{n-1} - A| < ... < \frac{1}{2^{n-1}} \cdot |a_1 - A|$. Hence, $\forall n \in \mathbb{N}$ $|a_n - A| \leq \frac{1}{2^{n-1}} \cdot |a_1 - A|$. $lim\frac{|a_1-A|}{2^{n-1}} = 0$. Hence, $lima_n = A$, where A is root of equation $(x-1)^3 = x$. :)

□ Let x and y are positive numbers such that $x+y = 1$. Find the minimum value of $\frac{x+7y}{\sqrt{1-x}} + \frac{y+7x}{\sqrt{1-y}}$.

Solution

Let $x = \cos^2\theta,\ y = \sin^2\theta \left(\theta \neq 0,\ \frac{\pi}{2}\right)$, we have

$P := \frac{x+7y}{\sqrt{1-x}} + \frac{y+7x}{\sqrt{1-y}}$

$= \frac{1+6\sin^2\theta}{|\sin\theta|} + \frac{1+6\cos^2\theta}{|\cos\theta|}$

$= \frac{1}{|\sin\theta|} + \frac{1}{|\cos\theta|} + 6(|\sin\theta| + |\cos\theta|)$. Set $|\sin\theta| + |\cos\theta| = t$, we have

$P := f(t) = \frac{2t}{t^2-1} + 6t$ $(0 < t \leq \sqrt{2})$.

$f'(t) = \frac{2}{(t^2-1)^2}(t+\sqrt{2})(\sqrt{3}t+1)(t-\sqrt{2})(\sqrt{3}t-1)$,

yielding the local minimum $f(\sqrt{2}) = 8\sqrt{2}$, which is the desired minimum value.

Another approach:

Let $\sqrt{1-x}=X,\ \sqrt{1-y}=Y$, by $x+y=1$, we have $X^2+Y^2=1$ and
$\frac{x+7y}{\sqrt{1-x}}+\frac{y+7x}{\sqrt{1-y}}$
$=6(X+Y)+\frac{X+Y}{XY}$ where $1<X+Y\le\sqrt{2}$.

Another way

Using the second form, WLOG let $a\ge b$. Since $a^2+b^2=2$, $a+b\le 2$ by RMS-AM, so $(a-1)^2\le(b-1)^2$. Also, $1-\frac{1}{a}\ge 1-\frac{1}{b}$. Hence, by Chebyshev,

$$(a-1)^2\left(1-\frac{1}{a}\right)+(b-1)^2\left(1-\frac{1}{b}\right)\le\frac{(a-1)^2+(b-1)^2}{2}\left(2-\frac{1}{a}-\frac{1}{b}\right)\quad(*)$$

Also, we have $\frac{1}{a}+\frac{1}{b}\ge 2$ which is evident from Holder:

$$(a^2+b^2)\left(\frac{1}{a}+\frac{1}{b}\right)^2\ge(1+1)^3$$

So the right hand side of $(*)$ is nonpositive; the original expression is as well.

□ Solve the inequation $\sqrt{x+1}>1+\sqrt{\frac{x-1}{x}}$

Solution

Obviously, $x>-1$. SImilarly, $[0,1)$ is undefined. However, values from $[-1,0)$ are clearly failures since the left hand side has an unattainable maximum of 1 while the right hand side has a minimum of $1+\sqrt{2}$. For $[1,\infty)$, we get $\frac{(x^2-x+1)^2}{4x^2-4x}>1$. Subtracting one, we get $\frac{x^4-2x^3-x^2+2x+1}{4x^2-4x}>0$. Now, that factors into $\frac{(x^2-x-1)^2}{4x^2-4x}>0$ which is true for all values of x greater than 1 except x=$\frac{1+\sqrt{5}}{2}$.

□ Prove that for all x, y and z the following inequality holds: $|x-y|+|y-z|+|z-x|\ge 2\sqrt{x^2+y^2+z^2-xy-xz-yz}$.

Solution

It is symmetric in x,y,z, since the modulus signs allow us to reverse the minus signs, creating symmetry. Therefore we may assume $x\ge y\ge z$. It becomes $2(x-z)\ge\sqrt{2(x-y)^2+2(y-z)^2+2(x-z)^2}$, but it is obvious that $2(x-y)^2+2(y-z)^2\le 2(x-z)^2$, so we are done. :)

Another way There is a nice geometric-algebraic interpretation to this: Because $|x-y|+|y-z|\ge|z-x|$ and symetrically, there exists a triangle with sides $|x-y|,|y-z|,|z-x|$.Let ABC be this triangle and let $AB=|x-y|,BC=|y-z|,CA=|z-x|$. Then our inequality is rewritten as

$$AB+BC+CA\ge\sqrt{2(AB^2+BC^2+CA^2)}$$

$$\iff 2(AB\cdot BC+BC\cdot CA+CA\cdot AB)\ge AB^2+BC^2+CA^2.$$

Now substitute $AB=a+b,BC=b+c,CA=c+a$, with $a,b,c\ge 0$, and you will get that the inequality is equivalent to $ab+bc+ca\ge 0$, which is obvious. :wink:

□ Let $A_1,A_2,...,A_{63}$ be the nonempty subsets of 1, 2, 3, 4, 5, 6. For each of these sets A_i, let $\pi(A_i)$ denote the product of all the elements in A_1. Then what is the value of $\pi(A_1)+\pi(A_2)+...+\pi(A_{63})$?

Solution

Let S_n be the set of all subsets of $\{1,2,3,4,5,6\}$ with exactly n elements.

The polynomial $\prod_{r=1}^{6}(rx+1)=1+\sum_{n=1}^{6}\left(\sum_{A_k\in S_n}\pi(A_k)\right)x^n$ because the x term contains the sum of 1 through 6, the x^2 term is the sum of all possible products of two of these integers, etc.

Substitute $x=1$ to get $1+\sum_{k=1}^{63}\pi(A_k)=\prod_{r=1}^{6}(r+1)=7!=5040\Rightarrow\boxed{5039}$.

□ The following equation: $x^4+4x^3+5x^2+2x=10+12\sqrt{(x+1)^2+4}$

Solution

$x^4+4x^3+5x^2+2x=10+12\sqrt{(x+1)^2+4}$ Rewrite this as $(x+1)^4-(x+1)^2=10+12\sqrt{(x+1)^2+4}$. Let $u=(x+1)^2 \Rightarrow u\geq 0$ We get $u^2-u=10+12\sqrt{u+4}$. Let $v=\sqrt{u+4} \Rightarrow v\geq 2$. $v^4-9v^2-12v+10=0$. We search for a,b,c such that $v^4-9v^2-12v+10=(v^2-av+b)(v^2+av+c)$. It's easy to find that $a=4, b=2, c=5$. $v^2+4v+5=0$ has no real roots, and the only root of $v^2-4v+2=0$ greater than 2 is $v=2+\sqrt{2}$. From this we obtain $x_{1,2}=-1\pm\sqrt{2+4\sqrt{2}}$.

□ Find the range of a for which the equation with respect to x, $a\cos^2 x+4\sin x-3a+2=0$ has real roots.

Solution

Using $\cos^2 x=1-\sin^2 x$ and substituting $\sin x=u$ we get a new problem find all a such that the equation $f(u)=au^2-4u+2(a-1)=0$ has a root(s) in $[-1;1]$. [color=darkred]Solution without calculus[/color]: if $a=0, u=-\frac{1}{2}$ so $a=0$ works. When $a\neq 0$ the discriminant is $D=-2(a+1)(a-2)$ so for $f(u)=0$ to have solutions $a\in[-1;2]$. When $a\in(0;2]$ the vertex of this parabola lies on $u=\frac{2}{a}\geq 1$. The bigger root is greater than 1, and for the smaller to be in $[-1;1]$ we need $f(1)\leq 0$ and $f(-1)\geq 0$. $f(1)=3a-6, f(-1)=3a+2$ so all $a\in(0;2]$ give solutions. When $a\in[-1;0]$ the vertex is less than -2, and so is the smaller root, now for the bigger root to be in $[-1;1]$ we need $f(-1)\geq 0$ and $f(1)\leq 0$ this gives $a\geq -\frac{2}{3}$. So $a\in[-\frac{2}{3};2]$. [color=brown]Calculus solution:[/color] From $au^2-4u+2(a-1)=0$ we get $a=\frac{4u+2}{u^2+2}=g(u)$ where $u\in[-1;1]$ calculating $g'(u)$ we see that on $[-1;1]$ $g(u)$ is increasing. As $g(-1)=-\frac{2}{3}$, and $g(1)=2$ the same result follows.

□ Find the set of primes that satisfy: $p+1=2a^2, p^2+1=2b^2$, where a and b are integers.

Solution

Subtract the first equation from the second to get $2(b+a)(b-a)=p(p-1)$. Since p must be equal to exactly one of the factors on the left and bigger than the product of the other factors, $p=b+a$, so that $2(b-a)=p-1$. Substituting p for $b+a$, $2(p-2a)=2p-4a=p-1 \Rightarrow p=4a-1$. Substitute $2a^2-1$ for p to get $2a^2-4a=2a(a-2)=0$. Because $p>0$, $a\neq 0$. Then $a=2$, so $p=7$ is the only possibility. Because $7+1=2*2^2$ and $7^2+1=2*5^2$, this checks. Then $p=7$ is only solution.

□ Prove this $\forall n\in N, n\geq 1$ $\sum\limits_{d|n}\frac{\mu^2(d)}{\phi(d)}=\frac{n}{\phi(n)}$

Solution

induction based on factorization:

Base case: n=1, which is simply 1=1.

Inductive step: Write $n=mp^a$, where p prime, $a\in\mathbb{N}$, and $p\not| m$. Let $f(d)=[\mu(d)]^2$.

$$\sum_{d|n}\frac{f(d)}{\varphi(d)}=\sum_{k=0}^{a}\sum_{d|m}\frac{f(dp^k)}{\varphi(dp^k)}$$

From f and φ multiplicative, this becomes

$$\sum_{k=0}^{a}\left(\frac{f(p^k)}{\varphi(p^k)}\sum_{d|m}\frac{f(d)}{\varphi(d)}\right)$$

From $f(1)=f(p)=1$, $f(p^k)=0$ for $k>1$, this simplifies to $\left(1+\frac{1}{p-1}\right)\left(\frac{m}{\varphi(m)}\right)$. The left fraction is $\frac{p}{p-1}=\frac{p^a}{p^{a-1}(p-1)}=\frac{p^a}{\varphi(m)}$, so multiplying the terms gives $\frac{n}{\varphi(n)}$.

□ Prove that

$$\sum_{i=0}^{k}(-1)^i\binom{n}{i}=(-1)^k\binom{n-1}{k}$$

Solution

For each subset S of $\{1, 2, 3, \cdots, n-1\}$ with at most $k-1$ elements, we can pair S with $S \cup \{n\}$. Here exactly one set in the pair has an even number of elements and the other has an odd number of elements.

The binomial sum on the left hand side is the number of subsets of $\{1, 2, 3, \cdots, n\}$ with at most k elements and of even parity minus those of odd parity. Of the subsets of $\{1, 2, 3, \cdots, n\}$ with at most k elements, the only ones that do not fall into one of the pairs are those with exactly k elements, all coming from $\{1, 2, 3, \cdots, n-1\}$. There are $\binom{n-1}{k}$ of them, and they contribute to the even number count (so added) if k even, or to the odd number count (so subtracted) if k odd. Therefore, the binomial sum is equal to $(-1)^k\binom{n-1}{k}$, as desired.

☐ Find all real solution of: $cos(x) + cos(x.\sqrt{2}) = 2$

Solution

$\cos x \leq 1 \Rightarrow \cos(x) + \cos(x\sqrt{2}) \leq 2$. Equality holds iff x and $x\sqrt{2}$ both have cosines of 1, so that both are integral multiples of 2π. But if $x = 2\pi n$, then $2\pi n\sqrt{2}$ is an integral multiple of $2\pi \Rightarrow n\sqrt{2}$ is an integer. Since $\sqrt{2}$ irrational, $n = 0$ and so $x = 0$.

☐ Suppose $f(x)$ is a polynomial with integer coefficients such that $f(0) = 11$ and $f(x_1) = f(x_2) = ... = f(x_n) = 2002$ for some distinct integers $x_1, x_2, ..., x_n$. Find the largest possible value of n.

Solution

For each x_k, an x_k can be factored out of $f(x_k) - f(0) = 1991 = 11 \cdot 181$. So x_k must be a divisor of 1991, and there exist 8 such divisors: $\pm 1, \pm 11, \pm 181, \pm 1991$.

In addition, $P(x) = 2002 + Q(x)\prod_k(x - x_k)$ for some polynomial Q, and Q will also have integer coefficients. Substituting 0 in for x gives

$$Q(0) = (-1)^{n+1}\frac{1991}{\prod_k x_k}$$

which must be an integer. So only one x_k can be divisible by 19 and only one can be divisible by 181. After using up 1 and -1 that means at most 4 for n.

The polynomial $P(x) = 2002 + (x+1)(x-1)(x-19)(x-181)$ does satisfy the conditions for $n = 4$, so the maximum is $\boxed{4}$.

☐ Find all pairs of natural pairs of natural numbers (n, k) such that $(n+1)^k - 1 = n!$

Solution

By inspection we find the solutions $\boxed{(1,1), (2,1), (4,2)}$. To see that there are no more solutions, note that k is uniquely determined by n and if n is odd and greater than 1, the left hand side is odd while the right hand side is even, so no solution here. So suppose there is a solution with $n = 2m,\ m \geq 3$.

Our equation is equivalent to

$$(2m-1)! = 1 + (2m+1) + (2m+1)^2 + \cdots + (2m+1)^{k-1}$$

Since $m \geq 3$, $2 < m \leq 2m-1$ so 2 and m appear as distinct factors in $(2m-1)!$, making the left hand side congruent to 0 (mod $2m$). On the other hand, the right hand side is congruent to k (mod $2m$), so that k is divisible by $2m$. In particular, $k \geq 2m = n$. But this means

$$(n+1)^k - 1 \geq (n+1)^n - 1 > (n+1)^{n-1} > 2 \cdot 3 \cdot \cdots \cdot n = n!$$

which is a contradiction, so there are no solutions other than the three given.

☐ Calculate $(tan(3\pi/11) + 4sin(2\pi/11))^2 = 11$

Solution

The generalization is this: if p is an odd prime, S is the set of the $\frac{p-1}{2}$ nonzero squares modulo p, and z is a primitive pth root of 1, then

$$\left(1+2\sum_{k\in S} z^k\right)^2 = p\cdot(-1)^{\frac{p-1}{2}}$$

I forget the proof, but it goes something like this: define f so that $f(0)=0$, $f(a)=1$ if a is in S, and $f(a)=-1$ otherwise. After multiplying out the left hand side, which is equal to

$$\left(\sum_{k=0}^{p-1} z^k f(k)\right)^2 = \sum_{k=0}^{p-1}\left(z^k \sum_{j=0}^{p-1} f(j)f(k-j)\right)$$

(and the index of 0 in the right hand sum can be replaced by 1 since $f(0)f(k)=0$), the expression $f(j)f(k-j)$ for $j\neq 0, k\neq 0$ is rewritten in some clever way [size=150]**(...)**[/size] (using things such as $f(a)f(b)=f(ab)$ and $a\neq 0 \Rightarrow f(a^2)=1$) to make it clear that

$$\sum_{j=1}^{p-1} f(j)f(k-j) = -f(-1) = -(-1)^{\frac{p-1}{2}}$$

for $k\neq 0$. For $k=0, j\neq 0$, $f(j)f(k-j)=f(-1)=(-1)^{\frac{p-1}{2}}$, which would make the sum equal to

$$(p-1)(-1)^{p-1}2-(z+z^2+\cdots+z^{p-1})(-1)^{\frac{p-1}{2}} = p\cdot(-1)^{\frac{p-1}{2}}$$

Now the fix the hole where the ... is ...

$\square$ Let n be a natural number and $f(n)=2n-1995\lfloor\frac{n}{1000}\rfloor$($\lfloor\ \rfloor$ denotes the floor function).

1. Show that if for some integer r: $f(f(f...f(n)...))=1995$ (where the function f is applied r times), then n is multiple of 1995.

2. Show that if n is multiple of 1995, then there exists r such that:$f(f(f...f(n)...))=1995$ (where the function f is applied r times). Determine r if $n=1995.500=997500$

Solution

For 1)

Let f^r denote $\underbrace{f\circ f\circ f\circ\cdots\circ f}_{r}$. Since $f(n)\equiv 2n \pmod{1995}$, $f^r(n)\equiv 2^r n \pmod{1995}$. Then if $f^r(n)$ is divisible by 1995, so is $2^r n$. But 1995 and 2 are relatively prime, so 1995 must divide n.

For 2)

If n is a multiple of 1995, then so is $f^r(n)$ for any r. Also, $\frac{a}{1000}-1<\left\lfloor\frac{a}{1000}\right\rfloor\leq\frac{a}{1000}$, so $\frac{a}{200}\leq f(a)<\frac{a}{200}+1995$ for any positive integer a.

Suppose there exists a positive integer n which is a multiple of 1995 but $f^r(n)\neq 1995$ for any n. Let m be the least positive value of $f^r(n)$. Then $m\geq 1995\cdot 2$ since m is a multiple of 1995. But the inequality gives

$$0<\frac{m}{200}\leq f(m)<\frac{m}{200}+1995\leq\frac{101m}{200}<m$$

so $0<f(m)<m$ which is a contradiction. Therefore the sequence must eventually reach 1995.

When $n=1997\cdot 500$, $\frac{5}{2}\cdot 1995\leq f(n)<\frac{7}{2}\cdot 1995$ so $f(n)$ must equal $3\cdot 1995$. Since $\frac{3\cdot 1995}{200}$ is clearly less than 1995, $f^2(n)=1995$, so $r=2$ in this case.

$\square$ find the greatest commond divisor of natural numbers a and b satisfying $(1+\sqrt{2})^{2007}=a+b\sqrt{2}$

Solution

$$(1+\sqrt{2})^{2007}=a+b\sqrt{2}\Rightarrow(1-\sqrt{2})^{2007}=a-b\sqrt{2}$$

Multiplying the two,

$$a^2 - 2b^2 = -1$$

Suppose d is a (positive) common divisor of a and b, $a = dx$ and $b = dy$ where x and y are positive integers. Then

$$d^2(x^2 - 2y^2) = -1$$

In particular, d divides 1, so d must equal 1 and $\gcd(a, b) = 1$.

□ Let $a_1, a_2, ...$ be positive numbers such that $a_{n+1} = a_n^2 - 2(n = 1, 2, ...)$. Prove that $a_n \geq 2$ for all $n \geq 1$.

Solution

If $a_1 \geq 2$, then a_n will always be ≥ 2 by induction (as $a_k \geq 2 \Rightarrow a_{k+1} = a_k^2 - 2 \geq 2$).

Suppose that $a_1 < 2$ but a_n is positive for all n. It follows by induction that $a_n < 2$ for all n, so we can write

$$a_n = 2\cos\alpha_n$$

where $0 < \alpha_n < \frac{\pi}{2}$. But we have

$$2\cos\alpha_{n+1} = a_{n+1} = a_n^2 - 2 = 4\cos^2\alpha_n - 2 = 2\cos 2\alpha_n$$

so $\alpha_{n+1} = 2\alpha_n$. Let $\theta = \alpha_1$; then it follows that

$$a_n = 2\cos 2^{n-1}\theta$$

for all n. Since $\theta > 0$ and 2^{n-1} gets arbitrarily large for large enough n, there will be some M such that $2^{M-1}\theta \geq \frac{\pi}{2}$. Consider the least such M. Then because $2^{M-2}\theta < \frac{\pi}{2}$, $2^{M-1}\theta < \pi$. It follows that

$$\frac{\pi}{2} \leq 2^{M-1}\theta < \pi \Rightarrow a_M \leq 0$$

but this is a contradiction!

□ Let a, b are two positive integers such that $a, bneq1$. Find all integer values of $\frac{a^2+ab+b^2}{ab-1}$

Solution

Let $n = \frac{a^2+ab+b^2}{ab-1}$. When $a = b = 2$ and $a = 11,\ b = 2$, we find $n = \boxed{4, 7}$. We now show that no other n are possible. Suppose some a, b produced an integer $n \neq 4, 7$. Then consider such a solution with least value of $\max\{a, b\}$.

If $a = b$, then $n = \frac{3a^2}{a^2-1} = 3 + \frac{3}{a^2-1}$. Then $a^2 - 1$ divides 3, so $a^2 - 1 \leq 3 \Rightarrow a \leq 2$. But a is positive and not 1, so a must equal 2 and $n = 4$, a contradiction.

Otherwise, $a \neq b$. WLOG let $a > b$. Our expression for n can be written as

$$a^2 - b(n-1)a + (b^2 + n) = 0$$

a quadratic in a. Let c be the other root of the quadratic. Then from Vieta's, $c = b(n-1) - a$, an integer, and $c = \frac{b^2+n}{a}$, which is positive. We now show $c < a$ which is equivalent to $b^2 + n < a^2$. After writing n in terms of a and b and multiplying by $\frac{ab-1}{a}$, it's equivalent to

$$b(a^2 - b^2) > 2a + b$$

Now $a > b$, so $b \leq a - 1$ and $b(a^2 - b^2) \geq b(a^2 - (a-1)^2) = b(2a - 1)$. Finally, $b(2a-1) > 2a + b$ is equivalent to $(a-1)(b-1) > 1$. But as $b \geq 2$ and $a \geq 3$, this is clearly true.

Now that $c < a$, consider the ordered pair (b, c). Since $\max\{b, c\} < a = \max\{a, b\}$, this ordered pair cannot be one that satisfies $\frac{b^2+bc+c^2}{bc-1} \neq 4, 7$ if $c \neq 1$. In this case $\frac{b^2+bc+c^2}{bc-1}$ either equals 4 or 7. But using the quadratic equation with a and c as roots, we find $\frac{b^2+bc+c^2}{bc-1} = n$, so $n = 4$ or $n = 7$, a contradiction.

Otherwise, $c = 1$ so that

$$n = \frac{b^2 + b + 1}{b - 1} = b + 2 + \frac{3}{b - 1}$$

Then $b - 1$ divides 3, so $b = 2$ or $b = 4$. Either way, $n = 7$, a contradiction.

So in any case, no n other than 4 or 7 are possible, as desired.

□ Let a, b, c, x, y, z be real numbers so that satifying the following system: $\begin{cases} aa + b + c = 0 \\ x + y + z = 0 \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0 \end{cases}$

Calculate value of expression $A = xa^2 + yb^2 + zc^2$

Solution

$$\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)(a^3 + b^3 + c^3) = (xa^2 + yb^2 + zc^2) + \sum_{\text{cyc}} \frac{x(b^3 + c^3)}{a}$$

Since the above equals 0 and $b^3 + c^3 = (b + c)(b^2 - bc + c^2) = -a(b^2 - bc + c^2)$, we have

$$(xa^2 + yb^2 + zc^2) = \sum_{\text{cyc}} x(b^2 - bc + c^2) = \sum_{\text{cyc}} x(b^2 + c^2)$$

(from $xbc + yca + zab = abc\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$). In particular,

$$xa^2 + yb^2 + zc^2 = \frac{1}{2}(x + y + z)(a^2 + b^2 + c^2) = \boxed{0}.$$

□ Show that there is a positive integer k such that, for every positive integer n, $k2^n + 1$ is composite.

Solution

By the Chinese Remainder Theorem, we can find a positive integer k such that $k \equiv 2 \pmod{641(2^{32}-1)}$ and $k \equiv 2^{33} \pmod{\frac{2^{32}+1}{641}}$. (It is a fact that $\frac{2^{32}+1}{641}$ is an integer not divisible by 641, and of course it's relatively prime with $2^{32} - 1$; also

$$2^{32} - 1 = (2 + 1)(2^2 + 1)(2^4 + 1)(2^8 + 1)(2^{16} + 1)$$

a product of pairwise relatively prime integers. Then we have

$$n \equiv 2^j - 1 \pmod{2^{j+1}} \Rightarrow k \cdot 2^n \equiv -1 \pmod{2^{2^j} + 1}$$

for each j between 0 and 4 inclusive,

$$n \equiv 31 \pmod{64} \Rightarrow k \cdot 2^n \equiv -1 \pmod{641}$$

$$n \equiv 63 \pmod{64} \Rightarrow k \cdot 2^n \equiv -1 \pmod{\frac{2^{32} + 1}{641}}$$

Since n must fall into one of the 7 cases above, $k \cdot 2^n + 1$ must always be divisible by at least one of the 7 factors, and $k \geq 2 + 641(2^{32} - 1)$ so $k \cdot 2^n + 1$ is always greater than each of the factors and is therefore composite.

$\square$ Find a formula for a_k if
$\sum_{k=1}^{n}\binom{n}{k}a_k = \frac{n}{n+1}$

Solution

$$\boxed{a_k = \frac{(-1)^{k+1}}{k+1}}$$

The base case $k=1$ is clearly true. For the inductive step, if we can show

$$\sum_{k=1}^{n}(-1)^{k+1}\cdot\frac{n+1}{k+1}\cdot\binom{n}{k} = n$$

then we will be done, because the right hand side is equal to

$$(n+1)\sum_{k=1}^{n}a_k\binom{n}{k} = (n+1)a_n + \sum_{k=1}^{n-1}(-1)^{k+1}\cdot\frac{n+1}{k+1}\cdot\binom{n}{k}$$

But we have $\frac{n+1}{k+1}\cdot\binom{n}{k} = \binom{n+1}{k+1}$, so that

$$\sum_{k=1}^{n}(-1)^{k+1}\cdot\frac{n+1}{k+1}\cdot\binom{n}{k} = \sum_{k=1}^{n}(-1)^{k+1}\binom{n+1}{k+1}$$

$$= -1 + (n+1) + (1-1)^{n+1} = n$$

as desired.

$\square$ Find all $n \in \mathbb{N}^*$ satisfy 3^n-1 is divisible by n^3.

Solution

We claim $\boxed{1, 2}$ are the only solutions (clearly they work). Suppose true for some $n > 2$. Then let p be the least prime divisor of n. We have $3^n \equiv 1 \pmod{p}$ and $3^{p-1} \equiv 1 \pmod{p}$. Let e be the order of 3 modulo p, then e divides both $p-1$ and n (which has no prime divisor less than or equal to $p-1$) so $e = 1$. But $3 \equiv 1 \pmod{p}$ implies $p = 2$.

Now write $n = 2^k \cdot m$ where k, m positive integers and m odd. Then $2^{3k}\cdot m^3$ divides $3^{2^k\cdot m} - 1$.

First, 2^{3k} divides $3^{2^k\cdot m} - 1$ which factors by difference of squares into

$$(3^m - 1)\prod_{j=0}^{k-1}(3^{2^j\cdot m} + 1)$$

$3^m - 1$ is congruent to 2 mod 4, $3^m + 1$ is congruent to 4 mod 8, and $3^{2^j\cdot m} + 1$ is congruent to 2 mod 4 for positive j. That means the above product is divisible by 2^{k+2} but not 2^{k+3}. This means $k + 2 \geq 3k$, so k must equal 1.

Now with $k = 1$, $n = 2m$ so $8m^3$ divides $3^{2m} - 1$. Now $n > 2$ means $m > 1$, and we do the same thing as before: let q be the smallest prime divisor of m, so that $3^{2m} \equiv 1 \pmod{q}$ and $3^{q-1} \equiv 1 \pmod{q}$, from which we deduce $3^2 \equiv 1 \pmod{q}$. But this means $q = 2$ which contradicts m odd! Therefore, no other solutions.

$\square$ $a \neq 0$, $b \neq 0$ and $c \neq 0$ such that $a^2(b+c-a) = b^2(c+a-b) = c^2(a+b-c)$. Prove that $a = b = c$.

Solution

Suppose a, b, c not all equal. At least one number is distinct from all others, so WLOG let $a \neq b, a \neq c$. We have

$$0 = a^2(b+c-a) - b^2(c+a-b) = c(a^2-b^2) + (a^2b - ab^2) - (a^3 - b^3)$$

$$= (a-b)((a+b)c + ab - (a^2+ab+b^2)) = (a-b)((a+b)c - a^2 - b^2) \Rightarrow (a+b)c = a^2+b^2$$

Similarly, $(a+c)b = a^2+c^2$. In particular,

$$0 = (a+c)b - (a^2+c^2) - (a+b)c + (a^2+b^2) = a(b-c) + (b^2-c^2) = (b-c)(a+b+c)$$

But either case leads to a contradiction when plugging into $(a+b)c = a^2+b^2$:

$$b = c \Rightarrow (a+b)b = a^2+b^2 \Rightarrow a(a-b) = 0$$

$$a+b+c = 0 \Rightarrow -(a+b)^2 = a^2+b^2 \Rightarrow a^2+b^2+(a+b)^2 = 0$$

So $a = b = c$.

□ For $|x| \leq 1$, $|ax^2+bx+c| \leq 1$. Prove

$$|cx^2+bx+a| \leq 2$$

for $|x| \leq 1$

Solution

Let $P(x) = ax^2+bx+c, Q(x) = cx^2+bx+a, j = P(-1), k = P(0), l = P(1)$. We must have

$$P(x) = j \cdot \frac{x^2-x}{2} + k \cdot (1-x^2) + l \cdot \frac{x^2+x}{2}$$

For $|x| \geq 1$, we have

$$|P(x)| \leq |j| \cdot \frac{|x^2-x|}{2} + |k| \cdot |1-x^2| + |l| \cdot \frac{|x^2+x|}{2}$$
$$\leq \frac{|x^2-x|}{2} + |1-x^2| + \frac{|x^2+x|}{2} = \frac{x^2-x}{2} + (x^2-1) + \frac{x^2+x}{2}$$
$$= 2x^2 - 1 < 2x^2$$

so that $|Q(1/x)| < 2$. As a result, $|Q(y)| < 2$ if $y \in [-1,1]$ and $y \neq 0$.

Similarly, $Q(0) = \frac{j}{2} - k + \frac{l}{2}$ so $|Q(0)| \leq 2$ as well. Another way

For contradiction, assume there is a point in $[-1,1]$ such that $|cx^2+bx+a| > 2$

Notice that at each of $x = \pm 1$, $|cx^2+bx+a| = |ax^2+bx+a| = |a \pm b + c| \leq 1$. Therefore $|cx^2+bx+a|$ takes its maximum at its vertex, which must lie in $(-1,1)$.

Thus the x coordinate of the vertex must lie in $(-1,1)$ so $|b| < 2|c|$. The $|y|$ value is > 2: $\left|\frac{b^2-4ac}{4c}\right| > 2$.

Going back to the original quadratic, $f(x) = ax^2+bx+c$, we know that $|f(-1)|, |f(0)|, |f(1)| \leq 1 \implies |c| \leq 1, |a \pm b + c| \leq 1$. By the triangle inequality, $|a \pm b| \leq |a \pm b + c| + |c| \leq 2$ thus $|a| + |b| \leq 2$.

Hence

$$|b^2| + |4ac| \geq |b^2 - 4ac| > 8|c| \iff \iff b^2 > 4(2-|a|)|c| \geq 4|b \| c| \geq 2b^2 \iff -b^2 > 0$$

contradiction.

We conclude $|cx^2+bx+a| \leq 2$ in $[-1,1]$ □

□ Let a and b are positive numbers such that $a^9+b^9 = 2$. Prove that

$$\frac{a^2}{b} + \frac{b^2}{a} \geq 2$$

Solution

The inequality $(a^3+b^3)^9 \geq 256a^9b^9(a^9+b^9)$ is equivalent to

$$(a^6+2a^3b^3+b^6)^4 \geq 256a^9b^9(a^6-a^3b^3+b^3)$$

If we let $X = a^3b^3$ and $Y = a^6-a^3b^3+b^6$, then AM-GM gives $(3X+Y)^4 \geq 256X^3Y$.

□ Suppose $(2+\sqrt{3})^{2r-1} = 1+m+n\sqrt{3}$, where m,n,r are positive integers. Then prove that m has an odd number of divisors.

Solution

Let $r = s+1$. Conjugating with respect to $\sqrt{3}$, we get $\left(2-\sqrt{3}\right)^{2s+1} = 1+m-n\sqrt{3}$ so that

$$m = \frac{\left(2+\sqrt{3}\right)^{2s+1}+\left(2-\sqrt{3}\right)^{2s+1}-2}{2}$$

$$= \frac{\left(2+\sqrt{3}\right)^{2s}\left(1+\sqrt{3}\right)^2+\left(2-\sqrt{3}\right)^{2s}\left(1-\sqrt{3}\right)^2-4}{4}$$

$$= \left(\frac{\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)^s+\left(1-\sqrt{3}\right)\left(2-\sqrt{3}\right)^s}{2}\right)^2$$

The element under the square is of the form $\frac{\alpha+\overline{\alpha}}{2}$ where $\alpha = a+b\sqrt{3}$ for integers a, b. So $m = a^2$. It then follows easily that m has an odd number of divisors, since each divisor less than a can be paired with $\frac{m}{a}$.

□ Let $\frac{\sin x+\sin y+\sin z}{\sin(x+y+z)} = \frac{\cos x+\cos y+\cos z}{\cos(x+y+z)} = 2\sqrt{2}$. Find $\cos x\cos y+\cos y\cos z+\cos z\cos x$

Solution

Let $a = \cos x+i\sin x,\ b = \cos y+i\sin y,\ c = \cos z+i\sin z$. Then a, b, c have absolute value 1 and satisfy the equation

$$a+b+c = 2\sqrt{2}abc$$

Conjugating both sides of the above (note that $|a| = 1 \Rightarrow \overline{a} = \frac{1}{a}$, etc.),

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{2\sqrt{2}}{abc} \Rightarrow ab+bc+ca = 2\sqrt{2}$$

Now we compute: $\cos x\cos y+\cos y\cos z+\cos z\cos x$ is half of $\sum_{\text{cyc}}\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)$, which is

$$\sum_{\text{cyc}}\frac{c(a^2+1)(b^2+1)}{abc}$$

$$= \frac{abc(ab+bc+ca)+(a+b+c)+(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)}{abc}$$

$$= \frac{abc(ab+bc+ca)+(a+b+c)+(a+b+c)(ab+bc+ca)-3abc}{abc}$$

$$= 2\sqrt{2}+2\sqrt{2}+8-3 = 4\sqrt{2}+5$$

$$\Rightarrow \cos x\cos y+\cos y\cos z+\cos z\cos x = \boxed{\frac{4\sqrt{2}+5}{2}}$$

Of course, having the sum much greater than 3 is ridiculous, so either I messed up horribly or there's no solution.

□ Let $k_1, k_2, .., k_n, k_i \neq k_j \forall i \neq j$. Prove that: $a_1 e^{k_1 x} + a_2 e^{k_2 x} + ... + a_n e^{k_n x} = 0, \forall x \in R$ if and only if $a_1 = a_2 = ... = a_n = 0$

Solution

First, analize the case $\max_i k_i > 0$ and $a_i = 0$. WLOG $k_1 = \max_i k_i$. Then $a_1 e^{k_1 x} + \cdots + a_n e^{k_n x} = e^{k_1 x}\left(a_1 + a_2 e^{(k_2 - k_1)x} + \cdots + a_n e^{(k_n - k_1)x}\right)$ and if we take $x \to \infty$ the function will diverge since the 2nd factor converges to a_1. This clearly acontradiction.

I think we can extend the proof if we assume that $|k_1| = \max_i |k_i|$ and then multiply each k by ± 1 so that the maximal k is positive. We can also just cut away every $a_i = 0$ and analize the remaining funtion, so $a_i = \ldots = a_n = 0$.

Another way If a function f is identically zero, $f^{(m)}(0) = 0$ for all nonnegative integers m. So we get

$$\sum_{j=1}^{n} a_j k_j^m = 0$$

By taking linear combinations,

$$\sum_{j=1}^{n} a_j P(k_j) = 0$$

for every polynomial P. But for each j, we can choose the polynomial

$$P_j(t) = \prod_{r \neq j} \frac{t - k_r}{k_j - k_r}$$

Plugging in P_j into the above gives a_j on the left hand side. So all a_j are zero.

□ Let $x, y, z \in R, x \neq y, y \neq z, z \neq x$ such that: $x^2 = 2 + y, y^2 = 2 + z, z^2 = 2 + x$ Find max and min: $P = x^2 + y^2 + z^2$

Solution

If $x > 2$, then $x < y < z < x$. If $x < -2$, then $y > 2$. Both lead to contradictions, so we must have $|x| \leq 2$.

Let $x = 2\cos\theta$ to get $y = 2\cos 2\theta, z = 2\cos 4\theta$, and

$$\cos 8\theta = \cos\theta \Rightarrow \sin\frac{9\theta}{2}\sin\frac{7\theta}{2} = 0$$

Now we cannot have $\theta \equiv 0 \pmod{2\pi}$ because then $x = y = 2$. So if $7\theta \equiv 0 \pmod{2\pi}$, then the unique solution up to cyclic permutation is

$$\left(2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{8\pi}{7}\right)$$

(This is the only solution because $\cos\frac{12\pi}{7} = \cos\frac{2\pi}{7}$ etc.). Now we have

$$z = e^{2\pi i}7 \Rightarrow 0 = z^3 + z^{-3} + z^2 + z^{-2} + z + z^{-1} + 1 = x^3 + x^2 - 2x - 1$$

etc. giving $x + y + z = -1, xy + yz + zx = -2 \Rightarrow x^2 + y^2 + z^2 = 5$.

Otherwise, $9\theta \equiv 0$. But $3\theta \not\equiv 0$ (otherwise we would get all variables equal to -1). Here the unique solution is

$$\left(2\cos\frac{2\pi}{9}, 2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9}\right)$$

up to cyclic permutation. In this case, $x^3 - 3x - 1 = 0$ giving $x + y + z = 0, xy + yz + zx = -3 \Rightarrow x^2 + y^2 + z^2 = 6$.

So the minimum and maximum values of P are 5 and 6, respectively (in fact, they're the only values).

$\square$ Let a, b, c and d are non-negative numbers such that

$$a^2 + b^2 + c^2 + d^2 = ab + ac + ad + bc + bd + cd$$

Prove that

$$(a + b + c + d)^3 \geq 21.6(abc + abd + acd + bcd)$$

When the equality holds?

Solution

Normalize to $a + b + c + d = 6, ab + ac + ad + bc + bd + cd = 12$, and WLOG let $a \leq b \leq c \leq d$. So the polynomial

$$t^4 - 6t^3 + 12t^2 - mt + n$$

(where $m = abc + abd + acd + bcd, n = abcd$) has four nonnegative roots, and we must show $m \leq 10$. By Rolle's Theorem (and the fact that roots of multiplicity have zero derivative at the root), the derivative of the above,

$$4t^3 - 18t^2 + 24t - m$$

has three nonnegative real roots x, y, z with sum $\frac{9}{2}$ and $a \leq x \leq b \leq y \leq c \leq z \leq d$. But this is equal to

$$2(t - 1)^2(2t - 5) - (m - 10)$$

so we indeed get $m \leq 10$ (otherwise $x, y, z > \frac{5}{2}$).

In order to have equality, we have $x = y = 1$, that means $b = 1$. So our original polynomial is

$$2\int_1^t (u - 1)^2(2(u - 1) - 3)\ du = (t - 1)^4 - 2(t - 1)^3 = (t - 1)^3(t - 3)$$

making the only equality case (1,1,1,3) (up to permutations and scaling).

$\square$ Let a, b and c are non-negative numbers such that $a^2 + b^2 + c^2 = 4(ab + ac + bc)$. Prove that

$$a^3 + b^3 + c^3 \geq 12abc$$

Solution

Normalize by letting $x = \frac{6a}{a+b+c}$ etc. Because the condition and inequality are homogenous, the inequality in a, b, c reduces to one in x, y, z where x, y, z nonnegative and $x+y+z = xy+yz+zx = 6$. This also means $x^3 + y^3 + z^3 - 3xyz = 108$.

Now the polynomial $P(t) = (t - x)(t - y)(t - z) = t^3 - 6t^2 + 6t - p$ (for $p = xyz$) has x, y, z as roots. But also,

$$t^3 - 6t^2 + 6t - 4\left(\sqrt{2} - 1\right) = \left(t - \left(2 - \sqrt{2}\right)\right)^2 \left(t - \left(2 + 2\sqrt{2}\right)\right)$$

So we must have $p \leq 4\left(\sqrt{2} - 1\right)$. Otherwise, the above expression is positive for x, y, z, which would mean $x, y, z > 2 + 2\sqrt{2}$ and then $x + y + z$ would be much greater than 6.

Now $xyz \leq 4\left(\sqrt{2}-1\right)$ and $x^3+y^3+z^3-3xyz=108$ implies

$$\frac{x^3+y^3+z^3}{xyz}=3+\frac{108}{xyz}\geq 30+27\sqrt{2}$$

So the best constant is $30+27\sqrt{2}$, with $x=2+2\sqrt{2}, y=z=2-\sqrt{2}$ producing equality.

□ Let b be an even positive integer. Assume that there exist integer $n>1$ such that $\frac{b^n-1}{b-1}$ is perfect square. Prove that b is divisible by 8.

Solution

We have

$$\frac{b^n-1}{b-1}=b^{n-1}+\ldots+b+1=k^2, \quad n,k\in\mathbb{N}, \ \ n\neq 1,$$

from which we can get

$$k^2-1=b\sum_{i=0}^{n-2}b^i.$$

Because $b\equiv 0 \pmod 2$, from the left side follows $k\equiv 1 \pmod 2$, so $k=2t-1$, $t\in\mathbb{N}$, $t\neq 1$ (because $b\geqslant 2$).

Now we get

$$4t\left(t-1\right)=b\sum_{i=0}^{n-2}b^i.$$

Notice that $4t\left(t-1\right)\equiv 0 \pmod 8$ and

$$\sum_{i=0}^{n-2}b^i\equiv 1 \pmod 2,$$

from which directly follows $b\equiv 0 \pmod 8$.

Q. E. D. Another way Let $\frac{b^n-1}{b-1}=k^2$, so that $1+b+\cdots+b^{n-1}=k^2$. The left side of this equation is odd, so k must be odd also. Since squares are 0, 1, or 4 mod 8, we have $k^2\equiv 1 \pmod 8$, so

$b+b^2+\cdots+b^{n-1}\equiv 0 \pmod 8$ $b(1+b+\cdots+b^{n-2})\equiv 0 \pmod 8$

Since $1+b+\cdots+b^{n-2}$ is odd, $8|b$.

□ A prime number p divides a^2+2 for a natural number a. Prove that p or $2p$ is of the form x^2+2y^2 for some natural numbers x, y.

Solution

Let S be the set of ordered pairs of integers (u,v) such that $0\leq u<\sqrt{p}\sqrt[4]{2}, 0\leq v<\frac{\sqrt{p}}{\sqrt[4]{2}}$. The number of elements in S is $\lceil\sqrt{p}\sqrt[4]{2}\rceil\lceil\frac{\sqrt{p}}{\sqrt[4]{2}}\rceil>p$, so by Pigeonhole, there exist u,u',v,v' such that $(u,v),(u',v')$ distinct elements in S and

$$u-av\equiv u'-av' \pmod p \Rightarrow x\equiv ay \pmod p$$

where $x=u-u', y=v-v'$. Then x, y not both 0, and

$$|x|<\sqrt{p}\sqrt[4]{2}, |y|<\frac{\sqrt{p}}{\sqrt[4]{2}}\Rightarrow 0<x^2+2y^2<2\sqrt{2}p$$

But $x\equiv ay \pmod p$ implies $x^2+2y^2=(x+ay)(x-ay)+(a^2+2)y^2$ is divisible by p, so either p or $2p$ is of the form x^2+2y^2. Of course, as mentioned above, this means p must be of that form.

□ The equation

$$x^{10}+\left(13x-1\right)^{10}=0$$

has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

Solution

We get $(13x-1)^{10} = -x^{10}$ or $13x - 1 = \theta_n x$ where $\theta_0, \theta 1, \cdots, \theta_9$ are the tenth roots of -1. Solving for x, we get the roots $x = \dfrac{1}{13-\theta_n}$.

Lemma 1: If r_k is the root corresponding to θ_k, then $\overline{r_k}$ is the root corresponding to $\overline{\theta_k}$. Proof: If $r_k = \dfrac{1}{13-\theta_k}$, then $\overline{r_k} = \overline{\dfrac{1}{13-\theta_k}} = \dfrac{\overline{1}}{\overline{13-\theta_k}} = \dfrac{1}{\overline{13-\theta_k}}$. Let $\theta_k = a+bi$. We get $\overline{\theta_k} = a - bi$. Now notice $\dfrac{1}{\overline{13-\theta_k}} = \dfrac{1}{\overline{13-a-bi}} = \dfrac{1}{13-a+bi} = \dfrac{1}{13-\overline{\theta_k}}$. ■

We desire $\dfrac{1}{r_1\overline{r_1}} + \dfrac{1}{r_2\overline{r_2}} + \dfrac{1}{r_3\overline{r_3}} + \dfrac{1}{r_4\overline{r_4}} + \dfrac{1}{r_5\overline{r_5}}$. From **Lemma 1**, this is

$$\sum_{i=0}^{4}\left(\frac{1}{\left(\frac{1}{13-\theta_i}\right)\left(\frac{1}{13-\overline{\theta_i}}\right)}\right) = \sum_{i=0}^{4}(13-\theta_i)(13-\overline{\theta_i} = \sum_{i=0}^{4}(169 - 13(\theta_i + \overline{\theta_i}) + \theta_i\overline{\theta_i})$$

This is

$$169\cdot 5 - 13(\theta_0 + \theta_1 + \theta_2 + \cdots + \theta_9) + \sum_{i=0}^{4}\theta_i\overline{\theta_1})$$

Since the θs are the roots of $x^{10} - 1 = 0$, their sum is, by Vietas, 0. Additionally, they are roots of unity (solutions to $x^{20} = 1$, so they have magnitude 1. We therefore get $\theta_n\overline{\theta_n} = |\theta_n|^2 = 1$ for all n. This means we can simplify our answer to

$$169\cdot 5 - 13\cdot 0 + 5 = 170\cdot 5 = \boxed{850}$$

Another appraoch Let $t = 1/x$. After multiplying the equation by t^{10}, $1 + (13-t)^{10} = 0 \Rightarrow (13-t)^{10} = -1$.

Using DeMoivre, $13 - t = e^{\frac{(2k+1)\pi}{10}}$ where k is an integer between 0 and 9.

$t = 13 - e^{\frac{(2k+1)\pi}{10}} \Rightarrow \bar{t} = 13 - e^{-\frac{(2k+1)\pi}{10}}$.

Since $e^{iy} + e^{-iy} = 2\cos y$, $t\bar{t} = 170 - 2\cos\frac{(2k+1)\pi}{10}$ after expanding. Here k ranges from 0 to 4 because two angles which sum to 2π are involved in the product..

The expression to find is $\sum t\bar{t} = 850 - 2\sum_{k=0}^{4}\cos\frac{(2k+1)\pi}{10}$.

But $\cos\frac{\pi}{10} + \cos\frac{9\pi}{10} = \cos\frac{3\pi}{10} + \cos\frac{7\pi}{10} = \cos\frac{\pi}{2} = 0$ so the sum is $\boxed{850}$.

□ Let integer $n \geq 2$. If for all integer k, satisfying $0 \leq k \leq \sqrt{\frac{n}{3}}$. $k, k^2 + k + n$ are all prime numbers. Prove that for all integer k, satifying $0 \leq k \leq n-2$ then k, k^2+k+n are all prime numbers.

Solution

Suppose $k^2 + k + n$ is not prime for some $0 \leq k \leq n-2$. Then the least nonnegative j for which $j^2 + j + n$ is not prime is at most $n-2$.

$j^2 + j + n$ cannot equal 1 since $j^2 + j + n = \left(j + \frac{1}{2}\right)^2 + n - \frac{1}{4} \geq \frac{7}{4}$, so $j^2 + j + n$ is composite. Let p be the smallest prime divisor of $j^2 + j + n$. Then $j^2 + j + n \geq p^2$, but also

$$0 \equiv j^2 + j + n \equiv j(j+1) + n \equiv (p-j)(p-(j+1)) + n$$

$$\equiv (p-1-j)^2 + (p-1-j) + n \pmod p$$

So p divides $(p-1-j)^2 + (p-1-j) + n = (p-1-j)^2 + (n-1-j) + p \geq p+1$, so also $(p-1-j)^2 + (p-1-j) + n$ is not prime.

Note also that the above expression is equal to $(j-p)^2 + (j-p) + n$. As $j - p < j$, $j - p$ must be negative, so $j \leq p - 1$ making $p - 1 - j$ nonnegative.

Therefore $p - 1 - j \geq j \Rightarrow j \leq \frac{p-1}{2}$, giving

$$p^2 \leq j^2 + j + n \leq \frac{p^2-1}{4} + n \Rightarrow p \leq \sqrt{\frac{4n-1}{3}}$$

$$\Rightarrow j \leq \frac{\sqrt{\frac{4n-1}{3}} - 1}{2} < \sqrt{\frac{n}{3}}$$

So $0 \leq j \leq \sqrt{\frac{n}{3}}$ and $j^2 + j + n$ is not prime. So if $k^2 + k + n$ are all prime for $0 \leq k \leq \sqrt{\frac{n}{3}}$, then they are all prime for $0 \leq k \leq n - 2$, as desired.

□ Solve in the natural numbers

$$x^2 + 615 = 2^n$$

Solution

$615 = 3 \cdot 205 = 3 \cdot 5 \cdot 41$. In particular, 615 is divisible by 3 so x cannot. Then $2^n \equiv x^2 \equiv 1 \pmod 3$ so n is even. Let $n = 2m$ so that

$$615 = (2^m + x)(2^m - x),\ 2^m = \frac{(2^m + x) + (2^m - x)}{2}, x = \frac{(2^m + x) - (2^m - x)}{2}$$

Now as $2^m + x$ is the bigger factor, it must be divisible by 41 since $41 > 3 \cdot 5$. The possible cases for the factors are

$$(41, 15), (123, 5), (205, 3), (615, 1)$$

of which only the second has arithmetic mean a power of 2. In this case, $m = 6$ making $n = 12$, and $x = 59$. So the unique solution is $\boxed{x = 59, y = 12}$.

□

Find the number of ways to tile a 5×2 grid with blue 1×1 tiles, red 2×1 tiles and green 2×2 tiles.

Solution

Let a_n be the number of ways to tile an $n \times 2$ grid, and b_n be the number of ways to tile an $n \times 2$ grid with a corner removed.

For a recursion for b_n, note that green tiles can't fill in the corner adjacent to the removed one. If it is filled blue, then a_{n-1} ways to fill in the rest, and if filled red, b_{n-1} ways to fill in the rest. So we get

$$b_n = a_{n-1} + b_{n-1}$$

Now for a recursion for a_n, if we align the grid such that there are 2 rows, consider what space fills in the upper right corner. If blue, there are b_n ways. If green, there are a_{n-2} ways. If red, either the red tile is horizontal or vertical. If vertical, there are a_{n-1} ways. If horizontal, the lower right corner can be blue for b_{n-1} ways, or red for a_{n-2} ways. This means

$$a_n = b_n + a_{n-1} + b_{n-1} + 2a_{n-2} = 2a_{n-1} + 2a_{n-2} + 2b_{n-1}$$

Then $b_{n-1} = \frac{a_n}{2} - a_{n-1} - a_{n-2}$, so $b_{n-2} = \frac{a_{n-1}}{2} - a_{n-2} - a_{n-3}$. Subtracting,

$$a_{n-2} = \frac{a_n}{2} - \frac{3a_{n-1}}{2} + a_{n-3}$$

$$\Rightarrow a_n = 3a_{n-1} + 2a_{n-2} - 2a_{n-3}$$

We find $a_1 = 2$ and $a_2 = 8$. Also, $b_1 = 1$, so for purposes of applying the recursive formula, $a_0 = 1$. Now we just compute:

$$a_3 = 26,\ a_4 = 90,\ a_5 = \boxed{306}$$

□ Let $F(x)$ represent the reciprocal of the cube root of x. Without using calculators or computers find the integral part of $F(4) + F(5) + F(6) + F(7) + \ldots + F(999999) + F(1000000)$

Solution

We have the inequality

$$\frac{3}{2}((n+1)^{2/3} - n^{2/3}) < \frac{1}{\sqrt[3]{n}} < \frac{3}{2}(n^{2/3} - (n-1)^{2/3})$$

This can be proven using difference of cubes; it is equivalent to

$$\frac{n + (n+1)}{n^{4/3} + n^{2/3}(n+1)^{2/3} + (n+1)^{4/3}} < \frac{2}{3} \cdot \frac{1}{\sqrt[3]{n}} < \frac{n + (n-1)}{n^{4/3} + n^{2/3}(n-1)^{2/3} + (n-1)^{4/3}}$$

or, after letting $r = \sqrt[3]{\frac{n+1}{n}} > 1$ and $s = \sqrt[3]{\frac{n-1}{n}} < 1$,

$$\frac{1+r^3}{1+r^2+r^4} < \frac{2}{3} < \frac{1+s^3}{1+s^2+s^4}$$

But $\frac{1+x^3}{1+x^2+x^4} = \frac{1+x}{1+x+x^2}$ after factoring out $x^2 - x + 1$, and

$$\frac{1+x}{1+x+x^2} = \frac{2}{3 + \frac{(x-1)(2x+1)}{(x+1)}}$$

which is less than $\frac{2}{3}$ when $x > 1$ and greater when $x < 1$. Summing from $n = 4$ to $n = 10^6$,

$$\frac{3}{2}((10^6+1)^{2/3} - 4^{2/3}) < F(4) + F(5) + \cdots + F(10^6) < \frac{3}{2}(1000 - 3^{2/3})$$

Clearly $3^{2/3} = \sqrt[3]{9} > 2$ and $(10^6+1)^{2/3} > 1000$. In addition, $4^{2/3} = \sqrt[3]{16} < \frac{8}{3}$, which is evident from cubing both sides; it reduces to $27 < 32$. So we end up with

$$\boxed{1496} < F(4) + F(5) + \cdots + F(10^6) < 1497$$

□ Let $a; b; c$ be numbers,all greater than or equal $-\frac{3}{2}$,such that $abc + ab + bc + ca + a + b + c \geq 0$ Prove that $a + b + c \geq 0$

Solution

Let $x = a+1$, $y = b+1$ and $z = c+1$. Hence, $xyz \geq 1$, where $x \geq -\frac{1}{2}$, $y \geq -\frac{1}{2}$ and $z \geq -\frac{1}{2}$. We must to prove that $x+y+z \geq 3$. If x, y and z are non-negative numbers then $x+y+z \geq 3\sqrt[3]{xyz} \geq 3$. Let $x < 0$, $y < 0$ and $x + y = p$. Hence, $z > 0$, $-1 \leq p < 0$ and $x + y + z \geq x + y + \frac{1}{xy} \geq x + y + \frac{4}{(x+y)^2}$. Thus, it remains to prove that $p + \frac{4}{p^2} \geq 3$. But $p + \frac{4}{p^2} \geq 3 \Leftrightarrow (p+1)(p-2)^2 \geq 0$. :)

□ Given a positive integer a_0, we construct a sequence as follows: If the unit digit of a_i does not exceed 5, then a_{i+1} is obtained by deleting this digit (If nothing remains upon this deletion, the sequence ends). Otherwise, $a_{i+1} = 9a_i$. Can the sequence be infinite?

Solution

The sequence cannot be infinite. Suppose there exists such an infinite sequence. Let L be the least positive integer such that $a_0 = L$ produces infinite sequence. The units digit of L must exceed 5, since the sequence ends automatically if $1 \leq L \leq 5$, and if $L > 5$ but has units digit at most 5, then

$$a_1 = \left\lfloor \frac{L}{10} \right\rfloor \leq \frac{L}{10} < L$$

This means the sequence a_{n+1} for nonnegative integers n starts at a positive integer less than L and is infinite, a contradiction.

Now that the units digit of L is at least 6, we have $a_1 = 9L$. In particular, $a_1 \equiv -L \pmod{10}$ so the units digit of a_1 is at most 4. Therefore, we get

$$a_2 = \left\lfloor \frac{a_1}{10} \right\rfloor = \left\lfloor \frac{9L}{10} \right\rfloor$$

and the sequence starting with a_2 is infinite, again a contradiction. So no infinite sequence exists.

☐ Find the number of ordered triples (x, y, z) of non-negative integers satisfying (i) $x \leq y \leq z$ (ii) $x + y + z \leq 100$.

Solution

Since $0 \leq x \leq y \leq z$, there are integer numbers $a, b, c \geq 0$ such that :

$x = a, y = a + b, z = a + b + c$

and the relation $x + y + z \leq 100$ becomes:

$3a + 2b + c \leq 100$ with $a, b, c \geq 0$ (*)

To find the number of solutions (a, b, c) of (*) we will use the following

Lemma. The number a_n of solutions (a, b, c) of the Diophantine equation $3a + 2b + c = n$, with $a, b, c \geq 0$ is given by

$a_0 = 1$

$a_n = \frac{1}{72}\left[6n^2 + 36n + 47 + 9\left(-1\right)^n + 8\theta_n\right]$ (1)

where $\theta_n = 2$ if $n \equiv 0 \,(\mathrm{mod} 3)$ and $\theta_n = -1$ if $n \not\equiv 0 \,(\mathrm{mod} 3)$.

Proof. The generating function of the sequence a_n is

$(1 + x^3 + x^6)\,(1 + x^2 + x^4)\,(1 + x + x^2) = \frac{1}{1-x^3} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x}$ (2)

The right side of (2) can be written as sum of partial fractions

$\frac{\frac{17}{72}}{1-x} + \frac{\frac{1}{4}}{(1-x)^2} + \frac{\frac{1}{6}}{(1-x)^3} + \frac{\frac{1}{8}}{1+x} + \frac{\frac{1}{9}}{1-\omega x} + \frac{\frac{1}{9}}{1-\omega^2 x}$

where $\omega = e^{2\pi i/3}$ is a complex cube root of 1 , satisfying $\omega + \omega^2 = -1$.

From the well known relations

$\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$

$\frac{1}{(1-x)^2} = \sum\limits_{n=0}^{\infty} (n+1)\, x^n$

$\frac{1}{(1-x)^3} = \sum\limits_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

$\frac{1}{1+x} = \sum\limits_{n=0}^{\infty} (-1)^n\, x^n$

$\frac{1}{1-\omega x} = \sum\limits_{n=0}^{\infty} \omega^n x^n$

$\frac{1}{1-\omega^2 x} = \sum\limits_{n=0}^{\infty} \omega^{2n} x^n$

we have :

$$\sum_{n=0}^{\infty} a_n x^n =$$
$$= \tfrac{17}{72}\left(\sum_{n=0}^{\infty} x^n\right) + \tfrac{1}{4}\left(\sum_{n=0}^{\infty}(n+1)\,x^n\right) + \tfrac{1}{6}\left(\sum_{n=0}^{\infty}\tfrac{(n+1)(n+2)}{2}x^n\right) + \tfrac{1}{8}\left(\sum_{n=0}^{\infty}(-1)^n\,x^n\right) + \tfrac{1}{9}\left(\sum_{n=0}^{\infty}(\omega^n+\omega^{2n})\,x^n\right)$$

Collecting all terms and equating the coefficients we get the formula (1). ■

Corollary. The number S of solutions (a, b, c) of $3a + 2b + c \le n$ with $a, b, c \ge 0$ is

$$S = 1 + \sum_{k=1}^{n} a_k = 1 + \tfrac{1}{72}\sum_{k=1}^{n}\left(6k^2 + 36k + 47 + 9\,(-1)^k + 8\theta_k\right) =$$
$$= 1 + \tfrac{1}{72}\sum_{k=1}^{n}\left[6\tfrac{k(k+1)(2k+1)}{6} + 36\tfrac{k(k+1)}{2} + 47k\right] + \tfrac{1}{72}\sum_{k=1}^{n}\left[9\,(-1)^k + 8\theta_k\right] =$$
$$= 1 + \tfrac{1}{72}\left(2n^3 + 21n^2 + 66n\right) + \tfrac{1}{72}\sum_{k=1}^{n}\left[9\,(-1)^k + 8\theta_k\right]$$

For $n = 100$ we find the number of solution of (*) : $S = 30787$

□ Find all prime numbers p and q such that p divides $q + 6$ and q divides $p + 7$

Solution

Clearly $p \ne q$ (otherwise p divides both $p+7$ and $p+6$), so p and q are relatively prime. Now observe that

$$p|(6p + 7q + 42),\ q|(6p + 7q + 42) \Rightarrow pq|(6p + 7q + 42)$$

Let $6p+7q+42 = kpq$. If k is even, then $q = 2$. This means $p|8$ so $p = 2$, but this fails because $2 \not| 9$. Similarly, if k is divisible by 3, then $q = 3$. This means $p|9$ so $p = 3$, but this fails because $3 \not| 10$. Therefore either $k = 1$ or $k \ge 5$.

If $k = 1$, then $6p + 7q + 42 = pq$ implies

$$(p - 7)(q - 6) = 84$$

Both factors are positive (if both negative, product less than 84). Then $q > 6$ and $p > 7$, so $q-6$ can't be divisible by 2 or 3, $p-7$ can't be divisible by 7. The only case that remains is $p-7 = 12, q-6 = 7$ which leads to $(19, 13)$.

If $k \ge 5$, then $6p + 7q + 42 \ge 5pq$ implies

$$(5p - 7)(5q - 6) \le 252$$

As shown before, $q = 2$ and $q = 3$ are bad, so $q \ge 5$. This means $p \le \frac{252}{19} < 18 \Rightarrow p \le 3$. In either case, $p|6$ so $p = q$, but this is a contradiction!

Therefore $\boxed{(19, 13)}$ only solution.

□ Solve the system of the equations: $\begin{cases} 3(x^2+y^2+z^2) = 1 \\ x^2y^2 + y^2z^2 + z^2x^2 = xyz(x+y+z)^3 \end{cases}$

Solution

$x^2y^2 + y^2z^2 + z^2x^2 = xyz(x+y+z)^3 \Leftrightarrow \Leftrightarrow 3(x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2x^2) = xyz(x+y+z)^3$. But $xyz(x+y+z) \ge 0$, $3(x^2+y^2+z^2) \ge (x+y+z)^2$ and $x^2y^2+y^2z^2+z^2x^2 \ge xyz(x+y+z)$. If $3(x^2+y^2+z^2) > (x+y+z)^2$ then $x^2y^2+y^2z^2+z^2x^2 = xyz(x+y+z) = 0$. Hence, $(x, y, z) \in \{(\pm\frac{1}{\sqrt{3}}, 0, 0), (0, \pm\frac{1}{\sqrt{3}}, 0), (0, 0, \pm\frac{1}{\sqrt{3}})\}$. If $3(x^2+y^2+z^2) = (x+y+z)^2$ then $x = y = z = \pm\frac{1}{3}$.

□ Let $a, b, c \in \mathbb{R}^+$ and $abc = 1$ Prove that

$$\sum_{cyc}\frac{4}{a^5(b+c)^2} \ge \frac{3\sqrt{3}}{\sqrt{a^2+b^2+c^2}}$$

Solution

This is my proof By Powermean ; $\sqrt{a^2+b^2+c^2} \geq \frac{a+b+c}{\sqrt{3}}$ It's remain to prove

$$\sum_{cyclic} \frac{4}{a^5(b+c)^2} \geq \frac{9}{a+b+c}$$

$$\leftrightarrow (a+b+c)(\sum_{cyclic} \frac{1}{a^5(b+c)^2}) \geq \frac{9}{4}$$

By Cauchy-Schwarz ;

$$(a+b+c)(\sum_{cyclic} \frac{1}{a^5(b+c)^2}) \geq (\sum_{cyclic} \frac{1}{a^2(b+c)})$$

It's equivalent to prove that $(\sum_{cyclic} \frac{1}{a^2(b+c)}) \geq \frac{3}{2}$ Substitute

$$a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$$

$\because abc = 1 \rightarrow xyz = 1$ It's equivalent to prove $\frac{x}{y+z} \geq \frac{3}{2}$ which is nessbit

$\square$ Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that for all real numbers x and y, $f(x^3+y^3) = (x+y)(f(x)^2 - f(x)f(y) + f(y)^2)$. Prove that for all real numbers x, $f(1996x) = 1996f(x)$

Solution

$x = y = 0 \rightarrow f(0) = 0$ $y = 0 \rightarrow f(x^3) = xf(x)^2$ $\therefore f(x) = x^{\frac{1}{3}}f(x^{\frac{1}{3}})^2$ Therefore $f(x)$ and x always have the same sign $\therefore f(x) \geq 0 \forall x \geq 0$ Let S be the set $S = \{\ a > 0 | f(ax) = af(x) \forall x \in \mathbb{R}\ \}$ Clearly $1 \in S$ $\because axf(x)^2 = af(x^3) = f(ax^3) = f((a^{\frac{1}{3}}x)^3) = a^{\frac{1}{3}}f(a^{\frac{1}{3}}x)^2$ since x and $f(x)$ have the same sign $\therefore f(a^{\frac{1}{3}}x) = a^{\frac{1}{3}}f(x)$ I will show that $a, b \in S$ impiles $a+b \in S$ $f((a+b)x) = f((a^{\frac{1}{3}}x^{\frac{1}{3}}) + (b^{\frac{1}{3}}x^{\frac{1}{3}}))$ $= (a^{\frac{1}{3}} + b^{\frac{1}{3}})[f(a^{\frac{1}{3}}x^{\frac{1}{3}})^2 - f(a^{\frac{1}{3}}x^{\frac{1}{3}})f(b^{\frac{1}{3}}x^{\frac{1}{3}}) + f(b^{\frac{1}{3}}x^{\frac{1}{3}})^2] = (a+b)f(x)$ By induction,we have $n \in S$ for each positive integer n,so in particular,$f(1996x) = 1996f(x)$ for all $x \in R$

$\square$ Let a, b, c be nonzero real numbers such that $a+b+c=0$ and $a^3+b^3+c^3 = a^5+b^5+c^5$. Find the value of $a^2+b^2+c^2$.

Solution

Let $k = a^2+b^2+c^2$. Then

$$k(a^5+b^5+c^5) = k(a^3+b^3+c^3) = (a^2+b^2+c^2)(a^3+b^3+c^3)$$

Thus,

$$k(a^5+b^5+c^5) = a^5+b^5+c^5 + a^2b^2(a+b) + b^2c^2(b+c) + a^2c^2(a+c)$$

and since $a+b+c=0$, we have that $a+b=-c$, $b+c=-a$, and $a+c=-b$. thus,

$$(k-1)(a^5+b^5+c^5) = -a^2b^2c - b^2c^2a - a^2c^2b = -abc(ab+bc+ac)$$

Also, since $a+b+c=0$, we have that $a^5+b^5+c^5 = a^3+b^3+c^3 = 3abc$. Plugging this in, we have that

$$3abc(k-1) = \frac{-abc[(a+b+c)^2 - (a^2+b^2+c^2)]}{2}$$

Hence, either $abc = 0$ (in which case a, b, or c must be 0, which contradicts the given) or $6(k-1) = -1(0^2-k) = k \implies 5k = 6$, which gives us that $k = \boxed{\frac{6}{5}}$.

□ Show that $\gcd(2^m - 1, 2^n - 1) = 2^d - 1$, where $d = \gcd(m, n)$

Solution

It's old problem and very well-known

$(n^a - 1, n^b - 1) = n^{gcd(a,b)}$,where $n, a, b \in \mathbb{N}$

Proof

Let $d = gcd(n^a - 1, n^b - 1)$ and $k = ord_d n$

It's easy to see that $n^{gcd(a,b)} - 1|d$

since $n^a \equiv n^b \equiv 1(modd)$,Hence $k|a, k|b$

so $k|gcd(a, b)$

thus $n^{gcd(a,b)} \equiv 1(modd) \to d|n^{gcd(a,b)} - 1$

so $d = n^{gcd(a,b)} - 1$

□ Solve the equation $a^3 + b^3 + c^3 = 2001$ in positive integers.

Solution

$\forall t \in \mathbb{N} t^3 \equiv 0, \pm 1 \pmod 9$. $2001 \equiv 3 \pmod 9$. Hence, $a = 3x + 1, b = 3y + 1, c = 3z + 1$ for $\{x, y, z\} \subset \mathbb{N}_0$. $a^3 \le 2001$. Hence, $x \le 3$ and $y \le 3, z \le 3$. Let $x \ge y \ge z$. Then $3a^3 \ge 2001 \Rightarrow x > 2$. Hence, $x = 3$. Hence, $b^3 + c^3 = 1001$. Hence, $2(3y + 1)^3 \ge 1001 \Rightarrow y > 2$. Hence, $2 < y \le 3$. Hence, $y = 3 \Rightarrow z = 0 \Rightarrow a = 10, b = 10, c = 1$. Well $\{(10, 10, 1), (10, 1, 10), (1, 10, 10)\}$. :)

□ Let $a, b, c \in \left[\frac{1}{3}, 3\right]$.Prove that

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \ge \frac{7}{5}$$

Solution

Let $a = \max\{a, b, c\}$. We obtain: $\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \ge \frac{7}{5} \Leftrightarrow \Leftrightarrow (3a - 2b)c^2 - (2a^2 - ab - 3b^2)c + 3a^2 b - 2ab^2 \ge 0$. Thus, it remains to prove that $(2a^2 - ab - 3b^2)^2 - 4ab(3a - 2b)^2 \le 0$, which equivalent to $(a - b)(a - 9b)(4a^2 + b^2) \le 0$, which obviously true.

□ x, y, z, a, b, c number $a = x + y - z, b = -x + y + z, c = x - y + z$ and $\frac{(a+b+c)^5 - a^5 - b^5 - c^5}{(a+b)(b+c)(c+a)} = \frac{(x+y+z)^5 - x^5 - y^5 - z^5}{(x+y)(y+z)(z+x)}$.

Prove: $a = b = c = x = y = z$

Solution

$(a+b+c)^5 - a^5 - b^5 - c^5 = 5\sum_{sym}(a^4 b + 2a^3 b^2 + 2a^3 bc + 3a^2 b^2 c) = = 5(a+b)(a+c)(b+c)(a^2 + b^2 + c^2 + ab + ac + bc)$. Id est, $\frac{(a+b+c)^5 - a^5 - b^5 - c^5}{(a+b)(b+c)(c+a)} = \frac{(x+y+z)^5 - x^5 - y^5 - z^5}{(x+y)(y+z)(z+x)} \Leftrightarrow \Leftrightarrow \sum_{cyc}(a^2 + ab - x^2 - xy) = 0 \Leftrightarrow \Leftrightarrow \sum_{cyc}((x+y-z)^2 + (x+y-z)(y+z-x) - x^2 - xy) = 0 \Leftrightarrow \Leftrightarrow \sum_{cyc}(x^2 - xy) = 0 \Leftrightarrow \sum_{cyc}(x-y)^2 = 0 \Leftrightarrow x = y = z$. It gives also $a = b = c = x$

□ Let a, b, c, x and y are positive numbers such that $ay + bx + \sqrt{3}(ab - xy) = 0$ and $a^2 + x^2 = b^2 + y^2 = (x - y)^2 + c^2$. Prove that $c = a + b$.

Solution

Since a, b, x, y are positive, $\sqrt{3}(xy - ab) = ay + bx > 0$. So $xy > ab$...

We have

$$4(ab - xy)^2 = (ay + bx)^2 + (ab - xy)^2 = (a^2 + x^2)(b^2 + y^2) = (a^2 + x^2)^2$$

Since $xy > ab$,

$2(xy - ab) = a^2 + x^2 = b^2 + y^2 = (x - y)^2 + c^2$

So

$a^2+x^2-2(xy-ab)+b^2+y^2=(x-y)^2+c^2$

Therefore $(a+b)^2+(x-y)^2=(x-y)^2+c^2$. c positive implies $c=a+b$.

EDIT: It turns out that $xy>ab$ must hold...

Let $k\in\mathbb{N}p\in\mathbb{N}\backslash\{0,1\}$ and $a,r\in(0,\infty)$ Consider the sequence $(a_n)_{n\geq 1}$ defined by $a_n=a+(n-1)\cdot r$, $\forall\ n\in\mathbb{N}$. Then:

■ 1° $\lim_{n\to\infty}\frac{a_{qn+k+1}\cdot a_{qn+k+1+p}\cdot\ldots\cdot a_{qn+k+1+s(n-1)p}}{a_{qn+k}\cdot a_{qn+k+p}\cdot\ldots\cdot a_{qn+k+s(n-1)p}}=\sqrt[p]{\frac{ps+q}{q}}$

■ 2° $\lim_{n\to\infty}\sqrt[n]{\frac{a_{qn+k}\cdot a_{qn+k+p}\cdot\ldots\cdot a_{qn+k+s(n-1)p}}{(n!)^s}}=$

■ 3° $\lim_{n\to\infty}\frac{\sqrt[n]{a_{qn+k}\cdot a_{qn+k+p}\cdot\ldots\cdot a_{qn+k+s(n-1)p}}}{n^s}=$

Applycations: $\lim_{n\to\infty}\frac{\binom{4n}{2n}}{4^n\cdot\binom{2n}{n}}=\frac{\sqrt{2}}{2}$; $\lim_{n\to\infty}\frac{5^{5n}\cdot\binom{2n}{n}^3}{\binom{10n}{5n}\cdot\binom{5n}{n}\cdot\binom{4n}{2n}}=4$

□ Let $a,b,c>0$ so that $a+b+c=1$. Prove that: $\frac{\sqrt{a^2+abc}}{c+ab}+\frac{\sqrt{b^2+abc}}{a+bc}+\frac{\sqrt{c^2+abc}}{b+ca}\leq\frac{1}{2\sqrt{abc}}$

Solution

Note that $\sum\frac{\sqrt{a^2+abc}}{c+ab}=\sum\frac{\sqrt{a(c+a)(a+b)}}{(b+c)(c+a)}$.

Therefore our inequality is equivalent to

$$\sum\frac{\sqrt{a(c+a)(a+b)}}{(b+c)(c+a)}\leq\frac{a+b+c}{2\sqrt{abc}}$$
$$\iff\sum a(a+b)\sqrt{bc(c+a)(a+b)}\leq\frac{1}{2}(a+b+c)(a+b)(b+c)(c+a)$$

By AM-GM,

$$\begin{aligned}suma(a+b)\cdot 2\sqrt{bc(c+a)(a+b)}&\leq\sum a(a+b)(b(c+a)+c(a+b))\\&=\sum a(a+b)(ab+2bc+ca)\end{aligned}$$

Now

$$\begin{aligned}\sum a(a+b)(ab+2bc+ca)&=\sum a^2(ab+bc+ca)+\sum a^2bc+\sum ab(ab+bc+ca)+\sum ab^2c\\&=(a^2+b^2+c^2+ab+bc+ca)(ab+bc+ca)+2abc(a+b+c)\\&=(a+b+c)^2(ab+bc+ca)-(ab+bc+ca)^2+2abc(a+b+c)\\&=(a+b+c)^2(ab+bc+ca)-(a^2b^2+b^2c^2+c^2a^2)\\&\leq(a+b+c)^2(ab+bc+ca)-abc(a+b+c)\\&=(a+b+c)(a+b)(b+c)(c+a)\end{aligned}$$

which was what we wanted.

□ Let a, b and c are non-negative numbers such that $a^2+b^2+c^2=3$. Prove that:

$$(3-a)(3-b)(3-c)\geq 8$$

Solution

Consider the function $f(x)=(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$. $f(x)=x^3-px^2+qx-r$ where $p=a+b+c$, $q=ab+bc+ca$, and $r=abc$. $p^2=(a+b+c)^2=$

$a^2+b^2+c^2+2ab+2bc+2ac$ and $2q=2(ab+bc+ca)=2ab+2bc+2ca$ so $p^2-2q=a^2+b^2+c^2=3$. Solving for q, we get $q=\dfrac{p^2-3}{2}$. $r=abc\le\left(\sqrt{\dfrac{a^2+b^2+c^2}{3}}\right)^3=\left(\sqrt{\dfrac{3}{3}}\right)^3=1$ by QM-GM. $f(3)=(3-a)(3-b)(3-c)=27-9p+3q-r=27-9p+3\left(\dfrac{p^2-3}{2}\right)-r=27-9p+\dfrac{3p^2}{2}-\dfrac{9}{2}-r$ $=\dfrac{45}{2}-9p+\dfrac{3p^2}{2}-r=9+\left(\dfrac{27}{2}-9p+\dfrac{3p^2}{2}\right)-r=9+\dfrac{3}{2}(9-6p+p^2)-r=9+\dfrac{3}{2}(p-3)^2-r$ We have $(3-a)(3-b)(3-c)=9+\dfrac{3}{2}(p-3)^2-r$. Since $\dfrac{3}{2}(p-3)^2\ge 0$ and $-r\ge -1$, and adding these inequalities together we have $\dfrac{3}{2}(p-3)^2-r\ge -1$, $9+\dfrac{3}{2}(p-3)^2-r\ge 8$. $(3-a)(3-b)(3-c)=9+\dfrac{3}{2}(p-3)^2-r\ge 8$, with equality when $a=b=c=1$, so we are done.

□ $x,y\in R^+$ $x^3+y^3=4x^2$

Find the Max of $x+y$

Solution

Let $x+y=k$. Hence, the equation $k(x^2-x(k-x)+(k-x)^2)=4x^2$ has real root. But $k(x^2-x(k-x)+(k-x)^2)=4x^2\Leftrightarrow(3k-4)x^2-3k^2x+k^3=0$. If $k=\frac{4}{3}$ so $x=\frac{4}{9}$ and $y=\frac{8}{9}$. Let $k\neq\frac{4}{3}$. Hence, $(3k^2)^2-4(3k-4)k^3\ge 0$, which gives $0\le k\le\frac{16}{3}$. For $k=\frac{16}{3}$ we obtain: $x=\frac{32}{9}$ and $y=\frac{16}{9}$. Hence, $\max_{x^3+y^3=4x^2}(x+y)=\frac{16}{3}$. Since $\frac{32}{9}>0$ and $\frac{16}{9}>0$, the answer is $\frac{16}{3}$.

□ Prove that: $cos(sin(x))>sin(cos(x))$

Solution

$\cos\sin x>\sin\cos x\Leftrightarrow\sin\left(\frac{\pi}{2}-\sin x\right)-\sin\cos x>0\Leftrightarrow\Leftrightarrow 2\sin\frac{\frac{\pi}{2}-\sin x-\cos x}{2}\cos\frac{\frac{\pi}{2}-\sin x+\cos x}{2}>0$, which is true because $|\sin x+\cos x|\le\sqrt{2}$ and $|\sin x-\cos x|\le\sqrt{2}$, which gives $0<\frac{\frac{\pi}{2}-\sqrt{2}}{2}\le\frac{\frac{\pi}{2}-\sin x-\cos x}{2}\le\frac{\frac{\pi}{2}+\sqrt{2}}{2}<\frac{\pi}{2}$ and $0<\frac{\frac{\pi}{2}-\sqrt{2}}{2}\le\frac{\frac{\pi}{2}-\sin x+\cos x}{2}\le\frac{\frac{\pi}{2}+\sqrt{2}}{2}<\frac{\pi}{2}$.

□ Solve system of equation

$$\begin{cases}2\sqrt{2x+3y}+\sqrt{5-x-y}=7\\3\sqrt{5-x-y}-\sqrt{2x+y-3}=1\end{cases}$$

Solution

The answer is $\{(3,1)\}$. Let $2x+3y=a^2$, $5-x-y=b^2$ and $2x+y-3=c^2$, where a, b and c are non-negatives. Hence, $2a+b=7$, $3b-c=1$ and $a^2+4b^2+c^2=17$, which gives $b=7-2a$, $c=20-6a$ and $a^2+4(7-2a)^2+(20-6a)^2=17$. From here we obtain $a=3$ and $x=3$, $y=1$.

□ Find all pairs of positive integers (x,y) such that

$$x^y=y^{x-y}.$$

Solution

let $y=\frac{m}{n}\cdot x$, where $(m,n)=1,\{m,n\}\subset\mathbb{N}$. Hence$x^{\frac{m}{n}\cdot x}=(\frac{m}{n}\cdot x)^{x-\frac{m}{n}\cdot x}$. Hence $x=(\frac{m}{n})^{\frac{n-m}{2m-n}}$ and $y=(\frac{m}{n})^{\frac{m}{2m-n}}$. 1) $\frac{n-m}{2m-n}>0$. Hence, $n=1$ and $\frac{1-m}{2m-1}>0$. Hence, $\frac{1}{2}<m<1$. This is contradiction. 2) $\frac{n-m}{2m-n}=0$. Hence $m=n$ and $x=y=1$ 3) $\frac{n-m}{2m-n}<0$. Hence,$x=(\frac{n}{m})^{\frac{n-m}{n-2m}}$, $y=(\frac{n}{m})^{\frac{m}{n-2m}}$, $n>2m$. Hence $m=1$ and $x=n^{\frac{n-1}{n-2}}$, $y=n^{\frac{1}{n-2}}$, $n\ge 3$. Let $f(t)=t^{\frac{1}{t-2}}$, $t>2$. Hence, $f'(t)=f(t)\cdot\frac{\frac{t-2}{t}-lnt}{(t-2)^2}<0$. Hence, $f(t)\le f(3)=3$ and $y\le 3$. Let $n=3$. Hence $y=3$ and $x=9$. Let $n=4$. Hence $y=2$ and $x=8$. Let $n>4$. Hence $1<y<2$. This is contradiction. Well, $\{(1,1),(9,3),(8,2)\}$.

□ If real numbers $a,b,c\in[0,2]$ and $a+b+c=3$, show that

$$a^2b+b^2c+c^2a\ge 2$$

Solution

Let $a=\frac{2y+2z-x}{3}$, $b=\frac{2x+2z-y}{3}$ and $c=\frac{2x+2y-z}{3}$. Hence, $x+y+z=3$ and $x+a=2$, which gives that x, y and z are non-negatives. Id est, we need to prove that $\sum_{cyc}(2y+2z-x)^2(2x+2z-y)\geq 2(x+y+z)^3$. Let $x=\min\{x,y,z\}$, $y=x+u$ and $z=x+v$. Hence, $\sum_{cyc}(2y+2z-x)^2(2x+2z-y)-2(x+y+z)^3=$ $=27x^3+27(u+v)x^2+9(u+v)^2x+(4u+v)(u-2v)^2\geq 0$. – Solve the following equation:

$$\sqrt{\frac{x^2-3x+2}{x^2+2x}}=1+x$$

Solution

Let $x^2+2x=a$ and $x-2=b$. Then we obtain $b^2+b=a^2+a$.

□ Solve the inequation $\sqrt{x^2-x-6}+7\sqrt{x}\leq\sqrt{6(x^2+5x-2)}$.

Solution

$x^2-x-6\geq 0$ gives $x\geq 3$. After squaring of the both sides we obtain $5x^2-18x-6\geq 14\sqrt{x(x^2-x-6)}$. $5x^2-18x-6\geq 0$ gives $x\geq\frac{9+\sqrt{111}}{5}$. After squaring we need to solve $(x^2-12x-18)(25x^2-76x-2)\geq 0$, which with $x\geq\frac{9+\sqrt{111}}{5}$ gives $x\geq 6+3\sqrt{6}$.

□ Let a,b,c be random real numbers and a+b+c=3 Prove that $a^2\ .(b-c)^2\ +b^2\ .(a-c)^2\ +\ c^2\ .(a-b)^2\geq\frac{9}{2}\ .abc(1-abc)$

Solution

Let $a+b+c=3u$, $ab+ac+bc=3v^2$ and $abc=w^3$. If $w^3(1-w^3)\leq 0$ then the inequality is obvious. Thus, we can assume $w^3(1-w^3)>0$, which is $0<w^3<1$. We see that you inequality is equivalent to $f(w^3)\geq 0$, where $f(w^3)=w^6-5u^3w^3+4u^2v^4$. But $f'(w^3)=2w^3-5u^3<0$. Hence, f is a decreasing function. Hence, by uvw it remains to check one case only: $b=c$, which after homogenization and assuming $b=c=1$ gives $(a-1)^2(3a^2+8a+16)\geq 0$, which is obvious.

□ Let a, b and c are non-negative numbers for which $a^2+b^2+c^2=2(ab+ac+bc)$. Prove that

$$a+b+c\geq 3\sqrt[3]{2abc}$$

Solution

Add $2(ab+ac+bc)$ to both sides of the condition. This gives $(a+b+c)^2=4(ab+bc+ca)$.

Consider the monic cubic polynomial with roots a,b,c. If this polynomial is x^3-px^2+qx-r, then we know $p^2=4q$ and we want to prove that $p^3\geq 54r$. Equivalently, we want to figure out how high the constant term can be for the polynomial to still have 3 real roots.

Using the condition, we consider the polynomial $x^3-px^2+p^2x/4-r$. Take the derivative and set it equal to 0. We get $3x^2-2px+p^2/4=0$, which has solutions $p/6$ and $p/2$. Since we have a positive leading coefficient, we get a local maximum at the lower critical point $p/6$. We need the polynomial at this local maximum to be greater than or equal to 0 in order to have 3 real roots. So $(p/6)^3-p(p/6)^2+p^2(p/6)/4-r\geq 0$, or $p^3/216-p^3/36+p^3/24\geq r$. When simplified, we have $p^3/54\geq r$ as desired.

□ a, b and c are real numbers such that $\{a,b,c\}=\{a^4-2b^2,b^4-2c^2,c^4-2a^2\}$ and $a+b+c=-3$. Find the values of a, b and c.

Solution

we have $\{a,b,c\}=\{a^4-2b^2,b^4-2c^2,c^4-2a^2\}$ so $\{a+1,b+1,c+1\}=\{a^4-2b^2+1,b^4-2c^2+1,c^4-2a^2+1\}$ so : $0=(a+1)+(b+1)+(c+1)=(a^4-2b^2+1)+(b^4-2c^2+1)+(c^4-2a^2+1)$

$= (a^2-1)^2 + (b^2-1)^2 + (c^2-1)^2$ therefore $a^2 = b^2 = c^2 = 1$ and since $a+b+c = -3$ we get $a = b = c = -1$. We have a similary problem: a, b and c are real numbers such that $\{a, b, c\} = \{a^6 - 2b^2, b^6 - 2c^2, c^6 - 2a^2\}$ and $a+b+c = -3$. Find the values of a, b and c. ?

$\square$ Let $a, b, c > 0, a+b+c = 1$. Prove that: $a^a b^b c^c + a^b b^c c^a + b^a c^b a^c \leq 1$

Solution

From weighted AM-GM, we have

1) $\frac{a^2+b^2+c^2}{a+b+c} \geq (a^a b^b c^c)^{\frac{1}{a+b+c}} \implies a^2+b^2+c^2 \geq a^a b^b c^c$

2) $\frac{ab+bc+ca}{a+b+c} \geq (a^b b^c c^a)^{\frac{1}{a+b+c}} \implies ab+bc+ca \geq a^b b^c c^a$

3) $\frac{ac+ba+cb}{a+b+c} \geq (a^c b^a c^b)^{\frac{1}{a+b+c}} \implies ab+bc+ca \geq a^c b^a c^b$

Adding the three inequalities we get $(a+b+c)^2 = 1 \geq a^a b^b c^c + a^b b^c c^a + a^c b^a c^b$

$\square$ Determine all pairs (a, b) in positive integers which satisfy next equation:

$LCM(a, b) + GCD(a, b) + a + b = ab,\ a \geq b.$

where LCM means the Least Common Multiple,and GCD does the Greatest Common Divisor of a, b

Solution

Let $(a, b) = g, a = gA, b = gB$. Then $[a, b] = gAB$ and our equation becomes

$$gAB + g + gA + gB = g^2 AB$$

which is equivalent to

$$(A+1)(B+1) = gAB$$

Hence $AB \mid (A+1)(B+1)$. But since $(A, A+1) = 1$ we conclude that $A \mid B+1$ and $B \mid A+1$. Let $B = kA - 1$. Then $kA - 1 \mid A + 1$ implies $kA - 1 \leq A + 1 \Leftrightarrow (k-1)A \leq 2$. Therefore $(k, A) = (3, 1), (2, 2), (2, 1), (1, 2)$. Hence $(k, A, B) = (1, 2, 1), (3, 1, 2), (2, 2, 3), (2, 1, 1)$. Thus we get the pairs $(a, b) = (3, 6), (4, 6), (4, 4)$.

Remark: I worked with $A \mid B+1$ so I got the pairs where $a \leq b$. For $a \geq b$, we can simply reverse each pair. –

Solution

phương trình

$$2(x^2 - 2x + 2) = 3\sqrt[3]{x^2 - 2}$$

By arqady

$\square$ Show that a and b have the same parity if and only if there exist integer c and d such that $a^2 + b^2 + c^2 + 1 = d^2$.

Solution

$a^2 + b^2 + 1$ can only have the form $4k+3, 4k+1, 4k+2$. The numbers $d+c, d-c$ have the same perity, so $(d+c)(d-c) = 4k+3,\ 4k+1$, or $4k$. This shows that a^2+b^2+1 can't be $4k+2$, so a, b have the same parity.

On the other hand, if they do have the same parity, then let $a^2 + b^2 + 1 = mn$, where m, n must be odd. Then the system of equations $d - c = m, d + c = n$ has a solution and we're done.

$\square$ The numbers p and q are primes and $p^2 + 1 \equiv 0 \pmod{q}$ and $q^2 - 1 \equiv 0 \pmod{p}$. Prove that p+q+1 is a composite.

Solution

$p|q^2 - 1 \Rightarrow p|p^2 + 2pq + q^2 - 1 = (p+q+1)(p+q-1)$. Assume $p+q+1$ is a prime. In this case, $p|p+q-1 \Rightarrow p|q-1 \Rightarrow q = mp+1,\ m \geq 1$. q also divides p^2+1, so $\frac{p^2+1}{q} = np+1,\ n \geq 0$. If $n > 0$, then it's clear that $(mp+1)(np+1) > p^2+1$, which is false, so $n = 0 \Rightarrow m = p$, so $q = p^2+1$, which means that $p+q+1 = p^2+p+2$, which is even and > 2, and thus a composite.

□ Is it possible to partition a (1sqrt(2)) rectangle into a finite number of squares?

Solution

Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function which takes the value 0 in x iff x is rational (such functions can easily be constructed using the axiom of choice; see the last few lines). Assume a rectangle is tiled with squares and one of its sides is 1. Define the function φ on the set of rectangles with sides parallel to the ones of the initial rectangle by setting $\varphi(\mathcal{R}) = f(a)f(b)$, where a, b are the lengths of the sides of the rectangle $\mathcal{R}$.

It's easy to see that φ is additive, in the sense that if a rectangle $\mathcal{R}$ is tiled with $\mathcal{R}_1, \ldots, \mathcal{R}_n$, then $\varphi(\mathcal{R}) = \varphi(\mathcal{R}_1) + \ldots + \varphi(\mathcal{R}_n)$. φ takes the value $f(a)^2$ on all squares of side a, so if at least one of the squares paving $\mathcal{R}$ had an irrational side, $\varphi(\mathcal{R})$ would be > 0 (because it's the sum of the squares of $f(a)$ for all sides a of the squares paving $\mathcal{R}$). However, $\varphi(\mathcal{R}) = 0$, because if its sides are 1 and t, then $\varphi(\mathcal{R}) = f(1)f(t) = 0$, since $f(1) = 0$.

The above means that all the squares have rational sides, so, in particular, t must also be rational. However, note that a lot more has been proved, namely that all the squares tiling a rectangle with a rational side have rational sides.

For the construction of f, consider a basis $(a_i)_{i\in I}$ of $\mathbb{R}$ as a vector space over $\mathbb{Q}$ which contains 1 (assume $a_{i_0} = 1$), and set $f(\sum \alpha_i a_i) = \sum_{i\neq i_0} \alpha_i a_i$ (for each real $\sum \alpha_i a_i$, only finitely many of the α_i's are non-null).

□ Let $x_1, \ldots, x_n$ and be positive numbers and n be a natural number. Prove that

$$\sqrt{\sum_{k=1}^{n} x_k} + \sqrt{\sum_{k=2}^{n} x_k} + \sqrt{\sum_{k=3}^{n} x_k} + \ldots \geq \sqrt{\sum_{k=1}^{n} k^2 \cdot x_k}$$

Solution

The Minkowski inequality (which actually follows from multiple application of the triangle inequality in $\mathbb{R}^n$) says that if $a_{i,j}$ are real numbers, for all natural i and j with $1 \leq i \leq k$ and $1 \leq j \leq n$, then

$\sqrt{a_{1,1}^2 + a_{1,2}^2 + ... + a_{1,n}^2} + \sqrt{a_{2,1}^2 + a_{2,2}^2 + ... + a_{2,n}^2} + ... + \sqrt{a_{k,1}^2 + a_{k,2}^2 + ... + a_{k,n}^2} \geq \sqrt{(a_{1,1} + a_{2,1} + ... + a_{k,1})}$

Apply this inequality for n = k and the numbers $a_{i,j}$ defined as follows:

$a_{i,j} = \sqrt{x_j}$ for $i \leq j$ and $a_{i,j} = 0$ for i $>$ j.

Then you get

$\sqrt{\left(\sqrt{x_1}\right)^2 + \left(\sqrt{x_2}\right)^2 + ... + \left(\sqrt{x_n}\right)^2} + \sqrt{\left(\sqrt{x_2}\right)^2 + \left(\sqrt{x_3}\right)^2 + ... + \left(\sqrt{x_n}\right)^2} + ... + \sqrt{\left(\sqrt{x_n}\right)^2} \geq \sqrt{\left(1\sqrt{x_1}\right)^2 + (}$

so that

$\sqrt{x_1 + x_2 + ... + x_n} + \sqrt{x_2 + x_3 + ... + x_n} + ... + \sqrt{x_n} \geq \sqrt{1x_1 + 2^2x_2 + ... + n^2x_n}$.

And this is enough.

Another approach:

Define $x_i = X_i^2$, $\forall i \in \{1, \ldots, n\}$, and plug that all in the provided inequalities:

$$\sqrt{\sum_{k=1}^{n} X_k^2} + \sqrt{\sum_{k=2}^{n} X_k^2} + \sqrt{\sum_{k=3}^{n} X_k^2} + \ldots \geq \sqrt{\sum_{k=1}^{n} (k \cdot X_k)^2}$$

This is the triangle inequality which states that the sum of the length of the vectors $(X_1, X_2, X_3, \ldots)$, $(0, X_2,$ is at least the lenght of the sum of these vectors: $(X_1, 2X_2, 3X_3, 4X_4, \ldots)$.

☐ Suppose $x_0, x_1, \ldots, x_n$ and $x_0 > x_1 > \ldots > x_n$. Prove that at least one of the numbers $|F(x_0)|, |F(x_1)|, |F(x_2)|, \ldots, |F(x_n), |$ where

$$F(x) = x^n + \sum_{k=1}^{n} a_k x^{n-k}, \quad a_k \in \mathbb{R},$$

is greater than $\dfrac{n!}{2^n}$.

Solution

It was in Crux proposed by Mohammed Aassila, with more conditions $a_0 = 1$ and $x_0, x_1, ..., x_n$ are integers Here a combination of solutions by M.Bataille, Kee-Wai Lau $P(x) = (x-x_0)(x-x_1)...(x-x_n)$ by considering the decomposition $\frac{F}{P}$ into partial fractions we get $F(x) = \sum_{k=0}^{n} F(x_k) \prod_{j\neq k} \frac{x-x_j}{x_k-x_j}$ The leading of coefficient of F is $a_0 = 1$ gives $\sum b_k = \sum_{k=0}^{n} \frac{F(x_k)}{\prod_{j\neq k}(x_k-x_j)} = 1$ $|\prod_{j\neq k}(x_k - x_j)| \geq \frac{n!}{C_n^k}$ $|b_k| \leq \frac{|F(x_k)|C_n^k}{n!}$ $1 =\leq \sum b_k \leq \sum |b_k| \leq (\max |F(x_k)|)/n!.(\sum C_n^k) = 2^n/n!(max|F(x_k)|)$

☐ Sequences $\{a_n\}$ and $\{b_n\}$ satisfy

$a_0 = 1$

$a_n = \dfrac{2a_{n-1}}{1+2a_{n-1}^2}$

$b_n = \dfrac{1}{1-2a_n^2}$

Prove that b_n is the square of an integer for all $n > 0$.

Furthermore, find a closed expression for a_n and b_n in terms of n.

Solution

$2a_n^2 = \dfrac{8a_{n-1}^2}{4a_{n-1}^4 + 4a_{n-1}^2 + 1}$

$1 - 2a_n^2 = \dfrac{4a_{n-1}^4 - 4a_{n-1}^2 + 1}{4a_{n-1}^4 + 4a_{n-1}^2 + 1} = \dfrac{(2a_{n-1}^2 - 1)^2}{(2a_{n-1}^2 + 1)^2}$

Therefore $b_n = \left[\dfrac{2a_{n-1}^2 + 1}{2a_{n-1}^2 - 1}\right]^2 = \left[1 + \dfrac{2}{2a_{n-1}^2 - 1}\right]^2 = (1 - 2b_{n-1})^2$.

So by induction, all of the b_n are integers.

☐ Evaluate

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{2^n - 1}.$$

Solution

Let $S_d = \dfrac{x^d}{1-x^d} = x^d + x^{2d} + x^{3d} + \ldots$ S_d contains x^n iff $d|n$, and since the coefficient of x^n in $\phi(d)S_d$ is $\phi(d)$, it means that when we sum up all the coefficients of x^n in all S_d, $d|n$ we get $\sum_{d|n} \phi(d)$.

☐ Prove the product of five consecutive numbers cannot be a perfect square

Solution

$(n-2)(n+2) = n^2 - 4$, $(n-1)(n+1) = n^2 - 1$. we have $gcd(n, n^2 - 1) = 1$, $gcd(n, n^2 - 4)|4$, $gcd(n^2 - 1, n^2 - 4)|3$. what do we make of this? first, n has to be either a k^2 or $2k^2$. in the first case, either $n^2 - 1 = l^2$, $n^2 - 4 = m^2$, contradiction, or $n^2 - 1 = 3l^2, n^2 - 4 = 3m^2$, contradiction(just look at the differences of both equations in each case). thus $n = 2k^2$. $n^2 - 1$ is odd, thus either $n^2 - 1 = 3l^2$, $n^2 - 4 = 6m^2$ or $n^2 - 1 = l^2$, $n^2 - 4 = 2m^2$. The second case is impossible, since $(n - l)(n + l) = 1$. thus $n^2 - 1 = 3l^2$, $n^2 - 4 = 6m^2$. the first equality gives $(2k^2 - 1)(2k^2 + 1) = 3l^2$. since $gcd(2k^2 - 1, 2k^2 + 1) = 1$, we have $2k^2 - 1 = 3p^2$, $2k^2 + 1 = q^2$, contradiction(look at the difference mod 8), or $2k^2-1=p^2$, $2k^2+1=3q^2$. the second equality gives $2(k^2-1)(k^2+1)=3m^2$, thus $m=2r$, consequently $(k^2-1)(k^2+1)=6r^2$. $gcd(k^2-1,k^2+1)=2$. k^2+1 is not divisible by 3 and 4, consequently $k^2+1=2a^2$ or $k^2+1=a^2$. in the first case $(2a)^2=2k^2+2=2k^2-1+3=p^2+3$, contradiction. Thus $k^2+1=a^2$, $k^2-1=6b^2$. then $3(2b)^2=2k^2-2=2k^2+1-3=3q^2-3$, contradiction.

☐ Let a,b,c,d be the areas of the triangular faces of a tetrahedron, and let h_a,h_b,h_c,h_d be the corresponding altitudes of the tetrahedron. If V denotes the volume of tetrahedron, prove that

$$(a+b+c+d)(h_a+h_b+h_c+h_d)\geq 48V$$

Solution

Since the volume of a tetrahedron equals $\frac{1}{3}\cdot$ face area $\cdot$ corresponding altitude, we have $V=\frac{1}{3}\cdot a\cdot h_a$, so that $h_a=\frac{3V}{a}$. Similarly, $h_b=\frac{3V}{b}$, $h_c=\frac{3V}{c}$ and $h_d=\frac{3V}{d}$. Thus,

$(a+b+c+d)\left(h_a+h_b+h_c+h_d\right)=(a+b+c+d)\left(\frac{3V}{a}+\frac{3V}{b}+\frac{3V}{c}+\frac{3V}{d}\right)=3V\cdot(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$

But by the Cauchy-Schwarz inequality,

$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq\left(\sqrt{a}\cdot\sqrt{\frac{1}{a}}+\sqrt{b}\cdot\sqrt{\frac{1}{b}}+\sqrt{c}\cdot\sqrt{\frac{1}{c}}+\sqrt{d}\cdot\sqrt{\frac{1}{d}}\right)^2=4^2=16.$

Thus,

$(a+b+c+d)\left(h_a+h_b+h_c+h_d\right)=3V\cdot(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq 3V\cdot 16=48V,$

and we are done.

☐

For positive integers, the sequence $a_1,a_2,a_3,...a_n,...$ is defined by

$a_1=1;\ a_n=\left(\frac{n+1}{n-1}\right)(a_1+a_2+a_3+\cdots+a_{n-1}),\ n>1.$

Determine the value of a_{1997}.

Solution

$a_n=\frac{n+1}{n-1}\cdot(a_1+a_2+...+a_{n-1})$.

There is no way to solve this equation without properly simplifying it. How to simplify it? Well, try to define the auxiliary sequence $s_n=a_1+a_2+...+a_n$. Then, $a_n=(a_1+a_2+...+a_n)-(a_1+a_2+...+a_{n-1})=s_n-s_{n-1}$, so the equation above becomes

$s_n-s_{n-1}=\frac{n+1}{n-1}\cdot s_{n-1}$.

Hence, $s_n=\frac{n+1}{n-1}\cdot s_{n-1}+s_{n-1}=\frac{2n}{n-1}\cdot s_{n-1}=2\cdot n\cdot\frac{s_{n-1}}{n-1}$. Division by n yields $\frac{s_n}{n}=2\cdot\frac{s_{n-1}}{n-1}$. Hence, the sequence $\frac{s_n}{n}$ is a geometrical progression with quotient 2. Its first member is $\frac{s_1}{1}=s_1=a_1=1$, and thus we can find any member of this geometrical progression by the formula $\frac{s_n}{n}=2^{n-1}\cdot\frac{s_1}{1}=2^{n-1}$. Hence, $s_n=n\cdot 2^{n-1}$. Consequently,

$a_n=s_n-s_{n-1}=n\cdot 2^{n-1}-(n-1)\cdot 2^{n-2}=n\cdot 2\cdot 2^{n-2}-(n-1)\cdot 2^{n-2}=(n\cdot 2-(n-1))\cdot 2^{n-2}=(n+1)\cdot 2^{n-2}$.

Thus, for n = 1997, we get $a_{1997}=1998\cdot 2^{1995}$.

☐ If Δ is the area and w_a, w_b, w_c are the angle bisectors of a triangle ABC, then prove the inequality

$(w_a^3+w_b^3+w_c^3)\cdot\left(\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\right)\geq 12\cdot\Delta\cdot\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right).$

Solution

The first thing to do is to tame the monstrous left hand side: By the Chebyshev inequality, applied to the oppositely sorted number arrays $(w_a^3;\ w_b^3;\ w_c^3)$ and $\left(\frac{1}{w_a^2};\ \frac{1}{w_b^2};\ \frac{1}{w_c^2}\right)$, we have

$\frac{w_a^3+w_b^3+w_c^3}{3}\cdot\frac{\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}}{3}\geq\frac{w_a^3\cdot\frac{1}{w_a^2}+w_b^3\cdot\frac{1}{w_b^2}+w_c^3\cdot\frac{1}{w_c^2}}{3}=\frac{w_a+w_b+w_c}{3},$

so that, after multiplication with 9, we have $(w_a^3+w_b^3+w_c^3)\cdot\left(\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\right)\geq 3\,(w_a+w_b+w_c)$. Thus, instead of proving the inequality

$(w_a^3+w_b^3+w_c^3)\cdot\left(\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\right)\geq 12\cdot\Delta\cdot\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$,

it is enough to show

$3\,(w_a+w_b+w_c)\geq 12\cdot\Delta\cdot\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$.

This simplifies to

$w_a+w_b+w_c\geq 4\cdot\Delta\cdot\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$.

But that's not all, of course. Since the shortest segment joining a given point to a point on a given line is the perpendicular from the point to the line, every cevian from the vertex A of triangle ABC, in particular the angle bisector w_a, is greater or equal to the altitude h_a from the vertex A. So we have $w_a\geq h_a$. Since the area of triangle ABC can be found by the formula $\Delta=\frac{1}{2}ah_a$, we have $h_a=\frac{2\Delta}{a}$, so that we get $w_a\geq\frac{2\Delta}{a}$. Similarly, $w_b\geq\frac{2\Delta}{b}$ and $w_c\geq\frac{2\Delta}{c}$. Thus,

$w_a+w_b+w_c\geq\frac{2\Delta}{a}+\frac{2\Delta}{b}+\frac{2\Delta}{c}=2\cdot\Delta\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=2\Delta\cdot\left(\frac{\frac{1}{b}+\frac{1}{c}}{2}+\frac{\frac{1}{c}+\frac{1}{a}}{2}+\frac{\frac{1}{a}+\frac{1}{b}}{2}\right)$.

By the AM-HM inequality, applied to the positive numbers $\frac{1}{b}$ and $\frac{1}{c}$, we have $\frac{\frac{1}{b}+\frac{1}{c}}{2}\geq\frac{2}{b+c}$; similarly, $\frac{\frac{1}{c}+\frac{1}{a}}{2}\geq\frac{2}{c+a}$ and $\frac{\frac{1}{a}+\frac{1}{b}}{2}\geq\frac{2}{a+b}$. Thus,

$w_a+w_b+w_c\geq 2\Delta\cdot\left(\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}\right)=4\cdot\Delta\cdot\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$.

And it's done. Needless to say that equality holds if and only if the triangle ABC is equilateral (what else would you expect from such a dumb inequality).

— For every positive integer n and every integer k, show that

$\binom{2^n-1}{k}\equiv(-1)^{\lceil k/2\rceil}\binom{2^{n-1}-1}{\lfloor k/2\rfloor}\pmod{2^n}$.

□ Given that 1002004008016032 has a prime factor $p>250000$, find p.

Solution

Note that: $x:=1000,\ y:=2\implies N=x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5=\frac{x^6-y^6}{x-y}$

Hence, $N=\frac{1000^6-2^6}{1000-2}=2^5\left(\frac{500^6-1}{500-1}\right)$

$=2^5\left[\frac{(500-1)(500^2+500+1)(500+1)(500^2-500+1)}{(500-1)}\right]$

$=2^5(500^2+500+1)(501)(500^2-500+1)$

The only one of these greater than 500^2 is $500^2+500+1=250501$, and hence $\boxed{p=250501}$.

– **Suppose** $0<\alpha<\beta<\gamma$ **and let** $a_k,b_k,c_k\ ,\ k\in\{1,2,...,n\}$, **positive numbers such that**

$$b_k^{\gamma-\alpha}\leq a_k^{\gamma-\beta}c_k^{\beta-\alpha}\qquad\forall k\in\{1,2,...,n\}.$$

Prove $\boxed{\left(\sum_{k=1}^{n}b_k\right)^{\gamma-\alpha}\leq\left(\sum_{k=1}^{n}a_k\right)^{\gamma-\beta}\left(\sum_{k=1}^{n}c_k\right)^{\beta-\alpha}}$. – For $p>0$ suppose that equation $x^3-px+q=0$ has the roots $-\infty<x_1<x_2<x_3<+\infty$. Find

$$\begin{cases}\alpha &= \dfrac{11}{21}\sqrt{3p}-\dfrac{9q}{14p}\\ \beta &= \dfrac{16}{27}\sqrt{3p}-\dfrac{q}{3p}\end{cases}.$$

Prove that $x_3\in(\alpha,\beta)$ and $x_1,x_2\notin(\alpha,\beta)$. – Consider that x_0 and x_1 are selected in $\mathbb{R}$ such that the sequence $(x_n)_{n=0}^{\infty}$ generated by $x_{n+1}=\frac{x_{n-1}x_n-1}{x_{n-1}+x_n}$, $n\in\{1,2,...\}$ is well defined . Let $(F_n)_{n=0}^{\infty}$ with $x_n=\cot F_n$, $F_n\in(0,\pi)$, $n\in\{0,1,2,...\}$. **Find the recurrence relation(s) satisfied by the terms of** $(F_n)_{n=0}^{\infty}$ **.**

☐ If P(x) is a polynomial of degree 998 such that P(k)=1/k is true for k=1, 2, 3,...999 then find the value of P(1001)

Solution

Try following generalization: suppose that $x_0, x_1, ..., x_n$ are mutual distinct real numbers, i.e. $x_\alpha \neq x_\beta$ for $0 \leq \alpha \neq \beta \leq n$. Let $\{y_0, y_1, ..., y_n\} \subset \mathbb{R}$ and $P(x)$ be (the) polynomial satisfying $P(x_j) = y_j$ for $j \in \{0, 1, \ldots, n\}$. **If** $w \notin \{x_0, x_1, ...x_n\}$, **find** $P(w)$ [/color] **Solution.** $P(x)$ is well-defined [i.e. exists and it's unique] , more precisely P(x) is the so called Lagrange (interpolation) polynomial :

$$(*) \qquad P(x) = \sum_{k=0}^{n} \frac{\omega(x)}{(x - x_k)\omega'(x_k)} y_k \ .$$

Further **select** $x := w$ **in (*)** . I have used notation : $\begin{cases} \omega(x) := \prod_{k=0}^{n} (x - x_k) \\ \omega'(x_k) := \prod_{j=0, j\neq k}^{n} (x_k - x_j) \end{cases}$.

☐ Find a_n if: (1) $a_{n-1} = 2a_n - n - 2$, (2) $a_1 = 3$

Solution

Let us try to solve a more general recurrence, namely

$$(*) \quad x_n = Ax_{n-1} + b_{n-1} \quad , \quad n \in \{2, 3, ...\} \ , \quad A \neq 0 \ , \ x_1 = \alpha,$$

where $(b_n)_{n=1}^{\infty}$ is a given sequence. In your case $x_1 = 3$, $A = \frac{1}{2}$ and $b_n = \frac{n+3}{2}$. From (*) one finds the equalities

$$\frac{x_k}{A^k} - \frac{x_{k-1}}{A^{k-1}} = \frac{b_{k-1}}{A^k} \quad , \quad k \in \{2, 3, ..., n, ...\} \ .$$

By summing , using the fact that [**telescoping-sum**] $\sum_{k=p}^{q} (T_k - T_{k-1}) = T_q - T_{p-1}$ we give $x_n = \alpha A^{n-1} + \sum_{k=2}^{n} b_{k-1} A^{n-k}$, $n \geq 2$. In your case $a_n = n + 1 + \dfrac{1}{2^{n-1}}$, $n \in \{1, 2, ...\}$.

☐ For $n = 2, 3, 4 \ldots$, prove $n! < \left(\frac{n+1}{2}\right)^n$.

Solution

The function $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$, is (strictly) concave on its domain. Therefore

$$(*) \quad f\left(\sum_{k=1}^{n} w_k x_k\right) > \sum_{k=1}^{n} w_k \cdot \ln x_k$$

for any system $\{x_1, x_2, ..., x_n\} \subset (0, \infty)^n$ and any positive weights $\{w_1, w_2, \cdots, w_n\}$ with $\sum_{k=1}^{n} w_k = 1$. For $k \in \{1, 2, ..., n\}$ consider $w_k = \frac{1}{n}$, $x_k = k$. Then from $(*)$

$$(*') \quad \ln\left(\frac{1}{n} \sum_{k=1}^{n} k\right) > \frac{1}{n} \sum_{k=1}^{n} \ln k \ ,$$

or $n \cdot \ln\left(\frac{n+1}{2}\right) > \ln n!$, that is $\left(\frac{n+1}{2}\right)^n > n!$ as desired.

☐ If $x + \frac{1}{x} = -1$ find $x^{999} + \frac{1}{x^{999}}$

Solution

Assume $x^n + \frac{1}{x^n} = \alpha$. Denote by $T_n(x)$ the Chebychev polynomial of degree n. We note that

$$(*) \quad \begin{cases} T_n(y) = \dfrac{\left(y+\sqrt{y^2-1}\right)^n + \left(y-\sqrt{y^2-1}\right)^n}{2} = \\ = \cos\left(n \cdot \arccos y\right) \ , \quad \text{when } \ |y| \leq 1 \ . \end{cases}$$

From $(*)$ we give

$$T_n\left(\frac{A+B}{2\sqrt{AB}}\right)=\frac{A^n+B^n}{2\,(AB)^{\frac{n}{2}}}$$

which imply ($A\rightsquigarrow x$, $B:\rightsquigarrow \frac{1}{x}$)

$$A^n+B^n=x^n+\tfrac{1}{x^n}=2\cdot T_n\left(\tfrac{\alpha}{2}\right) \quad .$$

For instance, when $\alpha=-1$, because $T_n(-z)=(-1)^nT_n(z)$, one finds

$$\boxed{x^n+\tfrac{1}{x^n}=2\cdot T_n\left(-\tfrac{1}{2}\right)=2(-1)^nT_n\left(\tfrac{1}{2}\right)=2(-1)^n\cos\left(\tfrac{n\pi}{3}\right)}\ ,$$

or (e.g. using first equality from $(*)$)

$$\boxed{x^n+\tfrac{1}{x^n}=2(-1)^n\cdot T_n\left(\tfrac{1}{2}\right)==(-1)^n\tfrac{\left(1+i\sqrt{3}\right)^n+\left(1-i\sqrt{3}\right)^n}{2^n}}\ .$$

Finally $(n\rightsquigarrow 999)$

$$\boxed{\begin{gathered}x^{999}+\tfrac{1}{x^{999}}=-2\tfrac{\left(1+i\sqrt{3}\right)^{999}+\left(1-i\sqrt{3}\right)^{999}}{2^{999}}=-2\cdot\cos\left(\tfrac{999\cdot\pi}{3}\right)=\\=-2\cdot\cos\left(333\cdot\pi\right)=1\end{gathered}}\ .$$

□ Let x_1 be the smallest and $x_n$the largest of the n real numbers $x_1,x_2,...,x_n$. Prove that if $x_1+x_2++x_n$=0 then $x_1^2+x_2^2+x_n^2+nx_1x_n$ is not positive.

Solution

For $k\in\{1,2,...,n\}$ we have $(x_k-x_1)\geq 0$ and $(x_k-x_n)\leq 0$.Therefore $(*)\quad (x_k-x_1)(x_k-x_n)\leq 0\ ,\ \forall k\in\{1,2,...,n\}$. By summing inequalities $(*)$ one finds

$$\sum_{k=1}^{n}x_k^2-(x_1+x_n)\underbrace{\sum_{k=1}^{n}x_k}_{=0}+x_1x_n\underbrace{\sum_{k=1}^{n}1}_{=n}\leq 0\ .$$

– For all real a, b, c prove the identity

$$(b-c)^2\,(b+c-2a)^2+(c-a)^2\,(c+a-2b)^2+(a-b)^2\,(a+b-2c)^2=\tfrac{1}{2}\left((b-c)^2+(c-a)^2+(a-b)^2\right)^2.$$

Solution

Let $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$.

□ If a,b,c are distinct positive numbers, prove

$$a^ab^bc^c>\left(a^{pc+qb}b^{pa+qc}c^{pb+qa}\right)^{\frac{1}{p+q}}\ ,\ \forall p,q\in(0,\infty).$$

Solution

Note: All summations are cyclic.

Take the log of both sides; it remains to show that

$p\sum a\log a + q\sum a\log a > p\sum c\log a + q\sum b\log a$.

But (a,b,c) and $(\log a,\log b,\log c)$ are similarly ordered, by rearrangement we have $\sum a\log a > \sum c\log a$ and $\sum a\log a > \sum b\log a$ (strict because they are distinct) so we just add up p times the first and q times the second to get the desired inequality.

□ tổ hợp

– Q.Find the number of possible real solutions to the following equation:

$(9+\sin x)^{f(x)} + (10+\sin x)^{f(x)} = (11+\sin x)^{f(x)}$ where $f(x) = \frac{x}{1-x}$

□

Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ satisfying : $f(x+f(y)) = f(x) - y$

Solution

Taking $y = a + f(b)$,

$$f(x+[f(a+f(b))]) = f(x) - [a+f(b)] \quad (1)$$

But we know

$$f(a+f(b)) = f(a) - b$$

so

$$f(x+[f(a+f(b))]) = f(x+f(a)-b) = f([x-b]+f(a)) = f(x-b) - a$$

Equating with (1),

$$f(x-b) = f(x) - f(b)$$

which is just Cauchy's functional equation. Over the integers, this has the unique solution $f(x) = xf(1)$; plugging this in and simplifying gives

$$[f(1)]^2 y = -y$$

for all y which is obviously false, so there is $\boxed{\textit{no solution}}$.

□ Evaluate $\dfrac{1}{2}\cot^{-1}\dfrac{2\sqrt[3]{4}+1}{\sqrt{3}} + \dfrac{1}{3}\tan^{-1}\dfrac{\sqrt[3]{4}+1}{\sqrt{3}}$.

Solution

Let $P = \frac{1}{2}\cot^{-1}\frac{2\sqrt[3]{4}+1}{\sqrt{3}} + \frac{1}{3}\tan^{-1}\frac{\sqrt[3]{4}+1}{\sqrt{3}}$ $6P = 3\tan^{-1}\frac{\sqrt{3}}{2\sqrt[3]{4}+1} + 2\tan^{-1}\frac{\sqrt[3]{4}+1}{\sqrt{3}} = \tan^{-1}\frac{3a-a^3}{1-3a^2} + \tan^{-1}\frac{2b}{1-b^2}$ $(a = \frac{\sqrt{3}}{2\sqrt[3]{4}+1}, b = \frac{\sqrt[3]{4}+1}{\sqrt{3}}) = \tan^{-1}\frac{\sqrt{3}(1+\sqrt[3]{4})}{\sqrt[3]{2}+\sqrt[3]{4}-1} + \tan^{-1}\frac{\sqrt{3}(1+\sqrt[3]{4})}{1-\sqrt[3]{2}-\sqrt[3]{4}} = \pi,\ P = \frac{\pi}{6}$.

□ a, b are positive and $ab = 8$ Find the range of $\sqrt{a^2+64} + \sqrt{b^2+1}$, **without calculus**.

Solution

Clearly if one of the variable is large enough the expression tends to infinity. So we looking for the minimum. By AM-GM: $\sqrt{a^2+64} = \sqrt{a^2+16+16+16+16} \geq \sqrt{5}\sqrt[10]{16^4a^2} = \sqrt{5}\sqrt[5]{256a}$

$\sqrt{b^2+1} = \sqrt{\frac{b^2}{4}+\frac{b^2}{4}+\frac{b^2}{4}+\frac{b^2}{4}+1} \geq \sqrt{5}\sqrt[5]{\frac{b^4}{16}}$

So,

$$\sqrt{a^2+64} + \sqrt{b^2+1} \geq \sqrt{5}\left(4\sqrt[5]{\frac{a}{4}} + \sqrt[5]{\frac{b^4}{16}}\right)$$

$$\overset{AM-GM}{\geq} 5\sqrt{5}\sqrt[25]{\frac{a^4b^4}{2^{12}}} = 5\sqrt{5}$$

Minimum occurs when $a = 4, b = 2$ Range: $[5\sqrt{5},\infty)$

□ $\sum_{n\geq j\geq i\geq 0}\frac{i}{i+j}$

Solution

$\sum_{n\geq j\geq i\geq 0}\frac{i}{i+j} = \sum_{n\geq j>i\geq 0}\frac{i}{i+j} + \sum_{n\geq i>j\geq 0}\frac{i}{i+j} + \sum_{n\geq i=j\geq 0}\frac{i}{i+j} = \sum_{n\geq j>i\geq 0}\frac{i}{i+j} + \sum_{n\geq i>j\geq 0}\frac{i}{i+j} + \frac{n}{2}$

Now for each pair (i,j) $\frac{i}{i+j} + \frac{j}{j+i} = 1$. $\therefore \sum_{n \geq j > i \geq 0} \frac{i}{i+j} + \sum_{n \geq i > j \geq 0} \frac{i}{i+j}$ is equal to the number of pairs (i,j) where $i > j$. Now there are $(n+1,2)$ pairs of (i,j) where $0 \leq i,j \leq n$ with i,j distinct. So there are $(n+1,2)/2$ pairs of (i,j) where $i > j$.

So $\sum_{n \geq j \geq i \geq 0} \frac{i}{i+j} = \frac{(n+1,2)}{2} + \frac{n}{2} = \frac{n(n+1)}{4} + \frac{n}{2} = \frac{n(n+3)}{4}$

$\square$ Prove that, for any prime p, it is possible to find integers x and y such that $x^2 + y^2 + 1$ is divisible by p.

Solution

Firstly note for $p = 2$ setting $x = 1, y = 0$ suffices. Else p is an odd prime.

Define for the set A_p as the set of squares (including 0) modulo p: i.e. the integers $n \in [0, p-1]$ such that there exists an integer a with $a^2 = n \mod p$.

It is well-know that $|A_p| = \frac{p+1}{2}$ since modulo p: $a^2 = b^2 \Leftrightarrow a \in \{-b, b\}$ Define the set B_p as follows: $B_p = \{-1 - s | s \in A_p\}$ so $|B_p| = |A_p| = \frac{p+1}{2}$. We also know that $|A_p \cup B_p| \leq p$

Now $|A_p \cup B_p| = |A_p| + |B_p| - |A_p \cap B_p| = p + 1 - |A_p \cap B_p| \leq p$ so $A_p \cap B_p$ is not empty.

$\therefore \exists x, y$ such that $x^2 + y^2 = -1 \mod p \Rightarrow p | x^2 + y^2 + 1$ as required.

$\square$ Let a,b,c be any numbers. Show that if $(a+b+c)^3 = a^3+b^3+c^3$ then $(a+b+c)^{17} = a^{17}+b^{17}+c^{17}$.

Solution

Define: $s_1 = a + b + c, s_2 = ab + bc + ca, s_3 = abc, T_k = a^k + b^k + c^k$

Then note $T_{k+3} = T_{k+2}s_1 - T_{k+1}s_2 + T_k s_3$ (just multipy out) (*)

Now $T_0 = 3, T_1 = s_1, T_2 = s_1^2 - 2s_2$ so we have:

$T_3 = s_1^3 - 3s_1s_2 + 3s_3$ but we know $T_3 = s_1^3$ so:

$s_3 = s_1 s_2$ subbing this into (*) gives:

$T_{k+3} = T_{k+2}s_1 - T_{k+1}s_2 + T_k s_1 s_2$

Now it is fairly easy to prove by induction that $T_{2k+1} = s_1^{2k+1}, T_{2k} = s_1^{2k} + (-1)^k 2 s_2^k$

So in particular $a^{17} + b^{17} + c^{17} = T_{17} = s_1^{17} = (a + b + c)^{17}$

$\square$ Let $0 < b < a \leqslant 2$ and $2ab \leq 2b + a$. Prove that: $a^2 + b^2 \leq 5$

Solution

As $a > b$ we note that for the intequality to be false we need $a^2 > \frac{5}{2} \Rightarrow a > 1.5$

Now $2ab \leq 2b + a \Rightarrow 2b(a-1) \leq a \Rightarrow b \leq \frac{a}{2(a-1)}$ as $a > 1.5$ $a - 1 > 0$

$\therefore a^2 + b^2 \leq a^2 + \frac{a^2}{4(a-1)^2} = \frac{a^2(4a^2-8a+5)}{4(a-1)^2}$

We need this to be less than or equal to 5 i.e. we need $a^2(4a^2 - 8a + 5) \leq 20(a-1)^2 \Rightarrow 4a^4 - 8a^3 - 15a^2 + 40a - 20 \leq 0$

$\Rightarrow (a-2)(4a^3 - 15a + 10) \leq 0$ but as $a \leq 2$ $a - 2 \leq 0$ so we need $4a^3 - 15a + 10 \geq 0$

Let $f(a) = 4a^3 - 15a + 10$ then $f'(a) = 12a^2 - 15 > 0$ as $a > 1.5$. So $f(a)$ is increasing in the range $(1.5, 2]$ and $f(1.5) = 1$ so $f(a) \geq 0$ as required.

$\square$ p is prime and n, m are natural number such that $p^n + 576 = m^2$ Find the maximum value of $m + n + p$

Solution

Obviously $p^n = (m-24)(m+24)$ so $\gcd(m-24, m+24) = p^k$. So $p^k | 48$. So $p = 2$ or $p = 3$. If $2^n + 576 = m^2$, then say $m - 24 = 2^l$ and $m + 24 = 2^{n-l}$. Then $2^{n-l} - 2^l = 48$ so $l = 4$ and $n - l = 6$. Hence $n = 10$ and $m = 40$. If instead $3^n + 576 = m^2$, let $m - 24 = 3^l$ and $m + 24 = 3^{n-l}$. Then $3^{n-l} - 3^l = 48$ so $l = 1$ but $3^x - 1 = 16$ has no solution, so this case is impossible.

So $m = 40$, $n = 10$ and $p = 2$ so $m + n + p = 52$.

□ Solve the following system $\begin{cases} \sin x+2\sin(x+y+z)=0 \\ \sin y+3\sin(x+y+z)=0 \\ \sin z+4\sin(x+y+z)=0 \end{cases}$

Solution

$\begin{cases} \sin x+2\sin(x+y+z)=0 \\ \sin y+3\sin(x+y+z)=0 \\ \sin z+4\sin(x+y+z)=0 \end{cases}$

$2\cdot I-III: 2\sin x-\sin z=0$

$2\cdot II-3\cdot III: 4\sin y-3\sin z=0$

let $\sin z=4a, a\in\left[-\frac{1}{4},\frac{1}{4}\right]$

$\sin x=2a\Rightarrow x=(-1)^m\arcsin 2a+m\pi, m\in Z$

$\sin y=3a\Rightarrow y=(-1)^n\arcsin 3a+n\pi, n\in Z$

So solutions are $\boxed{((-1)^m\arcsin 2a+m\pi,(-1)^n\arcsin 3a+n\pi,(-1)^p\arcsin 4a+p\pi)}$

$m,n,p\in Z,\ a\in\left[-\frac{1}{4},\frac{1}{4}\right]$

□ f(x)=$x^{13}+2x^{12}+3x^{11}+...+13x+14$ and w is 15th root of unity. Find $f(w).f(w^2)...f(w^{14})$

Solution

$f(x)=x^{13}+2x^{12}+3x^{11}+...+13x+14$ is a arithemtico geometric progression There exist a standard method for simplifying this Multiply $f(x)$ by $\frac{1}{x}$ and subtracting(Diagonnaly) will give you the geometric progression And you will get $f(x)\left(1-\frac{1}{x}\right)=\frac{x-x^{14}}{1-x}-\frac{14}{x}$ Since We find $f(x)$ only for 15 th roots of unity Hence $x^{15}=1$ And this substitution will yield $x^{14}=\frac{1}{x}$ And applying it here and a small simplification yeilds $f(x)\left(1-\frac{1}{x}\right)=-\frac{x+1}{x}-\frac{14}{x}$ And Hence $f(x)=-\frac{x+15}{x-1}$ Now we have reduced the given thing into another form where the calculation easy. Now the calculation depends on $x^{15}-1=(x-1)(x-\omega)\cdots(x-\omega^{14})$ Substitute $x=-15$ Similarly substitute $x=1$ (please do note that you should take the limit here i.e $x\to 1$) Dividing the two will yeild our desired result and final result is $\boxed{\dfrac{15^{15}+1}{15\cdot 16}}$ – Suppose that a,b,c are positive integers such that

$$a+b+c=32$$

$$\frac{b+c-a}{bc}+\frac{c+a-b}{ca}+\frac{a+b-c}{ab}=\frac{1}{4}$$

Is there exist a triangle with sidelengths $\sqrt{a},\sqrt{b},\sqrt{c}$? If there is, find its largest angle. –

□ Let $x,\ y,\ z$ be real numbers such that $x+y+z=0, xyz=-1$. Find the minimum value of $|x|+|y|+|z|$.

Solution

One may proceed as follows :

Let's denote by (i) the condition $x+y+z=0$ and by (ii) the second one $xyz=-1$.

Due to (ii) , $x\neq 0$, $y\neq 0$ and $z\neq 0$ which is equivalent to $|x|>0$, $|y|>0$ and $|z|>0$.

(ii) yields also that : $|x|+|y|+|z|=|x|+|y|+\dfrac{1}{|x||y|}$

AM-GM applied to $(|x|,|y|)$ gives : $\dfrac{1}{|xy|}\geq\dfrac{2}{|x|+|y|}$

Thus , combining previous facts , we get :

$|x|+|y|+|z|\geqslant|x|+|y|+\dfrac{2}{|x|+|y|}$

A quick study of the function $t\longmapsto t+\dfrac{2}{t}$ on $I=]0,+\infty[$ shows that it has a minimum on I which is $2\sqrt{2}$ reaching it at $t=\sqrt{2}$.

So :

$|x|+|y|+|z| \geqslant 2\sqrt{2}$

It remains to verify that this minimum value is reached for some value of the triplet (x,y,z).

According to the previous function study , this minimum is reached when $|x|+|y|=\sqrt{2}$

Let's find the exact value of (a) $z<0$ in the case.

$z<0$ together with (ii) imply that $xy>0$ which, in turn implies that $|xy|=xy \quad (iii)$

Besides $(i) \Longrightarrow x^2+y^2=z^2-2xy$

and $|x|+|y|=\sqrt{2} \Longrightarrow x^2+y^2=2-2|xy|=2-2xy$ (due to (iii))

identifying , we get : $z^2=2$ implying $(z<0)$ that $z=-\sqrt{2}$.

back to (i) and (ii) and solving the system :

$$\begin{cases} x+y=\sqrt{2} \\ xy=\dfrac{1}{\sqrt{2}} \end{cases}$$

we get :

$$x,y=\frac{-\sqrt{2}\pm\sqrt{2+\dfrac{4}{\sqrt{2}}}}{2}$$

which completes the proof .

□ solve the system : $\begin{cases} \frac{1}{x}+\frac{1}{y+z}=\frac{1}{2} \\ \frac{1}{y}+\frac{1}{z+x}=\frac{1}{3} \\ \frac{1}{z}+\frac{1}{x+y}=\frac{1}{4} \end{cases}$

Solution

First, clear the fractions, and let $S=x+y+z$ to get: $2S=xy+xz$ $3S=xy+yz$ $4S=xz+yz$

We may now solve this as a linear system to get: $xy=\frac{S}{2}$ $xz=\frac{3S}{2}$ $yz=\frac{5S}{2}$

Multiplying these equations gives:

$xyz=\sqrt{\frac{15S^3}{8}}$

...and after dividing by the above equations in turn we have:

$x=\sqrt{\frac{3S}{10}}$ $y=\sqrt{\frac{5S}{6}}$ $z=\sqrt{\frac{15S}{2}}$

Now as $x+y+z=S$, we have that either $S=0$, an impossibility as then we would get $x=y=z=0$, or else $\sqrt{S}=\sqrt{\frac{3}{10}}+\sqrt{\frac{5}{6}}+\sqrt{\frac{15}{2}}=\frac{23}{\sqrt{30}}$

Plugging this back into our expressions gives the exceptionally nice answer of: $x=\frac{23}{10}$, $y=\frac{23}{6}$, $z=\frac{23}{2}$, which works in the original equation.

□ If a and b re positive integers and $a^2+b^2=c$, prove trhat c does not end the digits 11.

Solution

First of all, we know that $x^2\equiv 0,1,4,5,6,$ or $9 \pmod{10}$ for all $x\in\mathbb{N}$. According to the problem, we see $a^2+b^2\equiv 1 \pmod{10}$. Checking the cases, we see that $\{a^2,b^2\}\equiv\{5,6\} \pmod{10}$. Suppose, WLOG, that $a^2\equiv 5 \pmod{10}$ and $b^2\equiv 6 \pmod{10}$. It is clear that we have $a\equiv 5 \pmod{10}$ and $b\equiv 6 \pmod{10}$ (why?). So, there are non-negative integers $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_m$ such that $a=\overline{a_na_{n-1}\ldots a_2a_15}$ and $b=\overline{b_mb_{m-1}\ldots b_2b_16}$. Try to compute a^2 and b^2 modulo 100:

$$\begin{aligned} a^2 &\equiv (5+10a_1+100a_2+\cdots+10^na_n)^2 \\ &\equiv (25+100a_1+\text{Some stuff which is divisible by } 100) \\ &\equiv 25 \pmod{100} \end{aligned}$$

$$b^2 \equiv (6 + 10b_1 + 100b_2 + \cdots + 10^m b_m)^2$$

$$\equiv (36 + 120a_1 + \text{Some stuff which is divisible by } 100)$$

$$\equiv 36 + 20b_1 \pmod{100}$$

So, $a^2 + b^2 \equiv 61 + 20b_1 \pmod{100}$, and the problem is equal to solving the equation $61 + 20b_1 \equiv 11 \pmod{100}$, and the conclusion follows.

□ solve in $\mathbb{Z}$ $\frac{y!+z!}{x!} = 3^x$

Solution

Obviously $y, z \geq x$. Now assume $z \geq y$ If $z \geq y + 3$ then $z!$ would obviously have an additional factor in 3 than $y!$ and hence it's impossible to have $\frac{y!+z!}{x!}$ only containing powers of 3 Therefore z can take values $y, y+1, y+2$ We have reduced this to simple cases and now try out each. Take the last We have $y!\frac{1+(y+1)(y+2)}{x!} = 3^x$ And This is only possible if $y = x$ or $y = x + 1 = 3^n$ And also if $y = 0$ and $x = 1$ As $0! = 1! = 1$ Take the first We have $y^2 + 3y + 3 = 3^x$ And possible only if $x = 1$ and $y = 0$ $z = 2$ Next. $y = x + 1 = 3^n$. Substitute and check it is not possible. Take second case. $z = y + 1$ Proceed similarly To get $y = x = 1$ $z = 2$ Third case also impossible. For the case $z \leq y$ Just flip the solutions of y and z we have got

So solutions. $(x = 1, y = 0, z = 2), (x = 1, y = 1, z = 2)$
$, (x = 1, y = 2, z = 1), (x = 1, z = 2, y = 0)$.

□ Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\binom{n}{1}$ and $a_{n-2} = \binom{n}{2}$, and that all the roots of P are real, find the remaining coefficients. Note that $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

Solution

Let the roots be $r_1, r_2, \ldots, r_n$. By Vieta's formula,

$$r_1 + r_2 + \ldots r_n = \binom{n}{1} = n$$

$$r_1r_2 + r_1r_3 + \ldots r_1r_n + r_2r_3 + r_2r_4 + \ldots r_2r_n + \ldots r_{n-1}r_n = \binom{n}{2}$$

Squaring the first and subtracting twice the second,

$$r_1^2 + r_2^2 + \ldots r_n^2 = n^2 - 2\binom{n}{2} = n$$

But by Cauchy-Schwarz,

$$n = r_1^2 + r_2^2 + \ldots r_n^2 \geq \frac{1}{n}(r_1 + r_2 + \ldots r_n)^2 = n$$

so equality holds, meaning $r_1 = r_2 = r_3 = \cdots = r_n = 1$ (since their sum is n), and

$$P(x) = (x-1)^n$$

It is easy to see that

$$[x^r]P(x) = (-1)^{n-r}\binom{n}{r}$$

□ let $f(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$ here all the coefficients are non zero integers. $f(n)$ is divisible by 11 whenever n is an integer. how many minimum values among a,b,c,d,e,f,g have to divisible by 11?

Solution

$f(0) = g \implies g$ must be a multiple of 11. both $f(1)$ and $f(-1)$ are multiples of 11 $\implies f(1)+f(-1)$ is also a multiple of 11 $\implies (a+b+c+d+e+f+g)+(a-b+c-d+e-f+g) = 2(a+c+e+g)$must be a multiple of 11 $\implies a+c+e$ must be a multiple of 11 (since g is already a multiple of 11) ....(1)

Similarly , $f(2)+f(-2)$ is also a multiple of 11 $\implies (64a+32b+16c+8d+4e+2f+g)+(64a-32b+16c-8d+4e-2f+g) = 8(16a+4c+e)+2g$ must also be a multiple of 11 $\implies 16a+4c+e$ is a multiple of 11 .... (2)

Similarly, $f(3)+f(-3)$ is a multiple of 11 $\implies (729a+243b+\ldots+g)+(729a-243b+\ldots+g) = 18(81a+9c+e)+2g$ must be a multiple of 11 $\implies 81a+9c+e$ is a multiple of 11....................(3)

Now, subtracting above equations: Equation (2) $-$ (1): $15a+3c$ should be a multiple of 11 $\implies 5a+c$ should be a multiple of 11.................. (4)

Equation (3) $-$ (1): $80a+8c$ must be a multiple of 11 $\implies 10a+c$ must be a multiple of 11..............(5)

Equation (5) $-$ (4) : $5a$ must be a multiple of 11 $\implies a$ must be a multiple of 11 Substituting back in equation (5) $\implies c$ must be a multiple of 11 And thus using equation (1) $\implies e$ must also be a multiple of 11

Similarly, if we observe $f(1)-f(-1)$ and $f(2)-f(-2)$, we obtain that b, d and f must also be multiples of 11 Thus, all the coefficients have to be multiples of 11, for the above polynomial

□ r, s, t are prime numbers, p and q are two numbers whose LCM is $r^2s^4t^2$ then find the number of possible pairs of (p, q)

Solution

Let the prime factorization of p, q be: $p = r^{a_1}s^{b_1}t^{c_1}$, where $0 \leq a_1 \leq 2, 0 \leq b_1 \leq 4, 0 \leq c_1 \leq 2$ $q = r^{a_2}s^{b_2}t^{c_2}$, where $0 \leq a_2 \leq 2, 0 \leq b_2 \leq 4, 0 \leq c_2 \leq 2$

Now $lcm[p, q] = r^{\max(a_1,a_2)}s^{\max(b_1,b_2)}t^{\max(c_1,c_2)}$.

Thus (a_1, a_2) can have a total of 5 combinations: $(0, 2), (1, 2), (2, 2), (2, 1), (2, 0)$. Similarly, (b_1, b_2) can have a total of 9 combinations, and (c_1, c_2) can have a total of 5 combinations Thus we have $5 \cdot 9 \cdot 5 = 225$ different combinations, and hence 225 possible pairs of (p, q), because each combination represents a different prime factorization of (p, q). – find all $(m, n) \in N^2$ which $\frac{m^2}{2mn^2-n^3+1} \in N$ – $a, b, c \in N$, $c^2+1|a+b$, $ab|c(c^2-c+1)$ prove that : $\{a, b\} = \{c, c^2-c+1\}$

□ Find all function $f : \mathbb{R} \to \mathbb{R}$ that are continuous in zero and satisfy

$$f(x+y) - f(x) - f(y) = xy(x+y)$$

Solution

Letting $x = y = 0$, we see that $-f(0) = 0$, or $f(0) = 0$. Choosing any $x \in \mathbb{R}$ and $y = -x$, we have $f(0) - f(x) - f(-x) = 0$, or $f(-x) = -f(x)$–that is, f must be an [i]odd[/i] function.

The right-hand side of the given relation suggests a cubic polynomial that vanishes at the origin. In fact, with a little experimentation, we find that $f(x) = x^3/3$ satisfies the equation.

Now suppose that f is any solution of the functional equation and consider $F(x) = f(x) - x^3/3$. Then $F(x+y) - F(x) - F(y) = f(x+y) - (x+y)^3/3 - (f(x) - x^3/3) - (f(y) - y^3/3) = [f(x+y) - f(x) - f(y)] - (x+y)^3/3 + x^3/3 + y^3/3 = xy(x+y) - xy(x+y) = 0$ for all $x, y \in \mathbb{R}$, or $F(x+y) = F(x) + F(y)$ for all $x, y \in \mathbb{R}$.

Thus F satisfies **Cauchy's equation**. The additivity of F plus the continuity at 0 implies that F is continuous at [i]every[/i] $x \in \mathbb{R}$. Under these conditions, the solution of Cauchy's equation is known to be $F(x) = cx$ for an arbitrary constant c.

Therefore, finally, we have $f(x) = F(x) + \frac{x^3}{3} = cx + \frac{x^3}{3}$ for any real number c.

$\square$

$\square$ Let $x, y, z, k, l, h \in \mathbb{R}^+$ such that $xy + yz + zx = 1$.Find the minimize value of the expression:

$$\mathrm{P} = kx^2 + ly^2 + hz^2$$

Solution

Suppose that there are positive real numbers a, b, c, m, n, p such that

$$P = kx^2 + ly^2 + hz^2 = (a+m)x^2 + (b+n)y^2 + (c+p)z^2$$

By AM-GM, we have $ax^2+by^2 \geq 2\sqrt{ab}xy, mx^2+cz^2 \geq 2\sqrt{mc}zx, ny^2+pz^2 \geq 2\sqrt{np}yz$. Equality holds when $ax = by$ for the first inequality, $cz = mx$ for the second, $ny = pz$ for the third. Multiplying, we get $acn = bmp$.

For the condition $xy + yz + zx = 1$ to be used, we impose that $ab = mc = np = t$, so that $acn = bmp = \sqrt{t^3}$. Eventually, the minimum value is $2\sqrt{ab} = 2\sqrt{t}$. Now, we are going to find the exact value of t.

Multiplying these equations $k = a + m, l = b + n, h = c + p$, we get

$$\begin{aligned} klh &= (a+m)(b+n)(c+p) = (ab + an + mb + mn)(c+p) \\ &= abc + mbc + mnc + abp + anp + mnp + mbp + anc = t(c+b+n+p+a+m) + 2\sqrt{t^3} \\ &= 2\sqrt{t^3} + t(k+l+h) \end{aligned}$$

Letting $q = \sqrt{t}$, it turns out that q is the positive root of the cubic equation $2q^3 + q^2(k+l+h) - klh = 0$. Therefore, the minimum value of P is $2\sqrt{t} = 2q$.

$\square$ Solve that $\begin{cases} (x-1)(2y-1) = x^3 + 20y - 28 \\ 2(\sqrt{x+2y} + y) = x^2 + x \end{cases}$

Solution

Note that

$$\begin{aligned} 2(\sqrt{x+2y} + y) = x^2 + x &\iff x + 2y + 2\sqrt{x+2y} = x^2 + 2x \\ &\iff x^2 - (x+2y) + 2(x - \sqrt{x+2y}) = 0 \\ &\iff (x - \sqrt{x+2y})(x + \sqrt{x+2y} + 2) = 0 \end{aligned}$$

Thus, $x = \sqrt{x+2y}$ $(x \geq 0)$ and $\sqrt{x+2y} = -x - 2$ $(x \geq -2)$. When $x = \sqrt{x+2y}$, we have $2y = x^2 - x$, so the first equation becomes $(x-1)(x^2 - x - 1) = x^3 + 10(x^2 - x) - 28$, which gives $12x^2 - 10x - 29 = 0$. Since $x \geq 0$, we get $x = \frac{5+\sqrt{373}}{12}$, which we get $y = \frac{169-\sqrt{373}}{144}$.

When $-x - 2 = \sqrt{x+2y}$, we have $2y = x^2 + 3x + 4$, so the first equation becomes $(x-1)(x^2 + 3x + 3) = x^3 + 10(x^2 + 3x + 4) - 28$, which simplifies to $8x^2 + 30x + 15 = 0$. Since $x \geq -2$, we get $x = -\frac{15+\sqrt{105}}{8}$, which we get $y = \frac{113+3\sqrt{105}}{64}$.

These solutions are the only real solutions.

$\square$ Determine the value of the sum $\sum_{n=1}^{\infty} \arctan \frac{1}{2n^2}$.

Solution

In the familiar trigonometric identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B},$$

we can replace A by $\arctan \alpha$ and B by $\arctan \beta$ and take the inverse tangent of both sides to get

$$\arctan \alpha - \arctan \beta = \arctan\left(\frac{\alpha-\beta}{1+\alpha\beta}\right) \quad \text{if } \alpha\beta < 1.$$

Letting $\alpha = \frac{1}{2n-1}$ and $\beta = \frac{1}{2n+1}$, we can write (after some simple algebra)

$$\arctan\left(\frac{1}{2n-1}\right) - \arctan\left(\frac{1}{2n+1}\right) = \arctan\left(\frac{\frac{1}{2n-1}-\frac{1}{2n+1}}{1+\frac{1}{2n-1}\frac{1}{2n+1}}\right) = \arctan\frac{1}{2n^2}.$$

But

$$\sum_{n=1}^{N}\left\{\arctan\left(\frac{1}{2n-1}\right) - \arctan\left(\frac{1}{2n+1}\right)\right\} = \arctan 1 - \arctan\left(\frac{1}{2N+1}\right) \to \arctan 1 \text{ as } N \to \infty.$$

Therefore,

$$\sum_{n=1}^{\infty} \arctan\frac{1}{2n^2} = \arctan 1 = \frac{\pi}{4}.$$

GENERALIZATION: In general, using telescoping cancellation in the same way, we can show that

$$\sum_{n=1}^{\infty} \arctan\frac{f(n)-f(n+1)}{1+f(n)f(n+1)} = \arctan f(1).$$

For example, letting $f(n) = 1/n$, we find that $\sum_{n=1}^{\infty} \arctan\frac{1}{n^2+n+1} = \frac{\pi}{4}$.

☐ Find all function $f : \mathbb{R} \to \mathbb{R}$ which satisfy equation $f(xf(y)+x) = xy + f(x)$ for each $x, y \in \mathbb{R}$

Solution

Set $x = 1$ and $y = -1 - f(1)$. The functional equation becomes

$$f\left(f(-1-f(1))+1\right) = -1 - f(1) + f(1) = -1.$$

Letting $f(-1-f(1)) + 1 = z$, we can write the last equation as

$$f(z) = -1. \quad (*)$$

If we let $y = z$ and $w = f(0)$ in the original functional equation and use $(*)$, we get

$$w = f(\overbrace{xf(z)+x}^{x(-1)+x}) = zx + f(x), \text{ or } f(x) = w - zx.$$

Substituting this last relation in the original functional equation, we get

$$z^2xy - zwx - zx + w = xy - zx + w.$$

Equating coefficients, we get $z = \pm 1$ and $w = 0$, so $\boxed{f(x) = x}$ or $\boxed{f(x) = -x}$. We can see that both functions are in fact solutions of our functional equation.

☐ Consider

$$S = \sqrt{2x + \sqrt{2x + \sqrt{2x + \sqrt{2x + \cdots}}}}$$

Given that x is nonnegative number. Describe the behaviour of S when x approaches 0. Determine if S represents a convergent or divergent series.

Solution

First we have to make sense of the expression $S = \sqrt{2x + \sqrt{2x + \sqrt{2x + \sqrt{2x + \cdots}}}}$.

Take $S_1 = \sqrt{2x}, S_2 = \sqrt{2x + \sqrt{2x}}$, and $S_{n+1} = \sqrt{2x + S_n}$ for $n \geq 2$. Therefore $S = \lim_{n\to\infty} S_n$, if this limit exists.

Since $x > 0$, it is easy to show by induction that $S_n < S_{n+1}$, so **the sequence $\{S_n\}$ is monotonically increasing**. Furthermore,we show that $S_n < \max\{2x, 2\}$ (so $\{S_n\}$ **is bounded**) , also by induction: $S_1 = \sqrt{2x} \leq \max\{2x, 1\} \leq \max\{2x, 2\}$. Now if $S_n \leq 2$, then $S_{n+1} \leq \sqrt{2x+2} \leq \sqrt{4} = 2$. Similarly, if $2 < S_n \leq 2x$, then $S_{n+1} = \sqrt{2x + S_n} \leq \sqrt{4x} \leq 2x$.

Since $\{S_n\}$ is a bounded monotonic sequence, the sequence converges. Thus we can write (since the square root function is continuous)

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} \sqrt{2x + S_n} = \sqrt{2x + \lim_{n\to\infty} S_n} = \sqrt{2x + S}.$$

Squaring the relation $S = \sqrt{2x + S}$, we get $S^2 - S - 2x = 0$. Solving for S, we find that

$$S = \frac{1 + \sqrt{1 + 8x}}{2},$$

where we take the positive solution of the quadratic equation. We can see that $S \to 1$ **as** $x \to 0$.

□ Find a closed form expression for :

$$\sum_{k=0}^{n} \binom{n}{k} \cos(ak + b).$$

where a, b are real numbers.

Solution

Let $z = e^{ia}, w = e^{ib}$ $\sum_{k=0}^{n} C_k^n z^k w = w(1+z)^n = w(1 + \cos a + i \sin a)^n = w(2\cos^2 \frac{a}{2} + i2 \sin \frac{a}{2} \cos \frac{a}{2})^n$
$= w(2\cos\frac{a}{2})^n (\cos\frac{a}{2} + i\sin\frac{a}{2})^n = w(2\cos\frac{a}{2})^n (\cos\frac{na}{2} + i\sin\frac{na}{2}) = (2\cos\frac{a}{2})^n (\cos(\frac{na}{2} + b) + i\sin(\frac{na}{2} + b))$
Required sum $= (2\cos\frac{a}{2})^n \cos(\frac{na}{2} + b)$

□ Find all solutions to $2^b = c^2 - b^2$, where $a, b, c \in N$

Solution

Maybe you mean $2^a = c^2 - b^2$? If not, I am going to solve this and then consider yours as a special case.

$(c-b)(c+b) = 2^a \implies c - b = 2^k, c + b = 2^m, m + k = a.$

We now have that $2b = 2^m - 2^k \implies b = 2^{m-1} - 2^{k-1}$ and $c = 2^{m-1} + 2^{k-1}$.

Now, we have your equation, yielding $b = 2^{m-1} - 2^{k-1} = m + k$.

In fact, $2^{k-1}(2^{m-k} - 1) = m + k$. Suppose $k > 2$. We see that $2^{k-1} > k$, and if $m \geq 6$ then $2^{k-1}(2^{m-k} - 1) > m + k$.

Thus, we try $k = 1, 2, 3, 4$ (the limit because $m \geq 6 \implies k < 6$).

Exhausting all cases, we find that $k = 2 \implies 2(2^{m-2} - 1) = m + 2 \implies m = 4 \implies b = 6, c = 10$.

So, the only solution is $\boxed{(b, c) = (6, 10)}$

Hết 2010-2013

$$\frac{1}{xy} = \frac{x}{z} + 1$$

□ $$\frac{1}{yz} = \frac{y}{x} + 1$$

$$\frac{1}{zx} = \frac{z}{y} + 1$$

Solution

We have,

$z = xy(x+z)$; $x = yz(y+x)$; $y = zx(z+y)$

$\Rightarrow z^2 = xyz(x+z)$-(i) ; $\Rightarrow x^2 = xyz(y+x)$-(ii) ; $\Rightarrow y^2 = xyz(z+y)$-(iii)

Subtracting 2 equations at a time we get:- $(z-x)(z+x) = xyz(z-y)$; $(x+y)(x-y) = xyz(x-z)$; $(y+z)(y-z) = (y-x)xyz$

Multiplying these 3 equations we get (for $x \neq y \neq z$), $x^3y^3z^3 = -(x+y)(y+z)(z+x)$-(iv)

Multiplying (i), (ii) and (iii) we get,

$1 = xyz(x+y)(y+z)(z+x)$-(v)

From (iv) and (v) we get, $x^4y^4z^4 = -1$

Hence, no real solutions.

For $x = y$, $\frac{1}{x^2} = \frac{x}{z} + 1$; $\frac{1}{xz} = 2$; $\frac{1}{zx} = \frac{z}{x} + 1$ $z = x^2(x+z)$; $xz = \frac{1}{2}$; $1 = z(z+x)$

We get $z^2 = x^2$ As $x \neq -z$ $x = z$

Hence, we get $x = y = z = \pm\frac{1}{\sqrt{2}}$

□ Let f be a function satisfying the following: 1) $f(ab) = f(a) + f(b)$, when $(a,b) = 1$ 2) $f(p+q) = f(p) + f(q)$, when p and q are primes

Find all values of $f(2002)$

Solution

We shall proof that $f(p) = 0$ for all prime p.

By Rule 2, $f(6) = f(3) + (3)$. But by Rule 1, $f(6) = f(2) + f(3)$. So $f(2) = f(3)$. By Rule 2, $f(5) = f(2) + f(3) = 2f(2)$. By Rule 2, $f(10) = f(5) + f(5) = 4f(2)$. But by Rule 1, $f(10) = f(2) + f(5) = 3f(2)$. So $f(2) = 0$.

For all prime p, by Rule 2, $f(2p) = f(p) + f(p)$. But by Rule 1, $f(2p) = f(2) + f(p)$. So $f(p) = f(2) = 0$.

$f(2002) = f(2 \times 7 \times 143) = f(2) + f(7) + f(143)$, by Rule 1. So the result is 0.

□ Find all positive integers n such that$\lfloor \frac{n^2}{5} \rfloor$ is a prime number

Solution

$p \leq \frac{n^2}{5} < p+1$, where p is prime. So $5p \leq n^2 < 5p+5 \implies 0 \leq n^2 - 5p < 5$. Solve all cases from 0 to 4. Eg, $n^2 - 5p = 1 \implies p = \frac{(n+1)(n-1)}{5}$. Since p is prime, either $5 = n+1$ or $5 = n-1$, which yields $n = 6, 4$.

Final conclusion is n=$4, 5, 6$.

Or another way:

The quadratic residues of n^2 are $0, \pm 1$.

Case one: $n^2 = 5a \implies \lfloor \frac{n^2}{5} \rfloor = a$ but $5|a$. So only solution is $n, a = 5$

Case two: $n^2 = 5a+1 \implies \lfloor \frac{n^2}{5} \rfloor = a$ So a needs to be prime. Note that then $(n+1)(n-1) = 5a$. $n+1 = 5 \implies n = 4, a = 3$ so we are good.

If $n-1 = 5 \implies n = 6, a = 7$ so we are good.

Case three: $n^2 = 5a+4 \implies \lfloor \frac{n^2}{5} \rfloor = a$ Again, a needs to be prime. $(n+2)(n-2) = 5a$.

$n+2 = 5 \implies n = 3, a = 1$ so we can throw it away. $n-2 = 5, n = 7, a = 9$ which is again incorrect.

Thus, $n = 4, 5, 6$

– find a and b such that this is an integer -

$\frac{1}{a} + \frac{1}{b} + \frac{a}{b+1}$. — For any positive integer n, prove that

$$\{\sqrt{n}\} = \lfloor \sqrt{n + \lfloor \sqrt{n} \rfloor} \rfloor$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, and $\{x\}$ denotes the integer closest to x.

[hide="my solution, is this correct? Is there a cleaner one?"] let $a^2 \le n < (a+1)^2$ where $a = \lfloor\sqrt{n}\rfloor \implies a \le \sqrt{n} < a+1$. Also, $(a+\frac{1}{2})^2 - \frac{1}{4} = a^2 + a \le n + a < (a+1)^2 + a = (a+\frac{3}{2})^2 - \frac{5}{4}$ $\implies a \le \lfloor\sqrt{n+a}\rfloor \le a+1$

Case 1: when $a \le \sqrt{n} < a+\frac{1}{2}$ $a \le \lfloor\sqrt{n+\lfloor\sqrt{n}\rfloor}\rfloor = \lfloor\sqrt{n+a}\rfloor < \lfloor\sqrt{(a+\frac{1}{2})^2+a}\rfloor = \lfloor\sqrt{(a+1)^2-\frac{3}{4}}\rfloor$ $\implies a \le \lfloor\sqrt{n+\lfloor\sqrt{n}\rfloor}\rfloor < a+1 \implies \lfloor\sqrt{n+\lfloor\sqrt{n}\rfloor}\rfloor = a = \{\sqrt{n}\}$

Case 2: when $a+1 > \sqrt{n} > a+\frac{1}{2}$, similarly we get $a+1 \ge \lfloor\sqrt{n+a}\rfloor = \lfloor\sqrt{n+\lfloor\sqrt{n}\rfloor}\rfloor > \lfloor\sqrt{(a+\frac{1}{2})^2+a}\rfloor = \lfloor\sqrt{(a+1)^2-\frac{3}{4}}\rfloor \implies a+1 \ge \lfloor\sqrt{n+\lfloor\sqrt{n}\rfloor} \ge a+1 \implies \lfloor\sqrt{n+\lfloor\sqrt{n}\rfloor}\rfloor = a+1 = \{\sqrt{n}\}$ Thus the result. – Find $f:\mathbb{R}\to\mathbb{R}$ satisfying : $f(\frac{x+y}{x-y}) = \frac{f(x)+f(y)}{f(x)-f(y)}$ – Find $f:\mathbb{R}\to\mathbb{R}$ satisfying : $f(f(x)+y) = f(x^2-y) + 4f(x)y$ – Find all the set of four (x,y,z,t) positive integers such that $1+5^x = 2^y + 2^z \cdot 5^t$. –

□ 1) Find $f:\mathbb{N}\to\mathbb{N}$ satisfying: $f(m^2+n^2) = f^2(m) + f^2(n)$ and $f(1) > 0$

2) Find $f:\mathbb{Z}\to\mathbb{Z}$ satisfying : $f(0) = 1, f(f(n)) = f[f(n+2)+2] = n, n \in \mathbb{Z}$

Solution

To the second one: From f(f(n))=n you have this: For distinct x,y integers $f(x) \ne f(y)$. Suppose it's not true, so f(x)=f(y) for some x,y. Then $x = f(f(x)) = f(f(y)) = y$ which is not true, contradiction (from now we can use $f(a) = f(b) \implies a = b$). We also know that $f(f(0)) = 0$ so f(1)=0. We can show by induction that f(2k+1) for k positive integer (also 0) is -2k. It's true for k=0. Suppose it's true for some k. Then set n=2k+1: f(f(2k+1))=f(f(2k+3)+2) From this we have f(2k+3)=f(2k+1)-2=-2(k+1) so we proved it. Very similary you can prove that $f(2k) = -(2k-1)fork \ge 0$. You can do similar things with negative integers. Let's prove that f(-(2k+1))=2k+2 Set n=-1. Then $f(f(-1)) = f(f(1)+2) \implies f(-1) = 2$ By setting n=-3,-5,... (induction) you can prove it for other values. It's similar also for even negative numbers. Sorry for writing so many times word similar, but it's similar and I am lazy to write it completely

□ P(x) is a polynomial with integer coefficients.

P(21) =17, P(32)= -247 and P(37) =33

Prove that if P(N) =N+51 for some integer N then N=26

Solution

Let N be such integer that $P(N) = N+51$. And let for another integer x be $P(x) = y$. Then we know that $(N-x)|(P(N)-P(x))$. We can modify it to: $(N-x)|(N+51-y) \implies (N-x)|(51-y+x)$.

Now set $x = 21, 37$. From this flows $(N-37)|55, (N-21)|55$. Divisors of 55 are -55,-11,-5,-1,1,5,11,55. $N-37$ and $N-21$ are divisors of 55 with distance 16. We can see that there are only 2 possibilities: $N-37 = -11, N-21 = 5 \implies N = 26$ or $N-37 = -5, N-21 = 11 \implies N = 32$. But second possibility can't be true, because we know that $P(32) = -247$.

□ Find all natural numbers X such that the product of the digits of X equals $X^2 - 10X - 22$.

Solution

Let X have $n > 1$ digits. Then $X^2 - 10X - 22 \ge 10^{2n-2} - 10^n - 22$. Maximum of product of digits is 9^n. But we can prove that for $n \ge 3$ is $X^2 - 10X - 22 \ge 10^{2n-2} - 10^n - 22 > 9^n$. So we have to consider just 1-digit and 2-digits numbers.

1-digit numbers are easy, for them must $x^2 = x^2 - 10x - 22$ which is impossible

For 2-digit numbers is product of digits maximaly 81, but for $X \ge 17$ is $X^2 - 10X - 22 > 81$ so it's just about considering numbers from 10,11,...,16. You can find out, that it sits only for 12.

□ a_1, a_2, a_3 are three different positive integer numbers, and such that $a_1|a_2+a_3+a_2a_3$ $a_2|a_3+$

$a_1 + a_3 a_1\ a_3 | a_1 + a_2 + a_1 a_2$ prove that a_1, a_2, a_3 can't all be *primes*

Solution

$(a_2+1)(a_3+1) = 1(moda_1) \to (a_1+1)(a_2+1)(a_3+1) = 1(moda_1)$ Similarly, $(a_1+1)(a_2+1)(a_3+1) = 1(moda_2)$ And, $(a_1+1)(a_2+1)(a_3+1) = 1(moda_3)$

If all primes: $(a_1+1)(a_2+1)(a_3+1) = 1(moda_1 a_2 a_3)$ Now, LHS $\geq 1, a_1 a_2 a_3 + 1$ but, $2a_1 a_2 a_3 + 1 > (a_1+1)(a_2+1)(a_3+1)$ for $a_1 \geq 7, a_2 \geq 5, a_3 \geq 3$

Now if WLOG, $a_1 = 2$, then $(a_2+1)(a_3+1) = 1(moda_1)$ would be false.

□ On the plane we have finite set of triangles. Figure F is sum of all this triangles. $P_f = 16$, where P is an area. Prove, that we can choose triangles, which are separable and sum of their areas is $\geqslant 1$.

Solution

Assume a_1 is the largest side of the equilateral triangle. we consider the area where there have equilateral triangles intersect with a_1 let the area is S_1,with not had work we can know(rememeber a_1 is the largest side): $S_1 \leq (\pi + \frac{\sqrt{3}}{4} + 3)(a_1)^2$ if $S_1 \geq 16$, we can know $\frac{\sqrt{3}}{4}(a_1)^2 > 1$ if $S_1 < 16$ we can chose the largest whicn does not intersect with a_1,assume its side is intersect with a_2,so record S_2 similar.we can get $S_2 \leq (\pi + \frac{\sqrt{3}}{4} + 3)(a_2)^2$,too.(if S_2 intersect with S_1,throw away these area) wo do this again and again.finally we will get $S_1 + S_2 \ldots\ldots + S_n \geq 16$ so we can get $\frac{\sqrt{3}}{4}(a_1^2 + a_2^2 + \ldots\ldots + a_n^2) > 1$ we are done. the chart.doc.is why $S_1 \leq (\pi + \frac{\sqrt{3}}{4} + 3)(a_1)^2$

□ How many p prime numbers can be found that $p^2 + 23$ has 14 positive divisor?

Solution

Firstly we look at number of divisors generally. Let $p_1, p_2, \ldots, p_m$ be all distinct prime divisors of n. Then we can write n as $n = p_1^{a_1} p_2^{a_2} \ldots p_m^{a_m}$ where $a_1, a_2, \ldots, a_m$ are positive integers. Now consider a divisor of n. We know that it don't have other prime divisors than n has. So divisor of n is in form $p_1^{b_1} p_2^{b_2} \ldots p_m^{b_m}$ where b_i is nonegative integer such that $b_i \leq a_i$ for each $i = 1, 2, \ldots m$. Now a little from combinatoics. We can choose exponent b_i as $0, 1, 2, \ldots, a_i$ so there's exactly $a_i + 1$ ways how to choose exponent b_i. So to choose proper exponents $b_1, b_2, \ldots, b_m$ we have exactly $(a_1+1)(a2+1)\ldots(a_m+1)$ ways and it's also number of divisors of n.

We know that $p^2 + 23$ has 14 divisors. We can try $p = 2, 3$ and it fails. So suppose that $p > 3$ (we will need it later). We know that $14 = (a_1+1)(a2+1)\ldots(a_m+1)$ (it's number of divisors). Because $14 = 2 * 7$ we know that there are only two ways:

1) $m = 1; a_1 + 1 = 14 \implies p^2 + 23$ has only one prime divisor and it's exponent is 13, so there exists such prime q such that $q^{13} = p^2 + 23$.

2) $m = 2; a_1 + 1 = 7; a_2 + 1 = 2 \implies p^2 + 23$ has two prime divisors with exponents $6, 1$, so there exists 2 prime q, r such that $p^2 + 23 = q^6 r$.

Now let's look at $p^2 + 23$. We exclude $p = 2, 3$ so it's known fact (and easy to prove) that p is in form $6k + 1$ or $6k - 1$:

a) $p^2 + 23 = (6k+1)^2 + 23 = 36k^2 + 12k + 1 + 24 = 12(3k^2 + k + 2)$ b) $p^2 + 23 = (6k-1)^2 + 23 = 36k^2 - 12k + 1 + 23 = 12(3k^2 - k + 2)$

In both cases $p^2 + 23$ is divisible by $12 = 2^2 * 3$ so it has at least two prime divisors so case 1) cannot happen. Now we see that it must be case 2). We also see that primes q, r are $2, 3$. Exponent of 2 is at least 2 so we know that $p^2 + 23 = q^6 r = 2^6 * 3$. From this equality we get $p = \sqrt{2^6 \cdot 3 - 23} = 13$.

Answer is: There's only one such prime $p = 13$.

□ A is a positive integer B and C are integers The equation $A(X^2) + BX + C = 0$ has 2 distinct roots in the interval $(0,1)$ Prove $A \geq 5$ Find a quadratic polynomial satisfying these conditions when A=5

Solution

Let $f(x) = A(x^2) + Bx + C$ and let x_1, x_2 be the roots of f(x). $f(x) = a(x - x1)(x - x2)$ $f(0) = C = A.x_1.x_2 > 0$ and $f(1) = A + B + C = A(1 - x_1)(1 - x_2) > 0$ $f(0)f(1) > 0$ so $f(0)f(1) \geq 1$,or $A^2.x1.(1 - x1).x2.(1 - x2) \geq 1$ $\quad (*)$ we know that for $0 < x < 1, x(1 - x) \leq \frac{1}{4}$ and equality holds iff $x = \frac{1}{2}$ since x1 and x2 are different we have $x_1.(1 - x_1).x_2.(1 - x_2) < \frac{1}{16}$ from (*) ; $A^2 > 16 \implies A > 4 \implies A \geq 5$ For $a = 5$, $5(x^2) - 5x + 1$ satisfies the conditions.

□ Solve this system of equalities in integers: $x(y + z + 1) = y^2 + z^2 - 5$
$y(z + x + 1) = z^2 + x^2 - 5$
$z(x + y + 1) = x^2 + y^2 - 5$

Solution

Let's call that equations (1),(2) and (3). Then: $(1) - (2)$ is $(x - y)(x + y + z + 1) = 0$

$(2) - (3)$ is $(y - z)(x + y + z + 1) = 0$

$(3) - (1)$ is $(z - x)(x + y + z + 1) = 0$

There are 2 cases:

first case: $x + y + z + 1$ is not 0. Then we easily see that there's only one possibility: x=y=z. Using this in (1) we will get $x = y = z = -5$

second case: $x + y + z + 1 = 0$. Now we will make small modification in (1): $y^2 + z^2 - 5 = x(y + z + 1) = x(x + y + z + 1 - x) = -x^2$ So: $x^2 + y^2 + z^2 = 5$ Equations are symmetric, so we can deduce that only possible solutions are: $(2, 1, 0), (2, -1, 0), (-2, 1, 0), (-2, -1, 0)$ and their permutations. But only for $(-2, 1, 0)$ holds $x + y + z + 1 = 0$ This solution and permutations are succesful.

□ Let $0 < a < b$ prove that

There exist $c \in [a, b]$
$\left(\prod_{k=1}^{n} \frac{e^{kb} - e^{ka}}{b - a}\right) = n! e^{\frac{n(n+1)c}{2}}$

Solution

Given that e^x is differentiable we can apply the mean value theorem to get that $\frac{e^{bk} - e^{ak}}{b - a} = k(\frac{1}{k})\frac{e^{bk} - e^{ak}}{b - a} = ke^{c_k}$ for some $c_k \in (ka, kb)$. Thus $\prod_{k=1}^{n} \frac{e^{bk} - e^{ak}}{b - a} = n! e^{\sum_{k=1}^{n} c_k}$

But $c_k \in (ka, kb) \Rightarrow \frac{n(n+1)a}{2} < \sum_{k=1}^{n} c_k < \frac{n(n+1)b}{2} \Rightarrow a < \frac{2\sum_{k=1}^{n} c_k}{n(n+1)} < b$. Now clearly letting $c = \frac{2\sum_{k=1}^{n} c_k}{n(n+1)}$ we get the desired result.

Here a generalisation of problem

$1 \leq k \leq p \in N^*$, $a < b$

$f_k : [a, b] \to R$ function class C^1

suppose each f_k' is strictly increasing

Then there exist $c \in [a, b]$ s.t.

$\prod_{k=1}^{p} \frac{f_k(b) - f_k(a)}{b - a} = \prod_{k=1}^{p} f_k'(c)$ – Prove that : with $n \geq 2$ then : $[\sqrt{n}] + [^3\sqrt{n}] + ... + [^n\sqrt{n}] = [\log_2 n] + [\log_3 n] + ... + [\log_n n]$

□ The real numbers s, t varies being satisfied with $s^2 + t^2 = 1, s \geqq 0, t \geqq 0$. Find the range of the value of the root for the following equation can be valued.

$$x^4 - 2(s + t)x^2 + (s - t)^2 = 0$$

Solution

$s \geqq 0, t \geqq 0, s^2 + t^2 = 1 \iff (s + t)^2 = 2(s^2 + t^2) - (s - t)^2 \leqq 2 \iff 1 \leqq s + t \leqq \sqrt{2},$

equality occurs when $(s,t)=(0,1),(1,0),\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.

Solving for quadratic equation to x^2, we have $x^2=s+t\pm\sqrt{4st}=s+t\pm\sqrt{(s+t)^2-(s-t)^2}$.

From $A.M.\geqq G.M.$ $0\leqq s+t-\sqrt{4st},s+t+\sqrt{4st}\leqq 2(s+t)\leqq 2\sqrt{2}$.

Therefore $0\leqq x^2\leqq 2\sqrt{2}$, the desired answer is $-2^{\frac{3}{4}}\leqq x\leqq 2^{\frac{3}{4}}$.

□ A triangle is called Heronian if its sides and area are integers. Determine all five Heronian triangles whose perimeter is numerically the same as its area.

Solution

Clearly, any triangle (with integer side lengths) with area equal to perimeter will be Heronian. Now, with s,a,b,c standing for the semiperimeter and the three side lengths, make the following substitutions: $s-a=x,s-b=y,s-c=z$. We know x,y,z are positive from the triangle inequality. Now, the constraint that the area is equal to the perimeter can be written (through Hero's formula) as follows: $2s=\sqrt{sxyz}\Rightarrow 4s=xyz$. Now, at the moment, we don't know if s,x,y,z are integers. It is possible that s is half an integer, in which case x,y,z would all be half-integers. However, in the equation $4s=xyz$ we know the left-hand side must be an integer as $4s=2(a+b+c)$, so if s were not an integer, neither would x,y,z be, so the right-hand side could not be an integer. Thus s,x,y,z are all integers. Now, note that $s=x+y+z$, so we substitute that in to find that we must have $4(x+y+z)=xyz,x,y,z\in\mathbb{Z}^+$. This is symmetric in x,y,z so let us assume without loss of generality that $x\geq y\geq z$. Then $xyz=4(x+y+z)\leq 4(x+x+x)=12x\Rightarrow yz\leq 12$. We could do some further analysis, such as noting that at least one of x,y,z must be even, but at this point, it's easy enough just to try all possible pairs (y,z) and see which of them yield integral $x=\frac{4(y+z)}{yz-4}$. We find the triples $(x,y,z)=(24,5,1),(14,6,1),(9,8,1),(10,3,2),(6,4,2)$. Now, to find (a,b,c) from these, we have the formulas $a=y+z,b=x+z,c=x+y$. Thus these five triples of (x,y,z) yield the following triangles: $(29,25,6),(20,15,7),(17,10,9),(13,12,5),(10,8,6)$. Incidentally, it appears that only the last two are right triangles.

□ A function $f:N\to N$ satisfies the following:

(i) $f(xy)=f(x)+f(y)-1$ (ii) there exist a finite number of x, such that $f(x)=1$ (iii) $f(30)=4$

Determine $f(14400)$.

Note: $N=\{1,2,3,...\}$

Solution

Suppose $k>1$ is such that $f(k)=1$. Then

$f(k^2)=f(k)+f(k)-1=1$, and if $f(k^n)=f(k)=1$, then $f(k^{n+1})=f(k^n)+f(k)-1=1$, so there are infinite k's satisfying $f(k)=1$. Then the only such k must be 1.

$$f(30)=44=f(2)+f(15)-14=f(2)+f(3)+f(5)-2f(2)+f(3)+f(5)=6$$

Then since they're greater than 1, each of them is at least 2 so

$$f(2)=f(3)=f(5)=2.$$

Then $f(14400)=f(16)+2f(30)-2=f(16)+6=4f(2)-3+6=8-3+6=11$.

□

□ The definition of a primitive root can be extended to composite numbers. Say w is a primitive root modulo n if $\phi(n)$ is the smallest power of w which is congruent to 1 modulo n.

a.) Find any primitive roots of 10. b.) Show that 12 has no primitive roots

Solution

$\phi(10) = \phi(12) = 4$ For a.) $w^4 \equiv 1 \mod 10$ and $w^3, w^2, w \neq 1 \mod 10$. Only odds will work, so, $\pm1, \pm3$ work with ±1 being eliminated because $(\pm1)^2 \equiv 1 \mod 10$. So, 3 and 7 work. For b.) $w^4 \equiv 1 \mod 12$ and $w^3, w^2, w \neq 1 \mod 12$. Again only odds will work, so, $\pm1, \pm3, \pm5$ may work. $\pm1, \pm5$ are eliminated because $(\pm1)^2 \equiv 1 \mod 12$ and $(\pm5)^2 \equiv 1 \mod 12$. ±3 is eliminated because $gcd(3, 12) = 3$ so any power of 3 will differ from 12 by a multiple of 3. So none can work.

In general: $w^{\phi(n)} \equiv 1 \mod n$ tells us that $w^{\phi(n)} - nk = 1$. If $gcd(w, n) = m$, $m \mid w^{\phi(n)} - nk$, therefore $w^{\phi(n)} \equiv 1 \mod n$ can only happen if $m = 1$.

□ For each positive integer $n \leq 49$ we define the numbers $a_n = 3n + \sqrt{n^2 - 1}$ and $b_n = 2(\sqrt{n^2 + n} + \sqrt{n^2 - n})$. Prove that there exist two integers A, B such that

$$\sqrt{a_1 - b_1} + \sqrt{a_2 - b_2} + \cdots + \sqrt{a_{49} - b_{49}} = A + B\sqrt{2}.$$

Solution

First I factored, $b_n = 2\sqrt{n}(\sqrt{n+1} + \sqrt{n-1})$. Then I noticed that $\sqrt{n+1}\sqrt{n-1} = \sqrt{n^2-1}$ which is part of a_n. So, I say $r = (\sqrt{n+1} + \sqrt{n-1})$ and square it to get $r^2 = 2n + 2\sqrt{n^2-1}$. Then I said $a_n = \dfrac{4n + r^2}{2}$ and noticed $4n = (2\sqrt{n})^2 = s^2$. From these it immediately follows that $a_n = \dfrac{s^2 + r^2}{2}$, $b_n = rs$, and $a_n - b_n = \dfrac{s^2 - 2rs + r^2}{2} \Rightarrow \sqrt{a_n - b_n} = \dfrac{s - r}{\sqrt{2}} = \dfrac{2\sqrt{n} - (\sqrt{n+1} + \sqrt{n-1})}{\sqrt{2}} = \dfrac{(\sqrt{n} - \sqrt{n-1}) - (\sqrt{n+1} - \sqrt{n})}{\sqrt{2}}$ So, $\displaystyle\sum_{n=1}^{49} \sqrt{a_n - b_n} = \sum_{n=1}^{49} \frac{(\sqrt{n} - \sqrt{n-1}) - (\sqrt{n+1} - \sqrt{n})}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2}} \left(\sum_{n=1}^{49} (\sqrt{n} - \sqrt{n-1}) - \sum_{n=1}^{49} (\sqrt{n+1} - \sqrt{n}) \right)$$
$$= \frac{1}{\sqrt{2}} \left(\left(\sum_{n=1}^{49} \sqrt{n} - \sum_{n=1}^{49} \sqrt{n-1} \right) - \left(\sum_{n=1}^{49} \sqrt{n+1} - \sum_{n=1}^{49} \sqrt{n} \right) \right)$$
$$= \frac{1}{\sqrt{2}} \left(\left(\sum_{n=1}^{49} \sqrt{n} - \sum_{n=0}^{48} \sqrt{n} \right) - \left(\sum_{n=2}^{50} \sqrt{n} - \sum_{n=1}^{49} \sqrt{n} \right) \right)$$
$$= \frac{1}{\sqrt{2}} \left(\left(\sqrt{49} - \sqrt{0} \right) - \left(\sqrt{50} - \sqrt{1} \right) \right)$$
$$= -5 + 4\sqrt{2}$$

□ Let x, y, z be positive integers, and let h denote their greatest common divisor. If $1/x - 1/y = 1/z$, prove that both $hxyz$ and $h(y - x)$ are perfect squares.

Solution

Let $x = ha, y = hb, z = hc$, so that $(a, b, c) = 1$. We have $(y - x)z = xy$, so dividing by h^2, $(b - a)c = ab$. $hxyz = h^4abc, h(y - x) = h^2(b - a)$, so we want to show that $abc, b - a$ are perfect squares.

Suppose that some prime p divides c, and let d be such that $p^d \| c$. Since $(b - a)c = ab$, we have $c \mid ab$; but p can't divide both a, b because $(a, b, c) = 1$. Then it divides just one of them, which means it doesn't divide $b - a$. So we have $p^d \| LHS, p^d \| RHS$, then $p^d \| ab$. We can do this for all primes that divide c. Then the highest power of p dividing abc is $2d$, which is even, and the highest power of p dividing $b - a$ is 0, which is also even.

Now suppose some prime q divides a but not c. Then it must also divide $b - a$; so it divides b. Similarly any prime which divides b but not c divides $b - a$ and thus a. Let e, f be such that $q^e \| a, q^f \| b$. If $e \neq f$, $q^{e+f} \| RHS$ whereas $q^{min(e,f)} \| LHS$, contradiction. So $e = f$. Then the highest power of q dividing abc is $e+f=2e$, which is even. We've characterized all prime divisors of abc as having even exponents, so it is a perfect square. Then as $ab=c(b-a), abc=c^2(b-a)$. Thus as abc is a square, $c^2(b-a)$ is a square and $b-a$ is a square.

– Let $n \geq 2$ be a positive integer. Suppose that $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ are $2n$ positive numbers such that $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$ and

$$a_i \geq 0, 0 \leq b_i \leq \frac{n-1}{n}, i=1,2,...,n$$

Prove that

$$b_1a_2a_3...a_n + a_1b_2a_3...a_n + ... + a_1a_2...a_{k-1}b_ka_{k+1}...a_n + ... + a_1a_2...a_{n-1}b_n \leq$$

$$\frac{1}{n(n-1)^{n-2}}$$

□ Let P be a point inside triangle ABC, and D, E, F be the feet of perpendiculars from P to the lines BC, CA, AB respectively. Prove that : (i) $EF = AP.\sin A$ (ii) $PA+PB+PC \geq 2(PE+PD+PF)$

Solution

Denote $\angle FAP = \alpha$ and $\angle EAP = \beta$ then $\alpha+\beta = A$ $EF^2 = PF^2+PE^2-2PE\cdot PF\cos FPE = PF^2+PE^2+2PE\cdot PF\cos A$ And $PF = AP\sin\alpha, PE = AP\sin\beta$ Thus we obtain $EF^2 = AP^2(\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta\cos A)$

In fact, $\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta\cos A = \frac{1-\cos 2\alpha}{2}+\frac{1-\cos 2\beta}{2}+(\cos(\alpha-\beta)-\cos(\alpha+\beta))\cos A = 1-\cos(\alpha-\beta)\cos(\alpha+\beta)+\cos(\alpha-\beta)\cos A-\cos^2 A = \sin^2 A$ for (ii)

Let E' and F' be the projection of E and F on BC, respectively. We have $EF \geq E'F'$, $DE' = PE\cos(\frac{\pi}{2}-C) = PE\sin C$ and similarly $DF' = PF\sin B$

hence $AP\sin A = EF \geq DE'+DF' = PE\sin C+PF\sin B$ and $AP \geq PE\frac{\sin C}{\sin A}+PF\frac{\sin B}{\sin A}$

Similarly, we have another two inequalities. Sum them up and we obtain that

$PA+PB+PC \geq PD\left(\frac{\sin C}{\sin B}+\frac{\sin B}{\sin C}\right)+PE\left(\frac{\sin C}{\sin A}+\frac{\sin A}{\sin C}\right)+PF\left(\frac{\sin B}{\sin A}+\frac{\sin A}{\sin B}\right) \geq 2(PD+PE+PF)$

□

□ In the plane, Γ is a circle with centre O and radius r, P and Q are distinct points on Γ, A is a point outside Γ, M and N are the midpoints of PQ and AO respectively. Suppose $OA = 2a$ and $\angle PAQ$ is a right angle. Find the length of MN in terms of r and a

Solution

$\sqrt{\frac{r^2}{2}-a^2}$. Anyway, it is unnecessary for A to be outside the circle.

Here is an analytic solution (luckily, there is no complicated calculation). Let O be origin and OM be x-axis. Suppose $P(x_0, y_0), Q(x_0, -y_0)$ and $N(x, y)$. Thus $M(x_0, 0)$ and $A(2x, 2y)$. Since AP is perpendicular to AQ, $\overrightarrow{AP}\cdot\overrightarrow{AQ} = 0$, which is $(2x-x_0)(2x-x_0)+(2y-y_0)(2y+y_0) = 0$ Expand it and apply $x^2+y^2 = a^2$ we get $2x_0x = \frac{4a^2+x_0^2-y_0^2}{2}$.

$MN^2 = (x-x_0)^2+y^2 = x^2-2x_0x+x_0^2+y^2 = a^2-2x_0x+x_0^2 = a^2-\frac{4a^2+x_0^2-y_0^2}{2}+x_0^2 = \frac{x_0^2+y_0^2}{2}-a^2 = \frac{r^2}{2}-a^2$.

□ Is there any composite number such that when its prime factors are listed in increasing order and viewed as one number, it's the same number?

To show you what I mean, let's say 123456789.

123456789 = 3*3*3607*3803 Which can be turned into the number 3336073803. So that doesn't work.

...or maybe you can take the prime factorization of THAT and it would equal the original number, or if not then do it again...

Any thoughts on this?

Solution

No composite number exists which has the property that if its prime factors are listed in increasing order, the original number is formed. This follows from the more general fact that I will prove: A number $n = \overline{N_1 N_2}$, where N_1 and N_2 are consecutive blocks of digits, is greater than the product $N_1 \cdot N_2$. Suppose that N_2 has k digits. Then we need to have $n = N_1 \cdot N_2$. But $n = 10^k \cdot N_1 + N_2 > 10^k \cdot N_1 > N_1 \cdot N_2$. The result regarding primes follows from the fact that we would have to write $n = \overline{p_{n_1} p_{n_2} \cdots p_{n_k}} = p_{n_1} \cdot p_{n_2} \cdots p_{n_k}$ which is impossible.

Moreover, we see that the said transformation takes to number $n = p_{n_1} \cdot p_{n_2} \cdots p_{n_k}$ to the number $f(n) = \overline{p_{n_1} p_{n_2} \cdots p_{n_k}}$, with the property that $n < f(n)$. It follows that $n < f(n) < f(f(n)) < \cdots$ so that the procedure never returns to the original number.

☐ Let n be a positive integer. Prove that there is no positive integer solution to the equation

$$(x+2)^n - x^n = 1 + 7^n$$

Solution

if $n = 1$, then obviously there's no solution. so assume $n \geq 2$.

notice that if x is odd, then one of $x, x+2$ is $1(\mod 4)$ and the other is $-1(\mod 4)$. so then if n is odd, the LHS is $\pm 2(\mod 4)$ while the RHS is $0(\mod 4)$, and if n is even, the LHS is $0(\mod 4)$ while the RHS is $2(\mod 4)$. in either case there's obviously no solution.

now, if x is even, one of $x, x+2$ is $0(\mod 4)$ and the other is $2(\mod 4)$, which imply that the LHS is $0(\mod 2^n)$, and in particular that it is divisible by 4 (because we assumed $n \geq 2$). now, for the RHS to also be divisible by 4, we need n to be odd. so $n \geq 3$. for n odd, however, the RHS is divisible by 8 but not by 16. in view of the fact that the LHS is $0(\mod 2^n)$, it follows that n can only be 3. but for n = 3, the equation becomes $(x+2)^3 - x^3 = 344$, and this can easily be seen to have no solutions... – If a,b,c are sides of triangle ABC. Inscribed circle is tangent to sides BC,AC,AB at points K,L,M and F is the intesection point of segments AK,BL,CM. Then prove that:

$6 \leq 4\sum \frac{(a-b)^2}{(b+c-a)(a+c-b)} + 6 \leq \frac{AF}{FK} + \frac{CF}{FM} + \frac{BF}{FL} \leq 6 + \frac{(a+b+c)^2\left((a-b)^2+(b-c)^2+(c-a)^2\right)}{8S^2}$

☐ find all pairs of (a, b) from positive integers, where $a^2b + a + b$ would be divisible by $ab^2 + b + 7$.

Solution

Let $ab^2 + b + 7 = x, a^2b + a + b = y$. Suppose $x \mid y$. Then $x \mid by - ax = b^2 - 7a$, so $ab^2 + b + 7 \mid b^2 - 7a$.

Now if $b^2 - 7a$ is positive, $ab^2 + b + 7 > b^2 + b > b^2 > b^2 - 7a$, contradiction. If $b^2 - 7a$ is zero, then $b^2 = 7a$, so a is of the form $7k^2$, and $b^2 = 49k^2, b = 7k$, so we have the solution $(7k^2, 7k)$ which we can see works as $7^3k^4 + 7k + 7 \mid 7^3k^5 + 7k^2 + 7k = k(7^3k^4 + 7k + 7)$.

Finally $b^2 - 7a$ may be negative. Then since $ab^2 + b + 7 \mid 7a - b^2$, we need $ab^2 < ab^2 + b + 7 \leq 7a - b^2 < 7a$, so $ab^2 \leq 7a, b^2 \leq 7$. So $b = 1$ or $b = 2$. If $b = 1$, then we need $a + 8 \mid a^2 + a + 1$; since $a + 8 \mid (a+8)(a-7) = a^2 + a - 56$, then $a + 8 \mid 57$. Also $a + 8$ is at least 9 so it is 19 or 57, giving $a = 11, a = 49$, both of which work. If $b = 2$, then $4a + 9 \mid 2a^2 + a + 2, 4a + 9 \mid 4a^2 + 2a + 4, 4a + 9 \mid 4a^2 + 9a, 4a + 9 \mid 7a - 4, 4a + 9 \mid 3a - 13$. So $3a - 13$ can't be positive, but it clearly can't be zero either because it's not divisible by 3. Then it must be negative; but then $13 - 3a$ is at most 10, while $4a + 9$ is at least 13. So there are no solutions in this case. – An integer sequence is related with the formula $(n-1)a_{n+1} = (n+1)a_n - 2(n-1)$ for $n \geq 1$ If $2000|a_{1999}$, determine the least value of n, such that $n \geq 2$ and $2000|a_n$ – Find all integer solutions of equation

$$(n+3)^n = \sum_{k=3}^{n+2} k^n$$

– Let $n \geq a_1 > a_2 > ... > a_k$ be positive integers such that $lcm(a_i, a_j) \leq n$ for all i, j. Prove that $ia_i \leq n$ for $i = 1, 2, ..., k$ – Let $f(x) = x^3 + 17$. Prove that for every natural number n, $n \geq 2$, there is natural number x such that $f(x)$ is divisible by 3^n but not by 3^{n+1} – Prove that if a and b are positive integers, then

$$(a + \frac{1}{2})^n + (b + \frac{1}{2})^n$$

is an integer for only finitely many positive integers n – Let $U =$ positive integer x such that x is not more than 2500 $A_n =$ nx with x in U and n is a positive integer. $B = A_8$ union A_9 union $A_{16} = b_1, b_2, ..., b_r$ with $b_i < b_j$ for $i < j$ (a) Find r when b_r is the biggest member of B (b) Find all primes n such that $b_{132} - b_{97}$ is a member of A_n – Given a real number a and $f_1, f_2, ..., f_n$ additive functions from reals to reals such that $f_1(x)f_2(x)...f_n(x) = ax^n$ for all real number x. Prove that there exists b, which is a real number, and i, which is in the set $1, 2, .., n$ such that $f_i(x) = bx$ for all real number x. – Solve the following system: $\begin{array}{c} x + 2log_{\frac{1}{3}} y = -1 \\ x^3 + y^3 = 28 \end{array}$.

□ Let a, b, c be positive integers such that $1 < a < b < c$. Suppose that $(ab-1)(bc-1)(ca-1)$ is divisible by abc. Find the values of a, b, c

Solution

By opening parenthises we obtain $abc \mid ab+bc+ca-1$. So if $a \geq 3$ then $abc \geq 3bc > ab+bc+ca-1$. Contradiction. Therefore, $a = 2$. Then $c \mid 2b-1$ (since we know $c \mid ab-1$). It follows that $c = 2b-1$ (since $c > b$). Then $b \mid 2(2b-1)-1$ (since $b \mid ac-1$), i.e. $b = 3$. And finally $c = 5$. Answer: $a = 2, b = 3, c = 5$.

□ Suppose $f(x)$ is a polynomial with integral coefficients. If $f(x) = 2$ for three different integers a, b, c, prove that $f(x)$ can never be equal to 3 for any integer x

Solution

Since $f(\alpha)$ is the remainder that $f(x)$ leaves when divided by $(x-\alpha)$, all $(x-a)$, $(x-b)$ and $(x-c)$ divide $f(x)-2$, so we can write $f(x)-2 = (x-a)(x-b)(x-c)g(x)$ (where $g(x)$ is another polynomial). Then if $f(x) = 3$, we would have that $1 = (x-a)(x-b)(x-c)g(x)$, but 1 cannot be written as a product of three different integers. Q.E.D.

□ Prove for any integer $n > 1$ that $(n-1)^2|n^{n-1}-1$

Solution

It is equivalent to proving $n-1 \mid n^{n-2}+n^{n-3}+...+n+1$. Now $n \equiv 1 \pmod{n-1}$ so all the terms are $n^k \equiv 1^k \equiv 1$, and there are $n-1$ terms. Then their sum is $S \equiv n-1 \equiv 0$.

□ Evaluate

$$\cot\left[\sum_{k=1}^{n} \cot^{-1}(1+k+k^2)\right]$$

Solution

Basically, I'll try to motivate how I discovered that $arccotn^2+n+1 = arccotn+1-arcotn$. Whenever you see a rather ugly sum, you should immediately be alert to the possibility of telescoping. But how do we split up an inverse cotagent into parts that would telescope? Well, let's try to use the addition formula. We know that $\cot(a+b) = \frac{1}{\tan(a+b)} = \frac{tanatanb-1}{\tan a+\tan a} = \frac{1-\cot a\cot b}{\cot a+\cot b}$. Right now, this isn't helpful. We want inverse cotagents. So let $x = \cot a, y = \cot b$ s.t. $a = arccotx, b = arccoty$. Now we have $\cot(arccotx + arccoty) = \frac{1-xy}{x+y}$. If we take arccot of both sides, we now have $arccotx + arccoty = arccot(\frac{1-xy}{x+y})$. This is much more helpful. Now we try to get this $arccot(k^2+k+1)$ as resembling the RHS of that identity. Now, either you're good at algebra and you can see that $k^2+k+1 = \frac{1-(-k)(k+1)}{-k+k+1}$, or you're not that good and you just guess that $x = k+1, y = -k$, as you want this thing to telescope and those choices work out well. (I did the latter). In any case, you can then find the identity $arccot(k^2+k+1) = arccot(k+1) - arccotk$ and finish the problem.

□Hay

□

□

□ Prove that the first thousand digits after the decimal point in the value of $(6+\sqrt{35})^{1980}$ are all 9

Solution

A related well known result, from which the problem can be easily derived:

We show that for any natural m and any $n \geq 1$, $(m+\sqrt{m^2-1})^n = k+\sqrt{k^2-1}$ for some natural k.

For $n = 1$ it is obvious; now assume it true for n and look at $(m+\sqrt{m^2-1})^{n+1} = (m+\sqrt{m^2-1})^n(m+\sqrt{m^2-1})$. It's easy to see by expanding $(m+\sqrt{m^2-1})^n$ that we will get something of the form $a+b\sqrt{m^2-1}$. By hypothesis, $b^2(m^2-1)+1 = a^2$. Now we have

$$(a+b\sqrt{m^2-1})(m+\sqrt{m^2-1}) = am+(m^2-1)b+(a+bm)\sqrt{m^2-1}.$$

Here $(am+b(m^2-1))^2 - (m^2-1)(a+bm)^2 = a^2 - b^2(m^2-1) = 1$ by hypothesis, so the induction is done.

□ If $a_i \geq 1$, prove that

$2^{n-1}(a_1a_2\dots a_n+1) \geq (1+a_1)(1+a_2)\dots(1+a_n)$

Solution

Induction on n.

Base case: the two sides are equal.

Assume $2^{n-1}(a_1a_2\dots a_n+1) \geq (1+a_1)(1+a_2)\dots(1+a_n)$.

Now:

$2^{n-1}(a_1a_2\dots a_n+1) \geq (1+a_1)(1+a_2)\dots(1+a_n)$

$\Rightarrow 2^{n-1}(a_1a_2\dots a_n+1)(1+a_{n+1}) \geq (1+a_1)(1+a_2)\dots(1+a_n)(1+a_{n+1})$

from whence it remains to prove $2(a_1a_2\dots a_na_{n+1}+1) \geq (a_1a_2\dots a_n+1)(1+a_{n+1})$.

Let $r = a_1a_2\dots a_n$, and $y = a_{n+1}$ for convenience.

Then we are required to prove $2(ry+1) \geq (r+1)(y+1)$. This is equivalent to $(r-1)(y-1) \geq 0$. Clearly, $y = a_{n+1} \geq 1$ as given. Also, any mean (such as r, the nth power of a geometric mean) cannot be less than the minimum of the elements involved, whereby $r^n \geq 1$ so that $r \geq 1$.

and the induction is complete.

Another way Write $f(a_n) = 2^{n-1}(a_1a_2\dots a_n+1) - (1+a_1)(1+a_2)\dots(1+a_n)$

We calculate $f'(a_n) = 2^{n-1}a_1a_2\dots a_{n-1} - (1+a_1)(1+a_2)\dots(1+a_{n-1})$

Clearly, this is ≥ 0, because

$a_1a_2\dots a_{n-1} \geq \left(\frac{1+a_1}{2}\right)\dots\left(\frac{1+a_{n-1}}{2}\right)$

since $a_i \geq \frac{1}{2}+\frac{a_i}{2} \geq 1$

So $f(a_n)$ is strictly increasing in a_n. Thus it's enough to show that the inequality holds for $a_n = 1$. By induction, we see it's enough to show that the inequality holds for all $a_i = 1$

But then we just get $2^n = 2^n$ which establishes the result.

□ A and B are odd positive integers and A<B.

The sum of all the integers greater that A and less than B is 1000.

Find A and B.

Solution

The problem clearly implies $1000 = \sum_{i=1}^{B-1} i - \sum_{j=1}^{A-1} j$.

Set $A = 2x + 1$, $B = 2y + 1$ where x,y are natural. (because A,B odd this can be done)

Then we have

$1000 = \sum_{i=1}^{2Y} i - \sum_{j=1}^{2X} j$

implying

$1000 = Y(2Y + 1) - (X)(2X + 1) = (Y - X)(2Y + 2X + 1)$

The parity of the second factor is odd. This implies 8 divides $y - x$. So the second factor can only be $125, 25, 5, 1$. We rule out the last 3 because the second factor is bigger than the first. So $Y - X = 8$, $2Y + 2X + 1 = 125$, from whence $Y = 35$, $X = 27$.

□ Let a, b, c be roots of $12x^3 - 985x - 1728 = 0$ Find $a^3 + b^3 + c^3$

Solution

Method 1) use the identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$, $abc = \frac{1728}{12}$, $a + b + c = 0$. so $a^3 + b^3 + c^3 = 3abc = 432$.

Method 2) the given equation $12x^3 - 985x - 1728 = 0$ is true for a, b, c. So solving for $x^3 = \frac{985x+1728}{12}$. then we plug in a, b, c to get 3 equations.

$a^3 = \frac{985a+1728}{12}$ $b^3 = \frac{985b+1728}{12}$ $c^3 = \frac{985c+1728}{12}$

we also know $a + b + c = 0$, so adding the 3 equations we get $a^3 + b^3 + c^3 = 3 \cdot \frac{1728}{12}$.

□ Note $M = \{x \in \mathbb{Q} \mid x(x^2 + 6) + \sqrt{3}(6x + 5)x^3 = \sqrt{3}(11x^2 + 10x + 2) + 6x^4 - 10x^2 - 1$ and $S = \sum_{x \in M} x$. Compute S.

Solution

group all the $\sqrt{3}$'s.

$\sqrt{3}(6x^4 + 5x^3 - 11x^2 - 10x - 2) + (-6x^4 + x^3 + 10x^2 + 6x + 1)$.

Factors into $\sqrt{3}(x^2 + 2)(2x + 1)(3x + 1) - (3x + 1)(2x + 1)(x^2 - x - 1)$.

Factors into $[(3x + 1)(2x + 1)][\sqrt{3}(x^2 + 2) - (x^2 - x - 1)]$.

rightmost factor does not have rational roots, so the only roots in S are $\frac{-1}{3}, \frac{-1}{2}$. so S is -5/6.

□ A non-negative integer $f(n)$ is assigned to each positive integer n in such a way that the following conditions are satisfied: (a) $f(mn) = f(m) + f(n)$ for all positive integers m, n (b) $f(n) = 0$ whenever n ends in a 3 (in base 10) (c) $f(10) = 0$ Prove that $f(n) = 0$ for all positive integers n.

Solution

It is easy to see that if $f(mn)$ is 0, then so are $f(m)$ and $f(n)$, because ie. $f(m) \geq 0$. (*)

Now $f(3) = 0$ by (b), and $f(2) = f(5) = 0$ by (*) and (c).

It is enough to prove $f(p) = 0$ for all primes $p > 5$. We show that there exists some numbers y [where y is 3 mod 10] for which $p|y$. In which case, $0 = f(y) = f(pk) = f(p) + f(k)$ implying $f(p) = 0$.

It is enough to show that $10k + 3 = 0$ (mod p) for some k, and any prime $p > 5$. Clearly, 10 does not divide p, in which case $10k$ cycles through the residues $1, 2, \ldots, p$. Thus it meets the residue -3. QED.

□ Let $m, n \in N$. Prove that $|36^m - 5^n| \geqslant 11$.

Solution

Let $f(m,n) = |36^m - 5^n|$. Note first that $36 - 5^2 = 11$, and second that since 36 is even and 5 is odd, $f(m,n)$ is always even-odd=odd. Therefore, if we can show that $f(m,n) \neq 1, 3, 5, 7, 9$ we'll be done. Observe that $f(m,n)$ cannot be a multiple of 3 since that would mean that $36^m - 5^n \equiv -5^n \equiv 0 \pmod 3$, contradiction. This rules out the possibilities of $f(m,n) = 3, 9$. Further, if $f(m,n)$ is a multiple of 5, we know that $f(m,n) \neq 5, 7$ since both yield a contradiction $\mod 5$. We are left with the possibility that $f(m,n) = 1$. However in this case, either $36^m - 5^n = 1$ or $36^m - 5^n = -1$. The first one means that

$$\begin{aligned} 36^m - 1 = 5^n &\iff (36-1)(36^{m-1} + 36^{m-2} + \cdots + 36^2 + 36 + 1) = 5^n \\ &\iff 35(36^{m-1} + 36^{m-2} + \cdots + 36^2 + 36 + 1) = 5^n \\ &\implies \text{contradiction.} \end{aligned}$$

(Since $7 \nmid 5$.)

Similarly, the second one ($f(m,n) = -1$) also yields a contradiction (take the resulting equation mod 9).

Thus $\boxed{f(m,n) \geq 11}$, as desired. Another way Notice that $36^m - 5^n \equiv 1 \pmod 5, 3 \pmod 4, 1, -1 \pmod 6$. Assume that $|36^m - 5^n| < 11$ From the first equation we arrive at $36^m - 5^n = -9, -4, 1, 6$. From the second equation we arrive at $36^m - 5^n = -9, -5, -1, 3, 7$ and from the third we arrive at $36^m - 5^n = -7, -5, -1, 1, 5, 7$.

It follows from the first two equations that the only place that satisfies both is $36^m - 5^n = -9$ however this does not satisfy the third therefore $|36^m - 5^n| < 11$ is false and $|36^m - 5^n| \geq 11$

□ Let a be an arbitrary constant number. Solve the following inequality.

$$a(x^2+1) < x(a^2+1)$$

Solution

For $a < -1$, we have $x < a, \frac{1}{a} < x$

For $a = -1$, we have $x \neq -1$

For $-1 < a < 0$, we have $x < \frac{1}{a}, a < x$

For $a = 0$, we have $x > 0$

For $0 < a < 1$, we have $a < x < \frac{1}{a}$

For $a = 1$, there don't exist the set of roots.

For $a > 1$, we have $\frac{1}{a} < x < a$

□ hình học – Find the least n such that whenever the elements of set 1,2,...,n are coloured red or blue, there always exist x, y, z, w (not necessarily distinct) of the same colour such that $x + y + z = w$

□ Find all integer solutions to $m^3 - n^3 = 2mn + 8$.

Solution

Let $m = n + p$ (with $p \in \mathbb{Z}$) $\Rightarrow (n+p)^3 - n^3 = 2(n+p)n + 8 \Rightarrow (3p-2)n^2 + p(3p-2)n + (p^3 - 8) = 0$

This is a second-degree equation in n, so in order to have (any) solutions, $D \geq 0 \quad \Rightarrow p^2(3p-2)^2 - 4(3p-2)(p^3-8) \geq 0 \quad \Rightarrow \ldots \quad \Rightarrow 0 < p < 3$ $p = 1 \quad \Rightarrow n^2 + n - 7 = 0 \quad \Rightarrow$ no solutions... $p = 2 \quad \Rightarrow 4n^2 + 8n = 0 \quad \Rightarrow (0, 2)$ and $(-2, 0)$

☐ Let $a_1 = 21$ and $a_2 = 90$, and for $n \geq 3$, let a_n be the last two digits of $a_{n+1} + a_{n+2}$. What is the remainder of $a_1^2 + a_2^2 + ... + a_{2005}^2$ when it is divided by 8 ?

Solution

Notice that $x^2 \equiv (x-4)^2 \mod 8$. So if $a_k + a_{k+1} \geq 100$, the new $a_{k+2} \equiv a_k + a_{k+1} - 4 \mod 8$ and thus the square is the same as if you just added and didn't take the last two digits.

So then we have the sequence (mod 8): $5, 2, 7, 1, 0, 1, 1, 2, 3, 5, 0, 5, 5, 2, ...$ which we find repeats every 12 numbers with a sum of $0 \mod 8$. Thus we take $2005 \equiv 1 \mod 12$ so the sum is $5^2 \equiv 1 \mod 8$.

☐ Let M be a set in the plane with area greater than 1. Show that M contains two distinct points (x_1, y_1) and (x_2, y_2) such that $x_2 - x_1$ and $y_2 - y_1$ are integers.

Solution

Obviously we can restrict our attention to $[0,1]^2$ because $x - y \in Z \Leftrightarrow x - y \in Z$.

Now if a point is "doubled" then clearly those 2 points can be used.

Else, no point is doubled and $(1,1), (0,0)$ can be used. – Let $x^2 + xy + y^2 = a, y^2 + 2yz + z^2 = b, z^2 + 2zx + x^2 = a + b$, where x, y, z are positive and a, b are positive parameters. Find $xy + yz + zx$. – Find a simple form of $\sin A \sin 2A + \sin 2A \sin 3A + ... + \sin{(n-2)A} \sin{(n-1)A}$, where $A = \frac{\pi}{n}$ – Let $f : \mathbb{R} \to \mathbb{R}$ be an injective function. If $f^-1(x) + f(x) = x \forall x \in R$ then prove that f is an odd function!

– Prove that the number of integral solutions (x, y) to $x^2 + y^2 = n$ is equal to $4(d_1 - d_3)$, where d_1 is the number of divisors of n of the form $4k + 1$ and d_3 is the number of divisors of n of the form $4k + 3$. – Let $f(x)$ be a cubic polynomial with roots r_1, r_2, r_3 such that $\frac{f(\frac{1}{2}) + f(\frac{-1}{2})}{f(0)} = 997$, find $\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}$ – Let α be the root, one of the cubic equation $x^3 + 3x^2 - 1 = 0$.

(1) Express $(2\alpha^2 + 5\alpha - 1)^2$ in the form of $a\alpha^2 + b\alpha + c$, where a, b, c are rational numbers.

(2) Express other two roots except α in the form of $a\alpha^2 + b\alpha + c$. –

Let $g(x) = 3x^2 - 2x(a+b+c) + ab + bc + ac$ and $-1 \leq a, b, c \leq 1$ and $y = \frac{a+b+c}{3}$.

Prove

(a) $|g(y)| + min[g(1), g(-1)] \leq 3 \leq |g(y)| + max[g(1), g(-1)]$

(b) $|g(y)| \leq \frac{1}{2} \cdot max[(g(1), g(-1)]$

(c) Find a, b, c if $|g(y)| = \frac{1}{2} \cdot \max[g(1), g(-1)]$

☐ tổ hợp

☐ số học

☐ Let f be a convex function and x_1, x_2, x_3 in its domain. Prove that $f(x_1) + f(x_2) + f(x_3) + \frac{f(x_1+x_2+x_3}{3} \geq \frac{4}{3}[f((x1 + x2)/2) + f((x2 + x3)/2) + \frac{f(x_3+x_1)}{2}$.

Solution

Suppose, without loss of generality, that $x_1 < x_2 < x_3$, and further that $x_2 < \frac{x_1+x_2+x_3}{3}$. Let

$$g_l(t) = \frac{1}{4}(f(x_1 + t) + f(x_2) + f(x_3 - t) + f(\frac{x_1 + x_2 + x_3}{3}))$$

and

$$g_r(t) = \frac{1}{3}(f(\frac{(x_1 + t) + x_2}{2}) + f(\frac{x_2 + (x_3 - t)}{2}) + f(\frac{x_3 - t + x_1 + t}{2}))$$

for $t \in [0, x_2 - x_1]$. We take the derivatives, and find

$$g_l'(t) = \frac{1}{4}(f'(x_1 + t) - f'(x_3 - t))$$

and

$$g_r'(t) = \frac{1}{6}(f'(\frac{(x_1 + t) + x_2}{2}) - f'(\frac{x_2 + (x_3 - t)}{2})).$$

By construction, $x_1 + t \le \frac{1}{2}(x_1 + x_2 + t) \le \frac{1}{2}(x_2 + x_3 - t) \le x_3$. But, by the convexity of f, f' is monotone increasing. So g_l' and g_r' are negative, and

$$g_l'(t) \le \frac{1}{4}(f'(\frac{(x_1+t)+x_2}{2}) - f'(\frac{x_2+(x_3-t)}{2})) \le g_r'(t).$$

Thus, we push the extreme two points in until one of the hits the third point, and the desired inequality (i.e $g_l \ge g_r$) only gets less true. Now we may suppose that $x_1 = x_2 \le x_3$. We do something similar to the above. Let

$$h_l(t) = \frac{1}{4}(f(x_1+t) + f(x_2+t) + f(x_3-2t) + f(\frac{x_1+x_2+x_3}{3}))$$

and

$$h_r(t) = \frac{1}{3}(f(\frac{(x_1+t)+(x_2+t)}{2}) + f(\frac{(x_2+t)+(x_3-2t)}{2}) + f(\frac{(x_3-2t)+x_1+t}{2}))$$

for $t \in [0, \frac{1}{3}(x_3 - x_1)]$. As above, we can show that $h_l'(t) \le h_r'(t)$. Then we are reduced to the case when $x_1 = x_2 = x_3$, and the desired inequality obviously holds then. Another approach f is convex iff: given any a, b,$(a \le b)$ in its domain, the graph of $f([a, b])$ lies completely under or touching the line segment connecting the points $(a, f(a))$ and $(b, f(b))$.

Let f be convex.

Lemma: Let $a, d(a \le d)$ be in the domain of f. Choose any $b, c \in [a, d]$. Then the slope of AB is less than or equal to the slope of CD, where $A = (a, f(a))$, etc..

Proof: By convexity, B and C lie under AD. So slope $AB \le$ slope $AD \le$ slope CD. QED.

Now, suppose we wish to prove that

$$f(x_1)+f(x_2)+f(x_3)+f((x_1+x_2+x_3)/3) \ge 4/3[f((x_1+x_2)/2)+f((x_2+x_3)/2)+f((x_3+x_1)/2).$$

WLOG, let $x_1 < x_2 < x_3$ and $x_2 < \dfrac{x_1+x_2+x_3}{3}$. Now we replace x_1 by x_2, and x_3 by $x_3 - x_2 + x_1$. We wish to prove that this makes the inequality less true, so we want:

$$\frac{1}{4}(f(x_2)-f(x_1)+f(x_3-x_2+x_1)-f(x_3)) \le \frac{1}{3}(f(x_2)-f((x_1+x_2)/2)+f((x_1+x_3)/2)-f((x_2+x_3)/2)).$$

We divide both sides by $x_2 - x_1$, and this becomes

$$\frac{1}{4}(\text{slope } X_1X_2 - \text{slope } X_3'X_3) \le \frac{1}{6}(\text{slope } M_1X_2 - \text{slope } M_3M_2)$$

where X_3', M_1, M_2, and M_3 are the points on the graph of f at $x_3 - x_2 + x_1$, $(x_1+x_2)/2$, $(x_2+x_3)/2)$, and $(x_1 + x_3)/2$, respectively.

By the lemma, both sides of the inequality are negative, so it suffices to prove

$$\text{slope } X_3'X_3 - \text{slope } X_1X_2 \ge \text{slope } M_3M_2 - \text{slope } M_1X_2.$$

But, by the lemma, slope $X_3'X_3 \ge$ slope M_3M_2 and slope $M_1X_2 \ge$ slope X_1X_2. The former is true because $(x_1 + x_3)/2 \le x_3 - x_2 + x_1$.

So we are reduced to the case where $x_1 = x_2$. The rest can be done in a similar manner.

$\square$ We are given the graph of a polynomial with integer coefficients. We choose two points in the graph with integer coordinates and such that their distance is an integer too. Prove that the segment joining these to points is parallel to the x-axis.

Solution

Let the polynomial in question be $P(x)$. If the two vertices are $(x_1, P(x_1))$ and $(x_2, P(x_2))$ then the integral distance d between these points satisfies $(P(x_1)-P(x_2))^2+(x_1-x_2)^2=d^2$ It is fairly well known that for a polynomial P that x_1-x_2 divides $P(x_1)-P(x_2)$ which means there exists an integer k such that $P(x_1)-P(x_2)=k(x_1-x_2)$. Substituting and factoring the LHS yields $(x_1-x_2)^2(k^2+1)=d^2$. This also implies d is divisible by (x_1-x_2) so there exists another integer m such that $d=m(x_1-x_2)$. So either $x_1=x_2$ (the degenerate case) or $k^2+1=m^2$. However it is an easy number theory practice to show that the only integral solutions to the above equation are $(k,m)=(0,-1)$ or $(0,-1)$. In either case $k=0$. Notice though that $k=\frac{P(x_1)-P(x_2)}{x_1-x_2}$ which is numerically equal to the slope connecting these points. Since this slope is zero it is therefore parallel to the x-axis.

□ số học

□ Solve $10(25^{cos\pi x}-4^{\cos\pi x})=7(5^{\cos\pi x}-2^{\cos\pi x})$ and find all solutions that satisfy the inequality $x^4-6x^2-1\le 0$

Solution

Making the substitution $u=\cos\pi x$ the equation becomes $10(5^{2u}-2^{2u})=7(5^u-2^u)$ which implies either $5^u=2^u$ or $5^u+2^u=\frac{7}{10}$. In the first case $u=0$ and thus $x=\frac{k}{2}$ for k odd. In the second case notice $\frac{7}{10}=\frac{1}{5}+\frac{1}{2}$ so $u=-1$ which means x is an odd integer. By the quadratic equation $-\sqrt{3+\sqrt{10}}\le x\le\sqrt{3+\sqrt{10}}$ which yields the only valid values for x to be $-\frac{3}{2},-1,-\frac{1}{2},\frac{1}{2},1,\frac{3}{2}$

□ Be a,b,c roots of $P(x)x^3+px^2+qx+r=0$. If $S_n=a^n+b^n+c^n$, n is integer and n>3, being $K=S_n+pS_{n-1}+qS_{n-2}$ Find K

Solution

K in terms of S? if thats the case figure out S_1,S_2,S_3 using relationships btwn roots and coefficients.

then write $x^3=-px^2-qx-r$ so $x^4=-px^3-qx^2-rx$.

so then $S_4=-pS_3-qS_2-rS_1$ and continue like this so you have

$S_n=-pS_{n-1}-qS_{n-2}-rS_{n-3}$, so $S_n+pS_{n-1}+qS_{n-2}=-rS_{n-3}=K$.

□ Let n is nature and n>2. Prove that $(n^{n^{n^n}}-n^{n^n})\vdots 9$

Solution

clearly $2|n^n-n$

so $n^{n^n-n}=n^2k=1\pmod 6$, or

$n^{n^n-n}-1=0\pmod 6$, and

$[n^n][n^{n^n-n}-1]=0\pmod 6$, thus

$n^{[n^n][n(n^n-n)-1]}=n^6j=1\pmod 9$, by Euler's Theorem, or

$n^{[n^n][n(n^n-n)-1]}-1=0\pmod 9$, hence

$[n(n^n)][n^{[n^n][n(n^n-n)-1]}-1]=0\pmod 9$, or

$[n(n^n)][n^{[}n(n^n)-n^n]-1]=0\pmod 9$, or

$n(n(n^n))-n(n^n)=0\pmod 9$

□ Find all primes p such that $\frac{2^{p-1}-1}{p}$ is a square.

Solution

Obviously $p=2$ doesn't work, so assume $p>2$, then $p=2k+1$. We have $2^{2k}-1=n^2p=(2^k+1)(2^k-1)$. Then one of the factors is a perfect square, and the other p times a perfect square, because they're coprime. If 2^k-1 is a perfect square we have $k=1$, as for bigger k it's a contradiction mod 4. Also $k=1,p=3$ works. If 2^k+1 is a perfect square, $2^k+1=a^2\Rightarrow 2^k=(a+1)(a-1)$, from which $a-1=2,a+1=4,2^k=8,k=3,p=7$ as $a-1,a+1$ are both powers of 2. $p=7$ is the only other solution.

□ ABC is an acute-angled triangle with $\measuredangle A=30$. H is the orthocenter and M is the midpoint of BC. T is a point on HM such that $HM = MT$. Show that $AT = 2BC$.

Solution

Of course you mean that T is such that $H \neq T$.

We use the following lemma: let ABC be a triangle, H its orthocenter, O the circumcenter and A' the point on the circumcircle opposite A. Then HA' and BC bisect each other, that is, $HBA'C$ is a parallelogram. To prove it, note that BH, CH are perpendicular to AC, AB respectively, and $A'C$ is perpendicular to AC because AA' is a diameter; similarly $A'B$ is perpendicular to AB. Then $BH, A'C$ and $CH, A'B$ are parallel.

The converse is clearly true, because if we take M such that $BM = MC, HM = MT$, then T coincides with A'. Then T is opposite A through O, that is, AT is a diameter. Also, since $\measuredangle A = 30$, we have $\measuredangle BOC = 60$ and $BO = OC$, from which BOC is equilateral and BC is a radius. We're done.

□ $\sqrt{\dfrac{x-1977}{23}} + \sqrt{\dfrac{x-1978}{22}} + \sqrt{\dfrac{x-1979}{21}} = \sqrt{\dfrac{x-23}{1977}} + \sqrt{\dfrac{x-22}{1978}} + \sqrt{\dfrac{x-21}{1979}}$

Solution

Set $y = 2000 + x$ and $g(y,t) = \sqrt{1 + x/t}$. It's clear that we need to show that $x = 0$ is the only solution to

$g(y,21) + g(y,22) + g(y,23) = g(y,1977) + g(y,1978) + g(y,1979)$

It is not difficult to show that $g(y,21) > g(y,1978)$ for $x > 0$ and the inequality is reversed for $x < 0$. The answer follows easily from this.

□ Prove that the Generalized Binomial Coefficients defined as:

$\binom{n}{k}_C = \dfrac{\prod_{i=1}^{n} C_i}{\left(\prod_{i=1}^{k} C_i\right)\left(\prod_{i=1}^{n-k} C_i\right)}$ for $1 \leq k \leq n$ are all integers,

where $\{C_n\}_{n=1}^{\infty}$ is a sequence of positive integers such that $\gcd(C_m, C_n) = C_{\gcd(m,n)}$.

Solution

Let p be an arbitrary prime. for each $i \geq 1$ let m_i (if it exists) be the smallest positive integer such that $p^i | C_{m_i}$. then if $p^i | C_k$, where $k = qm_i + r$, then $p^i | (C_{m_i}, C_{qm_i+r}) = C_{(m_i, qm_i+r)} = C_{(m_i,r)}$, so $r = 0$ (else (m_i, r) contradicts the minimality of m_i. hence the only k for which $p^i | C_k$ are the multiples of m_i, and, in general, the number of C_j with $j \leq N$ for which $p^i | C_j$ is $[\frac{N}{m_i}]$.

this means that, in general, the highest power of p dividing $\prod_{i=1}^{N} C_i$ is $\sum_{j=1}^{\infty} [\frac{N}{m_j}]$.

so we need to show $\sum_{j=1}^{\infty} [\frac{n}{m_j}] \geq \sum_{j=1}^{\infty} [\frac{k}{m_j}] + \sum_{j=1}^{\infty} [\frac{n-k}{m_j}]$

this is evident from the general fact that

$[\frac{n}{r}] \geq [\frac{k}{r}] + [\frac{n-k}{r}]$

□ What are the last three digits of $2003^{2002^{2001}}$?

Solution

The remainder of $2003^{2002^{2001}}$ when divided by 1000 is the same as the remainder of $3^{2002^{2001}}$ when divided by 1000, since $2003 \equiv 3 (\mathrm{mod} 1000)$. We will try to find positive integer n such that $3^n \equiv 1 (\mathrm{mod} 1000)$ and then express 2002^{2001} in the form of $nk + r$ so that

$$2003^{2002^{2001}} \equiv 3^{nk+r} \equiv (3^n)^k.3^r \equiv 3^r (\mathrm{mod} 1000)$$

Now,

$$3^{2m} = (10-1)^m = (-1)^m + 10m(-1)^{m-1} + 100\frac{m(m-1)}{2}(-1)^{m-2} + ... + 10^m$$

After the first 3 terms of the expansion, all the remaining terms are divisible by 1000, so letting $m = 2q$, we have

$$3^{4q} \equiv 1 - 20q + 100q(2q-1) (\mathrm{mod} 1000)$$

Using this, we can check that $3^{100} \equiv 1(\mod 1000)$, now we want to find the remainder when 2002^{2001} divided by 100. Now, $2002^{2001} \equiv 2^{2001}(\mathrm{mod}100) \equiv 4.2^{1999}(\mathrm{mod}4.25)$, so we will investigate powers of 2 modulo 25. Note that $2^{10} \equiv -1(\mathrm{mod}25)$, so we have

$$2^{1999} \equiv (2^{10})^{199}.2^9 \equiv 13(\mathrm{mod}25)$$

Thus,$2^{2001} = 4.13 \equiv 52(\mathrm{mod}100)$. Therefore, 2002^{2001} can be written as $100k + 52$ for some integer k, so

$$2003^{2002^{2001}} \equiv 3^{52}(\mathrm{mod}1000) \equiv 1 - 20.13 + 1300.25 \equiv 241(\mathrm{mod}1000)$$

using the equation above. Hence, the last 3 digits is 241

□ Evaluate $\lim_{x\to\infty} \frac{1}{n^4} \prod_{j=1}^{2n} (n^2 + j^2)^{\frac{1}{n}}$

Solution

Let $A = \lim_{n\to\infty} \frac{1}{n^4} \prod_{j=1}^{2n} (n^2 + j^2)^{\frac{1}{n}}$

Taking the factor in front and distributing it into the product we get,

$A = \lim_{n\to\infty} \prod_{j=1}^{2n} \left(1 + \frac{j^2}{n^2}\right)^{\frac{1}{n}}$

$\ln A = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{2n} \ln\left(1 + \frac{j^2}{n^2}\right)$

Interpreting this as a Riemann Sum we have

$\ln A = \int_0^2 \ln(1 + x^2)dx$

Using integration by parts (details omitted) we find that $\ln A = 2\ln(5) + 2\tan^{-1}(2) - 4$

So $A = 25e^{(2\tan^{-1}(2)-4)}$

– Let $a, b \in N^*$ and $n \in N$ with $n \geq 2$. Prove that there exists $n \in N^*$ so that $\dfrac{ab + x^n}{a + b + x} \in N^*$.

□ Let n, k be positive integers and let $F(n, k) = 1$ when $n|k$, $n < k = 2$ when $k|n$, $k < n = 3$, otherwise

Let m be a positive integer, find $\sum_{1\leq n,k\leq m} F(n, k)$

Solution

All the values where the function equals 1 are

n = 1, k = 2 to m ($\lfloor\frac{m}{1}\rfloor - 1$ times) n = 2, k = 4 to m or m - 1 ($\lfloor\frac{m}{2}\rfloor - 1$ times) $\vdots$

All the values where the function equals 2 can be obtained by switching the values of n and k when the function equals 1.

The total number of possibilities shown so far is $2(\lfloor\frac{m}{1}\rfloor + \lfloor\frac{m}{2}\rfloor + \cdots + \lfloor\frac{m}{m}\rfloor - m)$

To get the number of values where the function equals 3, we subtract this from the total, which is m^2.

Adding all these values, we get

$3(\lfloor\frac{m}{1}\rfloor + \lfloor\frac{m}{2}\rfloor + \cdots + \lfloor\frac{m}{m}\rfloor - m) + 3(m^2 - 2(\lfloor\frac{m}{1}\rfloor + \lfloor\frac{m}{2}\rfloor + \cdots + \lfloor\frac{m}{m}\rfloor - m) = 3(m^2 - (\lfloor\frac{m}{1}\rfloor + \lfloor\frac{m}{2}\rfloor + \cdots + \lfloor\frac{m}{m}\rfloor) + m)$

□ Determine the prime numbers a, b, c so that the number $A = a^4 + b^4 + c^4 - 3$ is also prime.

Solution

$3|A$ if $a \neq b \neq c \neq 3$ so WLOG $a = 3$. $4|A$ if $a \neq b \neq c \neq 2$ so WLOG $b = 2$. $A = 94 + c^4$ we know that if $c > 5$ then $c^4 \equiv 1 \mod 5$ and $5|A$, So $c = 5$ and $A = 719$. And I do'nt knwo wheter it is prime or not ;)

$\square$ Let n be a natural number. Prove that neither 10^n nor 10^n+3 can be written as a sum of the squares of three prime numbers.

Solution

$10^n \equiv 0 \mod 4$ if $n > 1$ but $p^2 \equiv 1 \mod 4$ when it is odd and 0 for 2 so ${q_1}^2+{q_2}^2+{q_3}^2 \equiv 3,2 \mod 4$ so 10^nt can't expressed as sum of squares of three prime nuber . for 10 is abvious . for 10^n+3 we consider both side mod $3.10^n+3 \equiv 1 \mod 3$, but ${q_1}^2+{q_2}^2+{q_3}^2 \equiv 0,2 \mod 3$ and so no. – Let $A_0B_0C_0$ be a triangle and P a point. Define a new triangle whose vertices $A_1B_1C_1$ as the feet of the perpendiculars from P to B_0C_0, C_0A_0, A_0B_0, respectively. Similarly, define the triangles $A_2B_2C_2$ and $A_3B_3C_3$. Show that $A_3B_3C_3$ is similar to $A_0B_0C_0$.

And if we do the same for a n-gon, what do we obtain? – Prove that $(p-1)!+1$ is not power of p where p is aprime number.(exept $2,3,5$) – Determine all the couples of positive integers (a,b) such that 2^a+3^b is a perfect square. – We have a regular n-gon $A_1A_2...A_n$. At each vertex, we write one of the numbers $1,2,3,...,n$ and no two vertices have the same number. Let the number written at A_n be called B_n. **(a)** Find the maximum of $\sum_{i=1}^{n}|B_i-B_{i+1}|$ where $B_{n+1}=B_1$ **(b)** For how many arrangements are the maximum in **(a)** attained? – Solve the $a^b+1=b^a$ in natural numbers. – Find all functions $f: N_0 \to N_0$, (where N_0 is the set of all non-negative integers) such that $f(f(n))=f(n)+1$ for all $n \in N_0$ and the minimum of the set $\{f(0), f(1), f(2)\cdots\}$ is 1. – Let a real number x such that $-1 \le x \le 1$ and a positive integer n. Show that the function $f_n(x)=cos(n.\arccos x)$ can be written as a polynomial $P(x)$ such that $\deg P(x)=n$ and the coefficient of highest degree monomial of $P(x)$ is equal to 2^{n-1}. – a) For which nonnegative integers a, b, c is $4^a+4^b+4^c$ a perfect square? b) For which nonnegative integers n is $n2^{n-1}+1$ a perfect square? — Let $X=[1,16]\cap\mathbb{N}$. Please divide X into 2 parts A, B such that $|A|=|B|$ and $\sum_{i\in A} i^2=\sum_{j\in B} j^2$.

$\square$ Solve the equation $x\sqrt{1-x^2}-y\sqrt{1-y^2}=1$

Solution

We have $|x|,|y|\le 1$.

Put $x=\cos X$, $y=\cos Y$. Then $\sqrt{1-x^2}=\sin X$.

It becomes $cosXsinX-cosYsinY=1$.

or $\sin 2X-\sin 2Y=2$.

Thus $\sin 2X=1$, $\sin 2Y=-1$, where it is easy to find X,Y and thus x,y as $2^{-1/2}$, $-2^{-1/2}$.

$\square$ Let A be a set of 20 integers chosen from the set $B=\{1,4,7,...,100\}$. Prove that there must be two distinct integers in A with sum 104.

Solution

We want to divide the set B of 34 elements into two distinct subset of 20 and 14 elements. Now suppose to put numbers for which the sum is 104 into different subset. There are 16 couples. This can be done for only 14 elements since in the subset A there will be at least 2 distinct integers for which the sum is 104.

$\square$ Let ΔABC an equilateral triangle and P a point on the circumscribed circle of the triangle. If the circle radius is $r=1$ show that $PA^2+PB^2+PC^2=6$.

Solution

Since P is on the circumcircle, say on minor arc AB, then $PACB$ is a cyclic quadrilateral. Applying Ptolemy's theorem we have

$$PC=PA+PB$$

Also because $PACB$ is cyclic opposite angles are supplementary implying $\angle APB=\frac{2\pi}{3}$. From special right triangles or otherwise we know $AB = \sqrt{3}$. Using the law of cosines on triangle APB we have

$$3 = PA^2 + PAPB + PB^2$$

which implies

$$6 = 2PA^2 + 2PAPB + 2PB^2 = PA^2 + PB^2 + (PA + PB)^2 = PA^2 + PB^2 + PC^2$$

as desired. – Let S be a set $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{50}\}$. Choose 7 distinct fraction in S such that the sum of the 7 fraction is 1 . –

Find all integers $n \geq 2$ and prime numbers p such that $n^{p^p} + p^p$ is prime. – We have n positive real numbers where their sum is 1976. What is the largest product of these n positive real numbers. – Let $C_n = (n+4)C_{n-1} - 4nC_{n-2} + (4n-8)C_{n-3}$ for $n \geq 3$ and $C_0 = 2, C_1 = 3, C_2 = 6$.

What is C_n? – let $p(n)$ be defined by the function that maps the positive integers n to the product of its digits (i.e $p(1123) = 1*1*2*3 = 6$, $p(31) = 3$, $p(2005) = 0$). find all postive integers n so that

$$11p(n) = n^2 - 2005$$

– Determine all functions $f : \mathbb{R} - [0, 1] \to \mathbb{R}$ such that

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}.$$

– Let n, k be positive integers such that $n^k > (k+1)!$ and consider the set

$$M = \{(x_1, x_2, \ldots, x_n), x_i \in \{1, 2, \ldots, n\}, i = \overline{1, k}\}.$$

Prove that if $A \subset M$ has $(k+1)! + 1$ elements, then there are two elements $\{\alpha, \beta\} \subset A$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ such that

$$(k+1)! \,|(\beta_1 - \alpha_1)(\beta_2 - \alpha_2) \cdots (\beta_k - \alpha_k) \ .$$

—- Show that

$$\sum_{i=1}^{n+1} \frac{2^i}{i} \cdot \binom{n}{i-1} = \frac{3^{n+1} - 1}{n+1}.$$

Solution

$\sum_{i=1}^{n+1} \frac{2^i}{i} \ {}_nC_{i-1} = \sum_{r=0}^{n} \frac{2^{r+1}}{r+1} \ {}_nC_r$ where $r = i - 1$

$= \sum_{r=0}^{n} 2^{r+1} \int_0^1 x^r dx \cdot \ {}_nC_r$

$= 2 \int_0^1 \sum_{r=0}^{n} \ {}_nC_r (2x)^r dx$

$= 2 \int_0^1 (1 + 2x)^n dx$

$= 2 \left[\frac{(1+2x)^{n+1}}{2(n+1)}\right]_0^1 = \frac{3^{n+1}-1}{n+1}$ Here is a standard solution.

$\sum_{i=1}^{n+1} \frac{2^i}{i} \ {}_nC_{i-1}$

$= \sum_{i=1}^{n+1} \frac{2^i}{i} \frac{n!}{(i-1)!(n-i+1)!}$

$= \sum_{i=1}^{n+1} 2^i \cdot \frac{(n+1)!}{i!(n-i+1)!} \cdot \frac{1}{n+1}$

$= \frac{1}{n+1} \sum_{i=1}^{n+1} \ {}_{n+1}C_i \ 2^i$

$= \frac{1}{n+1} \left(\sum_{i=0}^{n+1} \ {}_{n+1}C_i \ 2^i - \ {}_{n+1}C_0 \ 2^0\right)$

$= \frac{1}{n+1}\{(1+2)^{n+1} - 1\} = \frac{3^{n+1}-1}{n+1}$

n is a positive integer and $a_1 + a_2 + ... + a_n = 1$. $(a_1, a_2, ..., a_n > 0)$

Let A be the minimum value of n numbers $\frac{a_1}{1+a_1}$, $\frac{a_2}{1+a_1+a_2}$, ... $\frac{a_n}{1+a_1+a_2+...+a_n}$.

When a_1, a_2, ... a_n vary, what is the largest possible value of A? – If p is a prime number , l, a are natural numbers and the number mp is even prove

that $(1+p^a l)^{mp} = 1+mp \cdot p^a l + Mp^{2a}$ where M is sum of positive integers. – Let $N_0 = \{0, 1, 2 \cdots\}$. Find all functions: $N_0 \to N_0$ such that:

(1) $f(n) < f(n+1)$, all $n \in N_0$;

(2) $f(2) = 2$;

(3) $f(mn) = f(m)f(n)$, all $m, n \in N_0$. – Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that: For all a and b in $\mathbb{Z} - \{0\}$, $f(ab) \geq f(a) + f(b)$. Show that for all $a \in \mathbb{Z} - \{0\}$ we have $f(a^n) = nf(a)$ for all $n \in \mathbb{N}$ if and only if $f(a^2) = 2f(a)$ – Find all function $f : \Re \to \Re$ such that

$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$

for all $x, y, z, t \in \mathbb{R}$ – Let $\mathbb{N}_0 = \{0, 1, 2 \cdots\}$. Does there exist a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that:

$$f^{2003}(n) = 5n, \forall n \in \mathbb{N}_0$$

where we define: $f^1(n) = f(n)$ and $f^{k+1}(n) = f(f^k(n))$, $\forall k \in \mathbb{N}_0$? – Let F be the set of all fractions $\frac{m}{n}$, where m and n are positive integers such that $m+n \leq 2005$. Find the largest number $f \in F$ such that $f < \frac{16}{23}$. – Consider a real poylnomial $p(x) = a_n x^n + ... + a_1 x + a_0$. (a) If $\deg(p(x)) > 2$ prove that $\deg(p(x)) = 2 + deg(p(x+1) + p(x-1) - 2p(x))$. (b) Let $p(x)$ a polynomial for which there are real constants r, s so that for all real x we have

$$p(x+1) + p(x-1) - rp(x) - s = 0$$

Prove $\deg(p(x)) \leq 2$. (c) Show, in (b) that $s = 0$ implies $a_2 = 0$. – If $\{a_n\}_{n \geq 0}$ is an arithmetic sequence where the first term ant it's ratio are pozitive, then $\frac{1}{a_1 a_2} + \frac{1}{a_3 a_4} + ... + \frac{1}{a_{2n-1} a_{2n}} < \frac{n}{a_0 a_{2n}}$ for any $n \in \mathbb{N}^*$. – Find all integers $n \geq 2$ such that $x_1 x_2 + x_2 x_3 + ... + x_{n-1} x_n \leq \frac{n-1}{n}(x_1^2 + x_2^2 + ... + x_n^2)$ for all $x_1, x_2, ..., x_n \in R^+$ – Consider the equation $x^3 = 3x + p$ and define $f(p)$ as follows:

*if the equation has 3 real roots, $f(p)$ is the product of the greatest and smallest roots. *if the equation has 1 real root, $f(p)$ is the square of this root.

Determine the minimum of $f(p)$ as p ranges over all real numbers. —- If a, b, c are positive integer satisfying

$2ab + 2ac + 2bc = abc$

Find the ordered triples (a, b, c) – Let $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ s.t.

$$f(x)f(y) + f(x) + f(y) = f(xy) + a, \ \forall x, y \in \mathbb{R} .$$

Determine all simultaneously continuous and bijective functions which satisfy the above condition. —- Prove that $4x^3 - 2x^2 - 15x + 9$ and $12x^3 + 6x^2 - 7x + 1$ has three distinct real roots – Find the real numbers p, q, and t satisfying the following equality.

$$\{(p^2+1)t^2 - 4t + p^2 + 5\}^2 + \{t^2 - 2qt + q^2 + \sqrt{3}\}^2 = 4$$

— Let $c \geq 1$ be an integer, and define the sequence $a_1, a_2, ...$ by $a_1 = 2$ and

$$a_{n+1} = ca_n + \sqrt{(c^2-1)(a_n^2-4)}$$

for positive integer n. Prove that a_n is integer for all n – If $x_i > 0$ and $x_i y_i - z_i^2 > 0$ for $i \leq n$, then

$$\frac{n^3}{(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i) - (\sum_{i=1}^n z_i)^2} \le \sum_{i=1}^n \frac{1}{x_i y_i - z_i^2}$$

Prove this inequality for n=2, and then also generally. — Solve in $\mathbb{R}$ the following equation: $\left(2\cos^2\frac{\pi}{24}\right)^x + \left(4\cos\frac{5\pi}{12}\right)^2 \le \frac{5\sqrt{3}-3}{2\sqrt{2}}$.

Solution

Remember $2\cos^2 y - 1 = \cos 2y$

This lets us write pi/24 and 10pi/24 in terms of the cosines of pi/12 and 10pi/12, which we know (30, 300 deg). Find all integers $n \ge 2$ and prime numbers p such that $n^{p^p} + p^p$ is prime.

$\square$ If n divides one Fibonacci number (the sequence $1, 1, 2, 3, 5, 8, 13, 21, ...$), show that it will divide infinitely many of them

Solution

We can prove that $F_k | F_{lk}$ where F_x denote xth Fibonacci number. As we know

$$F_k = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^k - (\frac{1-\sqrt{5}}{2})^k)$$

and

$$F_{lk} = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{lk} - (\frac{1-\sqrt{5}}{2})^{lk})$$

in the futur we will use $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$ so as we know number like $a^x + b^x$ and $a^x b^x$ are integers ....($*$). Now let's divide F_{lk} and F_k that is equal to

$$\frac{a^{lk} - b^{lk}}{a^k - b^k}$$

wich gives

$$a^{kl-1} + a^{kl-2}b^k + ... + a^k b^{kl-2} + b^{kl-1}$$

wich is integer because of ($*$). So there you go... This is much stronger stuff :D

$\square$ Find 2^{2006} positive integers satisfying the following conditions. (i) Each positive integer has 2^{2005} digits. (ii) Each positive integer only has 7 or 8 in its digits. (iii) Among any two chosen integers, at most half of their corresponding digits are the same.

Solution

Define $S_1 = \{77, 78\}$ Define the inverse of an element to be 7 ->8, 8->7 Define the S_i' to be the inverse of S_i, e.g. $S_1' = \{88, 87\}$ Define S_i^2 to be writing itself again, e.g. $S_1^2 = \{7777, 7878\}$ Define $S_i + S_i'$ to be writing after corresponding element, e.g. $S_1 + S_1' = \{7788, 7887\}$ Define $T_i = \{x | x \in S_i or x \in S_i'\}$ Define $S_{i+1} = \{x | x \in S_i^2 or x \in S_i + S_i'\}$

Now I claim that T_{2005} is the required set

First, each no. has 2^{2005} digits, which is obvious. Having 7 or 8 as the only digits is trivial as well.

Now it suffices to prove that T_{2005} fulfills condition 3. We will proceed by induction.

in T_1 obviously (iii) is fulfilled Assume when i =k, T_i is true for condition iii, then when i=k+1, $T_{k+1} = S_i^2 \cup S_i + S_i'$ Since $T_i = S_i \cup S_i'$ and it fulfills condition 3, it means that for elements in T_{k+1}, The first half has at most half of the digits different , the second half has at most half of the digits different. So any two elements of T_i has at most half of the digits different Induction done. Thus T_{2005} fulfills the condition

Another way Construction: To find 2^{n+1} positive integers satisfying the following conditions: (i) Each positive integer has 2^n digits. (ii),(iii) Take the 2^n numbers satisfying the conditions (i) Each positive integer has 2^{n-1} digits. (ii),(iii)

First get 2^n numbers by replacing each "7" by "77" and each "8" by "88". Then get another 2^n numbers by replacing each "7" by "78" and each "8" by "87".

For the case where the numbers have only one digit, the numbers "7" and "8" satisfy the conditions.

Proof: Suppose we have 2^n-digit numbers A and B with exactly half their digits in common. If we perform the same replacement on both of them (e.g. 7–>78 and 8–>87) clearly the new numbers will share exactly half their digits. If we perform different replacements on each of them (e.g. 7-78, 8-87 on A; 7-77 and 8-88 on B), the previously agreeing digits will each contribute one agreement and the previously disagreeing digits will each contribute one agreement to the new pair of numbers. Thus the new numbers will share half their digits.

Suppose we have 2^n-digit numbers with no digits in common. Clearly if we perform the same replacement on both numbers the resulting numbers will share no digits, and if we perform different replacements, the resulting numbers will share half their digits.

For our two numers for the case n=0, no digits are in common. It follows from induction that when we carry out our construction, every pair of numbers will share exactly half their digits or none of their digits, so we can perform the construction 2005 times to get the desired set of numbers.

☐ Find the number of ways in which 5^n could be expressed as a product of 3 factors.

Solution

It is not hard to prove that the number of solutions of $x+y=n$ and $x \leq y$ is $\lfloor \frac{n}{2} \rfloor + 1$ we will going to need that.

We have to find number of triplets x, y, z of nonnegative integers such that $x \leq y \leq z$ and $x+y+z=n$, in that case (x, y, z) denote the powers of 5 in expression.Let S be set of all that triples. We see that $x \leq \lfloor \frac{n}{3} \rfloor$. Now for $k \in \{0, 1, ..., \lfloor \frac{n}{3} \rfloor\}$ we make set A_k of all triples $k = x \leq y \leq z$ such that $x+y+z=n$. So $S = A_1 \cup A_2 \cup ... \cup A_{\lfloor \frac{n}{3} \rfloor}$ and for $i \neq j$ stands $A_i \cap A_j = \phi$, so $|S| = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} |A_k|$,and from that all we need is to compute $|A_k|$.

Let $a = y - k$ and $b = z - k$ then $a + b = n - 3k$ and $0 \leq a \leq b$ so as we said on the beginning we have $\lfloor \frac{n-3k}{2} \rfloor + 1$ such pairs a, b and every pair define one pair y, z (and one solution $k = x \leq y \leq z$) so $|A_k| = \lfloor \frac{n-3k}{2} \rfloor + 1$ and from that we have

$$|S| = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} |A_k| = \lfloor \frac{n}{3} \rfloor + \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} \lfloor \frac{n-3i}{2} \rfloor$$

And that's it, I just can't calculate this last sum...

☐ Given $\sum_{i=1}^{n} x_i = n$ for $x_i \in \mathbb{R}$ and
$\sum_{i=1}^{n} x_i^4 = \sum_{i=1}^{n} x_i^3$
Solve the system of equation for x_i .

Solution

We have

$$\sqrt[4]{\frac{\sum_{i=1}^{n} x_i^4}{n}} \geq \sqrt[3]{\frac{\sum_{i=1}^{n} x_i^3}{n}}$$

by Power means, which gives

$$n \cdot \left(\sum_{i=1}^{n} x_i^4 \right)^3 \geq \left(\sum_{i=1}^{n} x_i^3 \right)^4$$

or

$$n \geq \sum_{i=1}^{n} x_i^3.$$

So,

$$1 = \frac{\sum_{i=1}^{n} x_i}{n} \geq \sqrt[3]{\frac{\sum_{i=1}^{n} x_i^3}{n}}$$

which is the power mean inequality "the wrong way around".

So, equality must hold, and all variables must be equal $x_i = 1$.

□ Let $F(x) \in Z[x]$, and $F(1), F(2), ..., F(n)$ is all not divisible by n. Is it necessary that $F(x)$ has no integer roots?

Solution

Suppose x is an integer root of f. Then $f(x) = 0$ mod n. However, consider the function modulo n. Then because x^k mod n $= r^k$ mod n, where r is the residue of x, it follows that F(r) is not zero mod n. Contradiction. So there are no integer roots of f.

□ Let $p(x)$ be a polynomial with integer coefficients. If $p(0)$ and $p(1)$ are odd then show that $p(x)$ does not have any integer root.!

Solution

$f(ax+b) \equiv f(b) \pmod{a}$ (basic properties of congruences) So $f(2x) \equiv f(0) \equiv 1 \pmod{2}$ $f(2x+1) \equiv f(1) \equiv 1 \pmod{2}$ So f(x) is odd for all integer x. But 0 is even. Contradiction. So there are no integer solutions. Another way The sum of the coefficents as well as the constant coefficent is odd.

Suppose $f(x) = \sum_{i=0}^{n} a_i x^i$ and that x is an integer root.

If x is even, then $f(x) = \sum_{i=0}^{n} a_i x^i = (\sum_{i=1}^{n} a_i x^i) + a_0$ which is odd. (every term in the brackets has a factor x which is even.)

If x is odd, then $f(x) = (\sum_{i=1}^{n} a_i x^i) + a_0$ is odd, (because, if a_i is odd, then $a_i x^i$ is odd, and if a_i is even, then $a_i x^i$ is even. So the parity of the stuff in the brackets is the same as the parity of $\sum_{i=1}^{n} a_i$, which is even. So an even + a_0 is odd as required.)

Contradiction.

□ Let D be the set of positive reals different from 1 and let n be a positive integer. If for $f : D \to \mathbb{R}$ we have $x^n f(x) = f(x^2)$, and if $f(x) = x^n$ for $0 < x < \frac{1}{1989}$ and for $x > 1989$, then prove that $f(x) = x^n$ for all $x \in D$.

Solution

By induction we have $f(x^{2^n}) = f(x) * x^{(2^n-1)n}$

if $1/1989 \leq x < 1$ there exists n $x^{2^n} < 1/1989$ and then $f(x^{2^n}) = x^{2^n n} * f(x) * x^{2^n-1}$ and then $f(x) = x^n$

The same way, if $1 < x < 1989$ ther existe n such as $1989 < x^n$

□ $(2+\sqrt{3})^k = 1 + m + n\sqrt{3}$ with m and n integrers and k odd.

Prove that m is a perfect square

Solution

We have $(2-\sqrt{3})^k(2+\sqrt{3})^k = 1 = (1+m+n\sqrt{3})(2-\sqrt{3})^k = (1+m+n\sqrt{3})(1+m-n\sqrt{3})$

$\Rightarrow (1+m)^2 - 3n^2 = 1 \Rightarrow m^2 + 2m = 3n^2$

Finally, observe $(1+m+n\sqrt{3})(2+\sqrt{3})^2 = 1 + (6+7m+12n) + (7n+4+4m)\sqrt{3}$.

So k -> k+2 makes (m,n) -> (6 + 7m + 12n , 7n + 4 + 4m)

Now we use induction.

We show that if m is a perfect square for k, the "new m" generated by k+2 will also be a perfect square *

Put $x = 6 + 7m + 4\sqrt{3m^2+6}$ (x is the "new m"). Then x is a root of the equation $x^2 - x(14m + 12) + (m-6)^2 = 0$.

Let x, y be the roots of the above equation. Then $\sqrt{x} + \sqrt{y} = \sqrt{(\sqrt{x}+\sqrt{y})^2} = \sqrt{x+y+2\sqrt{xy}} = 4\sqrt{m}$ which is an integer. But x is an integer root. Because the discriminant is an integer, so too must y be an integer root. We have, for some integer z, $y = (z-\sqrt{x})^2$ so that x must be a perfect square (otherwise, $(z-\sqrt{x})^2$ cannot be an integer)).

So x is a perfect square and the induction is complete.

☐ Prove the following inequality.

$$2^n < \frac{(2n)!}{(n!)^2} < 2^{2n} \ (n = 2, 3, \cdots)$$

Solution

The middle is 2n choose n.

We know $\sum_{r=0}^{n} \binom{n}{r} = 2^n$ by Binomial theorem.

Now, $\binom{2n}{n} < \sum_{r=0}^{2n} \binom{2n}{r} = 2^{2n}$

and $\binom{2n}{n} = \binom{2n-1}{n} + \binom{2n-1}{n-1} > \binom{2n-2}{n} + \binom{2n-2}{n-1} + \binom{2n-2}{n-2} > \cdots$

$\cdots > \sum_{r=0}^{n} \binom{n}{r} = 2^n$, as desired.

☐ Let n be a prime number. Find all $x \in N$ such that $(1^n + 2^n + ... + x^n) + (1^n + 2^n + ... + (n-1)^n) = 1^n + 2^n + ... + (2n-1)^n$

Solution

Well i think is easy to see that $(2n-1)^n$ is far much bigger than $1^n + 2^n + 3^n + ... + n^n$

lets do it by induction...for 2 it happens let say for n-1 happens

the $1^(n-1) + 2^(n-1) + ... + (n-1)^(n-1) < (2n-3)^(n-1)$ lets multiply it by (n-1) and we get:

$(n-1)(2n-3)^(n-1) > (n-1)(1^(n-1) + 2^(n-1) + ... + (n-1)^(n-1))$

$(n-1)(1^(n-1) + 2^(n-1) + ... + (n-1)^(n-1)) > 1^n + 2^n + 3^n + ... + (n-1)^n$

then $(n-1)(2n-1)^(n-1) > (n-1)(2n-3)^(n-1) > 1^n + 2^n + 3^n + ... + (n-1)^n$

and $(n-1)(2n-1)^(n-1) + n^n > 1^n + 2^n + 3^n + ... + n^n$

but $(n-1)(2n-1)^(n-1) < (2n-1)^n$

so the problem reduces that :

$1^n + 2^n + ... + (2n-2)^n + (2n-1)^n > 1^n + 2^n + ... + (2n-2)^n + 1^n + 2^n + 3^n + ... + n^n$

so $2n-1 > k > 2n-2$ but it can't be so for n>1 there's no solution.

the only solution is $n = 1, k = 0$

☐ Find all reals x satisfy $[x^2 - 2x] + 2[x] = [x]^2$

Solution

write $x = k + y$ where k is the integer part and y is the mantissa

$[x^2 - 2x] + 2[x] = [x]^2$

$[y^2 + 2ky + k^2 - 2y - 2k] = k^2 - 2k$

$k^2 - 2k + [y^2 + 2ky - 2y] = k^2 - 2k$

$[y^2 + 2ky - 2y] = 0$

Let $f(y) = y^2 + 2(k-1)y$. We need $f(y) \geq 0$ and $f(y) < 1$, and we want to determine the values of y that give this based on the parameter k.

The roots of f(y) are 0 and 2(k-1). So if $k \leq 0$ the roots are to the left of the y axis and we guarantee $f(y) \geq 0$. If k = 1 then all values of y work. If k=2 or higher then the curve is completely under the x axis. So $k \leq 1$.

For $f(y) < 1$, we require $y^2 + 2(k-1)y - 1 < 0$ in the interval $y\ in [0, 1)$. The quadratic opens up, and the roots of this quadratic are $-(k-1) \pm \sqrt{(k-1)^2 + 1}$. We want the positive root, so we see that y in $[0, \sqrt{(k-1)^2+1} - (k-1))$ is where y should be.

We combine these two conditions, so that $k \leq 1$ and $y \in [0, \sqrt{(k-1)^2+1} - (k-1))$ describes the full nature of x.

☐ Prove that, for $m \neq n$, $(F_m, F_n) = 1$, where $F_k = 2^{2^k} + 1$.

Furthermore, using this result, prove that there exist an infinite number of primes.

Solution

Lemma If $m > n$, then $2^{2^n} + 1$ divides $2^{2^m} - 1$. Proof of lemma: We will use mathmatical induction (by fixing n). For $m = n + 1$, $2^{2^m} - 1 = 2^{2^{n+1}} - 1 = (2^{2^n})^2 - 1 = (2^{2^n} + 1)(2^{2^n} - 1)$ is divisible by $2^{2^n} + 1$. Assume the lemma is true for $m = n + k$, i.e. $2^{2^n} + 1$ divides $2^{2^{n+k}} - 1$. Now for $m = n+k+1$, $2^{2^m} - 1 = 2^{2^{n+k+1}} - 1 = (2^{2^{n+k}})^2 - 1 = (2^{2^{n+k}} + 1)(2^{2^{n+k}} - 1)$ is divisble by $2^{2^{n+k}} - 1 \Rightarrow$ divisible by $2^{2^n} + 1$. Thus the lemma is proved.

Now we go back to the original problem: If $m \neq n$, then $(2^{2^m} + 1, 2^{2^n} + 1) = 1$. Proof Without loss of generality, let $m > n$. By Euclidean Algorithm, $(qx + r, x) = (r, x)$. By our lemma, we can let $x = 2^{2^n} + 1$ and $qx = 2^{2^m} - 1$ for some q. Then $(2^{2^m} + 1, 2^{2^n} + 1) = (qx + 2, x) = (2, x) = 1$ as x is an odd number. The conclusion follows.

Corollary There exists infinitely many primes. Proof of corollary Since $(F_m, F_n) = 1 \quad \forall m \neq n \in \mathbb{N}$, every F_m is either itself prime or has a prime factor other that the ones of other F_n. Therefore, there exists infinitely many primes.

☐ The non-negative real numbers a, b, c, d add up to 1. Prove the inequality $|ab - cd| \leq \frac{1}{4}$.

Solution

Assume the opposite. Let's say $ab - cd > \frac{1}{4}$.

Clearly, $ab > \frac{1}{4}$, which implies $\sqrt{ab} > \sqrt{\frac{1}{4}}$.

By AM-GM, $\frac{a+b}{2} \geq \sqrt{ab}$. The maximum of $a + b$ is 1.

So we have $\frac{1}{2} \geq \sqrt{ab} > \sqrt{\frac{1}{4}}$.

So, $\frac{1}{2} > \frac{1}{2}$. Contradiction.

If we remove the absolute value signs, we have also $cd - ab > \frac{1}{4}$, which leads back to the argument above.

The case for equality is when $a = .5$, $b = .5$, $c = 0$, $d = 0$, or $a = 0$, $b = 0$, $c = .5$, $d = .5$.

☐ How many five-element subsets S of set $A = \{0, 1, 2, ..., 9\}$ are there which satisfy $\{r(x+y)|x, y \in S, x \neq y\} = A$, where $r(n)$ denotes the remainder when n is divided by 10?

Solution

Let $B = \{r(x + y)|x, y \in S, x \neq y\}$. Suppose S contains 0. Then for $x \neq 0$, $r(x + y) = r(x + 0) = x$, so $x \in B$. For $x = 0$, $r(0 + y)|0$ no matter what y we pick since anything divides 0. So any subset containing 0 works. Now suppose S does not contain 0. Consider the case where $x = 1$. Then $r(1+y)|1$, so $r(1 + y) = 1$ and $r(y) = 0$. But, given that $y \leq 9$, this means $y = 0$, a contradiction. So, the number of 5-element subsets S which work is the number of 5-element subsets containing 0, of which there are $\binom{9}{4}$.

☐ P is any point inside a triangle ABC. The perimeter of the triangle $AB + BC + Ca = 2s$. Prove that $s < AP + BP + CP < 2s$.

Solution

Triangle inequality tells us: $AB < AP + BP$, $AC < AP + CP$, $BC < CP + BP$ So:

$$AB + BC + AC < 2AP + 2BP + 2CP \iff s < AP + BP + CP(1)$$

Then, we extend CP, and call D the intersection of CP and AB. Again, we use the triangle inequality: $CP + DP < AC + AD$, and $BP - PD < BD$ So:

$$BP + CP < AC + AB$$

On exactly the same way, we prove that $AP + CP < AB + CB$ and that $AP + BP < AC + BC$ So it follows that:

$$2AP + 2CP + 2BP < 2AB + 2BC + 2AC \iff AP + CP + BP < 2s(2)$$

Putting (1) and (2) together, we get the following inequality:

$$s < AP + BP + CP < 2s$$

□ Given that $P = \{p_1, p_2, ..., p_k\}$ is a set of distinct, not necessarily consecutive primes, prove that $\frac{1}{p_1} + \frac{1}{p_2} + ...\frac{1}{p_k}$ is never integer.

Solution

Take any prime from the list. Let that prime be x. Thn assume that the sum is an integer, and let that integer be equal to y. If you remove x from the list, the sum is now $\frac{yx-1}{x}$. Now take another prime from the list, and let that be z. If we remove this from the list, the sum is now $\frac{z(yx-1)-1}{xz}$. Continuing in this manner, eventually we will have one term left. Its sum must then have a denominator of $p_1p_2p_3 \dots p_k$, where one number is missing from the product. Let that term be called p_i. Then since this term is equal to the sum of the list, we have that $p_i = p_1p_2p_3 \dots p_k$. But since all p_k are prime, this is impossible because it would mean that the product of multiple primes is anothe prime, which cannot happen. Therefore our assumption was false and the prduct cannot be an integer.

□ Let $f : \mathbb{Z} \to \{-1, 1\}$ be a function such that

$$f(mn) = f(m)f(n), \ \forall m, n \in \mathbb{Z}.$$

Show that there exists a positive integer a such that $1 \leq a \leq 12$ and $f(a) = f(a + 1) = 1$.

Solution

Note that $f(1) = 1$. If $f(2) = 1$, we are done. So let $f(2) = -1$. If $f(3) = f(5) = 1$, we are done since $f(4) = [f(2)]^2 = 1$. So let $f(3) = f(5) = -1$. But then $f(9) = f(10) = 1$.

□ If $f(x) = x^4 + 3x^3 + 9x^2 + 12x + 20$ and $g(x) = x^4 + 3x^3 + 4x^2 - 3x - 5$,find the $a(x), b(x)$ of smallest degree such that $a(x)f(x) + b(x)g(x) = 0$

Solution

Firstly: $g(x) = x^4 + 3x^3 + 4x^2 - 3x - 5 = x^4 + 3x^3 + 5x^2 - x^2 - 3x - 5 = x^4 - x^2 + 3x^3 - 3x + 5x^2 - 5 = x^2(x^2 - 1) + 3x(x^2 - 1) + 5(x^2 - 1) = (x^2 - 1)(x^2 + 3x + 5)$ $g(1) = 0$ $g(-1) = 0$ $a(x)f(x) + b(x)g(x) = 0$ $a(1)f(1) + b(1)g(1) = 0$ $a(-1)f(-1) + b(-1)g(-1) = 0$ $a(1) = 0$ $a(-1) = 0$, so $a(x) = (x^2 - 1)a_1(x)$ Next $(x^2 - 1)a_1(x)f(x) + b(x)(x^2 - 1)(x^2 + 3x + 5) = 0$ If $x \neq 1$ and $x \neq -1$ then $a_1(x)f(x) + b(x)(x^2 + 3x + 5) = 0$ Notice, that $f(x) = (x^2 + 3x + 5)(x^2 + 4)$ $a_1(x)(x^2 + 3x + 5)(x^2 + 4) + b(x)(x^2 + 3x + 5) = 0$ $a_1(x)(x^2 + 4) + b(x) = 0$ Polynomial $a_1(x)$ must have as little degree as it possible, so $a_1(x) = c$, $c \neq 0$ $a(x) = c(x^2 - 1)$ $b(x) = -c(x^2 + 4)$ and $c \neq 0$

□ $ABCD$ is a quadrilateral and P, Q are the midpoints of CD, AB, AP, DQ meet at X and BP, CQ meet at Y. Prove that $A[ADX] + A[BCY] = A[PXOY]$.

Solution

First let find the area of BCY and AXD: doing: $\frac{BY}{BP}=m$ and $\frac{DX}{DQ}=n$ we have:

$\frac{\Delta BCY}{\Delta BCP}=\frac{\Delta BCY}{\frac{\Delta BCD}{2}}=m \implies \Delta BCY=m.\frac{\Delta BCD}{2}$

$\frac{\Delta AXD}{\Delta ADQ}=\frac{\Delta AXD}{\frac{\Delta ABD}{2}}=n \implies \Delta AXD=n.\frac{\Delta ABD}{2}$ thus:

$\Delta AXD+\Delta BCY=n.\frac{\Delta ABD}{2}+m.\frac{\Delta BCD}{2}$ and:

$\Box QYPX=\Box ABCD-\Delta CBQ-\Delta ADP-\Delta CYP-\Delta AQX \Rightarrow$

$\Box QYPX=\Box ABCD-\frac{\Delta ABC}{2}-\frac{\Delta ACD}{2}-(1-m).\frac{\Delta BCD}{2}-(1-n).\frac{\Delta ABD}{2}$

$\Rightarrow \Box QYPX=\Box ABCD-\frac{\Box ABCD}{2}-\frac{\Box ABCD}{2}+n.\frac{\Delta ABD}{2}+m.\frac{\Delta BCD}{2}$

$\Rightarrow \Box QYPX=n.\frac{\Delta ABD}{2}+m.\frac{\Delta BCD}{2}$ in consequence:

$\Box QYPX=\Delta AXD+\Delta BCY.$

$\Box$ let A=(1,2,...,99) be a set.50 number are chosen from A,inwich the sum of each two number isnt equal to 99 or 100.

prove that: the 50 chosen number should be : 50,51,...,99

Solution

Suppose we choose an element k $(k\neq 99)$ from A. We know that $99-k$ and $100-k$ cannot be chosen also. Therefore, the elements in A can be paired up as: $(1,98),(2,97),\dots,(49,50),(99)$. We can only take one element from each set of parentheses. Thus, we must choose the element 99 in order to end up with 50 numbers. But then we cannot choose 1 or else 99+1=100. Thus we have to choose 98. Similarily, we can't have 2. So we must take 97. This logic continues all the way down thus forcing us to choose $50,51,\dots 99$.

$\Box$ assume: $a_n=a_{n-1}+\frac{a_{n-2}^2}{a_{n-1}}$ and $b_n=\frac{a_{n+1}}{a_n}$. if b_n is convergant to L, prove: $1<L<\frac{3}{2}$

Solution

Dividing through by a_{n-1}, we get

$b_{n-1}=1+\frac{1}{b_{n-2}^2}$

Taking limits, we get

$L=1+\frac{1}{L^2}$ or $L^3-L-1=0$. This is a cubic, and hence it has a real root. We want to show that there's only one, and that it lies in the desired interval.

Now, $x^3-x=0$ has a maximum for $x\leq 0$ of $\frac{2\sqrt{3}}{9}<1$, so we must have $L>0$

$\frac{d}{dL}L^3-L-1=3L^2-1$, so that it is increasing for $|L|^2>\frac{1}{3}$.

Furthermore, $L^3-L<0$ for $0<L<1$. So $L>1$. Setting $L=\frac{3}{2}$, we see that $L^3-L-1=\frac{27}{8}-\frac{3}{2}-1>0$, so

$1<L<\frac{3}{2}$

as desired. The way to proceed is obvious here, there was just a bunch of boring grunt work to be done.

$\Box$ Let $P_1P_2P_3\dots P_{12}$ be a regular dodecagon. Show that

$$|P_1P_2|^2+|P_1P_4|^2+|P_1P_6|^2+|P_1P_8|^2+|P_1P_{10}|^2+|P_1P_{12}|^2$$

is equal to

$$|P_1P_3|^2+|P_1P_5|^2+|P_1P_7|^2+|P_1P_9|^2+|P_1P_{11}|^2.$$

Solution

Place the 12 points of the regular dodecagon on a circle. Notice P_1P_7 is the diameter which means triangle $P_1P_7P_k$ for $k\in\{2,3,4,5,6,8,9,10,11,12\}$ is a right triangle all with P_1P_7 as the hypotenuse.

So we have

$$|P_1P_2|^2 + |P_1P_8|^2 = |P_1P_3|^2 + |P_1P_9|^2$$

$$|P_1P_4|^2 + |P_1P_{10}|^2 = |P_1P_5|^2 + |P_1P_{11}|^2$$

After cancelling out the above terms we are left with showing

$$|P_1P_6|^2 + |P_1P_{12}|^2 = |P_1P_7|^2$$

which is true by the pythagorean theorem.

□ In how many ways can one choose distinct numbers a and b from 1, 2, 3, ..., 2005 such that a + b is a multiple of 5?

Solution

Consider the set $\mod 5$. First if $a \equiv 0 \mod 5$ then $b \equiv 0 \mod 5$. There are 401 multiples of 5 so there are $\binom{401}{2}$ ways to select a, b. If $a \equiv -1 \mod 5$ then $b \equiv 1 \mod 5$ however these two sets are disjoint so there are 401^2 more ways. Similarly there are 401^2 ways if either a or b are equivalent to $\pm 2 \mod 5$. This accounts for all possible residues so there is a total of $2 \cdot 401^2 + \binom{401}{2} = 401802$ ways to select a, b.

□ Consider an array of numbers of size 8×8. Each of the numbers in the array equals 1 or -1. "Doing a move" means that you pick any number in the array and you change the sign of all numbers which are in the same row or column as the number you picked. (This includes changing the sign of the "chosen" number itself.) Prove that one can transform any given array into an array containing numbers +1 only by performing this kind of moves repeatedly.

Solution

For each square X let the X-cross be the set of squares in the same row or column as X (including X), so that a move changes the sign of the squares of the X-cross for the chosen square X.

We say that a square X is odd if the number of minus signs on its cross is odd, and even otherwise. Now consider the following set of moves: for each square X, apply one move to X if it is odd, and none if it is even. This solves the problem.

To prove this gives the result, do the following: for each square X with a $-$, place a coin with an X written on it, on each square of the X-cross (so each square will, in the end, have as many coins as the number of $-$s on its cross). When you're done, for each square apply to it as many moves as the number of coins on it (and note that the above set of moves is the same as this taken mod 2, so it's equivalent). The point is that the moves of the X-coins add up to changing just X, since each square not on the X-cross has just 2 X-coins on its cross, each square on the X-cross but different from X has 8 X-coins on its cross, and X has 15 X-coins on its cross. Then it is obvious that this gives the required result.

□ Show that one can find 50 distinct positive integers such that the sum of each number and its digits is the same.

Solution

we build a number a_1 accordingly to a certain rule. It goes like this: its first digit to the right is 1. Then goes 9 and 0, so that the ending is 091. The next digit is again 1 and then there have to be a little more nines - exactly a thousand and finely again 0, so now we have a longer ending: $0\underbrace{9...9}_{1000}1091$.

Then the story goes from the begining, that means we add 1, an appropriate number of 9's and finely 0. The question is: what is the "appropriate" number of 9's? Well, that depends on what position is the 1 before the 9's. If it stands for $1 \cdot 10^k$ in a decimal system, then we add 10^k nines. We perform this operation (at least) 49 times, so that the number has (at least) 49 one's. Well, it's a bit long number to write... :oops: Now, let's say this is a_1 and let the sum of the digits of a_1 will be $S(a_1)$. Let's also write: $f(a_1) = a_1 + S(a_1)$. Note that: $f(a_1) = f(a_1 + 9)$, because the ending of $a_1 + 9$ is 100 and that means that $S(a_1 + 9) = S(a_1) - 9$. Similarly, $f(a_1 + 9000) = f(a_1)$, because the ending of $a_1 + 9000$ is $1\underbrace{0...00}_{1000}0091$ and so $S(a_1 + 9000) = S(a_1) - 9000$. Now its obvious how to continue this argument. We see that $f(a_1 + 9 \cdot 10^k) = f(a_1)$, where $k + 1$ is a number of a position for digit 1 in a_1. Of course, the solution above means that there are not only 50 numbers, but there are n distinct numbers such that the sum of each number and its digits is the same, for $n \in \mathbb{N}$.

□ Given 101 distinct non-negative integers less than 5050 show that one can choose four a, b, c, d such that $a + b - c - d$ is a multiple of 5050

Solution

We have those integers in an increasing order: $0 \leqslant a_1 < a_2 < ... < a_{101} < 5050$. Let us look at the differences between consequent terms of the sequence a_n. If we suppose all those differences to be pairwise different, we have: $a_{101} = a_1 + (a_2 - a_1) + (a_3 - a_2) + ... + (a_{101} - a_{100}) \geqslant 0 + 1 + 2 + 3 + ... + 100 = 5050$ - a contradiction. So there have to be such i, j that: $a_i - a_{i+1} = a_k - a_{k+1}$, whence $a_i + a_{k+1} - a_{i+1} - a_k = 0$. And that's it.

□ Find all $f : \mathbb{R} \to \mathbb{R}$

(1) there are only finitly many $s \in \mathbb{R} : f(s) = 0$ (2) $\forall x, y \in \mathbb{R} : f(x^4 + y) = x^3 f(x) + f(f(y))$

Solution

Taking $x = 0$, we get $f(f(y)) = f(y)$. Taking $x = 1, y = 0$ we get $f(0) = 0$. Taking $y = 0$, we get $f(x^4) = x^3 f(x)$. Hence, if x is a zero of f, then x^4 is a zero as well. Since there are only finitely many zeroes, we must have $f(x) = 0 \Rightarrow x \in \{0, 1\}$. But we can't have $f(1) = 0$ either since then $f(2) = f(1) + f(1) = 0$, $f(3) = f(1) + f(f(2)) = 0$, ... contradiction. So if $f(x) = 0$, then $x = 0$. Now take $t = f(x^4) - x^4$ (and keep x fixed). Then $f(x^4) = f(f(x^4)) = f(x^4 + t) = x^3 f(x) + f(t) = f(x^4) + f(t)$ and hence $f(t) = 0, t = 0, f(x^4) = x^4$. So for all positive x, we have $f(x) = x$. It's easy to see that we must have $f(x) = x$ for all negative x as well, and we're done. – Given $\triangle ABC$, let D, E, F be the points on AB, BC, and CA, respectively, such that $AD : DB = BE : EC = CF : FA = 2 : 1$. Next, take points X, Y, and Z on DE, EF, and FD, respectively, such that $DX : XE = EY : YF = FZ : ZD = 2 : 1$. Prove that $\triangle ABC$ and $\triangle XYZ$ are similar. – Given $n \equiv 3(\mod 6)$ objects $a_1, a_2, \ldots, a_n$, show one can find $\frac{\binom{n}{2}}{3}$ triples (a_i, a_j, a_k) such that every pair $(a_i, a_j)(i \neq j)$ appears in exactly one triple. – Call a number $a - b\sqrt{2}$ with a and b both positive integers tiny if it is closer to zero than any number $c - d\sqrt{2}$ such that c and d are positive integers with $c < a$ and $d < b$. Three numbers which are tiny are $1 - \sqrt{2}$, $3 - 2\sqrt{2}$, and $7 - 5\sqrt{2}$. Without using a calculator or computer, prove whether or not each of the following is tiny:

$$(a) 58 - 41\sqrt{2}, \qquad (b) 99 - 70\sqrt{2}.$$

□ can anyone find a relatively simple method (no calculus) to find the coordinates of a point Q (x,y) which is the rotation of point P (a,b) through an angle of l about the origin? (x,y) in terms of a,b,l

Solution

Let $a = r\cos\theta$, $b = r\sin\theta$ (where $r = \sqrt{a^2 + b^2}$). Then:

$$x = r\cos(\theta + l) = r\cos\theta\cos l - r\sin\theta\sin l = a\cos l - b\sin l$$

$$y = r\sin(\theta + l) = r\sin\theta\cos l + r\cos\theta\sin l = b\cos l + a\sin l$$

$\square$ Prove that $\forall a, b \in \mathbb{R}; \exists x, y \in [0,1]$:

$$|xy - ax - by| \geq \frac{1}{3}$$

If the right hand side $\frac{1}{3}$ change to 0.33334, is the inequality also true?

Solution

If $|a| \geq 1/3$ then we may take $x = 1$ and $y = 0$

So we may consider only $|a|, |b| < 1/3$

if $a, b \geq 0$ take $x = y = 1 : 1 - a - b \geq 1/3$ if $a < 0$ or $b < 0$ take $x = y = 1$ also.

1/3 is sharp by looking at case $a = b = 1/3$

$\square$ Find all the polynomials $P(x)$ with real coificients, such that $P(x^3 + 1) = (P(x + 1))^3$, for every real number x.

Solution

Looking for (eventually complex) roots : if $x + 1$ is a root then $x^3 + 1$ is also a root. $x \neq -1, 0$ would give an infinite number of roots.

So $P(x) = x^n(x - 1)^p$

Plugging back only possibility is $n = 0$

$P(x) = (x - 1)^p$ works and is the only solution.

$\square$ For certain ordered pairs (a, b) of real numbers, the system of equations

$$ax + by = 1$$

$$x^2 + y^2 = 50$$

has at least one solution, and each solution is an ordered pair (x, y) of integers. How many such ordered pairs (a, b) are there?

Solution

The equation $x^2 + y^2 = 50$ describes a circle centered around the origin with radius $5\sqrt{2}$. There are three points on the circle with integer coordinates in the first quadrant: (1,7), (5,5), (7,1), so there are twelve total points on the circle with integer coordinates.

The equation $ax + by = 1$ can describe any line on the plane. It can intersect the circle at one or two points.

Intersects at one point with integer coordinates: There are 12 points, so there are 12 lines.

Intersects at two points with integer coordinates: $\binom{12}{2} = 66$ lines.

So total there are 78 ordered pairs.

$\square$ Determine all triples of positive integers (x, y, z) with $x \leq y \leq z$ satisfying $xy + yz + zx - xyz = 2$.

Solution

For x, y, z such that $1 \leq x \leq y \leq z$, $xyz = xy + yz + zx - 2 \iff x = \frac{x}{z} + 1 + \frac{x}{y} \leq 1 + 1 + 1 = 3$, yielding $x = 1, 2$. Case 1. $x = 1$ Plugg this into the equation $xyz = xy + yz + zx - 2$, we have $y + z = 2$. $\therefore (y, z) = (1, 1)$.Similarly Case 2. $x = 2$, we have $yz = 2y + 2z - 2 \iff y = 2 \cdot \frac{y}{z} + 2 - \frac{2}{z} < 2 + 2 = 4$, yielding $y = 2, 3$. Plugg these into the equation $yz = 2y + 2z - 2$,we have that for $y = 2$, the equation $2z = 4 + 2z - 2$, which contradicts. Then for $y = 3$, the equation $3z = 2z + 4 \iff z = 4$. Therefore desired answer is $(x, y, z) = (1, 1, 1), (2, 3, 4)$.

$\square$ On sides $AB, BC, CA$of a triangle ABC we take points M, K, L with $ML//BC$ and $MK//AC$. Segments AK, ML meet at Q and segments BL, MK meet at P . Prove that $PQ//AB$.

Solution

Since $AC \,||\, MK$ then $\angle PMQ = \angle MLA$.

Since $ML \,||\, BC$ then $\angle PMQ = \angle MKB$.

Also $\angle CAB = \angle KMB$ and so $\Delta AML \sim \Delta KMB \sim \Delta ABC$.

$\frac{AC}{AB} = \frac{PM}{MB}$ and $\frac{BK}{QM} = \frac{AB}{AM}$.

Multiply these we get $\frac{AL}{AM} = \frac{PM}{MB} \cdot \frac{BK}{QM}$.

Also $\frac{AL}{AM} = \frac{MK}{MB} \Longrightarrow \frac{MK}{MB} = \frac{PM}{MB} \cdot \frac{BK}{QM}$.

Hence $\frac{MK}{BK} = \frac{PM}{QM}$ and then we get $\Delta PQM \sim \Delta MBK \sim \Delta ALM$.

So $\angle QPM = \angle PMB$ and $PQ \,||\, AB$.

□ Find all real solution of: $\sqrt{4x-8} + \sqrt[3]{14x-20} = \sqrt{24-4x} + \sqrt[3]{2092-182x}$

Solution

Rearrange the given equation:

$$\sqrt{4x-8} - \sqrt{24-4x} = \sqrt[3]{2092-182x} - \sqrt[3]{14x-20}$$

Let $f(x)$ be the LHS and $g(x)$ be the RHS above. We find that f is only real on $[2, 6]$. We can also fairly simply show that f is strictly increasing and g is strictly decreasing on $[2, 6]$. Thus if $f(6) < g(6)$, then there are no real solutions.

Simple calculation gives:

$f(6) = \sqrt{24-8} - \sqrt{24-24} = 4$

$g(6) = \sqrt[3]{2092-1092} - \sqrt[3]{84-20} = 6$

Thus $f(6) < g(6)$, so for any x in the domain of f, we must have $f(x) < g(x)$.

Suppose that x is not in the domain of f, then we will have a complex number on the LHS above. The cube root function, however, produces real values for all real numbers. Thus if x is real, the RHS above will be real. Thus the two sides can never be equal, completing the proof that there are no real solutions.

QED

□ In the Mathematical Competition of HMS (Hellenic Mathematical Society) take part boys and girls who are divided into two groups : [i]Juniors[/i] and [i]seniors.[/i]The number of the boys taking part of this year competition is 55

Solution

From the problem conditions we have $b = \frac{55}{100}(b+g)$ and $\frac{jb}{sb} = \frac{j}{s}$.

Hence $\frac{b}{g} = \frac{11}{9}$.

So $\frac{sb}{jb} = \frac{sb+sg}{jb+jg} \Longrightarrow \frac{b-jb}{jb} = \frac{100-(jb+jg)}{jb+jg}$.

$\frac{b}{jb} = \frac{100}{jb+jg} \Longrightarrow \frac{1}{b} = \frac{1}{100} + \frac{jg}{100jb}$.

$\frac{g}{100b} = \frac{jg}{100jb} \Longrightarrow \frac{jb}{jg} = \frac{b}{g} = \frac{11}{9}$.

□ Consider the following series:

$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

$T_n = S_1 + S_2 + \cdots + S_n$

$U_n = \frac{T_1}{2} + \frac{T_2}{3} + \cdots + \frac{T_n}{n+1}$.

Prove that $T_n + \ln(n+1) > U_n + n$.

Solution

Let $F : \mathbb{N} \to \mathbb{N}$, the function $F(n) = T_n + ln(n+1) > U_n + n$.

Lemma 1. $T_n = (n+1)(S_{n+1} - 1)$. Proof. In $(n+1)(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1})$ we overcount the number of 1's by 1, $\frac{1}{2}$'s by 2, and the number of $\frac{1}{k}$'s by k, so in total we overcounted by 1, $n+1$ times. Result follows.

Lemma 2. $F(n+1) - F(n) > 0$. Its equivalent to

$(T_{n+1} - T_n) + (\ln(n+2) - \ln(n+1)) + (U_n - U_{n+1}) + (n - (n+1)) > 0$.

$\Leftrightarrow S_{n+1} + \ln\frac{n+2}{n+1} > \frac{T_{n+1}}{n+2} + 1$

and using lemma 1, $\Leftrightarrow \ln\frac{n+2}{n+1} > S_{n+2} - S_{n+1}$

or $\Leftrightarrow \ln((\frac{n+2}{n+1})^{n+2}) > 1$, but $(\frac{n+2}{n+1})^{n+2} > e$ where the result follows. (e is the limit when n gets big, and the function is trivially decreasing using calculus)

We can verify $F(1) > 0$. Then $F(n+1) > F(n) > \cdots > F(1) > 0$, for all n. Result follows.

□ if the in circle of a quadrangle $ABCD$ has radius r, then prove that:

$AB + CD \geq 4r$

Solution

drawing the incircle and all the radii to the tangent points (call P the tangent point on AB, Q on BC, R on CD and S on DA). O is the incenter.

Call $\angle AOP = \angle AOS = \alpha_1$, $\angle BOQ = \angle BOP = \alpha_2$, $\angle COR = \angle COQ = \alpha_3$, and $\angle DOS = \angle DOR = \alpha_4$. Note that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 180$ and $0 < \alpha_i < 90$.

We find that $AB = r(\tan\alpha_1 + \tan\alpha_2)$, $BC = r(\tan\alpha_2 + \tan\alpha_3)$, $CD = r(\tan\alpha_3 + \tan\alpha_4)$, $DA = r(\tan\alpha_4 + \tan\alpha_1)$. So it remains to show that

$AB + BC + CD + DA = 2r\sum\tan\alpha_i \geq 8r$, or $\sum\tan\alpha_i \geq 4$.

But since $\tan x$ is convex on $(0, 90)$, by Jensen's with equal weights $w_i = \frac{1}{4}$ we get

$\tan\alpha_1 + \tan\alpha_2 + \tan\alpha_2 + \tan\alpha_2 \geq 4\tan\left(\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4}\right) = 4\tan 45 = 4$ as desired.

□ tổ hợp

□ Evaluate

$$\left\lfloor \sum_{n=1}^{10^9} n^{-2/3} \right\rfloor$$

Solution

Note that

$\int_1^N n^{-2/3}\, dn < \sum_{n=1}^N n^{-2/3} < 1 + \int_1^N n^{-2/3}\, dn$

$\Rightarrow 2997 < \sum_{n=1}^{10^9} n^{-2/3} < 2998$

so $\left\lfloor \sum_{n=1}^{10^9} n^{-2/3} \right\rfloor = 2997$.

□ Find all couples (p, q) of prime numbers for which $p^2 + q^2 + p^2q^2$ is a perfect square.

Solution

We have $k^2 = p^2 + q^2 + p^2q^2 = (p^2+1)(q^2+1) - 1$. If p, q are odd, then looking mod 4, we see that the LHS is 0 or 1 and the RHS is 3. Our other option is for either p or q to be 2. WLOG $p = 2$ giving $k^2 = 5q^2 + 4 \Rightarrow (k-2)(k+2) = 5q^2$. The possible pairs for $(k-2, k+2)$ are $(1, 5q^2), (5, q^2)$, and $(q, 5q)$.

Solving these, we get (discarding values we don't want) $q = 3$ as the only solution. Therefore, $(2, 3)$ is the only solution. (indeed $2^2 + 3^2 + 2^2 \cdot 3^2 = 7^2$). ANother way $2, 2$ is not a solution. $mod\ 4:$ p and q can't be both odd, wlog, $p = 2$ Rewriting $q(q+4) = A^2 - 4 = (A+2)(A-2)$ quickly gives $q = 3$

$2, 3$ only solution.

□ Let $a_1 = 25, a_2 = 48$ and for all $n \geq 1$, let a_{n+2} be the remainder from dividing $a_n + a_{n+1}$ by 100. Find the remainder from dividing $a_1^2 + a_2^2 + ... + a_{2000}^2$ by 8.

Solution

We have ${a_{n+2}}^2 = {a_{n+1}}^2 + {a_n}^2 + 2 \cdot a_{n+1} \cdot a_n \ mod \ 8$

period of $a_n \ mod \ 8$ is hopefully very short : $1, 0, 1, 1, 4, 1$ and $1 + 0 + 1 + 1 + 4 + 1 = 8$

$2000 \ pmod 6 = 2$ so $a_1^2 + a_2^2 + ... + a_{2000}^2 \pmod 8 = 1 + 0 = 1$

□ Let S be the set of polynomials $a^n x_n + a_{n-1} x_{n-1} + ... + a_0$ with non-negative real coefficients such that $a_0 = a_n \leq a_1 = a_{n-1} \leq a_2 = a_{n-2} \leq \ldots$

For example, $x_3 + 2.1x_2 + 2.1x + 1$ or $0.1x_2 + 15x + 0.1$. Show that the product of any two members of S belongs to S.

Solution

Let the first generalized member of S be $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_{n-1} x + a_n, a_k \leq a_{k-1}, k = 1, 2, 3...n$, and let the second generalized member of S be $g(x) = b_m x^m + b_{m-1} x^{m-1} + ... + b_{m-1} x + b_m, b_k \leq b_{k-1}, k = 1, 2, 3...m$.

We now consider the coefficients of $f(x)g(x) = c_{n+m} x^{n+m} + c_{n+m-1} x^{n+m-1} + ...$ The coefficient of the first term is $a_n b_m$. The coefficient of the second term is given by all terms of $f(x)$ and $g(x)$ that multiply out to give an exponent of $n+m-1$; there are two of these term pairs, $a_{n-1} x^{n-1} * b_m x^m$ and $a_n x^n * b_{m-1} x^{m-1}$, which add up to $a_{n-1} b_m + b_{m-1} a_n$. Because $a_{n-1} \geq a_n$, it follows that $a_{n-1} b_m \geq a_n b_m$. Applying the same argument to $b_{m-1} a_n$ gives $a_{n-1} b_m + b_{m-1} a_n \geq 2 a_n b_m$.

Repeating this analysis for the coefficient of the $(k+1)^{th}$ term from the beginning up to the center term of $f(x)g(x)$ gives $c_{n+m-k} x^{n+m-k} = \sum_{i=0}^{k} a_{n-i} b_{m-k+i} x^{n+m-k}$. Each subsequent term of c_{n+m-k} contains one more $a_{n-p} b_{m-k+p}$ than the previous; additionally, every term in the addition except the last satisfies this inequality: $a_{n-i} b_{m-(k+1)+i} \geq a_{n-i} b_{m-k+i}$, and since there is even an extra term, $c_{n+m-k} > c_{n+m-(k+1)}$ is guaranteed, thus satisfying one requirement of belonging to S.

The second requirement, that $c_{n+m-k} = c_k$, requires repeating the analysis starting from the constant term, c_0. From the expansion of $f(x)g(x)$, it is obvious that $c_0 = a_0 b_0 = a_n b_n$. In order to find c_1, it is necessary to find all terms in $f(x)$ and $g(x)$ that contribute a term of x to $f(x)g(x)$; this gives, similar to the previous analysis, a sum of $c_1 x^1 = a_1 b_0 x^1 + a_0 b_1 x^1 = (a_{n-1} b_m + a_n b_{m-1})x$.

Repeating this analysis for the coefficient of the $(k+1)^{th}$ term from the end gives $c_k x^k = \sum_{i=0}^{k} a_i b_{k-i} x^k = \sum_{i=0}^{k} a_{n-i} b_{m-k+i} x^k$; it is immediately obvious that $c_k = c_{n+m-k}$, satisfying the second requirement of belonging to S. Therefore:

$\forall f(x), g(x) \in S : f(x)g(x) \in S$

Q.E.D. – Prove or disprove: For any set of integers $a_1, a_2, ..., a_n$, there exists integers $b_1, b_2, ..., b_n$ such that $a_1 b_1 + a_2 b_2 + ... + a_n b_n = \gcd(a_1, a_2, ..., a_n)$

□ Prove if $n, d, k \in N$ and $d|\, n \ \varphi(nd^k) = d^k \varphi(n)$

Solution

Given that $d|k$, we have $\phi(nd^k) = (nd^k) \times \prod_{p|nd^k} \frac{p-1}{p}$, where p is prime. We have $d^k \phi(n) = d^k(n) \times \prod_{p|n} \frac{p-1}{p}$. Since the set of all p such that $p|nd^k$ is equal to the set of all p such that $p|n$, the $\sum$ in both equations is equal, and the factor nd^k is also equal, so we have equality and the desired result.

□ Let S be a set with 6 elements. How many pairs of subsets X and Y of S are there such that X is a subset of Y and $X \neq Y$?

Solution

For each subset Y of size k, there are $2^k - 1$ proper subsets X (i.e. subsets with $X \neq Y$).

So the total number of pairs of subsets is $\sum_{k=0}^{6}(2^k - 1)\binom{6}{k} = \sum_{k=0}^{6} 2^k \binom{6}{k} - \sum_{k=0}^{6} \binom{6}{k}$

Which by the binomial theorem is equal to $(1+2)^6 - (1+1)^6$, or $3^6 - 2^6 = 665$ ANother way Let's temporarily ignore the condition $X \neq Y$. Each of the 6 elements of S has 3 choices: that element is either a member of both X and Y, or a member of Y only, or a member of neither. Thus we have 3^6 choices. We need to subtract out the case $X = Y$; there are 2^6 subsets of S. Thus the final answer is $3^6 - 2^6$.

□ Find all positive primes p for which there exist integers m, n satisfying: **1.** $p = m^2 + n^2$ **2.** $m^3 + n^3 - 4$ is divisible by p

Solution

$m^2 + n^2 | m^3 + n^3 - 4 \Longrightarrow m^2 + n^2 | (m+n)(mn - m^2 - n^2) + 4 \Longrightarrow m^2 + n^2 | (m+n)mn + 4 \Longrightarrow m^2 + n^2 | m^2 n + n^2 m + 4 \Longrightarrow k(m^2 + n^2) = m^2 n + n^2 m + 4$

Suppose $n > m \geq 2, k \geq 2$ If $k > n$ then $(n+1)(m^2 + n^2) \leq k(m^2 + n^2) = m^2 n + n^2 m + 4 \Longrightarrow nm^2 + n^3 + m^2 + n^2 \leq m^2 n + n^2 m + 4 \Longrightarrow n^3 + m^2 + n^2 \leq n^2 m + 4 \leq n^3 + m^2$

If $k = n$ then: $n(m^2 + n^2) = m^2 n + n^2 m + 4 \Longrightarrow n^2(n - m) = 4 \Longrightarrow 9 \leq 4$

Hence $k < n$:

$k(m^2 + n^2) = m^2 n + n^2 m + 4 \Longrightarrow m^2(n-k) + m(n^2) + (4 - kn^2)$

Hence $(n^2)^2 - 4(n-k)(4 - kn^2)$ sould be a perfect square. $(n^2)^2 - 4(n-k)(4-kn^2) = n^4 + 4kn^3 + 4k^2 n^2 - 16n + 16k = A$

But $(n^2 + 2kn - 1)^2 < A < (n^2 + 2kn)^2$

We can easily check if $k = 1$ or $m = 1$

□ Find all functions $f : \Re \to \Re$ satisfying: $(x+y)(f(x) - f(y)) = (x-y)f(x+y)$

Solution

If we tak $x = -y$ and $x \neq 0$, then $f(0) = 0$. Now we can assume that $f(1) = k$ and $f(2) = a$, where a and k are reals. Taking $y = 1$ and $x = n+2$, where n is a natural, we have $f(n+3) = \frac{n+3}{n+1}(f(n+2) - k)$. And now we can prove by induction that $f(n+3) = \frac{a-2k}{2} n^2 + \frac{5a-8k}{2} n + (3a - 3k)$, for all integer n such tah $n \geq 0$. We can extending this result for all integer n. It's just obtain $f(-1)$ and $f(-2)$ in function of a and k, and make a new induction. To obtain $f(-1)$ we can take $y = -1$ and $x = 3$ and to obtain $f(-2)$ we can take $y = -2$ and $x = 3$. And now we have to extend this result for all racional n. We can take $x = qy$ and $y = \frac{p}{q}$ and we'll have $(q+1)(f(p) - f(\frac{p}{q})) = (q-1)f(p + \frac{p}{q})$. And taking now $x = -p$ and $y = p + \frac{p}{q}$ we can calculate $f(p + \frac{p}{q})$, and finally estend for all racional n. In this point it's just verify that f is a continuous function. And this is in fact a merely consequence of it's definition. We can prove that f is diferenciable just looking to this definition. And soon we can extend that formula for all reals n.

□ Does there exist an integer k which can be expressed as the sum of two factorials $k = m! + n!$ (with $m \leq n$) in two different ways?

Solution

$2 = 1! + 1! = 0! + 0! = 0! + 1!$ $1! + n! = 0! + n!$ Proof

For $m \leq n$ and $p \leq q$, with all of them > 1, let us denote $k = m! + n! = p! + q!$. WLOG, let $(n - m) < (q - p)$. Then $p < m, q > n$.

We know that $m!|k$. We therefore know that $m!|(p! + q!)$, which can only be true if $\frac{m!}{p!} | \frac{q!}{p!} + 1$. If $m > p + 1$, then $\frac{m!}{p!}$ is even. Then the equation can only hold true if $\frac{q!}{p!}$ is odd, which only occurs when $q = p$ or $q = p + 1$, both of which contradict our assumption that $p < m, q > n$.

This means $m \le p+1$, and because $p < m$, we have $m = p+1$ and $p+1 | \frac{q!}{p!}+1$. Because we have assumed $p > 1$ in order to prevent a trivial solution, $p+1 \ge 3$.

For $p+1 \ge 3$, then, we write our original equation as $k = (p+1)p! + n! = p! + q!$. Subtracting yields $p(p!) = q! - n! = n!(\frac{q!}{n!} - 1)$. This requires that $\frac{p(p!)}{n!}$ be an integer because $q > n$, but because we have $p < m \le n$, the fraction becomes $\frac{p}{(p+1)(p+2)...(n)}$ which is clearly never an integer.

Hence, only the trivial solutions work.

□ For $p, q \in \mathbb{N}$ where $p > q$ prove that
$\sum_{k=1}^{\infty} \frac{(pq)^k}{(p^k-q^k)(p^{k+1}-q^{k+1})} = \frac{q}{(p-q)^2}$

Solution

Let $r = \frac{q}{p}$ $(0 < r < 1)$, since $\lim_{n\to\infty} r^n = 0$, we have
$\sum_{k=1}^{n} \frac{(pq)^k}{(p^k-q^k)(p^{k+1}-q^{k+1})} = \frac{1}{p(1-r)} \sum_{k=1}^{n} \frac{r^k}{(1-r^k)(1-r^{k+1})} = \frac{1}{p(1-r)} \sum_{k=1}^{n} \left(\frac{1}{1-r^k} - \frac{1}{1-r^{k+1}}\right)$
$= \frac{1}{p(1-r)} \left(\frac{1}{1-r} - \frac{1}{1-r^{n+1}}\right) \longrightarrow \frac{1}{p(1-r)} \left(\frac{1}{1-r} - 1\right) \ (n \longrightarrow \infty)$
$= \frac{r}{p(1-r)^2} = \frac{q}{(p-q)^2}$. Q.E.D.

□ If

$$\frac{(a-b)(b-c)c-a)}{(a+b)(b+c)(c+a)} = \frac{1}{11},$$

find

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}.$$

Solution

Denote $\frac{a}{a+b} = x$, $\frac{b}{b+c} = y$, $\frac{c}{c+a} = z$. We have to calculate $s = x+y+z$. From the statement we have $(2x-1)(2y-1)(2z-1) = \frac{1}{11}$, hence (1) $4(2xyz - (xy+xz+yz)) + 2(x+y+z) - 1 = \frac{1}{11}$. OTOH, $\frac{1}{x} - 1 = \frac{b}{a}$, $\frac{1}{y} - 1 = \frac{c}{b}$, $\frac{1}{z} - 1 = \frac{a}{c}$. This yields to $(\frac{1}{x}-1)(\frac{1}{y}-1)(\frac{1}{z}-1) = 1$. We obtain $1 - (x+y+z) + xy + xz + yz - xyz = xyz$, hence $2xyz - (xy+xz+yz) = 1 - (x+y+z) = 1 - s$. Replace in (1) and obtain a simple equation in s. – $\sqrt[2]{x} + \sqrt[3]{x^2-1} + \sqrt[4]{x^3+15} = x^2+2$ – Given positive integer n and positive real number M. Among all arithmetic sequences $a_1, a_2, a_2 \cdots$ which satisfy $a_1^2 + a_{n+1}^2 \le M$, find the maximum of $S = a_{n+1} + a_{n+2} + \cdots + a_{2n+1}$. – Find the number of positive integer solutions to the equation (x_i and P are positive integers

$x_1x_2...x_k + x_{k+1}x_{k+2}....x_{2k} + ... + x_{nk+k}x_{nk+k-1}...x_{nk} = P$

□ For which c real numbers ,there can be found a line that intersects $y = x^4 + 9x^3 + cx^2 + 9x + 4$ curve at four distinct points?

Solution

If r_k denote the roots of the polynomial $x^4 + 9x^3 + cx^2 + (9-a)x + (4-b)$, then the r_k are all real and distinct. Then $\sum_{j<k}(r_j - r_k)^2 > 0$.

Expanding each, this is $3\sum_k r_k^2 > 2\sum_{j<k} r_j r_k = 2c$.

Adding on $6c = 6\sum_{j<k} r_j r_k$ to both sides to complete the square, $3\left(\sum_k r_k\right)^2 > 8c$. But $\sum_k r_k = -9 \Rightarrow c < \frac{243}{8}$.

So if $c \ge \frac{243}{8}$ it certainly won't work. But if $c < \frac{243}{8}$ then the sum $\sum_{j<k}(r_j - r_k)^2$, or $243 - 8c$, is positive.

Just suppose that $r_1 = r_2 + \alpha = r_3 + 2\alpha = r_4 + 3\alpha$. Then this sum is $(3\alpha)^2 + 2(2\alpha)^2 + 3\alpha^2 = 243 - 8c$.

Then $20\alpha^2 = 243 - 8c$, so $\alpha = \sqrt{\frac{243-8c}{20}}$.

Then from $\sum_k r_k = 4r_4 + 6\alpha = -9$, $r_4 = -\frac{9+6\alpha}{4}$. So a set of r_k can be constructed, which then determine the coefficients.

Therefore there is a line if and only if $\boxed{c < \frac{243}{8}}$.

$\square$ If A, B, C and D are consequent vertices of a regular (I don't know if it's the right word, polygon with all sides equal, and all angles equal) polygon, find the number of vertices if

$$\frac{1}{|AB|} = \frac{1}{|AC|} + \frac{1}{|AD|}$$

.

Solution

$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD} \iff \frac{1}{AB} = \frac{AC+AD}{AC \cdot AD} \iff \overline{AC} \cdot \overline{AD} = \overline{AB} \cdot \overline{AC} + \overline{AB} \cdot \overline{AD}$

Denote $V_1 = A$, $V_2 = B$,... We have $\overline{V_1V_3} \cdot \overline{V_1V_4} = \overline{V_1V_2} \cdot \overline{V_1V_3} + \overline{V_1V_2} \cdot \overline{V_1V_4}$ Then we can rewrite it as $\overline{V_1V_3} \cdot \overline{V_2V_5} = \overline{V_1V_2} \cdot \overline{V_3V_5} + \overline{V_2V_3} \cdot \overline{V_5V_8}$ (neglecting the number of vertices at this point).

However, since the polygon is regular, Ptolemy's Theorem must hold for any quadrilateral whose vertices are on this polygon. So we must have $\overline{V_1V_3} \cdot \overline{V_2V_5} = \overline{V_1V_2} \cdot \overline{V_3V_5} + \overline{V_2V_3} \cdot \overline{V_5V_1}$, which implies that $V_1 \equiv V_8$.

Therefore, the number of vertices is 7.

$\square$ From : $a+b+c+d = S$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = S$, we infer : $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} + \frac{1}{1-d} = S$. Find S ?

Solution

If the result holds for the real numbers a, b, c, d, then it must also hold for the real numbers $1/a$, $1/b$, $1/c$, $1/d$, so

$$\sum \frac{1}{1-1/a} = \sum \frac{a}{a-1} = S,$$

which implies that

$$\sum \frac{a}{1-a} = -S.$$

Then

$$S - (-S) = \sum \frac{1}{1-a} - \sum \frac{a}{1-a} = 4,$$

so $S = 2$.

Now to prove that $S = 2$ works. From the given conditions, $\sum a = 2$ and $\sum abc = 2abcd$. Then

$$\begin{aligned} \sum \frac{1}{1-a} &= \frac{4 - 3\sum a + 2\sum ab - \sum abc}{1 - \sum a + \sum ab - \sum abc + abcd} \\ &= \frac{4 - 6 + 2\sum ab - 2abcd}{1 - 2 + \sum ab - 2abcd + abcd} \\ &= \frac{-2 + 2\sum ab - 2abcd}{-1 + \sum ab - abcd} \\ &= 2. \end{aligned}$$

Hence, $S = 2$ works.

$\square$hình

$\square$ 9 people hold 5119 shares in a company. In every decision voting, any subset or all of the 9 people can participate.If a person participates in voting, he/she can either vote FOR or AGAINST the decision. The number of votes is equal to the number of shares a person holds. For every decision there shouldn't be a TIE between the two choices. In a decision, a group with smaller number of people should never win over higher number of people. What is the least number of shares a person can hold?

Solution

Let the number of shares for each person i be a_i where $1 \le i \le 9$ and i integral. WLOG assume that $a_1 < a_2 ... < a_9$

We know that $a_i \neq a_j$ otherwise there would be a tie. Since a smaller number of people should never outvote a higher number of people, we say that

$a_9 < a_1 + a_2$

But in order to minimise a_1 we need to maximise a_9 and this occurs when

$a_9 = a_1 + a_2 - 1$

By a similar argument, $a_8 = a_9 - 1 = a_1 + a_2 - 2$

This shows that all of $a_2, a_3 ... a_9$ are consecutive. With a little experimentation, one determines that $a_9 = 642, a_8 = 641, ...,$ and since the sum is 5119, we conclude that the smallest number of shares is 11

☐ is it possible to write the natrual numbers in a string such that after n numbers this string is a palendrone? i.e. $123456 \ldots n$ is a palendrone

Solution

First, definitions. For two strings of digits s_1, s_2 let us define $app(s_1, s_2)$ to be the result when s_2 is appended to the end of s_1 (ex.: $app(123, 456) = 123456$). For a string of digits s define $inv(s)$ to be the inversion of the string (ex.: $inv(123) = 321$). Finally, for a string of digits s define $first(s, l)$ to be the string that contains the first l digits of s.

Claim: $n = 1$.

Proof: Define the sequence $a_1 = 1, a_k = app(a_{k-1}, k)$, which is precisely the string in the problem. If $n > 1$ then n must be of the form $inv(a_k)$ for some integer $k > 1$ in order that $a_n = inv(a_n)$.

Now, consider the previous term to be appended to the string, $n-1$. Because $n = inv(a_k) = k...321$ we know $n-1 = k...320$. The string ends with $app(n-1, n) = k...320k...321$ and because $a_n = inv(a_n)$ we require that $inv(app(n-1, n)) = first(a_n, 2k)$. This is impossible - we can write $inv(app(n-1, n)) = 123...k023...k$, and because we append $k+1$ immediately after k when constructing a_n a 0 cannot be present there.

Hence, for $n > 1$ no such $n = a_k$ exists. Q.E.D.

☐ Let $2n > k$ natural numbers and $a_1, ..., a_n$ integers such that leaving different remainder when they are divided by k . Prove that for all integers l there exist index i, j from the set $\{1, 2, ..., n\}$ such that

$k|(a_i + a_j - l)$

Solution

first i want to state that the last condition is equvilent to

$$a_i + a_j - l \equiv 0 \pmod k \iff a_i \equiv l - a_j \pmod k$$

then considering how many a_i can exist so no i, j satisfy the condition we get $n < \frac{k-1}{2}$ which is contrary to $2n > k$ ... – The quadratic ineqaulity $ax^2 + bx + c \geq 0$ is true for all $x \in R$. If $b > a$, then find the minimum value of $\frac{a+b+c}{b-a}$.

☐ Given

$x^3 - 3x = y$

$y^3 - 3y = z$

$z^3 - 3z = x$

Find all sets of solutions $[x, y, z]$

Solution

Given: $a = 2\cos x$

Take $a^3 - 3a = b$ and we have

$b = 8\cos^3 x - 6\cos x = 2(4\cos^3 x - 3\cos x) = 2\cos 3x$

Try $|x| > 2$ and see that the equation can't work, so $|x| \leq 2$ so we can make a substitution.

So, set $x = 2\cos\theta \Rightarrow y = 2\cos 3\theta \Rightarrow z = 2\cos 9\theta \Rightarrow x = 2\cos 27\theta$

So, solve $2\cos\theta = 2\cos 27\theta$ and plug in for the set of values $(x, y, z) = (2\cos\theta, 2\cos 3\theta, 2\cos 9\theta)$

We get $27\theta \equiv \pm\theta \bmod 2\pi$

$\Rightarrow \theta = \frac{n\pi}{13}, \frac{m\pi}{14}$ for any m, n integers. – Find, with proof, all triples of real numbers (a, b, c) such that all four roots of the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + b$ are positive integers. (The four roots need not be distinct.) —- Find four distinct positive integers, a, b, c, and d, such that each of the four sums $a + b + c$, $a + b + d$,$a + c + d$, and $b + c + d$ is the square of an integer. Show that infinitely many quadruples (a, b, c, d) with this property can be created. – Let $\{a_n\}$ be a sequence such that $a_{n+1} = a_n^2 - na_n + 1$ with $n = 1, 2, 3\cdots$. When $a_1 \geq 3$, prove that for all $n \geq 1$:

(1): $a_n \geq n + 2$.

(2): $\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} \leq \frac{1}{2}$.

□ Find the positive integer solutions of the equation $3^x + 29 = 2^y$.

Solution

See that $x = 1, y = 5$ gives the first solution. There are no solutions for $x > 1, y > 5$.

Take the equation mod9. Since $x > 1$, this gives $2 \equiv 2^y \bmod 9 \Rightarrow 1 \equiv 2^{y-1} \bmod 9$. Euler's Theorem gives $2^{\phi(9)} = 2^6 \equiv 1 \bmod 9$, so $y = 6n + 1, n \geq 1$.

Take the equation mod32. Since $y > 5$, this gives $3^x - 3 \equiv 0 \bmod 32 \Rightarrow 3^x \equiv 3 \bmod 32 \Rightarrow 3^{x-1} \equiv 1 \bmod 32$. Euler's Theorem gives $3^{\phi(32)} = 3^{16} \equiv 1 \bmod 32$, so $x = 16m + 1, m \geq 1$.

Finally, take the equation mod7. This gives $3^{16m+1} + 1 \equiv 2^{6n+1} \bmod 7$. By Euler's Theorem, $2^6 \equiv 1 \bmod 7$, so $2^{6n+1} \equiv 2 \bmod 7$. This implies $3^{16m+1} \equiv 1 \bmod 7$, which cannot be true since, by Euler's Theorem, $3^6 \equiv 1 \bmod 7$, and $16m + 1$ cannot be a multiple of 6.

Therefore, there are no solutions $x > 1, y > 5$.

□ The equation

$$x^{10} + (13x - 1)^{10} = 0$$

has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

Solution

e devide both sides of the equation by x^{10} and we get (1) $(\frac{1}{x} - 13)^{10} = -1$. Denote $t = \frac{1}{x} - 13$. The equation becomes (2) $t^{10} = -1$. Let S be the sum in the original statement, $x_1, x_2, ..., x_{10}$ the solutions of equation (1) and $t_1, ..., t_{10}$ the solutions of the equivalent equation (2). Then $S = \frac{1}{2}\sum_{k=1}^{10} \frac{1}{x_k}\frac{1}{\overline{x_k}} = \frac{1}{2}\sum_{k=1}^{10}(13 + t_k)\overline{13 + t_k} = \frac{1}{2}(\sum_{k=1}^{10} 13^2 + 13\sum_{k=1}^{10} t_k + 13\sum_{k=1}^{10} \overline{t_k} + \sum_{k=1}^{10} t_k\overline{t_k})$. From (2) we get $\mid t_k \mid = 1$. Since in general $\mid z \mid^2 = z\overline{z}$, we get $t_k\overline{t_k} = 1$. The first Viete relation for equation (2) yields $\sum_{k=1}^{10} t_k = 0$, so we have also $\sum_{k=1}^{10} \overline{t_k} = 0$. We obtain $S = \frac{1}{2}(1690 + 0 + 0 + 10) = 850$

Another way Dividing the equation by x^{10}, we have $1 + \left(\frac{13x-1}{x}\right)^{10} = 0$, or $\left(13 - \frac{1}{x}\right)^{10} = -1$.

Let $y = 13 - \frac{1}{x}$. Then $y^{10} = 1$, so $y = \cos 18^\circ + i\sin 18^\circ, \cos 54^\circ + i\sin 54^\circ, ... \cos 342^\circ + i\sin 342^\circ$ (In general, $y = \cos(36n - 18)^\circ + i\sin(36n - 18)^\circ$, where $n = 1, 2, ...10$).

$\sum\left(\frac{1}{a_ib_i}\right) = \sum(13 - y_a)(13 - y_b)$

Since $(13 - y_a)(13 - y_b) = 169 - 13(y_a + y_b) + 1 = 170 - 13(y_a + y_b)$, the summation becomes $170 \cdot 5 - 13\sum_{k=1}^{10}(\cos(36n - 18)^\circ + i\sin(36n - 18)^\circ) = \boxed{850}$

– Find all solutions in integers m, n of the equation

$$(m-n)^2 = \frac{4mn}{m+n-1}.$$

□ Define a sequence (a_i) by $a_1 = 0$; $a_2 = 2$; $a_3 = 3$ and $a_n = \max_{1<d<n}\{a_d \cdot a_{n-d}\}$ for $n = 4, 5, 6, ...$ Find a_{1998}.

Solution

Consider the sequence of sets S_n definied inductively by $S_n = \{s_d \cdot s_{n-d}\}$, for all d and for all $s_d \in S_d$. We define $S_1 = , S_2 = 2, S_3 = 3$ If we wanted, we could continue this sequence $\{4\}, \{6\}, \{8, 9\}, \{12\}, \{16, 18\}$ Clearly, any element of S_n must be of the form $2^x 3^y$ Claim: if $a \in S_n$ and $a = 2^x 3^y$, then $n = 2x + 3y$ Proof: strong induction. Suppose it holds for $k < n$. Suppose $a \in S_n$. Then $a = s_d \cdot s_{n-d}$ for some d and s_d, s_{n-d} in S_d, S_{n-d}, respectively. Suppose $s_d = x^e y^f$ and $s_{n-d} = x^g y^h$. Then $a = x^{e+g} y^{f+h}$. By the induction hypothesis, $2e + 3f = d$ and $2g + 3h = n - 3$. Adding, we get $2(e+g) + 3(f+h) = n$. □ Claim: converse of previous claim: if $2x + 3y = n$, then $2^x 3^y \in S_n$. Proof: Induction. The claim holds for $n - 2$ and $n - 3$, and the result follows. □ From the last two claims, we see that S_n contains all numbers $2^x 3^y$ where $2x + 3y = n$. It is clear that due to the construction of the sequence $a_1, a_2, \cdots$ a_n is the largest element of S_n. Since $2^3 < 3^2$, $2^x 3^y < 2^{x-3} 3^{y+2}$, and $2x + 3y = 2(x-3) + 3(x-2)$. It follows that a_n is the element of S_n that isn't divisible by 2^3. $1998 = 3 \cdot 666$, so the answer is 3^{666}

□ When a biased coin is tossed the probability of a head is p Two players A and B alternately toss a coin until one of the sequences HHH, HTH occurs. A wins if HHH occurs first .B win s if HTH occurs first. For what values of p is the game fair that is such that Probability A wins = probability B wins

Solution

Lets forget p = 0, 1 - answer is obvious.

Obv. one of the two people win. So we just want chance of A winning to be 1/2. Say A wins a dollar by winning the game. Then, we want the expected value of the game to be 1/2.

The four states W,X,Y,Z correspond to the last two flips being HH HT TH TT.

W = p + (1-p)X X = 0 + (1-p)Z Y = pW + (1-p)X Z = pY + (1-p)Z

Then, $Z = Y$ from eqn 4, $X = (1-p)Y$ from eqn 2, $Y = pW + (1-p)^2 Y \Rightarrow Y(2-p) = W$ from eqn 3, and $Y(2-p) = p + (1-p)^2 Y$ implying $Y = \frac{p}{1+p-p^2}$

Finally, we want $\frac{W+X+Y+Z}{4} = \frac{1}{2}$.

So $2 = W + X + Y + Z = Y((2-p) + (1-p) + 1 + 1) = Y(5 - 2p) = \frac{(5-2p)p}{1+p-p^2}$

It comes to $-2p^2 + 5p = -2p^2 + 2p + 2$ implying $p = 2/3$.

□ tổ hợp

□ hình

□ Let a, b and c be real numbers such that $a^2 + b^2 = c^2$, solve the system:

$$z^2 = x^2 + y^2$$

$$(z+c)^2 = (x+a)^2 + (y+b)^2$$

in real numbers x, y and z.

Solution

The second equation is $c^2 + 2cz + z^2 = x^2 + 2ax + a^2 + y^2 + 2by + b^2$.

After subtracting the given equalities, $2ax + 2by = 2cz$ and $ax + by = cz$.

Multiplying the two equalities, $(a^2 + b^2)(x^2 + y^2) = c^2 z^2$.

But by the Cauchy-Schwartz Inequality, $(a^2+b^2)(x^2+y^2) \geq (ax+by)^2 = c^2z^2$.

We have equality here, so we must have $y^2 = \frac{b^2}{a^2} \cdot x^2$, and $z^2 = x^2+y^2$. These are all the solutions.

☐ find all sets of non-negative solution (m,n) such that $6^m + 2^n + 2$ is a square.

Solution

If $m, n \geq 2$ then $6^m + 2^n + 2$ is divisible by 2 but not 4 and cannot be a square.

So let $m = 1$. We then have $2^n + 8$. If $n \geq 4$, then it is divisible by 8 but not 16 and isn't a square. So we check $n < 4$. $n = 0$ and $n = 3$ work because $2^0 + 8 = 9$ and $2^3 + 8 = 16$.

Let $m = 0$. That gives $2^n + 3$. If $n \geq 2$, it is $3 \pmod 4$ and can't be a square. Checking $n = 0, 1$ gives no solutions.

Now let $n = 1$. We have $6^m + 4$. If $m \geq 2$, then $6^m + 4 = 4(2^{m-2} \cdot 3^m + 1)$ so $x^2 = 2^{m-2} \cdot 3^m + 1 \Rightarrow (x+1)(x-1) = 2^{m-2} \cdot 3^m$. Since $x+1$ and $x-1$ have the same parity, they must both be even. But since they differ by 2, the gcd of them is at most 2. And only one can be divisible by 3. So we must have one of them be $2 \cdot 3^m$ and the other 2^{m-3}. But $2 \cdot 3^m$ is way bigger, so there can't be any solutions.

For $n = 0$, we have $6^m + 3$ which is divisible by 3 but not 9 if $m \geq 2$. Checking $m = 0, 1$, we get the same solution as above.

So our only solutions are $(m, n) = (1, 0); (1, 3)$.

☐ Every positive integer k has a unique factorial base expansion $(f_1, f_2, f_3, \ldots, f_m)$, meaning that

$$k = 1! \cdot f_1 + 2! \cdot f_2 + 3! \cdot f_3 + \cdots + m! \cdot f_m,$$

where each f_i is an integer, $0 \leq f_i \leq i$, and $0 < f_m$. Given that $(f_1, f_2, f_3, \ldots, f_j)$ is the factorial base expansion of $16! - 32! + 48! - 64! + \cdots + 1968! - 1984! + 2000!$, find the value of $f_1 - f_2 + f_3 - f_4 + \cdots + (-1)^{j+1} f_j$.

Solution

Note that

$$(n+16)! - n! = n!([n+16][n+15]...[n+1] - 1)$$

$$= n * n! + (n+1) * (n+1)! + ... + (n+15)(n+15)!$$

Thus, $48! - 32! = 47!47 + 46!46 + 45!45 + ... + 32!32$. Thus, $f_{16} = 1$, and for all i, $32k \leq i \leq 32k+15$, $f_i = i$ and $f_i = 0$ for all other i. This continues all the way up to $k = 62$. Thus, our answer is $(-1) + (-32 + 33 - 34 + ... - 46 + 47) + (-64 + 65 - 66 + ...) + ...$ There are 62 such sums (like that in the parantheses), and each has value 8. Thus, the answer is $62 * 8 - 1 = \boxed{495}$

☐ x_1, x_2, x_3 roots of equation $x^3 + 3x^2 - 24x + 1 = 0$. Prove that $\sqrt[3]{x_1} + \sqrt[3]{x_2} + \sqrt[3]{x_3} = 0$.

Solution

We have: $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right], \forall a, b, c \in \mathbb{R}$

then, for: $a = \sqrt[3]{x_1}, b = \sqrt[3]{x_2}, c = \sqrt[3]{x_3}$

we have: $x_1 + x_2 + x_3 - 3 \cdot \sqrt[3]{x_1x_2x_3} = -3 - 3 \cdot (-1) = 0 \Rightarrow$

$\Rightarrow \sqrt[3]{x_1} + \sqrt[3]{x_2} + \sqrt[3]{x_3} = 0$

Problem Let x_1, x_2, x_3 be the roots of equation $x^3 - px^2 + qx - r = 0$ using Viete's relations, from: $(a+b+c)\left[(a+b+c)^2 - 3(ab+bc+ca)\right] = a^3 + b^3 + c^3 - 3abc \quad \forall a, b, c \in \mathbb{R}$ for: $A = \sqrt[3]{x_1} + \sqrt[3]{x_2} + \sqrt[3]{x_3}, B = \sqrt[3]{x_1x_2} + \sqrt[3]{x_2x_3} + \sqrt[3]{x_3x_1}$ we have: $\begin{cases} A(A^2 - 3B) = p - 3\sqrt[3]{r} \\ B(B^2 - 3\sqrt[3]{r}A) = q - 3\sqrt[3]{r^2} \end{cases}$ [/color]

For $x^3 - 4x^2 - 11x + 1 = 0$ we have: $\begin{cases} A(A^2 - 3B) = 7 \\ B(B^2 + 3A) = -14 \end{cases} \Leftrightarrow \begin{cases} A^3 - 3AB = 7 \\ B^3 + 3AB = -14 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} A^3 - 3AB = 7 \\ A^3 + B^3 = -7 \end{cases} \Leftrightarrow \begin{cases} A^3 - 3AB = 7 \\ B = \sqrt[3]{-A^3 - 7} \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} A^3 - 3A \cdot \sqrt[3]{-A^3 - 7} = 7 \\ B = \sqrt[3]{-A^3 - 7} \end{cases} \Leftrightarrow \begin{cases} A^3 + 3 \cdot \sqrt[3]{(A^3)^2 + 7A^3} = 7 \\ B = \sqrt[3]{-A^3 - 7} \end{cases}$

For $u = A^3$ we obtain equation: $u + 3\sqrt[3]{u^2 + 7u} = 7$

It is easy to prove that: $\exists! u \in \mathbb{R} : u + 3\sqrt[3]{u^2 + 7u} = 7$ and $u3\sqrt[3]{u^2 + 7u} = 7 \Leftrightarrow u = 1$

then: $A = \sqrt[3]{x_1} + \sqrt[3]{x_2} + \sqrt[3]{x_3} = 1$ and $B = \sqrt[3]{x_1x_2} + \sqrt[3]{x_2x_3} + \sqrt[3]{x_3x_1} = -2$

□ Suppose x,y,z are three integers which are in arithmetic progression. If x is of the form 8n + 4 where n is an integer and each of y,z is expressible as a sum of squares of two integers, show that **gcd** (x,y,z) cannot be odd.

Solution

$x = 8n + 4$ (*) $y = 8n + 4 + r = a^2 + b^2$ (**) $z = 8n + 4 + 2r = c^2 + d^2$ (***)

Now gcd is odd iff r is odd

$c^2 + d^2$ is even so mod 8 it must be 0 or 2 (looking quadratic residues mod 8). Only possibilty with (***) is $r = 1$ mod 8

Now an odd sum of two square (**) must be 1 or 5 mod 8.

$8n + 4 = c^2 + d^2 - 2(a^2 + b^2)$ would give $4 = 0$ mod 8 contradiction

□ hình

□ hình

□ Find the range of real number a, such that for all x and any $\theta \in \left[0, \frac{\pi}{2}\right]$, the inequality $(x + 3 + 2\sin\theta\cos\theta)^2 + (x + a\sin\theta + a\cos\theta)^2 \geq \frac{1}{8}$ is always true.

Solution

rewrite as

$(x + 3 + \sin 2\theta)^2 + (x + a\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right))^2 \geq \frac{1}{8}$

Since over $\theta \in [0, \frac{\pi}{2}]$, $\sin 2\theta \geq 0$, $\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) \geq 1$

Hence we only need to find $(x + 3)^2 + (x + a)^2 \geq \frac{1}{8}$. Expand and simplify we get

$2x^2 + 2(3 + a)x + a^2 + \frac{71}{8} \geq 0$

So the discriminant of x must be negative or zero

$\delta_x = (a + 3)^2 - 2a^2 - \frac{71}{4} \leq 0 \implies (2a - 5)(2a - 7) \geq 0 \implies a \in (-\infty, \frac{5}{2}] \cup [\frac{7}{2}, \infty)$

□ Let ABC be any triangle,P,Q,R points on [AB],[BC],[CA] sides respectively and these satisfy $\frac{AP}{AB} = \frac{BQ}{BC} = \frac{CR}{CA} = k < 1/2$ If G point is centroid of ABC triangle find ratio of Area(PQG)/Area(PQR)

Solution

Let $AF \cap BC = M$. So $\triangle ABM = \frac{1}{2}\triangle ABC$. Now observe triangle ABM, we see that:

$$\triangle ABM = \triangle APG + \triangle QMG + \triangle GPQ + \triangle BPQ$$

Now it is just simple area ratios:

$$\frac{\triangle BPQ}{\triangle BMA} = \frac{BP \cdot BQ}{BA \cdot BM} = 2k(1 - k)$$

$$\frac{\triangle QMG}{\triangle BMA} = \frac{QM \cdot MG}{BM \cdot MA} = \frac{1 - 2k}{3}$$

$$\frac{\triangle APG}{\triangle ABM} = \frac{AP \cdot AG}{AB \cdot AM} = \frac{2k}{3}$$

Thus we can find the area ratio between $\triangle GPQ$ and $\triangle ABC$. But the area ratio between $\triangle PQR$ and $\triangle ABC$ is a famous and easy conclution, so now the problem is easy to solve.

$\square$ Prove if $n = p_1^{\alpha_1}...p_s^{\alpha_s}$ and $n > 1$ $\tau(n)\varphi(n) \geq n$

Solution

I'll use (which are in fact harder to prove than is result

$\sum_{d|n} \varphi(d) = n$ $d|n \Rightarrow \varphi(d)|\varphi(n)$

Then $\sum_{d|n} \varphi(n) = \tau(n)\varphi(n) \geq \sum_{d|n} \varphi(d) = n$

$\square$ This is exactly application of the famous inequality of Arhimedes which says that if m, n are positive integers then

$1^m + 2^m + ... + (n-1)^m < \frac{n^{m+1}}{m+1} < 1^m + 2^m + ... + n^m$.

But it would be great if we have a proof for this.

Solution

We can use induction on n to prove $1^m + 2^m + ... + (n-1)^m < \frac{n^{m+1}}{m+1} < 1^m + 2^m + ... + n^m$. When $n = 2$ the inequality is obviously true. Assume the inequality is true when $n = k$, i.e. $1^m + 2^m + \cdots + (k-1)^m < \frac{k^{m+1}}{m+1} < 1^m + 2^m + ... + k^m$ When $n = k+1$, For the left side, $1^m + 2^m + \cdots + (k-1)^m + k^m < \frac{k^{m+1}}{m+1} + k^m = \frac{k^{m+1} + (m+1)k^m}{m+1} < \frac{(k+1)^{m+1}}{m+1}$ with the last inequality by binomial theorem. For the right side, $1^m + 2^m + \cdots + k^m + (k+1)^m > \frac{k^{m+1}}{m+1} + (k+1)^m = \frac{k^{m+1} + (m+1)(k+1)^m}{m+1}$ So it suffices to prove $k^{m+1} + (m+1)(k+1)^m > (k+1)^{m+1}$

$\Leftrightarrow (m+1)(k+1)^m > (k+1)^{m+1} - k^{m+1}$

$\Leftrightarrow \sum_{i=0}^m (m+1)\binom{m}{i} k^{m-i} > \sum_{i=0}^m \binom{m+1}{i+1} k^{m-i}$ This is true becasue $(m+1)\binom{m}{i} = \frac{(m+1)m(m-1)\cdots(m-i+1)}{i!} > \frac{(m+1)m(m-1)\cdots(m-i+1)}{(i+1)!} = \binom{m+1}{i+1}$

$\square$ Solve the equation in integer numbers: $x^2 + 3y^2 = 74x$

Solution

Wlog $y \geq 0$, we have $x \geq 0$. Looking mod 2, x and y must be both even. Looking mod 4 again, y must be $4Y$

Equation to solve is the $12Y^2 = X(37 - X)$, which implies $X \leq 37$ and $Y \leq 5$. This gives only 6 cases to check manually. Or Since $gcd(X, 37 - X) \neq 2, 3or4$ we must have $X = 0$ mod 12 or $X = 0$ mod 3 and $37 - X = 0 \mod 4$, ...

Putting everything together : $0, 0; 24, \pm 20; 50, \pm 20; 74, 0$

$\square$ A set of numbers is called [i]sum-free set[/i] if no two of them add up to a member of the same set and if no member of the set is double another member. How big could be a sum free subset of 1,2,3,...,2n+1)?

Solution

Taking $1, 3, ..., 2n+1$ we see that the number is $\geq n+1$

Let's we take a subset with (strictly) more then $n+1$ elements : $a_1 < ... < a_k$ The difference sequenc $a_k - a_{k-1}, ..., a_k - a_1$ takes at least $n+1$ different values, which, by pigeon-hole, can't be all different from the $a_1, .., a_{k-1}$.

So max is $n+1$.

$\square$ Find the number of positive integers which divide 10^{999} but not 10^{998}.

Solution

suppose $d_1|2^{999}5^{999}$, $d_2|2^{998}5^{998}$. We want to find the number of d_1 such that d_1 deosnt divide $2^{998}5^{998}$. We see that when d_1 has factor of 2^{999} then there are 1000 ways from $2^{999} \times 5^k$ for $0 \leq k \leq 999$. Simialrly if d_1 has the factor of 5^{999} then there are 999 ways since $2^k \times 5^{999}$ where $0 \leq k \leq 998$.

So total way is 1999

$\square$

Suppose that for any number a there is a point on the graph of $y = f(x)$ closest to the point $(a, 0)$ (this is guaranteed when f is continuous, but that's not important). Define $g(a)$ as the distance from $(a, 0)$ to that point, and prove that for all c and d, $g(c) - g(d) \leq |c - d|$.

Solution

We will call C and D the points of the graph of f achieving the required minimal distances. Assume the result to be false. Then $\exists c, d, g(c) > |c-d| + g(d)$. That means C must lie outside the circle with center $(c, 0)$ and radius $|c - d| + g(d)$. But every point D must lie inside this circle (by the triangle inequality); in particular, $d((c, 0), D) \leq |c - d| + g(d)$, which is absurd.

$\square$ There are six points on the plane, and no three points are collinear. Let G_1 be the centroid of a triangle formed by three points that are randomly chosen from the six points, and let G_2 be the centroid of a triangle formed by the other points. Prove that a line connecting G_1 and G_2 goes through a fix point regardless of how we choose the three points.

Solution

We can pick 2 different lines, so that there will be at-most one point.

Let the points be $P_i = (x_i, y_i)$ for $i \in \{1, 2, 3, 4, 5, 6\}$. We guess the "fixed point" we want is $P = (\frac{\sum_{i=1}^{6} x_i}{6}, \frac{\sum_{i=1}^{6} y_i}{6})$.

The easiest way is to look at the area G_1G_2P for any choice of triangle that will give G_1, G_2.

Wlog, P_1, P_2, P_3 has centroid $G_1 = (x_1, y_1)$ and likewise, $G_2 = (x_2, y_2)$.

Then we only need to verify $\frac{x_1+x_2}{2}y_1 + x_1y_2 + x_2\frac{y_1+y_2}{2} = \frac{x_1+x_2}{2}y_2 + x_1\frac{y_1+y_2}{2} + x_2y_1$. It works.

So we have found the unique point P. – The equation $(1 + x^3)^4 + (1 + x^2)^4 = 2x^4$ has real roots ?

$\square$ hình

$\square$ hình

$\square$ đa thức

$\square$ Given $B > 0$ that $\frac{x^3}{y} \geq A(x - y - B),\ x, y \in \mathbb{R}_+$. Find max A.

Solution

If $x < y + B$, then the RS is negative and the ineq must be true. This is the motivation for the substitution $x = y + B + k$. Assuming that the RS is positive, then k is positive.

Then $\frac{(y+B+k)^3}{yk} \geq A(*)$, so we want to find the minimum of the LS of (*). We have y,B,k positive.

Considering as a function in y, the derivative has a sign equal to the sign of $2y - B - k$. Then $y = \frac{B+k}{2}$ for the minimum.

Similarily, $k = \frac{B+y}{2}$. Solving, $y = k = B$. So the minimum of the LS of (*) is $27B$. This is the maximum for A.

ps this might be bad, i didnt check it

$\square$ During a certain lecture, the caterers didn't bring enough coffee, so each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that at some moment, some three were sleeping simultaneously.

Solution

Suppose not, for a contradiction.

Let A and B "share" sleeps, if for the pair of people (A,B), there exists a moment where they were both asleep. Let B "drop" A, if there is a time where A starts sleeping, then B starts and then wakes up, then A wakes up.

Case 1: There exists A, B so that B drops A. Then B shares with one person during that sleep, and on his other sleep must share with the other three people. It can only work if C drops B, for some $C \neq A$. Then C must share with the other 3 people on his other sleep, etc. so that we cannot fuffill it. Contradiction.

Case 2: There doesn't exist A, B so that B drops A. Then during one sleep, A needs to share with 2 other people (only 2 sleeps, maximum 2 shares per sleep). Consider the person A that woke up last. He can only share with one person. Contradiction.

☐ Here's a nice problem. There is an odd number of people in a plane. Their mutual distances are different. Everyone shoots his nearest neighbour. Prove that a) at least one person survives; b) No one is hit by more than 5 bullets; c) the path of the bullets do not cross; d) the set of segments formed by the bullet paths does not contain a closed polygon.

Solution

By induction. Suppose its true for n people (n odd). Add two people. Now the two people with shortest distance (say, A B) shoot each other. We revert to the "n" hypothesis, except that its possible for some people to shoot A or B. These shots are essentially "wasted shots." Since in hypothesis atleast one person survives, if we remove some guns, then atleast one person still survives.

Suppose ABCDEF shoot O. Consider angles AOB, BOC, ..., EOF, FOA. Atleast one is ≤ 60 deg. By "larger side larger angle" one of the AB's are smaller than the OA's. Contradiction.

Suppose A shoots B and C shoots D, with paths AB and CD crossing at O. We have a quad ACBD, with B closest to A and D closest to C. This means the main diagonal AB is shorter than AD or AC, and CD is shorter than AC or BC. So AC is longer than both of the main diagonals, contradiction.

Suppose we have polygon $A_1, A_2, \ldots, A_n$, with A_1 shooting A_2, ... until A_n shoots A_1. Consider circle with center A_1 and radius A_1A_2. It cannot have any points A_i inside. But for A_n to shoot A_1, A_1 must be closest to A_n, where the distance A_1A_n must be less than $A_{n-1}A_n$, which must be less than .... A_2A_1. It implies A_n is in the circle. Contradiction.

☐ Prove: From the set $\{1, 2, ..., n\}$, one can choose a subset with at most $2\lfloor\sqrt{n}\rfloor + 1$ elements such that the set of the pairwise differences from this subset is $\{1, 2, ..., n-1\}$. ($\lfloor x\rfloor$ means the greatest integer $\leq x$)

Solution

Let k be the largest integer so that $2^k < n$. Then choose 2^j for j = 0 ... k, and n. We have chosen at most $\lfloor log_2 n\rfloor + 1$ numbers.

To prove $\lfloor log_2 n\rfloor \leq 2\lfloor\sqrt{n}\rfloor$; we prove $log_2 n \leq 2\sqrt{n} + 2$, equiv. to $n \leq 4 * 4^{\sqrt{n}}$. Wrt n, the deriv. of the RS is $\frac{4\log(4)(\sqrt{n}+1)}{\sqrt{n}}$, of LS: 1. Then its clear the RS increases faster than the LS; also, RS > LS for small cases. It is enough.

☐ hình

☐ Find the number of unordered pairs $\{A, B\}$ of subsets of an n-element set X that satisfies the following: (a) $A \neq B$ (b) $A \cup B = X$

Solution

Little doubt this problem was posted before with so many problems in stock ..... But with so many it may be quicker to solve it ourselve instead of using search feature ;)

So easy to get wrong with combinatorics but I propose $\frac{3^n-1}{2}$

The number of ordered parts such as $A \cup B = X$ is $3^n = \sum \binom{n}{k} \cdot 2^k$ (choose k elements for A, elements for $B - A$ are imposed and we can take any subset of A to complete B).

Now we eliminate case $A = B$ (only one case!) and we divide by 2 to have ordered pairs.

☐ Show that there exists an equiangular hexagon in the plane, whose sides measure 5,8,11,14,23, and 29 units in some order.

Solution

The hexagon with side lengths 8, 29, 5, 14, 23, 11, in this order, is an equiangular hexagon.

Proof:

Consider equilateral triangle XYZ with side length 42. Take points A and B on XZ and YZ respectively such that $AB \parallel XY$ and $ZA = ZB = 8$. Take points C and D on YZ and XY respectively such that $CD \parallel XZ$ and $YC = YD = 5$. Take points E and F on XY and XZ respectively such that $EF \parallel YZ$ and $XE = XF = 23$.

Note that $\triangle ZAB$, $\triangle YCD$, and $\triangle XEF$ are equilateral triangles. By "cutting out" these three small equilateral triangles, we obtain a hexagon with side lengths $AB = 8$, $BC = 29$, $CD = 5$, $DE = 14$, $EF = 23$, and $FA = 11$. Since we have cut out small equilateral triangles from a large equilateral trianlge, each interior angle of the resultant hexagon is $120°$, and thus hexagon $ABCDEF$ is an equiangular hexagon.

$\square$ Let a, b, c be positive integers such that a divides b^2, b divides c^2 and c divides a^2. Prove that abc divides $(a+b+c)^7$.

Solution

$a = \prod p_i^{a_i}$, $b = \prod p_i^{b_i}$, $c = \prod p_i^{c_i}$

With p_i prime.

Condition is $a_i \leq 2 * b_i \leq 4 * c_i \leq 8 * a_i$

When expandind $(a+b+c)^7$ we only have to consider monoms like a^7,$a^6 * b$, ... If we take for instance $a^6 * b$: $6a_i + b_i \geq a_i + b_i + c_i$ so $abc | a^6 * b$ and so on.

$\square$ What is the maximum area of a rectangle circumscribed about a fixed rectangle of length 8 and width 4?

Solution

Let $ABCD$ and $JKLM$ be the circumscribing rectangle and the fixed rectangle, respectively. J lies on AB, K lies on BC, L lies on CD, and M lies on AD. AJ subtends $\angle\theta$, and BK subtends another angle which is also $\angle\theta$. We know that $|ML| = |JK| = 8$ and $|MJ| = |KL| = 4$. We use trigonometry and get, $|AJ| = 4\sin\theta$, $|BJ| = 8\cos\theta$, $|AM| = 4\cos\theta$, and $|MD| = |BK| = 8\sin\theta$.

It is now clear that the area of the circumscribing rectangle is $A = (|AM| + |MD|)(|AJ| + |BJ|)$. Let us define A as a function of $theta$.

$$A(\theta) = (4\cos\theta + 8\sin\theta)(4\sin\theta + 8\cos\theta) = 16\sin\theta\cos\theta + 32\sin^2\theta + 32\cos^2\theta + 64\sin\theta\cos\theta = 80\sin\theta\cos\theta + 32 =$$

$$A'(\theta) = 80\cos 2\theta 0 = \cos 2\theta 2\theta = \frac{\pi}{2}\theta = \frac{\pi}{4}$$

$|AJ| = 4\sin\frac{\pi}{4} = 2\sqrt{2}$
$|BJ| = 8\cos\frac{\pi}{4} = 4\sqrt{2}$
$|AM| = 4\cos\frac{\pi}{4} = 2\sqrt{2}$
$|MD| = 8\sin\frac{\pi}{4} = 4\sqrt{2}$

We now determine the area of the circumscribing rectangle. $A = (|AM| + |MD|)(|AJ| + |BJ|) = (2\sqrt{2} + 4\sqrt{2})(2\sqrt{2} + 4\sqrt{2}) = (6\sqrt{2})^2 = \boxed{72}$. We see that our dimensions satisfy teh dimensions of a square, which has the largest area. $\mathbb{QED}$

$\square$ Polynomial P(x)=x^3+ax^2+bx+c have three different real roots. Q(x)=$x^2+x+2001$ Polynomial P(Q(x)) have no real root. Prove P(2001)>$\frac{1}{64}$

Solution

The minimum of $Q(x)$ is for $x=-\frac{1}{2}$ and it is $Q(-\frac{1}{2})=2001-\frac{1}{4}$

So $Q(x)>2001-\frac{1}{4},\ \forall x\in R$ and $Q(x)$ goes to infinity when x does.

Let $x_0=2001-\frac{1}{4}$

There is no root of $P(x)$ greater than (or equal to) x_0. If $y\geq x_0$ then from continuity of $Q(x)$ we get that there is a x such that $Q(x)=y\Rightarrow P(y)=P(Q(x))\neq 0$.

Let $x_1<x_2<x_3$ the three roots of $P(x)$.

They are all less than x_0

Now, we will find the sign for $P(x_0),P'(x_0),P''(x_0),P'''(x_0)$

The coefficient of x^3 in $P(x)$ is $1>0$, hence $\forall x>x_3, P(x)>0\Rightarrow P(x_0)>0$

$P'(x)=3x^2+2ax+b$ (3>0)

From Rolle Theorem we have that, $P'(x)$ has a root in (x_1,x_2) and a root in (x_2,x_3)

And after the second root (for greater values of x), $P'(x)>0$ (because 3>0). So, $P'(x_0)>0$

$P''(x)=6x+2a$

$P''(x)$ is an increasing line, and it has a root at the midpoint of roots of $P'(x)$.

Since x_0 is greater that all of them, we have $P''(x_0)>0$

Finally, $P'''(x_0)=6$

Now, we take Taylor around of $x_0=2001-\frac{1}{4}$

$P(x_0+h)=P(x_0)+\frac{P'(x_0)}{1!}h+\frac{P''(x_0)}{2!}h^2+\frac{P'''(x_0)}{3!}h^3$

If we set $h=\frac{1}{4}$ we have

$P(x_0)>0$

$\frac{P'(x_0)}{1!}h>0$

$\frac{P''(x_0)}{2!}h^2>0$

$\frac{P'''(x_0)}{3!}h^3=\frac{1}{1}h^3=(\frac{1}{4})^3=\frac{1}{64}$

Finally $P(x_0+h)=P(2001)>\frac{1}{64}$

Another way Since P(x) has three roots ,say p,q r ,then $P(x)=(x-p)(x-r)(x-q)$ $P(Q(x))=(Q(x)-p)(Q(x)-r)(Q(x)-q)$ Since Q(x) has no real roots so the discriminant of of Q(x)-p is negative it means $p+1/4<2001$ Same for r and q. We have $P(2001)=P(Q(0))=(2001-p)(2001-r)(2001-q)$ and using the inequalities above we get 1/64<P(2001)

□ find all pairs (p;q) of positive integers such that p+q and pq+1are both powers of 2

Solution

we have :

$p+q=2^a$ $pq+1=2^b$

So, $(p+1)(q+1)=2^a+2^b$ and $(p-1)(q-1)=2^b-2^a$. The last one implies that $b>a$.

Hence, $2^a|(p+1)(q+1)$ and thus : $p+1=2^ir_1$ and $q+1=2^{a-i}r_2$ So $p-1=2(2^{i-1}r_1-1)$ and $q-1=2(2^{a-i-1}r_2-1)$

But $2^a|(p-1)(q-1)$. Then, for $a>2$, we have $a=i+1$ and $p=2^{a-1}r_1-1$. Hence, $q-1=2(r_2-1)$. But, $2^{a-1}|q-1$. So $r_2=2^{a-2}r_3+1$ and finally : $q=2^{a-1}r_3+1$.

Finally, $p=2^{a-1}r_1-1$ and $q=2^{a-1}r_3+1$.

With the first relation, we obtain that $r_1+r_3=2$ and then $r_1=r_3=1$

Then, if $a>2$, $p=2^{a-1}-1$ and $q=2^{a-1}+1$.

Now, if $a<3$, we check that the solution are the same.

Now, if $a=b$, $(p-1)(q-1)=0$, and thus $p=1$ and $q=2^a-1$ or $q=1$ and $p=2^a-1$.

The solution are then the following:

$p = 2^{a-1} - 1$ and $q = 2^{a-1} + 1$. $q = 2^{a-1} - 1$ and $p = 2^{a-1} + 1$. $p = 1$ and $q = 2^a - 1$ $q = 1$ and $p = 2^a - 1$

☐ Let a right-angled parallelogram ABCD. Let K the midpoint of BC and L the midpoint of AD. The perpendicular line from B to AK intersects the AK at E and the CL at Z. Prove that the AKZL is an isosceles trapezoid. Prove that: $(ABKZ) = \frac{1}{2}(ABCD)$ (We symbolize with (......) the area) If the ABCD is a square with $AB = BC = CD = AD = a$ find the area of isosceles trapezoid AKZL as a function of a.

Solution

Let F be the point on AK such that $AL \parallel FZ$. Then $AFZL$ is a parallelogram since $AL \parallel FZ$ and $AF \parallel LZ$. So $AL = FZ (= BK)$.

Since $FZ = BK$ and $FZ \parallel BK$, $\triangle FEZ \cong \triangle KEB$. Thus $FE = EK$. In $\triangle FKZ$, the perpendicular line from Z to FK (which is ZE) bisects FK, so FKZ is an isosceles triangle with $FZ = FK$, which implies $AKZL$ is an isosceles trapezoid with $AL = KZ$ ($\because AL = FZ = KZ$).

Note that $BFZK$ is a parallelogram. Since $BF = FZ$ and $\angle AFB = \angle AFZ$, $\triangle ABF \cong \triangle AZF$ So $(ABF) = (AZF) = (AKZ) = (DLZ)$ (parenthesis means area.) Also, $(BFZ) = (BKZ) = (CKZ)$

Therefore, $(ABKZ) = (ABF) + (AFZ) + (BFZ) + (BKZ) = (ALZ) + (DLZ) + (CKZ) + (BKZ)$ $= (ADZ) + (BCZ) = \frac{1}{2}(ABCD)$

For the last problem:

To find the ratio $AE : EK$, look at $\triangle ABK$. Since $\frac{AE}{EK} = \frac{AB^2}{BK^2} = 4$, $AE : EK = 4 : 1$. So $AF : FE : EK = 3 : 1 : 1$ ($\because FE = EK$)

$(AKZL) = (AFZL) + (FKZ) = \frac{3}{5}(AKCL) + \frac{1}{2}(FKCZ) = \frac{4}{5}(AKCL) = \frac{4}{5} \cdot \frac{1}{2}a^2 = \frac{2}{5}a^2$

☐ $\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 8x \rfloor = 2005$

Solution

we know

$$15k \leq 2005 \leq 15k + \lfloor 2y \rfloor + \lfloor 4y \rfloor + \lfloor 8y \rfloor$$

This implies k=133. Our problem is then reduced to solving

$$\lfloor 2y \rfloor + \lfloor 4y \rfloor + \lfloor 8y \rfloor = 10$$

Substitute $y = 1 - p, p > 0$ and using the identity $\lceil u \rceil = -\lfloor -u \rfloor$, we now must solve

$$\lceil 2p \rceil + \lceil 4p \rceil + \lceil 8p \rceil = 4$$

because $p > 0$ all terms on the LHS will be at least 1 implying that the LHS is at least 3. So we need for $\lceil 4p \rceil = 1$ and $\lceil 8p \rceil = 2$. For all real u it is known $u \leq \lceil u \rceil < u + 1$. For our first equation we have $4p \leq 1 < 4p + 1$ yielding $0 < p \leq \frac{1}{4}$. Similarly we have $8p \leq 2 < 8p + 1$ which yields $\frac{1}{8} < p \leq \frac{1}{4}$. This last range is also the intersection of the two and thus gives all possible values for p. After substitution we arrive at our solution set to the original problem: $133\frac{3}{4} \leq x < 133\frac{7}{8}$

☐ Prove that $\frac{gcd(m,n)}{n}\binom{n}{m} \in Z^+$ for all $n \geq m \in Z^+$

Solution

$\gcd(m, n)$ is a linear combination of m and n, and $\frac{m}{n}\binom{n}{m} = \binom{n-1}{m-1}$ I think, so the given number is some linear combination of $\binom{n-1}{m-1}$ and $\binom{n}{m}$ and hence an integer.

☐ Is there any formula for $\tan(x_1 + x_2 + ... + x_n)$?

Solution

$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$ (*)

Let $T(x)$ be the tangent function, and $a' = T(a)$ for all a.

We have the familiar $T(a+b+c)=\frac{a'+b'+c'-a'b'c'}{1-a'b'-b'c'-c'a'}$.

Using (*), we have $T(a+b+c+d)=\frac{\frac{a'+b'+c'-a'b'c'}{1-a'b'-b'c'-c'a'}+d'}{1-\frac{a'+b'+c'-a'b'c'}{1-a'b'-b'c'-c'a'}d'}$

$=\frac{a'+b'+c'-a'b'c'+d'-ab'd'-b'c'd'-a'c'd'}{1-a'b'-b'c'-c'a'-a'd'-b'd'-c'd'+a'b'c'd'}$

This strongly suggests the following. Suppose $a_1, a_2, \ldots a_n$ have symmetric polynomials $s_1, s_2, \ldots s_n$ (in example, $s_1=\sum a_i$, $s_2=\sum_{i<j} a_i a_j$, ..., $s_n=\prod a_i$). Define $s_0=1$.

We claim $T(a_1, a_2, \ldots, a_n)=\frac{s_1-s_3+s_5-\ldots}{s_0-s_2+s_4-\ldots}$. ($\Box$) We prove this by induction.

We use (*) to complete the induction step. Let s'_k be the kth symmetric polynomial of the terms $a_1 \ldots a_{n+1}$.

Then $T(a_1, \ldots, a_n, a_{n+1})=\frac{(s_1-s_3+s_5-\ldots)+a_{n+1}(s_0-s_2+s_4-\ldots)}{(s_0-s_2+s_4-\ldots)-(s_1-s_3+s_5-\ldots)a_{n+1}}$

where it remains to verify $(s'_1-s'_3+s'_5-\ldots)=(s_1-s_3+s_5-\ldots)+a_{n+1}(s_0-s_2+s_4-\ldots)$ (and $(s'_0-s'_2+s'_4-\ldots)=(s_0-s_2+s_4-\ldots)+(s_1-s_3+s_5-\ldots)a_{n+1}$ (

We verify

We verify

Thus, ($\Box$) is proved.

$\Box$ The least common multiple of positive integers a, b, c and d is equal to $a+b+c+d$. Prove that $abcd$ is divisible by at least one of 3 and 5.

Solution

The main idea is to limit the least common multiple value.

Say $a \geq b \geq c \geq d$. Then it is seen that $lcm=a+b+c+d \leq 4a$, but a must divide it, so it must be a, $2a$, $3a$,or $4a$.

The cases a, $3a$ and $4a$ can be dealt with easily. The case $2a$, which gives $a=b+c+d$, remains. Again, we limit the lcm value.

Hope you can continue from here

$\Box$ Find all $m \in N$ such that $(x+y)(y+z)(z+x)$ divides $x^m+y^m+z^m-(x+y+z)^m$

Solution

Not much to prove but if $(x+y)(y+z)(z+x)$ divides $f(x,y,z)=x^m+y^m+z^m-(x+y+z)^m$ then one of $(x+y),(y+z),(z+x)$ must be a zero of $f(x,y,z)$. WLOG assume $(x+y)$ is a zero. That is $f(x,-x,z)=0$. So $x^m+(-x)^m=0$ which is true only if m is odd.

$\Box$ Find all functions f which maps integer to integer such that $2000f(f(x))-3999f(x)+1999x=0$ for all integer x

Solution

It is $2000(f(f(x)-f(x))=1999(f(x)-x)$

Fix x.

So the RHS has a factor 2000. Thus, $f(x)=2000k_x+x$ for some k_x.

Then, $f(f(x))-f(x)=2000(k_{2000k_x+x}-k_x)$.

So the RHS has a factor 2000^2. Thus, $f(x)-x=2000^2k_x+x$, for some k_x,

implying the LHS has a factor 2000^3.. etc.

Contradiction. So $f(x)-x=0$.

$\Box$ Let a_n, a_{n+1}, which are two terms of the sequence $a_1, a_2, \cdots\cdots a_n, \cdots\cdots$, be the roots of the quadratic equation $x^2+3nx+C_n=0$.

If $a_1=1$, find $\sum_{n=1}^{2p} C_n$

Solution

$a_n+a_{n+1}=-3n$, where it is easy to get $a_{n+2}=a_n-3$.

Now $\sum_{k=1}^{2p} a_n a_{n+1}=\sum_{k=0}^{n-1} a_{2k+1}a_{2k+2}+a_{2k+2}a_{2k+3}$

$= \sum_{k=0}^{n-1}(1-3k)(-4-3k)+(-4-3k)(-2-3k) = \sum_{k=0}^{n-1} 18k^2+27k+4$
$= 4n+27(n)(n-1)/2+3(n-1)(n)(2n-1)$

☐ The vertices of a convex pentagon have integer coordinates. Find the least possible area of the pentagon.

Solution

Picks formula says $A = I + B/2 - 1$, where there are I lattice points inside the polygon and B on the edges.

We cant get I = 0 because its convex(*). The minimum of B is 5. We can acheieve I = 1 and B = 5 easily. Thus the area is 5/2.

(*): This requires some explanation: Basically, if there are no interior points, the polygon must fit on a horizontal or vertical strip of length 1. (If it doesn't, then it is atleast 2x2 which carries one point) But then 3 points are collinear, contradiction.

☐ Are there permutations a, b, c and d of $\{1, 2, \dots, 50\}$ such that

$$\sum_{i=1}^{50} a_i b_i = 2\sum_{i=1}^{50} c_i d_i \ ?$$

Solution

we make the LHS as large as possible and the RHS as small as possible and show they still dont meet.

By rearrangement, a_i and b_i are samesorted and c_i and d_i are oppositely sorted.

Thus we compare $\sum i^2$ to $2\sum i(n+1-i)$

The first is $(1/6)(n)(n+1)(2n+1)$. The second is $n^2(n+1)+n(n+1)-n(n+1)(2n+1)/3$

It is equivalent to compare $(1/2)(2n+1)$ to $n+1$. Obviously, the second one is bigger. So they never meet.

☐ Solve in integers the equation $(2x^2-5x+2)3^x = 1-4x^2$

Solution

The equation factorises into $(1-2x)((2-x)3^x-(1+2x)) = 0$ Therefore, $(1-2x) = 0, x = \frac{1}{2}$, or $(2-x)3^x = 2x+1$ We see that $x = 1$ is a solution For $x \geq 2, (2-x)3^x \leq 0$, but $2x+1 > 0$ For $x \leq -\frac{1}{2}, 2x+1 \leq 0$, but $(2-x)3^x > 0$, and 0 is not solution

Hence, the only solution in integer is $x = 1$

☐ số học

☐ Find primes p, q, r and positive integer a that satisfies $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = \frac{1}{a}$

Solution

Suppose p,q,r are distinct. $\frac{pq+pr+qr}{pqr} = \frac{1}{a}$ Because the numerator is congruent to $qr \not\equiv 0 \pmod{p}$, the numerator shares no factor of p with the denominator. Doing the same for q and r, we find that the LHS is fully simplified. If p and q are the same but r is different, the denominator has a factor of r but the numerator doesn't, so we cannot possibly simplify (pq+pr+qr)/(pqr) so that the numerator is 1. Thus $p = q = r$. $3p^2/p^3 = 3/p = 1/a$. The only prime that is a multiple of 3 is p=3. So (3,3,3,1) is our only solution.

Another way

we easily get $a(pq+pr+qr) = pqr$ because $p, q, r \in \mathbb{P}$ there are WLOG 4 possible values for a: 1, p, pq, pqr

1. $a = 1$: $pq+pr+qr = pqr$ $r(p+q) = pq(r-1)$ since $\gcd(r, r-1) = 1$ we get WLOG either $r = p$ or $r = pq$ 1.1. $r = p$: $r(r+q) = rq(r-1)$ $r+2q = qr$ hence r be divisable by q, and since

$q, r \in \mathbb{P}$, $r = q$ $q + 2q = q^2$ $q(q-3) = 0$ hence $p = q = r = 3$ and $a = 1$, works out in the equation as well. 1.2 $r = pq$: this is not possible since r is prime.

2. $a = p$: $p(pq + pr + qr) = pqr$ $pq + pr = 0$ is not possible since all terms of the sum are positive
3. $a = pq$: $pq(pq + pr + qr) = pqr$ $pq + r(p + q - 1) = 0$ all terms positiv again
4. $a = pqr$: $pqr(pq + pr + qr) = pqr$ how should that be possible ??

so we get: only solution is $(p, q, r, a) = (3, 3, 3, 1)$

□ find all integral solutions a, b, c (that means a, b, c are integers):

$$a^2 + b^2 + c^2 = a^2b^2$$

Solution

First case: a, b and c are even and $abc \neq 0$ Let k be the greatest integer such that $2^k|a, 2^k|b, 2^k|c$ ($k \geq 1$ since $abc \neq 0$), we note $a = 2^ka', b = 2^kb', c = 2^kc'$ $a^2 + b^2 + c^2 = (ab)^2 \Leftrightarrow a'^2 + b'^2 + c'^2 = a'^2b^2$ 2^{2k} divide b^2 then $a'^2 + b'^2 + c'^2 \equiv 0(4)$ implies that a', b', c' are even, absurd

Second case: a, b and c are even and $abc = 0$ If a or b equals 0 then $a = b = c = 0$ If $c = 0$ then $a^2 + b^2 = (ab)^2$, we set $d = gcd(a, b)$ and $a = da', b = db'$ then $a'^2 + b'^2 = a'^2b^2$ then a' divide b' thus $a' = 1$ (or -1), we have $1 + b'^2 = d^2b'^2 \Rightarrow 1 = b'^2(d-1)(d+1)$ impossible

Third case: a, b, c are not all even $(ab)^2 \equiv 0(4)$ implies that a, b, c are even then $(ab)^2 \equiv 1(4)$. Consequently a and b are odd then we have $1 + 1 + c^2 \equiv 1(4) \Rightarrow c^2 \equiv 3(4)$ absurd

There is only the solution $(0, 0, 0)$

□ Find the pair of positive integer such that $(1 + x + y)^2 = 1 + x^3 + y^3$.

Solution

Put $x + y = a$ and $xy = b$ then the given equality becomes $(1 + a)^2 = 1 + a^3 - 3ab$ doing calculations one obtains $a(a + 2) = a(a^2 - 3b)$ so either $a = 0$ or $a + 2 = a^2 - 3b$ for $a = 0$ we have $x + y = 0$ which is impossible because $x, y > 0$

if $a + 2 = a^2 - 3b$ then replacing a and b we get $x^2 + y^2 - xy - x - y - 2 = 0$ upon multiplication with 2 and completing squares we get $(x - y)^2 + (x - 1)^2 + (y - 1)^2 = 6$

obviously $(x - 1)^2 = 1$ or 4 and then we easily find the pair..

□ Fix k, n. We have k = 0,1,2 is trivial so we dont consider.

Choose largest s so that $k(2n - s) \leq (4n - s)$. Then it is obvious $s \geq \frac{2nk-4n}{k-1} = 2n - \frac{2n}{k-1} = 2n(\frac{k-2}{k-1})$ as well as $2(k-2)(k) \geq (k-1)^2$. The result follows from $(A_i \cap A_j) \geq s$.

Solution

Set L has $4n$ elements. Sets $A_0, A_1, ..., A_k$ each have $2n$ elements, and $A_i \subset L$, $\forall i = 0, 1, ..., k$. Prove that $\exists i, j \in \{1, 2, ..., n\}$, $n(A_i \cap A_j) \geq (1 - \frac{1}{k})n$

□ Seven students in a class compare their marks in 12 subjects studied and observe that no two of the students have identical marks in all 12 subjects. Prove that we can choose 6 subjects such that any two of the students have different marks in at least one of these subjects.

Solution

Let $\{X_i\}_{i=1}^{12}$ be the subjects, and (a, b, c, d, e, f, g) be the students.

If (in example) $(a, b) \in X_1$, then one of $(a, j), (b, j)$ is in X_1, for j = c,d,e,f,g.

Our strategy is as follows. Pick a subject containing (a,b) [there is atleast one]. From our lemma above, we guarantee 5 other unique pairs in that subject. Now pick a subject containing a pair we havent got to yet. We guarantee 4 other unique pairs, because only one can repeat (it is easy to show **) Continuing in this fashion, we guarantee 6+5+4+3+2+1 = 21 unique pairs, and we are done.

A short proof with handwaving: if in example we used (a,b), then we have also used either (x,a) or (x,b). So when we use (i, j), we use either (y,i) or (y,j). So if (x,a) = (y,i) in example, we have a,b,i,j,x fixed, so that when y varies it can only match up with (x,a) at most once.

□ Let $a, b, c \in Q$ such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$

Show that $A = \sqrt{a^2 + b^2 + c^2} \in Q$

Solution

Multiplying $a^2 + b^2 + c^2$ with $(a + b)^2$ yields $(a^2 + b^2)(a + b)^2 + c^2(a + b)^2 = (a^2 + b^2)(a + b)^2 + (ab)^2$ $= (a^2+b^2)^2+2(ab)(a^2+b^2)+(ab)^2 = [(a^2+b^2)^2+(ab)(a^2+b^2)]+[(ab)(a^2+b^2)+(ab)^2] = (a^2+b^2+ab)^2$ since a,b is rational, we have $a^2 + b^2 + c^2 = \frac{(a^2+b^2+ab)^2}{(a+b)^2}$ so $\sqrt{a^2 + b^2 + c^2} = \frac{(a^2+b^2+ab)}{(a+b)}$ ANother way We have:

$a^2 + b^2 + (\frac{ab}{a+b})^2$

where it remains to prove $\sqrt{(a^2 + b^2)(a + b)^2 + (ab)^2} \in Q$

Put s = a+b, p = ab. It becomes $\sqrt{(s^2 - p)^2}$.

□ Prove that $\frac{1}{l_a} + \frac{1}{l_b} + \frac{1}{l_c} \leq \frac{1}{r}$

Solution

I don't know this inequality before, but after I saw your post, I found out that I've just proved it!

Let I be the incentre of the $\triangle ABC$ and D be the intersection of the angle bisector of $\angle A$ and BC. Hence:

$$\frac{ID}{AD} = \frac{\triangle BIC}{\triangle ABC}$$

Let $\angle A = 2x$, $\angle B = 2y$, $\angle C = 2z$, we will have: $ID = \frac{r}{\sin{(x+2y)}}$. Sum up:

$$\sum \frac{ID}{AD} = \frac{\triangle BIC + \triangle BIA + \triangle CIA}{\triangle ABC}$$

$$\sum \frac{r}{l_a} \cdot \frac{1}{\sin{(x + 2y)}} = 1$$

$$\frac{1}{r} = \sum \frac{1}{l_a} \cdot \frac{1}{\sin{(x + 2y)}} \geq \sum \frac{1}{l_a}$$

With equality holds if $x + 2y = y + 2z = z + 2x = 90^o$, which gives: $x = y = z = 30^o$.

□ to hợp

□ For $a, b, c \in \mathbb{Q},\ a \neq b \neq c \neq a$ show:

$\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} = d^2,\ d \in \mathbb{Q}$

Solution

set $p = 1/(a - b); q = 1/(b - c); r = 1/(c - a)$. We know $1/p + 1/q + 1/r = 0 \implies pq + qr + pr = 0$ $\implies p^2 + q^2 + r^2 = p^2 + q^2 + r^2 + 2pq + 2qr + 2pr = (p + q + r)^2$ so take $d = p + q + r$, and d is obviously rational.

□ Solve in R the following equation $x^8y^4 + 2y^8 + 4x^4 - 6x^4y^4 = 0$

Solution

let $a = x^4$ and $b = y^4$ that $a, b \in \mathbb{R}_0^+$ then we know: $6ab = a^2b + 2b^2 + 4a \geq 3 \cdot \sqrt[3]{ab^2 \cdot 2b^2 \cdot 4a} = 6ab$ so $6ab \geq 6ab$ with equality only if $a^2b = 2b^2 = 4a$ one trival solution is $(0, 0)$ and we see if a or b is zero the othe is zero as well. otherwise we can conclude: $a^2b = 2b^2 \Leftrightarrow a^2 = 2b$ $2b^2 = 4a \Leftrightarrow b^2 = 2a$ $a^2b = 4a \Leftrightarrow ab = 4$ and $a^2 \cdot ab = 2b \cdot 4 \Leftrightarrow a^3 = 8$ so $a = 2$ and $b = 2$

the solutions are $\{(x, y)|(0, 0), (\sqrt[4]{2}, \sqrt[4]{2}), (-\sqrt[4]{2}, \sqrt[4]{2}), (\sqrt[4]{2}, \sqrt[4]{2}), (-\sqrt[4]{2}, -\sqrt[4]{2})\}$ – $f(x) \geq 0$ for all $x > 0$ and $y > 0$ and $f(x + y) = f(x) + f(y) + 2\sqrt{f(x)f(y)}$ Solve for $f(x)$. – Find $n \in N$ such that $\phi(n)$ divides n

□ Find all positive integers n such that $n(n+60)$ is a perfect square.

Solution

Let $n(n+60) = (n+k)^2$.

$n^2 + 60n = n^2 + 2kn + k^2$ $60n - 2kn = k^2$

$n = \frac{k^2}{60-2k}$

$n > 0$ and $k^2 > 0$, so $60 - 2k > 0$ or $k < 30$.

Then, rewrite $n = \frac{k^2}{60-2k} = -\frac{1}{2}k - 15 + \frac{900}{60-2k}$, we see that $60 - 2k$ must divide 900.

From here, we can try some even numbers for k ($k < 30$).

When $k = 12, 20, 24, 28$, we obtain $n = 4, 20, 48, 196$ respectively, which are our desired answers.

Another way Suppose $n(n+60) = k^2$. Then we will have $(n+30)^2 = k^2 + 30^2$. Now we see that it is just like pytagoras theorem and we also have the pytagorean triplet satisfy $(p^2+q^2, 2pq, p^2-q^2)$ for integer $p \geq q$.

Firstly , if all the side of the right angle triangle a, b, c have $gcd(a, b, c) = 1$. Then we can say that $30 = 2pq$ which is equivalent to $pq = 15$. WLOG , we can say $(p, q) = (15, 1)$ or $(5, 3)$ which gives us $n + 30 = 226$ and 34 . Hence , $n = 196, 4$

Secondly , if $gcd(a, b, c) > 1$: there are 3 such cases

(i) when $gcd(a, b, c) = 3$, divide all the three side a, b, c by 3 then we know that one side of it is 10 . So from $2pq = 10$ The only possible solution is $(p, q) = (5, 1)$. This gives us the hypotenus is $26 \times 3 = 78 = n + 30$. Hence $n = 48$

(ii)When $gcd(a, b, c) = 5$, divide the three side by 5 , one of the side is 6 . So $pq = 3$ and $(p, q) = (3, 1)$ which yields $n = 20$

(iii)When $gcd(a, b, c) = 2$, one side is 15 . So $(p+q)(p-q) = 15$ and gives us $(p, q) = (4, 1), (8, 7)$. But both this value gives us $n = 4, 196$ which is same with above .

So all the posible solution is $n = 4, 20, 48, 196$

□ Prove that for every integer $n > 0$ there exists an integer $k > 0$ such that $2^n k$ can be written in decimal notation using only the digits 1 and 2. we can generalize the problem: Prove that for every integer $n > 0$ there exists an integer $k > 0$ such that $2^n k$ can be written in decimal notation using only the digits 1 and 2,and it has just n digits.

Solution

Suppose that for some n there exists k such that $2^n \cdot k$ has only 1's and 2's in it's decimal expansion. Also assume that it's decimal expansion has n digits. We know that $2^n \cdot k = 0$ or $2^n \pmod{2}^(n+1)$. Also, $10^n = 0 \pmod{2}^n$, and thus we can find that $10^n = 2^n \pmod{2}^(n+1)$. Suppose $2^P n \cdot k = 2^n \pmod{2}^(n+1)$. Then $10^n + 2^n \cdot k \equiv 2 \cdot 2^n \equiv 0 \pmod{2}^(n+1)$. Thus $10^n + 2^n \cdot k$ is a $n+1$- digit number divisible by $2^(n+1)$ made up of only 1's and 2's. Now suppose $2^n \cdot k \equiv 0 \pmod{2}^(n+1)$. Then $2 \cdot 10^n \equiv 0 \pmod{2}^(n+1)$. So $2 \cdot 10^n + 2^n \cdot k \equiv 0 \pmod{2}^(n+1)$. $2 \cdot 10^n + 2^n \cdot k$ is an n+1 digit number divisible by $2^(n+1)$ made up of only 1's and 2's. However, $2^1 \cdot 1$ is a 1-digit number with only 1's and 2's in it's decimal expansion. So by induction, we are done.

Note how this gives us an algorithm to generate such numbers:

1. Start with an n-digit number $2^n \cdot k$. 2. Find $a = 2^n \cdot k \pmod{2}^(n+1)$. 3. If$a = 0$, then 2 folllowed by $2^n \cdot k$ is the $n+1$-digit number we are looking for. If $a = 2^n$, then 1 followed by $2^n \cdot k$ is our $n+1$-digit number.

Example:

12 is a 2-digit number divisible by 2^2. $12 \pmod{2}^3 = 4 = 2^2$, so 112 is divisible by 2^3.

□ Let S denote the set of all nonnegative integers whose base-10 representation contains no 1s.

Compute

$$\prod_{k\in S}\frac{10k+2}{10k+1}$$

or show that it diverges.

Solution

Convergence Let $f(x)=\frac{x}{x-1}$, so we are examining

$$P=[f(22)f(32)f(42)...f(92)][f(202)f(222)f(232)...f(992)][f(2002)f(2022)...]...$$

where there are $8\cdot 9^{k-2}$ arguments with k digits.

Because $f(x)$ is decreasing, $P<(f(22))^8(f(202))^{72}(f(2002))^{648}...$

Therefore $\log P<8\log f(22)+72\log f(202)+648\log f(2002)+...$

Now $\log f(x)<\frac{2}{x}$ for $x>2$, so $\frac{9}{8}\log P<9\cdot\frac{1}{11}+9^2\cdot\frac{1}{101}+9^3\cdot\frac{1}{1001}+...<\frac{9}{10}+\frac{81}{100}+\frac{729}{1000}+...$ which implies $\frac{9}{8}\log P<\sum_{i=1}^{\infty}(\frac{9}{10})^i=9\implies\log P<8\implies P<e^8$. – The arithmetic progression series $\{a_n\}$ and $\{b_n\}$ has sum of first n term as S_n and T_n respectively. If $\frac{S_n}{T_n}=\frac{2005n}{2006n+1}$, find $\frac{a_n}{b_n}$. — Let a_n be an arithmetic progression containing only natural numbers. Prove that for any p in the sequence $\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3},....$ there exist p terms in arithemtic progression. – n, a, b, c and d are non-negative integers such that $a^2+b^2+c^2+d^2=n^2-6$, $a+b+c+d\le n$, $a\ge b\ge c\ge d$. Find all ordered pairs (n,a,b,c,d). —— A triangle ABC has $\angle ACB>\angle ABC$. The internal bisector of $\angle BAC$ meets BC at D. The point E on AB is such that $\angle EDB=90°$. The point F on AC is such that $\angle BED=\angle DEF$. Show that $\angle BAD=\angle FDC$. – An operation displaying either the sign + or x are made on a computer display repeatedly. In each operation, assume that the probability displaying the same sign as the eve of one succesively in regardless of the course until then is p. First the sign x is displayed on the display. Let P_n be the probability such that n's the sign + appears before the three sign X will appear including the first. Note that the operation is over at the stage of appearing n's the sign +.

(1) Express P_2 in terms of p.

(2) For $n\ge 3$, express P_n in terms of p and n.

□ Determine all triples of positive integers (a,b,c) such that $a\le b\le c$ and $a+b+c+ab+bc+ca=abc+1$.

Solution

Since $2\le a\le b\le c$, we have $\frac{a}{b}\le 1,\ \frac{a}{c}\le 1$ and $\frac{2}{c}\le 1$.

Thus $a+b+c+ab+bc+ca=abc+1\ \cdots[*]\iff abc=a+b+c+ab+bc+ca-1$

$\iff a=\frac{a}{b}\cdot\frac{1}{c}+\frac{1}{c}+\frac{1}{b}+\frac{a}{c}+1+\frac{a}{b}\le\frac{1}{b}\cdot 1+\frac{1}{c}+\frac{1}{b}+1+1+1$

$=\frac{2}{b}+\frac{1}{c}+3\le 1+\frac{1}{2}+3=4.5$, yielding $a=2,3,4$.

Case 1: $a=2$

From $[*]$, we have $bc=3b+3c+1\iff(b-3)(c-3)=10$. Since $-1\le b-3\le c-3$, we have $(b-3,c-3)=(1,10),\ (2,5)$, yielding $(b,c)=(4,13),\ (5,8)$.

Similarly,

Case 2: $a=3$; $2bc-4b-4c=2\iff(b-2)(c-2)=5,\ 0\le b-2\le c-2$, yielding $(b,c)=(3,7)$.

Case 3: $a=4$; $3bc-5b-5c=3\iff(3b-5)(3c-5)=34,\ 1\le b-2\le c-2$, yielding $(b,c)=(2,13)$, which isn't suitable for $a\le b\le c$.

Therefore desired answer is $(a,b,c)=(2,4,13),\ (2,5,8),\ (3,3,7)$.

□ Let $ABCD$ be an orthodiagonal trapezoid such that $\measuredangle A=90°$ and AB is the larger base. The diagonals intersect at O, $(OE$ is the bisector of $\measuredangle AOD$, $E\in(AD)$ and $EF\|AB$, $F\in(BC)$. Let P,Q the intersections of the segment EF with AC,BD. Prove that:

(a) $EP = QF$;

(b) $EF = AD$.

Solution

a) $\Delta AEP \sim \Delta ACD \implies \frac{EP}{AE} = \frac{CD}{AD} \implies CD = EP.\frac{AD}{AE}$

$\Delta BFQ \sim \Delta BCD \implies \frac{FQ}{BF} = \frac{CD}{BC} \implies CD = FQ.\frac{BC}{BF}$

$\iff EP.\frac{AD}{AE} = QF.\frac{BC}{BF}$ considering that $EF \| AB$ by Thale's theorem, $\frac{BC}{BF} = \frac{AD}{AE}$ consequently: $EP = FQ$.

b) $\Delta DEQ \sim ABD \implies \frac{EQ}{DE} = \frac{AB}{AD} \implies EQ = \frac{DE.AB}{AD}$ and we know that $EP = FQ = \frac{AE.CD}{AD}$ so ,

$EF = \frac{DE.AB+AE.CD}{AD}$

Since the trapezoid is orthodiagonal we have that, $\Delta ADE \sim \Delta ACD \sim \Delta ABD$, simutaneously,with the bisector theorem, we get:

$\frac{AE}{AO} = \frac{DE}{DO} \implies \frac{AE}{DE} = \frac{AO}{DO} = \frac{AD}{CD} \implies CD.AE = AD.AE$ analogously $DE.AB = AD.DE$ thus: $EF = \frac{DE.AB+AE.CD}{AD} = AE + DE = AD$

□ The sidelengths of a triangle are a, b, c.

(a) Prove that there is a triangle which has the sidelengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$.

(b) Prove that $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le a + b + c < 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ca}$.

Solution

(a) a, b, c are sidelengths of a triangle iff $a < b + c$, $b < a + c$, $c < a + b$. So we are to prove that $\sqrt{a} < \sqrt{b} + \sqrt{c}$ for all variables.

Because b, c are sidelengths, their value has to be positive. So I assume it will not shock you if I say that $0 < 2\sqrt{bc}$. Nor will the following statement be of any surprising content:

$$b + c < b + c + 2\sqrt{bc}$$

Both sides are positive, so we can take the square root:

$$\sqrt{b + c} < \sqrt{b} + \sqrt{c}$$

Now using the fact that a, b, c are sidelengths:

$$\sqrt{a} < \sqrt{b + c} < \sqrt{b} + \sqrt{c}$$

Analogue for the other variables.

(b) The left inequality has no need of the fact that a, b, c are sidelengths. Just use AM-GM and add up:

$$\frac{a+b}{2} + \frac{c+b}{2} + \frac{a+c}{2} \ge \sqrt{ab} + \sqrt{ac} + \sqrt{bc}$$

The right inequality:

$$RHS = \sqrt{a}.\left(\sqrt{b} + \sqrt{c}\right) + \sqrt{b}.\left(\sqrt{a} + \sqrt{c}\right) + \sqrt{c}.\left(\sqrt{a} + \sqrt{b}\right) > a + b + c$$

The last inequality uses lemma (a). Done!

□ Solve the system: $\begin{cases} max\{x + 2y, 2x - 3y\} = 4 \\ min\{-2x + 4y, 10y - 3x\} = 4 \end{cases}$

Solution

We can obtain four possible systems:

$$\begin{cases} x + 2y = 4 \\ -2x + 4y = 4 \end{cases}$$

Cramer gives

$$(x, y) = \left(1, \frac{3}{2}\right)$$

$$\begin{cases} 2x - 3y = 4 \\ -2x + 4y = 4 \end{cases}$$

Giving

$$(x, y) = (14, 8)$$

But this can not be a solution because $max\{x + 2y, 2x - 3y\} = x + 2y = 30$

$$\begin{cases} x + 2y = 4 \\ 10y - 3x = 4 \end{cases}$$

Giving

$$(x, y) = (2, 1)$$

But this is not valid since $min\{-2x + 4y, 10y - 3x\} = -2x + 4y = 0$

$$\begin{cases} 2x - 3y = 4 \\ 10y - 3x = 4 \end{cases}$$

Giving

$$(x, y) = \left(\frac{52}{11}, \frac{20}{11}\right)$$

But this is not valid since $min\{-2x + 4y, 10y - 3x\} = -2x + 4y < 0$

So this gives us only one valid solution:

$$(x, y) = \left(1, \frac{3}{2}\right)$$

– Given one hundred positive real numbers such that: $\sum_{i=1}^{100} a_i = 300$ and $\sum_{i=1}^{100} a_i^2 > 10000$. Show that there exist three numbers with sum more that 100.

$\square$ Consider a standard twelve-hour clock whose hour and minute hands move continuously. Let m be an integer, with $1 \leq m \leq 720$. At precisely m minutes after 12:00, the angle made by the hour hand and minute hand is exactly $1°$. Determine all possible values of m.

Solution

Consider the angles starting at 12:00, going clockwise, after m minutes, the minute hand will make an angle of $6m$, and the hour hand will make an angle of $\frac{m}{2}$, we then have that $|6m - \frac{m}{2}| = \pm 1 + 360k, m, k \subset \mathbb{Z}$

$$11m = \pm 2 + 720k$$
$$11m = \pm 2 + 5k + 11 * 65 * k$$
$$11(m - 65k) = \pm 2 + 5k$$

we get the solutions: $k = 4$, and $m - 65k = 2$ or $\boxed{m = 262}$ or $k = 7$, $m - 65k = 3$ then $\boxed{m = 458}$ by CRT these are the unique solutions mod 55 (the numbers being k and m-65k) if we add 55, the minutes are not in the given range so that is all the solutions

☐ For each permutation $a_1, a_2, a_3, \ldots, a_{10}$ of the integers $1, 2, 3, \ldots, 10$, form the sum

$$|a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8| + |a_9 - a_{10}|.$$

The average value of all such sums can be written in the form p/q, where p and q are relatively prime positive integers. Find $p + q$.

Solution

$\boxed{\frac{55}{3}}$ - I checked it with the official solutions too -

Because of symmetry, we may find all the possible values for $|a_n - a_{n-1}|$ and multiply by the number of times it appears $5*8!$ and take that over $10!$ since that's the number of total permutations.

To find all possible values for $|a_n - a_{n-1}|$ we have $(1-10) + (1-9) + \ldots + (1-2) + (2-1) + (2-3) + \ldots + (2-10) + \ldots (10-9)$

This is equivalent to

$$2\sum_{k=1}^{k=9}\sum_{j=1}^{k} j = 330$$

Now we multiply it by $5 * 8!$ because if you fix a_n and a_{n+1} there are still 8! spots for the others and you do this 5 times because there are 5 places a_n and a_{n+1} can be.

Therefore, the answer is $\frac{330*8!*5}{10!} = \frac{55}{3}$ – Let ABC and $A'B'C'$ are equilateral triangle inscribed in a same circle (with a center O). Let $X = AB \cap A'C'$. Prove that ABC and $A'B'C'$ are symetric compared to OX.

☐ Given $x_1, x_2, x_3..., x_n$ are sets of random numbers selected from an interval with length of 1. Let $x = \frac{1}{n}\sum_{j=1}^{n} x_j, y = \frac{1}{n}\sum_{j=1}^{n} x_j^2$. Find the maximum value of $y - x^2$

Solution

First Case: when n is even we let $n = 2k$, thus from the question we get, $\frac{2k(x_1^2+x_2^2+x_3^2+...+x_{2k}^2)-(x_1+x_2....+x_{2k})^2}{(2k)^2}$ we can easily see that it would be equal to $\frac{(2k-1)(x_1^2+x_2^2+x_3^2+...+x_{2k}^2)-2(x_1x_2+x_1x_3...+x_{2k-1}x_{2k})}{4k^2}$ We can see that $(x_1x_2 + x_1x_3... + x_{2k-1}x_{2k})$ contain each x_n $2(2k-1)$times The top part of the fraction would equal to $(x_1 - x_2)^2 + (x_1 - x_3)^2 + ... + (x_1 - x_{2k})^2 + (x_2 - x_3)^2 + ...(x_{2k-1} - x_{2k})^2$ since we wanted to find the maximum, we use the greatest difference between the numbers; thus we set first k terms be the largest number in the interval and second k terms be the smallest number in the interval; In $(x_1 - x_2)^2 + (x_1 - x_3)^2 + ... + (x_1 - x_{2k})^2 + (x_2 - x_3)^2 + ...(x_{2k-1} - x_{2k})^2$

when x_l is chosen from first k term, and x_m is chosen from second k terms. the term $(x_l - x_m)^2$ would be 1, otherwise it would be 0. Thus there is k choices for x_l, and k choices for x_m, so the sum of th top will be k^2 Therefore when n is an even number, the maximum will be $\frac{k^2}{4k^2}$=$\frac{1}{4}$ Second Case: When n is odd we let $n = 2k + 1$ we can let either $(2k + 1)$th term equal 1 or 0, if $(2k + 1)$th term is 0 there is k choices for x_l, and $(k + 1)$ choices for x_m, the sum would be $k(k + 1)$. if $(2k + 1)$th term is 1 there is $(k + 1)$ choices for x_l, and k choices for x_m, the sum would be $(k + 1)k$, thus either value of $(2k + 1)$th term would yield the same sum for the top. thus the value of the expression will be $\frac{(k+1)k}{(2k+1)^2}$ Since $n = 2k + 1$,k=$\frac{n-1}{2}$, subtitute into the expression thus we get the maximum value for n when n is odd is $\frac{n^2-1}{4n^2}$

☐ Prove that : $n^{kn} \geq (n^k + n^{k-1} + ... + 1)^{n-1}$ $k, n \in \mathbb{N}$

Solution

Start with $f(x) = \frac{\ln x}{x-1}$. Easy to see that it is strictly decreasing for $x > 0$. ($x = 1$ is no irregularity). So is $e^{f(x)} = x^{\frac{1}{x-1}}$.

$\implies (n-1)^{\frac{1}{n-2}} > n^{\frac{1}{n-1}}$ for $n > 1$. (forget n=1) $\implies (n-1)^{n-1} > n^{n-2} \geq n^{n-k-1}$ as $k \geq 1$. $\implies n^{kn}(n-1)^{n-1} > n^{(k+1)(n-1)} \implies n^{kn} > \left(\frac{n^{k+1}}{n-1}\right)^{n-1} > \left(\frac{n^{k+1}-1}{n-1}\right)^{n-1} = RHS$ – Let a_n be a sequence, $a_1 = 1$, for $n \geq 1$ $a_{n+1} = a_n + (1/a_n)$ Find the $[a_{1000}]$ [ ] denotes greatest integer function

☐ Given a sequence: 1, 0, 1, 0, 1, $0 \cdots$ From the 7th term, each term equals to the **last digit** of the sum of the 6 numbers before it. Show that there cannot be another series of 1, 0, 1, 0, 1, 0 in that sequence.

Solution

Let the series described be series A. Let's assume that a series of $1, 0, 1, 0, 1, 0$ can be repeated. So looking back at the 6 previous entries, we have: $a, b, c, d, e, f, 1, 0, 1, 0, 1, 0$. Let x and y be nonegative arbitrary integers. Let y be the number that is excluded from consecutive series. Basically $a+b+c+d+e+f = 10x+1$ (Because first term of series that's present is 1). $b+c+d+e+f+1 = 10x+2-y = 10z$. $\Rightarrow y = 2$. $c+d+e+f+1+0 = 10x-y = 10z+1$. $\Rightarrow y = 9$. $10x-9+1-y = 10z \Rightarrow y = 2$. $10x-10-y = 10z+1 \Rightarrow y = 9$. $10x-19+1-y = 10z \Rightarrow y = 2$. $10x-20-y = 10z+1$.

So $2, 9, 2, 9, 2, 9, 1, 0, 1, 0, 1, 0$.

But $3(2)+3(9) = 33$. Contradition.

QED

☐ prove that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ is never an integer for any n.

Solution

We'll consider the number:$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

The lemma: In the sequence of numbers: $1, 2, 3, \cdots, n$ there exists a number k, which is divisible by such power of 2, that does not divide any other element of the sequence.

Lemma 2$lcp(a_1, a_2, \cdots, a_n) = \sum_i p_i^{max(e_1, e_2, \cdots, e_t)}$, where $a_k = p_1^{e_1} p_2^{e_2} \cdots$

Now let

$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = k$, $k \in \mathbb{Z}$ Write our number using the least common denominator=the least common multiple. This multiple will look like this :$2^{max(e_1, e_2, e_3, \ldots, e_n)} \cdot somethingodd$ If we multiply both sides by this number, we will obtain even number on the right side and odd on the left side, because of the first lemma Contradiction

☐ Let $A(x)$ and $B(x)$ polynomials with degree greater than 1 and assume that exists polynomials $C(x)$ and $D(x)$ such that:

$$A(x) \cdot C(x) + B(x) \cdot D(x) = 1, \ \forall x \in \mathbb{R}.$$

Prove that $A(x)$ isn't divisible by $B(x)$.

Solution

Assume for the sake of contradiction that $B(x)|A(x)$. Then $A(x) = Q(x)B(x)$ so $A(x)C(x) + B(x)D(x) = B(x)[Q(x)C(x) + D(x)] = 1 \Rightarrow B(x)|1$. But since B has degree greater than 1, it obviously cannot divide something of degree 0. Contradiction.

☐ Let $0 < a_0 < a_1 < \ldots < a_n$ and $a_i \in Z$.(i=0,1,...,n). Prove that $\sum \frac{1}{[a_i, a_{i+1}]} \leq 1 - \frac{1}{2^n}$

Solution

To maximize $\frac{1}{[a_i, a_{i+1}]}$, we must minimize $[a_i, a_{i+1}]$. Clearly, this is na_i, where n is a natural number.

Since $a_i < a_{i+1}$, $n \neq 1$, the minimum of n is $n = 2$, which gives us $a_{i+1} = 2a_i$. Thus

$$\begin{aligned}\sum_{i=0}^{n} \frac{1}{[a_i, a_{i+1}]} &\leq \sum_{i=0}^{n} \frac{1}{2 \cdot 2^i} \\ &= \frac{1}{2}\left(\frac{1-\frac{1}{2^n}}{\frac{1}{2}}\right) \\ &= 1 - \frac{1}{2^n}\end{aligned}$$

as desired.

□ Let F_n be a Fibonacci sequence. If $n|m$, then $F_n|F_m$[/color] Gåi $n|m \iff m = nk$ I will use inducion on k For $k = 1$ it's obvious. $F_n|F_{kn} \Rightarrow F_n|F_{kn+n}$

Formula: $F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$ By the formula we have: $F_{kn+n} = F_{kn-1}F_n + F_{kn}F_{n+1} \Rightarrow F_n|F_{kn+n}$

□ Let a, b, c be three positive real numbers such that $a + b + c = 1$. Let $\lambda = \min\{a^3 + a^2bc, b^3 + b^2ac, c^3 + abc^2\}$ Prove that the roots of $x^2 + x + 4\lambda = 0$ are real.

Solution

Real roots condition is $\frac{1}{16} \geq \lambda$

Notice that $b^3 + b^2ac \geq a^3 + a^2bc \Leftrightarrow (b - a)(a^2 + b^2 + ab + abc) \Leftrightarrow b \geq a$.

Wlog, a is the smallest of (a, b, c). Fix a. It fixes $b + c$, which fixes the maximum of bc as $(\frac{1-a}{2})^2$.

So we just need $a^3 + a^2(\frac{1-a}{2})^2 \leq \frac{1}{16}$

becoming $a(a + 1) \leq 1/2$, which is true in $a \in [0, 1/3]$.

– $\begin{cases}\log_4(x^2+y^2) - \log_4(2x+1) = \log_4(x+y) \\ \log_4(xy+1) - \log_4(4x^2+2y-2x+4) = \log_4\frac{x}{y} - 1\end{cases}$ $x, y \in \mathbb{R}$ – Let a, b, c be positive numbers such that $3a = b^3,\ 5a = c^2$. Assume that a positive integer is limited to $d = 1$ such that a is divisible by d^6.

(1) Prove that a is divisible by 3 and 5. (2) Prove that the prime factor of a are limited to 3 and 5. (3) Find a. –

Determin $f : \mathbb{N} \to \mathbb{R}$ such that $f(1) = 1$ and

$$f(n) = \begin{cases} 1 + f\left(\frac{n-1}{2}\right), & n \text{ odd}, \\ 1 + f\left(\frac{n}{2}\right), & n \text{ even} \end{cases}$$

— For a natural number k, let $p(k)$ denote the smallest prime number which does not divide k. If $p(k) > 2$, define $q(k)$ to be the product of all primes less than $p(k)$, otherwise let $q(k) = 1$. Consider the sequence

$$x_0 = 1, \qquad x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}, \qquad n \in \mathbb{Z}^+ \cup \{0\}$$

Determine all natural numbers n such that $x_n = 111111$.

□

Find all pairs of positive integers (a, b) such that $5a^b - b = 2004$.

Solution

when $b = 1$, $a = 401$

clearly $a = 1$ yields no solution . So when $a, b \geq 2$, by Bernoulli Ineq

$2004 = 5a^b - b \geq 5 \cdot 2^b - b = 5 \cdot (1 + 1)^b - b \geq 5(1 + b) - b = 5 + 4b$

$\iff 499.75 \geq b$ so $499 \geq b \geq 2$

Hence this also gives us $5a^b = 2004 + b \leq 2004 + 499 = 2503 \iff a^b \leq 500$

So $500 \geq a^b \geq 2^b$ gives us $b \leq 8$

Now taking mod 5 at the original equation , we see that $b \equiv 1 \mod 5$ .So we only have $b = 6$ (since $2 \leq b \leq 8$) . But checking this value yields no integer solution for a . Hence the only solution are

$(a, b) = (401, 1)$

□ Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition: $\frac{f(x)+f(y)}{2} \geq f(\frac{x+y}{2}) + |x - y|$

Solution

we'll prove by induction that

$\frac{f(x)+f(y)}{2} - f(\frac{x+y}{2}) \geq 2^n|x - y|$ for any $n \in \mathbb{N}$ (*)

from which it is obvious there can be no such function because if $x \neq y$ are fixed then the RHS can be as large as we want while the LHS is fixed.

the case $n = 0$ is the assumption of the problem. assume (*) holds for $n \geq 0$. then

$\frac{f(x)+f(\frac{x+y}{2})}{2} - f(\frac{3x+y}{4}) \geq 2^n|\frac{x-y}{2}|$

$2[\frac{f(\frac{x+3y}{4})+f(\frac{3x+y}{4})}{2} - f(\frac{x+y}{2})] \geq 2 \cdot 2^n|\frac{x-y}{2}|$

$\frac{f(\frac{x+y}{2})+f(y)}{2} - f(\frac{x+3y}{4}) \geq 2^n|\frac{x-y}{2}$

adding gives precisely

$\frac{f(x)+f(y)}{2} - f(\frac{x+y}{2}) \geq 2^{n+1}|x - y|$

□số học

□ Let $a, b, c \in R$, $|a| \geq |b + c|$, $|b| \geq |c + a|$, $|c| \geq |a + b|$. Prove: $a + b + c = 0$

Solution

WLOG, let $|a| \leq |b| \leq |c|$. We have $|b + c| \leq |a| \leq |b|$, so b, c have opposite sign (either can be zero also). Similarly, $|a + c| \leq |b| \leq |c|$ so a, c have opposite sign (or zero). WLOG, assume $a, b \leq 0$ and $c \geq 0$ (multiplying each term by -1 does not change the problem).

CASE 1: $a + b + c > 0$

Then $a + c > -b$ and both sides are positive so $|a + c| > |b|$, contradiction.

CASE 2: $a + b + c < 0$

Then $a + b < -c$ and both sides are negative so $|a + b| > |c|$, contradiction.

Hence we must have $a + b + c = 0$.

Another way: Squaring $|a| \geq |b + c|$, we get $a^2 \geq b^2 + 2bc + c^2$. Similarly, $b^2 \geq a^2 + 2ac + c^2$ and $c^2 \geq a^2 + 2ab + b^2$. Adding, we get $0 \geq a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = (a + b + c)^2$. Therefore, $a + b + c = 0$.

□ Using the area of a regular pentagon, prove that $4 \sin \frac{2\pi}{5} + \tan \frac{2\pi}{5} = 5 \cot \frac{\pi}{5}$.

Solution

Let the regular pentagon $ABCDE$ have center O and sides length 2. Drop a perpendicular from O to AB, with intersection H. It's easy to calculate $m\angle HOB$; it is $\frac{\pi}{5}$. Since HB is half of $AB = 2$, then the area of $\triangle HOB$ is half the area of $\triangle HOA$. Since $OH = \cot\left(\frac{\pi}{5}\right)$, then the area of $\triangle BOA$ is simply $OH = \cot\left(\frac{\pi}{5}\right)$ via one half times the product of base and height. Finally, the area of the pentagon is $5 \cot \frac{\pi}{5}$.

Now, triangulate the pentagon by drawing segments DA and DB to form $\triangle DEA$, $\triangle DAB$, and $\triangle DBC$. Again, it is easy to calculate $m\angle DAB = \frac{2\pi}{5}$, so then DH has length $\tan \frac{2\pi}{5}$. Thus, the area of $\triangle DAB$ is $\tan \frac{2\pi}{5}$ through

$$A = \frac{1}{2}(bh).$$

Drop a perpendicular from E to DA to intersect at P. Then $m\angle EAP = \frac{\pi}{5}$. Hence,

$$AP = 2\cos\frac{\pi}{5}$$

$$EP = 2\sin\frac{\pi}{5}.$$

The area of $\triangle PEA$ is then

$$\frac{1}{2}\left(2\cos\frac{\pi}{5}\right)\left(2\sin\frac{\pi}{5}\right) = \sin\frac{2\pi}{5}.$$

Thus, the area of $\triangle DEA$ is $2\sin\frac{2\pi}{5}$, as is the area of $\triangle DBC$. Then the area of the pentagon is $4\sin\frac{2\pi}{5} + \tan\frac{2\pi}{5}$.

The pentagon never changed, so its area surely could not have. Thus,

$$4\sin\frac{2\pi}{5} + \tan\frac{2\pi}{5} = 5\cot\frac{\pi}{5}.$$

$\square$ Given a, b, c are three real numbers such that : $a < b < c$; $a+b+c = 6$;$ab+bc+ac = 9$. Prove that: $0 < a < 1 < b < 3 < c < 4$

Solution

As $a < b < c$ it is clear that $c > 2$ $a + b = 6 - c$ $ab + c(a + b) = 9$ we see that a, b satisfy $a + b = 6 - c$ and $ab = (c - 3)^2$ so a and b are the roots of $f(x) = x^2 + (c - 6)x + (c - 3)^2 = 0$ the discriminant is $D = -3c(c - 4)$ $D > 0$ when $c < 4$ c is outside the roots of $f(x) = 0$ so $f(c) > 0$ but $f(c) = 3(c - 1)(c - 3)$ so $c > 3$ As $f(1) = (c - 1)(c - 4)$ we see that $f(1) < 0$ so $a < 1 < b$ $ab = (c - 3)^2 > 0$ and $b > 1$ we conclude that $a > 0$ and as $a + c > 3$ $b < 3$

– Let $f : \mathbb{R} \to \mathbb{R}$ satisfy: $f(2x) = f(\sin\frac{x+y}{2}\pi) + f(\sin\frac{x-y}{2}\pi)$ Find $f(2007 + \sqrt{2007})$ – Let $A_1, A_2, \ldots, A_n$ be finite sets. Prove that

$$\left|\bigcup_{i=1}^{n} A_i\right|\left(\sum_{i=1}^{n}|A_i| + 2\sum_{1\le i<j\le n}|A_i \cap A_j|\right) \ge \left(\sum_{i=1}^{n}|A_i|\right)^2$$

where $|E|$ denotes the number of elements in set E. —— Let $A_1, A_2, \ldots, A_n$ be finite sets, and let k be a positive integer. Prove that

$$\left|\bigcup_{i=1}^{n} A_i\right| \ge \frac{2}{k+1}\sum_{i=1}^{n}|A_i| - \frac{2}{k(k+1)}\sum_{1\le i<j\le n}|A_i \cap A_j|$$

where $|E|$ denotes the number of elements in set E.

Note that if $k = 1$, it can be easily deduced from PIE. – a, b, c are real numbers with $ac < 0$ and $\sqrt{2}a + \sqrt{3}b + \sqrt{5}c = 0$. Prove that the second degree equation $ax^2 + bx + c = 0$ has root(s) in the interval of $\left(\frac{3}{4}, 1\right)$. – Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy $f'(x) > f(x), \forall x \in \mathbb{R}^+$. Find all values of $\boxed{k}$ such that the inequation $f(x) > e^{kx}$ has root(s) for all x which is enough large —— Let $f(m) = n + \lfloor\sqrt{n}\rfloor$, where $\lfloor\ \rfloor$ denotes the greatest integer function. Prove that, for every positive integer m, the sequence $m, f(m), f(f(m)), f(f(f(m)))...$ contains the square of an integer. (Art and Craft of Problem Solving 2.2.5). – Let a, b, c positive integers such that the numbers

$k = b^c + a$, $l = a^b + c$, $m = c^a + b$ are primes.

Prove that at least two of the numbers k, l, m are equal – Let $\{x_n\}$ be a positive geometric progression and $x_1 = 0.5$. Let $S_n = \sum_{k=1}^{n} x_k$, then

$$2^{10}S_{30} - (2^{10} + 1)S_{20} + S_{10} = 0$$

(1) Find general term of $\{x_n\}$. (2) Determine the sum of $\{nS_n\}$ to n terms. – Solve:

$$(\frac{1}{2})^{2\sin^2 x} + \frac{1}{2} = \cos 2x + \log_4(4\cos^3 x - \cos 6x - 1)$$

□ Prove that
$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n^2}} \geq 2n + \frac{1}{2n} - \frac{3}{2}$, $\forall n \in \mathbb{N}$

Solution

First we have this inequality, using Cauchy theorem: $\frac{1}{\sqrt{n^2+a}} + \frac{1}{\sqrt{n^2+2n-a}} \geq \frac{2}{\sqrt[4]{(n^2+a)(n^2+2n-a)}} \geq \frac{2}{\sqrt{n^2+n}}$

Using this inequality, we have: $\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+2n}} = (\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2+2n}}) + (\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2n-1}}) + \cdots + \frac{1}{\sqrt{n^2+n}} \geq \frac{2}{\sqrt{n^2+n}} + \frac{2}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}} = \frac{2n+1}{\sqrt{n^2+n}} \geq 2$

Now we have: $1 \geq 1$ $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \geq 2$ $\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots + \frac{1}{\sqrt{8}} \geq 2$

$\cdots$ $\cdots$

$\frac{1}{\sqrt{(n-1)^2}} + \frac{1}{\sqrt{(n-1)^2+1}} + \cdots + \frac{1}{\sqrt{(n-1)^2+2(n-1)}} \geq 2$ $\frac{1}{n} + \frac{1}{2} \geq \frac{1}{2n}$

We now just add them together to have "Square root inequality" Another way using the idea of AM-HM and telescoping

$\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq \frac{4}{\sqrt{k+1}+\sqrt{k}} = 4\left(\sqrt{k+1} - \sqrt{k}\right)$

□ số học

□ Let S be a finite set of points on a line, with the property: if P and Q are two points of S, then exist a point R such that R is the midpoint of PQ, Q is the midpoint of PR, **[size=150]OR[/size]** P is the midpoint of QR. Determine the greatest possible number of points of S.

Solution

Here is a simple proof for the fact that S cannot have more than 5 elements. We may assume, without loss of generality, that 0 is the smallest element of S and that 1 is the greatest element of S. That means that $\frac{1}{2} \in S$. Since S has more than 5 elements (by assumption) there exists an $x_1 \in S$ such that $x_1 \notin \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$. Assume, for example, that $\frac{1}{3} < x_1 < \frac{1}{2}$. (In the other cases, we can give a similar proof.) There must exist an $x_2 \in S$ such that x_2 is the midpoint of x_1 and 1, and $\frac{2}{3} < x_2 < 1$. Now there must exist some number $x_3 \in S$ such that x_3 is the midpoint of 0 and x_2, and $\frac{1}{3} < x_3 < \frac{1}{2}$. It is easy to see that $\frac{1}{3} < x_3 < x_1$. Continuing this way, we get a decreasing sequence $x_1, x_3, x_5, \cdots$ of real numbers which converges to $\frac{1}{3}$, all terms of which are greater than $\frac{1}{3}$. This means that S must have infinitely many elements. Contradiction. So we're done: S can have at most 5 elements.

□

We know that when we rationalize the denominator of $\frac{1}{\sqrt{a}+\sqrt{b}}$, we can do it like that

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$$

But how to rationalize the denominator of $\frac{1}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}$?

Solution

Substitute $\sqrt[3]{a} = u$, $\sqrt[3]{b} = v$ and $\sqrt[3]{c} = w$. With the formulae $a^3 + b^3 + c^3 - 3abc = (a + b + c)\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)$ and $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, we get that

$$\begin{aligned}\frac{1}{u+v+w} &= \frac{(u-v)^2+(v-w)^2+(w-u)^2}{u^3+v^3+w^3-3uvw} \\ &= \frac{((u-v)^2+(v-w)^2+(w-u)^2)\left((u^3+v^3+w^3)^2+3uvw(u^3+v^3+w^3)+(3uvw)^2\right)}{(u^3+v^3+w^3)^3-(3uvw)^3} \\ &= \frac{\left(\left(\sqrt[3]{a}-\sqrt[3]{b}\right)^2+\left(\sqrt[3]{b}-\sqrt[3]{c}\right)^2+(\sqrt[3]{c}-\sqrt[3]{a})^2\right)\cdot\left((a+b+c)^2+3\sqrt[3]{abc}(a+b+c)+9\sqrt[3]{a^2b^2c^2}\right)}{(a+b+c)^3-27abc}\end{aligned}$$

and we're done.

☐ What number can be written in the form $x+y^2$, where x and y are positive integers no larger than 100, in the largest number of ways?

Solution

Try replacing "100" with smaller numbers. You probably want to look at 100 replaced by 1, 4 or 9. [hide="Solution"]Following the hint, we get the idea that the answer is probably 101. We can write $101 = 1+10^2 = 20+9^2 = 37+8^2 = 52+7^2 = 65+6^2 = 76+5^2 = 85+4^2 = 92+3^2 = 97+2^2 = 100+1^2$ for a total of 10 different ways. Suppose there were some number n which could be expressed in 11 different ways. Then $n = x_i + y_i^2$ for $i = 1, 2, \ldots, 11$. Now, without loss of generality we have $y_1 > y_2 > \ldots > y_{11}$, and since each is a positive integer, $y_1 > y_{11} + 10$. But then $x_{11} + y_{11}^2 = x_1 + y_1^2 > x_1 + y_{11}^2 + 20y_{11} + 100 > x_1 + y_{11}^2 + 100$ and so $x_{11} > x_1 + 100$, clearly a contradiction. So 101 is definitely [i]a[/i] mode of the set. Following the same argument through with 11 replaced by 10 will leave us in the final stage not with a contradiction but with the unique solution $a_1 = 10, a_{10} = 1$ and so $n = 101$.

☐ hình

☐ For all positive integers n, define $a_n = 0$ if n has an even number of distinct prime divisors and $a_n = 1$ otherwise. Is the number $0.a_1a_2a_3\cdots$ rational or irrational?

Solution

Suppose that our number is rational and denote $a_n = f(n)$. There exists a positive integer M and a positive integer a (the period) such that for all $x > M$, we have $f(x) = f(x+a)$. Choose $t > 0$ such that $a^t > M$. We have $f(a^t) = f(a^t + na)$ for all positive integers n. Now choose $n = (p-1)\cdot a^{t-1}$ where p is a prime number which does not divide a. Then $f(a^t) = f(p\cdot a^t)$, and that's a contradiction. ■

☐ find all pairs of positive integers (n,k), which satisfies: $\binom{n}{k} = k^3+1$

Solution

The only solutions are $(n,k) = (n,0), (2,1), (9,5)$ and $(14,10)$. For $k=0$, it is clear any n suffices. For $k=1$ we have $\binom{n}{1} = 2$ so $n=2$. For $k=2$ we have $\binom{n}{2} = 7$ which is readily seen to have no solutions. $(k^3+1) - \binom{k+3}{k} = \frac{1}{6}(5k^3-6k^2-11k) = \frac{1}{6}k(k+1)(5k-11)$, so $k^3+1 > \binom{k+3}{k}$ for $k \geq 3$. Thus, we must have $n > k+3$ when $k \geq 3$. Also note that $\binom{k+4}{k} - (k^3+1) = \frac{1}{24}(k+1)((k+4)(k+3)(k+2) - 24(k^2-k+1)) = \frac{1}{24}k(k+1)(k-5)(k-10)$ This final expression is positive when $k > 10$, so in those cases we must have $n < k+4$. Combining this result with that of the previous line, there are no solutions for $k > 10$. We also note from this factorization that $k = 5, 10$ give us solutions. The only cases left to check are $5 < k < 10$, and in these it suffices to note that $\binom{11}{6} > 6^3+1$, $\binom{12}{7} > 7^3+1$, $\binom{13}{8} > 8^3+1$ and $\binom{14}{9} > 9^3+1$, so there are no solutions in these cases, either. – Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)^3 = -\frac{x}{12} \cdot \left(x^2 + 7x \cdot f(x) + 16 \cdot f(x)^2\right), \ \forall x \in \mathbb{R}.$$

□ $x_1, x_2, ..., x_{1993}$ are real numbers satisfying $|x_1 - x_2| + |x_2 - x_3| + ... + |x_{1992} - x_{1993}| = 1993$, $y_k = \frac{x_1+x_2+...+x_k}{k}$ for $k = 1, 2, ..., 1993$. What is the maximum possible value of $|y_1 - y_2| + |y_2 - y_3| + ... + |y_{1992} - y_{1993}|$?

Solution

For the solution, (with skimping on the details) set $1992 = n$ and let $x_i - x_{i+1} = d_i$ and then $|y_i - y_{i+1}| = \frac{1}{i(i+1)} |\sum_{j=1}^{i} j \cdot d_j| \leq \sum_{j=1}^{i} \frac{j}{i(i+1)} |d_j|$ where the first equality results from just writing the thing out and appropriate adding/subtracting, while the inequality is the triangle inequality. Summing this for i from 1 to n gives us $|y_1-y_2|+|y_2-y_3|+\ldots+|y_n-y_{n+1}| \leq \frac{n}{n+1}|d_1|+b_2|d_2|+\ldots+b_n|d_n|$ where $0 < b_i \leq \frac{n}{n+1}$ for each $i > 1$. But this expression is at most $\frac{n}{n+1}(|d_1| + \ldots |d_n|)$. But the second part of this product is just $n + 1$, so this shows that the maximum is at most n. We can achieve n by setting $x_1 = n, x_2 = x_3 = \ldots = x_{n+1} = 0$.

□ Suppose that the coefficients of the equation $x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ are real and satisfy $0 < a_0 \leq a_1 \leq \ldots \leq a_{n-1} \leq 1$. Let z be a complex root of the equation with $|z| \geq 1$. Show that $z^{n+1} = 1$.

Solution

Let's construct a genuine argument. What follows still feels a little awkward, so I assume there is room for improvement of this proof.

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 x + a_0$, as above.

Define $Q(z) = (z - 1)P(z) = z^{n+1} - b_n z^2 - b_{n-1}b^{n-1} - \cdots - b_1 z - b_0$.

Here, $b_0 = a_0$, $b_1 = a_1 - a_0, \ldots, b_{n-1} = a_{n-1} - a_{n-2}, b_n = 1 - a_n$.

We want to note that $b_k \geq 0$ for all k and that $\sum_{k=0}^{n} b_k = 1$.

Now suppose $|z| \geq 1$. Then

$|Q(z)| \geq |z|^{n+1} - \sum_{k=0}^{n} b_k |z|^k$

$\geq |z|^{n+1} - \sum_{k=0}^{n} b_k |z|^n \geq |z|^n \left(|z| - \sum_{k=0}^{n} b_k\right)$.

Since $\sum_{k=0}^{n} b_k = 1$, this can only possibly be zero if $|z| = 1$. That means that $Q(z)$ (hence also $P(z)$) can have no roots with $|z| > 1$.

Now suppose that $|z| = 1$ and $Q(z) = 0$. Divide by z^{n+1}.

$1 - b_n z^{-1} - b_{n-1}z^{-2} - \cdots - b_1 z^{-n} - b_0 z^{-(n+1)} = 0$.

But $|z^{-1}| = |z^{-2}| = \cdots = |z^{-(n+1)}| = 1$.

We have that $\sum_{k=0}^{n} b_k z^{k-n-1} = \sum_{k=0}^{n} b_k |z^{k-n-1}| = 1$.

By the equality case in the triangle inequality, the only way for this to happen would be for $b_k z^{k-n-1}$ to be a nonnegative real number for each k. There are two ways for this to happen: $b_k = 0$ or $z^{k-n-1} = 1$.

But $a_0 > 0$ and hence $b_0 > 0$. So we must conclude that $z^{-n-1} = 1$ and hence $z^{n+1} = 1$.

□ số học

□ số học

□ tổ hợp

□ tooe hợp

□ to hợp khó

□ đại số, hình học

□ hình

$\square$ For $a, b, c > 0$ we have:

$$-1 < \left(\frac{a-b}{a+b}\right)^{1993} + \left(\frac{b-c}{b+c}\right)^{1993} + \left(\frac{c-a}{c+a}\right)^{1993} < 1$$

Solution

Without loss of generality, we can assume that $a = \max\{a, b, c\}$. Define

$$f(a,b,c) = \left(\frac{a-b}{a+b}\right)^{1993} + \left(\frac{b-c}{b+c}\right)^{1993} + \left(\frac{c-a}{c+a}\right)^{1993}.$$

It is easy to check that $f(a,b,c) = -f(a,c,b)$. Hence $-1 < f(a,b,c) < 1 \Leftrightarrow -1 < f(a,c,b) < 1$. This allows us to assume that $b \geq c$, so that $a \geq b \geq c > 0$. It is obvious that

$$\begin{aligned} 0 \leq & \left(\frac{a-b}{a+b}\right)^{1993} & < 1 \\ 0 \leq & \left(\frac{b-c}{b+c}\right)^{1993} & < 1 \\ -1 < & \left(\frac{c-a}{c+a}\right)^{1993} & \leq 0 \end{aligned}$$

and the only thing which we still need to show is

$$\left(\frac{b-c}{b+c}\right)^{1993} + \left(\frac{c-a}{c+a}\right)^{1993} \leq 0.$$

This reduces to

$$((b-c)(c+a))^{1993} + ((c-a)(b+c))^{1993} \leq 0$$

or

$$(b-c)(c+a) \leq -(c-a)(b+c)$$

or

$$bc + ab - c^2 - ac \leq -bc - c^2 + ab + ac$$

or

$$2c(a-b) \geq 0$$

which is obviously true. $\blacksquare$

$\square$ Prove that an infinite number of triangles each having a given interior point as centroid can be inscribed in a given circle.

Solution

Method 1 (constructive) The naive approach is to guess that any point on the circle can be a vertex of a triangle with its centroid at any given interior point. It turns out this doesn't work (see Method 2), but it comes suprisingly close: the second-most naive approach is to just take any chord passing through the given internal point G. Say it has endpoints A and P on the circle, and without loss of generality $AG < GP$. (If G happens to be the midpoint of segment AP, either G is the center of the circle, for which the problem is trivial (all equilateral triangles work) or G is not the center, in which case we happened to pick the one chord of which G is the midpoint, and we can pick any other chord instead.) Let M be the point on GP such that $2GM = AG$, and let BC be a chord(actually "the chord," unless M happens to be the center of the circle) passing through M such that $BM = MC$. Then G is the centroid of triangle ABC, and since the original chord AP was arbitrary, we can in fact repeat this infinitely many times to get infinitely many such triangles, Q.E.D.

Method 2 (non-constructive) Take any point A on the circle. The set of points which are the centroid of some triangle inscribed in the circle with a vertex at A is a disk whose boundary is internally tangent to the given circle at A, with diameter two thirds that of the diameter of the given circle. Any interior point of the given circle is covered by infinitely many of these disks. The fact that each triangle is counted 3 times obviously doesn't matter, and we're done.

□ hình

□ Let f(t) be a real valued function satisfying the differential equation

$f^{`}\left(1-\frac{1}{t}=t^2(\lambda-f^{`}t\right)$ Where λ is any real number $t \neq \{0,1\}$ Find all values of t for which the slope of the tangent line to the graph of $f(t)$ is $\frac{\lambda}{2}$.

Solution

Let $g(t)=1-\frac{1}{t}$. Then $g(g(t))=1-\frac{1}{1-\frac{1}{t}}=1-\frac{t}{t-1}=\frac{1}{1-t}$ and $g(g(g(t)))=t$.

Substitute $x=t$, $x=1-\frac{1}{t}$, and $x=\frac{1}{1-t}$ into the differential equation. Let $a=f'(x)$, $b=f'\left(1-\frac{1}{x}\right)$, $c=f'\left(\frac{1}{1-x}\right)$. You will have a system of three linear equations in a, b, and c.

Solve the system of equations for a (in terms of x and λ) to get an explicit formula for $f'(x)$. Then just set it equal to $\frac{\lambda}{2}$ and solve for x.

□ Let $n>3$ be a positive integer. Consider n sets, each having two elements, such that the intersection of any two of them is a set with one element. Prove that the intersection of all sets is non-empty.

Solution

Let the sets be $A_1, A_2, \ldots, A_n$. Let $A_1=\{a,b\}$ and suppose $A_2=\{a,c\}$ where a,b,c are distinct elements. Now, any other A_i must contain either a or b (and not both) or a or c (and not both). Thus, $A_i=\{b,c\}$ or $\{a,x\}$ for some element x. Because $n>3$, no A_i can be $\{b,c\}$ or else some other $A_j=\{a,x\}$ with $x\neq b,c$ and has no common element with $A_i=\{b,c\}$. Thus, all the sets contain a... qed

□

In the acute-angle triangle ABC we have $\angle ACB=45°$. The points A_1 and B_1 are the feet of the altitudes from A and B, and H is the orthocenter of the triangle. We consider the points D and E on the segments AA_1 and BC such that $A_1D=A_1E=A_1B_1$. Prove that

a) $A_1B_1=\sqrt{\frac{A_1B^2+A_1C^2}{2}}$;

b) $CH=DE$.

Solution

a) Considering that $\angle A_1CA=\angle A_1AC=\angle AHB_1=\angle BHA_1=\angle B_1BA_1=45^o$ the triangles AA_1C, AHB_1, BA_1H, BCB_1 are all right isosceles.Defining K as the feet of the altitude that pass thought A_1 and intersect the side AC, we have that,

$A_1K=KC=\sqrt{\frac{A_1C^2}{2}}=\frac{\sqrt{2}}{2}.A_1C$ and

$B_1K=AK-AB_1$ but we know that $AB_1\equiv B_1H$, therefore, $AH=\sqrt{2AB_1^2}=\sqrt{2}AB_1$ and,

$A_1B\equiv A_1H$ therefore $\sqrt{2}AB_1+A_1B=AH+A_1H=AA_1=A_1C \iff AB_1=\frac{\sqrt{2}}{2}(A_1C-A_1B)$, finally, we can note that $\angle AA_1K=\angle A_1AK$ it implies $AK=A_1K$ consequently,

$B_1K=\frac{\sqrt{2}}{2}.A_1C-\frac{\sqrt{2}}{2}(A_1C-A_1B)=\frac{\sqrt{2}}{2}.A_1B$ thus,

$A_1B_1^2 = A_1K^2 + B_1K^2 = \frac{2}{4}(A_1B^2 + A_1C^2) \implies A_1B_1 = \sqrt{\frac{A_1B^2+A_1C^2}{2}}$
b)$DE^2 = A_1D^2 + A_1E^2 = 2A_1B_1^2 = A_1B^2 + A_1C^2 = A_1H^2 + A_1C^2 = CH^2$

☐ Prove that 11 divides $10^{2n+1} + 3 \cdot 2^{10n+2}$.

Solution

Proof using induction (i) When $n = 1$, $1000 + 3 \cdot 4096 = 13288 = 11 \cdot 1208$

(ii) Assume true for $n = k$. We can let $10^{2k+1} + 3 \cdot 2^{10k+2} = 11a$ for some integer a.

(iii) When $n = k + 1$,
$10^{2(k+1)+1} + 3 \cdot 2^{10(k+1)+2}$
$= 10^{2k+3} + 3 \cdot 2^{10k+12}$
$= 100 \cdot 10^{2k+1} + 1024 \cdot 3 \cdot 2^{10k+2}$
$= 100 \cdot 10^{2k+1} + 1024(11a - 10^{2k+1})$
$= 100 \cdot 10^{2k+1} + 1024 \cdot 11a - 1024 \cdot 10^{2k+1}$
$= 1024 \cdot 11a - 924 \cdot 10^{2k+1}$
$= 11(1024a - 84 \cdot 10^{2k+1})$

Therefore, the expression is divisible by 11 for all natural numbers n. — Let $ABCDA_1B_1C_1D_1$ be a cube and P a variable point on the side $[AB]$. The perpendicular plane on AB which passes through P intersects the line AC' in Q. Let M and N be the midpoints of the segments $A'P$ and BQ respectively.

a) Prove that the lines MN and BC' are perpendicular if and only if P is the midpoint of AB.

b) Find the minimal value of the angle between the lines MN and BC'. – Solve the system in positive integers $x^2 = 2(y + z)$ and $x^6 = y^6 + z^6 + 31(y^2 + z^2)$ – Let $\{x_k\}_{k\geq 1}$ be a sequence of reals such that $x_1 = 1$ and $x_kx_{k+1} = k$ for $k \geq 1$. Prove that:

$$\sum_{k=1}^{n} \frac{1}{x_k} \geq 2\sqrt{n} - 1.$$

–

☐ số

☐xác xuất khó

☐ số học

☐ số học

☐ Solve the system of equation for all $a, b, c \in \mathbb{R}^+$ satisfying
$a + b + c = 1$
$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} = 1$

Solution

There are no solutions for the system of equations: $1 = (a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab})^2 \leq (a^2 + b^2 + c^2)(ca + ab + ca) = (1 - 2u)u$ if we denote $u = ab + bc + ca = -2u^2 + u = -2(u - \frac{1}{4})^2 + \frac{1}{8} \leq \frac{1}{8}$, which is a contradiction!

Problem $ab + bc + ca = 1$
$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} = 1$

Solution

Set $x = bc$ etc.
we get $\sum x = 1 = \sum \sqrt{yz}$.
But $x + y \geq 2sqrtxy$ whence $1 = \sum x \geq \sum \sqrt{yz} = 1$ and x=y=z for equality

Hence ab=bc=ca whence it is clear a=b=c

hence $a = \sqrt{3}/3$ – Prove that $\frac{gcd(m,n)}{n}C(n,m)$ is an integer.

$\square$ For $n \in \mathbb{Z}^+$, $n > 1$, prove that

$\frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{n^2-1} + \frac{1}{n^2} > 1$

Solution

For n > 1: Apply AM-HM inequality: $\frac{n+(n+1)+...+n^2}{n^2-n+1} > \frac{n^2-n+1}{\frac{1}{n}+\frac{1}{n+1}+...+\frac{1}{n^2}}$ Re-arrange terms: $\frac{1}{n}+\frac{1}{n+1}+...+\frac{1}{n^2}$ $> \frac{(n^2-n+1)^2}{n+(n+1)+...+n^2} = \frac{(n^2-n+1)^2}{\frac{(n+n^2)(n^2-n+1)}{2}} = \frac{2(n^2-n+1)}{n^2+n} \geq 1$ for $n \geq 2$ because $2(n^2-n+1)-(n^2+n) = n^2-3n+2 = (n-1)(n-2) \geq 0$ for $n \geq 2$.

Therefore, $\frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{n^2} > 1$ for all natural numbers $n > 1$. Proof 2

With Cauchy's:

let $S = \frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{n^2-1} + \frac{1}{n^2}$

$S(n+n+1+n+2+...+n^2-1+n^2) \geq (n^2-n+1)^2$

But we know the sum of n to n^2 is $(n+n^2)\cdot\frac{1}{2}(n^2-n+1)$

$S \geq \frac{2(n^2-n+1)^2}{(n+n^2)(n^2-n+1)}$

$= \frac{2n^2-2n+2}{n^2+n}$

$\frac{2(n^2-n+1)}{n(n+1)} > 1$

$\implies 2n^2-2n+2 > n^2+n \implies n^2-3n+2 > 0$

$\implies (n-1)(n-2) > 0$ which is clearly true for every postivie integer greater than 2, but easy just to check $n = 2$ with the original conditions. $S \geq \frac{2(n^2-n+1)}{n(n+1)} > 1$

Proof 3

$n > 1 \Rightarrow \frac{1}{n^2} < \frac{1}{n^2-1} < ... < \frac{1}{n+1} < \frac{1}{n}$

$\Rightarrow \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n^2-1} + \frac{1}{n^2} > (n^2-n)\frac{1}{n^2}$

$\Rightarrow \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n^2-1} + \frac{1}{n^2} > \frac{1}{n} + (1-\frac{1}{n}) = 1$

$\square$ Suppose that $a+b+c = 0$. Prove that:

$$\frac{a^3+b^3+c^3}{3}\cdot\frac{a^4+b^4+c^4}{2} = \frac{a^7+b^7+c^7}{7} = \frac{a^2+b^2+c^2}{2}\cdot\frac{a^5+b^5+c^5}{5}$$

Solution

Denote $A = ab+bc+ca, B = abc$. Then a, b, c are the roots of $x^3+Ax-B = 0$.

Define $S_n = a^n+b^n+c^n$ for $n = 0, 1, 2, 3, ...$ Then $S_0 = 3$, $S_1 = 0$,$S_2 = (a+b+c)^2-2(ab+bc+ca) = -2A$.

Note that $S_{n+3} = -AS_{n+1}+BS_n$ for $n = 0, 1, 2, ...$ $S_3 = -AS_1+BS_0 = 3B$, $S_4 = -AS_2+BS_1 = 2A^2$, $S_5 = -AS_3+BS_2 = -5AB$, and $S_7 = -AS_5+BS_4 = 7A^2B$.

Direct verification shows that $\frac{S_3}{3}\cdot\frac{S_4}{2} = \frac{S_7}{7} = \frac{S_2}{2}\cdot\frac{S_5}{5}$, which is equivalent to what we need to show.

$\square$ Prove that the product of 8 consecutive integers cannot be the square of a perfect square (a perfect fourth power).

Solution

Let $p(x) = x(x+1)(x+2)(x++3)(x+4)(x+5)(x+6)(x+7)$ Then $p(x) = (x^2+7x)(x^2+7x+6)(x^2+6x+10)(x^2+7x+12)$ As stated by JBL, $p(x) < (x^2+7x+7)^4$. If we write $u = (x^2+7x)$ Then $p(x) = u^4+28u^3+252u^2+720u$ While $(u+6)^4 = u^4+24u^3+216u^2+864u$ $p(x) > (u+6)^4$ except for a finite number of values of u.....

In fact, $p(x)-(u+6)^4 = 4u^3+36u^2-144u = 4u(u+12)(u-3)$ Therefore, $p(x) > (u+6)^4$ except for $u \leq 3$. That is, $x^2+7x \leq 3$, which is impossible for positive integer x.

In conlusion, $(x^2+7x+6)^4 < p(x) < (x^2+7x+7)^4$ for positive integer x, and hence the problem is solved.

Question Is there any generalization to this problem?

□ Let $S = \sum_{k=1}^{777} \frac{1}{\sqrt{2k+\sqrt{4k^2-1}}}$. S can be expressed as $\sqrt{p+\frac{m}{n}} - \frac{\sqrt{q}}{r}$, where $\gcd(m,n) = 1$. Find $p+m+n+q+r$. $p,q,r,m,n \in \mathbb{N}$, $\frac{\sqrt{q}}{r}$ is in its simplest form, and $\frac{m}{n} < 1$.

Solution

$$\begin{aligned}
\frac{1}{\sqrt{2k+\sqrt{4k^2-1}}} &= \sqrt{2k-\sqrt{4k^2-1}} \\
&= \frac{\sqrt{(2k+1)-2\sqrt{(2k+1)(2k-1)}+(2k-1)}}{\sqrt{2}} \\
&= \frac{\sqrt{2k+1}-\sqrt{2k-1}}{\sqrt{2}}
\end{aligned}$$

The sum telescopes. $S = \frac{\sqrt{1555}-1}{\sqrt{2}} = \sqrt{777+\frac{1}{2}} - \frac{\sqrt{2}}{2}$.

$p+q+r+m+n = \boxed{784}$. I think. (You should probably specify $p,q,r,m,n \in \mathbb{N}$ instead of $\mathbb{Z}$.)

□ $0 < a < 1, x^2+y = 0,$

Prove that $log_a(a^x+a^y) \leq log_a 2 + \frac{1}{8}$

Solution

from $\left(x-\frac{1}{2}\right)^2 \geq 0$ we have $x^2 - x + \frac{1}{4} \geq 0$.

$\iff x+y \leq \frac{1}{4}$ $(\because x^2 = -y)$

$\implies a^{x+y} \geq a^{\frac{1}{4}}$ $(\because 0 < a < 1)$

By AM-GM , $\frac{a^x+a^y}{2} \geq \sqrt{a^{x+y}} \geq \sqrt{a^{\frac{1}{4}}} = a^{\frac{1}{8}}$

$\iff \log_a(a^x+a^y) \leq \log_a 2 + \frac{1}{8}$

□ đại số – Given that $S = \{a,b,c,d,e,f,g,h,i\}$ is a set of nine elememts. $A_1 = \{a,b,c\}$, $A_2 = \{d,e,f\}$ and $A_3 = \{g,h,i\}$ are subsets of S. And $F : S \times S \to S$ is a function satisyfing

(1) $F(A_m \times A_n) = S$ for all $m,n \in \{1,2,3\}$, (2) $F(\{r\} \times S) = F(S \times \{s\}) = S$ for all $r,s \in S$, (3) $F(a,a) = F(b,h) = F(e,b) = F(g,c) = F(i,i) = a$, $F(c,e) = F(d,b) = F(i,d) = b$, $F(d,f) = F(f,h) = c$, $F(a,b) = F(d,g) = F(i,h) = d$, $F(e,d) = F(h,b) = e$, $F(a,e) = F(e,f) = F(g,h) = f$, $F(e,h) = F(f,d) = F(i,e) = g$, $F(a,f) = F(c,g) = F(f,c) = h$, and $F(a,g) = F(c,b) = F(g,f) = F(i,c) = i$.

Find the value of $F(F(g,i),F(i,g))$.

□ If real numbers x and y satisfy the condition $x^2+xy+y^2 = 1$, find the minimun and maximum value of:

$K = x^3y + xy^3$

Solution

Use polar transform, let $x = r\cos\theta$ and $y = r\sin\theta$. $x^2+xy+y^2 = 1 \Leftrightarrow r^2 + r^2\cos\theta\sin\theta = 1$ Then $\sin\theta\cos\theta = \frac{1}{r^2} - 1$ $sin2\theta = 2(\frac{1}{r^2}-1) \implies -1 \leq 2(\frac{1}{r^2}-1) \leq 1 \implies \frac{2}{3} \leq r^2 \leq 2 \quad (*)$

$K = (x^2+y^2)(xy) = r^2(r^2\cos\theta\sin\theta) = r^4(\frac{1}{r^2}-1) = r^2(1-r^2)$.

Making use of $(*)$ and consider the graph of **a part of parabola** open downwards(or the function $f(t) = t(1-t)$ is strictly decreasing for $t \geq \frac{1}{2}$):

Maximum value of $K = (\frac{2}{3})(1-\frac{2}{3}) = \frac{2}{9}$, and mimimum value of $K = 2(1-2) = -2$. – Let $p(x)$

be a polynomial with real coefficients. Prove that if

$$p(x) - p'(x) - p''(x) + p'''(x) \geq 0$$

for every real x, then $p(x) \geq 0$ for every real x – Let a, b, c be the lengths of three sides of a triangle and Δ be the area of the triangle.

Prove that for any $p > 0$, $\Delta \leq \frac{\sqrt{3}}{4}\left(\frac{a^p+b^p+c^p}{3}\right)^{\frac{2}{p}}$,

the equality sign holds if and only if $a = b = c$. – Let a trapez $ABCD(AB//CD)$ and the midpoint M of AB .Segment MD meets segment AC at point N .Let P the foot of the perpendicular of point N on line BC . Prove that $\angle MPN = \angle DPN$ — Let a, b, c, d be integers such that $ad - bc = k > 0$ and

$$\gcd(a, b) = \gcd(c, d) = 1$$

Prove that there are exactly k ordered pairs of real numbers (x_1, x_2) satisfying $0 \leq x_1, x_2 < 1$ and both $ax_1 + bx_2$ and $cx_1 + dx_2$ are integers – Let ABC be a triangle inscribed in circle R .Also let the angle bisector line of A, B, C intersect the circumference of circle R at$A', B,' C'$ respectively . Let $AA' \cap BC = N$, $C'A' \cap BB' = P$. Also denote the orthocenter as I . Now , let O be the circumcenter of triangle IPC' , and $OP \cap BC = M$. If $BM = MN$, and $\angle BAC = 2\angle ABC$, find all the angle $\angle A, \angle B, \angle C$ in the triangle ABC .

□ The semicircle with centre O and the diameter AC is divided in two arcs AB and BC with ratio $1:3$. M is the midpoint of the radium OC. Let T be the point of arc BC such that the area of the cuadrylateral $OBTM$ is maximum. Find such area in fuction of the radium.

Solution

$[OBTM] = [OBM] + [MBT]$ $[OBM]$ is always fixed it is not dependent on point T. Draw altitude TH of $\triangle MBT$ $[MBT] = \dfrac{1}{2}(BM)(TH)$ If we choose T such that TH is a maximum, O, H, and T are collinear, meaning that the tangent line containing T is perpendicular to TH and parallel to BM. This is because if we extend BM past M to meet circle O at point D, the T that maximizes the height of $\triangle MBT$ bisects arc BD.

Now $[OBTM]$ can be expressed as $[OBTM] = \frac{1}{2}[BM][OT]$, since the diagonals are perpendicular. Using Pythagoras, $[OBTM] = \frac{1}{2}\left(r\frac{\sqrt{5+2\sqrt{2}}}{2}\right)(r) = \frac{\sqrt{5+2\sqrt{2}}}{4}r^2$

□ For every triple of functions $f, g, h : [0, 1] \to \mathbb{R}$, prove that there are numbers x, y, z in $[0, 1]$ such that $|f(x) + g(y) + h(z) - xyz| \geq \frac{1}{3}$.

Solution

Suppose a counter-example exists. Then: $f(1) + g(1) + h(0) < \frac{1}{3}$ and $\frac{2}{3} < f(1) + g(1) + h(1)$ so $\frac{1}{3} < h(1) - h(0)$ and similarly $\frac{1}{3} < g(1) - g(0)$ but $-\frac{1}{3} < f(0) + g(0) + h(0)$ so adding $\frac{1}{3} < f(0) + g(1) + h(1)$ and we lose.

The functions $f(t) = g(t) = h(t) = \frac{3t-1}{9}$ gives us an error never larger than $\frac{1}{3}$.

□ A fenced, rectangular field measures 24 meters by 52 meters. An agricultural researcher has 1994 meters of fence that can be used for internal fencing to partition the field into congruent, square test plots. The entire field must be partitioned, and the sides of the squares must be parallel to the edges of the field. What is the largest number of square test plots into which the field can be partitioned using all or some of the 1994 meters of fence?

Solution

Call the side length of the squares a, we then have the inequality:

$$\left(\frac{52}{a} - 1\right) \times 24 + \left(\frac{24}{a} - 1\right) \times 52 \leq 1994$$

Simplifying the inequality, we have:

$a \geq \frac{416}{345}$

Because the resulting plots must all be identical, we have the restriction that $\frac{52}{a} = n$ and $\frac{24}{a} = m$ for some integers n and m.

The ratio of $\frac{n}{m}$ must then equal $\frac{52}{24} = \frac{13}{6}$, so $n = 13k$ and $m = 6k$ for some integer k.

Putting $13k$ into the expression $\frac{52}{n}$ (which equals a), we have $\frac{52}{13k} \geq \frac{416}{345}$. To minimize a, we want to maximize k, so $k = 3$. The value of a is then equal to $\dfrac{4}{3}$.

The total number of square plots is equal to $\dfrac{24 \times 52}{a^2} = \dfrac{24 \times 52 \times 9}{16} = \boxed{702}$

☐ Solve this equation with x,y are integer numbers: $x^{x+y} = y^{12}(1)\quad y^{x+y} = x^3(2)$

Solution

taking the natural log of both sides we get:

$(x+y)\ln x = 12\ln y$

$(x+y)\ln y = 3\ln x$

multiplying these together we get:

$(x+y)^2 = 6^2 \implies (x+y) = 6$ or $(x+y) = -6$

the latter gives no integral solutions whereas the former gives:

$(6-y+y)\ln(6-y) = 12\ln y \implies 6-y = y^2$ which has roots $y = 2$ and $y = -3$

this then gives solutions $(4,2)$ and $(9,-3)$

now we just have incorporate the solutions where x and y equal ± 1, and we get $(1,1)$ and $(1,-1)$

☐ Find n belong to N satisfying $\frac{n-37}{n+43}$ is a squared of a rational number.

Solution

Let $m = n+43$, so that we have $\frac{m-80}{m} = \frac{p^2}{q^2}$ for relatively prime positive integers p and q. Since $n \geq 1$, we have $m \geq 44$. Multiplying the equation by mq^2, we obtain $mq^2 - 80q^2 = p^2m$. Therefore, $m = \frac{80q^2}{q^2-p^2} = \frac{80q^2}{(q-p)(q+p)}$. By the Euclidean Algorithm, we have $\gcd(q,q-p) = \gcd(p,q-p) = \gcd(p,q) = 1$ and $\gcd(q,q+p) = \gcd(q,p) = 1$. Therefore, $(q-p)(q+p)$ divides 80. The possibilities are

$(q-p,q+p) = (1,1),(1,5),(2,2),(2,4),(2,8),(2,10),(2,20),(2,40),(4,4),(4,10),(4,20),(8,10)$ $(q,p) = (1,0),(3,2),(3,1),(2,0),(5,3),(6,4),(11,9),(21,19),(4,0),(7,3),(12,8),(9,1)$

Discarding all (q,p) such that q and p are not relatively prime, we have

$(q,p) = (1,0),(3,2),(3,1),(5,3),(11,9),(21,19),(7,3),(9,1)$ $m = 80,144,90,125,242,441,98,81$ $n = \boxed{37,101,47,82,199,398,55,38}$

☐ ABC is a triangle such that:$AB = 9, BC = 15, CA = 16$. D is a point in AC such that $\measuredangle ABD = 2\measuredangle DBC$. Find $cos\measuredangle ADB$

Solution

Extend BA to E such that $AE = 16$ $(BE = 25)$. $AC = AE$, so we can let $\angle AEC = \angle ACE = \alpha$ and $\angle BAC = 2\alpha$. Note that $\triangle ABC \sim \triangle CBE$ by SAS. Hence $\angle BAC = \angle BCE = 2\alpha$. But $\angle ACE = \alpha$, so $\angle ACB = \alpha$. Therefore, $\angle BAC = 2\angle BCA$.

Now let $\angle ABD = 2\beta$ and $\angle DBC = \beta$. We have $\angle BAC + \angle ACB + \angle ABC = 3\alpha + 3\beta = 180^\circ \iff \alpha + \beta = 60^\circ$. Then $\angle ADB = \angle DCB + \angle DBC = \alpha + \beta = 60^\circ$.

Hence $\cos\angle ADB = \boxed{\dfrac{1}{2}}$

☐ Consider the following sequence: $x_1 = 1$ and $x_{n+1} = x_n + \frac{1}{x_n}$.

Prove that $x_{100} > 14$

Solution

We have $(x_1)^2 = 1$ $(x_2)^2 = (x_1)^2 + \frac{1}{(x_1)^2} + 2$ $(x_3)^2 = (x_2)^2 + \frac{1}{(x_2)^2} + 2$ .... $(x_{100})^2 = (x_{99})^2 + \frac{1}{(x_{99})^2} + 2$ Shorten two sides we have $(x_{100})^2 = 1 + 2*99 + \frac{1}{(x_1)^2} + \frac{1}{(x_2)^2} + ... \frac{1}{(x_{99})^2} > 199 \implies x_{100} > 14(\ x_{100} > 0)$

□ Find all positive integers x and y such that $x^2 + xy = y^2 + 1$

Solution

Let's suppose that there are other solutions, and choose such (x, y) that x is minimal.

First we prove that $y > x$. If $y \leq x$ then $y^2 + 1 \leq x^2 + 1 < x^2 + xy$ (unless $xy = 1$, giving as solution $(1, 1)$ which is of the form (F_{2n-1}, F_{2n}). Then we prove that $2x > y$ in a similar manner.

Now we observe that $(2x - y, y - x)$ is also a solution of the original equation. Since $x < y$, $2x - y < x$, hence $2x - y = F_{2n-1}$ and $y - x = F_{2n}$ for some positive integer n. But now $x = 2x - y + y - x = F_{2n-1} + F_{2n} = F_{2n+1}$ and $y = y - x + x = F_{2n} + F_{2n+1} = F_{2n+2}$. Contradiction.

□ Find all four digit numbers of the form $aabb$ such that they are squares.

Solution

$11(100a + b) = p^2 \implies 100a + b = 11 \cdot x^2$

Now $100a + b \leq 909$ otherwise, $11(100a + b) > 9999$

$\implies x \leq 9$

check all possibilties $\implies 100a + b = 11 \cdot 8^2 = 704$

only soln is $11 \cdot 704 = 7744$

□ consider all natural numbers $1, 2, 3, ..., n$. Now take all possible products of them by pairs, so $1 \cdot 2, 1 \cdot 3, ...1 \cdot n, 2 \cdot 3, 2 \cdot 4, ...2 \cdot n...(n - 1) \cdot n$.

Find an expression in function of n for the sum of all those products.

Solution

Consider a polynomial $f(x) = (x - 1)(x - 2)(x - 3) \cdots (x - n)$. Clearly this polynomial has roots $1, 2, 3 \cdots, n$. Consider the case where n is even. $f(x) = (x - 1)(x - 2)(x - 3) \cdots (x - n) = x^n + s_1 x^{n-1} + s_2 x^{n-2} \cdots + s_{n-1} x + s_n$ Where each s_i represents the sum of the roots taken i at a time. However, the s_i 's signs will alternate from positive to negative, so the quantity we want is $-s_1 + s_2 - s_3 + s_4 \cdots + s_n$ which happens to be $f(-1) = (n + 1)!$ but including the coefficient of x^n. Finally we must subtract one though due to the first coefficient, so our sum is $(n + 1)! - 1$. A similar case work will yield the same answer for odd n.

□ Let be given two reals a, b such that $a - 2b + 2 = 0$. Prove that:

$$\sqrt{(a - 3)^2 + (b - 5)^2} + \sqrt{(a - 5)^2 + (b - 7)^2} \geq 6.$$

Solution

Note that if we change $\geq$ to $=$, we have an ellipse on the ab coordinate plane. If the line $a - 2b + 2 = 0$ intersects it at only one point, then it must be tangent. If so, all other points on $a - 2b + 2 = 0$ are outside the ellipse and consequently the LHS would be greater than 6.

$a = 2b - 2$, so substiution yields $\sqrt{(2b - 5)^2 + (b - 5)^2} + \sqrt{(2b - 7)^2 + (b - 7)^2} = 6 \Rightarrow \sqrt{5b^2 - 30b + 50} + \sqrt{5b^2 - 42b + 98} = 6$ Isolating the left radical and then squaring gives: $\sqrt{5b^2 - 30b + 50}^2 = \left(6 - \sqrt{(5b^2 - 42b +\right.$ $5b^2 - 30b + 50 = 36 - 12\sqrt{(5b^2 - 42b + 98} + 5b^2 - 42b + 98$ $0 = -12b + 84 - 12\sqrt{(5b^2 - 42b + 98}$ Isolating the right radical and then squaring gives: $\sqrt{5b^2 - 42b + 98}^2 = (-b + 7)^2$ $5b^2 - 42b + 98 = b^2 - 14b + 49$ $4b^2 - 28b + 49 = 0$ The discriminant $(-28)^2 - 4(4)(49) = 784 - 784 = 0$, so there is only one solution for b. Since the line is not orthogonal, there is only one intersection, proving the desired inequality.

□ Suppose that n people each know exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything that A knows, while B tells A nothing.

What is the minimum number of phone calls between pairs of people needed for everyone to know everything? Prove your answer is a minimum.

Solution

$2n-2$. We can see that this can always be done by induction: with $n=1$, we need zero calls. WOLOG, let A be the first caller, and let B be the last reciever (A=B is possible). For each new person added, we need only two additional calls: the new person calls A on the first call, and B calls the new person as the last call.

To see that this is optimal: it is clear that with $n>1$, each person needs to recieve a call at least once. Thus a minimum of n calls is necessary. WOLOG, let A be the first reciever, B be the second, and so on. It is clear that with $n>2$, A must recieve at least one additional call. With $n>3$, B must recieve at least one additional call, etc. Therefore we have n original calls, plus $n-2$ additional calls. $n+n-2=2n-2$ – During a certain election campaign, p different kinds of promises are made by the different political parties ($p>0$). While several political parties may make the same promise, any two parties have at least one promise in common; no two parties have exactly the same set of promises. Prove that there are no more than 2^{p-1} parties. – Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Define M to be the sum of all products of pairs a_ia_j $(i<j)$, *i.e.*,

$$M = a_1(a_2+a_3+\cdots+a_n) + a_2(a_3+a_4+\cdots+a_n) + \cdots + a_{n-1}a_n.$$

Prove that the square of at least one of the numbers $a_1, a_2, \ldots, a_n$ does not exceed $2M/n(n-1)$.

$\square$ Let a, b, c be nonzero real numbers such that $a+b+c=0$ and $a^3+b^3+c^3=a^5+b^5+c^5$. Prove that $a^2+b^2+c^2=\frac{6}{5}$.

Solution

Use identity $(m+n)^3 = m^3+n^3+3mn(m+n)$ to get

$$\left(\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45-\sqrt{2}}\right)^3 = (45+29\sqrt{2})+(45-\sqrt{2})+3\sqrt[3]{(45+29\sqrt{2})(45-29\sqrt{2})}\left(\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45}\right.$$

(to use less LaTeX, I'll let $t=\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45-\sqrt{2}}$) or

$$t^3 = 90+21t$$

By the rational roots theorem, any rational root of this equation must be $\pm$ a factor of 90. $2\sqrt[3]{45+29\sqrt{2}} < 10$ so we only need to look at factors <10, i.e. $\pm1, \pm2, \pm3, \pm5, \pm6, \pm9$. $t=6$ works. So the cubic expression is actually 6, which is rational.

$\square$ Find all real numbers x for which

$$10^x + 11^x + 12^x = 13^x + 14^x.$$

Solution

The way I would do it is to show that $10^x+11^x+12^x$ does not grow as fast as 13^x+14^x because

$10^y+11^y+12^y < 12^{y-x}(10^x+11^x+12^x)$

while

$13^y+14^y > 13^{y-x}(13^x+14^x)$

and clearly $13^{y-x} > 12^{y-x}$ for $y>x$ (both sides are also strictly positive). So once we have equality (easily seen to be $x=2$), the RHS will continue to grow faster than the LHS and we cannot have any more solutions. The same argument shows that if there existed a solution with $x < 2$ then the solution $x = 2$ wouldn't exist.

□ Let $\mathcal{U} = \{(x, y)\,|x, y \in \mathbb{Z},\ 0 \leq x, y < 4\}$.

(a) Prove that we can choose 6 points from $\mathcal{U}$ such that there are no isosceles triangles with the vertices among the chosen points.

(b) Prove that no matter how we choose 7 points from $\mathcal{U}$, there are always three which form an isosceles triangle.

Solution

the solution to (a) is $(0, 0), (0, 1), (2, 0), (3, 1), (2, 3), (3, 3)$

To prove (b), we first need to note that the total number of different distances from one of the four center points to any leagal point is only five: $1, \sqrt{2}, \sqrt{5}, 2, 2\sqrt{2}$ By the pigeon hole theorem, no point can be placed in the four center places. Notice that a maximum of two points can be placed in one of the four corners. This leaves five points to be placed on edges, and these will always form an isosceles triangle with the size $\sqrt{5}$ appearing twice. To see this, look at the two squares $(0, 1), (1, 3), (2, 0), (3, 2)$ and $(0, 2), (1, 0), (2, 3), (3, 1)$. By the pigeon hole theorem, at least one of them will contain three or more points, creating an isosceles triangle.

□ Find all natural numbers n such that the equation

$a_{n+1}x^2 - 2x\sqrt{a_1^2 + a_2^2 + ... + a_{n+1}^2} + a_1 + a_2 + ... + a_n = 0$

has real solutions for all real numbers $a_1, a_2, ..., a_{n+1}$.

Solution

There is no real solution if $4(a_1^2 + a_2^2 + ... + a_{n+1}^2) - 4(a_{n+1})(a_1 + a_2 + ... + a_n) < 0$ This can be expressed as $a_1^2 - a_{n+1}a_1 + a_2^2 - a_{n+1}a_2 + ... + a_n^2 - a_{n+1}a_n + a_{n+1}^2 < 0$ Each $a_i^2 - a_{n+1}a_i$ is lowest when $a_i = \frac{a_{n+1}}{2}$. Then $a_i^2 - a_{n+1}a_i$ comes out to be $-\frac{a_{n+1}^2}{4}$. Now it's clear that n=1, 2, 3, 4. – N is odd and $N \geq 15$. There are N cards such that on each card is written his index. Jack chooses any card from the N cards. There are 3 magicians: The first and the second magicians get $\frac{N-1}{2}$ cards(Any of them). Any of them is looking on his cards and gives 2 cards to the third magician that he decide. The third magician is looking on his 4 cards now, and decides what card was chosen by Jack. Find a strategy for the magicians to do this. — We have positive reals $a_0, a_1, a_2, a_3, a_4, a_5 \in [0, 10]$.

Also, $\sum a_i = 10$, $\sum ia_i \geq 25$.

Prove $\sum i(i-1)a_i \geq 40$.

□ Let x, y, z be real numbers whose sum is $\neq 0$. Prove that

$$\frac{x(y-z)}{y+z} + \frac{y(z-x)}{z+x} + \frac{z(x-y)}{x+y} = 0$$

holds if and only if two of the numbers are equal.

Solution

The proof uses the following well known trick to simplify the expression on the left hand side (LHS) of the given equation: $x - y = (x - z) + (z - y)$

$LHS = x \cdot \frac{y-z}{y+z} + y \cdot \frac{z-x}{z+x} + z \cdot \frac{(x-z)+(z-y)}{x+y} = (y-z)(\frac{x}{y+z} - \frac{z}{x+y}) + (z-x)(\frac{y}{z+x} - \frac{z}{x+y}) = (y-z)\frac{(x^2+xy-yz-z^2)}{(x+y)(y+z)} + (z-x)\frac{(xy+y^2-z^2-zx)}{(z+x)(x+y)} = (y-z)\left(\frac{(x-z)(x+z)+y(x-z)}{(x+y)(y+z)}\right) + (z-x)\left(\frac{(y-z)(y+z)+x(y-z)}{(z+x)(x+y)}\right) = (y-z)(x-z)\frac{(x+y+z)}{(x+y)(y+z)} + (z-x)(y-z)\frac{(x+y+z)}{(z+x)(x+y)} = \frac{(y-z)(x-z)(x+y+z)}{(x+y)} \cdot \left(\frac{1}{(y+z)} - \frac{1}{(z+x)}\right) = -\frac{(x-y)(y-z)(z-x)(x+y+z)}{(x+y)(y+z)(z+x)}$

Now, we know that $x+y+z \neq 0$. So, $LHS = 0 \iff \frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)} = 0 \iff (x-y)(y-z)(z-x) = 0$ $\iff$ if any two of x, y, z are equal. Another way $f(x) = (x+y)(y+z)(z+x)\left(\frac{x(y-z)}{y+z} + \frac{y(z-x)}{z+x} + \frac{z(x-y)}{x+y}\right)$ say $f(x)$ is a polynomial about x, where y, z are constants, then through substitution, $f(y)=f(-x-y)=0$, then by factor theorem

$f(x)=(x-y)(x+y+z)q(x)$, ($q(x)$ is a polynomial) we note that $f(x)$ is 3rd degree, so $q(x)$ is degree 1, also by symmetry, we have the other variables as factors, so

$f(x)=(x-y)(y-z)(z-x)(x+y+z)q_2(x)$

then $q_2(x)$ is constant due to degrees

then we can plug in arbitrary values of x,y,z to find $q_2(x)=1$

the rest follows by zero product property

$\square$ For a real parameter $p\neq 0$, let x_1,x_2 be the roots of the equation $x^2+px-\frac{1}{2p^2}=0$. Prove that $x_1^4+x_2^4\geq 2+\sqrt{2}$.

Solution

Since $p\neq 0$, $x^4=\left(x^2+px-\frac{1}{2p^2}\right)\left(x^2-px+p^2+\frac{1}{2p^2}\right)-\left(p^3+\frac{1}{p}\right)x+\frac{1}{4p^4}+\frac{1}{2}$.

Since x_i $(i=1,2)$ is the solution of the given quadratic equation, we have $x_i^2+px_i-\frac{1}{2p^2}=0$. Thus $x_1^4+x_2^4=-\left(p^3+\frac{1}{p}\right)(x_1+x_2)+2\left(\frac{1}{4p^4+\frac{1}{2}}\right)=-\left(p^3+\frac{1}{p}\right)(-p)+2\left(\frac{1}{4p^4}+\frac{1}{2}\right)$ $\because x_1+x_2=-p$ $=p^4+\frac{1}{2p^4}+2$. As sen pointed, we can use $A.M.-G.M.$ inequality.

$\square$

Let x,y,z be integers such that

$(x-y)^2+(y-z)^2+(z-x)^2=xyz$

Prove that $x^3+y^3+z^3$ is divisible by $x+y+z+6$

Solution

We know that $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)\Rightarrow x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)[(x-y)^2+(y-z)^2+(z-x)^2]\Rightarrow x^3+y^3+z^3=3xyz+\frac{1}{2}(x+y+z)xyz$ ...(Using the given information.) $\Rightarrow x^3+y^3+z^3=\frac{1}{2}(x+y+z+6)xyz\Rightarrow x^3+y^3+z^3=(x+y+z+6)(x^2+y^2+z^2-xy-yz-zx)$ Since x,y,z are all integers, we conclude $x^3+y^3+z^3$ is divisible by $x+y+z+6$.

Anyone have solutions that don't use

$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$ – Consider the tetrahedron $ABCD$ of volume 1 and the points M,N,P,Q,R,S on the edges AB,BC,CD,DA,AC,BD. If MP,NQ,RS are concurrent, then prove that the volume of $MNPQRS$ is $\leq\frac{1}{2}$. — Find the integer numbers x,y,z,t which satisfy $x+y+z=t^2$, $x^2+y^2+z^2=t^3$.

$\square$ The bisectors of the angles of $\triangle ABC$ cut BC,CA,AB in D,E,F. Prove that

$$\frac{1}{AB\cdot CE}+\frac{1}{BC\cdot AF}+\frac{1}{CA\cdot BD}=\frac{1}{r\cdot R}.$$

Solution

Let $AB=c, BC=a, AC=b$, then using the angle bisector theorem about the ratio of the segments in a triangle whenan angle bisector intersects the side: $CE=\frac{ab}{a+c}$ $AF=\frac{bc}{a+b}$ $BD=\frac{ac}{b+c}$ Also: $Area=pr=\frac{abc}{4R}$ So: $Rr=\frac{abc}{2(a+b+c)}$ Substituting the above identities in the expression, get: LS= $\frac{a+c}{abc}+\frac{a+b}{abc}+\frac{b+c}{abc}=\frac{2(a+b+c)}{abc}$ =RS. QED – Let $P(z)$ be a polynomial with complex coefficients which is of degree 1992 and has distinct zeros. Prove that there exist complex numbers $a_1,a_2,\ldots,a_{1992}$ such that $P(z)$ divides the polynomial

$$\left(\cdots\left((z-a_1)^2-a_2\right)^2\cdots-a_{1991}\right)^2-a_{1992}.$$

—— n is a given positive integer. For what $m\in[0,n]\cap\mathbb{Z}$ does the identity

$\sum_{k=m}^{n}\binom{n}{k}\binom{n}{n+m-k}=\binom{2n}{n-m}$ hold?

$\square$ Define a sequence (a_n) by $a_1 = 1, a_2 = 2$ and $a_{n+2} = 2a_{n+1} - a_n + 2, n \geq 1$. Prove that for any $m, a_m a_{m+1}$ is also a term of the sequence.

Solution

Now by inspection of the first couple of terms:

conjecture

$a_k = (k-1)^2 + 1$

inductive proof

$a_1 = 0^2 + 1 = 1$ yes.

$a_2 = 1^2 + 1 = 2$ yes

assume true for $n = k$ and $n = k + 1$

$\implies a_k = (k-1)^2 + 1$ and $a_{k+1} = k^2 + 1$

for $n = k + 2$

$a_{k+2} = 2a_{k+1} - a_k + 2$

$\implies a_{k+2} = 2(k^2+1) - ((k-1)^2+1) + 2$

$\implies a_{k+2} = 2k^2 + 2 - k^2 + 2k - 2 + 2 = k^2 + 2k + 2 = (k+1)^2 + 1$

completion of induction.

$a_m \cdot a_{m+1} = ((m-1)^2 + 1)(m^2 + 1)$

$= m^4 - 2m^3 + 3m^2 - 2m + 2 = (m^2 - m + 1)^2 + 1 = a_{m^2-m+2}$

$\square$ Find all pairs (x, y) of nonnegative integers such that $x^2 + 3y$ and $y^2 + 3x$ are simultaneously perfect squares.

Solution

Since the expressions x^2+3y and y^2+3x are symmetric in x and y, we may without loss of generality assume $x \geq y$. Consider the Diophantine system of equations

(1) $x^2 + 3y = a^2$

(2) $y^2 + 3x = b^2$.

where a and b are natural numbers. For $y = 0$ we obtain the solutions $(x, y) = (3t^2, 0)$. Suppose $y > 0$. Then

$x^2 < x^2 + 3y < x^2 + 3x < (x+2)^2$,

hence $a = x + 1$ by (1). This implies that

$x^2 + 3y = (x+1)^2$,

i.e.

$3y = 2x + 1$.

Consequently $x = 3s + 1$ and $y = 2s + 1$ for a non-negative integer s. So according to (2)

$b^2 = y^2 + 3x = (2s+1)^2 + 3(3s+1) = 4s^2 + 13s + 4$,

which is equivalent to

$(8s + 13 - 4b)(8s + 13 + 4b) = 105$.

Therefore

(3) $8s + 13 - 4b = d$,

(4) $8s + 13 + 4b = \frac{105}{d}$,

where $d \leq \sqrt{105}$ is a positive divisor of $105 = 3 \cdot 5 \cdot 7$, i.e. $d \in \{1, 3, 5, 7\}$. Adding (3) and (4), the result is

$2(8s + 13) = d + \frac{105}{d}$,

thus

$s = \frac{d + \frac{105}{d} - 26}{16}$.

This formula gives $(d,s)=(1,5),\ (3,\frac{3}{4}),\ (5,0),(7,-\frac{1}{4})$ as possible solutions. Seeing that s is an integer, we are left with two solutions,

$s=0 \ \Leftrightarrow \ (x,y)=(1,1),$

$s=5 \ \Leftrightarrow \ (x,y)=(16,11),$

$\square$ R is a solution to $x+\frac{1}{x}=\frac{\sin 210^\circ}{\sin 285^\circ}$. Suppose that $\frac{1}{R^{2006}}+R^{2006}=A$ find $\lfloor A^{10}\rfloor$ where $\lfloor x\rfloor$ is the greatest integer less than or equal to x.

Solution

$=\frac{\sin 210^\circ}{\sin 285^\circ}=\frac{-\sin 150^\circ}{-\sin 75}=\frac{2\sin 75^\circ\cos 75^\circ}{\sin 75}=2\cos 75$

Then using the quadratic formula, $x=\text{cis}75^\circ$. Now we can easily evaluate A using cis exponent rules.

$A=\text{cis}75*2006+\text{cis}-75*2006=2\cos 330=\sqrt{3}$

It follows that $A^{10}=\boxed{243}$

$\square$ $\triangle VA_0A_1$ is isosceles with base $\overline{A_1A_0}$. Construct A_2 on segment $\overline{A_0V}$ such that $\overline{A_0A_1}=\overline{A_1A_2}=b$. Construct A_3 on $\overline{A_1V}$ such that $b=\overline{A_2A_3}$. Contiue this pattern: construct $\overline{A_{2n}A_{2n+1}}=b$ with A_{2n+1} on segment $\overline{VA_1}$ and $\overline{A_{2n+1}A_{2n+2}}=b$ with A_{2n+2} on segment $\overline{VA_0}$. The points A_n do not coincide and $\angle VA_1A_0=90-\frac{1}{2006}$. Suppose A_k is the last point you can construct on the perimeter of the triangle. Find the remainder when k is divided by 1000.

Solution

Let $\angle VA_1A_0=\theta$. After some initial plodding and (not-very-rigorous) inductive reasoning, we reach the conclusion that we can construct A_n iff $(2n-1)\theta-(n-1)\pi>0$.

$\therefore$ If A_k is the last point we can construct on the perimeter of $\triangle VA_1A_0$, then the following two conditions must be satisfied: $(2k-1)\theta-(k-1)\pi>0$, and $(2k+1)\theta-k\pi\leq 0$

After pluggin in $\theta=90-\frac{1}{2006}$ (in degrees), we get the following two inequalities: $k<(90)(1003)+\frac{1}{2}$, and $k\geq(90)(2003)-\frac{1}{2}$

This implies $k=(90)(1003)$, so when k is divided by 1000, the remainder is $(90)(3)=270$. And, we are done.

$\square$ Find a way to generate all integral solutions to $x^2+2y^2=z^2$.

Solution

Use pythagorean triples. Transform your equation into $u^2+y^2=z^2$, where $u^2=x^2+y^2$. Pythagorean triples are generated as follows:

$\gcd(u,y)=\gcd(u,z)=\gcd(y,z)=1 \implies \exists a,b\in\mathbb{Z}$ such that

$u=2ab,\ y=|a^2-b^2|,\ z=a^2+b^2$. Substitution and Done! Another way We will proceed using Algebraic Number Theory and Unique Factorization in $\mathbb{Z}[i]$. Hence, $x^2+2y^2=(x+iy\sqrt{2})(x-iy\sqrt{2})=z^2$. Thus, $\exists a,b\in\mathbb{R}$ such that $x+iy\sqrt{2}=(a+ib)^2=a^2-b^2+2abi \implies x=a^2-b^2$ and $y=\sqrt{2}ab$. From these, we have that $z=a^2+b^2$. Now, if we find $c\in\mathbb{Z}$ such that $a=\sqrt{2}c$, then we have $y=2cb, x=2c^2-b^2, y=2c^2+b^2$. Thus, we have that $(x,y,z)=(2u^2-v^2,2uv,2u^2+v^2),\ u,v\in\mathbb{Z}$. Q.E.D

$\square$ The sequence $a_0,a_1,a_2,...$ satisfies $a_{m+n}+a_{m-n}=2(a_m+a_n)$ for all nonnegative integers m and n with $m\geq n$. If $a_1=1$, determine a_{2006}.

Solution

well to complete the inductive step.

$a_1=1^2, a_2=2^2$ true

assume true for a_k and a_{k+1}

$a_k=k^2$ and $a_{k+1}=(k+1)^2$

now we have $a_{k+2}+a_k=2(a_{k+1}+a_1)$ from initial condition.

$\implies a_{k+2}=2(a_{k+1}+1)-a_k$ since $a_1=1$

$a_{k+2}=2(k^2+2k+2)-k^2$

$\implies a_{k+2}=k^2+4k+4=(k+2)^2$

hence by MI, $a_m=m^2$ –

— Find all postive integer x and $n\in\mathbb{N}$ such that $x^n+2^n+1 \mid x^{n+2}+2^{n+2}+1$.

☐ Let $P_0P_1\dots P_{n-1}$ be a regular polygon inscribed in a unit circle. Prove that $P_0P_1{\cdot}P_0P_2\dots P_0P_{n-1}=n$.

Solution

Work in the complex plane. WLOG $P_k=e^{i\frac{2\pi k}{n}}$.

Let $P(x)=x^n-1$ be the polynomial with complex roots P_k. Let $Q(x)=\frac{x^n-1}{x-1}=1+x+x^2+...+x^{n-1}$ be the polynomial with those roots $P_k, k\neq 0$.

Now $P_0P_k=(1-P_k)$. Our product is then $\prod_{k=1}^{n-1}(1-e^{i\frac{2\pi k}{n}})=Q(1)=n$. QED.

☐ Prove that for each prime p the equation $2^p+3^p=a^n$ has no intger solutions (a,n) with $a,n>1$.

Solution

Equation obviously has no solutions with $p=2$, so we can say that p is an odd number.

$$2^p+3^p=(2+3)\cdot\sum_{i=0}^{p-1}2^{p-1-i}(-3)^i=a^n$$

Thus a must be divisible by 5, but since $n>1$, LHS is divible by 25, and it means that $\sum_{i=0}^{p-1}2^{p-1-i}(-3)^i$ is a multiple of 5.

$$\sum_{i=0}^{p-1}2^{p-1-i}(-3)^i$$

$$\equiv\sum_{i=0}^{p-1}2^{p-1-i}(2)^i \mod 5$$

$$\equiv\sum_{i=0}^{p-1}2^{p-1} \mod 5$$

$$\equiv p\cdot 2^{p-1} \mod 5$$

$$\equiv 0 \mod 5 \iff p=5$$

But our initial equation doesn't have solutions with $p=5$.

☐ Let $z_1,z_2,z_3\in\mathbb{C}$ such that $|z_1|=|z_2|=|z_3|=R$ and $z_2\neq z_3$. Prove that

$$\min_{a\in\mathbb{R}}|az_2+(1-a)z_3-z_1|=\frac{1}{2R}|z_1-z_2|\cdot|z_1-z_3|.$$

Solution

Let z_1,z_2,z_3 be three vectors in complex plane that start at the origin. Connect the three endpoints we get a triangle with circumcenter at the origin. Let the end point of z_1 be A, z_2 be B, and z_3 be C, then by simple analytic geometry (and simple algebra arrangement) the given equation is the same as:

$\min|k(z_2-z_3)+z_3-z_1|=\frac{bc}{2R}$

(for notation sake, I change the a in the original equation to k, because a will stand for a side of my triangle).

Now look at the expression in the min. We see that $z_3 - z_1$ is a vector parallel to and the same length as the side b, and $z_2 - z_3$ is parallel and the same length as the side a. using tip to tail vector addition, we see that the minimum expression becomes the minimum length from A to BC, which is the length of the altitude from A to BC by simple geometry.

So, we wish to prove that $h_A = \frac{bc}{2R}$. Recall that $\frac{ah_A}{2} = \triangle$, and also $\triangle = \frac{abc}{4R}$, equate and solve we see that the equation indeed is true.

If z_1 equals to either z_2 or z_3, plug in $k = 1$ we get both sides to be zero, so the equation holds all the time. – Let $k \circ m$ mean $k \geq m+2$. Show that every positive integer n has a unique representation of the form $n = F_{k_1} + ... + F_{k_r}$, where F_{k_i} are Fibonacci numbers and $k_1 \circ k_2 \circ \ldots \circ k_r \circ 0$. – A rectangular prism with dimensions $\ell \times w \times h$ has 2 planes connecting the opposite sides (forming an X) from top. There is a sphere with radius x inside the figure. What is the probability that this sphere neither touches the planes nor the sides of the rectangular prism?

□ Positive integers are written on all the faces of a cube, one on each. At each corner of the cube, the product of the numbers on the faces that meet at the vertex is written. The sum of the numbers written on the corners is 2004. If T denotes the sum of the numbers on all the faces, find the possible values of T.

Solution

Let the numbers on the "walls" of the cube be a, b, c and d such that a is "opposite" c and b is "opposite" d. Also, let the number on the "top" face be e and the number on the "bottom" face be f (e and f are "opposite" each other.)

Then the products at the eight corners are $abe, ade, bce, cde, abf, adf, bcf$ and cdf. We now have $abe+ade+bce+cde+abf+adf+bcf+cdf = 2004 \Rightarrow ab(e+f)+ad(e+f)+bc(e+f)+cd(e+f) = 2004$ $\Rightarrow (e+f)(ab+ad+bc+cd) = 2004 \Rightarrow (a+c)(b+d)(e+f) = 2^2 \cdot 3 \cdot 167$

Now, note that each of the three terms on the LHS is greater than or equal to 2.

The rest is just a matter of evaluating individual cases, such as $a + c = 2^2, b + d = 3$ and $e + f = 167$, in which case $T = 4 + 3 + 167 = 174$, and so on. – Let $n \geq 2$ be a given integer. How many solutions does the system of equations $x_1 + x_n^2 = 4x_n$, $x_2 + x_1^2 = 4x_1$, $\cdots$, $x_n + x_{n-1}^2 = 4x_{n-1}$ have in nonnegative real numbers $x_1, \cdots, x_n$? ——— Let $n \geq 3$ be an integer. Prove that for positive numbers $x_1 \leq x_2 \leq \cdots \leq x_n$, $\frac{x_n x_1}{x_2} + \frac{x_1 x_2}{x_3} + \cdots + \frac{x_{n-1} x_n}{x_1} \geq x_1 + x_2 + \cdots + x_n$.

□ Find minimum value of expression $\frac{1}{r}(\frac{4p}{u} + \frac{q}{\sqrt{1-v^2}})$, where p, q, r, u, v - positive numbers satisfying conditions: $pv + q\sqrt{1-u^2} \leq r$,

$p^2 + 2qr\sqrt{1-u^2} \geq q^2 + r^2$,

$2qr\sqrt{1-u^2} + q^2 \frac{1-v^2-u^2}{v^2-1} \geq r^2$

Solution

First we try to simplify problem a bit. Make a new variables $a = \frac{p}{r}, b = \frac{q}{r}$. After this problem looks easier: Find minimum value of function $f = \frac{4a}{u} + \frac{b}{\sqrt{1-v^2}}$, where a, b, u, v - positive numbers satisfying conditions: $av + b\sqrt{1-u^2} \leq 1$,

$a^2 + 2b\sqrt{1-u^2} \geq b^2 + 1$,

$2b\sqrt{1-u^2} + b^2 \frac{u^2}{1-v^2} \geq b^2 + 1$

After this lake a look at variables u and v: they satisfy inequalities $0 < u, v < 1$, its possible to make such angles $0 < \alpha, \beta < \frac{\pi}{2}$, what $u = \cos\alpha, v = \sin\beta$. After this problems looks like:

$f=\frac{4a}{\cos\alpha}+\frac{b}{\cos\beta}$, where variables satisfy: $a\sin\beta+b\sin\alpha\le 1$,

$a^2+2b\sin\alpha\ge b^2+1$,

$2b\sin\alpha+b^2\frac{\cos^2\alpha}{\cos^2\beta}\ge b^2+1$

Then notice what second condition allows to lower a until next equation will be satisfied: $a^2+2b\sin\alpha=b^2+1\Leftrightarrow a^2-b^2\cos^2\alpha=(1-b\sin\alpha)^2$ Then lower b until: $2b\sin\alpha+b^2\frac{\cos^2\alpha}{\cos^2\beta}=b^2+1\Leftrightarrow b\tan\beta\cos\alpha=1-b\sin\alpha\Leftrightarrow b\sin(\alpha+\beta)=\cos\beta$ From last two conditions what became equations we get what $a^2-b^2\cos^2\alpha=b^2\tan^2\beta\cos^2\alpha\Leftrightarrow a\cos\beta=b\cos\alpha$ Then $f=\frac{4a\cos\beta+b\cos\alpha}{\cos\beta\cos\alpha}=\frac{5b}{\cos\beta}=\frac{5}{\sin(\alpha+\beta)}\ge 5$ So the answer is 5.

□ In acute triangle ABC, E, F are on side BC such that <BAE=<CAF. Construct $FM\perp AB$ and $FN\perp AC$, extend AE to meet the circumcircle of ABC at D. Show that the area of AMDN is equal to the area of triangle ABC.

Solution

Let $\angle BAC=\alpha$ and $\angle BAE=\angle CAF=x$.

Then we have to prove that $\frac{1}{2}AB\cdot AC\sin\alpha=\frac{1}{2}AD(AN\sin(\alpha-x)+AM\sin x)$ (1)

We have $AM=AF\cos(\alpha-x)$ and $AN=AF\cos x$.

So (1) becomes $AB\cdot AC\sin\alpha=AD\cdot AF(\cos x\sin(\alpha-x)+\sin x\cos(\alpha-x))=AD\cdot AF\sin\alpha$.

But $\angle BAD=\angle FAC$ and $\angle ACF=\angle ADB$, from which $\Delta ABD\sim\Delta AFC$ and $\frac{AB}{AD}=\frac{AF}{AC}$.

Another way Let $\angle BAE=\angle CAF=\alpha$, $\angle EAF=\beta$,then

$$S_{\triangle ABC}=\frac{1}{2}AB\cdot AF\cdot\sin(\alpha+\beta)+\frac{1}{2}AC\cdot AF\cdot\sin\alpha=\frac{AF}{4R}(AB\cdot CD+AC\cdot BD)$$

Where R is the radius of the circumcircle. Since we also have:

$$\begin{aligned}S_{AMDN}&=\frac{1}{2}AM\cdot AD\cdot\sin\alpha+\frac{1}{2}AD\cdot AN\sin(\alpha+\beta)\\&=\frac{1}{2}AD[AF\cdot\cos(\alpha+\beta)\sin\alpha+AF\cos\alpha\sin(\alpha+\beta)]\\&=\frac{1}{2}AD\cdot AF\cdot\sin(2\alpha+\beta)\\&=\frac{AF}{4R}AD\cdot BC.\end{aligned}$$

By Ptolemy theorem we know that $AB\cdot CD+AC\cdot BC=AD\cdot BC$. Thus $S_{AMDN}=S_{\triangle ABC}$.

□ What is the last non-zero digit in N! ?

Note that, N is big enough.

Solution

Let $L(n)$ be the last nonzero digit in $n!$.

Suppose that $n=5q+r$ where $q\ge 1$ and r is from 0 to 4. Then I believe we get the recurrence

$$L(n)=2^qL(q)L(r)\bmod 10\,.$$

That recurrence comes pulling out the terms 5, 10, ..., $5q$ from the product $n!=1\cdot 2\cdot\ldots\cdot n$.

Using this recurrence, we can quickly calculate $L(n)$ ANother way To check the mod 10 recurrence, we need to check mod 2 and mod 5. Mod 2 is clear because both sides are even. To check mod 5, we can use the identity

$$(5q)!=10^qq!\prod_{i=0}^{q-1}\frac{(5i+1)(5i+2)(5i+3)(5i+4)}{2}\,.$$

The fraction $\frac{(5i+1)(5i+2)(5i+3)(5i+4)}{2}$ is 2 mod 5, so we get the recurrence

$$L(5q)\equiv 2^qL(q)\pmod 5\,.$$

Finally, it is easy to see that

$$L(n) \equiv L(5q)L(r) \pmod 5 .$$

$\square$ What is the rightmost nonzero digit of 2006!?

Solution

To calculate 2^{500} mod 10 is to observe that the powers of 2 mod 10 repeat in a cycle of 4. Thus $2^{500} \equiv 2^4 \equiv 6 \pmod{10}$.

Note that the numbers in your calculation arise because 2006! has 500 terminating zeros, and because the base 5 representation of 2006 is $(31011)_5$.

$\square$ In triangle ABC, $\angle A = 60^o$, AB>AC. O is the circumcentre. Two altitudes BE and CF meet at H. M and N are on BH and HF such that BM=CN, find the value of $\frac{MH+NH}{OH}$. (2002 China League)

Solution

Construct $M‘$ on BE such that $BM‘ = CH$ so $MM‘ = NH$. It‘s easy to show that $OM‘B$ is congruent to OCH proving that the angles $\angle OBM‘ = \angle OCH$ (it‘s appears after some operations with the 60 degrees angle, isogonals,...), and more, exist a rotation of 120 degrees between, seeing that $\angle CHB = 120$.

So, the triangle $OHM‘$ is isosceles and $\angle HOM‘ = 120 \Rightarrow \frac{NH+HM}{OH} = \frac{HM'}{OH} = \sqrt{3}$

$\square$ Prove $1 \cdot 2 \cdot 3 \cdots (p-2) \equiv 1 \mod p$ where p is prime.

Solution

For each $x \in \mathfrak{T} = [2; p-2]$ there is only one $\sigma(x) = x^{-1}(\mod p)$ $(\sigma : \mathfrak{T} \to \mathfrak{T})$ (it follows from the Bezout's theorem for linear cominations and continue divisions); so let's take two factors such that their product is $\equiv 1(\mod p)$ and multiply by 1: it will give $\equiv 1^{\frac{p-3}{2}+1} \equiv 1(\mod p)$

$\square$ In triangle ABC, a,b,c be its sides. If the measure of angles A, B and C forms a geometrical sequence, and $b^2 - a^2 = ac$, then find angle B. (1985 China League)

Solution

$\measuredangle A = a; \measuredangle B = ar; \measuredangle C = ar^2$. We have D in ray CB such that $BD = DC$. Then triangles ABC and DAC are similars.

Hence:$\frac{ar}{2} = a \implies r = 2$

$ar^2 + ar + a = \pi$:arrow: $7a = \pi \implies \measuredangle B = \frac{2\pi}{7}$ Another solution: Through point C construct $CD \parallel AB$ to meet the circumcircle of $\triangle ABC$ at D. Connect AD, then $ABCD$ is an isocelces trapzoid. By Ptolemy, we have: $b^2 = a^2 + c \cdot CD$. From $b^2 - a^2 = ac$, we get $CD = a$, thus: $AD = DC = CB$, thus $\angle B = 2\angle ABC$.

In triangle ABC, since the mearuse of angles A, B and C forms a geometrical sequence, so the common ratio q is 2, thus $\angle A + \angle B + \angle C = 7\angle A = \pi$, thus we get $\angle A = \frac{\pi}{7}$, thus $\angle B = \frac{2\pi}{7}$. – At each lattice point of a finite grid paper, we draw an arrow parallel to one of the sides of the paper (no arrows on the boundary can point outwards.) Show that there exist two neighboring points (horizontally, vertically, or diagonally) which the arrows point to opposite directions. – Find all n, a positive integer such that $(n-1)50^n < 51^n$

generalise this for $(n-1)x^n < (x+1)^n$ – Suppose p is a prime greater than 3. Find all pairs of integers (a, b) satisfying the equation

$a^2 + 3ab + 2p(a+b) + p^2 = 0.$

$\square$ Let $x_1 = x_2 = 1$, $x_3 = 4$, and $x_{n+3} = 2x_{n+2} + 2x_{n+1} - x_n$ for all $n \geq 1$. Prove that x_n is a square for all $n \geq 1$.

Solution

Here is a more "natural" way of solving the problem. We first compute by hand the first few terms of the sequence. We then note that $x_1 = 1^2, x_2 = 1^2, x_3 = 2^2, x_4 = 3^2, x_5 = 5^2, x_6 = 8^2, x_7 = 13^2$, and so on. We, now, suspect a surprising pattern here, and this should help us determine our next step.

<u>Claim</u>: $x_n = {F_n}^2$, where (F_n) is the Fibonacci sequence, with $F_1 = F_2 = 1$. **<u>Proof</u>**: We use strong induction on n to show our claim is true.

Base case: We have $x_1 = {F_1}^2, x_2 = {F_2}^2, x_3 = {F_3}^2$, as shown above.

Induction case: Let $x_m = {F_m}^2$ be true for all $m, 4 \le m \le n$. Now, $x_{n+1} = {F_{n+1}}^2 \Leftrightarrow 2x_n + 2x_{n-1} - x_{n-2} = {F_{n+1}}^2 \Leftrightarrow 2{F_n}^2 + 2{F_{n-1}}^2 - {F_{n-2}}^2 = \left(F_n + F_{n-1}\right)^2 \Leftrightarrow {F_n}^2 + F_{n-1} - {F_{n-2}}^2 = 2F_nF_{n-1}$ $\Leftrightarrow {F_n}^2 + F_{n-1} - 2F_nF_{n-1} = {F_{n-2}}^2 \Leftrightarrow \left(F_n - F_{n-1}\right)^2 = {F_{n-2}}^2 \Leftrightarrow {F_{n-2}}^2 = {F_{n-2}}^2$, which is true. Thus, $x_{n+1} = {F_{n+1}}^2$ is true, and this completes our inductive proof.

Hence, $x_n = {F_n}^2$, for all natural $n \ge 1$, where (F_n) is the Fibonacci sequence (with $F_1 = F_2 = 1$). And we are done. – Find all pairs $(a; b)$ of positive integers for which the numbers $a^3 + 6ab + 1$ and $b^3 + 6ab + 1$ are cubes of positive integers. – If a square is partitioned into n convex polygons, determine the maximum number of edges present in the resulting figure.

[You may find it helpful to use Euler's theroem: If a polygon is partioned into n polygons, then $v - e + n = 1$, where v is the number of vertices and e is the number of edges in the resulting figure.] —– Given a graph with n vertices and q edges numbered $1, ..., q$, show that there exists a chain of m edges, $m \ge \frac{2q}{n}$, each two consecutive edges having a common vertex, arranged monotically with respect to the numbering. — A cyclic quadrilateral $ABCD$ is given. The lines AD and BC intersect at E, with C between B and E; the diagonals AC and BD intersect at F. Let M be the midpoint of the side CD, and let N(different from M) be a point on the circuncircle of the triangle ABM such that $\frac{AN}{BN}$=$\frac{AM}{BM}$. Prove that the points E, F, and N are collinear.

□ Show that no integer of the form $xyxy$ in base 10 (where x and y are digits) can be the cube of an integer. Find the smallest base $b > 1$ for which there is a perfect cube of the form $xyxy$ in base b.

Solution

$xyxy = 101(10x + y) = k^3$

$\implies 10x + y = 101^2 \cdot m^3$ as 101 is prime

minimum is when $m = 1 \implies 10x + y = 10201$ but $0 \le x, y \le 9$

now let it be in base b

$\implies x \cdot b^3 + y \cdot b^2 + x \cdot b + y = xb(b^2 + 1) + y(b^2 + 1) = (xb + y)(b^2 + 1)$

from here, i just substituted in $b = 2, 3, 4, 5, 6$, and arrived at no solutions for x, y, where $0 \le x, y \le (b - 1)$

$b = 7$

$\implies (7x + y)50 = k^3$ and a solution can be found $x = 2, y = 6$

hence base 7 is smallest.

□ Show that if x is a non-zero real number, then $x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \ge 0$.

Solution

$x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \ge 0 \Leftrightarrow (x^3 - 1)\left(x^5 - \frac{1}{x^4}\right) \ge 0 \Leftrightarrow \frac{(x^3-1)^2(x^6+x^3+1)}{x^4} \ge 0$ Another way Multiply all by x^4, which is positive and won't affect sign.

Factorizes as $(x^9 - 1)(x^3 - 1)$, and both factors are the same sign.

Check:

$x \ge 1$ implies both terms non negative. $x < 1$ implies both terms negative.

Thus product is always non-negative. – Let x, y be positive integers with $y > 3$ and $x^2 + y^4 =$

$2\left[(x-6)^2+(y+1)^2\right]$. Prove that $x^2+y^4=1994$.

$\square$ P1, P2, and P3 are polynomials defined by: $P1(x) = 1+x+x^3+x^4+......+x^{96}+x^{97}+x^{99}+x^{100}$

$P2(x) = 1-x+x^2-......-x^{99}+x^{100}$

$P3(x) = 1+x+x^2+......+x^{66}+x^{67}$.

Find the number of distinct complex roots of P1 * P2 * P3.

Solution

well look at that, it **is** missing every third power. so factor out a $1+x$ to get $P1(x) = (x+1)(1+x^3+x^6...x^{99}) = (x+1)\left(\frac{x^{102}-1}{x^3-1}\right)$. so $P1$ has 100 roots, -1 and the 102nd roots of unity, not counting the cube roots of unity. so -1 is a double root, so there are only 99 distinct ones.

now some of those will be the same as the 68th roots from that other polynomial. the roots of $P1$ are in the form $e^{\frac{2k\pi}{102}}$ (k is an integer, $1 \le k \le 101$, k is not 34 or 68), the roots of $P2$ are in the form $e^{\frac{2k\pi}{101}}$ (k is an integer from 1 to 100), and the roots of $P3$ are $e^{\frac{2k\pi}{68}}$ (k is an integer from 1 to 67).

since 101 is relatively prime to 102 and 68, none of the roots of $P2$ overlap the other ones. let k_1 be the k from a root of $P1$ and k_3 be a k from $P3$. so the roots that aren't distinct are the ones where $\frac{k_1}{102} = \frac{k_3}{68}$

mulitplying by 34, we get $\frac{k_1}{3} = \frac{k_3}{2}$, which means $2k_1 = 3k_3$. so k_3 is divisible by 2, and k_1 is divisible by 3. i think any even number from 2 to 66 works for k_3, so there's 33 of them.

so we have $99+100+67-33 = 233$. so hopefully 233 is the right answer

$\square$

3 blue marbles and 4 red marbles are placed in an opaque bag. Bob takes one marble out at a time until he has taken out an equal number of red and blue marbles, or has taken all the marbles. Find the probability that Bob takes all of the marbles.

Solution

Note that Bob can take out equal number of both marbles only if he took out even number of marbles. We'll count the number of ways to take equal number of marbles and subtract.

case 1: $n=2$ we have 2 ways, ab and ba, to take out the marbles

case 2: $n=4$ we have 2 ways also, $aabb$ and $bbaa$. (because if it starts with ab or ba, by case 2 it'll end already).

case 3: $n=6$ we need to start with $aaba$ $aaab$, or the complement, then end with two of the same. So we have 4 ways to do it.

So we get: $1-\left(2*\frac{3*4}{7*6}+2*\frac{3*2*4*3}{7*6*5*4}+4*\frac{3*2*1*4*3*2}{7*6*5*4*3*2}\right) = \frac{1}{7}$

$\square$ In a quadrilateral $ABCD$, it is given that AB is parallel to CD and the diagonals AC and BD are perpendicular to each other. Show that (a) $AD \cdot BC \ge AB \cdot CD$ (b) $AD+BC \ge AB+CD$.

Solution

(a) Let $a = AP, b = BP, c = CP, d = DP$. Assume WLOG that $AB \le CD$. Since $AB \| CD$, $\triangle ABP \sim \triangle CDP$. So,

$$\frac{a}{c} = \frac{b}{d} = k$$

where $k \in (0,1]$. Thus, with $a = kc$ and $b = kd$,

$$(1-k^2)^2 \geq 0$$
$$c^2(1-k^2)d^2(1-k^2) \geq 0$$
$$(c^2-a^2)(d^2-b^2) \geq 0$$
$$a^2b^2 + c^2d^2 \geq a^2d^2 + b^2c^2$$
$$a^2b^2 + c^2d^2 + (a^2c^2 + b^2d^2) \geq a^2d^2 + b^2c^2 + (a^2c^2 + b^2d^2)$$
$$(a^2+d^2)(b^2+c^2) \geq (a^2+b^2)(c^2+d^2)$$
$$AD^2 \cdot BC^2 \geq AB^2 \cdot CD^2$$
$$AD \cdot BC \geq AB \cdot CD$$

(b) From part (a) we have

$$AD \cdot BC \geq AB \cdot CD$$
$$2AD \cdot BC \geq 2AB \cdot CD$$
$$(a^2+b^2+c^2+d^2) + 2AD \cdot BC \geq 2AB \cdot CD + (a^2+b^2+c^2+d^2)$$
$$(a^2+d^2) + 2AD \cdot BC + (b^2+c^2) \geq (a^2+b^2) + 2AB \cdot CD + (c^2+d^2)$$
$$AD^2 + 2AD \cdot BC + BC^2 \geq AB^2 + 2AB \cdot CD + CD^2$$
$$(AD+BC)^2 \geq (AB+CD)^2$$
$$AD + BC \geq AB + CD$$

□

Let a, b, c, d, r be natural numbers and we have $ab = cd$,$T = a^r + b^r + c^r + d^r$ then prove that $2^T - 1$ is composite number.

Solution

we proceed as follows.

$a = \frac{cd}{b}$. So, $c = hm, d = kn, b = hk$.

$T = (mn)^r + (hk)^r + (hm)^r + (kn)^r \Rightarrow T = (m^r + k^r)(n^r + h^r)$

Note that both the factors on the RHS above are greater than 2.

Now, we know that $(a^q - 1) = (a-1)(a^{q-1} + a^{q-2} + ... + 1)$ for any two naturals a and q.

Just put $a = 2^{m^r + k^r}$ and $q = n^r + h^r$ and the result follows.

□ Prove: $\frac{1989}{2} - \frac{1988}{3} + \frac{1987}{4} - ... + \frac{1}{1990} = \frac{1}{996} + \frac{3}{997} + \frac{5}{998} + ... + \frac{1989}{1990}$

Solution

$h_n = \sum_{i=1}^{n} \frac{1}{i}$ $\frac{1}{2}h_n = \sum_{i=1}^{n} \frac{1}{2i}$ $h_{2n} - \frac{1}{2}h_n = \sum_{i=1}^{n} \frac{1}{2i-1}$ $h_{2n} - h_n = -\sum_{i=1}^{2n} \frac{(-1)^n}{i}$

$\sum_{i=2}^{2n} (-1)^i \frac{2n-i+1}{i} = -1 + \sum_{i=2}^{2n} (-1)^i \frac{2n+1}{i} = -1 + (2n+1)(h_n - h_{2n} + 1)$

$\sum_{i=1}^{n} \frac{2i-1}{n+i} = \sum_{i=1}^{n} 2 - \frac{2n+1}{n+i} = 2n - (2n+1)(h_{2n} - h_n)$

□ Solve the equation $\sin x \cos y + \sin y \cos z + \sin z \cos x = \frac{3}{2}$.

Solution

Using AM-GM $LHS \geq 3.\sqrt[3]{\frac{sin(2x)sin(2y)sin(2z)}{8}} \geq \frac{3}{2}$, equality holds only if the sines of double argument are all 1 which means all x,y,z being of the form $\frac{(2k+1)\pi}{4}$. However, for this estimate I need all the terms on LHS to be positive. If one is negative (wlog the first one), we,ll have $sinz + cosz \geq 3/2$, contradiction.

□ Let d be a divisor of n. Let $d(n)$ be the number of divisors of n. Prove that if n is not a product of two primes then, for every d, the number of divisors that are not relatively prime to d is at least

$\frac{d(n)}{2}$.

Solution

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$ where the p_i are prime. It follows that $d(n) = (e_1+1)(e_2+1)\cdots(e_n+1)$. Also let $d = p_1^{f_1} p_2^{f_2} \cdots p_n^{f_n}$.

The statement which we want to prove is equivalent to proving that the number of divisors relatively prime to d is at most $\frac{d(n)}{2}$.

A divisor of n can be relatively prime to d only if at least one of the f_i are 0. So let $f_{k_1}, f_{k_2}, \dots f_{k_j}$ all be 0. (Note that not all of the f_i can be zero since taht would make $d = 1$). Then the number of divisors of n which are relatively prime to d is

$$\prod_{i=1}^{j} (f_{k_i} + 1).$$

However, the total number of divisors of n can be written as $d(n) = \prod_{i=1}^{n} (f_{k_i} + 1)$. We then have

$$\prod_{i=1}^{j} (f_{k_i} + 1) \cdot \prod_{i=j}^{n} (f_{k_i} + 1) = d(n) \Rightarrow \prod_{i=1}^{j} (f_{k_i} + 1) = \frac{d(n)}{\prod_{i=j}^{n} (f_{k_i} + 1)}.$$

Since $f_i \geq 1$ we arrive at our desired result:

$$\prod_{i=1}^{j} (f_{k_i} + 1) \leq \frac{d(n)}{2}.$$

□ Show that every positive integer can be written as a sum of distinct Fibonacci numbers.

Solution

Consider a number n. Clearly, for some F_i, where F_n is the nth Fibonacci number, we will have $F_i \leq n < F_{i+1}$. Now we want that $n - F_i$ is a Fibonacci number. Consider $F_{i+1} - F_i = F_{i-1}$ This number can be written as $F_{i-1} = F_{i-2} + F_{i-3}$. If one beween F_{i-2} and F_{i-3} is equal to $n - F_i$, we have done. If it isn't, we have 2 possible cases:$F_i < n < F_i + F_{i-2}$ or $F_i + F_{i-2} < n < F_{n+1}$ If n is in the first case, consider that $F_{i-2} = F_{i-3} + F_{i-4}$, if n is in the second, consider that $F_{i-3} = F_{i-4} + F_{i-5}$ and repeat the reasoning. Being our succession of F_n monotonic strict decreasing, it will have a lower bound in a finite number of passages. Therefore n can be written as a finite sum of distinct F_n.

□ Let $a, b \in \mathbb{R}^*$. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x - \frac{b}{a}\right) + 2x \leq \frac{a}{b} \cdot x^2 + \frac{2b}{a} \leq f\left(x + \frac{b}{a}\right) - 2x, \forall x \in \mathbb{R}.$$

Solution

Replacing x by $x + \frac{b}{a}$ in the first inequality yields

$f(x) + 2(x + \frac{b}{a}) \leq \frac{a}{b}(x + \frac{b}{a})^2 + \frac{2b}{a}$

$\Rightarrow f(x) \leq \frac{a}{b}x^2 + \frac{b}{a}$ ... (I)

Again, replacing x by $x - \frac{b}{a}$ in the second inequality yields $\frac{a}{b}(x - \frac{b}{a})^2 + \frac{2b}{a} \leq f(x) - 2(x - \frac{b}{a})$

$\Rightarrow \frac{a}{b}x^2 + \frac{b}{a} \leq f(x)$ ... (II)

Combining (I) and (II) gives $f(x) = \frac{a}{b}x^2 + \frac{b}{a}, \forall x \in \mathbb{R}$ – Let A, B, C, D be four points, not all in the same plane. Let H_A, H_B be the orthocenters of BDC and ACD, respectively. Prove that A, B, H_A, H_B are in the same plane if and only if they are concyclic. – Let p be a prime number such that $2p - 1$ is also prime. Find all pairs of natural numbers (x, y) such that

$$(xy - p)^2 = x^2 + y^2.$$

– Let ABC be a equilateral triangle. On the perpendiculars in A, C to the plane (ABC), we consider the points M, N (on the same side of (ABC)), such that $AM = AB = a$ and $MN = BN$.

(a) Find the distance from A to the plane (MNB);

(b) Determine $\sin\angle(MN, BC)$. – $\frac{a^2+b^2+c^2+1}{abc} = k \in \mathbb{Z}$

if $a, b, c > 0 \in \mathbb{Z}$ find all values of k.

□ số học

□ số học

□ $ABCD$ is a quadrilateral with $AD = BC$. If $\angle ADC$ is greater than $\angle BCD$, prove that $AC > BD$.

Solution

$AC^2 = AD^2 + CD^2 - 2AD \cdot CD \cos\angle ADC$

$BD^2 = BC^2 + CD^2 - 2AD \cdot BC \cos\angle BCD$

Since $f(x) = \cos x$ is monotonically decreasing in $[0, \pi]$, so $AC > BD$.

□ In the plane Oxy, two points $A \in xOx'$ and $b \in yOy'$ move and always satisfy the condition $AB = 3$. Given $J \in AB$ with $\vec{AJ} = 2\vec{JB}$. Find locus of point J.

Solution

Let's consider this locus only for the upperhalf plane.Then if $A = (t, 0)$, we have

$$B = (0, \sqrt{9 - t^2}).$$

It means that

$$\vec{OA} = [t, 0]$$

and

$$\vec{OB} = \left[0, \sqrt{9 - t^2}\right],$$

so

$$\vec{OJ} = \frac{1}{3}\vec{OA} + \frac{2}{3}\vec{OB} = \left[\frac{1}{3}t, \frac{2}{3}\sqrt{9 - t^2}\right].$$

Substituting $x = \frac{1}{3}t$ we get

$$\vec{OJ} = \left[x, \frac{2}{3}\sqrt{9 - 9x^2}\right] \iff J = \left(x, \frac{2}{3}\sqrt{9 - 9x^2}\right).$$

It means that this locus in the upperhalf plane is graph of function

$$f(x) = 2\sqrt{1 - x^2}$$

(for lowerhalf it's symmetric), so in fact this is an ellipse.

□ Let $n \geq 3$ be a positive integer. Prove that the sum of the cubes of all natural numbers, coprime and less than n, is divisible by n.

Solution

We wish to show that the sum of the cubes of the members of U_n is zero.

This is rather simple. The additive inverse of the cube of a member of U_n is the cube of its additive inverse (edit: in other words, $(a+b)|(a^3+b^3)$), so we add each number to its additive inverse (invariant under cubing) for a sum of zero.

For $n \geq 3$ we must have $2|\varphi(n)$ and so such a pairing always exists. (Or to be more explicit: For $n \geq 3$ we have $\frac{n}{2} \notin U_n$, so every member of U_n has an additive inverse different from itself.)

□ hình học

$\square$ Let $x, y, z > 0$ such that $xyz = 1$. Prove that:

$$x + y^2 + z^3 > 2.5.$$

Solution

$x + y^2 + z^3 = \frac{1}{6}x + \frac{1}{6}x + \frac{1}{6}x + \frac{1}{6}\ x + \frac{1}{6}x + \frac{1}{6}x + \frac{1}{3}y^2 + \frac{1}{3}y^2 + \frac{1}{3}y^2 + \frac{1}{2}z^3 + \frac{1}{2}z^3 \geq 11\sqrt[11]{\frac{x^6y^6z^6}{6^6 \cdot 3^3 \cdot 2^2}} > 2.5$

$\square$

show that the system

$xe^{x^2} + ye^{y^2} = 3$

$x^2 + y^2 = 1$

hasn't solution

Solution

We will prove that if $x^2 + y^2 = 1$, then $xe^{x^2} + ye^{y^2} < 3$. When $x < 0$ we have $xe^{x^2} + ye^{y^2} < (-x)e^{(-x)^2} + ye^{y^2} <$ (analogous for y), so we can assume that $x, y \geq 0$. Applying Cauchy-Schwarz we get $xe^{x^2} + ye^{y^2} \leq \sqrt{x^2 + y^2}\sqrt{e^{2x^2} + e^{2y^2}} = \sqrt{e^{2x^2} + e^{2y^2}}$, so we have to prove that $e^{2x^2} + e^{2y^2} < 9$. Let $a = x^2$. Then $0 \leq a \leq 1$ and we have to prove that $e^{2a} + e^{2(1-a)} < 9$. Let $t = e^{2a}$, $1 \leq t \leq e^2$. Then the inequality simplifies to $t + \frac{e^2}{t} < 9 \iff t^2 - 9t + e^2 < 0$. By the quadratic formula it is equivalent to $t \in \left(\frac{9-\sqrt{81-4e^2}}{2}, \frac{9+\sqrt{81-4e^2}}{2}\right)$. We have to prove that the interval $< 1, e^2 >$ enclosed in the interval $\left(\frac{9-\sqrt{81-4e^2}}{2}, \frac{9+\sqrt{81-4e^2}}{2}\right)$. It is equivalent to two inequalities: 1. $\frac{9-\sqrt{81-4e^2}}{2} < 1$ 2. $e^2 < \frac{9+\sqrt{81-4e^2}}{2}$. Proof 1. The inequality is equivalent to $\sqrt{81 - 4e^2} > 7 \iff e^2 < 8$ which is obvious. Proof 2. It is equivalent to $\sqrt{81 - 4e^2} > 2e^2 - 9$. Both sides are positive, so squaring we get $81 - e^2 > 4e^4 - 36e^2 + 81 \iff 4e^4 - 35e^2 < 0 \iff 4e^2 - 35 < 0 \iff e^2 < \frac{35}{4}$ which is obvious because $e^2 < 8$ and $8 < \frac{35}{4}$. The proof of $xe^{x^2} + ye^{y^2} < 3$ is ended. Of course $xe^{x^2} + ye^{y^2} < 3$ implies $xe^{x^2} + ye^{y^2} \neq 3$ so we are done.

$\square$ Show that the product of k consecutive positive integers can't be the kth power of an integer

Solution

It's clear that: $n(n + 1)...(n + k - 1) > n^k$ and $n(n + 1)...(n + k - 1) < (n + k)^k$ So we must have: $n(n + 1)...(n + k - 1) = (n + r)^k$ where $r \in 1, 2, ..., k - 1$ But then: $\frac{(n+r)^k}{n+r-1} = \frac{n(n+1)...(n+k-1)}{n+r-1}$ is an integer. It's an obvious contradiction because $(n + r, n + r - 1) = 1$.

$\square$ Solve the following trigonometric equation:

$cos 12x = 5 sen 3x + 9(tan x)^2 + (cot x)^2$

How many solutions does it have in $[0; 2\pi]$

Solution

We first rewrite our equation: $\cos 12x - 5\sin 3x = 9\tan^2 x + \cot^2 x$.

If $\tan x = 0$ then $\cot x$ is undefined, and if $\cot x = 0$ then $\tan x$ is undefined. So, we assume $\tan x \neq 0$ and $\cot x \neq 0$. Applying $AM - GM$ to the RHS, we get

$\frac{9\tan^2 x + \cot^2 x}{2} \geq \sqrt{9\tan^2 x \cdot \cot^2 x} \Rightarrow RHS \geq 6$, with equality occurring iff $9\tan^2 x = \cot^2 x$.

Now, note that $|LHS| \leq 6$. So, the only solution possible is when the following is satisfied: $LHS = RHS = 6$. So, $\cos 12x - 5\sin 3x = 9\tan^2 x + \cot^2 x = 6$, in which case we must have $9\tan^2 x = \cot^2 x$.

Solving $9\tan^2 x = \cot^2 x$, we obtain $\tan x = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$. This yields the following possible values for x: $\frac{\pi}{6}, \pi + \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$.

Out of the possible values for x above, only $x = \pi + \frac{\pi}{6}$ and $x = 2\pi - \frac{\pi}{6}$ satisfy the equation $\cos 12x - 5\sin 3x = 6$.

Answer: $\pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$.

□ **Bertrand's Theorem** states that for every $x > 1$, there exists a prime number between x and $2x$. Use this fact to show that every positive integer can be written as the sum of distinct primes. (For this result, assume that one is a prime.)

Solution

Let $x = x_0$. If it is prime, we are done.

Suppose it is odd. Then $r_0 = \lfloor \frac{x_0}{2} \rfloor + 1$ has the property $2r_0 - x_0 = 1$, so there exists a prime $r_0 < p_0 < 2r_0$ such that $2p_0 > x_0$. Let $x_1 = x_0 - p_0$, which is less than half of x_0, and repeat the algorithm.

Suppose it is even. Then there exists a prime $\frac{x_0}{2} < p_0 < x_0$, and then let $x_1 = x_0 - p_0$ as before. Repeat the algorithm.

In each case we have $2x_{k+1} < x_k$, so every new prime p_k generated is guaranteed to be distinct. The algorithm is guaranteed to terminate because each successive term is at most half the previous.

Once it terminates, $\sum p_k = x_0$. QED.

Another way The inductive hypothesis holds true for 1. Suppose it holds true for $1, 2, 3, 4, ...k$.

Then there exists a prime p such that $\lfloor \frac{k+1}{2} + 1 \rfloor < p < 2\lfloor \frac{k+1}{2} \rfloor + 2$, and since the inductive hypothesis is true for $(k+1) - p < p$, which cannot have p in its unique prime representation, it holds true for $k+1$. Hence our inductive hypothesis is true. QED.

□ Prove that: $\sin\theta + \sin(\theta+\alpha) + \sin(\theta+2\alpha) + ... + \sin(\theta+n\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\sin(\theta+\frac{n\alpha}{2})}{\sin\frac{\alpha}{2}}$

Solution

Below we use the identity $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin\theta + \sin(\theta+\alpha) + \sin(\theta+2\alpha) + ... + \sin(\theta+n\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\sin(\theta+\frac{n\alpha}{2})}{\sin\frac{\alpha}{2}}$

$\Leftrightarrow 2\sin\frac{\alpha}{2}[\sin\theta+\sin(\theta+\alpha)+\sin(\theta+2\alpha)+.......+\sin(\theta+n\alpha)] = 2\sin\frac{(n+1)\alpha}{2}\sin(\theta+\frac{n\alpha}{2} \Leftrightarrow [\cos(\theta-\frac{\alpha}{2}) - \cos(\theta+\frac{\alpha}{2})] + [\cos(\theta+\frac{\alpha}{2}) - \cos(\theta+\frac{3\alpha}{2})] + [\cos(\theta+\frac{3\alpha}{2}) - \cos(\theta+\frac{5\alpha}{2})] + ... + [\cos(\theta+(n-\frac{1}{2})\alpha) - \cos(\theta+(n+\frac{1}{2})\alpha)] = \cos(\theta-\frac{\alpha}{2}) - \cos(\theta+(n+\frac{1}{2})\alpha) \Leftrightarrow 0 = 0$

Note that if $\alpha = 0$, the identity is still true in the limit sense.

$\lim_{\alpha\to 0} LHS = (n+1)\sin\theta$

$\lim_{\alpha\to 0} RHS = (n+1)\sin\theta$

□ Let $P(x)$ be any polynomial with integer coefficients such that $P(21) = 17$, $P(32) = -247$, $P(37) = 33$. Prove that if $P(N) = N + 51$, for some integer N, then $N = 26$.

Solution

since $P(x)$ has integer coefficients, we know that $(a-b)|(P(a)-P(b))$ So, $21 - N| - 34 - N \to -\frac{55}{21-N} + 1 = k$, with k integer

Using those conditions we get

$21 - N|55$ $32 - N|330$ $37 - N|55$

So just try possible values of N – Let $0 \le a, b, c, d \le \pi$ such that $2\cos a + 6\cos b + 7\cos c + 9\cos d = 0$ and $2\sin a - 6\sin b + 7\sin c - 9\sin d = 0$. Prove that $3\cos(a+d) = 7\cos(b+c)$. —- coinsider the following system

$ax + by = e$ $cx + dy = f$

where $a, b, c, d, e, f \in \mathbb{Z}$. Suppose to choose a, b, c, d among all relative number whose absolute value is $\le n$, with $n \in \mathbb{N}$. Call p the probability that the system has exactly one solution (not necessary integer). Prove that $1 - \frac{1}{2n} \le p \le 1 - \frac{1}{3n^2}$

□ Let $T = 9^k$: k is an integer, $0 \le k \le 4000$. Given that 9^{4000} has 3817 digits and that its first(leftmost) digit is 9, how many elements of T have 9 as their leftmost digit?

Solution

Note that if 9^{n+1} begins with 9, necessarily 9^n begins with 1, and these 2 numbers must have the same length: it is impossible otherwise to have 2 successive powers of 9 the same length. Thus if 9^n has m digits and doesn't start with 1, there must have been $n-m+1$ pairs of successive powers with the same length. Thus, for up to 9^{4000}, there are 4000-3817+1=184 powers of 9 less with 9 as the first digit. :) –

– Let x and y be positive integers with x $<$ y. Find all possible integer values of $P = x^3 - y/1 + xy$ — For natural numbers n, sequence $\{a_n\}$ is defined recursively as follows:

$$a_1 + 2a_2 + 3a_3 + ... + na_n = a_{n+1}$$

$(a_1 = 1)$

For natural numbers n, sequence $\{b_n\}$ is defined recursively as follows:

$$b_1 + \frac{b_2}{2} + \frac{b_3}{3} + ... + \frac{b_n}{n} = b_{n+1}$$

$(b_1 = 1)$

Express $\sum_{k=1}^{n} a_k b_k$ in terms of n. Let x and y be positive integers with x $<$ y. Find all possible integer values of $P = x^3 - y/1 + xy$ ——— For natural numbers n, sequence $\{a_n\}$ is defined recursively as follows:

$$a_1 + 2a_2 + 3a_3 + ... + na_n = a_{n+1}$$

$(a_1 = 1)$

For natural numbers n, sequence $\{b_n\}$ is defined recursively as follows:

$$b_1 + \frac{b_2}{2} + \frac{b_3}{3} + ... + \frac{b_n}{n} = b_{n+1}$$

$(b_1 = 1)$

Express $\sum_{k=1}^{n} a_k b_k$ in terms of n.

☐ số học

☐ số học

☐ đại số

☐ số học

☐ số học

☐ số

☐ số

☐ giới hạn

☐

☐

☐ An ordinary deck of 52 cards with 4 aces is shuffled, and then the cards are drawn one by one until the first ace appears. On the average, how many cards are drawn?

Solution

Consider the probability that $n+1$ cards are select until an ace appears. The first n cards must be non-ace cards and the last must be an ace. The probablity of selecting n non-ace cards is $\frac{48}{52}\frac{47}{51}\frac{46}{50}\cdots\frac{48-n+1}{52-n+1} = \frac{48!(52-n)!}{(48-n)!52!}$ The probgability that an ace is drawn after that is $\frac{4}{52-n}$. Multiplying $n+1$ (the number of cards drawn) by the two probabilities yields the expression which is summed from $n=0$ to $n=48$.

The sum can actually be simplified to: $\sum_{n=0}^{48}\left((n+1)\frac{(51-n)(50-n)(49-n)}{1624350}\right)$

$\square$ Find an integer x such that $\left(1+\frac{1}{x}\right)^{x+1} = \left(1+\frac{1}{2003}\right)^{2003}$.

Solution

if x is positive, there are no solutions, as $x = 2003$ is too big, and $x = 2002$ is too small. x cannot be 0, so it must be negative. Ignoring $x = -1$, we get $\left(\frac{x+1}{x}\right)^{x+1} = \left(\frac{2004}{2003}\right)^{2003}$ let $k = |x|$ $\left(\frac{k}{k-1}\right)^{k-1} = \left(\frac{2004}{2003}\right)^{2003}$ which leads to $k = 2004$, so $x = -2004$

$\square$ find the nth term for the sequence 1, 2, 10, 67, 467, 3268, 22876

Solution

Can you say it more strictly? Every sequnce $\{a_n\}$ satisfying $a_1 = 1, a_2 = 2, a_3 = 10, a_4 = 67, a_5 = 467, a_6 = 3268, a_7 = 22876$ is an answer for your question, for example $\{a_n\}$ defined as $a_1 = 1, a_2 = 2, a_3 = 10, a_4 = 67, a_5 = 467, a_6 = 3268, a_7 = 22876$ and $a_n = \frac{\pi^n}{e}$ for $n \geq 8$. I hope that you can write version of your problem which doesn't allow sequence defined above.

$\square$ Let $a, b \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$, $a < b$, a does not divide b. Solve the equation

$$a\lfloor x\rfloor - b\left(x - \lfloor x\rfloor\right) = 0.$$

Solution

Since $a\lfloor x\rfloor \in \mathbb{Z}$, $b\{x\} \in \mathbb{Z}$, so $x = \lfloor x\rfloor + \frac{y}{b}$, $0 \leq y < b$, $y \in \mathbb{Z}$

So $a\lfloor x\rfloor = y$, so $a|y$, so the solutions are $x = t + \frac{at}{b}$ or $\boxed{x = \dfrac{(a+b)t}{b}}$ where $t \in \mathbb{Z}$ and $0 \leq t \leq \left\lfloor \frac{b}{a}\right\rfloor$ (a does not divide b)

$\square$ Show that

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$$

Solution

In $\triangle ABC$, let $m\angle A = \frac{\pi}{7}$ and let $m\angle B = m\angle C = \frac{3\pi}{7}$. Let $BC = x$. Choose D on $\overline{AC}$ such that $BD = x$, and E on $\overline{AB}$ such that $DE = x$. After some angle chasing, see that $\triangle AED$ is isosceles, with $AE = DE (= x)$. Note $AB = 2x\cos\frac{2\pi}{7} + x$ and $AC = 2x(\cos\frac{\pi}{7} + \cos\frac{3\pi}{7})$. Since $AB = AC$, equate the two, divide by $2x$, and rearrange to get the desired result.

$\square$ Let a, b, c be distinct reals. Prove that the following cannot occur.
$(a-b)^{\frac{1}{3}} + (b-c)^{\frac{1}{3}} + (c-a)^{\frac{1}{3}} = 0$

Solution

We prove the result by contradiction. Let $x^3 = a-b, y^3 = b-c$, and $z^3 = c-a$. (This is possible since the cube of a real can be positive, negative or zero.) Now, if $(a-b)^{\frac{1}{3}} + (b-c)^{\frac{1}{3}} + (c-a)^{\frac{1}{3}} = 0$ is true, we have $x+y+z = 0 \Rightarrow x^3+y^3+z^3 = 3xyz$ (This is a fairly elementary result, I believe.) $\Rightarrow 3xyz = 0 \Rightarrow (a-b)^{\frac{1}{3}}(b-c)^{\frac{1}{3}}(c-a)^{\frac{1}{3}} = 0 \Rightarrow (a-b)(b-c)(c-a) = 0 \Rightarrow$ At least two of the numbers a, b, and c are equal, which leads to a contradiction since a, b, and c are distinct reals. And, we are done.

$\square$ Proove that for any positive integer, the sum of the reciprocals of all of the integer's factors is equal to:
$\frac{\text{the sum of all of the factors}}{\text{the integer}}$

Solution

Let N be the number with factors $d_1, d_2, \cdots, d_{k-1}, d_k$. Note that $d_i \times d_{k-i+1} = N$. The sum of the reciprocals of the factors is $\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{k-1}} + \frac{1}{d_k}$, and we let our common denominator be N. So multiply $\frac{1}{d_i} \times \frac{d_{k-i+1}}{d_{k-i+1}} = \frac{d_{k-i+1}}{N}$,and we obtain $\frac{d_k+d_{k-1}+\cdots+d_2+d_1}{N}$, as desired.

☐ Show that every power of $\sqrt{2}-1$ can be written in the form $\sqrt{k+1}-\sqrt{k}$.

Solution

Let s_n be the nth power of $\sqrt{2}-1$. We will proceed by induction. For the base case, $\sqrt{2}-1=\sqrt{2}-\sqrt{1}$ so that works. Suppose

$s_n = a+b\sqrt{2} = \sqrt{k+1}-\sqrt{k}$.

Then $|a^2-2b^2|=1$. So

$s_{n+1} = (\sqrt{2}-1)(a+b\sqrt{2}) = (2b-a)+\sqrt{2}(a-b)$.

But since $|(2b-a)^2-2(a-b)^2| = |2b^2-a^2| = |a^2-2b^2| = 1$, we know

$s_{n+1} = (2b-a)+\sqrt{2}(a-b)$ can be written as $\sqrt{k+1}-\sqrt{k}$ for some k as well, completing the induction.

EDIT: Boo, someone beat me to it again.

☐ Solve for $x \geq 0$: $x = \frac{1}{x-1}+\frac{1}{x-2}+...+1 = \sum_{k=1}^{x-1}(\frac{1}{x-k})$

Solution

note $x \in \mathbb{Z}^+$, and $x > 1$ for the summation to be possible

this is taking a partial sum of the harmonic series up to $x-1$, that sum being equal to x

let $S_n = \sum_1^n \frac{1}{i}$ (this is your summation, just a bit clearer)

we want $S_{n-1} = n$

it is pretty easy to show that $S_n - 1 < \log_2 n$, so we want n such that

$n-1 < \log_2(n-1)$ $2^n < 2(n-1)$

which is true for no positive integers > 1...(easily shown by induction)

thus there are no solutions

☐ Prove that among any 39 consecutive natural numbers it's always possible to find one whose sum of digits is divisible by 11.

Solution

We will proceed by contradiction. Assume there exists a set of 39 natural numbers such that none of the 39 numbers have a sum of digits divisible by 11. Let these numbers be $a_1, a_2, a_3 \cdots a_{39}$.

If the last digit of a_n is 0, and $n \leq 30$, then the sum of the digits of a_n must be equivalent to 1 (mod 11). Otherwise, the sum of the digits of one of the next nine numbers would be divisible by 11. There are exactly three numbers with ones digit 0 among $a_1, a_2, a_3 \cdots a_{30}$, and these three numbers are consecutive multiples of 10. Let these numbers be $10n, 10(n+1), 10(n+2)$. The sum of digits of these three numbers is the same as the sum of digits of $n, n+1, n+2$. Therefore, $n, n+1, n+2$ each have a sum of digits equivalent to $1 \pmod{11}$.

For either $\{n, n+1\}$ or $\{n+1, n+2\}$, the only digit that differs between the two numbers is the ones digit. Therefore, it is impossible for the sum of digits of each of the three numbers to be equivalent to $1 \pmod{11}$. This is a contradiction, and our proof is complete. – Solve the inequation: $2x^2-3x\lfloor x-1\rfloor+\lfloor x-1\rfloor^2 \leq 0$ —– Find all polynomials f satisfying $f(x^2)+f(x)f(x+1)=0$.

☐ In 1593, the Belgian mathematician Adriaan van Roomen proposed the following problem:

Find the positive roots of the equation $x^{45}-45x^{43}+945x^{41}-12300x^{39}+111150x^{37}-740459x^{35}+3746565x^{33}-14945040x^{31}+469557800x^{29}-117679100x^{27}+236030652x^{25}-378658800x^{23}+483841800x^{21}-488494125x^{19}+384942375x^{17}-232676280x^{15}+105306075x^{13}-34512074x^{11}+7811375x^{9}-1138500x^{7}+95634x^{5}-3795x^{3}+45x = \sqrt{\frac{7}{4}-\sqrt{\frac{5}{16}}-\sqrt{\frac{15}{8}-\sqrt{\frac{45}{64}}}}$.

The French mathematician Viète was able to solve the equation. By hand. In just a few minutes, too, supposedly (Anecdote! One of the Bernoullis claimed to have summed the first 1000 10th powers in half of 15 minutes. My analysis professor did it in just over 8, but he was explaining it to us as he went).

Anyway, anyone here want to give it a try? Or is their 16th century intellect beyond us?

Solution

RHS is of course $2\sin 12^o$ Left side is what you get if you expand $2\sin(45y)$ in terms of $\sin y$ and you put $2\sin y = x$

so you "easily" :) get all the roots (For example one such root is $sin(12/45)^o$ and rest you can get by adding $k \cdot 8^o$ $(k = 0, 1, 2 \cdots$ etc.(remember 360/45=8) .

(To be honest, RHS was not that difficult to guess for any one who worked in the old days as us as I tell my kids .. in those days we have to do all calculations by hand, have to remember times tables up to 100, know log tables by heart and walk 10 miles uphill both ways in 40^oC (It looks even more terrible in Fahrenheit :) $= 104^oF$) heat and 5 feet of snow ..:))

(and of course only thing to keep in mind for LHS was to do all middle steps of calculations on slate so not to waste too much paper) :)

(Actually if you know $sin3x = 3sinx - 4sin^3x$ and $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x$ all you have to know and you apply first formula twice and second once.)

□ We consider the number A= 111...111222...222...999...999 − 123456789, where the number of digits of $1, 2, 9$ are equal whith 2003.Prove that 2003 divides A.

Solution

Let $A = a_1 + a_2 + ... + a_9$ where

$a_1 =$ 111....000... − 100000000, $a_2 =$ 000...222...000... − 020000000,

etc. (That is, we divide out A into its digit components.) Now, there are 8×2003 zeroes in the large part of a_1, 7×2003 in a_2, etc. We can write 111... (with 2003 ones) as $\frac{10^{2003}-1}{9}$. Then

$a_k = k\frac{10^{2003}-1}{9} \times 10^{(9-k)2003} - k \times 10^{9-k}$

Now, by Fermat's Little Theorem, $10^{2002} \equiv 1 \bmod 2003$. We therefore write

$a_k \equiv k \times 10^{9-k} - k \times 10^{9-k} \equiv 0 \bmod 2003$,

Thus completing the proof. QED.

□ Let $P(x)$ be a polynomial of degree n, so that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, ..., n$. Find $P(n+1)$.

Solution

Define the polynomial $Q(x)$ by

$$Q(x) := (x+1)P(x) - x,$$

so that $\deg(Q(x)) = n+1$. Furthermore, $Q(x)$ has roots at $x = 0, 1, \ldots, n$. Clearly, $Q(x)$ cannot have anymore roots, so then

$$Q(x) = C \cdot (x-0)(x-1)\cdots(x-n),$$

for some constant C. Consider $Q(-1)$. By definition,

$$Q(-1) = C \cdot (-1-0)(-1-1)\cdots(-1-n),$$

but at the same time,

$$Q(-1) := (-1+1)P(-1) - (-1) = 1,$$

so

$$C = \frac{(-1)^{n+1}}{(n+1)!}.$$

Therefore,

$$\frac{(-1)^{n+1}}{(n+1)!} \cdot (x-0)(x-1)\cdots(x-n) = (x+1)P(x) - x.$$

Then plugging in $n+1$ for x yields

$$\frac{(-1)^{n+1}}{(n+1)!}\cdot(n+1-0)(n+1-1)\cdots(n+1-n)=((n+1)+1)P(n+1)-(n+1),$$

$$(-1)^{n+1}=(n+2)P(n+1)-(n+1).$$

If n is even, then

$$-1=(n+2)P(n+1)-(n+1),$$
$$P(n+1)=\frac{n}{n+2}.$$

If n is odd, then

$$1=(n+2)P(n+1)-(n+1),$$
$$P(n+1)=\frac{n+2}{n+2}=1.$$

Hence,

$$P(n+1)=\begin{cases}1, & 2\not|n\\ \frac{n}{n+2}, & 2|n\end{cases}$$

□ Prove that there are no positive integers x and y such that x^2+y+2 and y^2+4x are perfect squares

Solution

The next perfect square after y^2 is $(y+1)^2=y^2+2y+1$.

We are given that x and y are positive integers, so $y+2>0$. Assuming (for the sake of reaching a contradiction) that x^2+y+2 was a perfect square, $y+2\geq 2x+1$.

Similarly the next perfect square after x^2 is $(x+1)^2=x^2+2x+1$. Thus, (as $4x>0$) again assuming that y^2+4x is a perfect square $4x\geq 2y+1$.

We have inequations $y+2\geq 2x+1$ and $4x\geq 2y+1$. Manipulating, $2y+2\geq 4x\geq 2y+1$. $4x$ is obviously even, so $4x=2y+2$. But then $y^2+4x=y^2+2y+2=(y+1)^2+1$ is not a perfect square for integer y. Contradiction. Initial assumption that both x^2+y+2 and y^2+4x could be perfect squares for positive integer x and y is false.

□ The sequence $a_1,a_2,...$ of natural numbers satisfies $\gcd(a_i,a_j)=\gcd(i,j)$ for all i not equal to j. Prove that $a_i=i$ for all i

Solution

Put $n=\prod_{i=1}^{k}p_i^{e_i}$, the prime factorization of n.

For any i, set $r=p_i^{e_i}$. Then $r=(r,n)=(a_r,a_n)$ implying a_n has a factor r. Thus, a_n is a multiple of n. So there exists a sequence $(b_1,b_2,\dots)$ with $a_n=nb_n$ for all natural n.

Now $k=(k,kb_k)=(a_k,a_{kb_k})=(kb_k,kb_kb_{kb_k})=kb_k$ implying $b_k=1$ for all k. The result follows.

□

□ Find all functions $f:R\to R$ where $f(x+y)=f(x).f(y).f(xy)$ for all real x, y

Solution

If there is any x such that $f(x)=0$, then $f(x+y)=0$ for all y so $f\equiv 0$. Assume f has no zeros. Then setting $x=y=0$ we have $f(0)=f(0)^3$ so $f(0)=\pm 1$. Note that if f is a solution, so is $-f$, so assume $f(0)=1$. Then setting $y=-x$ we have (1) $1=f(x)f(-x)f(-x^2)$ and setting $y=x$ gives us (2) $f(2x)=f(x)^2f(x^2)$ Setting $y=-2x$ gives $f(-x)=f(x)f(-2x)f(-2x^2)=f(x)f(-2x^2)\cdot(f(-x)^2f(x^2))$ (by (2)) and so $1=f(x)f(-2x^2)f(-x)f(x^2)$. Then by (1), $f(x)f(-x)f(-x^2)=f(x)f(-2x^2)f(-x)f(x^2)$ so $f(-x^2)=f(-2x^2)f(x^2)$ or, for positive t, $f(-t)=f(t)f(-2t)$. Then with (2) this gives us $f(-t)=f(t)f(-t)^2f(t^2)$ or $1=f(t)f(-t)f(t^2)$ for any positive t. Thus by (1), for any positive x we have $f(x)f(-x)f(-x^2)=f(x)f(-x)f(x^2)$ so $f(-x^2)=f(x^2)$ so in general $f(x)=f(-x)$.

But then $f(x+y)=f(x)f(y)f(xy)=f(-x)f(y)f(-xy)=f(-x+y)$, and since x,y are arbitary we have $f\equiv c$. Since $f(0)=1$, this gives us the solution $f\equiv 1$, and we also have the negative of this, $f\equiv -1$.

☐ Let $\{3,4,12\}$ be a set. In each step of conversion, you may choose two numbers $a,b\in\{3,4,12\}$ and convert them into $0.6a-0.8b$ and $0.8a+0.6b$. Is it possible to acquire the set $\{4,6,12\}$ through a finite number of conversions? Is it possible to reach $\{x,y,z\}$ such that $|x-4|,|y-6|,|z-12|\in\left[0,\frac{1}{\sqrt{3}}\right]$?

Solution

It is easy to see $a^2+b^2+c^2$ is an invariant. $\{4,6,12\}$ has a different value of the invariant, hence it is not reachable.

The actual value of the invariant with our given problem condition is 13^2. Because $\{4,6,12\}$ has too large a value, we wish to determine whether $\{4-k,6-k,12-k\}$ satisfies the condition where $k\in\left[0,\frac{1}{\sqrt{3}}\right]$.

$(4-k)^2+(6-k)^2+(12-k)^2=14^2-44k+3k^2$

Clearly a decreasing function in k for the relevant domain. For maximal k, it gives the value $14^2-\frac{44}{\sqrt{3}}+1>13^2$ (I can't think of a neat way to show this but it's true), so no such value is possible.

☐ Find all natural numbers n such that it is possible to construct a sequence in which each number $1,2,3,\ldots,n$ appears twice, the second of the appearances of each integer r being r places beyond the first appearance. For instance, for $n=4$,

$$4,2,3,2,4,3,1,1.$$

Also, for $n=5$,

$$3,5,2,3,2,4,5,1,1,4.$$

Solution

Let a_k be the place of the first appearance of k. Example: in 42324311, $a_4=1,a_2=2,a_3=3,a_1=7$.

Then $\sum_{j=1}^{2n}=\sum_{j=1}^{n}2a_j+j$ implying $\sum_{j=1}^{n}a_j=\frac{3n^2+n}{4}$.

If n is 2 or 3 mod 4, then $\sum a_j$ is not an integer, contradiction.

Now we just need an example for n = 0, 1 mod 4.

☐ Polynomial P is such that for all real x we have $P(sinx)+P(cosx)=1$. What can the degree of this polynomial be?

Solution

Lemma: $P(x)$ is an even polynomial.

Proof: $P(\sin x)+P(\cos x)=1$ $P(\sin(-x))+P(\cos(-x))=1$ $P(-\sin x)+P(\cos x)=1$ $P(\sin x)=P(-\sin x)$, or, phrased in another way, $P(y)=P(-y)\,\forall y\in[-1,1]$

So $P(x)$ must be an even function, and have no odd terms. :)

Let $P(x)=Q(x^2)$ for some polynomial $Q(x)$. The problem condition becomes

$Q(\sin^2 x)+Q(1-\sin^2 x)=1\forall x\in\mathbb{R}$

Let us substitute $u=\sin^2 x-\frac{1}{2}$, and we can write

$Q(u+\frac{1}{2})+Q(-u+\frac{1}{2})=1\forall u\in[-\frac{1}{2},\frac{1}{2}]$

Finally, let us substitute $R(x)=Q(x+\frac{1}{2})-\frac{1}{2}$. The problem becomes

$R(x)+R(-x)=0$

And so $R(x)$ can be any odd polynomial or $R(x)=0$. Plugging all the way back in, $P(x)$ can have degree $0, 2(2k-1), k\in\mathbb{N}$.

☐ Find every positive integer n such that n^3+n^2+n+1 is square

Solution

$\gcd(n+1,n^2+1)=\gcd(n+1,1-n)=\gcd(2,1-n)$ Therefore, $\gcd(n+1,n^2+1)=2$ when n is odd and $\gcd(n+1,n^2+1)=1$ when n is even.

Case 1 when n is odd Let n = 2m -1, where m is a positive integer. $(n+1)(n^2+1)=2m(4m^2-4m+2)=(2)^2\cdot m(2m^2-2m+1)$ Note that $\gcd(m,2m^2-2m+1)=\gcd(m,1)=1$. Therefore, both m and $2m^2-2m+1$ have to be perfect squares. Let $m=k^2$ for some positive integer k. Then $2m^2-2m+1=2k^4-2k^2+1=(k^2)^2+(k^2-1)^2$ ...... and I don't know how to proceed here. In my knowlege, I only know that $k^2=1$ and $k^2=4$ are possible solutions. That is, k = 1 or k = 2, showing n = 1 or n = 7. **Someone please help to finish this part.** :lol:

Edit:Further explanation for this part The general solution for $x^2+y^2=z^2$ is of the form $x=2tuv, y=2t(u^2-v^2), z=2t(u^2+v^2)$ where t, u and v are integers. Note that $\gcd(k^2,k^2-1)=\gcd(1,k^2-1)=1$. Either $(k^2,k^2-1)=(2uv,u^2-v^2)$ or $(k^2,k^2-1)=(u^2-v^2,2uv)$.

Case 2 when n is even Let n = 2m, where m is a positive integer $(n+1)(n^2+1)$ with $n+1$ and n^2+1 being relatively prime. Then both $n+1$ and n^2+1 should be perfect squares.(in view of the unique factorization theorem) However, $(2m)^2<4m^2+1=n^2+1<4m^2+4m+1<(2m+1)^2$, which proves that n^2+1 cannot be a perfect square. Therefore, n^3+n^2+n+1 cannont be a perfect square in this case.

☐ Prove that $\binom{n}{r}$ is an integer (without stating that it's a. the number of ways to choose, or b. a binomial coefficient).

Hint: show that more or equal powers of any prime $p<n$ divide the numerator $(n!)$ than the denominator $(r!(n-r)!)$.

Solution

This can be considered as a number theory problem.

Let me quote the following useful theorem about $n!$:

Let p be a prime factor of $n!$ and k be the power of p in the [i]prime factorization[/i] of $n!$, then $k=\sum_{r=1}^{\infty}[\frac{n}{p^r}]$, where [x] denotes the floor function of x, and the sum is indeed a finite sum. For example, take n = 10. The prime factors of 10! are 2, 3, 5 and 7. $[\frac{10}{2}]+[\frac{10}{2^2}]+[\frac{10}{2^3}]=5+2+1=8$, $[\frac{10}{3}]+[\frac{10}{3^2}]=3+1=4$, $[\frac{10}{5}]=2$, and $[\frac{10}{7}]=1$. Therefore, $10!=2^8\cdot3^4\cdot5^2\cdot7^1$

We can then solve the problem of this thread with this theorem.

☐ Find all $x,y\in\mathbb{R}$ such that:
$\sqrt{2x(y+1)}+\sqrt{(x+1)y}+\sqrt{(x-1)(y-2)}=\sqrt{4x(3y-1)}$

Solution

By Cauchy

$(2x+(x+1)+(x-1))((y+1)+y+(y-2))\geq(\sqrt{2x(y+1)}+\sqrt{(x+1)y}+\sqrt{(x-1)(y-2)})^2$
$\iff\sqrt{2x(y+1)}+\sqrt{(x+1)y}+\sqrt{(x-1)(y-2)}\leq\sqrt{4x(3y-1}$

Equality holds when $\frac{2x}{y+1}=\frac{x+1}{y}=\frac{x-1}{y-2}$

Solve it to get $(x,y)=(2,3)$

☐phuong rtrinhf

☐

If $0 < a < b < c < 1$, how can i verify that $(c-a)/(1-ca), (b-a)/(1-ab), (c-b)/(1-cb)$ are the three sides of a triangle?

Solution

Letting $\frac{c-a}{1-ac} = x$, $\frac{c-b}{1-bc} = y$, $\frac{b-a}{1-ab} = z$

First we show that $\max\{x, y, z\} = x$

or same as $x - y = \frac{(b-a)(1-c^2)}{(1-ac)(1-bc)} > 0$ and $x - z = \frac{(c-b)(1-a^2)}{(1-ac)(1-ab)} > 0$

so $x > y$, $x > z$

Then we will show $y + z > x$ which is equivalent to

$y + z - x = \frac{(c-b)(b-a)(c-a)}{(1-ac)(1-ab)(1-bc)} > 0$.

Hence x, y, z are three side of a triangle .

□ Find all real numbers x, y, z and w such that $\sqrt{x-y} + \sqrt{y-z} + \sqrt{z-w} + \sqrt{w+x} = x + 2$.

Solution

$$\sqrt{x-y} + \sqrt{y-z} + \sqrt{z-w} + \sqrt{w+x} = x + 2$$
$$\iff (\sqrt{x-y} - 1)^2 + (\sqrt{y-z} - 1)^2 + (\sqrt{z-w} - 1)^2 + (\sqrt{w+x} - 1)^2 = 0$$

Or we can Cauchy it

$(1+1+1+1)((x-y)+(y-z)+(z-w)+(w+x)) \geq (\sqrt{x-y}+\sqrt{y-z}+\sqrt{z-w}+\sqrt{w+x})^2 = (x+2)^2$
$\iff (x-2)^2 \leq 0 \implies x = 2$. Equality holds when $x - y = y - z = z - w = w + x = 1$ .Hence $(x, y, z, w) = (2, 1, 0, -1)$

□ Prove that the equation:

$x(x+1)(x+2)...(x+n) = 1$ has a postive solution that is less then $\frac{1}{n!}$

Solution

Let $f(x) = x(x+1)(x+2)\dots(x+n) - 1$. Clearly, $f(x)$ is an increasing function for $x \geq 0$, $f(0) = -1 < 0$, and $f(1) = (n+1)! > 0$, so $f(x)$ has exactly one positive root r.

Furthermore,
$$r = \frac{1}{(r+1)(r+2)\dots(r+n)} < \frac{1}{1 \cdot 2 \dots n} = \frac{1}{n!}.$$

□ Let $Q_n = 12^n + 43^n + 1950^n + 1981^n$.

Then $Q_1 = 12 + 43 + 1950 + 1981 = 1993 \cdot 2$,

$Q_2 = 144 + 1849 + 3802500 + 3924361 = 7728854 = 1993 \cdot 3878$,

$Q_3 = 1728 + 79507 + 714875000 + 7774159141 = 15189115376 = 1993 \cdot 7621232$.

Determine all the positive integers n for which Q_n are divisible by 1993.

Solution

$Q_n \equiv 12^n + 43^n + (-43)^n + (-12)^n \pmod{1993}$

If n is odd, we clearly have $12^n + (-12)^n \equiv 0 \pmod{1993}$ and $43^n + (-43)^n \equiv 0 \pmod{1993}$ so it is divisible by 1993.

If $n = 2k$ with k odd, we have $Q_n \equiv 2 \cdot 144^k + 2 \cdot 1849^k \equiv 2(144^k + (-144)^k) \equiv 0 \pmod{1993}$ so it is also divisible by 1993.

If $n = 2k$ with k even, we have $Q_n \equiv 4 \cdot 144^k \pmod{1993}$, which is never 0, so it is not divisible by 1993.

Hence Q_n is divisible by 1993 iff n is odd or $n = 2k$ for k odd.

□ Find all functions $f : R \to R$ which satisfys:

$f(x^2 + y) + f(f(x) - y) = 2f(f(x)) + 2y^2$

Solution

$y = -x^2 \implies f(0) + f(f(x) + x^2) = 2f(f(x)) + 2x^4$ $y = f(x) \implies f(0) + f(f(x) + x^2) = 2f(f(x)) + 2f(x)^2$

Hence

$f(x)^2 = x^4$

Clearly then $f(0) = 0$, so setting $x = 0$, we have $f(y) + f(-y) = 2y^2$.

Suppose $f(c) = -c^2$. Then we have $f(-c) = 2c^2 - f(c) = 3c^2$. But then we have $f(-c)^2 = 9c^4 \neq c^4$ unless $c = 0$. Hence $f(x) = x^2$ for all x.

□ Solve the system of equations below for $x_1, x_2, ..., x_n$: $\begin{cases} x_1 + x_2 + ... + x_n = a \\ x_1^2 + x_2^2 + ... + x_n^2 = a^2 \\ \cdots\cdots\cdots \\ x_1^n + x_2^n + ... + x_n^n = a^n \end{cases}$

Solution

Note that this argument works for $x_i \in \mathbb{C}$ as an added bonus!

Let $S_k = \sum_{i=1}^n x_i^k$. We are given $a^k = S_k$. Let $\theta_k = \frac{1}{(n-k)!} \sum_{sym} \prod_{i=1}^k x_i$, where by convention $\theta_0 = 1$.

Consider $f(y) = \prod_{i=1}^n (y - x_i)$. It expands to $f(y) = \sum_{j=0}^n (-1)^j \theta_j x^{n-j}$.

Since $f(y)$ is a polynomial with at most n roots (namely, $\{x_i\}_{i=1}^n$), it follows we have

$0 = \sum_{j=0}^n (-1)^j \theta_j x_i^{n-j}$ for $i = 1, 2, \dots, n$ (*)

$\Rightarrow 0 = \sum_{i=1}^n \sum_{j=0}^n (-1)^j \theta_j x_i^{n-j}$

$\Rightarrow 0 = \sum_{j=0}^n (-1)^j \theta_j S_{n-j}$

$\Rightarrow 0 = \sum_{j=0}^n (-1)^j \theta_j a^{n-j}$

$\Rightarrow f(a) = 0$

So a is a root. But $x_1, \dots, x_n$ are the roots. Therefore, one of the x_i's equals a. Wlog, $x_1 = a$.

From (*), we again have: $0 = \sum_{i=2}^n \sum_{j=0}^n (-1)^j \theta_j x_i^{n-j} \Rightarrow 0 = (-1)^n \theta_n + \sum_{j=0}^{n-1} (-1)^j \theta_j (0) \Rightarrow$ one of the $x_i = 0$, wlog x_2

The same argument shows that $x_2 = x_3 = x_4 = \cdots = x_n = 0$.

Hence, the solutions are $(a, 0, 0, \dots, 0)$ and it's permutations.

— Consider a non-empty set $S = \{1, 2, 3, ..., n\}$. Let us define a function $f(A)$ on a non-empty set A of S as follows: Arrange the elements of A in a decreasing order, say, $a_k, a_{k-1}, a_{k-2}, a_{k-3} ..., a_2, a_1$ where $1 \leq k \leq n$. Then, $f(A) = \frac{a_k}{a_{k-1}} \frac{a_{k-2}}{a_{k-3}} ...$ [For example, $f(\{1, 2, 3\}) = \frac{3}{2} 1$, and $f(\{1, 2, 3, 4, 5, 6\}) = \frac{6}{5} \frac{4}{3} \frac{2}{1}$.] Find $\frac{1}{n} \prod_{A \in S} f(A)$ where A is a non-empty subset of S. – Prove that if $2^p + 3^p = a^n$, with p a prime and a and n positive integers, then $n = 1$. – Show that $2^{3^k} + 1$ is divisible by 3^k for all positive integers k.

Can anyone solve it without induction —- You have 2006 beads on a (closed) necklace in positions $0, 1, 2, 3, ...2005$. You are attempting to color the beads such that they satisfy the following property:

If beads in position i and position j, $i > j$ have the same color, then neither the bead in position $i + (i - j)$ nor $j - (i - j)$ (taken mod2006) can have that color.

What is the minimum number of colors needed?

Can you generalize? – Find all primes $p > 0$ and all integers $q \geq 0$ such that $p^2 \geq q \geq p$ and $\binom{p^2}{q} - \binom{q}{p} = 1$. – The number $2000 = 2^4 \cdot 5^3$ is the product of seven not necessarily distinct prime factors. Let x be the smallest integer greater than 2000 with this property and let y be the largest integer less than 2000 with this property. Find $x - y$ — Prove that among 39 consecutive natural numbers, it is always possible to find a number such that the digits sum to a number divisible by 11. — Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(n + 1) > f(n)$ and $f(f(n)) = 3n$ for all $n \in \mathbb{N}$. Determine $f(1992)$. –

Prove that

$$\left(2^{2n} + 2^{n+m} + 2^{2m}\right)!$$

is divisible by

$$\left(2^n!\right)^{2^n + 2^{m-1}} \cdot \left(2^m!\right)^{2^m + 2^{n-1}},$$

for all $m, n \in \mathbb{N}^*$.

☐ prove that if both p and p^2+2 are primes, then p^3+2 is also prime. If $p = 2$, then $p^2+2 = 6$, which is composite. So we exclude this case.

If $p = 3$, then $p^2+2 = 11$ and $p^3+2 = 29$. So the statement holds in this case.

If $p > 3$, then two cases are possible. Either $p \equiv 1(\mod 3)$ or $p \equiv -1(\mod 3)$, so that in either case $p^2+2 \equiv 1+2 \equiv 0(\mod 3)$. So, $3|(p^2+2)$, and so there is nothing to consider here.

And, we are done. Another way well, p^2+2 can never be a prime except for $p = 1, 3$. (but in this case, we exclude $p = 1$, since 1 isn't a prime number.) Since every prime numbers can be expressed as $6a \pm 1$ for a positive integer a, we have $p^2+2 = 36a^2 \pm 12a + 3$ which is divisible by 3.

☐ số học – Prove that for $n \geq 1$

$\frac{1}{\sqrt{4n}} \leq \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\ldots\left(\frac{2n-1}{2n}\right) < \frac{1}{\sqrt{2n}}$

☐ Find all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ to $n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599$

Solution

To avoid any checking approach give something general:

When a is even, then $a^k \equiv 0 \mod 2^k$, but not necessary $\mod 2^{k+1}$. When a is odd, then $a^{2^k} \equiv 1 \mod 2^{k+2}$, but not necessary $\mod 2^{k+3}$ for $k \geq 1$.

Setting $k = 4$ into the first and $k = 2$ into the second, the result follows.

☐ If $a_1, a_2, ..., a_n$ are n distinct odd natural numbers not divisible by any prime greater than 5, show that $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_n} < 2$.

Solution

let S be our subset of the inverses of odd natural numbers. Now consider

$U = \left\{\left(1 + \frac{1}{3} + \frac{1}{3^2} + ... + \frac{1}{3^n} + ...\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + ... + \frac{1}{5^n} + ...\right)\right\}$

Clearly, $S \supset U$. But the sum of the element of U is equal to $\frac{\left(\frac{1}{3}\right)^{n+1} - 1}{\frac{1}{3} - 1} \cdot \frac{\left(\frac{1}{5}\right)^{n+1} - 1}{\frac{1}{5} - 1}$

but when $n \to +\infty$, the sum of the elements of $U \to \frac{15}{8} < 2$

☐ dãy số

☐ tổ hợp

☐ số học A sequence $(x_n)_{n\geq 1}$ is defined by the rules $x_1 = 2$ and $nx_n = 2(2n-1)x_{n-1}$ for $n \geq 2$. Prove that x_n is an integer for every positive integer n. – Show that the quadratic equation $x^2 + 7x - 14(q^2+1) = 0$, where q is an integer, has no integer root. – Find all real parameters p for which the equation:

$x^3 - 2p(p+1)x^2 + (p^4 + 4p^3 - 1)x - 3p^3 = 0$

has three distinct real roots which are sides of a right triangle.

☐ a)There are more chess masters in New York City than in the rest of U.S. combined. A chess tournament is planned to which all American masters are expected to come. It is agreed that the tournament should be held at the site which minimizes the total intercity traveling done by the contestants. The New York masters claim that, by this criterion, the site chosen should be their city. The West Coast masters argue that a city at or near the center of gravity of the players would be better. Where should the tournament be held?

Solution

Both the West Coast and New York city are considered as points on a 2-D coordinate plane. Now, draw a straight line between the two locations, and label the West Coast and New York by the coordinates $(0,0)$ and $(a,0)$, where $a>0, a\in\mathbb{R}$. Also, let the number of chess players in West Coast be n_1 and that in New York be n_2 $(n_2>n_1.)$

In the solution below, we will use the fact that if k points have masses $m_1, m_2, ..., m_k$ and coordinates $(x_1,y_1),(x_2,y_2),...,(x_k,y_k)$, respectively, then the center of mass of these k points is $(\frac{x_1m_1+x_2m_2+...+x_km_k}{m_1+m_2+...+m_k}, \frac{y_1m_1+y_2m_2+...+y_km_k}{m_1+m_2+...+m_k})$.

Lastly, we assume that each chess master has the same mass m (a fair assumption for our purposes.)

So, the center of mass (we only need calculate the x coordinate) of all the chess players from the West Coast and New York $=\frac{n_1(m\cdot 0)+n_2(m\cdot a)}{(n_1+n_2)m}=\frac{n_2a}{n_1+n_2}$.

Now, if we heed the claim of the West Coast players, then the amount of intercity travel needed to be done by the chess players will be $D_1=\frac{n_2a}{n_1+n_2}\cdot n_1+(a-\frac{n_2a}{n_1+n_2})\cdot n_2=\frac{2n_1n_2a}{n_1+n_2}$.

And, if we heed the claim of the New York masters, then the amount of intercity travel needed to be done by the chess players will be $D_2=an_1$.

Therefore, $D_1>D_2$

$\Leftrightarrow \frac{2n_1n_2a}{n_1+n_2}>an_1 \Leftrightarrow 2n_2>n_1+n_2 \Leftrightarrow n_2>n_1$, which is true from our assumptions.

Hence, the tournament should be held in New York.

□ Prove that a circle centered at point $(\sqrt{2},\sqrt{3})$ in the cartesian plane passes through at most one point with integer coordinates.

Solution

Let us assume, for the sake of contradiction, there is a circle with center $O(\sqrt{2},\sqrt{3})$ such that it passes through two points with integer coordinates. Let these two points be $A(a,b)$ and $B(c,d)$, where $a,b,c,d\in\mathbb{Z}$. Note that the line AB is a chord of the circle.

Let the midpoint of AB be C, where $C\equiv(\frac{a+c}{2},\frac{b+d}{2})$. Let L denote the line passing through C and perpendicular to AB. Now, slope of line AB equals $\frac{b-d}{a-c}\Rightarrow$ Slope of the line L equals $\frac{a-c}{d-b}$.

So, the equation of line L is given by

$y-\frac{b+d}{2}=(\frac{a-c}{d-b})(x-\frac{a+c}{2})$

Since L, passes through the point O, we must have $\sqrt{3}-m=n(\sqrt{2}-p)$, where $m=\frac{b+d}{2}, n=\frac{a-c}{d-b}$ and $p=\frac{a+c}{2}$. Note that m,n and p are all rational.

So, we get $\sqrt{3}-n\sqrt{2}=m-np$

$\Rightarrow 3+2n^2-2n\sqrt{6}=(m-np)^2$

$\Rightarrow \sqrt{6}$ is rational, which is clearly a contradiction.

Hence no such circle exists.

□ tập hợp

[Note: If you know the solution, please don't write it. Just provide a hint or two to those who wish to attempt to solve it.]

The numbers $1,2,...,2002$ are written in order on a blackboard. Then the $1st, 4th, 7th, ..., 3k+1th, ...$ numbers in the list are erased. Then the $1st, 4th, 7th, ...3k+1th$ numbers in the remaining list are erased (leaving $3,5,8,9,12,...$). This process is carried out repeatedly until there are no numbers left. What is the last number to be erased? – Prove that the number of binary n-words with exactly m 01-blocks is $\binom{n+1}{2m+1}$. ——- n is a positive integer, $f(n)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

(1) When $n>1$, show that $f(2^n)>\frac{1}{2}(n+2)$.

(2) When $n>1$, define $A_n=f(1)+f(2)+\cdots+f(n)$ and $B_n=n[f(n)-1]$. Find which one

of A_n and B_n is larger. – Given a sequence $\{a_n\}$ such that $a_{n+1} = \frac{p_1 a_n + p_2}{p_3 a_n + p_4}$ and $a_1 = p_0$. Find the general term of a_n. — assume $a_0 = 1, a_1 = 2$ and for each $n \geq 1 : a_{n+1} = a_n + \frac{a_{n-1}}{1+a_{n-1}^2}$ then prove this inequality for each $n \geq 0 : \sqrt{2n+1} \leq a_n < \sqrt{3n+2}$ – The points $z_1, \ldots, z_5$ form a convex pentagon in the complex plane. The origin and the points $\alpha z_1, \ldots, \alpha z_5$ all lie inside the pentagon. Show that $|\alpha| \leq 1$ and $\Re(\alpha) + \Im(\alpha) \tan \frac{\pi}{5} \leq 1$. – Let $f(x) = 3x^4 + 4x^3$. Show that $f(f(...(f(9))))$, (with f repeated 10 times), has more than one thousand $9's$ when expressed in decimal notation. – Let $P(x) = x^{2n} + c_1 x^{2n-1} + c_2 x^{2n-2} + ... + c_{2n-1} x + c_{2n}$ be a polynomial that can be expressed as the product of n cuadratic polynomials $x^2 + a_1 x + b_1, x^2 + a_2 x + b_2, ..., x^2 + a_n x + b_n$. If $c_1, c_2, ...c_{2n}$ are positives, prove that a_k and b_k are positives for some $k (1 \leq k \leq n)$. – Let $S(n)$ be the sum of the digits of n. If for some integer n we have that:

$S(n) = 50$ and $S(15n) = 300$

Find $S(4n)$ – Let S_1 denote the sequence of positive integers $1, 2, 3, 4, 5, 6...$ and define the sequence S_{n+1} in terms of S_n by adding 1 to those integers in S_n which are divisible by n. Thus, for example, S_2 is $2, 3, 4, ...$ and S_3 is $3, 3, 5, 5....$ Determine those integers n with the property that the first n-1 integers in S_n are n. —- Supose you have two circles,A and B, with equal ray,and they both has 200 sectors that are painted with white and black.You now that the circle A has 100 sectors painted with white and 100 painted with black.Now we put the circle A over the circle B. By turning the circle A over the B is possible that there are at least 100 sectors in commom? – Find all positive integer solutions x,y,z,p (p is a prime) to $x^p + y^p = p^z$ – Let a, b, c and d be a reels numbers wish satisfy

$a = \sqrt{4 - \sqrt{5 - a}}$
$b = \sqrt{4 + \sqrt{5 - b}}$
$c = \sqrt{4 - \sqrt{5 - c}}$
$d = \sqrt{4 + \sqrt{5 - d}}$

Find the value of $abcd$ – Given $\{a_n\}$, $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$. If $b_n = \sqrt{2 - a_n}$, then find b_n. – **Generalization.** Let $a_1, a_2, \cdots, a_n$ be a sequence of positive real numbers such that their sum equals $A \in \mathbb{R}^+$. If $b_1, b_2, \cdots, b_n$ are positive integers with sum B, then

$$\max\left(\prod_{i=1}^{n} a_i^{b_i}\right) = \left(\frac{A}{B}\right)^B \left(\prod_{i=1}^{n} b_i^{b_i}\right).$$

Equality is achieved when $\frac{a_i}{b_i}$ is constant. – Prove that there exists a rational number $\frac{c}{d}$ with $d < 100$ such that

$\left\lfloor k \frac{c}{d} \right\rfloor = \left\lfloor k \frac{73}{100} \right\rfloor$.

for for $k = 1, 2, 3...100$. —— A regular (n+2)-gon is inscribed in a circle. Let T_n denote the number of ways it is possible to join its vertices in pairs so that the resulting segments do not intersect one another. If we set $T_0 = 1$, show that

$$T_n = T_0 T_{n-1} + T_1 T_{n-2} + ... T_{n-1} T_0$$

. – Find, with proof, all natural numbers n such that $n^4 + 7^n + 47$ is a perfect square. – Find all values of m, n, p such that m, n are positive integers and p is a prime number that satisfy:

$p^n + 144 = m^2$. — How many ways are there to place k marbles in any of the positions $1, 2, \ldots, n$ (which are evenly spaced around a circle) such that no two marbles are neighboring each other? (Of course, $k \leq \left\lfloor \frac{n}{2} \right\rfloor$, and each position can have at most one marble.) – Determine all positive integers whose squares end in 196. – Let a, n be positive integers such that $(a, n) = 1$.

Show that $n|\phi(a^n-1)$. – Given that $\{a_n\}$ is an arithmetic sequence (Common difference $d \neq 0$). The sequence a_{k_1}, a_{k_2}, $\cdots$, a_{k_n} formed by some terms of $\{a_n\}$ is geometrical. If $k_1 = 1$, $k_2 = 5$ and $k_3 = 17$, then find the value of $\sum_{i=1}^{n} k_i$.

– Given n a natural number greater than 1 and p a prime, where $n|p-1$ and $p|n^3-1$, show that $4p-3$ is a square number. — Determine all positive integers whose squares end in 196. – – Prove that

$\forall n \in \mathbb{N}, \forall p \in \mathbb{P} : p \equiv 3 \bmod 4 \implies \nexists x \in \mathbb{Z} : p^n|(x^2+1)$

$\forall n \in \mathbb{N}, \forall p \in \mathbb{P} : p \equiv 1 \bmod 4 \implies \exists x \in \mathbb{Z} : p^n|(x^2+1)$ – In a country, there are 101 towns, and to get from any town to any other town, there is no more than one one-way path. Each town has 40 paths entering it, and 40 paths going out. Prove that it's possible to reach any town from any other through no more than two other towns.

– prove that ,the number of the non-isomere triangles, such that its lenghts have an integer mesures and primetre n , is $[\frac{n^2+3n+21+(-1)^{n-1}\cdot 3n}{48}]$:oops: such that $[x]$ represente sa parie entiere..

☐ tổ hợp

☐ số học

☐số học

☐ tổ hợp hay

☐ số học

☐ tổ hợp

☐ số học

☐ pt hàm

☐ pt hàm

☐ pt hàm

☐ số học

☐ số học

☐ đa thức

☐ đại số

☐ xác suất khó

☐ dãy số

☐ tổ hợp

☐ tổ hợp

☐ dãy số

☐ pt hàm

☐ hình

☐hình

☐ số học

☐ đa thức

☐ số học

☐ tập hợp – LMN is an isosceles triangle in the complex plane

$\angle LMN = \angle LNM = \alpha$

Complex numbers corresponding to the vertices L,M and N are Z_1,Z_2 and Z_3 respectively

Prove that $(Z_3-Z_2)^2 = 4(Z_1-Z_2)(Z_3-Z_1)(cos\alpha)^2$ – Find the number of solutions for the equation $|2|2|2x-1|-1|-1| = x^2$ $(0 < x < 1)$.

☐ tổ hợp

☐ hình học
☐ đại số
☐ số học
☐ tổ hợp
☐ tổ hợp
☐ số học
☐ đại số
☐ dãy sô
☐ tổ hợp
☐ tổ hợp, lý thuyết số rất khó Prove that the equation

$x^4 - 18x^2 + 4dx + 9 = 0$ has four real roots if $d^4 \leq 1728$ – Determine all functions $f : \mathbb{N} \to \mathbb{R}$ such that $\forall m, n, k$,

$$f(km) + f(kn) - f(k)f(mn) \geq 1.$$

– Find all positive integers n such that

$$n = d_6^2 + d_7^2 - 1$$

where $1 = d_1 < d_2 < ... < d_k = n$ are all positive divisors of the number n. – Let a_1, a_2, a_3 be tree different real numbers. Define numbers b_1, b_2, b_3 as follwing: $b_1 = \left(1 + \frac{a_1 a_2}{a_1 - a_2}\right)\left(1 + \frac{a_1 a_3}{a_1 - a_3}\right)$, and b_2, b_3 is defined assemble. Prove that: $1 + |a_1 b_1 + a_2 b_2 + a_3 b_3| \leq (1 + |a_1|)(1 + |a_2|)(1 + a_3|)$ – Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a function such that $f(n+1) > f(f(n))\ \forall\ n \in \mathbb{Z}^+$. Prove that $f(n) = n$. – Consider an arbitrary parallelogram $ABCD$ with center O and let P be a point different from O. $(PA)(PC) = (OA)(OC)$ and $(PB)(PD) = (OB)(OD)$. Show that the sum of lengths of two of the segments PA, PB, PC, PD equals the sum of lengths of the other two. – Let $A = 1!2!...1002!$ and $B = 1004!1005!...2006!$. Show that $2AB$ is a square and $A + B$ is not a square. – Let F_n represent any Fibonacci number.

Prove that: $\frac{3F_n \pm \sqrt{5{F_n}^2 \pm 4}}{2}$ provides two other Fibonacci numbers. – Find

$$\lim_{n \longrightarrow \infty} \left[\frac{1}{n^5} \sum_{h=1}^{n} \sum_{k=1}^{n} (5h^4 - 18h^2k^2 + 5k^4)\right]$$

Solution

the expression is the right hand approximation of $\int_0^1 \int_0^1 5x^4 - 18x^2y^2 + 5y^4\, dx\, dy$, this is easily determined with basic integration rules... – If $f(x) = x^{20} - 4x^{19} + 9x^{18} - 16x^{17} + ... + 441 = 0$ and $z_1, z_2, ..., z_{20}$ are the roots of $f(x)$ find the value of $cot\left(\sum_{k=1}^{20} cot^{-1} z_k\right)$ – Let ABC be a triangle with orthocentre H. Prove that the Euler Lines of triangles ABC, ABH, BCH, ACH are concurrent. –

$(a) 1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^n \binom{n}{n}$

$(b) 1 \cdot 2\binom{n}{2} + 2 \cdot 3\binom{n}{3} ... + (n-1)n\binom{n}{n}$

$(c) \binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} ... n^2\binom{n}{n}$

$(d) \binom{n}{1} - 2^2\binom{n}{2} + 3^2\binom{n}{3} ... + (-1)^{n+1} n^2\binom{n}{n}$

$(e) \binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} ... + (-1)^n \frac{1}{n+1}\binom{n}{n}$

$(f) \sum_{j\geq 1}(-1)^j \frac{\binom{n}{j-1}}{\sum_{1\leq k\leq j} k}$ – Consider all of the permutations of $\{1, 2, \ldots, n\}$ (where n is a positive integer). Let A be the set of those permutations such that each number in the permutation is either greater than all the numbers to its left or less than all the numbers to its right. Let B denote the set of those permutations $a_1, a_2, \ldots, a_n$ such that for $1 \leq i \leq n-1$, there is a $j > i$ such that $|a_j - a_i| = 1$. Show that $|A| = |B|$. – Find all strictly increasing functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(2) = 2$ and whenever $\gcd(m, n) = 1$ then $f(mn) = f(m)f(n)$.

□ hình – The altitudes of ΔABC are extended externally to points A', B', and C' respectively, where $AA' = k/h_a$, $BB' = k/h_B$, and $CC' = k/h_c$. Prove that the centroid of the triangle $A'B'C'$ coincides with the centroid of ABC.

□ lý thuyết trò chơi – Let

$$\prod_{n=1}^{1996}(1 + nx^{3n}) = 1 + a_1x^{k_1} + a_2x^{k_2} + ... + a_mx^{k_m}$$

where $a_1, a_2, ...a_m$ are nonzero and $k_1 < k_2 < .. < k_m$. Find a_{1996}. — Let $t(n)$ be the maximum number of different areas that you can divide a circle into when you place n points on the circumference and draw all the possible line segments connecting the points. Find a formula for $t(n)$. – Given a set of lattice points, we can perform one of the following operations (note that we still keep the original point in each case): 1. $(x, y) \to (x + 1, y + 1)$ (note that we still keep x, y 2. If x and y are both even, $(x, y) \to (x/2, y/2)$ 3. $(x, y), (y, z) \to (x, z)$ If we start with $7, 29$, can we get to $3, 1999$? – Solve the equation $(x^2 + y)(x + y^2) = (x - y)^3$ on the set of integers. – Let k be a positive integer. find all polynomials $P(x)$ with real coefficients s.t. $P(P(x)) = [P(x)]^k$ – in a triangle ABC the following relation is given: $2a^4 + b^4 + c^4 + 18b^2c^2 = 2a^2(4bc + b^2 + c^2)$. Find the measure of the triangle angles. – Solve in rational numbers the equation : $4x^2 - y^2 = 36$

Solution

If x and y are rational then so are $2x \pm y$. Let $2x - y = \frac{6p}{q}$ for any coprime p and q. From the system $2x - y = \frac{6p}{q}, \quad 2x + y = \frac{6q}{p}$ all the solutions could be found.

□ tổ hợp

□ số học – Let p be any prime. Prove that

$$\binom{2p}{p} \equiv 2 \pmod{p^2}.$$

— A harder problem: Prove that $\binom{2p}{p} \equiv 2 \pmod{p^3}$ (without using Wolstenholme of course). — It is given that x and y are positive integers and $3x^2+x = 4y^4+y$. Show that: $x-y$, $3x+3y+1$ and $4x+4y+1$ are squares of integers. —– Using congruences: $100 \equiv 1(mod\ 11), 1000 \equiv -1(mod\ 13), 1000 \equiv 1(mod\ 27)$ Derivation of a formula of attributes (features) devisibility by 11, 13 and 27. – A collection of n planes is given in a space such that no four planes intersect at the same point and each three planes intersect exactly at one point. What is the total number of points where three planes intersect? To how many parts these planes divide the whole space? how many of these parts are unbounded? – Calculation of real x in $x = \left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right]$

Solution

Since x is obviously integer, put $x = 30k + r$ where $k, r \in \mathbb{Z}$ and $0 \leqslant r \leqslant 29$. Then
$30k + r = 15k + \left[\frac{r}{2}\right] + 10k + \left[\frac{r}{3}\right] + 6k + \left[\frac{r}{5}\right]$
$k = r - \left[\frac{r}{2}\right] - \left[\frac{r}{3}\right] - \left[\frac{r}{5}\right]$

$x = 30k + r = 31r - 30\left(\left[\frac{r}{2}\right] + \left[\frac{r}{3}\right] + \left[\frac{r}{5}\right]\right)$
Running r through the designated range, we get all the solutions:
$x \in \{0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 47, 49, 53, 59\}$[/

$\square$ $\left[\frac{1}{3}\right] + \left[\frac{2}{3}\right] + \left[\frac{2^2}{3}\right] + \dots\dots\dots\dots + \left[\frac{2^{2013}}{3}\right] =$
where $[x] =$ Integer part of x

Solution

Note that $2^{2n} \equiv 1 \pmod 3), 2^{2n+1} \equiv 2 \pmod 3)$. This directly translates to even powers of 2 ending in 1 base 3 and odd powers of 2 ending in 2. Finally, the last key observation we need is that when taking the integer part, we drop the last digit in base 3.

Keeping this in mind, we can then proceed by adding all of the powers of 2

$$2^0 + \cdots + 2^{2013} = 2^{2014} - 1$$

Now we subtract $1007 \cdot 1$ and $1007 \cdot 2$

$$2^{2014} - 1007 - 1007 \cdot 2 - 1$$
$$= 2^{2014} - 3 \cdot 1007 - 1$$

Then our answer is $\frac{2^{2014}-1}{3} - 1007$.

$\square$ Let i and j be positive integers with $i \geq 1$ and $1 \leq j \leq i+1$. Define $a_{i,j}$ as follows: $a_{1,1} = a_{1,2} = a_{2,1} = a_{2,3} = 1$ $a_{2,2} = 2$ $a_{i,1} = a_{i,i+1} = a_{i-1,1} + a_{i-2,1}$ for $i \geq 3$ $a_{i,j} = \max(a_{i-1,j-1} + a_{i-2,j-1}, a_{i-1,j} + a_{i-2,j-1})$ for all $i \geq 3$ and $2 \leq j \leq i$ Find a closed closed form expression for $a_{i,j}$.

Solution

From the symmetry of the initial coniditions and the rule for generating $a_{i,j}$, it is clear that $a_{i,k} = a_{i,i+1-k}$. Therefore, WLOG, we need only worry about the $a_{i,j}$ for which $j \leq \lfloor \frac{i+1}{2} \rfloor$. I have listed the first values of this sequence below with i denoting the row number and j denoting the column number.

1					
1	2				
2	3				
3	4	5			
5	6	8			
8	9	12	13		
13	14	18	21		
21	22	27	33	34	
34	35	41	51	55	
55	56	63	78	88	89
89	90	98	119	139	144

Several patterns become apparent immediately. If F_n is the nth term of the fibonacci sequence, then $a_{i,1} = F_i$ and $a_{i,\lfloor \frac{i+1}{2} \rfloor} = F_{i+1}$. Also, $\max(a_{i-1,j-1} + a_{i-2,j-1}, a_{i-1,j} + a_{i-2,j-1}) = a_{i-1,j} + a_{i-2,j-1}$, so we have $a_{i,j} = a_{i-1,j} + a_{i-2,j-1}$ in this simplified case. All of these are easily proveable by induction, but

I will leave this out. We consider a new sequence $b_{i,j}$ such that $b_{i,j} = a_{i,j+1} - a_{i,j}$ The first few values of this sequence are listed below.

$$\begin{array}{lllll} 0 \\ 1 \\ 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 5 & 6 & 1 \\ 1 & 6 & 10 & 4 \\ 1 & 7 & 15 & 10 & 1 \\ 1 & 8 & 21 & 20 & 5 \end{array}$$

A simple formula for $b_{i,j}$ becomes apparent: $b_{i,j} = \begin{pmatrix} i-j-1 \\ j-1 \end{pmatrix}$ We now have a formula for $a_{i,j}$ with $j \leq \lfloor \frac{i+1}{2} \rfloor$ which we will prove by induction. $a_{i,j} = F_i + \sum_{k=1}^{j-1} \begin{pmatrix} i-k-1 \\ k-1 \end{pmatrix}$ We can easily check that this formula is satisfied for $i \leq 2$. Now assume that for some $i \geq 3$ and $1 < j < \lfloor \frac{i+1}{2} \rfloor$, the formula holds true for $a_{i-1,j}$ and $a_{i-2,j-1}$. Then we have $a_{i,j} = a_{i-1,j} + a_{i-2,j-1}$ $a_{i,j} = F_{i-1} + F_{i-2} + \sum_{k=1}^{j-1} \begin{pmatrix} i-k-2 \\ k-1 \end{pmatrix} + \sum_{k=1}^{j-2} \begin{pmatrix} i-k-3 \\ k-1 \end{pmatrix}$ By pascal's identity and the definition of the fibonacci sequence, this reduces to $a_{i,j} = F_i + \sum_{k=1}^{j-1} \begin{pmatrix} i-k-2 \\ k-1 \end{pmatrix} + \sum_{k=2}^{j-1} \begin{pmatrix} i-k-2 \\ k-2 \end{pmatrix}$ $a_{i,j} = F_i + \sum_{k=2}^{j-1} \begin{pmatrix} i-k-1 \\ k-1 \end{pmatrix} + 1$ $a_{i,j} = F_i + \sum_{k=1}^{j-1} \begin{pmatrix} i-k-1 \\ k-1 \end{pmatrix}$ Now that we have proven the formula for $j \leq \lfloor \frac{i+1}{2} \rfloor$, it easy to find the extension that for $j \geq \lfloor \frac{i+1}{2} \rfloor$, we have $a_{i,j} = F_i + \sum_{k=1}^{i-j} \begin{pmatrix} i-k-1 \\ k-1 \end{pmatrix}$

$\square$ Solve this system of equations for positive real numbers. $x^4+y^4+(2xy-1)(x^2+y^2)+2x^2y^2 = 0$ $\frac{1}{x} + \frac{1}{y} = 4(x+y)^5$

Solution

$x^4+y^4+(2xy-1)(x^2+y^2)+2x^2y^2 = (x^2+y^2)^2+(2xy-1)(x^2+y^2) = (x^2+y^2)(x^2+2xy+y^2-1) = (x^2+y^2)((x+y)^2-1) = (x+y-1)(x+y+1)(x^2+y^2) = 0$. There are no real solutions when $x^2+y^2 = 0 \iff x^2 = -y^2$ unless $x = y = 0$, but this would involve dividing by 0 [in the second equation] which isn't allowed. Thus, there are only two cases to consider.

Case 1: $x+y = 1$. In this case, the second equation is $\frac{1}{x} + \frac{1}{y} = 4 \iff \frac{x+y}{xy} = 4 \iff xy = \frac{1}{4}$. By Vieta's, x, y are the solutions to the quadratic $a^2 - a + \frac{1}{4} = a^2 + 2(a)(-\frac{1}{2}) + (-\frac{1}{2})^2 = (a-\frac{1}{2})^2 = 0$, which yields $x = y = \frac{1}{2}$.

Case 2: $x+y = -1$. In this case, the second equation is $\frac{1}{x} + \frac{1}{y} = -4 \iff \frac{x+y}{xy} = -4 \iff -xy = -\frac{1}{4} \iff xy = \frac{1}{4}$. By Vieta's, x, y are the solutions to the quadratic $a^2 + a + \frac{1}{4} = a^2 + 2(a)(\frac{1}{2}) + (\frac{1}{2})^2 = (a+\frac{1}{2})^2 = 0$, which yields $x = y = -\frac{1}{2}$.

It follows that $\boxed{(x,y) = (\pm\frac{1}{2}, \pm\frac{1}{2})}$.

$\square$ Show that there is exactly one pair of positive integers m, n, with $n < 200$, such that

$$\frac{59}{80} < \frac{m}{n} < \frac{45}{61}.$$

Solution

Lemma: If fractions satisfy the inequality $\frac{a}{b} < \frac{m}{n} < \frac{c}{d}$, then there exists a unique y such that $\frac{a+yc}{b+yd} = \frac{m}{n}$. **Proof:** This is a matter of solving linear equations. Notice that no solution occurs if a/b = c/d, which cannot happen. Notice that y is rational if the fractions are rational.

With this in mind, if we have the inequality $\frac{59}{80} < \frac{m}{n} < \frac{45}{61}$, then we can write $m = 59x + 45y$ and $n = 80x + 61y$ for some relatively prime positive integers x and y, by the lemma above. We wish to find the conditions in which $\frac{59x+45y}{80x+61y}$ simplifies. To do this, we use the Euclidean Algorithm:

$$\begin{aligned}(59x+45y, 80x+61y) &= (59x+45y, 21x+16y)\\ &= (17x+13y, 21x+16y)\\ &= (17x+13y, 4x+3y)\\ &= (x+y, 4x+3y)\\ &= (x+y, x)\\ &= (x, y)\end{aligned}$$

By our choice to make x and y relatively prime, this fraction will not simplify, so we must have that $80x + 61y = 200$. There is clearly only one solution to this in the positive integers, so $(m, n) = (104, 141)$ is unique.

$\square$ Let S denote the set of all nonnegative integers whose base-10 representation contains no 1s. Compute

$$\prod_{k\in S} \frac{10k+2}{10k+1}$$

or show that it diverges.

Solution

Convergence Let $f(x) = \frac{x}{x-1}$, so we are examining

$$P = [f(22)f(32)f(42)...f(92)][f(202)f(222)f(232)...f(992)][f(2002)f(2022)...]...$$

where there are $8 \cdot 9^{k-2}$ arguments with k digits.

Because $f(x)$ is decreasing, $P < (f(22))^8(f(202))^{72}(f(2002))^{648}...$

Therefore $\log P < 8\log f(22) + 72\log f(202) + 648\log f(2002) + ...$

Now $\log f(x) < \frac{2}{x}$ for $x > 2$, so $\frac{9}{8}\log P < 9 \cdot \frac{1}{11} + 9^2 \cdot \frac{1}{101} + 9^3 \cdot \frac{1}{1001} + ... < \frac{9}{10} + \frac{81}{100} + \frac{729}{1000} + ...$ which implies $\frac{9}{8}\log P < \sum_{i=1}^{\infty}(\frac{9}{10})^i = 9 \implies \log P < 8 \implies P < e^8$.[/hide]

$\square$ Find all positive integers n such that $\lfloor \frac{n^2}{5} \rfloor$ is a prime number $p \le \frac{n^2}{5} < p+1$, where p is prime. So $5p \le n^2 < 5p+5 \implies 0 \le n^2 - 5p < 5$. Solve all cases from 0 to 4. Eg, $n^2 - 5p = 1 \implies p = \frac{(n+1)(n-1)}{5}$. Since p is prime, either $5 = n+1$ or $5 = n-1$, which yields $n = 6, 4$.

Final conclusion is n=$4, 5, 6$. Or another way:

The quadratic residues of n^2 are $0, \pm 1$.

Case one: $n^2 = 5a \implies \lfloor \frac{n^2}{5} \rfloor = a$ but $5|a$. So only solution is $n, a = 5$

Case two: $n^2 = 5a+1 \implies \lfloor \frac{n^2}{5} \rfloor = a$ So a needs to be prime. Note that then $(n+1)(n-1) = 5a$. $n+1 = 5 \implies n = 4, a = 3$ so we are good.

If $n-1=5 \implies n=6, a=7$ so we are good.

Case three: $n^2=5a+4 \implies \lfloor \frac{n^2}{5} \rfloor = a$ Again, a needs to be prime. $(n+2)(n-2)=5a$.

$n+2=5 \implies n=3, a=1$ so we can throw it away. $n-2=5, n=7, a=9$ which is again incorrect.

Thus, $n=4,5,6$

$\square$ Let A be a set with at least two members. Show that there exists a bijective function $f: A \to A$ such that $f(x) \neq x$ for all $x \in A$.

Solution

For finite A take a cyclic permutation. For infinite A, we may need the axiom of choice to prove that A may be partitioned in disjoint pairs (x,y) of elements, and then define $f(x)=y, f(y)=x$ on each pair. A proof follows.

Consider the family $\mathcal{P}$ of all sets having as elements disjoint pairs from A, ordered by inclusion. This is a poset, with the property that any chain $\mathcal{C}$ is has a majorant - we may take $\bigcup_{C\in\mathcal{C}} C \in \mathcal{P}$, clearly a set having as elements disjoint pairs from A. By Zorn's Lemma (equivalent of the Choice Axiom), there exists a maximal element $M \in \mathcal{P}$. If $\bigcup_{P\in M} = A$, we are done. We cannot have more than one element in $A \setminus \left(\bigcup_{P\in M}\right)$, since that will contradict the maximality of M. Finally, if $A \setminus \left(\bigcup_{P\in M}\right) = \{a\}$, take out a countable subset of pairs from M, say (x_n, y_n), for $n \geq 1$, rearrange as $(a, x_1), (y_1, x_2), (y_2, x_3), \ldots$, and put them back, to obtain a set M' with $\bigcup_{P\in M'} = A$.

$\square$ Find $x \in \mathbb{R}$ satisfy $\sqrt{x^2+(1-\sqrt{3})x+2}+\sqrt{x^2+(1+\sqrt{3})x+2} \leq 3\sqrt{2}-\sqrt{x^2-2x+2}$

Solution

Let $T(x,0)$, $A\left(\frac{\sqrt{3}-1}{2}, \frac{-\sqrt{3}-1}{2}\right)$, $B\left(\frac{-\sqrt{3}-1}{2}, \frac{\sqrt{3}-1}{2}\right)$ and $C(1,1)$. Easy to show that ΔABC is a regular triangle with center $O(0,0)$, which is the Torricelli point of the triangle. Thus, $\sqrt{x^2+(1-\sqrt{3})x+2}+\sqrt{x^2+(1+\sqrt{3})x+2}+\sqrt{x^2-2x+2} = = TA+TB+TC \geq OA+OB+OC = 3\sqrt{2}$. The equality occurs, when $T \equiv O$, which happens for $x=0$. Id est, the answer is $\{0\}$.[/hide]

$\square$ Show that
$(z-e^{i\theta})(z-e^{-i\theta}) = z^2-2z+1$.

Solution

$$\left(z-e^{i\theta}\right)\left(z-e^{-i\theta}\right) = z^2-2z\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)+1 = z^2-2z\cos\theta+1$$

$$z^{2n}+1=0 \Leftrightarrow z=e^{\left(\frac{(2k-1)\pi}{2n}\right)i} = \cos\left(\frac{(2k-1)\pi}{2n}\right)+i\sin\left(\frac{(2k-1)\pi}{2n}\right)$$

$$z^{2n}+1 = \left(z-e^{-\frac{(2n-1)\pi i}{2n}}\right)\ldots\left(z-e^{-\frac{3\pi i}{2n}}\right)\left(z-e^{-\frac{\pi i}{2n}}\right)\left(z-e^{\frac{\pi i}{2n}}\right)\left(z-e^{\frac{3\pi i}{2n}}\right)\ldots\left(z-e^{\frac{(2n-1)\pi i}{2n}}\right)$$

$$= \prod_{k=-n}^{n}\left(z-e^{\left(\frac{(2k-1)\pi}{2n}\right)i}\right) = \prod_{k=1}^{n}\left(z-e^{-\frac{(2k-1)\pi i}{2n}}\right)\left(z-e^{\frac{(2k-1)\pi i}{2n}}\right)$$

$$= \prod_{k=1}^{n}\left(z^2-z\left(e^{\frac{(2k-1)\pi i}{2n}}+e^{-\frac{(2k-1)\pi i}{2n}}\right)+1\right) = \prod_{k=1}^{n}\left(z^2-2z\cos\left(\frac{(2k-1)\pi}{2n}\right)+1\right)$$

then $z^{2n}+1 = \prod_{k=1}^{n}\left(z^2-2z\cos\left(\frac{(2k-1)\pi}{2n}\right)+1\right)$

put z=i $\prod_{k=1}^{n}\left(i^2-2i\cos\left(\frac{(2k-1)\pi}{2n}\right)+1\right) = \prod_{k=1}^{n}\left(-2i\cos\left(\frac{(2k-1)\pi}{2n}\right)\right)$

$= (-1)^n 2^n i^n \prod_{k=1}^{n}\cos\left(\frac{(2k-1)\pi}{2n}\right) = i^{2n}+1$

now $P = \prod_{k=1}^{n}\cos\left(\frac{(2k-1)\pi}{2n}\right) = (-1)^n \frac{i^n+i^{-n}}{2^n} = (-1)^n \frac{2\cos\left(\frac{n\pi}{2}\right)}{2^n} = (-1)^n 2^{1-n}\cos\left(\frac{n\pi}{2}\right)$

$\square$

Pat leaves a small town A at 10:18am and walking at uniform speed arrives in town B at 1:30pm. On the same day Chris leaves town B at 9:00am and walking along the same route at uniform speed arrives at A at 11:40am. The road crosses a wide river and they both arrive at the bridge on their respective sides at exactly the same momne t. Pat leaves the bridge one minute later than Chris. When did they arrive at the bridge?

Solution

Let's say that the distance from A to B is 1000 units. Pat travels this in 192 minutes and so he is travelling at $\frac{125}{24}$ units per minute. Similarly, Chris is travelling at $\frac{25}{4}$ units per minute. So we see that they are 512.5 units apart at 10:18.

Let's say that the bridge is b units long, and we know that Chris traversed it 1 minute quicker. Hence, $\frac{24b}{125} - \frac{4b}{24} = 1$. Therefore $b = 31.25$.

Finally, lets call t the time (in minutes) elapsed since 10:18. We know the combined distance travelled is $s = 512.5 - b = 481.25$ units, and their combined speed is $u = \frac{25}{4} + \frac{125}{24} = \frac{275}{24}$. As there is no acceleration, we know that $s = ut$, hence $t = 42$ minutes. Hence, they arrived at the bridge at $10 : 18 + 42 = 11$ am.

$\square$

Find all ordered pairs of real numbers (x, y) for which:

$$(1 + x)(1 + x^2)(1 + x^4) = 1 + y^7$$

and

$$(1 + y)(1 + y^2)(1 + y^4) = 1 + x^7.$$

Solution

The equations are

$$\begin{aligned} 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 &= 1 + y^7 \\ 1 + y + y^2 + y^3 + y^4 + y^5 + y^6 + y^7 &= 1 + x^7. \end{aligned}$$

Substituting the first equation into the second,

$$y + y^2 + \ldots + y^5 + y^6 + x + x^2 + \ldots + x^5 + x^6 = 0.$$

Suppose, by way of contradiction, that $x = 1$. Then

$$(1 + x)(1 + x^2)(1 + x^4) = 1 + y^7$$

yields that $y > 1$ and

$$(1 + y)(1 + y^2)(1 + y^4) = 1 + x^7$$

returns $y < 1$, a contradiction. Hence, $x \neq 1$. Similarly, $y \neq 1$. Since $x = 1$ or $y = 1$ can't possibly be solutions,

$$y\left(\frac{y^6 - 1}{y - 1}\right) + x\left(\frac{x^6 - 1}{x - 1}\right) = 0.$$

Set

$$f(x) = x\left(\frac{x^6 - 1}{x - 1}\right).$$

A sign analysis of $f(x)$ gives

$$f(x)\begin{cases} >0 & \text{if} \quad x>0 \\ <0 & \text{if} \quad 0>x>-1 \\ >0 & \text{if} \quad -1>x. \end{cases}$$

Becuase

$$f(y)+f(x)=0,$$

then for $x, y \neq 0, -1$ exactly one of $\{x, y\}$ must be between 0 and -1. Suppose, without loss of generality,

$$0>x>-1.$$

Hence,

$$\boxed{y>0 \qquad \text{or} \qquad -1>y}.$$

Then

$$0>x^7>-1,$$

so

$$1>1+x^7>0.$$

Thus,

$$1>(1+y)(1+y^2)(1+y^4)>0$$

Since

$$y^2+1>1 \qquad \text{and} \qquad y^4+1>1,$$

then

$$0<(1+y)<1,$$

so

$$\boxed{-1<y<0,}$$

a contradiction to the bounds on y.

Therefore, the only solutions are $\boxed{x=y=0}$ and $\boxed{x=y=-1}$.

Problem : If a and b are two roots of $x^4+x^3-1=0$, prove that ab is a root of $x^6+x^4+x^3-x^2-1=0$.

Solution

Let the roots be a, b, c, d. Then we have

$$(x-a)(x-b)(x-c)(x-d)=0.$$

This means that

$$\begin{aligned} a+b+c+d &= -1 \\ abcd &= -1 \\ abc+abd+acd+bcd &= 0 \\ ab+ac+ad+bc+bd+cd &= 0 \end{aligned}$$

Due to (4), there is no x^5 term in the 6th degree polynomial. Also, since $abcd=-1$, the constant term is -1 as well because the product of the terms in (4) is $(-1)^3=-1$. We also find the x^4, x^3, and $-x^2$ terms as well due to the same reasoning that gave equations $(1)-(4)$ and the following equation:

$$(x-ab)(x-ac)(x-ad)(x-bc)(x-bd)(x-cd)=0.$$

$\square$

Given and ellepse with focus F_1 and F_2. P is a mobile point on the ellipse. Through F_1 construct a perpendicular line to the exterior angle bisector of $\angle F_1PF_2$. Find the locus of the projection.

Solution

Let H be the projection of F_1 onto the exterior angle bisector of $\angle F_1PF_2$ Let PD be the internal angle bisector of $\angle F_1PF_2$.Then of course $F_1H \parallel PD$

Let $Z \in F_1H \cap F_2P$. Then it is known that the triangle PZF_1 is isosceles (because $\angle PF_1Z = \angle PZF_1 = \angle F_1PD = \angle F_2PD$)

Hence $PZ = PF_1 \Rightarrow F_2Z = F_2P + F_1P = 2a = constant$

H is the midpoint of F_1Z, so the parallel line to F_2Z through H meets F_1F_2 on its midpoint O.

So $OH = \frac{F_2Z}{2} = a \Rightarrow H$ lies on the circle $w = (O, a)$

Inversely, for any point $H \in w$, we can see (using the same steps) that $F_1P + F_2P = 2a$

So, the locus of H is the circle w

$\square$

Let $ABCD$ be a quadrilateral with $AB \parallel CD$, $AB > CD$. Prove that the line passing through $AC \cap BD$ and $AD \cap BC$ passes through the midpoints of AB and CD.

Solution

Consider a triangle ABC with X on CB and Y on CA so that XY is parallel to AB. (*)

CXY is similar to CBA, so that CX/CB = CY/CA. It follows BX/XC = AY/YC. (**)

Suppose AX and BY intersect in J. and CJ intersects AB at Z. From our result (**) + Ceva, it follows AZ = ZB.

Finally, because our triangles are similar, the (collinear) line CJZ also hits the midpoint of XY.

So we have proved with the above configuration (*) that C, the midpoint of XY, the intersection of AX and BY, and the midpoint of AB are collinear.

Now putting $A', B', C', X', Y' = A, B, AD \cap BC, C, D$ we can appeal to the result above. here is another proof

Let $P = AD \cap BC$ and $O = AC \cap BD$

From similar triangles PDC, PAB we have:

$\frac{PD}{PA} = \frac{DC}{AB}$

Also, the triangles OAB, OCD are similar ($\angle OAB = \angle OCD, \angle DOC = \angle AOB$)

So, $\frac{OC}{OA} = \frac{DC}{AB} \Rightarrow$

$\frac{OC}{OA} = \frac{PD}{PA}$

Take a point $F \in OA$, such that $OF = OC$ (since $CD < AB \Rightarrow OC < OA \Rightarrow F$ is inside the segment OA)

Now, $\frac{OF}{OA} = \frac{OC}{OA} \Rightarrow$

$\frac{OF}{OA} = \frac{PD}{PA}$

Hence $DF \parallel PO$

In the triangle CDF, O is the midpoint of CF and $PO \parallel DF \Rightarrow PO$ intersects DC at its midpoint H.

Finally, from similar triangles PDC, PAB we find that the line PH is also median for $\triangle PAB$. Another solution We will use a theorem (I don't remember this theorem's name):

Lemma Let A, B, E be three collinear points and a point $P \notin AB$. Let a line $d \parallel AB$. d intersects the lines PA, PB, PE at the points A', B', E' respectively.

Then $\frac{EA}{EB} = \frac{E'A'}{E'B'}$

Proof of Lemma $\triangle PAE \sim \triangle PA'E' \Rightarrow \frac{PA}{PA'} = \frac{EA}{E'A'}$

$\triangle PBE \sim \triangle PB'E' \Rightarrow \frac{PB}{PB'} = \frac{EB}{E'B'}$

But $\frac{PA}{PA'} = \frac{PB}{PB'} \Rightarrow \frac{EA}{E'A'} = \frac{EB}{E'B'} \Rightarrow \frac{EA}{EB} = \frac{E'A'}{E'B'}$

So, if E is the midpoint of AB then the line PE intersects the segment $A'B'$ on its midpoint E' This means that the line EE' passes through $P = AA' \cap BB'$.

Let M, N be the midpoints of AB, CD respectively.

$A' = C, B' = D \Rightarrow MN$ passes through $AC \cap BD$

$A' = D, B' = C \Rightarrow MN$ passes through $AD \cap BC$

$\square$

Define $\mu(k)$ as the following:

- $\mu(1) = 1$ - $\mu(k) = (-1)^n$ for k a product of n distinct primes - $\mu(k) = 0$ otherwise

One.

Given an integer n, let $\mathcal{D}$ be the set of its positive integral divisors. Show that

$\sum_{d \in \mathcal{D}} \mu(d) = 0$

Two.

Show that

$\sum_{d \in \mathcal{D}} \mu(d) \cdot \frac{n}{d} = \varphi(n)$

Solution

For 1) Let $f(n) = \sum_{k|n} \mu(k)$. Note that $f(p^e) = \mu(1) + \mu(p) + \mu(p^2) + ... + \mu(p^{k-1} = 0$. Since $\mu(k)$ is multiplicative, also $f(n)$ is. Hence $\sum_{k|n} \mu(k) = 0$.

For 2)$\sum_{d \in \mathcal{D}} \mu(d) \cdot \frac{n}{d} = n \sum_{d \in \mathcal{D}} \frac{\mu(d)}{d} = 1 - \frac{1}{p_1} - ... - \frac{1}{p_k} + \frac{1}{p_1 p_2} + ... + \frac{1}{p_{k-1} p_k} + ... \pm \frac{1}{p_1 p_2 \cdots p_k} = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$.

$\square$

Show that if $3 \leq d \leq 2^{n+1}$, then $d \nmid (a^{2^n} + 1)$ for all positive integers a.

Solution

Suppose d divedes the expression (for contrary). Then a and d are obviously coprime. Order of a mod d divides 2^{n+1} but doesn't divide 2^n, so the order is 2^{n+1} and it must divide the totient of d, so $2^{n+1} < d$, contrary.

$\square$

What's the greatest integer and positive number such that it can't be expressed as the sum of two compossite odd numbers?

Solution

We know that 38 satisfies our conditions.

Suppose we have $n > 38$. Consider $n - 3, n - 9, n - 21, n - 27, n - 33$ mod 5. They are distinct, so one is divisible by 5. For $n > 38$ we have $n - 33 > 5$ so if one is divisible by 5, it will be odd and composite.

Equivalently, consider $n - 5, n - 25, n - 35$ mod 3.

$\square$

Let $a > -1$ and $r \in (0, 1)$ be reals. Prove that:

$$(1 + a)^r \leq 1 + ra.$$

Solution

We'll prove the equivalent statement $(1 + a)^r \geqslant 1 + ar$ for $a > -1$ and $\mathbb{R} \ni r > 1$.

Put $c = 1 + a$ and $s = r - 1$. Then the inequality becomes

$$\begin{aligned} c^{s+1} \geqslant 1 + (c-1)(1+s) \quad &\iff \quad c^{s+1} \geqslant c + (c-1)s \\ &\iff \quad c(c^s - 1) \geqslant (c-1)s \end{aligned}$$

for $c > 0, s > 0$

For $c = 1$ (1) is trivially satisfied, hence we'll deal with $c \neq 1$.

Part 1. s $\in \mathbb{Q}$ Put $s = \frac{p}{q}, p, q \in \mathbb{N}$

Case 1.1. c $>$ 1. Then we can write

$$\begin{aligned} c(c^{\frac{p}{q}} - 1) \geqslant (c-1)\frac{p}{q} \quad &\iff \quad \frac{c(c^{\frac{p}{q}} - 1)}{c - 1} \geqslant \frac{p}{q} \\ &\iff \quad \frac{c(c^{\frac{1}{q}} - 1)(c^{\frac{p-1}{q}} + c^{\frac{p-2}{q}} + \cdots + 1)}{(c^{\frac{1}{q}} - 1)(c^{\frac{q-1}{q}} + c^{\frac{q-2}{q}} + \cdots + 1)} \geqslant \frac{p}{q} \\ &\iff \quad \frac{c(c^{\frac{p-1}{q}} + c^{\frac{p-2}{q}} + \cdots + 1)}{c^{\frac{q-1}{q}} + c^{\frac{q-2}{q}} + \cdots + 1} \geqslant \frac{p}{q} \end{aligned}$$

The numerator on the LHS is not less than $c(\underbrace{1 + 1 + \cdots + 1}_{p}) = pc$, and the denominator is not greater than $\underbrace{c^{\frac{q-1}{q}} + c^{\frac{q-1}{q}} + \cdots + c^{\frac{q-1}{q}}}_{q} = qc^{\frac{q-1}{q}}$, hence we have

$\frac{c(c^{\frac{p-1}{q}} + c^{\frac{p-2}{q}} + \cdots + 1)}{c^{\frac{q-1}{q}} + c^{\frac{q-2}{q}} + \cdots + 1} \geqslant \frac{pc}{qc^{\frac{q-1}{q}}} = \frac{p}{q} c^{\frac{1}{q}} > \frac{p}{q}$

Case 1.2. c $<$ 1. Then we can write

$$\begin{aligned} c(c^{\frac{p}{q}} - 1) \geqslant (c-1)\frac{p}{q} \quad &\iff \quad \frac{c(c^{\frac{p}{q}} - 1)}{c - 1} \leqslant \frac{p}{q} \\ &\iff \quad \frac{c(c^{\frac{1}{q}} - 1)(c^{\frac{p-1}{q}} + c^{\frac{p-2}{q}} + \cdots + 1)}{(c^{\frac{1}{q}} - 1)(c^{\frac{q-1}{q}} + c^{\frac{q-2}{q}} + \cdots + 1)} \leqslant \frac{p}{q} \\ &\iff \quad \frac{c(c^{\frac{p-1}{q}} + c^{\frac{p-2}{q}} + \cdots + 1)}{c^{\frac{q-1}{q}} + c^{\frac{q-2}{q}} + \cdots + 1} \leqslant \frac{p}{q} \end{aligned}$$

The numerator on the LHS is not greater than $c(\underbrace{1 + 1 + \cdots + 1}_{p}) = pc$, and the denominator is not less than $\underbrace{c^{\frac{q-1}{q}} + c^{\frac{q-1}{q}} + \cdots + c^{\frac{q-1}{q}}}_{q} = qc^{\frac{q-1}{q}}$, hence we have

$\frac{c(c^{\frac{p-1}{q}} + c^{\frac{p-2}{q}} + \cdots + 1)}{c^{\frac{q-1}{q}} + c^{\frac{q-2}{q}} + \cdots + 1} \leqslant \frac{pc}{qc^{\frac{q-1}{q}}} = \frac{p}{q} c^{\frac{1}{q}} < \frac{p}{q}$

Conclusion 1. $(1 + a)^r \geqslant 1 + ar$ is satisfied for $a > -1$ and $\mathbb{Q} \ni r > 1$

Part 2. $\mathbf{s} \in \mathbb{I}$. Then we can generalize Conclusion 1 by using Dedekind cuts.

Conclusion. $(1+a)^r \geqslant 1 + ar$ is satisfied for $a > -1$ and $\mathbb{R} \ni r > 1$

$\square$

Let a Set X of 2003 points in the plane, and a unit circle be given. Prove that there is a point on the unit circle such that the sum of the distances from it to the 2003 points is at least 2003.

Solution

With vectors. Let the 2003 points be $A_1, A_2, \ldots, A_{2003}$. Choose a point A on the circle at let B be the antipodal point of the circle. Then: $4006 = 2003|AB| = |(AA_1 + A_1B) + (AA_2 + A_2B) + \ldots + (AA_{2003} + A_{2003}B)| \leq (|AA_1| + \ldots + |AA_{2003}|) + (|BA_1| + \ldots + |BA_{2003}|)$. Thus, the sum of (the sum of the distances from A to the members of X) and (the sum of the distances from B to the members of X) is 4006, so at least one summand is 2003, and we're done. This in fact shows that "most of" the circle must have the desired property. It also shows that the restriction to the plane was arbitrary: this works for k points with an m-dimensional sphere in n-dimensional space for any k, $m \leq n$. Although generalizing the dimension of the sphere upwards isn't very interesting (since the proof relied only on a sphere in 1-D space, that is two points, and every lower-dimensional sphere is contained in every higher-dimensional sphere).

$\square$

Let $x,\ y,\ z$ be positive integers that are coprime each other such that $\frac{1}{x} + \frac{1}{y} = \frac{2}{z}$. If z is a odd number, prove that xyz is a square number.

Solution

Are you sure that the numbers are "coprime each other" instead of "coprime"? The problem becomes much easier.

The former case $\frac{1}{x} + \frac{1}{y} = \frac{2}{z} \Rightarrow z(x+y) = 2xy \Rightarrow z|2xy$ Since x, y, z are pairwise relatively prime, it follows that $z|2$. So $z = 1$. Now $x + y = 2xy \Rightarrow (2x-1)(2y-1)) = 1$ So $x = y = 1$. Therefore $xyz = 1$.

The latter case $z(x+y) = 2xy$ Let $x = dx_0, y = dy_0$ where $gcd(x_0, y_0) = 1$. We also have $gcd(d, z) = 1$. Substituting above, $z(x_0 + y_0) = 2dx_0y_0$ Since $gcd(x_0, y_0) = 1$, x_0 and y_0 cannot divide $x_0 + y_0$, and so $x_0y_0|z$ Let $z = kx_0y_0$, where k is odd. Substituting $k(x_0 + y_0) = 2d \Rightarrow k|2d \Rightarrow k|d$ But $k|z$, so $k = 1$. Therefore $xyz = (dx_0)(dy_0)(x_0y_0) = (dx_0y_0)^2$.

$\square$

Find all function in $\mathbb{R}$ wish satisfay $yf(x) - xf(2y) = 8xy(x^2 - y^2)$.

Solution

$$\begin{aligned} yf(x) - xf(2y) &= 8xy(x^2 - y^2) = -(8yx(y^2 - x^2)) = -(xf(y) - yf(2x)) \\ &= yf(2x) - xf(y). \end{aligned}$$

Therefore, $-y\left(f(2x) - f(x)\right) = x\left(f(2y) - f(y)\right)$. Let $g(x) = f(2x) - f(x)$ for all real number x. Then $-yg(x) = xg(y)$. Letting $x = y$, we get $2xg(x) = 0$, so $g(x) = 0$ (because equality holds for all real numbers), i.e. $f(2x) = f(x)$ for all reals.

Our initial equation becomes $yf(x) - xf(y) = 8xy(x^2 - y^2)$. Now let $y = 2x$ and get $xf(x) = 2xf(x) - xf(2x) = 16x^2(x^2 - 4x^2) = -48x^4$, i.e. $f(x) = -48x^3$. Let $y = 4x$ and get $3xf(x) = 4xf(x) - xf(4x) = 32x^2(x^2 - 16x^2) = -15 \cdot 32x^4$, so $f(x) = -160x^3$. Contradiction.

$\square$

Polyhedron $ABCDEFG$ has six faces. Face $ABCD$ is a square with $AB = 12$; face $ABFG$ is a trapezoid with $\overline{AB}$ parallel to $\overline{GF}$, $BF = AG = 8$, and $GF = 6$; and face CDE has $CE = DE = 14$.

The other three faces are $ADEG, BCEF$, and EFG. The distance from E to face $ABCD$ is 12. Given that $EG^2 = p - q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$.

Solution

Set up $ABCDEFG$ in a 3-D coordinate system. Define $A(0,0,0)$, $B(12,0,0)$, $C(12,12,0)$, and $D(0,12,0)$. Let H be the midpoint of $\overline{CD}$, and let J be the point on plane $ABCD$ such that $EJ = 12$. H is therefore $H(6,12,0)$. Because triangle CDE is isosceles, it can be determined that $EH = \sqrt{EC^2 - CH^2} = \sqrt{196 - 36} = \sqrt{160}$. Triangle EJH is a right triangle with hypotenuse $\overline{EH}$; therefore, $JH = \sqrt{EH^2 - EJ^2} = \sqrt{160 - 144} = 4$. Because the figure is symmetric with respect to the plane $x = 6$, we can assume that the x-coordinate of point E is 6. Furthermore, since the distance from J to $\overline{CD}$ is 4, the distance from J to $\overline{AB}$ is 8, and the y-coordinate of point E is 8. Since $\overline{EJ}$ is perpendicular to plane $ABCD$, the z-coordinate of point E is 12.

We have now determined that $A = (0,0,0)$, $D = (0,12,0)$, and $E = (6,8,12)$. However, it is given that ADEG is a plane. For A, D, E, and G to lie in the same plane, the triple product of vectors AD, AE, and AG must be 0 (as in, the volume of the parallelopiped formed by those vectors is 0). Let $G = (a,b,c)$. Therefore,

$$\begin{aligned}
AG \cdot (AD \times AE) &= 0 \\
< a,b,c > \cdot(< 0,12,0 > \times < 6,8,12 >) &= 0 \\
< a,b,c > \cdot < (12)(12) - (0)(8), (0)(6) - (0)(12), (0)(12) - (12)(6) > &= 0 \\
< a,b,c > \cdot < 144, 0, -72 > &= 0 \\
144a - 72c &= 0 \\
144a &= 72c \\
2a &= c
\end{aligned}$$

Furthermore, since $FG = 6$ and the polyhedron is symmetrical with respect to the plane x = 6, the x-coordinate of F must be 9 and the x-coordinate of G must be 3. Therefore, $a = 3$ and $c = 2(3) = 6$. It is given that $AG = 8$, so, applying the distance formula: $3^2 + y^2 + 6^2 = 8^2 \Rightarrow y = \sqrt{19}$. Hence, $G = (3, \sqrt{19}, 6)$ and $E = (6,8,12)$.

Applying the distance formula one last time,

$$\begin{aligned}
EG^2 &= (6-3)^2 + (8 - \sqrt{19})^2 + (12-6)^2 \\
&= 9 + 8^2 - 2(8)(\sqrt{19}) + (\sqrt{19})^2 + 36 \\
&= 128 - 16\sqrt{19}
\end{aligned}$$

.

And so $p + q + r = 128 + 16 + 19 = \boxed{163}$

$\square$

Let f be a function from the set of non-negative integers into itself such that for all $n \geq 0$ we have that

$$(f(2n+1))^2 - (f(2n))^2 = 6f(n) + 1 \quad \text{and} \quad f(2n) \geq f(n).$$

How many numbers less than 2003 are there in the image of f?

Solution

By the functional equation

(1) $[f(2n+1)]^2 \ - \ [f(2n)]^2 \ = \ 6f(n) \ + \ 1$

we have

(2) $f(2n+1) \ > \ f(2n)$.

Furthermore

$[f(2n+1)]^2 \ = \ [f(2n)]^2 \ + \ 6f(n) \ + \ 1 \ \leq \ [f(2n)]^2 \ + \ 6f(2n) \ + \ 1 \ < \ [f(2n) \ + \ 3]^2$

since $f(2n) \geq f(n)$. Combining (1) and (2) we obtain $f(2n+1) = f(2n) + k$ where $k \in \{1, 2\}$. If $k = 2$, then

$4\,[f(2n) \ + \ 1] \ = \ 6f(n) \ + \ 1,$

hence $2|1$. This contradiction gives $k = 1$. Thus $f(2n+1) = f(2n) + 1$, which implies that

(3) $f(2n) = 3f(n)$

according to (1). Next assume $f(k) > f(k-1)(\cdot)$ for alle $k \leq m$. We observe that by (2) the induction hypothesis $(\cdot)$ is true for $m = 1$ and $f(m+1) > f(m)$ when m is even. When m is odd, i.e. $m = 2i - 1$, then

$$\begin{aligned}
f(m+1) \ - \ f(m) \ &= \ f(2i) \ - \ f(2i-1) \\
&= \ 3f(i) \ - \ [3f(i-1) \ + \ 1] \\
&= \ 3[f(i) - f(i-1)] - 1 \\
&\geq \ 3 \ - \ 1 \ = \ 2 \ > \ 0.
\end{aligned}$$

since $i \ = \ \frac{m+1}{2} \ \leq \ m$ for all $m \ > \ 0$. This induction step shows that f is a strictly increasing function. The implication of this is that the number of non-negative integers less than 2003 which are in the image of f, is given by the unique number N which satisfies the inequalities

(4) $f(N-1) \ < \ 2003 \ \leq f(N)$.

By setting $n = 0$ in (3), we get $f(0) = 0$. Therefore $f(1) = f(0) + 1 = 0 + 1 = 1$. Using induction on formula (3), we find that

$f(2^k) \ = \ 3^k f(1) \ = \ 3^k.$

Setting $k = 7$, the result is

$f(2^7) \ = \ 3^7 \ = \ 2187.$

Moreover, setting $m = 2^k - 1$ in the formula $f(2m+1) = f(2m) + 1 = 3f(m) + 1$, the result is $f(2^k - 1) = 3f(2^{k-1} - 1) + 1$. Consequently

$$\begin{aligned}
f(2^7 - 1) &= 3\,f(2^6-1) + 1 \\
&= 3\,[3f(2^5-1) + 1] + 1 \\
&= 3^2\,f(2^5-1) + 4 \\
&= 3\,[3^2f(2^4-1) + 4] + 1 \\
&= 3^3\,f(2^4-1) + 13 \\
&= 3\,[3^3f(2^3-1) + 13] + 1 \\
&= 3^4\,f(2^3-1) + 40 \\
&= 3\,[3^4f(2^2-1) + 40] + 1 \\
&= 3^5\,f(2^2-1) + 121 \\
&= 3\,[3^5\,f(2^1-1) + 121] + 1 \\
&= 3^6\,f(1) + 364 \\
&= 729\cdot 1 + 364 \\
&= 1093
\end{aligned}$$

Hence $1093 = f(2^7-1) < 2003 \leq f(2^7) = 2187$, thus by (4)
$N = 2^7 = 128$.

$\square$

Let a,b,c,d is the reals number satisfying that$a^2+b^2=1$ and $\frac{a^4}{b}+\frac{c^4}{d}=\frac{1}{b+d}$ Prove that $\frac{a^{2004}}{b^{1002}}+\frac{c^{2004}}{d^{1002}}=\frac{2}{(b+d)^{1002}}$

Solution

Replace 2004 with $2k$ Prove: $\frac{a^{2k}}{b^k}+\frac{c^{2k}}{d^k}=\frac{2}{(b+d)^k}$ Are you sure it's not $a^2+c^2=1$? Because then by Cauchy we have: $\frac{a^4}{b}+\frac{c^4}{d}\geq\frac{(a^2+c^2)^2}{b+d}$ But if $a^2+c^2=1$ then we have equality; so $b:a^2=c:d^2$ Notice $\sqrt[k]{\left(\frac{a^2}{b}\right)^k+\left(\frac{c^2}{d}\right)^k}\geq 2^{\frac{1}{k}-1}\left(\frac{a^2}{b}+\frac{c^2}{d}\right)$ from AM-GM (actually powermean or generalization whatever its called) but in fact, we have equality (we established $b:a^2=c:d^2$ from Cauchy) and that is the equality condition we need for powermean So $LHS=\left(2^{\frac{1}{k}-1}\left(\frac{a^2}{b}+\frac{c^2}{d}\right)\right)^k=2\frac{\left(\frac{a^2}{b}+\frac{c^2}{d}\right)^k}{2^k}\geq 2\frac{\left(2\frac{a^2+c^2}{b+c}\right)^k}{2^k}=\frac{2}{(b+d)^k}$ and I'm pretty sure we have equality in the last inequality (i don't know too much about the last inequality but I've heard about it) so $\frac{a^{2k}}{b^k}+\frac{c^{2k}}{d^k}=\frac{2}{(b+d)^k}$ as desired

$\square$

Find the positive numbers n such that n^4 is the multiple of $3n+7$.

Solution

If $3n+7|n^4$, then $3n+7|(3n)^4$, so $0\equiv(3n)^4\equiv(-7)^4 \pmod{3n+7}$. Thus we really should have that $3n+7|7^4$. Now, n is a positive integer, so $3n+7\geq 3\cdot 1+7=10$, so only $3n+7\in\{7^2,7^3,7^4\}$ is possible, i.e. $n\in\{14,112,798\}$. For all those numbers, it is easily checked that $3n+7|n^4$.

Another way $3n+7|3n^4$ and $3n+7|3n^4+7n^3$
$\implies 3n+7|7n^3 \implies 3n+7|21n^3$ but $3n+7|21n^3+49n^2$
$\implies 3n+7|49n^2 \implies 3n+7|147n^2$ but $3n+7|147n^2+343n$
$\implies 3n+7|343n \implies 3n+7|1029n$ but $3n+7|1029n+7^4$
$\implies 3n+7|7^4$

$\square$

If $\gcd(m,n)=1$, then by the Euclidean Algorithm we can find integers such that $nx=my+1$. So pick the least number, and choose the first n, then shift over one spot and do the same thing, etc. until you've done it x times, at which point you've increased that first number by 1 more than everything else.

Solution

For the necessary part, suppose that $\gcd(m, n) > 1$, and suppose one of our initial m integers is one less than the rest. Assume for the sake of contradiction that there is a sequence of moves that increases the smaller integer by $k+1$ and the rest of the integers by k. Then all the integers increase by a total of $k+1+k(m-1) = km+1$. Since each move increases the sum by n, we have $km+1 = jn$, and $\gcd(m, n) = 1$, a contradiction.

$\square$

Let A be a set. Prove that there is no onto function $f : A \to P(A)$

Solution

Consider $A_F = \{x \in A \mid x \notin f(x)\} \subset P(A)$. Assume that $f : A \to P(A)$ is onto. This means that there exists $y \in A$ such that $f(y) = A_F$. If $y \in A_F$, then $y \notin f(y) = A_F$, so $y \notin A_F$. But by the definition of A_F, $y \notin A_F \implies y \in f(y) = A_F$, which is a contradiction. Therefore there doesn't exist such a function.

Also, it should be clear that an onto function doesn't exist when A is finite, as $|A| = n$ and $|P(A)| = 2^n$. The proof above is needed when dealing with $|A| = \infty$. Since we have shown that an infinite set has a smaller size than its power set, we have shown that there are different sizes of infinity.

$\square$

Let $a_1, a_2, ...a_n$ be a permutation of the set $S_n = 1, 2, 3...n$. An element i in S_n is called a fixed point of this permutation if $a_i = i$. Let g_n be the number of derangements of S_n. Let f_n be the number of permutations of S_n with exactly one fixed point. Show that $|f_n - g_n| = 1$.

Solution

f_n=$n(g_{n-1})$ This can be seen if one tries to visualize it. (There's n ways to make one fixed point and since everything else is different, it's g_{n-1}). g_n=$(n-1)(g_{n-1}+g_{n-2})$. I will prove this later. $n(g_{n-1}) - (n-1)(g_{n-1}+g_{n-2}) = g_{n-1} - (n-1)(g_{n-2}) = g_{n-1} - f_{n-1}$ Since these are in absolute value signs, the multiplication by -1 won't matter. $|f_n - g_n| = |f_{n-1} - g_{n-1}| = ... = |f_1 - g_1| = 1$ Now, the proof that I held off. Recall the other formula for derangement, $n!/0!-n!/1!+n!/2!-...+\frac{(-1)^n n!}{n!}$. When one applies this formula to the three gs and multiplies through, $n!/0! - n!/1! + n!/2! - ... + \frac{(-1)^n n!}{n!} = (n-1)(n-1)!/0! - (n-1)(n-1)!/1! + (n-1)(n-1)!/2! - ... + \frac{(-1)^{n-1}(n-1)(n-1)!}{(n-1)!} + (n-1)!/0! - (n-1)!/1! + (n-1)!/2! - ... + \frac{(-1)^{n-2}(n-1)!}{(n-2)!}$. When all the equal terms on the left and right sides are subtracted off, $\frac{(-1)^{n-1}(n)!}{(n-1)!} + \frac{(-1)^n n!}{n!} = \frac{(-1)^{n-1}(n-1)(n-1)!}{(n-1)!}$ or $n-1 = n-1$ or $1-n = 1-n$.

$\square$

Find all integer values of a such that the quadratic expression $(x+a)(x+1991)+1$ can be factored as a product $(x+b)(x+c)$ where b, c are integers.

Solution

We have $b+c = a+1991$ and $bc = 1991a+1$. Let $b = a+k$, where k is an integer. Then $c = 1991-k$ from the first equation. Also, we have $(a+k)(1991-k) = 1991a+1$ from the second equation. Expanding and simplifying, we obtain $(1991-a)k - k^2 = 1$. Then $k(1991-a-k) = 1$. Therefore, $k = \pm 1$. From $k = 1$ we get $a = 1989$, and from $k = -1$, we get $a = 1993$. These are the only solutions.

$\square$

Twelve people are seated around a circular table. In how many ways can six pairs of people engage in handshakes so that no arms cross?

(Nobody is allowed to shake hands with more than one person at once.)

Solution

Clearly, there must be an even number of people around a table in order for each person to shake hands with someone else. So let $t(n)$ be a table with n pairs of people.

We can quickly write out the cases for smaller tables, and see that $t(1) = 1$, $t(2) = 2$.

To find the number of cases for larger tables, we can use recursion.

For a table with 6 people, label them in order A, B, C, D, E, F. We can't have A shake hands with a person an even number of seats away, or else the table would be divided into two sections with an odd number of people in them. So A can only shake hands with B, D, F.

If A shakes hands with B or F, the case breaks down into the 4 person case; if A shakes hands with D, it breaks down into two 2 two person cases.

The total number of possible arrangements is equal to

$$2t(2) + t(1)t(1)$$

So $t(3) = 2 \times 2 + 1 = 5$.

We can solve for $t(4)$ and $t(5)$ in the same way; $t(4) = 14$, $t(5) = 42$.

For 12 people

$$t(6) = t(5) + t(1)t(4) + t(2)t(3) + t(3)t(2) + t(4)t(1) + t(5)$$

Plugging in the corresponding values, we get $t(6) = 132$.

$\square$

Let $x_i > 0$ and $\sum_{i=1}^{2007} x_i = a, \sum_{i=1}^{2007} x_i^3 = a^2, \sum_{i=1}^{2007} x_i^5 = a^3$. Find a.

Solution

Use $\sum = \sum_{i=0}^{2007}$. By Cauchy

$$a^4 = \left(\sum x_i\right)\left(\sum x_i^5\right) \geq \left(\sum x_i^3\right)^2 = a^4,$$

so equality occurs, so $\frac{\sqrt{a_i}}{\sqrt{a_i^5}} = \frac{1}{a_i^2}$ is constant, i.e. all a_i are equal.

$\square$

For every positive integer k let $a(k)$ be the largest integer such that $2^{a(k)}$ divides k. For every positive integer n, determine $a(1) + a(2) + a(3) + ... + a(2^n)$.

Solution

$\sum_{i=1}^{2^n} a(i)$ is simply the sum of all the factors of 2 in all integers $\leq 2^n$. Thus $\sum_{i=1}^{2^n} a(i)$ =(number of integers $\leq 2^n$ which are divisable by 2^1) + (number of integers $\leq 2^n$ which are divisable by 2^2) + ... + (number of integers $\leq 2^n$ which are divisable by 2^n) $= \frac{2^n}{2^1} + \frac{2^n}{2^2} + ... + \frac{2^n}{2^n} = 2^{n-1} + ... + 2^0 = \frac{2^n-1}{2-1}$ (because $(a-1)(a^{n-1} + a^{n-2} + ... + a^1 + 1) = a^n - 1$)

$= 2^n - 1$

$\square$

Let S be a set of rational numbers with the following properties:

$\frac{1}{2} \in S$

If $x \in S$, then both $\frac{1}{x} \in S$ and $\frac{x}{x+1} \in S$

Prove that S contains all rational numbers in the interval $0 < x < 1$

Solution

We will prove the result by strong induction on the denominator of the fractions. Assume $p < q$.

Base case: $\frac{p}{q}$ with $q=2$. $\frac{1}{2} \in S$ so we're good.

Other useful case: We want to show all integers $n \geq 2$ are also in S. We can simply do $\frac{1}{m} \to \frac{\frac{1}{m}}{\frac{1}{m}+1} = \frac{1}{m+1}$ so $\frac{1}{2} \to \frac{1}{3} \to \cdots$ and $\frac{1}{m} \to m$ so we have all the integers.

Induction step: Suppose we know that all fractions $\frac{p}{q}$ with $2 \leq q \leq k$ are in S as well as all integers $n \geq 2$. We want to show $\frac{p'}{q'} \in S$ with $q' = k+1$ and any $p' < q'$. But we see that we can get $\frac{p'}{q'}$ as long as we have

$\frac{p'}{q'-p'} \to \frac{\frac{p'}{q'-p'}}{\frac{p'}{q'-p'}+1} = \frac{p'}{q'}$.

But $p' \geq 1$ so $q' - p' \leq k$, which means by our strong induction $\frac{p'}{q'-p'} \in S$. Hence we have shown that all $\frac{p'}{q'} \in S$, completing the induction. Another way Let $Q(x) = \frac{x}{x+1}$ Let $P(x) + \frac{1}{x}$ Let $F^n(x)$ refer to the function $F(x)$ applied n times. We have that $Q(P(x)) = \frac{1}{1+x}$. By induction then $Q^n(P(x)) = \frac{1}{n+x}$.

Now we just need to show that it is possible to get $2 \ldots n+x-1 = k$ in the numerator where $x=2$. This is clearly possible as we just go back to $1/(n+2-k)$ and apply $Q(x)$. Since k is at most $n+1$ it is always possible to apply this operation and we're done.

□

Find all triples of positive integers (p, q, n) with p and q primes satisfying:

$p(p+3) + q(q+3) = n(n+3)$.

Solution

Rearrange to get $p^2 + q^2 - n^2 = 3(n-p-q)$

Some case checking.

$1^\circ : p, q > 3$.

Then, $p^2 + q^2 \equiv 2 \pmod 3$ and $n^2 \equiv 0, 1 \pmod 3 \Rightarrow LHS \equiv 1, 2 \pmod 3$ whereas $RHS \equiv 0 \pmod 3$. So no solutions.

$2^\circ : p = q = 2$.

Then, $n^2 + 3n - 20 = 0 \Rightarrow$ discriminant is not a perfect square. So, no solutions again.

$3^\circ : p = q = 3$.

Then, $n^2 + 3n - 36 = 0 \Rightarrow$ discriminant is not a perfect square. So, no solutions again.

$4^\circ : p = 2, q = 3$.

Then, $n^2 + 3n - 28 = 0 \Rightarrow (n+7)(n-4) = 0$.

So all possible solutions are $(2, 3, 4)$ and $(3, 2, 4)$.

□

Is this equation where x,y are integers solvable: $8y^2 - x^2 = 7$

Solution

Let $z = 2y$. We are trying to find solutions to

$x^2 - 2z^2 = -7$

Such that z is even (ignore this condition until the end). We can reconsider $\in \mathbb{Z}[\sqrt{2}]$:

$(x + z\sqrt{2})(x - z\sqrt{2}) = -7$

We found a base solution and now we want to find the general one, which we do by finding a small unit $\in \mathbb{Z}[\sqrt{2}]$.

As it so happens,

$(3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 1$

So our solutions are given by the coefficients of

$x_k + z_k\sqrt{2} = (x + z\sqrt{2})(3 + 2\sqrt{2})^k, k \in \mathbb{Z}$

Note that

$(x_k + 2y_k\sqrt{2})(3+2\sqrt{2}) = (3x_k + 8y_k) + (6y_k + 2x_k)\sqrt{2}$

So z_k will always be even; hence y_k is an integer.

$\square$

It says at a school $90\%take maths 85\%$take Science 80%take English Then at least how many(

Solution

First, for the sake of clarity, draw a Venn diagram containing three intersecting circles and label these Math, Science and English.

Now, let $a =$ proportion of students taking all three subjects, $b =$ proportion of students taking exactly Math and English, $c =$ proportion of students taking exactly Math and Science, and $d =$ proportion of students taking exactly Science and English.

So, proportion of students taking only Math $= 0.9 - (a+b+c)$, proportion of students taking only Science $= 0.85 - (a+c+d)$, and, proportion of students taking only English $= 0.8 - (a+b+d)$.

Now, we must have $a+b+c+d+(0.9-(a+b+c))+(0.85-(a+c+d))+(0.8-(a+b+d)) = 1$ $\Rightarrow a = \frac{1.55-(b+c+d)}{2}$

But note that $b+c+d \le 1$. So, $a \ge \frac{1.55-1}{2} = 27.5\%$, which is our answer.

$\square$

if ;$a > b > 0$:and $A{=}\frac{1+a+\cdots+a^{2005}}{1+a+\cdots+a^{2006}}$ $B{=}\frac{1+b+\cdots+b^{2005}}{1+b+\cdots+b^{2006}}$ campar,$A$and$B$.

Solution

Let $p = 1+a+a^2+\cdots+a^{2005}$ and $q = 1+b+b^2+\cdots+b^{2005}$.

Then, $A = \frac{1}{1+\frac{a^{2006}}{p}}$ and $B = \frac{1}{1+\frac{b^{2006}}{q}}$.

$B > A \Leftrightarrow \frac{1}{1+\frac{b^{2006}}{q}} > \frac{1}{1+\frac{a^{2006}}{p}} \Leftrightarrow \frac{a^{2006}}{p} > \frac{b^{2006}}{q} \Leftrightarrow a^{2006}(1+b+b^2+\cdots+b^{2005}) > b^{2006}(1+a+a^2+\cdots+a^{2005}) \Leftrightarrow \sum_{i=0}^{2005}(ab)^i(a^{2006-i}-b^{2006-i}) > 0$

The last inequality is true, and we are done.

$\square$

x and y are nonnegative integers. Prove that the equation $14x^2+15y^2 = 7^{1990}$ has no solutions.

Solution

assume on the contrary that there is a solution, and let it be (x_1, y_1). Then since $14x_1^2$ and 7^{1990} are both divisible by 7, $15y_1^2$ must also be divisible by 7. So $y_1 = 7y_2$. Substituting and dividing both sides by 7 we obtain $2x_1^2 + 105y_2^2 = 7^{1889}$. Similarly, x_1 must be divisible by 7 so $x_1 = 7x_2$, and again we get $14x_2^2+15y_2^2 = 7^{1888}$. Continuing we see we'll arrive at $2x_n^2+105y_n^2 = 7$, which has no solutions, contradiction. Another way this is easy with modulo 3 this is equal to $2x^2 = 1 \pmod 3$ which has no solution

$\square$

Let a be a real number such that $|a| > 1$. Solve the system of equations:

$x_1^2 = ax_2 + 1$

$x_2^2 = ax_3 + 1$

$\dots$

$x_{999}^2 = ax_{1000} + 1$

$x_{1000}^2 = ax_1 + 1$

Solution

Case 1: Suppose $a > 1$ and, by way of contradiction, not all the x_i are equal. Then all x_i must be positive since perfect squares are non-negative. Moreover, there exists an index i such that

$$x_i > x_{i+1},$$

given $x_{999+n} = x_n$. Since $a > 0$,

$$ax_i + 1 > ax_{i+1} + 1 \qquad \Leftrightarrow \qquad x_{i-1}^2 > x_i.$$

Recall that all x_i are positive, so then

$$x_{i-1} > x_i.$$

By induction, this relationship holds for all i. But then

$$x_1 > x_2 > \ldots > x_{1000} > x_{1001} = x_1,$$

which is a contradiction. Hence, all x_i must be equal. From the quadratic formula,

$$\boxed{x_1 = \frac{a \pm \sqrt{a^2+4}}{2}.}$$

Case 2: Suppose $a < 1$ and again, by way of contradiction, that not all x_i are equal. Then each x_i must be negative since perfect squares are non-negative. Again, there exists an index i such that

$$x_i > x_{i+1}.$$

Since a is negative, then

$$ax_i + 1 < ax_{i+1} + 1 \qquad \Leftrightarrow \qquad x_{i-1}^2 < x_i^2.$$

Because all x_i are negative, then

$$x_{i-1} > x_i,$$

a relationship that holds for all i, through induction. Again,

$$x_1 > x_2 > \ldots > x_{1000} > x_{1001} = x_1,$$

which is a contradiction, so the initial assumption must have been false. Therefore, all x_i must be equal and from the quadratic formula,

$$\boxed{x_1 = \frac{a \pm \sqrt{a^2+4}}{2}.}$$

Overview Thus, regardless of a, the solution to the system of equations is

$$\boxed{x_1 = \frac{a \pm \sqrt{a^2+4}}{2}}$$

$$\boxed{x_1 = x_2 = \ldots = x_i = \ldots = x_{999} = x_{1000}}$$

□

Let

$$\prod_{n=1}^{1996} \left(1 + nx^{3^n}\right) = 1 + a_1 x^{k_1} + a_2 x^{k_2} + \ldots + a_m x^{k_m},$$

where $a_1, a_2, ..., a_m$ are nonzero and $k_1 < k_2 < ... < k_m$ Find a_{1996}

Solution

First, let's examine what the product looks like:

$$(1+x^3)(1+2x^9)(1+3x^{27})(1+4x^{81})\ldots = 1+x^3+2x^9+2x^{12}+3x^{27}+\ldots$$

Then

$$\begin{array}{rcl} k_1 & = & 3 \\ k_2 & = & 9 \\ k_3 & = & 12 \\ k_4 & = & 27 \\ k_5 & = & 30 \\ \vdots & \vdots & \vdots \end{array}$$

Let $(s_1 s_2 \ldots s_n)_2$ be the binary representation of n. Then

$$k_n = (s_1 s_2 \ldots s_n 0)_3,$$

through a simple induction argument. Since

$$1996 = (1111001100)_2$$

then

$$k_{1996} = (11110011000)_3,$$

$$k_{1996} = 3^{11} + 3^{10} + 3^9 + 3^8 + 3^7 + 3^4 + 3^3.$$

Therefore,

$$a_{1996}x^{k_{1996}} = (11x^{3^{11}})(10x^{3^{10}})(9x^{3^9})(8x^{3^8})(7x^{3^7})(4x^{3^4})(3x^{3^3}),$$

so then

$$a_{1996} = (11)(10)(9)(8)(7)(4)(3) = 665280.$$

□

if each$x_1, x_2, ..., x_n = +1 \vee -1$and we have this:$x_1x_2x_3x_4 + x_2x_3x_4x_5 + ... + x_{n-1}x_nx_1x_2 + x_nx_1x_2x_3 = 0$then prove it $4 \mid n$

Solution

Let $P = x_1x_2x_3x_4 + x_2x_3x_4x_5 + ... + x_nx_1x_2x_3 = 0$. Denote a replacement of $p \to q$ by $p \circ q$.

$x_i \circ -x_i$ does not change P mod 4 since 4 terms in P change their sign. If three of the four selected terms have the same sign, then a replacement $x_i \circ -x_i$ changes P by ± 4. If two of these four selected terms are >0 and the other two <0, then a replacement does not change P. If all four have the same sign, then P changes by ± 8. Initially, $P \equiv 0 \pmod 4$, thus $P \equiv \pm 4 \equiv \pm 8 \equiv 0 \pmod 4$ remains invaraint. Since a move does not change the congruency modulo 4, after a finite number of steps, $P = n \equiv 0 \pmod 4$.

□ Find all positive integer solutions to $abc - 2 = a + b + c$

Solution

WOLOG $a \geq b \geq c$

first observe that if we increase a variable (let's say a) by 1, then we increase the left by a value of bc and increase the right by 1. Since all variables are positive integers, $bc \geq 1$ with equality iff

$b = c = 1$. Therefore, in general, increasing a variable affects the left more than the right, so There's an upperbound for a (and therefore an upperbound for b and c).

Lemma: $c \le 2$.

If $c = 3$, then $a \ge b \ge 3$. The minimal solution (here minimal means the minimal value of the left subtract the right) we can have is $a = b = c = 3$ (because from above increasing a or b increase the gap between the left and the right). So plug in this minimal solution we have $25 > 9$, so $c = 3$ or above have no solutions.

case 1:$c = 2$. plug in the minimal solution in this case (which is $a = b = c = 2$), we have $6 = 6$, a solution. Since increasing any variable increase the gap between the left and the right, this is the only solution in this case.

finally, $c = 1$. we have $ab - 2 = a + b$, using Simon's favorite trick we obtain $(a-1)(b-1) = 3$, so another solution is $(4, 2, 1)$.

Now, removing the ordering imposed upon the variables, we have solutions $(2, 2, 2)$ and all cyclic permutations of $(4, 2, 1)$. QED.

$\square$

Show (and if you can, find) that there exists exactly one positive integer n such that $2^8 + 2^11 + 2^n$ is a perfect square.

Solution

We have that $2^n = p^2 - 2^8 - 2^{11} \implies 2^n = (p-48)(p+48)$. Let $n = u+v$. Then By unique factorization, $2^u = p - 48$ and $2^v = p + 48$. Subtracting the first from the second, we have $2^u - 2^v = 96 = 2^5 \cdot 3 \implies 2^v(2^{u-v} - 1) = 2^5 \cdot 3$. By unique factorization, $2^v = 2^5 \implies v = 5$ and $2^{u-5} - 1 = 3 \implies u = 7$. Thus $n = 7 + 5 = 12$. Q.E.D.

$\square$

Points M and N are given on sides AD and BC of a rhombus ABCD. Line MC meets the segment BD at T and line MN meets the segment BD at U. Line CU intersects the side AB at Q and the line QT intersects the side CD at P.

Show that $\triangle QCP$ and $\triangle MCN$ have equal area.

Solution

$\frac{\triangle POC}{\triangle TQC} = \frac{PQ}{TQ} \Longrightarrow \triangle PQC = \frac{PQ}{TQ} \triangle TQC$

Since AD is parallel to BC, so $\frac{PQ}{TQ} = \frac{BD}{BT}$, thus we must have:$\triangle PQC = \frac{BD}{BT} \triangle TQC$.

Similarly we have:

$\frac{\triangle TQC}{\triangle TUC} = \frac{QC}{UC} \Longrightarrow \triangle TQC = \frac{BD}{UD} \triangle TUC$

$\frac{\triangle TUC}{\triangle MUC} = \frac{TC}{MC} \Longrightarrow \triangle TUC = \frac{BT}{BD} \triangle MUC$

$\frac{\triangle MUC}{\triangle MNC} = \frac{MU}{MN} \Longrightarrow \triangle MUC = \frac{DU}{DB} \triangle MNC$

Pluge all the thing into our original equality, we shall have:

$\triangle PQC = \frac{BD}{BT} \cdot \frac{BD}{DU} \cdot \frac{BT}{BD} \cdot \frac{DU}{BD} \cdot \triangle MNC = \triangle MNC$ $\square$

Let ABC be an equilateral triangle, and P be an arbitrary point within the triangle. Perpendiculars PD, PE, PF are drawn to the three sides of the triangle. Show that, no matter where P is chosen,

$$\frac{PD + PE + PF}{AB + BC + CA} = \frac{1}{2\sqrt{3}}.$$

Solution

Draw ΔABC with point P in the centre. Draw perpendiculars to each of the three sides with D on

AB, E on BC, and F on AC. Draw AP, BP, CP. We find the area of ABC in terms of the three triangles ABP, BPC, APC. We have:

$[ABC] = \frac{1}{2}(AB \cdot DP + AC \cdot PF + BC \cdot EP)$

But $[ABC] = AB \cdot AB \cdot \sin 60$. So, we have:

$AB \cdot DP + AC \cdot PF + BC \cdot EP = AB \cdot AB \cdot \sin 60$

Let $AB = AC = BC = s$.

$DP + PF + DE = s \cdot \frac{\sqrt{3}}{2}$

Dividing by $3s$,

$\frac{DP+PF+DE}{AB+BC+CA} = \frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$ $\square$

Let a and b be positive integers such that $a|b^2, b^2|a^3, a^3|b^4, b^4|a^5, \ldots$. Prove that $a = b$.

Solution

Clearly, a and b have the same prime divisors, say $p_1, p_2, \cdots p_n$. Let $a = \prod_{i=1}^{n} p_i^{\alpha_i}$ and $b = \prod_{i=1}^{n} p_i^{\beta_i}$. It's given that $a^{2k-1}|b^{2k}$ and $b^{2k}|a^{2k+1}$ for all positive integers k, which is equivalent with $(2k-1)\alpha_i \leq 2k\beta_i$ and $2k\beta_i \leq (2k+1)\alpha_i$ for all $i \in \{1, 2, \cdots, n\}$. Therefore, $1 - \frac{1}{2k} \leq \frac{\beta_i}{\alpha_i} \leq 1 + \frac{1}{2k}$. Let $k \to \infty$ to get that $1 \leq \frac{\beta_i}{\alpha_i} \leq 1$, so $\alpha_i = \beta_i$, which proves the statement.

Another approach:Without UPF

$n|m \implies n \leq m$

So we have

$b^{2k} < a^{2k-1}$ $a^{2k} < b^{2k+1}$

Let $r = \frac{a}{b}$. Then

$r^{2k} < a \forall k$ $r^{2k} > \frac{1}{b} \forall k$

$\lim_{k\to\infty} r^{2k} = 0, 1, \infty$. Clearly only 1 works, which implies $r = 1$. QED. $\square$

Medians divide a triangle into 6 smaller ones. 6 circles are inscribed in the smaller triangles, 4 of which are equal. Prove that the triangle is equilateral.

Solution

Let the triangle be $\triangle ABC$ with medians AR, CQ, BS which are concurrent at P. We start by examining $\triangle APQ$ and $\triangle PBQ$ which have equal areas since their bases are equal ($AQ = QB$) and they share the same altitude. We are given that atleast four circles are equal, so we choose two of them by letting the incircles of $\triangle APQ$ and $\triangle BPQ$ be equal. Using the formula $rs = A$, because the radii of the incircles are the same and the two triangles have the same area, their perimeters must be the same, implying that $AP = PB$. This implies that PQ is the altitude of both triangles, and thus CQ is the altitude of $\triangle ABC$, making $\triangle ABC$ iscoceles with $CB = CA$. Thus the medians AR and BS are equal. We now use the two "remaining" circles on $\triangle PCS$ and $\triangle PSA$ and find that PS is the altitude of $\triangle CPA$ and thus BS is the altitude of $\triangle ABC$ as well. Thus, BS, CQ, AR are all three altitudes and medians to $\triangle ABC$ implying that $\triangle ABC$ is equilateral. QED.

$\square$

A bus ticket has six digits on it. It's considered to be lucky if the sum of the first three digits equals to the sum of the last three. Prove that the sum of all the lucky numbers is divisible by 13.

Solution

Let R_k be the set of all 3 digit numbers (include leading 0) that sum to k. Essentially, we require that

$\sum_k \sum_{x \in R_k, y \in R_k} 1000x + y \equiv 0 \mod 13$

$\Leftrightarrow \sum_k \sum_{x \in R_k} 1001|R_k|x \equiv 0 \mod 13$

$\Leftrightarrow 13 \cdot \left(77 \cdot \sum_k \sum_{x\in R_k} |R_k| x\right) \equiv 0 \mod 13$
where the result follows. $\square$

Of the first 100 natural numbers, 'k' numbers are randomly chosen, if the sum of the 'k' numbers is even A wins, if it is odd 'B' wins, find the values of 'k' for which the game is fair.

Solution

The game is always fair when k is odd.

Pair the number x with $100 - x$. If we chose the k numbers $x_1, \dots x_k$, then modulo 2, $\sum_k x_i \equiv -\sum_k 100 - x_i$, so by changing each number with it's pair, the player that wins changes.

However, this is not exactly correct: some choices of k numbers do not change when you change each number with it's pair. This occurs only when k is divisible by 2. When k is 0 mod 4, then we have extra (unpaired) "even" wins, and the game is biased for A. When k is 2 mod 4, we have extra (unpaired) "odd" wins, and the game is biased for B.

In total, the game is fair when k is odd.

$\square$ Without calculator, prove that

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = \frac{-1}{2}$$

Let $\omega = \exp\left(\frac{2i\pi}{5}\right)$. Then, $\sum_{k=1}^5 \omega^k = \omega \cdot \frac{\omega^5-1}{\omega-1} = 0$, so $\operatorname{Re}\left(\sum_{k=1}^5 \omega^k\right) = 0$, i.e. $\sum_{k=1}^5 \cos\left(\frac{2k\pi}{5}\right) = 0$. That means

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + \cos\left(\frac{6\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right) + \cos\left(\frac{10\pi}{5}\right) = 0,$$

i.e.

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + \cos\left(\frac{6\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right) = -1.$$

Since $\cos x = \cos(2\pi - x)$, that can be rewritten as

$$2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = -1,$$

which yields

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}.$$

$\square$ Suppose not all four integers, a, b, c, d, are equal. Start with (a, b, c, d) and repeatedly replace (a, b, c, d) by $(a-b, b-c, c-d, d-a)$. Prove that at least one number of the quadruple will eventually become arbitrarily large.

Solution

We obviously have $2(a_n^2 + b_n^2 + c_n^2 + d_n^2) + (a_n + c_n)^2 + (b_n + d_n)^2 \geq 2(a_n^2 + b_n^2 + c_n^2 + d_n^2)$.

Now since $(a_n + b_n + c_n + d_n)^2 = 0$, the equality marked (1) in the book subtracted from the above inequality gives

$2(a_n^2 + b_n^2 + c_n^2 + d_n^2) - 2a_nb_n - 2b_nc_n - 2c_nd_n - 2d_na_n \geq 2(a_n^2 + b_n^2 + c_n^2 + d_n^2)$

Finally, the first equality in the book gives

$a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2 \geq 2(a_n^2 + b_n^2 + c_n^2 + d_n^2)$.

Now, by induction we get the last inequality.

$\square$

Let q, r, p be three prime numbers.If $\frac{p}{q}+\frac{q}{r}+\frac{r}{p}$ is a natural number prove that $p=q=r$

Solution

$\frac{p^2r+pq^2+qr^2}{pqr}$ is an integer. $p^2r \equiv qr^2 \equiv 0 \mod r \implies r|pq^2 \implies r=p \vee (q^2=r^2 \implies q=r)$

case 1) $r=p \rightarrow \frac{r^2+q^2+qr}{pq}$ is integer. $qr \equiv q^2 \equiv 0 \mod q \implies q|r^2 \implies q=r \implies q=p=r$

case 2) $q=r \rightarrow \frac{p^2+pr+qr}{pq}$ is an integer. $p^2 \equiv pr \equiv 0 \mod p \implies p|qr \implies (p=q \vee p=r) \implies p=q=r$

ANother way Assume, without loss of generality, that $p \neq q$ and $p \neq r$. Since p, q, and r are prime, $(p, q)=1$ and $(p, r)=1$ Let $\frac{p}{q}+\frac{q}{r}+\frac{r}{p}=x$ Multiplying by a common denominator, we get $p^2r+r^2q+q^2p=xpqr$ Taking this mod p, we have that $r^2q \equiv 0 \bmod p$ Since r and q share no common factors with p, this is impossible because the left side cannot be divisible by p. Thus either q or r is equal to p, a contradiction. WLOG, assume $q=p$ By using this same argument on r, we find that either q or p is equal to r, which means that all three must be equal.

□ Let the incircle (with center I) of triangle ABC touch the side BC at X, and let A' be the midpoint of this side. Prove the line $A'I$ (extended) bisects AX.

Solution

It's very easy with barycentric coordinates.

Let W be the midpoint of AX and $p=\frac{BC+CA+AB}{2}=\frac{a+b+c}{2}$. The homogenous barycentric coordinates are as follows: $A'(0,1,1)$, $I(a,b,c)$, $A(1,0,0)$, $X(0,p-c,p-b)$. The normalized barycentric coordinates of X are $\left(0, \frac{p-c}{a}, \frac{p-b}{a}\right)$, so the normalized coordinates of W are $\left(\frac{1}{2}, \frac{p-c}{2a}, \frac{p-b}{2a}\right)$. Thus, W has homogenous barycentric coordinates $(a, p-c, p-b)$. We want to prove that A', I and W are collinear. That's equivalent to $\begin{vmatrix} 0 & 1 & 1 \\ a & b & c \\ a & p-c & p-b \end{vmatrix}=0 \iff \begin{vmatrix} 0 & 0 & 1 \\ a & b-c & c \\ a & (p-c)-(p-b) & p-b \end{vmatrix}=0$

$\iff \begin{vmatrix} 0 & 0 & 1 \\ a & b-c & c \\ a & b-c & p-b \end{vmatrix}=0 \iff \begin{vmatrix} a & b-c \\ a & b-c \end{vmatrix}=0 \iff a(b-c)-a(b-c)=0$, which is obviously true.

Another approach Draw B', C' on AB, AC such that $B'C'//BC$ and tangent to Incircle (I, r) at K', if K is the touch point of the excircle (I_a, r_a) at BC. So K' is the image by homothety $h=-\frac{r}{r_a}$ with center A besides $K'X$ is the diameter of (I) and A, K', K are collinears. Then if A' is the midpoint of BC also A' is midpoint of XK, hence $A'I$ is the midline in the triangle $K'XK$, now $A'I$ passes through the midpoint of AX too.

□ Prove that for any non-negative integer n then numbers

$$2^n-F_{n+3}+1=\sum_{k=3}^{m+1}(F_k-1)2^{n-k}$$

Where F_x is the x-th Fibonacci number.

Solution

I'm going to define the Fibonnaci sequence by $F_0=0, F_1=1$ and $F_{n+2}=F_{n+1}+F_n$ for $n \geq 0$ since that seems to fit the identity. We will proceed by induction.

Base case: $n=0,1,2$ - the RHS is zero, so we verify that $2^0-F_3+1=0$ since $F_3=2$, $2^1-F_4+1=0$ since $F_4=3$, and $2^2-F_5+1=0$ since $F_5=5$.

Before we move on to the induction step, notice the identity:

$F_{a+3}+F_{a+3}=F_{a+1}+F_{a+2}+F_{a+3}=F_{a+1}+F_{a+4}$

Induction step:
Suppose $2^m - F_{m+3} + 1 = \sum_{k=3}^{m}(F_k - 1)2^{m-k}$ for some $m \geq 2$. Then
$2^{m+1} - F_{m+4} + 1 = 2(2^m - F_{m+3} + 1) + (F_{m+1} - 1)$
by the identity, so

$$\begin{aligned} 2^{m+1} - F_{m+4} + 1 &= 2 \cdot \sum_{k=3}^{m}(F_k - 1)2^{m-k} + (F_{m+1} - 1) \\ &= \sum_{k=3}^{m}(F_k - 1)2^{(m+1)-k} + (F_{m+1} - 1) \cdot 2^{(m+1)-(m+1)} \\ &= \sum_{k=3}^{m+1}(F_k - 1)2^{(m+1)-k}, \end{aligned}$$

completing the induction.
□ Prove that

$$\sum_{k=0}^{n}[(n-2k)\binom{n}{k}]^2 = 2n\sum_{k=0}^{n-1}\binom{n-1}{k}^2$$

Solution

Divide by 2n and expand. Notice that if n is even, there exists some 2k where n-2k=0. If n is odd, the sum can be split into 2 equal halves since $\sum_{k=0}^{\frac{n-1}{2}}[(n-2k)\binom{n}{k}]^2 = \sum_{k=\frac{n-1}{2}}^{n}[(n-2k)\binom{n}{k}]^2$. This means that $\lfloor\frac{n-1}{2}\rfloor$ will cover both of these cases. This means that $n + (n-2)^2 n + (n-4)^2\frac{n(n-1)^2}{2!^2} + ... + (n-2\lfloor\frac{n-1}{2}\rfloor)^2\frac{n(n-1)^2(n-2)^2...(n-\lfloor\frac{n-1}{2}\rfloor+1)^2}{(\lfloor\frac{n-1}{2}\rfloor!)^2} = 1 + (n-1)^2 + \frac{(n-1)^2(n-2)^2}{(2!)^2} + ... + 1$ Now, subtract everything from the right hand side to the left, a pair at a time until all the pairs are removed, or there's only one number left on the right hand side. Notice that $n - 2 + (n-2)^2 n = (n-2)(n-1)^2$ and that $(n-2)(n-1)^2 + (n-4)^2\frac{n(n-1)^2}{(2!)^2} - 2(n-1)^2 = \frac{(n-1)^2(n-2)^2(n-4)}{(2!)^2}$ It seems that if n=c, then n-c is always a factor in the numerator of the fraction that equals 0 and everything else are squared factors with the subtracted constants ranging from 1 to $\lceil\frac{n-1}{2}\rceil$ which is all divided by $(\lceil\frac{n-1}{2}\rceil!)^2$. Now, let's show this pattern by induction. The base n=1 is already known. Assume it true for a given n. Let's show it true for n+1. There's 2 cases; n is even or n is odd. Case 1: Even Since it is true for n, one must realize that replacing n with n+1 is equally valid provided that one includes the parts not included in that sum as well. Another thing to remember is that both the denominator and the terms being subtracted do not change. (Think of the parts being subtracted in terms of k as it was done originally in the problem.) Hence, $\frac{(n)^2(n-1)^2...(n/2+2)^2}{((\frac{n}{2}-1)!)^2} + \frac{(n+1)(n^2)(n-1)^2...(n/2+2)^2}{(\frac{n}{2}!)^2} = \frac{(n)^2(n-1)^2...(n/2+1)^2}{(\frac{n}{2}!)^2}$ If one subtracts this and puts it under one denominator, $\frac{(n^2)(n-1)^2...(n/2+2)^2}{((\frac{n}{2})!)^2}((n/2)^2 + n + 1 - (n/2+1)^2)$. Obviously, the terms in the parenthesis equal 0. Case 2: Odd $\frac{(n)^2(n-1)^2...(\frac{n+1}{2}+1)^2}{(\frac{n-1}{2}!)^2} = \frac{n^2(n-1)^2...(\frac{n+1}{2}+1)^2}{(\frac{n-1}{2}!)^2}$ It's trivial because when odd advances to even, the change in the left hand side of the equation is nothing because one moves from n to 1^2 to n+1 to 2^2. That can be accounted for just by shifting the already known equation up 1. The only factor not accounted for is the right hand side's extra factor due to the fact that the n+1th row of the Pascal triangle has 1 more number than the nth row and that the numbers accounted for in the upwards shift are numbers that can be paired up. For example, $\binom{5-1}{0}$, $\binom{5-1}{1}$, $\binom{5-1}{2}$, $\binom{5-1}{3}$, $\binom{5-1}{4}$. To pair this up, we put the equal pairs together and find their counterparts on the fifth row. This means $\binom{5}{n}$ if n is less than 2 and $\binom{5-1}{4-n}$ corresponds to $\binom{5}{5-n}$. By doing so, we see that $\binom{5}{3}$ is not represented and is the number that's put on the left hand side.

□ Let $x,\ y$ be real numbers such that $\sin x \cos y = \sin y + \cos x = k$. Find the maximum and minimum value of k.

Solution

$\sin x \cos y = \sin y + \cos x \quad (1)$

Set $a = \sin y$ and $b = \cos x$. Of course $a, b \in [-1, 1]$

Then $\cos y = A\sqrt{1-a^2}$ and $\sin x = B\sqrt{1-b^2}$, where $A, B = \pm 1$

$(1) \Rightarrow$

$B\sqrt{1-b^2} \cdot A\sqrt{1-a^2} = a + b \Rightarrow$

$B^2(1-b^2)A^2(1-a^2) = (a+b)^2 \Rightarrow$

$1 - a^2 - b^2 + a^2b^2 = a^2 + b^2 + 2ab \Rightarrow$

$2a^2 + 2b^2 - a^2b^2 + 2ab - 1 = 0 \Rightarrow$

$\boxed{(2-a^2)b^2 + (2a) \cdot b + (2a^2 - 1) = 0}$

$D = (2a)^2 - 4(2a^2-1)(2-a^2) = 8(a^2-1)^2$

$b = \frac{-a \pm \sqrt{2}(a^2-1)}{2-a^2}$

$f(a) = a + b = a + \frac{-a \pm \sqrt{2}(a^2-1)}{2-a^2}$

Using the (+) or (-) sign we will get two functions, f_1 and f_2

Taking the (+) sign $f_1(a) = a + \frac{\sqrt{2}a^2 - a - \sqrt{2}}{2-a^2} = a + \frac{(a-\sqrt{2})(\sqrt{2}a+1)}{(\sqrt{2}-a)(\sqrt{2}+a)} = = a - \frac{\sqrt{2}a+1}{a+\sqrt{2}} = \boxed{\frac{a^2-1}{a+\sqrt{2}}}$

$f_1(-1) = f_1(1) = 0$ and $f_1(a) < 0,\ \forall a \in (-1, 1)$

We have $f_1'(a) = \frac{a^2 + 2\sqrt{2}a + 1}{(a+\sqrt{2})^2}$

This gives roots $-\sqrt{2} \pm 1$ for f_1', the only acceptable is $-\sqrt{2}+1$, and if we check the signs left and right we'll see that this is a local minimum $\boxed{f_1(-\sqrt{2}+1) = 2(-\sqrt{2}+1)} \Rightarrow a + b = 2a \Rightarrow a = b$

Taking the (-) sign $f_2(a) = a + \frac{-\sqrt{2}a^2 - a + \sqrt{2}}{2-a^2} = a - \frac{(a+\sqrt{2})(\sqrt{2}a-1)}{(\sqrt{2}+a)(\sqrt{2}-a)} = = a + \frac{\sqrt{2}a-1}{a-\sqrt{2}} = \boxed{\frac{a^2-1}{a-\sqrt{2}}}$

$f_2(-1) = f_2(1) = 0$ and $f_2(a) > 0,\ \forall a \in (-1, 1)$

$f_2'(a) = \frac{a^2 - 2\sqrt{2}a + 1}{(a+\sqrt{2})^2}$

So the roots are $\sqrt{2} \pm 1$ for f_2', the only acceptable is $\sqrt{2}-1$, and this is a local maximum $\boxed{f_2(\sqrt{2}-1) = 2(\sqrt{2}-1)} \Rightarrow a + b = 2a \Rightarrow a = b$

Summary:

Maximum $2(\sqrt{2}-1)$

Minimum $-2(\sqrt{2}-1)$

Notice that the min-max occurs when $a = b$

□ Solve the equation in $\mathbb{Z}$ $x^y = y^{x^2}$

Solution

$x^y = y^x$ if $p|x$ with p prime so $p|y$ and we have $xV_p(y) = yV_p(x) \implies V_p(x) = k\frac{x}{GCD(x,y)}$ et $V_p(y) = k\frac{y}{GCD(x,y)}$ So exist N, $x = N^{\frac{x}{GCD(x,y)}}$ et $y = N^{\frac{y}{GCD(x,y)}} \implies PGCD(x,y) = N^{\frac{min(x,y)}{GCD(x,y)}}$ if $x > y$ $=> y|x => x = ny$ $(n > 1)$ $(ny)^y = y^{ny}$ $n^y = y^{(n-1)y}$ $n = y^{n-1} => x = y^n$ $y^{ny} = y^{y^n}$ $ny = y^n$ $y^{n-1} = n$ if y>1 and n>2, $y^{n-1} \geq 2^{n-1} > n$ only possibilities i $n = 2$ give $y = 2$ and $x = 4$ if $x = y$, the equation is checked

$S =(2,4), (4,2), (x,x), x \in \mathbb{R}$

□ A sequence a_n of positive integers is given by $a_0 = m$ and $a_{n+1} = a_n^5 + 487$.

Find all values of m for which this sequence contains the maximum possible number of squares.

Solution

$a_0 = 3^2$ produces $a_1 = 244^2$ a square, whereupon no other values are square.

We investigate the sequence $\bmod 16$. Recall that $0, 1, 4, 9$ are the possible quadratic residues $\bmod 16$. The possible values of the RHS $\bmod 16$ are

$7, 8, 10, 12, 14, 0, 2$

Of which the RHS is 0 precisely when $a_n \equiv 9 \bmod 16$. There are two very simple cases.

Case: $m \equiv 9 \bmod 16$. Then a_1 might be a square, but no subsequent terms are $\equiv 0, 1, 4, 9 \bmod 16$ or can be (since that would require that the previous term be $\equiv 9 \bmod 16$.)

Case: $m \not\equiv 9 \bmod 16$. Then $a_1 \not\equiv 0, 1, 4, 9 \bmod 16 \implies a_n \not\equiv 0, 1, 4, 9 \bmod 16$.

Hence we desire m a square such that

$m^5 + 487$

Is a square. Let $m = p^2$. We want to solve

$(p^5)^2 + 487 = q^2$

The LHS is a square plus 487; the RHS is a square. Hence no solutions can have $q > 244$ (since $244^2 - 243^2 = 487$). We already discovered that solution, so our unique solution is $m = \boxed{9}$.

☐ Let H be a heptagon in the plane centered at the origin such that $(1, 0)$ is a vertex. Calculate the product of the distances from $(1, 1)$ to each vertex of H.

Solution

Since the heptagon is centred at the origin and a vertex is on $(1, 0)$, you could easily consider the seventh root of unity, in other words, the polynomial $f(z) = z^7 - 1 = (z-1)(z-\zeta)(z-\zeta^2)\dots(z-\zeta^6)$, where $\zeta = e^{i\frac{2\pi}{7}}$. We are to find the product of the distances between $(1, 1) = 1 + i$ and $1, \zeta, \zeta^2, ..., \zeta^6$, i.e. $|f(1 + i)| = |(1 + i)^7 - 1|$.

$$|(1+i)^7 - 1| = \left| \left(\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right)^7 - 1 \right| = |7 - i8| = \sqrt{113}$$

☐ Prove that the equation

$x^2 + x + 1 = py$ has integer solutions (x, y) for infinitely many primes p.

Solution

Suppose there are only finite p such that $p|(x^2 + x + 1)$ for some integer x. Let the set of all such p be $S = \{p_1, p_2, \dots, p_k\}$. Then take

$x = p_1 p_2 \cdots p_k \Rightarrow x^2 + x + 1 = p_1 p_2 \cdots p_k (p_1 p_2 \cdots p_k + 1) + 1.$

This, however, means that $x^2 + x + 1$ is not divisible by $p_1, p_2, \dots, p_k$ which implies that it is divisible by some other prime p_{k+1}, contradicting the fact that S was the set of all primes dividing $x^2 + x + 1$ for any integer x. Hence S is infinite.

☐ Let x, y be reals satisfying:

$\sin x + \cos y = 1$ $\sin y + \cos x = -1$ Prove $\cos 2x = \cos 2y$

Solution

Let $z = \frac{\pi}{2} - y$. Then

$\sin x + \sin z = 1$ $\cos x + \cos z = -1$

And we want to prove

$\cos 2x = \cos(\pi - 2z) = -\cos(2z)$

Well, let $u = e^{ix}, v = e^{iz}$. Then

$u + v = i - 1$

Of course, geometrically, we can only have two cases (there are only two right triangles with side lengths $1, 1, \sqrt{2}$ such that the hypotenuse is the line from $(0, 0)$ to $(-1, 1)$):

$u = i, v = -1$

Or

$u=-1, v=i$

Both of which satisfy our condition. QED.

☐ Let p be positive. Let C be the curve $y=2x^3$ and $P(p,2p^3)$ a point on C. Let l_1 be the tangent line at P and l_2 be another tangent line of C which passes through P.

(1) Express the slope of l_2 in term of p.

(2) Find $tanx$, where x is the angle formed between l_1 and l_2 and not more than 90 degrees.

(3) Find the maximum value of $tanx$

Solution

For 1) $f(x)=2x^3 \Rightarrow f'(x)=6x^2$

Let $Q(q,2q^3)$ be a point on C. The tangent at Q has equation $y-f(q)=f'(q)(x-q) \Leftrightarrow$

$y-2q^3=6q^2(x-q)$

This tangent line intersects C at the point Q and (possibly) at the point P

In order to find the coordinates of P, we set $y=2x^3$ in the above equation and we get:

$2x^3-2q^3=6q^2(x-q) \Leftrightarrow$

$x^3-q^3=3q^2(x-q) \Leftrightarrow$

$(x-q)(x^2+qx+q^2)=3q^2(x-q) \Leftrightarrow$

$(x-q)(x^2+qx+q^2-3q^2)=0 \Leftrightarrow$

$(x-q)(x^2+qx-2q^2)=0 \Leftrightarrow$

$(x-q)(x-q)(x+2q)=0 \Leftrightarrow$

$\boxed{(x-q)^2(x+2q)=0}$

The last equation has two roots, one is q (double) and the other is $-2q$ (if $q=0$ then it is a triple root)

So the other intersecting point P exist always and we can find it from $p=-2q$. Solving for q we get unique solution $q=-\frac{p}{2}$. Notice that $q<0$

The tangent at Q has slope $6q^2=6\left(-\frac{p}{2}\right)^2=\frac{6}{4}\cdot p^2$

For 2) If s_1, s_2 is the slope of l_1, l_2 respectively then $s_1=6p^2, s_2=\frac{6}{4}\cdot p^2$

The angle between the line l_i and the x-axis is θ_i. Then $s_i=\tan\theta_i$. Since $\tan\theta_i$ are positive, we can suppose that $\theta_i \in (0,\frac{\pi}{2})$. Notice that $s_1>s_2 \Rightarrow \theta_1>\theta_2$

The angle between the two lines is x, where $\tan x=\tan(\theta_1-\theta_2)=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}=\frac{6p^2-\frac{6}{4}p^2}{1+6p^2\cdot\frac{6}{4}p^2}=$
$=\frac{24p^2-6p^2}{4+36p^4}=\frac{18p^2}{4(1+9p^4)}=\frac{3}{2}\cdot\frac{3p^2}{1+9p^4}$ For 3) If my calculations are correct, then $\tan x\geq 0$ with equality only when $p=0$. But we have $p>0$, so there is not a minimum.

I suppose that the problem is asking for the maximum of $\tan x$

At this point, I won't use derivatives

$(3p^2-1)\geq 0$, with equality only for $p^2=\frac{1}{3} \Leftrightarrow p=\frac{\sqrt{3}}{3}$

$(3p^2-1)\geq 0 \Leftrightarrow$

$(3p)^2-2(3p)+1\geq 0 \Leftrightarrow$

$(3p)^2+1\geq 2(3p) \Leftrightarrow$

$\frac{1}{2}\geq\frac{3p}{1+9p^4}$

$\frac{3p}{1+9p^4}\leq\frac{1}{2} \Rightarrow \tan x=\frac{3}{2}\cdot\frac{3p}{1+9p^4}\leq\frac{9}{4}$, with equality when $p=\frac{\sqrt{3}}{3}$

☐ Let a,b,c be positive numbers with $\sqrt{a}+\sqrt{b}+\sqrt{c}=\frac{\sqrt{3}}{2}$ Prove that the system of equations

$$\begin{cases}\sqrt{y-a}+\sqrt{z-a}=1\\ \sqrt{z-b}+\sqrt{x-b}=1\\ \sqrt{x-c}+\sqrt{y-c}=1\end{cases}$$ has exactly one solution (x,y,z) in real numbers.

Solution

We make the following substitutions, since all the numbers involved are positive. Let

$p=\sqrt{a}, q=\sqrt{b}, r=\sqrt{c}$

And

$u=\sqrt{x}, v=\sqrt{y}, w=\sqrt{z}$

Then our condition becomes $p+q+r=\frac{\sqrt{3}}{2}$, and our equations become

$\sqrt{v^2-p^2}+\sqrt{w^2-p^2}=1$, etc.

Notice that if we let p, q, r be the altitudes to some triangles and let v, w, etc. be their side lengths, then by Pythagorean Theorem we have produced the following triangles:

$(v, w, 1)$ $(w, u, 1)$ $(u, v, 1)$

And now the real geometric meaning behind the problem is apparent!

Let there be a point P inside an equilateral triangle of side length 1. Let p, q, r be the length of the perpendiculars from P to each side. It's well-known that $p+q+r=\frac{\sqrt{3}}{2}$ (we can sum up some areas).

This tells us that u, v, w are simply the distances from P to the vertices, and then the proof is obvious:

Consider the locus of points that are a distance p away from one of the given sides. They form a line within the triangle parallel to that side. Consider the intersection of that locus and the locus of points a distance q away from another side. There is precisely one point which satisfies our conditions (P), which, by the above argument, must be a distance r from the third side. Then u, v, w exist and are unique.

QED.

□ Prove that any prime number $2^{2^n}-1$ cannot be represented as the difference of two 5th powers of integers. n is a positive integers.

Solution

Let $2^{2^n}-1=a^5-b^5=p$. Now, because $(a-b)|(a^5-b^5)$, if $a-b>1$, then $(a-b)|p$, so p won't be a prime. So we should have $a-b=1$. Then: $a^5-b^5=(b+1)^5-b^5=5b^4+10b^3+10b^2+5b+1$. So: $2^{2^n}-1=5b^4+10b^3+10b^2+5b+1$ $(1)2^{2^n}=5b^4+10b^3+10b^2+5b+2$ Consider the general equation: $2^w=5k^4+10k^3+10k^2+5k+2$ We must have: $(2)2^w\equiv 2(\mod 5)$ $2^0\equiv 1(\mod 5)$ $2^1\equiv 2(\mod 5)$ $2^2\equiv 4(\mod 5)$ $2^3\equiv 3(\mod 5)$ $2^4\equiv 1(\mod 5)$ $2^5\equiv 2(\mod 5)$ ... So (2) is true only when $w=4q+1$ for $q\geq 0$

Now, because 2^n is not of the form $4q+1$, (1) has no solutions.

□ N dwarfs of heights $1, 2, ..., N$ are arranged in a circle. For each pair of neighbouring dwarfs the positive difference between the heights is calculated: the sum of these N differences is called the "total variance" V of the arrangement. Find (with proof) the maximum and minimum possible values of V.

Solution

Let V_m be the minimum V when $N=m$. Clearly $V_2=2$. Now, let's examine what happens if we have N dwarves in optimal order and add another: Let's say we add the new dwarf in between dwarves of heights a and $a+b$. Let the new dwarf have height $a+b+c$. $V_{n+1}=V_n+(a+b+c-a)+(a+b+c-a+b)-b=V_n+2c$. So we need to minimize c, which is the difference between the new dwarf's height and that of his greatest neighbour. If we put the new dwarf beside the second tallest dwarf, this will be 1. Hence, $V_{n+1}=V_n+2$. From here we can see that $V_n=2(n-1)$.

□

For $x, y, u, v > 0$ prove that

$$\frac{xy + xv + uy + uv}{x + y + u + v} \geq \frac{xy}{x + y} + \frac{uv}{u + v}$$

Solution

$f(a, b) = af\left(1, \frac{b}{a}\right)$, so put $g(r) := f(1, r) = \frac{r}{1+r}$, and that function is definitely concave. Then, if $y = px, v = qu$, we have

$$\begin{aligned} f(x + u, y + v) &= f(x + u, px + qu) \\ &= (x + u)g\left(\frac{px + qu}{x + u}\right) \\ &\geqslant (x + u)\left[\frac{x}{x + u}g(p) + \frac{u}{x + u}g(q)\right] \\ &= xg(p) + ug(q) \\ &= xf\left(1, \frac{y}{x}\right) + uf\left(1, \frac{v}{u}\right) \\ &= f(x, y) + f(u, v) \end{aligned}$$

$\square$ Find all pairs (k, m) of positive integers such that $k^2 + 4m$ and $m^2 + 5k$ are both perfect squares.

Solution

Let's put $k^2 + 4m = (k + p)^2$, $m^2 + 5k = (m + q)^2$. Then $p, q \in \mathbb{N}^+$. Transforming these equations we get: $4m = 2kp + p^2$ $5k = 2mq + q^2$. Plugging the first into the second and then the second into the first gives $5k = 2q \cdot \frac{2kp+p^2}{4} + q^2$ $4m = 2p \cdot \frac{2mq+q^2}{5} + p^2$ which is equivalent to $(5 - pq)k = \frac{p^2q}{2} + q^2$. $\left(4 - \frac{4}{5}pq\right)m = \frac{2pq^2}{5} + p^2$ Since k and $\frac{p^2q}{2} + q^2$ are positive numbers, also $5 - pq$ is positive, i.e. $pq \leqslant 4$. Thus $pq \in \{1, 2, 3, 4\}$. From the second equation: $m = \frac{2pq^2+5p^2}{20-4pq}$, so $2|5p^2$. Hence $2|p$. The only possible pairs (p, q) are $\{2, 1\}$, $\{2, 2\}$, $\{4, 1\}$. After evaluating (k, m) for each pair we see that the pairs (k, m) we got satisfy the given conditions. Therefore the final result is $(k, m) \in \{(1, 2), (8, 9), (9, 22)\}$.

Another way **1.** $k \geq m$

$k^2 < k^2 + 4m \leq k^2 + 4k < k^2 + 4k + 4 = (k + 2)^2$

$\Rightarrow k^2 < k^2 + 4m < (k + 2)^2 \Rightarrow k^2 + 4m = (k + 1)^2$

$\Rightarrow k = \frac{4m-1}{2} = 2m - \frac{1}{2}$, so k can't be intger.

2. $k < m$

$m^2 < m^2 + 5k < m^2 + 5m < m^2 + 5m + 6.25 = (m + 2.5)^2$

$\Rightarrow m^2 + 5k = (m + 1)^2$ or $m^2 + 5k = (m + 2)^2$. $\Rightarrow 5k = 2m + 1$ or $5k = 4m + 4$.

2.1. $5k = 2m + 1$

So $k^2 + 4m = k^2 + 10k - 2$ and

$k^2 < k^2 + 10k - 2 < k^2 + 10k + 25 = (k + 5)^2 \Rightarrow k^2 < k^2 + 10k - 2 < (k + 5)$ and we have to consider 4 cases, but only two of them gives us a solution: for $k^2 + 10k - 2 = (k + 2)^2$ $k = 1; m = 2$ and for $k^2 + 10k - 2 = (k + 4)^2$ $k = 9; m = 22$

2.2. $5k = 4m + 4$

So $k^2 + 4m = k^2 + 5k - 4$ and

$k^2 < k^2 + 5k - 4 < k^2 + 5k + 6.25 = (k + 2.5) \Rightarrow k^2 < k^2 + 5k - 4 < (k + 2.5)^2$ and we have to consider 2 cases, but only one of them gives us a solution: for $k^2 + 5k - 4 = (k + 2)^2$ $k = 8; m = 9$.

☐ Let $p(n)$ be the number of partitions of n, and let $p(n, m)$ be the number of partitions of n containing m terms. Show that

$$p(n) = p(2n, n).$$

Solution

Let $n = a_1 + a_2 + \cdots + a_k$ be any partition of n (by the definition of a partition, we have $a_i \geq 1$). Rewrite this as

$n = a_1 + a_2 + \cdots + a_k + a_{k+1} + \cdots + a_n$

where $a_i = 0$ for $i > k$. Then

$2n = (a_1 + 1) + (a_2 + 1) + \cdots + (a_k + 1) + (a_{k+1} + 1) + \cdots + (a_n + 1)$

is a valid partition for $2n$ with exactly n terms. Clearly, the reverse works as well, giving us a one-to-one correspondence.

☐ Prove that the only solution (>1) to the following equation is: (a,b,c)=(2,2,3)

$3^a - b^c = 1$

Solution

Lemma: Only (m,n) (>1) satisfying $3^m - 2^n = 1$ is (2,3)

Proof: if m>2 then n>3 $\Rightarrow 3^m \equiv 1 \pmod 8 \Rightarrow m = 2k$ then $2^n = (3^k - 1)(3^k + 1)$ by unique factorization: $3^k + 1 = 2^r$ but this is impossible since, then $3^k - 2^r = -1 \Rightarrow 3^k \equiv -1 \pmod 8$ contradiction, since $3^k \equiv 1, 3 \pmod 8$ depending if k is even or odd. ■ ANother approach $b^c = 3^a - 1 \cong -1 mod(3)$, so $b \cong -1 mod(3)$ and c is odd. Then, $b + 1 | b^c + 1 = 3^a$ and there exists a positive integer n such that $b = 3^n - 1$. Because the post of amirhtlusa, we can suppose $n \geq 2$. $3^a = (3^n - 1)^c + 1 > (2.3^{n-1})^c > 3^{(n-1)c+1}$ Let d, k nonnegative integers such that $c = d.3^k$ and $(d, 3) = 1$. We can prove: (by induction over k) $(3^n - 1)^c \cong d.3^{n+k} - 1 mod(3^{2n+k})$ Then, $n + k \geq a > (n-1)c + 1 \geq (n-1)(k+1) + 1 = n + (n-1)k \geq n + k$ wich is absurd.

☐ Find all positive integers n and d such that both of the following are true: i) d divides $2n^2$. ii) $n^2 + d$ is a perfect square.

Solution

If $p > 2$ is a prime divisor of d and e, α positive integers such that $d = e.p^\alpha$, $(p, e) = 1$; we have $p|n$ and $n = m.p^\beta$ for some positive integers m ,β with $(m, p) = 1$, $\alpha \leq 2\beta$; Now, because $n^2 + d = p^\alpha(m.p^{2\beta-\alpha} + e)$ is a perfect square, α is even. Then $d = 2^r a^2$ with a odd, and $n = 2^s ab$ with b odd and $r \leq 2s+1$. We have two situations: If r is odd, we have $n^2 + d = 2^{r-1}a^2(2^{2s-r+1}b^2 + 2)$, and, because there are not two perfect squares with diference 2, there are not such n, d. If r is even, we have $n^2 + d = 2^r a^2(2^{2s-r}b^2 + 1)$, and because there are not two positive perfect squares with diference 1, there are not such n, d.

☐ a_n is a sequence such that $4 \cdot a_n = a_{2n}$ and $a_{2n} = 2 \cdot a_{2n-1} + \frac{1}{4}$ for all $n \in \mathbb{N}$. Find the sum $S = a_1 + a_2 + ... + a_{31}$

Solution

We know that $4 \cdot a_n = a_{2n} = 2 \cdot a_{2n-1} + 1/4$. Letting $n = 1$, $4a_1 = 2a_1 + 1/4 \implies a_1 = \frac{1}{8}$.

We solve $2 \cdot a_{2n-1} + 1/4 = 4 \cdot a_n \implies \boxed{a_{2n-1} = 2 \cdot a_n - \frac{1}{8}}$. And it is given that $\boxed{a_{2n} = 4 \cdot a_n}$.

Consider $\sum_{k=1}^{2^n} a_k$. This is the same as taking the sum of the individual sums of the odd k and the even k, so is the same as $= \sum_{k=1}^{2^{n-1}} (a_{2k-1} + a_{2k}) = \sum_{k=1}^{2^{n-1}} (6a_k - \frac{1}{8})$.

Repeating this over and over $\sum_{k=1}^{32} a_k = \sum_{k=1}^{16} (6a_k - \frac{1}{8}) = 6 \sum_{k=1}^{8} (6a_k - \frac{1}{8}) - 2 = 36 \sum_{k=1}^{4} (6a_k - \frac{1}{8}) - 6 - 2 = 216 \sum_{k=1}^{2} (6a_k - \frac{1}{8}) - 18 - 6 - 2 = 1296 \sum_{k=1}^{1} (6a_k - \frac{1}{8}) - 54 - 18 - 6 - 2$. The summation part now has only one term, $a_1 = \frac{1}{8}. = 1296(6(1/8) - 1/8) - 62 = 730$.

We need to get rid of a_{32} from that sum. $a_1 = \frac{1}{8} \implies a_2 = 1/2 \implies ... \implies a_{32} = 128$.

So the sum is $730 - 128 = \boxed{602}$.

☐ Let n and k be positive integers, and let set $S = \{1, 2, \ldots, n\}$. A subset of S is called 'skipping' if it doesn't contain consecutive integers. How many $k-$ element subsets of S are there? Also, how many skipping subsets of S are there total?

Solution

The amount of total skipping subsets is equal to $\sum_{i=0}^{n} F_n$ where F_n are the Fibonacci numbers. This is easy to prove by induction. Show the base case. THe skipping subsets of 1=1 which is equal to the number we got in our sum. Assume it true for all F_n up to n. To prove it true for F_{n+1}, let's analyze our subsets. On top of our subsets for that doesn't include n+1, we have $(\sum_{i=0}^{n-1} F_n) + 1$ that does include n+1. Listing out the sums, 1,1,2,3,...,F_n 1,1,2,3,...,F_{n-1},1. By adding diagonally and putting the end one at the beginning, one gets 1,1,2,3,...,F_{n+1}=$\sum_{i=0}^{n+1} F_n$

☐ Let p be a prime $= 1(\mod 3)$ and q be the integer part of $\frac{2p}{3}$. If $\frac{1}{(1)(2)} + \frac{1}{(3)(4)} + ... + \frac{1}{(q-1)(q)} = \frac{m}{n}$, for integers m, n, show that m is divisible by p.

Solution

if $p = 1 + 3a$, $q = 2a$ we take $H_n = \sum_{k=1}^{n} \frac{1}{k}$ $S = \sum_{k=1}^{q/2} \frac{1}{(2k-1)2k} = \sum_{k=1}^{q/2} \frac{1}{2k-1} - \frac{1}{2k} = H_q - \frac{1}{2}H_{q/2} - \frac{1}{2}H_{q/2} = \sum_{k=q/2+1}^{q} \frac{1}{k} = \frac{m}{n}$

$q = 2a$ and $p = 3a + 1$ $2S = 2\sum_{k=q/2+1}^{q} \frac{1}{k} = \sum_{k=a+1}^{2a} \frac{1}{k} + \sum_{k=a+1}^{2a} \frac{1}{p-k} = \sum_{k=a+1}^{2a} \frac{p}{k(p-k)}$ so, there exist $(c, d) \in N^2/;\ gcd(pc, d) = 1\ ;\ 2S = p\frac{c}{d} \implies p\frac{c}{d} = \frac{2m}{n} \implies 2md = pnc \implies p|m$ (because $p > 2$, $\gcd(p, d) = 1$)

☐ Find all positive integers that can be written as $1/a_1 + 2/a_2 + ... + 9/a_9$, where a_i are positive integers.

Solution

All the integers from 1 to 45 are attainable. For any n, all the integers from 1 to $n(n+1)/2$ are attainable. Let's prove the general case by induction. Base case, n=1. 1/1=1 is the only possible case. Similarly, n=2 works. 1/3+2/3=1, 1/1+2/2=2, and 1/1+2/1=3. Assume this true for all positive integers up to n. Let's prove this n+1. $1/a_1 + 2/a_2 + ... + n/a_n + \frac{n+1}{a_{n+1}}$. Now, the sum of the first n terms can be anything from 1 to $n(n+1)/2$. Since one can always get a total sum of 1 and n+1 by setting all the denominators equal to $(n+1)(n+2)/2$ and the last one to n+1 while everything else to 1, respectively, all the integers from 1 to $(n)(n+1)/2 + 1$ are attainable. Now, one can also add n+1 to the sum of the previous n by setting the denominator a_{n+1} as 1. This will yield all integers from n+2 to $(n+1)(n+2)/2$. Now, we need to show that for all positive n, the union of these two sets will be all integers from 1 to $(n+1)(n+2)/2$. $n(n+1)/2 + 1 \geq n + 2$ or $n^2 - n - 2 \geq 0$ or $(n-2)(n+1) \geq 0$ which is only "false" for n+1 case where n is 1 or n=2, which we already shown is true.

☐ Let $a_1, ...a_n$ be $n > 1$ distinct real numbers. Set

$S = a_1^2 + ... + a_n^2, M = \min_{1 \leq i \leq j \leq n}(a_i - a_j)^2$

Prove that

$\frac{S}{M} \geq \frac{n(n-1)(n+1)}{12}$

Hint If we let $r = \sqrt{M}$ and assume WLOG that $a_1 \leq a_2 \leq ... \leq a_n$, then

$a_{1+k} \geq a_1 + kr$

☐ Evaluate $\sum_{k\equiv 1(\mod 3)} \binom{n}{k}$ and if $k \equiv 2(\mod 3)$

Solution

Let $\omega = e^{\frac{2\pi i}{3}}$. It is easy to see that for any integer l the value of expression $\frac{1+\omega^l+\omega^{2l}}{3}$ is 1 when $n \mid l$ and 0 when $n \nmid l$. Thus

$$\begin{aligned}
\sum_{k\equiv j \pmod 3} \binom{n}{k} &= \sum_{n\mid k-j} \binom{n}{k} \\
&= \frac{1}{3} \cdot \sum_{n\mid k-j} \binom{n}{k} \cdot \left(1+\omega^{k-j}+\omega^{2(k-j)}\right) \\
&= \frac{1}{3} \cdot \sum_{k} \binom{n}{k} \cdot \left(1+\omega^{k-j}+\omega^{2(k-j)}\right) \\
&= \frac{1}{3} \cdot \left(\sum_{k} \binom{n}{k}\right) + \frac{1}{3} \cdot \left(\omega^{-j} \cdot \sum_{k} \binom{n}{k}\omega^k\right) + \frac{1}{3} \cdot \left(\omega^{-2j} \cdot \sum_{k} \binom{n}{k}\omega^{2k}\right) \\
&= \frac{1}{3} \cdot \left(2^n + \omega^{-j} \cdot (1+\omega)^n + \omega^{-2j} \cdot \left(1+\omega^2\right)^n\right) \\
&= \frac{1}{3} \cdot \left(2^n + \omega^{-j} \cdot \left(-\omega^2\right)^n + \omega^{-2j} \cdot (-\omega)^n\right) \\
&= \frac{1}{3} \cdot \left(2^n + (-1)^n \cdot \left(\omega^{2n-j} + \omega^{n-2j}\right)\right) \\
&= \frac{1}{3} \cdot \left(2^n + (-1)^n \cdot \left(\omega^{-(j+n)} + \omega^{j+n}\right)\right) \\
&= \frac{1}{3} \cdot \left(2^n + (-1)^n \cdot 2\mathrm{Re}\left(\omega^{j+n}\right)\right) \\
&= \frac{1}{3} \cdot \left(2^n + (-1)^n \cdot 2\cos\left(\frac{2\pi}{3} \cdot (j+n)\right)\right).
\end{aligned}$$

The last expressions allows us to evaluate the desired sum if we know residues of n and j modulo 3:

$$\begin{cases} \sum_{k\equiv j \pmod 3} \binom{n}{k} = \frac{1}{3} \cdot (2^n + 2 \cdot (-1)^n) & \text{for } 3 \mid n+j \\ \sum_{k\equiv j \pmod 3} \binom{n}{k} = \frac{1}{3} \cdot (2^n - (-1)^n) & \text{for } 3 \nmid n+j. \end{cases}$$

□ show that any number of the form $n^4 + 4^n$ are not prime for n $>$2.

2. for a, b integers suh that $a + b = 1$, show that $[a + (a/1)]^2 + [b + (1/b)]^2 \geq 25/2$.

3.let a,b any two positive integers, show that $2^{1/2}$ is alwayys lies between a/b and (a+2b)/(a+b).

4. prove that the equations $x^2 - 2y^2 + 8z = 3$ has no solutions for any positive integers x,y,z.

5. let a,b,c be integers such that $a + b + c = 0$, prove that $2a^2 + 2b^4 + 2c^4$ is a perfect square.

5. let x,y be two positive odd integers. show that it is imposibble that teh value of $x^2 + y^2$ to be a perfect integer.

7. without calcuylator, prove that $\cos(2.pi/5) + \cos(4pi/5) = -1/2$.

8.prove that $(a + b)^n \leq 2^n - 1(a^n + b^n)$ for positive integer n.

9. prove that $n^2 + 11n + 2$ is not divisible by 12769 for all integers n.

Solution

For 1) n must be odd, so $n = 2k + 1$, $k \in \mathbb{N}$

$(2k+1)^4 + 4^{2k+1} = (2k+1)^4 + 2^2 \cdot (4^k)^2 = (2k+1)^4 + 2^2 \cdot (4^k)^2 + 4 \cdot 4^k(2k+1)^2 - 4 \cdot 4^k(2k+1)^2 = ((2k+1)^2 + 2 \cdot 4^k)^2 - 4 \cdot 4^k(2k+1)^2 = ((2k+1)^2 + 2 \cdot 4^k + 2 \cdot 2^k(2k+1))((2k+1)^2 + 2 \cdot 4^k - 2 \cdot 2^k(2k+1))$

if this product is equal to a prime p, then one of the 2 factor is 1 and the other is p, or one is -1 and the other $-p$. This last possibility is not possible, because $(2k+1)^2 + 2 \cdot 4^k + 2 \cdot 2^k(2k+1)$ is always positive and greater than 1. So the only possibility is that

$(2k+1)^2 + 2 \cdot 4^k + 2 \cdot 2^k(2k+1) = p$ $(2k+1)^2 + 2 \cdot 4^k - 2 \cdot 2^k(2k+1) = 1$

Let's show that the second one is not possible:

$(2k+1)^2+2\cdot 4^k-2\cdot 2^k(2k+1)=1 \Longleftrightarrow 2\cdot 2^k(2^k-2k-1)=-2k(2k+2) \Longleftrightarrow 2^k(2^k-2k-1)=-k(k+1)$

since $(k,k+1)=1$, and if $k\geq 2$, 2^k contains at least two factors 2, it must be or $2^k=k$, that is impossible, or $2^k=k+1$, that is also impossible

☐ Prove that if $(N,10)=1$ then N^{101} ends with three digits, which are also the last thee three digits of N.

Solution

We wish to show that $N^{101}\equiv N \mod 1000$ for $gcd(N,10)=1$. Of course, this suggests the Totient Theorem. However, $\varphi(1000)=400$.

Note that $\varphi(125)=100$, however. We therefore know that $N^{100}\equiv 1 \mod 125$. Moreover, $\varphi(8)=4$. It follows that $N^{100}\equiv 1 \mod 8$.

By CRT we know that $N^{100}\equiv 1 \mod 1000$. QED.

☐ Prove that each two numbers in the sequence $2+1, 2^2+1, ..., 2^{2^n}+1$ are relative prime numbers

Solution

Let $F_n=2^{2^n}+1$, then for m<n we have $F_n-2=F_{n-1}F_{n-2}...F_0$. It mean $F_m|F_n-2$, therefore $(F_m,F_n)=(F_m,2)=1$.

☐ Given $x,y,z\geq 0$, prove:

$$\frac{xy}{\sqrt{xy+2z^2}}+\frac{yz}{\sqrt{yz+2x^2}}+\frac{zx}{\sqrt{zx+2y^2}}\geq\sqrt{xy+yz+zx}.$$

Solution

Because $f(x)=\frac{1}{\sqrt{x}}$ is a convex function on $(0,+\infty)$ (because $f''(x)=\frac{3}{4\sqrt{x^5}}$) we can apply the Weighted Jensen's inequality:

$$\frac{\frac{xy}{\sqrt{xy+2z^2}}+\frac{xz}{\sqrt{xz+2y^2}}+\frac{yz}{\sqrt{zy+2x^2}}}{xy+yz+xz}=\frac{xy.f(xy+2z^2)+xz.f(xz+2y^2)+yz.f(zy+2x^2)}{xy+yz+xz}$$

$$\geq f(\frac{xy^2+2xyz^2+y^2z^2+2x^2yz+x^2z^2+2xy^2z}{xy+yz+xz})=f(\frac{(xy+yz+xz)^2}{xy+xz+yz})$$

$$=\frac{1}{\sqrt{xy+xz+yz}}$$

☐ For which real numbers a does the sequence defined by the initial condition $u_0=a$ and the recursion $u_{n+1}=2u_n-n^2$ have $u_n>0$ for all $n\geq 0$?

Solution

$u_{n+1}=2u_n-n^2 \Longleftrightarrow u_{n+1}-(n+1)^2-2(n+1)-3=2(u_n-n^2-2n-3)$, yielding $u_n=(a-3)2^n+n^2+2n+3$. Since $n^2+2n+3=(n+1)^2+2>0$, thus the answer is $a\geq 3$.

☐ Prove that: $\forall n\in\mathbb{Z}, n>0$, we have: $\frac{1}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)\geq\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$

Solution

Adding $\frac{1}{n+1}\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)$ to both sides, and letting H_k be the kth harmonic sum or w/e, it would suffice to prove that

$\frac{H_{2n}}{n+1}\geq\frac{(2n+1)H_n}{2n(n+1)}$

or

$2nH_{2n}\geq(2n+1)H_n$

This is true since for $1\leq k\leq n$,

$2n(n+2k) = 2n^2 + 4nk \geq 2n^2 + 2nk + n + k = (n+k)(2n+1)$

$2n\left(\frac{1}{k} + \frac{1}{n+k}\right) = \frac{2n(n+2k)}{k(n+k)} \geq \frac{2n+1}{k}$

Probably there's an easier way but this was the first thing I thought of and it seems to have worked.

□ Prove that

$P_{n,r}(x) = \frac{(1-x^{n+1})(1-x^{n+2})...(1-x^{n+r})}{(1-x)(1-x^2)...(1-x^r)}$

Is a polynomial in x of degree nr, where n and r are nonnegative integers. (When $r = 0$ the empty product is understood to be 1 and we have $P_{n,0} = 1$ for all $n \geq 0$.)

Solution

We will show that the roots of the denominator are a subset of the roots of the numerator. Now, the roots of the denominator are simply the k^{th} roots of unity, $k = 1, 2, 3, ...r$. With what multiplicity do they occur?

A particular (primitive) k^{th} root of unity occurs as a root of $(1 - x^m)$ if and only if $k|m$. Hence each k^{th} root occurs $\lfloor \frac{r}{k} \rfloor$ times.

Applying the same logic to the numerator, each k^{th} root occurs $\lfloor \frac{(n+r)-(n+1)+1}{k} \rfloor$ times, which is, of course, the same number. QED.

□ Let z be a real number greater than 1 and let $z_1, z_2, ...z_n$ be the n roots of unity ($z_k = re^{2\pi(k-1)i/n}$). Show that

$$\prod_{k=1}^{n} |z - z_k| = z^n - 1$$

Solution

We have $\prod_{k=1}^{n} |z - z_k|$ Using the identity $|a \parallel b| = |ab|$ this becomes $|\prod_{k=1}^{n}(z - z_k)|$ And since z_k is the kth root of unity this simplifies to $|z^n - 1|$ Since $z > 1$ we know $z^n > 1$ for all natural n and so we have $z^n - 1$ and we're done.

□ Let S be a set of real numbers which is closed under multiplication. Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Solution

Suppose that neither T nor U is closed under multiplication. Then there exists $a, b \in T$ and $c, d \in U$ such that $ab \notin T$ and $cd \notin U$. Since $a, b, c, d \in S$ and S is still closed under multiplation, $ab \in U$ and $cd \in T$.

Consider $abcd$. If it is in U, then $\{a, b, cd\}$ are three numbers in T whose product is not in T, contradicting a given condition, so $abcd \notin U$. Since $abcd \in S$ it thus must be in T; yet by similar argument to the above it cannot be in T.

Therefore we have a contradiction, and at least one of T and U is closed under multiplication.

□ Let $\{x\}$ denote the closest integer to x (using the standard rounding conventions). Define $f(n) := n + \{\sqrt{n}\}$. Prove that, for every positive integer m, the sequence

$$f(m), f(f(m)), f(f(f(m)))$$

never contains the square of an integer.

Solution

It is sufficient to show that f(k) can't be a square for any integer k.

$n^2 \leq k \leq n^2 + 2n + 1$

$n^2+n \le f(k) \le n^2+3n+2$ the only perfect square in that range is $n^2+2n+1=(n+1)^2$

now if $k \ge n+1/2$ we round up and if its less we round down, however: $(n+1/2)^2 = n^2+n+1/4$ so $f(n^2+n)=n^2+2n$ and $f(n^2+n+1)=n^2+2n+2$

so the perfect square is skipped, and thus f(k) can never be a perfect square

□ Prove that: $x+\frac{4x^3}{(x-1)(x+1)^3} > 3 \quad \forall x>1$

Solution

By AM-GM we have:

$$x+\frac{4x^3}{(x-1)(x+1)^3} \ge 2\sqrt{\frac{4x^4}{(x-1)(x+1)^3}}.$$

Therefore, it suffices to prove that $4\sqrt{\frac{x^4}{(x-1)(x+1)^3}} > 3$, i.e.

$$16x^4 > 9(x-1)(x+1)^3 = 9(x^4+2x^3-2x-1) \iff 7x^4+18x+9 > 18x^3$$

But according to AM-GM:

$$\begin{aligned}7x^4+18x+9 &= x^4+x^4+x^4+x^4+x^4+x^4+x^4+9x+9x+9\\ &\ge 10\sqrt[10]{9^3x^{30}} = 10x^3\sqrt[5]{27}.\end{aligned}$$

But $10\sqrt[5]{27} \approx 19.33$, so $7x^4+18x+9 \ge 10x^3\sqrt[5]{27} > 18x^3$, and we're done. Another approach $x+\frac{4x^3}{(x-1)(x+1)^3}+1 = \frac{x^2-1}{x}+\frac{x+1}{2x}+\frac{x+1}{2x}+\frac{4x^3}{(x-1)(x+1)^3} \ge 4$

□ In a triangle ABC, AB is smaller then BC and BC is smaller than AC. The points A', B', C' are such that AA' is perpendicular to BC and $AA'=BC$, BB' is perpendicular to AC and $BB'=AC, CC'$ perpendicular to AB and $CC'=AB$. If $< AC'B = 90$ degrees, prove that A', B' and C' are colinear. (lies on a straight line)

Solution

Angle-Chasing Method

If we draw out , we can see that C' lies in ABC while A', B' lie outside ABC . Let the orthocenter be H and let BB' and AC' intersect at M ; AA' and BC' intersect at N . Also let $\angle BC'A' = x$,$\angle AC'B' = y$,$\angle C'AH = a$,$\angle C'BH = b$,then

$\angle AA'C' = 180° - \angle C'AH - \angle AC'A'$
$= 180° - a - (\angle AC'B + \angle BC'A')$
$= 180° - a - (90° + x)$
$= 90° - a - x$ .....(i)

$\angle BB'C' = 180° - \angle C'BH - \angle B'C'B$
$= 180 - b - (\angle B'C'A + \angle AC'B)$
$= 180° - b - (y + 90°)$
$= 90° - b - y$.....(ii)

$\angle C'MB = 90° - \angle C'BM = 90° - b$ $\angle C'NA = 90° - \angle C'AN = 90° - a$

$\implies \angle B'HA' = 360 - \angle C'MB - \angle C'NA - \angle AC'B$
$= 360 - (90° - b) - (90° - a) - 90°$
$= 90° + a + b$ .....(iii)

Also $\angle AA'C' + \angle BB'C' + \angle B'HA' = 180°$ so from (i),(ii),(iii) $(90° - a - x) + (90° - b - y) + (90° + a + b) = 180 \iff x + y = 90° \implies \angle BC'A' + \angle B'C'A + \angle AC'B = x + y + 90° = 180°$.

Hence A', B', C' collinear .

$\square$ Prove that there are no pairs of positive integers a, b that solve the equation $4a(a+1) = b(b+3)$.

Solution

$(2a+1)^2 = 4a(a+1)+1 = b(b+3)+1$, so $b(b+3)+1$ must be a perfect square. But clearly, $(b+1)^2 < b(b+3)+1 < (b+2)^2$. No solutions. ANother way Expand to get $4a^2+4a = b^2+3b$ which is equivalent to $4a^2+4a-(b^2+3b) = 0$. Solving for a we get $a = \frac{-4\pm\sqrt{16+4(b^2+3b)}}{8}$ which simplifies to $-\frac{1}{2} \pm \frac{\sqrt{b^2+3b+4}}{4}$. For a to be an integer we must have $\sqrt{b^2+3b+4}$ be an integer. But, $(b+\frac{3}{2})^2 < b^2+3b+4$. And, since $b > 0$ we have $b^2+3b+4 < (b+2)^2$ implying that $b+\frac{3}{2} < \sqrt{b^2+3b+4} < b+2$. So we have it that $\sqrt{b^2+3b+4}$ is not an integer for integer b implying that a is not an integer implying that there are no solutions.

$\square$ Let n be a natural number. Define $t(n)$ as the number of positive divisors of n (including 1 and n) en define $\sigma(n)$ as the sum of these numbers. Show that

$$\sigma(n) \geq \sqrt{n}.t(n)$$

Solution

Let the divisors be $1 = d_0 < d_1 < \cdots < d_k = n$. Clearly $d_0 d_k = n$ and in general $d_i d_{k-i} = n$. Then we clearly have

$d_0 d_1 \cdots d_k = n^{\frac{k+1}{2}}$.

By AM-GM, we know

$\frac{d_0+d_1+\cdots+d_k}{k+1} \geq \sqrt[k+1]{d_0 d_1 \cdots d_k} = \sqrt[k+1]{n^{\frac{k+1}{2}}} = \sqrt{n}$.

But $\sigma(n) = d_0 + d_1 + \cdots + d_k$ and $\tau(n) = k+1$ so

$\frac{\sigma(n)}{\tau(n)} \geq \sqrt{n} \Rightarrow \sigma(n) \geq \tau(n)\sqrt{n}$.

$\square$ Consider 8 integers $x_1, x_2, ...x_1$ around a circle. An operation consists of replacing them x_1 with $|x_1 - x_2|$, x_2 with $|x_2 - x_3|$, $...x_8$ with $|x_8 - x_1|$. For what starting sequences will all the numbers eventually become 0 after a finite number of operations? Generalize.

Solution

Consider the following sequence $00000001 \to 00000011 \to 00000101 \to 00001111 \to 00010001 \to 00110011 \to 01010101 \to 11111111 \to 00000000, 11111110 \to 00000011$;

It takes at most 8 steps for all numbers to become divisible by 2. In other words, it takes at most $8k$ steps for all numbers in the 8-tuple to be divisible by 2^k. As soon as the maximum number in the initial 8-tuple S follows $\max S < 2^k \implies k \geq \lceil \log_2(\max S) \rceil$, the 8-tuple has all entries as zero. The condition holds for all 2^m-tuples.

$\square$ $\prod_{k=0}^{2^{1999}} (4sin^2 \frac{k\pi}{2^{2000}} - 3)$

Solution

$\sin(x) = \cos(\pi/2 - x)$ so every possible x from the original equation are used for $(4\sin^2(\pi/2-x) - 3)(4\cos^2(\pi/2-x)-3)$. Multiplying through, we get $16\sin^2(\pi/2-x)\cos^2(\pi/2-x)-3$. SEt this equal to $4\sin^2 z - 3$. One gets $2\sin(\pi/2-x)\cos(\pi/2-x) = \sin(z)$. This looks like the double angle formula, so $z = \pi - 2x = 2x$ since taking the sine of either will have the same value. The only factor that isn't paired is the median, which then becomes the maximum in the new group of factors. However, the signs keep alternating because $\cos(\pi - 2x) = -\cos(2x)$. Noting that if this entire equation was reduced down to k=2 and the denominator being 4, the product is 3, so it should similarly follow that for k=2^{1999}, the product is also 3.

$\square$ Suppose $(a_i)_{i\geq 1}$ is a sequence of positive integers satisfying $\gcd(a_i, a_j) = \gcd(i, j)$ for $i \neq j$. Show that $a_i = i$ for each i.

Solution

Assume to the contrary, so let's say $a_m \neq m$. Note that $m \mid a_m$ as $gcd(a_m, a_{2m}) = m$. So, $a_m = km$ for some k. Note that $mk \mid a_{mk}$ (for the same reason as above). $gcd(m, mk) = m$. But $gcd(a_m, a_{mk}) = mk$. Contradiction.

□ Find positive integers such that: $(m/n)^m = (mn)^n$

Solution

Evidently $n|m$. Let $m = dn$. Then we have $d^{d-1} = n^2$. If d is odd then $n = d^{\frac{d-1}{2}}, \quad m = d^{\frac{d+1}{2}}$ If d is even then also $n = d^{\frac{d-1}{2}}, \quad m = d^{\frac{d+1}{2}}$ but d perfect square.

□ Show that the probability of two randomly chosen positive integers are relatively prime is $\frac{6}{\pi^2}$.

Solution

The probability that 2 does not divide both of them is $1 - \frac{1}{2^2}$. In fact, the probability that any prime p does not divide both of them is $1 - \frac{1}{p^2}$. So the desired probability is

$\prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^2}\right) = \frac{1}{\prod_{i=1}^{\infty} \frac{1}{\left(1-\frac{1}{p_i^2}\right)}} = \frac{1}{\prod_{i=1}^{\infty}(1+p_i^2+p_i^4+\cdots)} = \frac{1}{1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots} = \frac{6}{\pi^2}$.

□ Let $p(x)$ be a 1999-degree polynomial with integer coeffficients that is equal to ± 1 for 1999 different integer values of x. Show that $p(x)$ cannot be factored into the product of two plynomials with integer coefficients.

Solution

suppose the polynomial can be factored into a product of polynomials with integer coefficients, $A(x)B(x)$ wolog, let $A(x)$ have the lower degree. Then the degree of $A(x)$ is at most 999. For each of the 1999 values, $A(x)$ and $B(x)$ both give integer results, and the product of these results is 1 or -1. This means that for each of the 1999 values, $A(x)$ gives 1 or -1. Then $A(x)$ gives 1 at least 1000 times, or $A(x)$ gives -1 at least 1000 times, which is impossible for a polynomial of degree 999.

□

Given the numbers $1, 2, 2^2, \ldots, 2^{n-1}$, for a specific permutation $\sigma = x_1, x_2, \ldots, x_n$ of these numbers we define $S_1(\sigma) = x_1, S_2(\sigma) = x_1 + x_2, \ldots$ and $Q(\sigma) = S_1(\sigma)S_2(\sigma)\cdots S_n(\sigma)$. Evaluate $\sum \frac{1}{Q(\sigma)}$, where the sum is taken over all possible permutations.

Solution

Claim: $\sum \frac{1}{Q(\sigma)} = \prod_{i=1}^{n} \frac{1}{x_i}$.

Proof, by induction: Base Case: $n = 1$. Then $S_1(\sigma) = 1$ and $\sum \frac{1}{Q(\sigma)} = \frac{1}{1} = \prod_{i=1}^{1} \frac{1}{x_i}$. Inductive Step: Take an n for which $\sum \frac{1}{Q(\sigma)} = \prod_{i=1}^{n} \frac{1}{x_i}$. We want to show this relationship is satisfed for $n+1$ as well.

Suppose we take the sum over all permutations of $\sigma = x_1, \ldots, x_n, x_{n+1}$. Set $S = \sum_{i=1}^{n+1} x_i$, $P = \prod_{i=1}^{n+1} x_i$ and define $\psi(k) = \sum \frac{1}{Q(\sigma)}$, where the sum is taken over all permutations of $\{x_1, \ldots, x_n, x_{n+1}\} \setminus \{x_k\}$. Thus,

$$\sum \frac{1}{Q(\sigma)} = \left(\frac{1}{S}\right) \cdot \left(\sum_{i=1}^{n+1} \psi(i)\right).$$

From the inductive hypothesis, the sum taken over all permutations of a set of n elements is just the reciprocal of the product of the elements. That is,

$$\psi(k) = \frac{x_k}{x_1 \cdot x_2 \cdots x_n \cdot x_{n+1}} = \frac{x_k}{P}.$$

Hence,

$$\sum \frac{1}{Q(\sigma)} = \left(\frac{1}{S}\right) \cdot \left(\sum_{i=1}^{n+1} \frac{x_i}{P}\right).$$

$$\sum \frac{1}{Q(\sigma)} = \left(\frac{1}{S \cdot P}\right) \cdot \left(\sum_{i=1}^{n+1} x_i\right).$$

$$\sum \frac{1}{Q(\sigma)} = \left(\frac{1}{S \cdot P}\right) \cdot (S) = \frac{1}{P} \prod_{i=1}^{n+1} \frac{1}{x_i},$$

and the prove is completed through induction. Therefore,

$$\sum \frac{1}{Q(\sigma)} = \prod_{i=0}^{n} \frac{1}{2^{i-1}}.$$

□ If in a triangle ABC $2m(B) = m(A) + m(C)$ then $2b \geq a + c$.

Solution

By the Law of Sines, we have $a = \left(\frac{sin(A)}{sin(B)}\right) b$ and $c = \left(\frac{sin(C)}{sin(B)}\right) b$. Plugging these into the inequality yields

$sin(A) + sin(C) \leq 2sin(B)$.

Since $2m\angle B = m\angle A + m\angle C$, we have $m\angle B = 60$. Thus the above inequality turns into

$sin(A) + sin(C) \leq \sqrt{3}$.

Now since $m\angle A + m\angle C = 120$, let $A = 60 - \theta$ and $C = 60 + \theta$. Then after the use of the $sin(\alpha+\beta)$ identity, we have

$\sqrt{3}cos(\theta) \leq \sqrt{3}$

which is definitely true.

□ Let P_n be the set of subsets of $\{1, 2, ..., n\}$. Let $c(n, m)$ be the number of functions $f : P_n \to \{1, 2, ..., m\}$ such that $f(A \cap B) = min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n$$

Solution

Let $S = \{1, 2, ...n\}$. Let $s_i = \{i\}, i = 1, 2, ...n$. For a set $a \in P_n$, let a^{-1} denote the complement of a with respect to S.

Consider $f(S)$. For any $A \in P_n$ we have

$f(S \cap A) = f(A) = \min\{f(S), f(A)\}$

So that we have $f(S) \geq f(A) \forall A \in P_n$. In other words, if $f(S) = j$ then there are only j possibilities $1, 2, ...j$ for any other $f(A)$. This suggests that we group our functions f according to their value of $f(S)$, which sets a maximum.

Given a value of $f(S) = j$, I claim that the values of $f(a_i^{-1}), i = 1, 2, ...n$ uniquely determine f. There are n values and they can take on j different values for a total of j^n possibilities as j ranges from 1 to m - precisely our desired summation.

We have defined f for subsets missing no members (S) and for subsets missing one member (a_i^{-1}). The values of f at subsets missing two members can be defined in terms of the subsets missing each of the two members; in other words,

$f(\{x, y\}^{-1}) = \min\{f(a_x^{-1}), f(a_y^{-1})\}$

Similarly, it's obvious from induction that for any $A \in P_n$ we have

$f(A^{-1}) = \min\{f(a_x^{-1}) \,|\, x \in A\}$

Because A ranges across all of P_n, so does A^{-1}, and so every value in the domain of f is uniquely defined.

Hence for a given value of $f(S)=j$ we have j^n possible functions, so that we can conclude that across all possible values of j there are

$c(n,m)=\sum_{j=1}^{m} j^n$

Possible functions $f: P_n \to \{1,2,...m\}$. QED.

$\square$ Let $S_n=\sum_{k=1}^{n} \frac{F_k}{2^k}$. Where $F_1=F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for $n \geq 3$.

Find a formula for S_n without using induction.

Solution

$$\sum_{k=1}^{n}\frac{F_k}{2^k}=\sum_{k=1}^{n}\frac{\phi^k-(1-\phi)^k}{2^k\sqrt{5}}=\frac{1}{\sqrt{5}}\left[\sum_{k=1}^{n}\left(\frac{\phi}{2}\right)^k-\sum_{k=1}^{n}\left(\frac{1-\phi}{2}\right)^k\right]=\frac{1}{\sqrt{5}}\left[\left(\frac{\phi}{2}\right)\left(\frac{\left(\frac{\phi}{2}\right)^n-1}{\left(\frac{\phi}{2}\right)-1}\right)-\left(\frac{1-\phi}{2}\right)\left(\frac{(1\ldots}{(\ldots}\right.\right.$$

where $\phi=\frac{1+\sqrt{5}}{2}$. You can prove the substitution for F_k using either induction or the characteristic equation of the fibonacci sequence. The rest requires some simplification which is left to the reader.

Another way

$$S_n=\sum_{k=1}^{n}\frac{F_k}{2^k}$$

$$S_n+2S_n=\sum_{k=1}^{n}\frac{F_k}{2^k}+\sum_{k=0}^{n-1}\frac{F_{k+1}}{2^k}=1+\sum_{k=1}^{n-1}\frac{F_{k+2}}{2^k}+\frac{F_n}{2^n}=1+\frac{F_n}{2^n}+(4S_n+\frac{F_{n+1}}{2^{n-1}}-\frac{F_1}{2^{-1}}-\frac{F_2}{2^0})=4S_n+\frac{F_n}{2^n}+\frac{2F_{n+1}}{2^n}-2=4\ldots$$

Therefore $S_n=2-\frac{F_{n+3}}{2^n}$

$\square$ Let a,b,c be positive real numbers with $a^2+b^2+c^2=3$.Prove that the following inequality occurs:

$$4(a^3+b^3+c^3)\geq 3(a+b+c+abc)$$

Solution

Let the power mean of order t of numbers a,b,c be μ_t, i.e. $\mu_t=\left(\frac{a^t+b^t+c^t}{3}\right)^{\frac{1}{t}}$ for $t\neq 0$ and $\mu_0=\sqrt[3]{abc}$. Then the ineqality rewrites as

$$4\cdot 3\mu_3^3\geqslant 3\cdot\left(3\mu_1+\mu_0^3\right)$$

for $\mu_2=1$. Then cancelling 3 at both sides and homogenising, we get

$$4\mu_3^3\geqslant 3\mu_1\cdot\mu_2^2+\mu_0^3.$$

But from the power mean inequality we have $3\mu_3^3\geqslant 3\mu_1\mu_2^2$ and $\mu_3^3\geqslant\mu_0^3$ which ends the proof.

$\square$ Prove that the equation $m^4+n^4=a^2$ is not possible in integers m,n,a all of which are different from zero.

Solution

Assume to the contrary that there exists a solution to the equation $m^4+n^4=a^2$, and that there exists a minimal a. Using Gaussian integers $\mathbb{Z}[i]$, or simply by knowing Pythagorean triples,

$$m^2=2uvn^2=|u^2-v^2|a=u^2+v^2.$$

Since $\gcd(u,v)=1$, one of u,v must be odd, but if v is even, we get that $n^2\equiv 3 \pmod 4$. Contradiction. So u is even, that is, $u=2w$. As a result, $m^2=4vw \implies v=x^2,\ w=y^2,\ (v,w)=1$. So, $n^2=x^4-4y^4 \implies 2y^2=2\alpha\beta,\ n=|\alpha^2-\beta^2|,\ x^2=\alpha^2+\beta^2$. From $y^2=\alpha\beta,\ (\alpha,\beta)=1$, we have that $\alpha=\gamma^2,\ \beta=\delta^2 \implies x^2=\delta^4+\gamma^4$, contradicting the minimality of a.■

Remark. This also leads to Fermat's Last Theorem $x^n + y^n = z^n$ for $n = 4$.

$\square$ If $n > 1$, find the two smallest integral values of n for which $x^2 + x + 1$ is a factor of $(x+1)^n - x^n - 1$, over the set of polynomials with integer coefficients.

Solution

Solution. If $x^2+x+1|(x+1)^n - x^n - 1$, then $(\omega+1)^n - \omega^n - 1 = 0$, where $\omega = e^{i\frac{2\pi}{3}}$ is the third root of unity. Since $\omega^2+\omega+1 = 0$, we have $(-1)^n\omega^{2n} - \omega^n - 1 = 0 \implies (-1)^{n+1}\omega^{2n} + \omega^n + 1 = 0$.

Lemma. Let ζ be the n-th root of unity. Then $\sum_{j=0}^{n-1} \zeta^{jk} = n$ if $n|k$, and equal to 0 otherwise.

Proof. A direct application of geometric summation. The proof is left as an exercise.

Ipso facto of the above [i]lemma[/i], n must be odd and $3 \not| n$. The two smallest such n are 5 and 7. ■

Remark. The method above can easily be generalized.

$\square$ Find the biggest n that divides $a^{25} - a$ for all a.

Solution

We consider the set of primes p such that $\varphi(p)|24$. (We do not consider the prime powers because $p^{25} - p$ cannot be divisible by any powers of p.)

Firstly, the divisors of 24:

$1, 2, 3, 4, 6, 8, 12, 24$

This means we have

$p = 2, 3, 5, 7, 13$

Our maximal n is therefore

$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = \boxed{2730}$.

$\square$ The graph of $f(x) = x^4 + 4x^3 - 16x^2 + 6x - 5$ has a common tangent line at $x = p$ and $x = q$. Compute the product pq.

Solution

Say the tangent line has equation $y = Ax + B$. Then the polynomial $f(x) - Ax - B$ would have the x-axis as its common tangent, so then p, q would both be double roots.

Ergo,

$x^4 + 4x^3 - 16x^2 + (6 - A)x - (5 + B) = (x - p)(x - p)(x - q)(x - q)$

Vieta's sums give

$4 = -2p - 2q \Leftrightarrow p + q = -2 \implies p^2 + 2pq + q^2 = 4$ $-16 = p^2 + q^2 + 4pq \implies -16 = 4 + 2pq \Leftrightarrow pq = \boxed{-10}$.

Remark: The statement to find the product pq should scream Vieta's to a seasoned problem-solver. The problem is to figure out what kind of polynomial would have roots p, q.

Another approach $x^4 + 4x^3 - 16x^2 + 6x - 5 = (x^2 + 2x - 10)^2 + 24x - 105 \Longleftrightarrow x^4 + 4x^3 - 16x^2 + 6x - 5 - (24x - 105) = (x^2 + 2x - 10)^2$.

$\square$ For a positive integer n, let $r(n)$ denote the sum of the remainders when n is divided by $1, 2, ..., n$ respectively. Prove that $r(k) = r(k-1)$ for infinitely many integers.

Solution

Solution sketch"]For any $p \leq 2^n - 1$ which is not a power of 2, the remainder when p divides 2^n is exactly one more than the remainder when p divides $2^n - 1$.

For $p = 2^m \leq 2^n - 1$, the remainder when p divides 2^n is 0 and the remainder when p divides $2^n - 1$ is $2^m - 1$.

Thus $r(2^n) - r(2^n - 1) = 1 \cdot (2^n - n - 1) - \sum_{m=0}^{n-1} 2^m - 1 = 0$, so we're done.

At least up to $n = 4096$ the powers of 2 are the only numbers with this property.

$\square$ Prove that for every natural number $n \geq 4$, there exists at least one natural number m, such that

$$n! < m < (n+1)!$$

and $n^3|m$.

Solution

A non-constructive proof is much simpler. Consider the set $A = \{n!+1, n!+2, \ldots, (n+1)!-1\}$. If the size of A is at least n^3, then since all the integers in A are consecutive, some multiple of n^3 must belong to A. This means that

$n \cdot n! - 1 \geq n^3$

for all integers $n \geq 4$. This is obvious, but if necessary, this inequality can be rigorously proved by a few cases of induction.

$\square$ Find all integers n for which $2^{1994} + 2^{1998} + 2^{1999} + 2^{2000} + 2^{2002} + 2^n$ is a perfect square.

Solution

$2^{1994}+2^{1998}+2^{1999}+2^{2000}+2^{2002}+2^n = x^2$ $1+2^4+2^5+2^6+2^8+2^{(n-1994)} = \frac{x^2}{2^{1994}}$ $369+2^{(n-1994)} = \left(\frac{x}{2^{997}}\right)^2$ $a = n-1994$ $b^2 = \left(\frac{x}{2^{997}}\right)^2$ $369+2^a = b^2$ $0+1, 2 = 0, 1 \pmod 3$, so looking at the multiplicative group mod 2, we see that $2 \mid a$ and that $3 \not| \, b$ $a = 2c$ $369+2^{2c} = b^2$ $(b+2^c)(b-2^c) = 41 \cdot 3^2$ $41-9$ is the only set of factors that will form a power of 2, so:

$b+2^c = 41$ $b-2^c = 9$ $2^{c+1} = 32$ $c = 4$ $2c = a = n-1994$, so:

$n = 2002$ $x = 25 \cdot 2^{997}$

$\square$ Let O be a given point, let $P_1, P_2, ...P_n$ be vertices of a regular n-gon, and let $Q_1, Q_2, ...Q_n$ be given by

$$\vec{OQ_i} = \vec{OP_i} + \vec{P_{i+1}P_{i+2}}$$

where we interpret $P_{n+1} = P_1$, etc. Prove that $Q_1, Q_2, ...Q_n$ are vertices of a regular n-gon.

Solution

We can use complex numbers. Let O be 0, P_1 be 1. Then by putting $\omega := e^{\frac{2\pi i}{n}}$ we get that the vector $O\vec{P}_k$ correspons with ω^k. Thus vector $O\vec{Q}_k$ is represented by $\omega^k + \omega^{k+2} - \omega^{k+1} = \omega^k(1-\omega+\omega^2)$, but it is just ω^k, i.e. $O\vec{P}_k$ after roation and scaling equivalent to multiplying by $1-\omega+\omega^2$. Hence the polygon $Q_1 \ldots Q_n$ is similar to the polygon $P_1 \ldots P_n$, so it is regular.

$\square$ Prove that $\frac{R}{r} > \frac{b}{a} + \frac{a}{b}$, where a, b are different sides of a triangle.

Solution

we will use the fact that : $l_a \geq h_a$, so : $l_a^2 = \frac{4pbc(p-a)}{(b+c)^2}$ but $(b+c)^2 \geq 4bc$, therefore: $l_a^2 \leq p(p-a)$

but from $l_a \geq h_a$ we have : $\frac{2}{b} + h_c^2 \leq l_b^2 + l_c^2 \leq ap$, but $h_b = \frac{2S}{b}$ and $h_c = \frac{2S}{c}$

$\Rightarrow 4S^2(\frac{1}{b^2} + \frac{1}{c^2}) \leq ap$, and multypling by $\frac{bc}{4S} \iff \frac{b}{c} + \frac{c}{b} \leq \frac{R}{r}$

$\square$ A finite sequence $a_1, a_2, \ldots, a_n$ is called p-balanced if any sum of the form $a_k + a_{k+p} + a_{k+2p} + \cdots$ is the same for $k = 1, 2, \ldots, p$. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal to zero. Hint Denote $P(x) = a_{50}x^{49} + a_{49}x^{48} + \cdots + a_2x + a_1$. Let $\omega_p = e^{\frac{2\pi i}{p}}$. We know that

$\frac{1}{p}\sum_{i=0}^{p-1} P(\omega_p^i) = a_1 + a_{1+p} + a_{1+2p} + \cdots$.

$\square$ Prove

$$\sqrt[n]{n!} \leq \prod_{p|n!} p^{\frac{1}{p-1}}$$

Solution

Lemma 1: $\lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + ... \leq \frac{n}{p-1}$

Proof: remove the floor functions to get

$$\frac{n}{p}+\frac{n}{p^2}+...=\frac{n}{p-1}$$

Now, we note that

$$\prod_{p|n} p^{\frac{n}{p-1}} \geq \prod_{p|n} p^{\lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \dots} = n!$$

Simply take the nth root and we are done. Make the solution as simple as possible

☐ Find all primes a, b, c, d, e, not necessarily distinct, such that: $a^2+b^2=c^2+d^2+e^2$

Solution

If p is an odd prime, $p \in \{1,3,5,7\} \implies p^2 \equiv 1 \pmod 8$. If p is an even prime, $p=2 \implies p^2 \equiv 4 \pmod 8$. Therefore all primes p are $1, 4 \pmod 8$. Our equation becomes

$$\{1,4\}+\{1,4\} \equiv \{1,4\}+\{1,4\}+\{1,4\}$$

where $\{a,b\}$ denote exactly one of a and b. The left hand side is either 2, 5, or $8 \equiv 0$ in modulo 8. The right hand side is either 3, 6, $9 \equiv 1$, or $12 \equiv 4$ in modulo 8. Therefore, there are no possible solutions. ■

☐ Let $(m,n)=p$, where p is a prime. Prove $\varphi(mn)=\frac{p}{p-1}\cdot\varphi(m)\cdot\varphi(n)$.

Solution

Let $m=pa, n=pb; \gcd(a,b)=1; a=\prod q_i^{a_i}, r=\prod r_i^{b_i}; p_i, r_i$ are primes.

Then:

$$\begin{aligned}
\phi(mn)=\phi(p^2ab)&=p(p-1)\cdot a\prod\frac{q_i-1}{q_i}\cdot b\prod\frac{r_i-1}{r_i}\\
&=\frac{p^3}{(p-1)}\cdot a\prod\frac{q_i-1}{q_i}\cdot\frac{p-1}{p}\cdot b\prod\frac{r_i-1}{r_i}\cdot\frac{p-1}{p}\\
&=\frac{p}{(p-1)}\cdot ap\prod\frac{q_i-1}{q_i}\cdot\frac{p-1}{p}\cdot bp\prod\frac{r_i-1}{r_i}\cdot\frac{p-1}{p}\\
&=\frac{p}{\phi p}\phi(m)\cdot\phi(n)
\end{aligned}$$

☐ Calculation of positive integer ordered pairs (x,y,z) in $3^x-5^y=z^2$

Solution

Since y is positive integer number then $5^y \equiv 1 \pmod 4$. If x odd then $3^x \equiv 3 \pmod 4$. Therefore $3^x-5^y \equiv 2 \pmod 4$, a contradiction because $z^2 \equiv 0,1 \pmod 4$. Thus, x is even. Let $x=2m$ with $m \in \mathbb{N}^*$. Since x is even then $3^x \equiv 1 \pmod 8$. We also have $z^2 \equiv 0,1,4 \pmod 8$. It follows that y is even. Let $y=2n$ with $n \in \mathbb{N}^*$. From the equation we have $(3^m-5^n)(3^m+5^n)=z^2 \quad (1)$. Let $\gcd(3^m-5^n, 3^m+5^n)=d$ with $d \in \mathbb{N}^*$. Therefore $d|(3^m+5^n)-(3^m-5^n)$ or $d|2\cdot5^n$. But $5 \nmid 3^m-5^n$ so $d|2$. We have $d \in \{1;2\}$.

If $d=1$ then we have $3^m-5^n=p^2, 3^m+5^n=q^2$ implies $(q-p)(q+p)=2\cdot5^n$. Since $p+q-(p-q)=2q$ is even then we implies $p+q$ and $p-q$ are both even, thus $4|(p-q)(p+q)$, a contradiction.

If $d=2$ then $2|z$. Let $z=2z_1, 3^m-5^n=2r, 3^m+5^n=2h$ with $(r,h)=1,\ r,h,z_1 \in \mathbb{N}$. From (1) we have $rh=z_1^2$. Since $(r,h)=1$ then $r=a^2, h=b^2$ with $a,b \in \mathbb{N}^*$. Therefore $2b^2-2a^2=2\cdot5^n$. It follows that $(b-a)(b+a)=5^n$.

☐ Calculate $(tan\frac{\pi}{7})^2+(tan\frac{2\pi}{7})^2+(tan\frac{4\pi}{7})^2$

Solution

We have, by the [b]de Moivre [/b]formula, $\left(\cos\dfrac{k\pi}{7}+\mathrm{i}\sin\dfrac{k\pi}{7}\right)^7=\cos k\pi+\mathrm{i}\sin k\pi=(-1)^k$, for all $1\le k\le 6$. Denoting $s=\sin\dfrac{k\pi}{7}$ and $c=\cos\dfrac{k\pi}{7}$, and expanding by the [b]Newton[/b]'s binomial formula, and equalling the imaginary parts we get $s^7-\binom{7}{5}s^5c^2+\binom{7}{3}s^3c^4-\binom{7}{1}sc^6=0$. Now divide by sc^6 to get the equation $t^6-21t^4+35t^2-7=0$, of roots $t=\tan\dfrac{k\pi}{7}$, for all $1\le k\le 6$. By [b]Viète[/b]'s relations, $\sum_{k=1}^6\tan^2\dfrac{k\pi}{7}=\left(\sum_{k=1}^6\tan\dfrac{k\pi}{7}\right)^2-2\sum_{1\le p<q\le 6}\tan\dfrac{p\pi}{7}\tan\dfrac{q\pi}{7}=0+42=42$. But $\tan^2\dfrac{3\pi}{7}=\tan^2\dfrac{4\pi}{7}$, $\tan^2\dfrac{5\pi}{7}=\tan^2\dfrac{2\pi}{7}$, and $\tan^2\dfrac{6\pi}{7}=\tan^2\dfrac{\pi}{7}$, therefore $\tan^2\dfrac{\pi}{7}+\tan^2\dfrac{2\pi}{7}+\tan^2\dfrac{4\pi}{7}=\dfrac{1}{2}\sum_{k=1}^6\tan^2\dfrac{k\pi}{7}=21$.

– Solve the equation $x^2+2=\sqrt{2^x}+4\log_2 x$ – □ $x,y,z\in\mathbb{Z}$ Solve the equation $x^2+3y^2=z^2$?

Solution

Rearranging yields $3y^2=(z-x)(z+x)$. There exist integers n and q such that $z-x=n$, $z+x=3q$ and $y^2=qn$. There also exist integers such that $y=rs=tu$, $q=tu$ and $n=us$. Again, there exist integers such that $r=ab$, $s=cd$, $t=ac$ and $u=db$. Combining the equations, it follows that

$$x=\frac{3a^2bc-bcd^2}{2},$$

$$y=abcd$$

and

$$z=\frac{3a^2bc+bcd^2}{2}.$$

□ to hợp □□□□ số học □ tổ hợp hay □ số học □ tổ hợp

□ Determine the real values of the parameter m so that inequality $mx^2+(m+1)x+m-1>0$ hasn't real solutions.

Solution

Observe that $mx^2+(m+1)x+m-1>0 \iff f(x)<m$, where $f(x)=\frac{1-x}{x^2+x+1}$. Prove easily that the range of f is $\Im(f)=\left[1-\frac{2}{\sqrt3},1+\frac{2}{\sqrt3}\right]$. Therefore, the inequality $mx^2+(m+1)x+m-1>0$ hasn't real solutions $\iff$ the inequality $f(x)<m$ hasn't real solutions $\iff$ $\Im(f)\subset(m,\infty)$, i.e. $\boxed{m<1-\dfrac{2}{\sqrt3}}$.

□ Find all the prime numbers $p_1,p_2,p_3,...,p_n$ such that $p_1p_2p_3.....p_n=10(p_1+p_2+p_3+....+p_n)$

Solution

Since $10\mid RHS$ we must have WLOG $(p_1,p_2)=(2,5)$. Plugging in and dividing by 10 yields $\prod_{i=1}^n q_i=7+\sum_{i=1}^n q_i$ for $q_i\in\mathbf{P}$. Clearly $n=1$ cannot work, so we try $n=2$ to get $q_1q_2=7+q_1+q_2\iff q_1q_2-q_2-q_2=7\iff(q_1-1)(q_2-1)=8=2^3$, and trying possible factors of 8 we find the unique solution $(q_1,q_2)=(3,5)$, or $(p_1,p_2,p_3,p_4)=(2,3,5,5)$. Now let $q_1\le q_2\le\cdots\le q_n$. Note that $\prod_{i=1}^n x_i\ge\sum_{i=1}^n x_i$ for any sequence of $\{x_i\}_{i=1}^n$ such that each $x_i\ge 2$. This follows from the fact that $x_1x_2\ge x_1+x_2\iff(x_2-1)(x_2-1)-1\ge 0$, and applying this fact repeatedly easily yields the result for higher n. Applying this to our q_i equation, we see that $7+\sum_{i=1}^n q_i=\prod_{i=1}^n q_i\ge\left(\sum_{i=1}^{n-1}q_i\right)q_n$. Letting $P=\prod_{i=1}^{n-1}q_i$ and $S=\sum_{i=1}^{n-1}q_i$ we get $7+S+q_n\ge Sq_n\iff(q_n-1)(S-1)\le 8$. Testing the (highly limited) possible values of q_n while keeping in mind that $n\ge 3\iff S\ge 2+2+2=8$ (since we already did the case $n=2$ above), we see that $(q_n-1)(S-1)\ge 7(q_n-1)\ge 7(1)=7<8$, however this assumes that $q_1=\cdots=q_4=2$ which (checking by hand) cannot happen, so $q_n\ge 3$

and $7(q_n-1) \geq 7(3-2) = 7(2) = 14 > 8$, contradiction. Thus, the only solution is the one we found above, namely $\boxed{(p_1,p_2,p_3,p_4) = (2,3,5,5)}$.

☐ Solve for x in terms of c, $c < 1$,

$$\frac{1+\ln x}{x} = c$$

Solution

$$\begin{aligned} 1+\ln x &= cx \\ e^{-1-\ln x} &= e^{-cx} \\ \tfrac{1}{ex} &= e^{-cx} \\ -\tfrac{c}{e} &= -cxe^{-cx} \\ \mathrm{W}(-\tfrac{c}{e}) &= -cx \\ \therefore\ -\tfrac{1}{c}\mathrm{W}(-\tfrac{c}{e}) &= x \end{aligned}$$

Be aware that $\mathrm{W}(z)$ may take zero, one, or two real values. The condition $0 < c < 1$ ensures there will be two solutions. For $c \leq 0$, there will be one.

☐ Determine all integers x, y that satisfy the equation

$$x^3 = y^2 + 2.$$

Solution

Consider the UFD $\mathbb{Z}[\sqrt{-2}]$. We get $x^3 = (y+i\sqrt{2})(y-i\sqrt{2})$. Let $d = (y+i\sqrt{2}, y-i\sqrt{2})$. This means $d|2i\sqrt{2}$. Let $\zeta(a+bi\sqrt{2}) = a^2+2b^2$. It is easy to check that ζ is multiplicative.

Lemma 1: $i\sqrt{2}$ is irreducible. Proof: Let $i\sqrt{2} = cd$ for $c \in \mathbb{Z}[\sqrt{-2}]$. We know $\zeta(i\sqrt{2}) = \zeta(c)\zeta(d) = 0^2 + 2\cdot 1^2 = 2$. Therefore, one of $\zeta(c), \zeta(d)$ is one, meaning one of c, d is a unit.

This means d is either $1, 2, i\sqrt{2}$, or $2i\sqrt{2}$. Clearly $2 \not| (y+i\sqrt{2})$ so we can narrow d down to 1 or $i\sqrt{2}$.

If $i\sqrt{2}|(y+i\sqrt{2})$, then $i\sqrt{2}|y$, meaning y is even, so $x^3 \equiv 2 \pmod 4$, which is impossible. Therefore, d must equal 1.

Now we know $y+i\sqrt{2} = u(p+iq\sqrt{2})^3$, $y-i\sqrt{2} = v(r+is\sqrt{2})^3$ for $p,q,r,s \in \mathbb{Z}$ and $u, v = \pm 1$

Comparing the imaginary parts of the first equation, we get $\pm 1 = 3p^2q - 2q^3 = q(3p^2-2q^2)$. It is clear that this means $p, q = \pm 1$ so $y+i\sqrt{2} = (1+i\sqrt{2})^3 = -5+i\sqrt{2}$ or $y+i\sqrt{2} = (-1+i\sqrt{2})^3 = -5+i\sqrt{2}$ (their negatives would have a negative $i\sqrt{2}$ term). Therefore, $y = \pm 5$ and $x = 3$. ■

☐ Suppose that f is bounded and for $a \leq x \leq b$ and, for every pair of values x_1, x_2, with $a \leq x_1 \leq x_2 \leq b$,

$$f(\frac{1}{2}(x_1+x_2)) \leq \frac{1}{2}(f(x_1)+f(x_2)).$$

Prove that

$$f(x+\delta) - f(x) \leq \frac{1}{2}(f(x+2\delta)-f(x)) \leq \cdots \leq \frac{1}{2^n}(f(x+2^n\delta)-f(x))$$

, $a \leq x + 2^n\delta \leq b$.

Solution

To prove $\frac{1}{2^{n-1}}(f(x+2^{n-1}\delta)-f(x)) \le \frac{1}{2^n}(f(x+2^n\delta)-f(x))$, note it is equivalent to $\frac{f(x)+f(x+2^n\delta)}{2} \ge f(x+2^{n-1}\delta)$. And this is true by what we are given, $x_1 = x, x_2 = x+2^n\delta$. Now we're done.

□ Find all the integers written as $\overline{abcd}$ in decimal representation and $\overline{dcba}$ in base 7.

Solution

First of all $a,b,c,d \in \{0,1,2,3,4,5,6\},\ a,d \neq 0$.

$1000a+100b+10c+d = a+7b+49c+343d \implies 333a+31b = 13c+114d$. The maximum for the right-hand side is $6(13+114) = 762$, so $0 < a < 3$.

If $a = 2$ then $666+31b = 13c+114d$. (Notice that $b \equiv c \pmod 6$.) The minimum d is found by minimizing b and maximizing c: $666 = 78+114d \implies d > 5$. So $d = 6 \implies 31b = 13c+18 \implies 13(b-c) = 18(1-b) \implies b = c = 1$, giving the solution $2116_{10} = 6112_7$.

If $a = 1$ then $333+31b = 13c+114d$, so $b+3 \equiv c \pmod 6$ and $c = b \pm 3$. If $c = b+3$, then $333+31b = 13b+39+114d \implies 148 = 3(19d-3b)$, no solution. Otherwise say $c = b-3$ and $333+31b = 13b-39+114d \implies 62 = 19d-3b$. Then $d > 3,\ 2 \equiv d \pmod 3 \implies d = 5$, no solution.

□ Let ABC be a triangle with the incircle $C(I,r)$. Prove that $(\forall)\ E \in (AB)$ and $(\forall)\ F \in (AC)$ so that $I \in EF$

there are the inequalities $\left\{\begin{array}{ll} 1 \blacktriangleright & AE+AF \ge 4r \\ & \\ 2 \blacktriangleright & \frac{1}{AE}+\frac{1}{AF} \le \frac{1}{r} \end{array}\right\|$. When each from these inequalities comes an equality ?

Solution

Let D be the intersection of lines AI and BC. Then, $\frac{DI}{IA} \cdot BC = \frac{BE}{EA} \cdot DC + \frac{CF}{FA} \cdot DB$. If $AE = x$ and $AF = y$, since $\frac{DI}{IA} = \frac{BC}{AB+AC} = \frac{a}{b+c}$, $DB = \frac{ac}{b+c}$ and $DC = \frac{ab}{b+c}$, we have $\frac{a}{b+c} \cdot a = \frac{c-x}{x} \cdot \frac{ab}{b+c} + \frac{b-y}{y} \cdot \frac{ac}{b+c} \Rightarrow a = \frac{bc}{x} - b + \frac{bc}{y} - c \Rightarrow bc(\frac{1}{x}+\frac{1}{y}) = a+b+c \Rightarrow \frac{1}{x}+\frac{1}{y} = \frac{a+b+c}{bc}$. (Alternatively, we could have used vectors) By applying Cauchy's inequality, we obtain $\frac{a+b+c}{bc} = \frac{1}{x}+\frac{1}{y} \ge \frac{4}{x+y} \Rightarrow x+y \ge \frac{4bc}{a+b+c} = \frac{4 \cdot \frac{2pr}{\sin A}}{2p} = \frac{4r}{\sin A} \ge 4r$, since $\sin A \in (0,1]$, so the first inequality is proven. Similarly, for the second inequality, $\frac{1}{x}+\frac{1}{y} = \frac{a+b+c}{bc} = \frac{2p}{\frac{2pr}{\sin A}} = \frac{\sin A}{r} \le \frac{1}{r}$. For both inequalities, equality holds if and only if $\sin A = 1$, so when triangle ABC is right-angled.

□ For a triangle ABC let its circumcircle be (O) and a point P be on the small arc AB .

A line passing through P and perpendicular to OA meets AB , CA at D , E respectively .

A line passing through P and perpendicular to OB meets AB , BC at F , G respectively.

Prove that $DP = DE \iff FP = FG \iff$ the line AP is the C-symmedian in $\triangle ABC$. **Lemma.**

In $\triangle\ ABC$ consider a point $M \in [BC]$ and denote $\delta_d(X)$ - the **distance** from X to the line d . Then $MB = MC \iff$

$\delta_{AM}(B) = \delta_{AM}(C) \iff AB \cdot \sin\widehat{MAB} = AC \cdot \sin\widehat{MAC} \iff \sin C \cdot \sin\widehat{MAB} = \sin B \cdot \sin\widehat{MAC}$

.

Proof of the proposed problem.

Denote the midpoint M of the side $[AB]$, the intersections $\left\|\begin{array}{c} X \in PE \cap OA \\ Y \in PG \cap OB \\ S \in CP \cap AB \end{array}\right\|$ and $\left\|\begin{array}{c} m(\angle PAB) = x \\ m(\angle PBA) = y \end{array}\right\|$ .Observe that $x+y = C$

and the quadrilaterals $OXDM$, $OYFM$ are cyclically, i.e. $m(\angle ADE) = m(\angle PDF) = m(\angle PFD) = m(\angle BFG) = C$. Therefore,

$PD = PF$, $\left\|\begin{array}{c} m(\angle APE) = C-x = y \\ m(\angle BPG) = C-y = x \end{array}\right\|$ and $\left\|\begin{array}{c} m(\angle AEP) = B \\ m(\angle BGP) = A \end{array}\right\|$ (lines DE , FG are **an-**

tiparallels to BC , AC in $\triangle\ ABC$).

Apply the upper lemma in the triangles PAE and PBG to the cevians AD , BF respectively :

$$\left\|\begin{array}{ccccc} DE = DP & \iff & \sin\widehat{APE}\cdot\sin\widehat{DAE} = \sin\widehat{AEP}\cdot\sin\widehat{DAP} & \iff & \sin y\cdot\sin A = \sin B\cdot\sin x \\ \\ FG = FP & \iff & \sin\widehat{BPG}\cdot\sin\widehat{FBG} = \sin\widehat{BGP}\cdot\sin\widehat{FBP} & \iff & \sin x\cdot\sin B = \sin A\cdot\sin y \end{array}\right\|$$

.

In conclusion, $\boxed{DE = DP \iff b\cdot\sin x = a\cdot\sin y \iff FG = FP}$. Observe that **in this case**

$\frac{SA}{SB} = \frac{CA}{CB}\cdot\frac{\sin\widehat{SCA}}{\sin\widehat{SCB}} = \frac{b}{a}\cdot\frac{\sin y}{\sin x} = \frac{b^2}{a^2}$, i.e. **in this case** the point S is the foot of the C-symmedian in the triangle ABC .

☐ Solve the equation $\sqrt[3]{1-x}+\sqrt[3]{1+x} = \frac{x^2+2}{\sqrt{x^2+1}}$ (**without derivatives**).

Solution

Denote the set S of the zeroes for our equation. Thus, $0\in S$ and $x\in S \iff -x\in S$. We can suppose w.l.o.g. that $x>0$. Observe that $\frac{x^2+2}{\sqrt{x^2+1}} = \sqrt{x^2+1}+\frac{1}{\sqrt{x^2+1}}\ge 2$, $(\forall)x\in\mathbb{R}$, particularly and for $x>0$, with equality iff $x=0$.

Since for $x\in(-1,1)$, $\sqrt[3]{1-x}+\sqrt[3]{1+x}<2$, obtain that our equation hasn't zeroes in $(-1,1)^*$. For $x\ge 1$ have $\frac{x^2+2}{\sqrt{x^2+1}} =$

$\sqrt{x^2+1}+\frac{1}{\sqrt{x^2+1}} > \sqrt{x^2+1} > \sqrt[3]{x^2+1} \ge \sqrt[3]{x+1} \ge \sqrt[3]{1-x}+\sqrt[3]{1+x} \implies \sqrt[3]{1-x}+\sqrt[3]{1+x} < \frac{x^2+2}{\sqrt{x^2+1}}$

$\implies x\notin S$. In conclusion our equation has an unique zero, $x=0$, i.e. $\boxed{\sqrt[3]{1-x}+\sqrt[3]{1+x} = \frac{x^2+2}{\sqrt{x^2+1}} \iff}$

.

Another way Perhaps it would be easier if you do this:

$\sqrt[3]{1-x}\le\frac{1+1+1-x}{3}$

and

$\sqrt[3]{1+x}\le\frac{1+1+1+x}{3}$

so left side is at most 2 and this is exactly when $x=0$.

On the other side you have:

$\frac{x^2+1}{\sqrt{x^2+1}}+\frac{1}{\sqrt{x^2+1}}\ge 2\sqrt{\frac{x^2+1}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}}} = 2$

so right side is at least 2 and this is exactly when $x=0$.

Thus only solution is $x=0$

☐ A equation $f(x)\equiv ax^3+bx^2+cx+d=0$, $a\ne 0$ has three real roots x_k , $k\overline{1,3}$. Prove that the tangent TT to G_f at the point $T\in G_f$ with $x_T = \frac{x_1+x_2}{2}$ cut the X-axis in the point $R(x_3,0)$.

Lemma. Let $f(x) = ax^3+bx^2+cx+d$, $x\in\mathbb{R}$ be a real polynomial function, where $a\ne 0$. and the points $P_k\left(x_k, f\left(x_k\right)\right)$, $k\in\overline{1,3}$. Then $P_3\in P_1P_2 \iff x_1+x_2+x_3 = -\frac{b}{a}$.

Proof

$$P_3\in P_1P_2 \iff \begin{vmatrix} x_1 & ax_1^3+bx_1^2+cx_1+d & 1 \\ x_2 & ax_2^3+bx_2^2+cx_2+d & 1 \\ x_3 & ax_3^3+bx_3^2+cx_3+d & 1 \end{vmatrix} = 0 \iff$$

$(x_1-x_2)(x_2-x_3)(x_3-x_1)\left[a(x_1+x_2+x_3)+b\right] = 0 \iff x_1+x_2+x_3 = -\frac{b}{a}$.

Particular case. Using the above lemma in the proposed problem, $R\in TT \iff 2\cdot\frac{x_1+x_2}{2}+x_3 = -\frac{b}{a}$, what is truly.

An easy extension. Let $f(x) = ax^3+bx^2+cx+d$, $x\in\mathbb{R}$ be a real polynomial function, where $a\ne 0$.

Let d be a line which cut the graph G_f of the function f in the points P_k , $k\in\overline{1,3}$. For any $k\in\overline{1,3}$ the tangent in the point $P_k \in G_f$ cut again G_f in the point Q_k . Prove that the points Q_k , $k \in \overline{1,3}$ are collinearly.

Proof. Denote $P_k\,(x_k, f(x_k)) \in G_f$ and $Q_k\,(y_k, f(y_k)) \in G_f$, $k \in \overline{1,3}$. Thus, from the upper lemma,

$P_3 \in P_1P_2 \iff x_1 + x_2 + x_3 = -\frac{b}{a}$ and for any $k \in \overline{1,3}$ we have $Q_k \in P_kP_k \iff 2x_k + y_k = -\frac{b}{a}$.

Observe that $\sum_{k=1}^{3}(2x_k + y_k) = -\frac{3b}{a}$, i.e. $y_1 + y_2 + y_3 = -\frac{b}{a}$ what means from the same lemma that $Q_3 \in Q_1Q_2$.

Particular case. The extremum points, if they exist ($f'(x) = 0$) and the inflexion point of G_f ($f''(x) = 0$) are collinearly.

□ Let ABC be a triangle with the incircle $w = C(I, r)$ which touches $\triangle ABC$ in $Y \in CA$, $Z \in AB$.

Denote the midpoint M of $[BC]$ and $P \in YZ \cap AM$. Prove that $m(\angle BPC) > 90^\circ$.

Solution

Denote $X \in w \cap BC$, the orthocenter H of $\triangle ABC$, $L \in AH \cap MI$ and $D \in AH \cap BC$.

I"ll use two well-known properties (we can show easily them !) : $P \in IX$ and $AL = r$. Therefore, $\frac{PX}{IX} = \frac{AD}{LD} \iff PX = \frac{rh_a}{h_a - r} = \frac{r \cdot ah_a}{ah_a - ar} = \frac{r \cdot 2pr}{2pr - ar} \implies \boxed{PX = \dfrac{2S}{b+c}}$ (nice !). Otherwise

(**without the second mentioned properties**), $\frac{PX}{AD} = \frac{MX}{MD} = \frac{\frac{|b-c|}{2}}{\frac{|b^2-c^2|}{2a}} = \frac{a}{b+c} \implies PX = \frac{2S}{b+c}$.

Thus, $\boxed{m(\angle BPC) > 90^\circ \iff IX^2 < XB \cdot XC} \implies \frac{4S^2}{(b+c)^2} < (p-b)(p-c) \iff$
$4p(p-a) < (b+c)^2 \iff (b+c)^2 - a^2 < (b+c)^2 \iff 0 < a^2$, what is truly.

Remark. $\frac{PZ}{PY} = \frac{AZ}{AY} \cdot \frac{MB}{MC} \cdot \frac{AC}{AB} \implies \frac{PZ}{b} = \frac{PY}{c} = \frac{YZ}{b+c}$. From another well-known relation $[ZBC] \cdot PY + [YBC] \cdot PZ = [PBC] \cdot YZ \implies [ZBC] \cdot c + [YBC] \cdot b = [PBC] \cdot (b+c)$. Therefore, $a(p-b)\sin B \cdot c + a(p-c)\sin C \cdot b = a \cdot PX \cdot (b+c) \implies PX = \frac{2S}{b+c}$ because $h_a = b\sin C = c\sin B$
.

□ Let be $G = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 3x - 4y\}$ and $H = \{(x, y) \in \mathbb{R}^2 \mid 2xy = 4x + 3y\}$. Determine :

$M = \{z = x^2 + y^2 \mid (x, y) \in G \cap H\}.$

Solution

If $y = 0$, then $(0, 0) \in G \cap H$. Suppose $y \neq 0$. Thus, $(x, y) \in G \cap H \iff \left\| \begin{array}{rcl} x^2 - y^2 & = & 3x - 4y \\ 2xy & = & 4x + 3y \end{array} \right\| \implies$
$\frac{x^2 - y^2}{2xy} = \frac{3x - 4y}{4x + 3y}$ $(*)$.

Denote $t = \frac{x}{y}$. The relation $(*)$ becomes $\frac{t^2 - 1}{2t} = \frac{3t - 4}{4t + 3} \iff 4t^3 - 3t^2 + 4t - 3 = 0 \iff (4t - 3)(t^2 + 1) = 0 \iff t = \frac{3}{4}$.

Therefore, $\left\| \begin{array}{c} x = 3\lambda \\ y = 4\lambda \end{array} \right\| \implies 24\lambda^2 = 24\lambda\ ,\ \lambda \neq 0 \implies \lambda = 1 \implies \left\| \begin{array}{c} x = 3 \\ y = 4 \end{array} \right\|$. In conclusion, $G \cap H = \{(0, 0)\ ;\ (3, 4)\}$ and $M = \{0, 25\}$.

□ Solve the trigonometrical equation $32\cos^6 x - \cos 6x = 1$.

Solution

Method 1. $32\cos^6 x - \cos 6x = 1 \iff 16\cos^6 x = \cos^2 3x \iff 4\cos^3 x = \cos 3x \vee 4\cos^3 x + \cos 3x = 0 \iff$

$\cos x = 0 \vee 8\cos^2 x = 3 \iff \cos x = 0 \vee \cos 2x = -\frac{1}{4}$ a.s.o. I used the relations $\left| \begin{array}{c} 2\cos^2 \frac{\phi}{2} = 1 + \cos\phi \\ \cos 3\phi = \cos\phi \cdot (4\cos^2\phi - \end{array} \right.$
.

Method 2. Denote $z=\cos x+i\cdot\sin x$. Prove easily that for any $n\in\mathbb{N}$ we have $\cos nx=\frac{z^{2n}+1}{2z^n}$. Therefore,

$32\cos^6 x-\cos 6x=1 \iff \left(z^2+1\right)^6=\left(z^6+1\right)^2 \iff \left(z^2+1\right)^6=\left(z^2+1\right)^2\left(z^4-z^2+1\right)^2 \iff$
$\left(z^2+1\right)^2\left[\left(z^2+1\right)^4-\left(z^4-z^2+1\right)^2\right]=0 \iff \left(z^2+1\right)\left[\left(z^2+1\right)^2+\left(z^4-z^2+1\right)\right]\left[\left(z^2+1\right)^2-\left(z^4-z^2+\right.\right.$
$0 \iff$

$z^2\left(z^2+1\right)\left(2z^4+z^2+2\right)=0 \iff z\left(z^2+1\right)\left[2\left(z^2+1\right)^2-3z^2\right]=0 \iff$

$z\left(z^2+1\right)\left(z^2\sqrt{2}-z\sqrt{3}+\sqrt{2}\right)\left(z^2\sqrt{2}+z\sqrt{3}+\sqrt{2}\right)=0 \iff z\in\left\{\ 0\ ;\ \pm i\ ;\ \frac{\pm\sqrt{3}\pm i\sqrt{5}}{2\sqrt{2}}\ \right\}$ a.s.o.

☐ $ABCD$ is a parallelogram. Consider the points $X\in(BC)$ and $Y\in(CD)$. The areas of triangles ADY, XYC and ABX are 6 , 17 and 29 respectively. What is the area of the parallelogram ?

Solution

Denote $S=[ABCD]$, $AD=BC=a$, $AB=CD=b$ and $BX=x$, $DY=y$, i.e. $CX=a-x$, $CY=b-y$. Therefore,

$$\Longrightarrow\left\|\begin{array}{lll}\frac{[ABX]}{[ABC]}=\frac{BX}{BC} & \Longrightarrow & \frac{58}{S}=\frac{x}{a}\\ \frac{[ADY]}{[ADC]}=\frac{DY}{DC} & \Longrightarrow & \frac{12}{S}=\frac{y}{b}\\ \frac{[XCY]}{[BCD]}=\frac{CX\cdot CY}{CB\cdot CD} & \Longrightarrow & \frac{34}{S}=\frac{(a-x)(b-y)}{ab}\end{array}\right\|\Longrightarrow \frac{34}{S}=\frac{(a-x)(b-y)}{ab}=\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)=\left(1-\frac{58}{S}\right)\left(1-\frac{12}{S}\right)$$

$34\cdot S=(S-58)(S-12) \implies f(S)\equiv S^2-104\cdot S+12\cdot 58=0$. Since $f(29)<0$ and $29<S$ obtain $\boxed{S=52+2\sqrt{502}}$.

Remark. Observe that $[AXY]=2\sqrt{502}$.

If $[ABX]=m$, $[ADY]=n$, $[XCY]=p$ then $S^2-2(m+n+p)\cdot S+4mn=0$, i.e. $\boxed{S=m+n+p+\sqrt{(m+n+p)^2-4mn}}$.

For example, $m=8$, $n=2$, $p=7 \implies S=32$ and $[XAY]=15$. Thus the solution of our problem is a integer number.

☐ Consider two squares $ABCD$, $CEFG$, where $E\in(BC)$ and the line BC separates the points A , F . Denote $AB=a\ge b=CE$ and $H\in BG\cap DF$. Ascertain the area $[BDH]$.

Solution

Method I. Denote $I\in BC\cap DF$. Observe that $I\in AG$. In the trapezoid $ADFG$ exists the relation $\frac{1}{IC}=\frac{1}{AD}+\frac{1}{FG}$,

i.e. $IC=\frac{ab}{a+b}$ and $\boxed{IB=\frac{a^2}{a+b}}$. Thus, $\frac{[BDH]}{[BDG]}=\frac{BH}{BG}=\frac{BI}{BI+FG} \implies \implies \boxed{[BDH]=\frac{a^3(a+b)}{2(a^2+ab+b^2)}}$.

Remark. Denote $P\in AB\cap FG$. Thus, $\boxed{\frac{IC}{b}}=\frac{DC}{DG}=\frac{AI}{AG}=\frac{IB}{PG}=\boxed{\frac{IB}{a}}=\frac{IB+IC}{b+a}=\boxed{\frac{a}{a+b}}$.

In conclusion, $\frac{IC}{b}=\frac{IB}{a}=\frac{a}{a+b}$. Show easily that $[BDF]=\frac{a^2}{2}$ and $[BDH]\ge\frac{a^2}{3}$ for any $0<b\le a$.

Method II. I'll use same notations from first method. Denote $S\in AG\cap BD$. Observe that $[DBG]=\frac{a(a+b)}{2}$ and $I\in AG$. Apply the **Ceva's theorem** to the point I and the triangle BDG : $\frac{SB}{SD}\cdot\frac{CD}{CG}\cdot\frac{HG}{HB}=1 \implies$

$\frac{a}{a+b}\cdot\frac{a}{b}\cdot\frac{HG}{HB}=1 \implies \frac{HG}{b(a+b)}=\frac{HB}{a^2}=\frac{BG}{a^2+ab+b^2}$. Thus, $\frac{[BDH]}{[DBG]}=\frac{BH}{BG}=\frac{a^2}{a^2+ab+b^2} \implies$

$[BDH]=\frac{a^2}{a^2+ab+b^2}\cdot[DBG] \implies \boxed{[BDH]=\frac{a^3(a+b)}{2(a^2+ab+b^2)}}$.

Remark. You can solve similarly this problem if $ABCD$, $CEFG$ are two rhombus. Appears only the factor $\sin\phi$, where $\phi = m\left(\widehat{ABC}\right)$, i.e. $[BDH] = \frac{a^3(a+b)}{2(a^2+ab+b^2)} \cdot \sin\phi$.

$\square$ Let $ABCD$ be a convex quadrilateral . Denote $O \in AC \cap BD$. Prove that if the perimeters of AOB , BOC , COD , AOD are equally, then $ABCD$ is a rhombus.

Solution

$$\left\|\begin{array}{ll} AB+AO=CB+CO & (1) \\ \\ BC+BO=DC+DO & (2) \\ \\ CD+CO=AD+AO & (3) \\ \\ DA+DO=BA+BO & (4) \end{array}\right\|$$

. Adding the relations (1) , (3) obtain $AB+CD=AD+CB$, i.e. $ABCD$ is a tangential quadrilateral. Denote the tangent points $M \in (AB)$, $N \in (BC)$, $P \in (CD)$,

$R \in (DA)$ of the incircle of $ABCD$ with the its sides. It is well-known $O \in MP \cap NR$. Since $BM = BN$,

the relation (1) becomes $AM + AO = CN + CO$. But $AM = AR$ and for $\triangle AOR$ si $\triangle CON$ we have

$\widehat{AOR} \equiv \widehat{CON}$ and $m\left(\widehat{ARO}\right) + m\left(\widehat{CNO}\right) = 180^\circ$. Hence $\frac{AO}{CO} = \frac{AR}{CN} = \frac{AM}{CN} = \frac{AM+AO}{CN+CO} = 1$.

In conclusion, $OA = OC$. Prove analogously that $OB = OD$, i.e. $ABCD$ is a parallelogram.

$\square$

An easy extension. Prove that for any $x \in \left(0, \frac{\pi}{2}\right)$, $\frac{\sin^{p+2}x}{\cos^p x} + \frac{\cos^{p+2}x}{\sin^p x} \ge 1$, where $p \in \mathbb{N}^*$.

Solution

I"ll apply the well-known **Chebyshev's inequality** for $n = 2$:

$\boxed{a \le b \ \wedge \ x \le y \implies (a+b)(x+y) \le 2(ax+by)}$. Indeed,

$(\forall)\ x \in \left(0, \frac{\pi}{2}\right)$, $\tan^p x \le \cot^p x \iff \tan x \le \cot x \iff \sin^2 x \le \cos^2 x$. Therefore,

$\tan^p x + \cot^p x = \left(\sin^2 x + \cos^2 x\right)\left(\tan^p x + \cot^p x\right) \le 2\left(\sin^2 x \tan^p x + \cos^2 x \cot^p x\right) \implies$

$\frac{\sin^{p+2}x}{\cos^p x} + \frac{\cos^{p+2}x}{\sin^p x} = \sin^2 x \tan^p x + \cos^2 x \cot^p x \ge \frac{1}{2} \cdot (\tan^p x + \cot^p x) \ge 1$.

Remark. $\frac{\sin^{p+2}x}{\cos^p x} + \frac{\cos^{p+2}x}{\sin^p x} \ge \frac{\tan^p x + \cot^p x}{2} \ge \left(\frac{\tan x + \cot x}{2}\right)^p \ge 1$ . .

$\square$

ABC is a triangle, O is the midpoint of its side $[BC]$ and $A = \frac{4\pi}{7}$, $C = \frac{2\pi}{7}$. Calculate $m(\angle AOC)$.

Solution

Denote $m(\angle AOC) = x$. From the well-known property $1 = \frac{OB}{OC} = \frac{AB}{AC} \cdot \frac{\sin\widehat{OAB}}{\sin\widehat{OAC}} = \frac{\sin C}{\sin B} \cdot \frac{\sin(x-B)}{\sin(A+B-x)} \iff \sin B \sin(C+x) = \sin C \sin(x-B) \iff \cos(C-B+x) - \cos(B+C+x) = \cos(B+C-x) - \cos(C+x-B) \iff 2\cos(C-B+x) = \cos(B+C+x) + \cos(B+C-x) \iff \cos(C-B+x) = \cos(B+C)\cos x \iff \cos(C-B+x) = -\cos A \cos x \iff \cos(C-B) - \sin(C-B)\tan x = -\cos A \iff \tan x = \frac{\cos(C-B)-\cos(B+C)}{\sin(C-B)} \iff \boxed{\tan x = \frac{2\sin B \sin C}{\sin(C-B)} = \frac{2\tan B \tan C}{\tan C - \tan B}}$. **Our case :** $\tan\widehat{BOC} = 2\sin\frac{2\pi}{7}$.

$\square$

Prove that $\frac{1}{2\sqrt{n}} < \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$. In fact the stronger inequality $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$ holds for $n > 1$.

Solution

Denote $a_n = \frac{1\cdot3\cdot5\cdot\ \dots\ \cdot(2n-1)}{2\cdot4\cdot6\cdot\ \dots\ \cdot2n}$ and $b_n = \frac{2\cdot4\cdot6\cdot\ \dots\ \cdot2n}{3\cdot5\cdot7\cdot\ \dots\ \cdot(2n+1)}$, where $n \in \mathbb{N}^*$. Observe that $a_n < b_n$

because $\frac{k}{k+1} < \frac{k+1}{k+2}$ for any $k \in \overline{1, 2n-1}$ and $a_n b_n = \frac{1}{2n+1}$. Hence $a_n^2 < a_n b_n = \frac{1}{2n+1}$ from where obtain that

$\boxed{a_n < \dfrac{1}{\sqrt{2n+1}}}$. From the relation $(2k+1)^2 > 4k(k+1)$ obtain that $\frac{2k+1}{2\sqrt{k(k+1)}} > 1$, $(\forall)\ k \in \overline{1, n-1} \implies$

$\frac{3\cdot5\cdot\ \dots\ \cdot(2n-1)}{2^{n-1}\cdot2\cdot3\cdot4\cdot\ \dots\ \cdot(n-1)\cdot\sqrt{n}} > 1$, i.e. $\frac{1\cdot3\cdot5\cdot\ \dots\ \cdot(2n-1)}{4\cdot6\cdot8\cdot\ \dots\ \cdot(2n-2)\cdot\sqrt{n}} > 1 \iff \frac{2n}{\sqrt{n}}\cdot a_n > 1 \iff \boxed{a_n > \dfrac{1}{2\sqrt{n}}}$.

Observe that $a_n < \frac{1}{\sqrt{3n+1}} \implies a_{n+1} = \frac{2n+1}{2n+2}\cdot a_n < \frac{2n+1}{2n+2}\cdot\frac{1}{\sqrt{3n+1}}$ and $\frac{2n+1}{2n+2}\cdot\frac{1}{\sqrt{3n+1}} < \frac{1}{\sqrt{3n+4}} \iff$

$(2n+1)^2(3n+4) < (2n+2)^2(3n+1) \iff 12n^3+28n^2+19n+4 < 12n^3+28n^2+20n+4 \iff 0 < n$.

In conclusion, $a_2 < \frac{1}{\sqrt{7}}$ and $a_n < \frac{1}{\sqrt{3n+1}} \implies a_{n+1} < \frac{1}{\sqrt{3n+4}}$ for any $n \geq 2$, i.e. $a_n < \frac{1}{\sqrt{3n+1}}$ for any $n \in \mathbb{N}$, $n \geq 2$.

Similar proposed problems. Prove that :

- ► $2\cdot\left(\sqrt{n+1}-1\right) < \sum_{k=1}^{n}\frac{1}{\sqrt{k}} < 2\sqrt{n}$, $(\forall)\ n \in \mathbb{N}$, $n \geq 2$.
- ► $\sum_{k=0}^{n}\frac{1}{k!} < 3 - \frac{n+2}{(n+1)!(n+1)}$, $(\forall)\ n \in \mathbb{N}^*$.

□

Let ABD be a triangle with $AB = 1$. Suppose that $K \in AD$ such that $KD = 1$, $BK \perp BA$ and $m(< DBK) = 30^\circ$. Determine AD .

Solution

► **Case** $K \in (AD)$. Denote $AK = x$ and $BD = y \implies \frac{KA}{KD} = \frac{BA}{BD}\cdot\frac{\sin\widehat{KBA}}{\sin\widehat{KBD}} \iff \frac{x}{1} = \frac{1}{y}\cdot\frac{\sin 90^\circ}{\sin 30^\circ} \iff$ $\boxed{xy = 2}$ $(*)$.

Apply the generalized **Pytagoras' theorem** in $\triangle ABD$: $AD^2 = BA^2 + BD^2 + BD\cdot BA \iff (x+1)^2 = y^2+y+1 \iff$

$x^2+2x = y^2+y \overset{(*)}{\iff} x^2+x^2y = y^2+y \iff x^2(y+1) = y(y+1) \iff x^2 = y \overset{(*)}{\iff} x = \sqrt[3]{2} \iff AD = 1+\sqrt[3]{2}$.

► **Case** $D \in (AK)$. Denote $AD = x$ and $BK = y \implies \frac{DA}{DK} = \frac{BA}{BK}\cdot\frac{\sin\widehat{DBA}}{\sin\widehat{DBK}} \iff \frac{x}{1} = \frac{1}{y}\cdot\sqrt{3} \iff$ $\boxed{xy = \sqrt{3}}$ $(*)$.

Apply the **Pytagoras' theorem** in $\triangle ABK$: $AK^2 = BA^2 + BK^2 \iff (x+1)^2 = y^2+1 \iff x^2+2x = y^2 \overset{(*)}{\iff}$

$x^2+2x = \frac{3}{x^2} \iff x^4+2x^2-3 = 0 \iff (x-1)\left(x^3+3x^2+3x+3\right) = 0 \iff x = 1 \iff AD = 1$.

□

Let ABC be the C-right-angled isosceles triangle whose equal sides have length 1 . For $P \in [AB]$ denote the feet of the

perpendiculars from P to the other sides are $Q \in CA$ and $R \in CB$. Consider the areas of the triangles APQ and PBR

and the area of the rectangle $QCRP$. Prove that regardless of how P is chosen, the largest of these three areas is at least $\frac{2}{9}$

Solution

Denote $\left\{\begin{array}{c} QA=QP=CR=x \\ \\ RB=RP=CQ=1-x \end{array}\right\|$ and $\left\{\begin{array}{ccc} m=[AQP] & = & \frac{x^2}{2} \\ \\ n=[BRP] & = & \frac{(1-x)^2}{2} \\ \\ p=[CQPR] & = & x(1-x) \end{array}\right\|$. Prove easily that $\max\{m,n,p\} =$

$\left\{\begin{array}{lll} n & \text{if} & 0\le x\le \frac{1}{3} \\ \\ p & \text{if} & \frac{1}{3}\le x\le \frac{2}{3} \\ \\ m & \text{if} & \frac{2}{3}\le x\le 1 \end{array}\right\|$ and $\left\{\begin{array}{ccccc} n\ge \frac{2}{9} & \iff & 9x^2-18x+5\ge 0 & \iff & x\le \frac{1}{3} \\ \\ p\ge \frac{2}{9} & \iff & 9x^2-9x+2\le 0 & \iff & \frac{1}{3}\le x\le \frac{2}{3} \\ \\ m\ge \frac{2}{9} & \iff & 9x^2-4\ge 0 & \iff & \frac{2}{3}\le x \end{array}\right\|$. Thus, $\max\{m,n,p\}\ge \frac{2}{9}$.

□

Prove or disprove that $[ABC]=\frac{R}{2}\cdot\sum a\cdot\cos A$.

Solution

Proof 1 (metric). Denote the distance $\delta_{XY}(P)$ of the point P to the line XY and the circumcenter O of $\triangle ABC$. Prove easily that

$\left\{\begin{array}{lllllll} A\le 90^\circ & \implies & m\left(\widehat{BOC}\right)=2A & \implies & \delta_{BC}(O)=R\cdot\cos A & \implies & [BOC]=\frac{1}{2}\cdot a \\ \\ A>90^\circ & \implies & m\left(\widehat{BOC}\right)=360-2A & \implies & \delta_{BC}(O)=R\cdot\cos\left(180^\circ-A\right) & \implies & [BOC]=-\frac{1}{2}\cdot a \end{array}\right.$

In conclusion: if ABC is acute or right, then $[ABC]=\sum[BOC]=\frac{R}{2}\cdot\sum a\cdot\cos A$; if ABC is obtuse in A , then

$[ABC]=[AOB]+[AOC]-[BOC]=\frac{R}{2}\cdot c\cdot\cos C+\frac{R}{2}\cdot b\cdot\cos B-\left(-\frac{R}{2}\cdot a\cdot\cos A\right)=\frac{R}{2}\cdot\sum a\cdot\cos A$

.

Proof 2 (trig). $\frac{R}{2}\sum a\cdot\cos A=\frac{R}{2}\sum 2R\sin A\cdot\cos A=\frac{R^2}{2}\sum\sin 2A=2R^2\prod\sin A=2R^2\prod\frac{a}{2R}=\frac{abc}{4R}=[ABC]$.

I used the well-known identity $\boxed{\sum\sin 2A=4\prod\sin A}$.

□

Let $\triangle ABC$ be an C-isosceles and $P\in(AB)$ be a point so that $m\left(\widehat{PCB}\right)=\phi$. Express AP in terms of C , c and $\tan\phi$.

Solution

Apply an well-known relation $\frac{PA}{PB}=\frac{CA}{CB}\cdot\frac{\sin\left(\widehat{PCA}\right)}{\sin\left(\widehat{PCB}\right)}=\frac{\sin(C-\phi)}{\sin\phi}=\frac{\sin C-\cos C\cdot\tan\phi}{\tan\phi}\implies$

$\frac{PA}{\sin C-\cos C\cdot\tan\phi}=\frac{PB}{\tan\phi}=\frac{c}{\sin C+(1-\cos C)\cdot\tan\phi}\implies\boxed{PA=c\cdot\frac{\sin C-\cos C\cdot\tan\phi}{\sin C+(1-\cos C)\tan\phi}}$.

Particular case. $C=90^\circ\implies PA=\frac{c}{1+\tan\phi}$.

□ Find the smallest natural $n>11$ such that exists a polynomial $p(x)$ with degree n that verifies: i) $p(k)=k^n$, for $k=1,2,\ldots,n$. ii) $p(0)\in\mathbb{Z}$. iii) $p(-1)=2003$.

Solution

$P(x)=\lambda\prod_{r=1}^{n}(x-r)+x^n$

$\implies P(0)=\lambda(-1)^n n!\implies\lambda\in\mathbb{Q}$

And $P(-1)=\lambda(-1)^n(n+1)!+(-1)^n=2003$

For $n\in$ even

$\lambda(n+1)!=2002=2\times 7\times 11\times 13$

$\implies\min n=12,\ \lambda=\frac{2002}{13!}$

For $n\in$ odd

$-\lambda(n+1)! = 2004 = 4\times 3\times 167$
$\implies \min n = 333,\ \lambda = -\frac{2004}{334!}$
Hence smallest $n=12$ and $P(x) = \frac{2002}{13!}\prod_{r=1}^{12}(x-r) + x^{12}$

$\square$
Prove that

$$\sum_{i=1}^{n}(-1)^{n+i}\binom{n}{i}\binom{ni}{n} = n^n$$

I've been thinking that the $(-1)^{n+i}$ comes from a use of the Principle of Inclusion and Exclusion, but I have no idea how to actually come up with that particular solution.

Solution

$\sum_{i=1}^{n}(-1)^{n+i}\binom{n}{i}\binom{ni}{n} = (-1)^n\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{ni}{n}$
$= (-1)^n$coefficient of x^n in $\sum_{i=0}^{n}(-1)^i\binom{n}{i}(1+x)^{ni}$
$= (-1)^n$coefficient of x^n in $(1-(1+x)^n)^n$
$= (-1)^n$coefficient of x^n in $\left(-nx-\binom{n}{2}x^2-\binom{n}{3}x^3-\cdots\right)^n$
$= (-1)^n(-n)^n$
$= n^n$

$\square$ $0\le c\le b\le a$
Show that $\frac{a^2-b^2}{c}+\frac{c^2-b^2}{a}+\frac{a^2-c^2}{b}\ge 3a-4b+c$

Solution

The inequality can be written as

$$(a-b)\left(\frac{a+b}{c}-2\right)+(a-c)\left(\frac{a+c-b}{b}\right)+(b-c)\left(\frac{2a-b-c}{a}\right)\ge 0$$

Having checked all the single expressions non-negative, we can confirm ourselves that the inequality holds and hence, the proof is completed.

$\square$ Let $M\subset N($ set of natural number$)$. Assume that for $x\in M$ we have $4x, [\sqrt{x}]\in M$ Prove that $M=N$

Solution

By well-ordering property of $\mathbb{N}$, M should have the smallest element $k\ge 1$ and since $\sqrt{k}<k$ when $k>1$ then we must have $k=[\sqrt{k}]$ hence $k=1\in M\to 4\in M$. Therefore, $\{4^t=2^{2t}; t\in\mathbb{N}\}\subset M$, and then $\{[\sqrt{2^{2t}}]=2^t; t\in\mathbb{N}\}\subset M$. If we prove that M contains the square of any odd number then since any natural number can be written $2^t s$ where s is an odd number, we reach the assertion simply because we will have $\{[\sqrt{4^t s^2}]=2^t s\ : t\in\mathbb{N}, s \text{ is an odd number }\}=\mathbb{N}$.

As above, $\{4^n \mid n=0,1,2,\dots\}\subset M$. For a fixed $x\in\mathbb{N}$, $x>1$, consider the interval $\left[2^k\dfrac{\ln x}{\ln 4}, 2^k\dfrac{\ln(x+1)}{\ln 4}\right)$, of length $2^k\dfrac{\ln(1+1/x)}{\ln 4}>1$ for large enough k. That means there exists a positive integer n in that interval, so $x^{2^k}\le 4^n<(x+1)^{2^k}$.

Then $x=\left\lfloor 4^{n/2^k}\right\rfloor$, so $x\in M$, by repeated application of the $m\to\lfloor\sqrt{m}\rfloor$ rule.

$\square$ If n> is a composite number with r distinct prime factors, then $\phi(n)\ge\frac{n}{2^r}$

Solution

Let $n=p_1^{k_1}p_2^{k_2}...p_r^{k_r}$ be the factorization of n into distinct prime numbers p_i, then $\phi(n)=n(1-\frac{1}{p_1})(1-\frac{1}{p_2})...(1-\frac{1}{p_r})$. Since $p_i\ge 2\to 1-\frac{1}{p_i}\ge\frac{1}{2}$, therefore, $\phi(n)\ge\frac{n}{2^r}$.

$\square$ Determine $x, y, z \in \mathbb{R}$ such that $2x^2 + y^2 + 2z^2 - 8x + 2y - 2xy + 2xz - 16z + 35 = 0$

Solution

$2x^2 + y^2 + 2z^2 - 8x + 2y - 2xy + 2xz - 16z + 35 = y^2 + 2y(1-x) + (x^2 - 2x + 1) + x^2 + 2x(z - 3) + (z^2 - 6z + 9) + z^2 - 10z + 25 = (y + 1 - x)^2 + (x + z - 3)^2 + (z - 5)^2 = 0$

$z = 5; y = -3; x = -2$

$\square$ $2 < P$ prime number . Prove : $[(2 + \sqrt{5})]^p - 2^{p+1} \vdots p$

Solution

Somehow I believe it is not as written: $\lfloor 2 + \sqrt{5} \rfloor^p - 2^{p+1}$, for this is trivially $4^p - 2^{p+1} \equiv 4 - 2^2 = 0 \pmod{p}$.

Rather, I think it is meant to be: $\lfloor (2 + \sqrt{5})^p \rfloor - 2^{p+1}$. But $0 < \sqrt{5} - 2 < 1$, and $(\sqrt{5} + 2)^p - (\sqrt{5} - 2)^p$ is a positive integer, thus it is precisely $\lfloor (2 + \sqrt{5})^p \rfloor$. Expanding, it is readily seen that $(\sqrt{5} + 2)^p - (\sqrt{5} - 2)^p \equiv 2 \cdot 2^p \pmod{p}$ (since all $\binom{p}{k} \equiv 0 \pmod{p}$ for $0 < k < p$), thus the thesis.

$\square$ If f a continuous function in $\mathbb{R}$ such that $\forall x \in \mathbb{R}$ and $\forall c \in (0, 1)$ holds: $f(x)f(cx) = e^x$ then find f.

Solution

Should that $\forall c \in (0, 1)$ be just $c \in (0, 1)$ (a constant, not a variable)?

If so, define $g(x) = (\ln f(x)) - \frac{x}{1+c}$.

f cannot have a zero, for if $f(z) = 0$, $f(z)f(cz) = 0 = e^z$ is contradictory. Thus, since f is continuous, f must always be positive. Therefore, g is also continuous.

From the equation, we have $g(x) + g(cx) = 0$, and $g(0) = 0$. Therefore, if $g(x) = m$, $g(c^2x) = g(c^4x) = g(c^6x) = \cdots = m$ as well, and $g(cx) = g(c^3x) = g(c^5x) = \cdots = -m$. This establishes $\lim_{x \to 0} g(x) = m = -m$ since g is continuous. So, $m = 0$. Thus g must be always zero.

Plugging back and solving, $f(x) = e^{\frac{x}{1+c}}$.

$\square$ a, b, c are positive integers. Find (a, b, c) satisfying $abc + ab + c = a^3$

Solution

From $abc + ab + c = a^3$ follows $a \mid c$, hence $c = ad$. Now the relation writes $abd + b + d = a^2$, so it follows $a \mid b + d$, hence $b + d = ae \geq a$. Finally, the relation now writes $bd + e = a \leq b + d$, thus $(b - 1)(d - 1) + e \leq 1$. This is only possible if $e = 1$ and $b = 1$ or $d = 1$ (or both).

If $e = b = 1$, it follows $d = a - 1$, so $c = a(a - 1)$. If $e = d = 1$, it follows $b = a - 1$ and $c = a$. Both coincide on $e = b = d = 1$, for $(a, b, c) = (2, 1, 2)$.

$\square$ Solve the functional equation $f : R \to R$ such that $f(n + 1) = f(n) + 1$

Solution

Let us define a function from reals to reals such that

$$q(n) = f(n) - n \Longrightarrow f(n) = q(n) + n$$

We have

$$f(n + 1) = f(n) + 1 \Longrightarrow q(n + 1) + n + 1 = q(n) + n + 1$$

$$\Longrightarrow q(n + 1) = q(n) \forall n \in \mathbb{R}$$

So, q is a constant function. Let

$$q(n) = k \forall n \in \mathbb{R} \Longrightarrow \boxed{f(n) = n + k}$$

$\square$ Find the constant term of the expression: $(x^2 + \frac{1}{x^2} + y + \frac{1}{y})^8$

Solution

Any element, after the expansion, is in the form $\frac{8!}{t_1!t_2!t_3!t_4!}(x^2)^{t_1}(\frac{1}{x^2})^{t_2}y^{t_3}(\frac{1}{y})^{t_4}$ where $t_1+t_2+t_3+t_4=8$ and $t_i \geq 0$. The constant terms occur when $t_1=t_2=a$ and $t_3=t_4=b$. Therefore, $2a+2b=8 \to a+b=4$. Then, $(a,b)=(0,4),(1,3),(2,2),(3,1),(4,0)$. Hence the constant term is $\frac{8!}{0!0!4!4!}+\frac{8!}{1!1!3!3!}+\frac{8!}{2!2!2!2!}+\frac{8!}{3!3!1!1!}+\frac{8!}{4!4!0!0!}$.

□ Solve the following inequation, for $0 \leq x < 2\pi$:

$$\frac{3\sin^2 x+2\cos^2 x+4\sin x-(1+4\sqrt{2})\sin x\cos x+4\cos x-(2+2\sqrt{2})}{2\sin x-2\sqrt{2}\sin x\cos x+2\cos x-\sqrt{2}}>2$$

Solution

if you let $sinx=a$ and $cosx=b$, then the denominator can be factored as $(2a-\sqrt{2})(1-b\sqrt{2})$ Also, subtracting 2 from each side and giving a common denominator cancels a lot of things quite nicely. the numerator is originally $3a^2+2b^2+4a-(1+4\sqrt{2})ab+4b-(2+2\sqrt{2})$ but once you subtract 2 from each side, it becomes $3a^2+2b^2+4a-(1+4\sqrt{2})ab+4b-(2+2\sqrt{2})-2(2a-2\sqrt{2}ab+2b-\sqrt{2})$ cancelling lots of things, we get $3a^2+2b^2-ab-2$ Now, $a^2+b^2=1$ (you see why?) so we write $a^2-ab=a(a-b)$ Therefore the whole inequality we wish to solve becomes $\frac{sinx(sinx-cosx)}{(2sinx-\sqrt{2})(1-\sqrt{2}cosx)}>0$ To do this, we consider the signs of each part of the numerator and the denominator $sinx>0$ on the interval $(0,\pi)$ $sinx>cosx$ on the interval $(\pi/4,5\pi/4)$ $sinx>\sqrt{2}/2$ on the interval $(\pi/4,3\pi/4)$ $cosx<\sqrt{2}/2$ on the interval $(\pi/4,7\pi/4)$ So we must find the intervals such that each point is in all of the above sets, none of the above sets, or exactly 2 of the above sets. for none: $(7\pi/4,2\pi)$ for all: $(\pi/4,3\pi/4)$ for exactly 2: $(\pi,5\pi/4)$ That's hopefully right Edit: Just to clarify, those are intervals, so $\pi/4<x<3\pi/4$ or $\pi<x<5\pi/4$ or $7\pi/4<x<2\pi$

□ Is $\cos\frac{\pi}{2010}$ rational?

Solution

Denote by $T_n(x)$ the Chebyshev polynomial of the first kind. We use the known fact that these polynomials have integer coefficients.

Assume that $\cos\frac{\pi}{2010}\in\mathbb{Q}$
$\Longrightarrow T_{67}(\cos\frac{\pi}{2010})=\cos\frac{\pi}{30}\in\mathbb{Q}$
$\Longrightarrow T_5(\cos\frac{\pi}{30})=\cos\frac{\pi}{6}\in\mathbb{Q}$
$\Longrightarrow \frac{\sqrt{3}}{2}\in\mathbb{Q}$
Which is clearly false, so our intial assumption was also false.

□ Let $d(n)$ be the number of divisors of n. Show that:
$\sum_{k=1}^{n+1}\lfloor\frac{n+1}{k}\rfloor-\sum_{k=1}^{n}\lfloor\frac{n}{k}\rfloor=d(n+1)$

Solution

Dunno if it's right: $\lfloor\frac{n+1}{k}\rfloor=\lfloor\frac{n}{k}\rfloor$ when $n\not\equiv -1 \pmod{k}$. And, if $n\equiv -1 \pmod{k}$ then $\lfloor\frac{n+1}{k}\rfloor=\lfloor\frac{n}{k}\rfloor+1$. So, when $k\mid n+1$ we have that $\lfloor\frac{n+1}{k}\rfloor-\lfloor\frac{n}{k}\rfloor=1$. When $k\nmid n+1$ we have that $\lfloor\frac{n+1}{k}\rfloor-\lfloor\frac{n}{k}\rfloor=0$. Thus, the sum is equal to the number of divisors of $n+1$.

□ Find all integers (m,n) such that m^2+n^2 and $m^2+(n-2)^2$ are both perfect squares.

Solution

if $m=0$ then all n works

if $m\neq 0$, WLOG we just need to consider $m>0$, $n>2$.

Let ABC be a right angle triangle at B, AB = m, BC = n. Let D be a point on BC such that BD = n-2. Observe that the length of AC is the solution for $m^2+n^2=k^2$, while AD is the solution for $m^2+(n-2)^2=j^2$

By triangle inequality, AD + DC $>$ AC . So $j+2>k$. Since k is integer, we have $k \leq j+1$. But $k>j$ so we must have $k=j+1$.

However, note that k and j have the same odd even parity, so $k \neq j+1$. There is no solution for $m \neq 0$

$\square$ Solve the equation $4[x]=25\{x\}-4,5$

Solution

$4[x]=25\{x\}-4,5 \implies 8[x]=50\{x\}-9 \implies 8[x]=50(x-[x])-9 \implies x=\frac{58[x]+9}{50} \implies [x]=t \in Z \implies t \leq x < t+1 \implies t \leq \frac{58t+9}{50} < t+1 \implies 50t \leq 58t+9 < 50t+50 \implies 0 \leq 8t+9 < 50 \implies -\frac{9}{8} \leq t < \frac{41}{8} \implies t=-1;0;1;2;3;4;5 \implies$

$x=-\frac{49}{50};\frac{9}{50};\frac{67}{50};\frac{5}{2};\frac{183}{50};\frac{241}{50};\frac{299}{50}$

$\square$ Let $\omega(n)$ denote the number of distinct prime divisors of $n>1$, with $\omega(1)=0$ For example, $\omega(360)=\omega(2^3\cdot 3^2\cdot 5)=3$

For $n \in \mathbb{Z}^+$ prove that : $\tau(n^2)=\sum_{d|n} 2^{\omega(d)}$

;where $\tau(n)$ denote the number of divisors of n

Solution

We can prove it by induction on $n \geq 1$.

The assertion is obviously true for $n=1$ $(\tau(1^2)=1=2^0)$.

Suppose the assertion is valid for all $n \leq n_0$. If n_0+1 is a prime number, thus the result is true $(\tau(p^2)=3$, and the divisors of p are 1 and p, with $\omega(1)=0$ and $\omega(p)=1$. We have $2^0+2^1=3$.)

If n_0+1 isn't a prime number, we will write it $n_0+1=p^k \times q$ with $gcd(p,q)=1$. So we have $q \leq n_0$, and by induction hypothesis : $\tau(q^2)=\sum_{d|q} 2^{\omega(d)}$ (1). We also have $\tau((n_0+1)^2)=\tau(q^2)\tau(p^{2k})=(2k+1)\tau(q^2)$ (2) because of $gcd(p,q)=1$. (1) and (2) imply that $(k+1)\tau(q^2)=(2k+1)\sum_{d|q} 2^{\omega(d)}$. Now among the divisors of n_0+1, we can partition them into k+1 groups G_i $(0 \leq i \leq k)$: the first one G_0 with divisors d with $p \nmid d$, and the k groups with in the i-1-th group all the divisors d so as $p^i|d$ and $p^{i+1} \nmid d$ for $1 \leq i \leq k$. We deduce from that : $\sum_{d|n_0+1} 2^{\omega(d)}=\sum_{i=0}^{k}\sum_{d\in G_i} 2^{\omega(d)}$. But $\sum_{d\in G_0} 2^{\omega(d)}=\sum_{d|q} 2^{\omega(d)}$, and $\sum_{d\in G_i} 2^{\omega(d)}=\sum_{d\in G_0} 2\times 2^{\omega(d)}=2\sum_{d|q} 2^{\omega(d)}$ for $1 \leq i \leq k$. So we also find $\sum_{d|n_0+1} 2^{\omega(d)}=(2k+1)\sum_{d|q} 2^{\omega(d)}$.

So the result is valid for all $n \in \mathbb{Z}^+$

$\square$ Prove if $1,\sqrt{2},2\sqrt{2}$ can be members of an arithmetic progression.

Solution

I will use proof by contradiction to show that they cannot be members of an A.P. Suppose on the contrary that they can be members of an A.P. Then there exists non-zero integers m and n and a real number d (the common difference between consecutive terms of the A.P.) such that $\sqrt{2}=1+md$ and $2\sqrt{2}=1+nd$ Then $nd=2\sqrt{2}-1$ and $md=\sqrt{2}-1$ Then $\frac{n}{m}=\frac{nd}{md}=\frac{2\sqrt{2}-1}{\sqrt{2}-1}$ Then $\frac{n}{m}=\frac{2\sqrt{2}-1}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}=3+\sqrt{2}$ Then $\sqrt{2}=\frac{n}{m}-3=\frac{n-3m}{m}$ m and n are non-zero integers so $n-3m$ is an integer and m is a non-zero integer. Then $\frac{n-3m}{m}$ is a rational number. Then $\sqrt{2}$ is a rational number, so we arrive to a contradiction. Thus $1,\sqrt{2}$ and $2\sqrt{2}$ cannot be members of an arithmetic progression.

$\square$ Find the value of c, where $c>0$, such that $\sin x=cx$ has exactly 5 solutions. If X is the largest of the five solutions of the equation, explain why $\tan X=X$.

Solution

Prove easily that $X \in \left(2\pi, 2\pi+\frac{\pi}{2}\right)$ and $\left\| \begin{array}{c} \sin X = cX \\ (\sin x)'\|_X = (cx)'\|_X \end{array} \right\|$, i.e. $\left\| \begin{array}{c} \sin X = cX \\ \cos X = c \end{array} \right\|$.

In conclusion $X \in \left(2\pi, 2\pi + \frac{\pi}{2}\right)$ and $\left\| \begin{array}{ccc} \boxed{\tan X = X} & \Longrightarrow & \sin X = \frac{X}{\sqrt{1+X^2}} \\ c = \frac{\sin X}{X} & \Longrightarrow & \boxed{c = \dfrac{1}{\sqrt{1+X^2}}} \end{array} \right\|$.

☐ Prove: if gcd(a,b)=1 ,then $gcd(a^3 + b^3, a^2 + b^2)|(a - b)$.

Solution

If $d = (a^3 + b^3, a^2 + b^2)$, we have $-b^3 \equiv a^3 \equiv a(-b^2) \pmod d$ and hence $a \equiv 1 \pmod d$. Symmetry gives $b \equiv 1 \pmod d$ and so, $d|a^2 + b^2 \equiv 2$. If one of a, b is even and the other is odd, we have $d = 1|a - b$. If both are odd, we have $d = 2|a - b$ as required.

☐ Prove that if $a_1, a_2, a_3, b_1, b_2, b_3$ are positive real number such that $a_1 + a_2 + a_3 = 3$ and $b_1 + b_2 + b_3 = 1$ then $\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \frac{a_3^2}{b_3} \geq 5$.

Solution

Well, from $AM - GM$ we have $\frac{a_1^2}{b_1} + b_1 \geqslant 2a_1$; $\frac{a_2^2}{b_2} + b_2 \geqslant 2a_2$; $\frac{a_3^2}{b_3} + b_3 \geqslant 2a_3$ Summing these inequalities side by side we get

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \frac{a_3^2}{b_3} + (b_1 + b_2 + b_3) \geqslant 2(a_1 + a_2 + a_3)$$

Using $b_1 + b_2 + b_3 = 1$ and $a_1 + a_2 + a_3 = 3$, we get

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \frac{a_3^2}{b_3} \geqslant 5$$

However, the equality in the above inequalities holds for $\frac{a_i^2}{b_i} = b_i$ or $a_i = b_i$. But in that case we would have $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 1$, which contradicts $a_1 + a_2 + a_3 = 3$. The proposed inequality is still true, but there's no equality. ☐ Given n numbers $a_1, a_2, \ldots, a_n$.The supersum of these numbers S is defined as

$S = \frac{s_1 + s_2 + \ldots + s_n}{n}$ where

$s_k = a_1 + a_2 + \ldots + a_k, (k = 1, 2, \ldots, n)$

If the supersum of $a_1, a_2, \ldots, a_{99}$ is equal to 1000,find the supersum of $1, a_1, a_2, \ldots, a_{99}$.

Solution

According to the question we get: $\frac{(a_1)+(a_1+a_2)+\cdots+(a_1+a_2+\cdots+a_{99})}{99} = 1000$

$\Longrightarrow (a_1) + (a_1 + a_2) + \cdots + (a_1 + a_2 + \cdots + a_{99}) = 99 \cdot 1000$

We want to find: $\frac{1+(1+a_1)+(1+a_2+a_3)+\cdots+(1+a_1+a_2+\cdots+a_{99})}{100}$

$\Longleftrightarrow \frac{100+(a_1)+(a_1+a_2)+\cdots+(a_1+a_2+\cdots+a_{99})}{100}$

$\Longleftrightarrow 1 + \frac{(a_1)+(a_1+a_2)+\cdots+(a_1+a_2+\cdots+a_{99})}{100}$

$= 1 + \frac{99 \cdot 1000}{100} = 991$

☐ Prove that $n > \frac{n^2}{\sigma(n)} > \phi(n)$

Solution

Let $n = {p_1}^{k_1} \cdot {p_2}^{k_2} \cdots {p_r}^{k_r}$ $n > \frac{n^2}{\sigma(n)} > \phi(n) \Longleftrightarrow 1 > \frac{n}{\sigma(n)} > \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right)$

☐ Twenty points, which form a regular 20-gon, are chosen on a circle. Then they are split into ten pairs, and the points in each pair are connected by a chord. Prove that some pair of these chords have the same length.

Solution

Name the points $P_1 \ldots P_{20}$.

Suppose we have an arrangement in which there are 10 pairs (P_i, P_j). There are 10 possible distances between P_i and P_j, so each pair must have a different distance.

We know that if P_i and P_j are d points apart (counting in the shortest direction), then $j-i = \pm d \equiv d \pmod 2$. Note now that $j-i = j+i-(2i) \equiv j+i \pmod 2$, thus transitively, $j+i \equiv d \pmod 2$.

Thus, $\sum_{\text{pairs}} j+i \equiv \sum_{\text{pairs}} d \equiv 1+\ldots+10 \equiv 55 \equiv 1 \pmod 2$.

However, we know that $\sum_{textpairs} j+i = 1+\ldots+20 = 210 \equiv 0 \pmod 2$, a contradiction.

Thus, there can be no such arrangement.

□ prove that inequality holds for for any real x,in $1^x+2^x+6^x+12^x \geq 4^x+8^x+9^x$ find x when equality holds.

Solution

$1^x+2^x+6^x+12^x \geq 4^x+8^x+9^x \Leftrightarrow 1^x+2^x+6^x+12^x-4^x-8^x-9^x \geq 0 \Leftrightarrow (4^x-3^x-1)(3^x-2^x-1) \geq 0$. The final statement is true since for $x>1$, both factors on the LHS are positive, while for $x<1$, both are negative otherwise. At $x=1$, equality holds.

□ Given that $\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$, prove that $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots = \frac{\pi^2}{8}$ and find the value of $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$.

Solution

Given that $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots = \frac{\pi^2}{6}$ then $(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots)+\frac{1}{2^2}(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots) = \frac{\pi^2}{6}$ $\implies (1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots) = \frac{\pi^2}{6}-\frac{\pi^2}{24} = \frac{\pi^2}{8}$

for $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots = (1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots)-\frac{1}{2^2}(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots) = \frac{\pi^2}{8}-\frac{\pi^2}{24} = \frac{\pi^2}{12}$

□ Let m be a natural number, and let $q=2m+1$. Then prove that $\sum_{k=1}^{m}\left(\tan\frac{k\pi}{q}\right)^{2n}$, $n = 1,2,3,\ldots$ are natural mumbers.

Solution

$\tan q\theta = \frac{C_1^q t - C_3^q t^3+\ldots}{1-C_2^q t^2+\ldots}$ $(t=\tan\theta,\ \theta=\frac{\pi}{q})$ Hence $C_1^q - C_3^q t^2 + \ldots = 0$

This is an equation in t^2 with integral coefficients and the last term's coefficient is either +1 or -1. So the sum of roots and the sum of product of roots are all integers.

The sum of the n-th power of roots can be an integral combinations of the sum of product of roots.

Since $\tan^2\theta = \tan^2(\pi-\theta)$, $\sum_{k=1}^{m}\left(\tan\frac{k\pi}{q}\right)^{2n}$ is half the sum which is even, so the above term is an integer.

□ Show that there is no term independent of x in the expansion of $(x^6+3x^5)^{\frac{1}{2}}$ in powers of x for $\mid x \mid < 3$.

Solution

For $x \in (0,3) \implies (x^6+3x^5)^{\frac{1}{2}} = \sqrt{3}x^{\frac{5}{2}}\left(1+\frac{x}{3}\right)^{\frac{1}{2}}$ Clearly there is no term independent of x

For $x \in (-3,0)$

$x^6+3x^5 = x^5(x+3) < 0 \implies (x^6+3x^5)^{\frac{1}{2}}$ is not defined.

□ If a function f is such that:

$f(x,y) = \sqrt{x^2+y^2}+\sqrt{x^2+y^2-2x+1}+\sqrt{x^2+y^2-2y+1}+\sqrt{x^2+y^2-6x-8y+25}$. Then, find the minimum value of the function.

Solution

$f(x,y) = \sqrt{x^2+y^2}+\sqrt{(x-1)^2+y^2}+\sqrt{x^2+(y-1)^2}+\sqrt{(x-3)^2+(y-4)^2}$

now let $P(x,y)$, $A(0,0)$, $B(1,0)$, $C(3,4)$, $D(0,1)$

YOU have to find minimum of $PA+PB+PC+PD$

For any point P, by triangle inequality

$PA+PC \geq AC$, $PB+PD \geq BD$

Where equality holds ,when P is intersection of diagonals hence $PA+PB+PC+PD \geq AC+BD=5+\sqrt{2}$

$\square$ a and b are naturals such as b^2+ab+1 divide a^2+ab+1 .Proof that $a=b$

Solution

we have that: $b^2+ab+1 \mid a^2+ab+1$ Obviously: $b^2+ab+1 \mid b^2+ab+1$

Subtracting gives: $b^2+ab+1 \mid a^2-b^2$

Since we have that $b^2+ab+1=b(a+b)+1$, so $gcd(a+b, b^2+ab+1)=1$.

So by Euclid's lemma we have that: $b^2+ab+1 \mid a-b$

So we have two cases: Case 1: $a-b=0$; which gives $a=b$ Case 2: $|b^2+ab+1| \leq |a-b|$ Or equivalently: $b^2+(a-1)(b-1) \leq -2b < 0$ which is Obviously a contradiction.

$\square$ Suppose that we have 27 odd numbers less than 100. Prove that there is at least one pair of these numbers such that their sum is 102.

Solution

Let $A=\{1,3,5,...,99\}$ be the set of odd numbers from 1 to 100, then $|A|=50$. Excluding 1 and 51, we can partition the remaining elements into pairs such that the sum of the numbers in each pair is 102, so there are $\frac{50-2}{2}=24$ of these pairs. In other words,

$$A=\{1\}\cup\{51\}\cup\underbrace{\{3,99\}\cup\{5,97\}\cup...\cup\{49,53\}}_{24 \text{ pairs}}$$

By Pigeonhole Principle, 2 of the 27 numbers we choose will fall in the same set, e.i. form a pair, so we're done.

$\square$ $x, y \in R$, where $x^2+y^2=1$ Find the Max (Min) of $\quad (x+2y)^2+(3x+2y)^2$ No Calculus. Do there exist solving ways of generalized patterns for this kind of problems?

Solution

$(x+2y)^2+(3x+2y)^2=10x^2+16xy+8y^2$, set a $p>0$, then from Am-Gm ineq, we have

$$\begin{aligned}
10x^2+16xy+8y^2 &= 10x^2+16px\frac{y}{p}+8y^2 \\
&\leqslant 10x^2+8\left(p^2x^2+\frac{y^2}{p^2}\right)+8y^2 \\
&= (10+8p^2)x^2+\left(\frac{8}{p^2}+8\right)y^2.
\end{aligned}$$

we need $10+8p^2=\frac{8}{p^2}+8=k$, get $p^2=\frac{\sqrt{65}-1}{8}$ and $k=9+\sqrt{65}$, so we have

$$10x^2+16xy+8y^2 \leqslant (9+\sqrt{65})(x^2+y^2)=9+\sqrt{65}.$$

equality holds when $x=\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{65}}}, y=\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{65}}}$ or $x=-\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{65}}}, y=-\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{65}}}$, so the max is $9+\sqrt{65}$.

the min is similar, using $2px\frac{y}{p} \geqslant -\left(p^2x^2+\frac{y^2}{p^2}\right)$ can solve it, the answer is $9-\sqrt{65}$.

$\square$ Prove that if integers a, b and c satisfy $a^2+b^2=c^2$, then a and b cannot both be odd.

Solution

let's prove by contradiction. We have $a^2=c^2-b^2=(c+b)(c-b)$. Since a is odd, a^2 is odd and therefore $(c+b)(c-b)$ must be odd, so $c+b$ and $c-b$ are both odd and c is even.

If $a \equiv 1 \mod 4$, then we have $a^2 \equiv 1 \mod 4$. If $b \equiv 1 \mod 4$, then we have $b^2 \equiv 1 \mod 4$, so $c^2 \equiv 2 \mod 4$ but c must be even, and c^2 is therefore $0 \mod 4$ and we have a contradiction.

If $a \equiv 1 \mod 4$, then we have $a^2 \equiv 1 \mod 4$. If $b \equiv 3 \mod 4$, then we have $b^2 \equiv 1 \mod 4$, so $c^2 \equiv 2 \mod 4$ but c must be even, and c^2 is therefore $0 \mod 4$ and we have another conradiction. This is symmetric with the case $a \equiv 3 \mod 4$ and $b \equiv 1 \mod 4$.

If $a \equiv 3 \mod 4$, then we have $a^2 \equiv 1 \mod 4$. If $b \equiv 3 \mod 4$, then we have $b^2 \equiv 1 \mod 4$, so $c^2 \equiv 2 \mod 4$ but c must be even, and c^2 is therefore $0 \mod 4$ and we have another contradiction.

So a and b cannot both be odd.

$\square$ Find three real numbers x, y, z, such that $x < y < z$ and they form a geometric sequence satisfying

$x + y + z = \frac{19}{18}$

$x^2 + y^2 + z^2 = \frac{133}{324}$

Solution

We have

$$(x + y + z)^2 - (x^2 + y^2 + z^2) = 2(xy + yz + zx).$$

With the choice $x = y/r$, $z = yr$, we immediately find

$$xy + yz + zx = y\left(\frac{y}{r} + y + yr\right) = y(x + y + z),$$

from which we obtain

$$\left(\frac{19}{18}\right)^2 - \frac{133}{324} = 2 \cdot \frac{19}{18} y,$$

or $y = 1/3$. Then $1/r + 1 + r = \frac{19}{6}$, which easily gives us $r = 3/2$ (since $x < y < z$ implies $r > 1$). Therefore, $x = 2/9$, $y = 1/3$, and $z = 1/2$.

$\square$ Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the property: $f(x + y) = f(x)e^{f(y)-1}$ for all $x, y \in \mathbb{R}$

Solution

Let $P(x, y)$ be the assertion that $f(x + y) = f(x)e^{f(y)-1}$ for all $x, y \in \mathbb{R}$ $P(0, 0) \implies f(0) = f(0)e^{f(0)-1} \implies f(0) = 0$ or 1

Suppose that $f(0) = 0$ $P(x, 0) \implies f(x) = \frac{f(x)}{e} \implies f(x) = 0, \forall x \in \mathbb{R}$

If $f(0) \neq 0$ then $f(0) = 1$. $P(0, x) \implies f(x) = e^{f(x)-1}, \forall x \in \mathbb{R}$ Then $f(x + y) = f(x)e^{f(y)-1} = f(x)f(y), \forall x, y \in \mathbb{R}$ Then $e^{f(x+y)-1} = f(x + y) = f(x)f(y) = e^{f(x)-1}e^{f(y)-1} \implies f(x + y) = f(x)+f(y)-1, \forall x, y \in \mathbb{R}$ Then $f(x)f(y) = f(x+y) = f(x)+f(y)-1, \forall x, y \in \mathbb{R}$ By letting $y = x$ in this new equation: $f(x)^2 = 2f(x) - 1, \forall x \in \mathbb{R}$ Then $[f(x) - 1]^2 = 0, \forall x \in \mathbb{R}$ Then $f(x) = 1, \forall x \in \mathbb{R}$

Therefore $f(x) = 0, \forall x \in \mathbb{R}$ or $f(x) = 1, \forall x \in \mathbb{R}$

$\square$ Find $x, y, z \in \mathbb{R}$ satisfying $\frac{4\sqrt{x^2+1}}{x} = \frac{4\sqrt{y^2+1}}{y} = \frac{4\sqrt{z^2+1}}{z}$, and $xyz = x + y + z$, where $x, y, z > 0$.

Solution

$\frac{4\sqrt{x^2+1}}{x} = \frac{4\sqrt{y^2+1}}{y} = \frac{4\sqrt{z^2+1}}{z}$

$\frac{16x^2+16}{x^2} = \frac{16y^2+16}{y^2} = \frac{16z^2+16}{z^2}$

$16 + \frac{16}{x^2} = 16 + \frac{16}{y^2} = 16 + \frac{16}{z^2}$

$\frac{16}{x^2} = \frac{16}{y^2} = \frac{16}{z^2}$

$x^2 = y^2 = z^2$

$x = y = z = a$ (Because all three values are positive)

$xyz = x + y + z$ $a \times a \times a = a + a + a$

$a^3 = 3a$

a isn't zero, so we can divide both sides by it.

$a^2 = 3$ $\sqrt{3} = a = x = y = z$

$\square$ Let $[\sqrt{x}] = 10$ and $[\sqrt{y}] = 14. Find[\sqrt[4]{x + y}]$ where $[x]$is the floor part.

Solution

$\lfloor\sqrt{x}\rfloor = 10 \implies 10 \le \sqrt{x} < 11 \implies 100 \le x < 121$ $\lfloor\sqrt{y}\rfloor = 14 \implies 14 \le \sqrt{y} < 15 \implies 196 \le y < 225$ So $4^4 < 296 \le x + y < 346 < 5^4$ So $4 < \sqrt[4]{x+y} < 5$ So $\lfloor\sqrt[4]{x+y}\rfloor = 4$

□ Given the equation: sin (kx) = sin (x) Find the value of k for which this equation and the equation cos (3x) = cos (2x) have, within the range (0, 360] (degrees), one and only one common solution

Solution

Angles will be in degrees.

When is $\sin a = \sin b$? When either $a \equiv b \pmod{360}$ or $a \equiv 180 - b \pmod{360}$.

When is $\cos a = \cos b$? When either $a \equiv b \pmod{360}$ or $a \equiv -b \pmod{360}$.

So the equation $\sin kx = \sin x$ can be written as $kx \equiv x \pmod{360}$ or $kx \equiv 180 - x \pmod{360}$. That gets us two families of solutions:

$x = \frac{360j}{k-1}$ or $x = \frac{180+360j}{k+1}$ for $j \in \mathbb{Z}$.

The equation $\cos 3x = \cos 2x$ can be solved as follows:

$3x \equiv 2x \pmod{360}$ which implies $x \equiv 0 \pmod{360}$ or $x = 360n$.

or

$3x \equiv -2x \pmod{360}$, which implies $5x \equiv 0 \pmod{360}$ or $x = \frac{360n}{5}$.

That second equation includes the first.

So, when do solutions coincide?

Either $\frac{360n}{5} = \frac{360j}{k-1}$ or $\frac{360n}{5} = \frac{180+360j}{k+1}$.

Take the first equation, divide by 360 and multiply by $5(k-1)$ to get $(k-1)n = 5j$.

This always has $n = 0, j = 0$ as a solution. We also have solutions whenever $5 \mid n$ (but that's the same place on the circle). If 5 doesn't divide n, then we would need $5 \mid (k-1)$ or $k \equiv 1 \pmod{5}$. Then $j = \frac{(k-1)n}{5}$, and as n ranges over all integers not equivalent to 5, then j will always be an integer.

Now let's look at the other equation. This time, divide by 180 and multiply by $5(k+1)$. That leaves $2(k+1)n = 5 + 10j$.

If 5 divides $k+1$, we get no solution, as one side is divisible by 10 and the other side is $\equiv 5 \pmod{10}$. But if 5 doesn't divide $k+1$, then we would have $5 \mid n$, which gets us back to $x \equiv 0 \pmod{360}$.

So:

If $k \not\equiv 1 \pmod{5}$, then the only solution in the circle is $x \equiv 0 \pmod{360}$. However, if $k \equiv 1 \pmod{5}$, then $\{0, 72, 144, 216, 288\}$ and their equivalents mod 360 are all solutions.

The question asked for the k that produce a unique solution in the circle; that would be $\{k : k \not\equiv 1 \pmod{5}\}$.

□ (x) is a polynomial of degree 998.p(k)=1/k for K is integral varying from 1 to 999. Find the value of P (1001).

Solution

a. 1 b. 1001 c. 1/1001 d.1/(1001!)

Your definition is equivalent to $kP(k) = 1$ for all the integers between 1 and 999. So, $kP(k) - 1 = A(k-1)(k-2)...(k-999)$, where A is some unknown constant. For $k = 0$, we have that $-1 = -A(999!)$, so $A = \frac{1}{999!}$. Now, $1001P(1001) - 1 = \frac{1000!}{999!}$. $1001P(1001) = 1001$, so $\boxed{P(1001) = 1}$. The answer: $\boxed{A}$.

□ Given a, b, c and $\frac{ab+bc+ac}{\sqrt{abc}}$ are all positive integers, does that imply that $\sqrt{\frac{ac}{b}}, \sqrt{\frac{ab}{c}}, \sqrt{\frac{bc}{a}}$ must all be integers?

Solution

Clearly $\sqrt{abc} \in \mathbb{N}$ so $abc = k^2, k \in \mathbb{N}$

Write $M = (a, b, c) = (\alpha^2 xy, \beta^2 yz, \gamma^2 zx)$

With $\gcd(\alpha, \beta) = \gcd(\beta, \gamma) = \gcd(\gamma, \alpha) = 1$

[hide="constructive proof"] Take $M = (a, b, c)$ and let $\gcd(a, b) = y \Longrightarrow M = (a'y, b'y, c)$

Let $\gcd(a', c) = x \Longrightarrow M = (a''yx, b'y, c'x)$

Let $\gcd(b', c') = z \Longrightarrow M = (a''xy, b''yz, c''zx)$

Since $\gcd(a'', b'') = \gcd(b'', c'') = \gcd(c'', a'') = 1$ it follows that a'', b'', c'' are perfect squares.

$\therefore M = (\alpha^2 xy, \beta^2 yz, \gamma^2 zx)$

This gives

$\frac{ab+bc+ca}{\sqrt{abc}} = \frac{\sum \alpha^2\beta^2 y}{\alpha\beta\gamma}$

Hence $\alpha | z, \beta | x$ and $\gamma | y$

Therefore

$\sqrt{\frac{ab}{c}} = \sqrt{\frac{\alpha^2 xy\beta^2 yz}{\gamma^2 zx}} = \frac{\alpha\beta y}{\gamma} \in \mathbb{N}$ because $\gamma | y$

$\square$ Prove that every $f : \mathbb{N} \to \mathbb{N}$ which is a bijection can be written as the sum of two involutions.

Solution

I assume that should read "composition of two involutions".

Let $X_1 = \mathbb{N}$. We define X_n iteratively as follows: let $S_n = \{x : \exists n \in \mathbb{Z}, f^n(x) = \min(X_n)\}$, and set $X_{n+1} = X_n \setminus S_n$; thus, $\bigcup S_n = \mathbb{N}$. (here f^n refers to the composition of f, n times)

Suppose $|S_n| = k \in \mathbb{N}$. If $k = 1$, then define $g_n(x) = h_n(x) = x$ where $x \in S_n$. Otherwise, $S_n = \{x_1, \ldots, x_k\}$ where $f(x_i) = x_{i+1}, x_{k+1} := x_1$, define the involutions $g_n, h_n : S_n \to S_n$ as follows: $g_n(x_i) = x_{k+2-i}, h_n(x_i) = x_{k+3-i}$ (they are involutions due to the definition of x_{k+1}, though this is shown in more detail in the hidden tag); obviously $f_n(x_i) = h_n(g_n(x_i))$. [hide="More specifically,"]$g_n(x_1) = x_1, g_n(x_i) = x_{k+2-i}$ for $2 \leq i \leq k$; and $h_n(x_1) = x_2, h_n(x_2) = x_1$, and $h_n(x_i) = x_{k+3-i}$ for $3 \leq i \leq k$. Observe that

$$h_n(g_n(x_i)) = \begin{cases} x_{k+3-(k+2-i)} = x_{i+1}, & 2 \leq i \leq k-1 \\ x_2, & i = 1 \\ x_1, & i = k \end{cases} = f_n(x_i).$$

Here's an example, for $k = 5$ and $S_n = \{1, 2, 3, 4, 5\}$:

x	g(x)	h(g(x))
1	1	2
2	5	3
3	4	4
4	3	5
5	2	1

where $g(1) = 1$, and the remaining elements are 'reflected' by g; and all the elements are 'reflected' by h. If S_n is countably infinite, select an arbitrary element $x_1 \in S_n$, and let $S_n = \{x_1, \ldots\}$ where $x_{2k+1} = f^k(x_1)$ and $f^k(x_{2k}) = x_1$, $k \in \mathbb{N}$. Then define the involutions $g_n, h_n : S_n \to S_n$ as follows: $g_n(x_1) = x_1, g_n(x_{2k}) = x_{2k+1}, g_n(x_{2k+1}) = x_{2k}$; and $h_n(x_1) = x_3, h_n(x_3) = x_1, h_n(x_{2k}) = x_{2k+3}, h_n(x_{2k+3}) = x_{2k}$. Verify, much like above, that $f_n(x_i) = h_n(g_n(x_i))$.

Then, naturally, we have $f = h(g(x))$, where $g(x) = g_n(x)$ if $x \in S_n$ and $h(x) = h_n(x)$ if $x \in S_n$.

Note in essence that the involutions defined are similar to slightly shifted reflections; will post a more informal explanation.

□ Let p is a prime. Prove $p^2 \equiv 1 \pmod{30}$ or $p^2 \equiv 19 \pmod{30}$

Solution

It's only true for $p > 5$. We have to show that either $p^2 - 1$ is divisible by 30 or $p^2 - 19$ is. Both are even for $p > 5$. Since p is either 1 or 2 mod 3 for p >5, both are divisible by 3. So we have to show 5 divides one of the two. If $p > 5$ then it is either 1,2,3, or 4 mod 5. If it is 1 or 4 mod 5, then 5 divides $p^2 - 1$. If it is 2 or 3 mod 5, then 5 divides $p^2 - 19$. So since 2,3, and 5 all divide one of $p^2 - 1$ or $p^2 - 19$, one of them must be divisible by 30.

□ Find the smallest natural number n,such that there exist positive integrs $x_1, x_2, ..., x_n$, such that $x_1^3 + x_2^3 + ... + x_n^3 = 2008$

Solution

assume there are two positive integers a, b such that $a^3 + b^3 = 2008$

Then $2008 = a^3 + b^3 \geq \frac{(a+b)^3}{4} \implies a + b < 2\sqrt[3]{1004} < 2 \cdot 11 = 22$

Since $2008 = 2^3 \cdot 251$ we have $a + b = 1, 2, 4$ or 8

But $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ so $a^2 - ab + b^2 \geq 251$ but $a^2 - ab + b^2 = (a+b)^2 - 3ab < 8^2 = 64$ contradiction

□ prove: $lcm(1, 2, ..., 2n) = lcm(n + 1, n + 2, ..., n + n)$

Solution

This is obvious, since for every number $a \in \{1, 2, 3, \ldots, n\}$ there exist a number $b \in \{n + 1, n + 2, \ldots, 2n\}$ such that $a \mid b$. The claim easily follows.

□ Prove that: in eight integers have three digits,$\exists$ $\overline{a_1a_2a_3}$ and $\overline{b_1b_2b_3}$ satisfy $a_1a_2a_3b_1b_2b_3 \equiv 0 \pmod{7}$

Solution

just note that $10^3 \equiv -1 \mod 7$, By the box principle there are two integers a_i, a_j with the same residue $\mod 7$ so $10^3a_j + a_i \equiv a_i - a_j \equiv 0 \mod 7$

□ $a_1, a_2, ..., a_n$ are positive numbers such that their sum is one. Find the minimum of: $a_1/(1 + a_2 + ... + n) + a_2/(1 + a_1 + a_3 + ... + a_n) + ... + a_n/(1 + a_1 + ... + a_{n-1})$(and please prove it!).

Solution

Assuming you meant to have $1 + a_2 + \cdots + a_n$ in the denominator of the first term, Let $S = \frac{a_1}{1 + a_2 + \cdots + a_n} + \frac{a_2}{1 + a_1 + \cdots + a_n} + \cdots + \frac{a_n}{1 + a_1 + \cdots + a_{n-1}}$. We have that $a_1 + a_2 + \cdots + a_n = 1$, Thus we can rewrite the original expression as,

$$S = \sum_{i=1}^{n} \frac{a_i}{2 - a_i}$$

We can then add one to each term then subtract n to get,

$$S = -n + \sum_{i=1}^{n} \frac{2}{2 - a_1}$$

Take out a factor of 2 from the sum,

$$S = -n + 2\left(\sum_{i=1}^{n} \frac{1}{2 - a_1}\right)$$

Use Cauchy-Schwarz to show that,

$$(2n-1)\left(\sum_{i=1}^{n}\frac{1}{2-a_1}\right)\geq n^2 \implies \sum_{i=1}^{n}\frac{1}{2-a_1}\geq\frac{n^2}{2n-1}$$

Hence,

$$S=-n+2\left(\sum_{i=1}^{n}\frac{1}{2-a_1}\right)\geq -n+2\left(\frac{n^2}{2n-1}\right)=\frac{2n^2}{2n-1}-n=\boxed{\frac{n}{2n-1}}$$

And that's our answer. Equality occurs when $a_1=a_2=\cdots=a_n=\frac{1}{n}$

□ Find all x such that:

$\sqrt{cos2x-sin4x}=sinx-cosx$

Solution

$\sqrt{cos2x-sin4x}=sinx-cosx$

$$\Leftrightarrow\begin{cases}\sin x-\cos x\geq 0\\ \cos 2x-\sin 4x=1-\sin 2x\end{cases}$$

$$\Leftrightarrow\begin{cases}\sin(x-\frac{\pi}{4})\geq 0\\ (\cos 2x+\sin 2x)(\cos 2x+\sin 2x-1)=0\end{cases}$$

$$\Leftrightarrow\begin{cases}k2\pi\leq x-\frac{\pi}{4}\leq\pi+k2\pi\,(k\in Z)\quad(*)\\ \left[\begin{array}{l}\sin(2x+\frac{\pi}{4})=0\qquad(1)\\ \sin(2x+\frac{\pi}{4})=\frac{\sqrt{2}}{2}=\sin\frac{\pi}{4}\qquad(2)\end{array}\right.\end{cases}$$

We have

$(1):\sin(2x+\frac{\pi}{4})=0$

$\Leftrightarrow x=-\frac{\pi}{8}+\frac{l\pi}{2}\ \ (l\in Z)$

Because the condition (*) must be satisfied by x, therefore :

$k2\pi\leq -\frac{\pi}{8}+\frac{l\pi}{2}-\frac{\pi}{4}\leq\pi+k2\pi\ (l,k\in Z)$

$\Rightarrow\frac{3\pi}{8}\leq\frac{l\pi}{2}-k2\pi\leq\frac{11\pi}{8}$

$\Rightarrow\frac{3}{8}\leq\frac{l}{2}-2k\leq\frac{11}{8}$

$\Rightarrow l=2(2k+1)=2a(a\in Z)$

$\Rightarrow x=-\frac{\pi}{8}+a\pi(a\in Z)$

We have

$(2):sin(2x+\frac{\pi}{4})=sin\frac{\pi}{4}$

$$\Leftrightarrow\left[\begin{array}{l}x=m\pi\\ x=\frac{\pi}{4}+m\pi\end{array}\right.\quad(m\in Z)$$

Similarly, we obtain $x=\frac{\pi}{4}+m\pi$ and $x=(2k+1)\pi$ where $k,m\in Z$

Conclusion, the solutions for the given equation are: $x=-\frac{\pi}{8}+a\pi,x=\frac{\pi}{4}+m\pi,x=(2k+1)\pi$ where $a,m,k\in Z$.

□ I mean that the number of digits of a, plus the number of digits of a^n equals 361

Solution

$\lfloor\log_{10}a\rfloor+\lfloor n\log_{10}a\rfloor=359\qquad(1)$

so $(n+1)\lfloor\log_{10}a\rfloor\leq\lfloor\log_{10}a\rfloor+\lfloor n\log_{10}a\rfloor\leq\lfloor(n+1)\log_{10}a\rfloor$

let $\log_{10}a=p+r$ with $p\in\mathbb{N}$ and $0<r<1$ then

$(n+1)p\leq 359\leq(n+1)p+(n+1)r<(n+1)(p+1)\implies p\leq\frac{359}{n+1}<p+1$

so $p=\left\lfloor\frac{356}{n+1}\right\rfloor$

from (1): $p+np+\lfloor nr\rfloor=359$ but since $0\leq\lfloor nr\rfloor<n$ we have

$359<(n+1)\left\lfloor\frac{356}{n+1}\right\rfloor+n\qquad(2)$

but the only value of $n \in \{1, 2, \ldots, 9\}$ for which (2) is true is $\boxed{n = 6}$

□ Solve the equation $x^x + y^y = \overline{xy} + 3$ where $\overline{xy} = 10x + y$

Solution

$\overline{xy} + 3 \leq 99 + 3 \leq 102 \implies x^x + y^y \leq 102 \implies x, y \leq 3.$

Furthermore, 0^0 is undefined so neither digit can be 0.

Case $x = 1$: $1 + y^y = 13 + y \implies y^y - y = 12 \implies y \notin \mathbb{N}$.

Case $x = 2$: $4 + y^y = 23 + y \implies y^y - y = 19 \implies y \notin \mathbb{N}$.

Case $x = 3$: $27 + y^y = 33 + y \implies y^y - y = 6 \implies y \notin \mathbb{N}$.

So, there are no solutions in $\mathbb{N}$.

□ Find all integer solutions (n,m) to - $n^4 + 2n^3 + 2n^2 + 2n + 1 = m^2$

Solution

we factor the left side of the equation, we obtain

$(n + 1)^2(n^2 + 1) = m^2$

Now $n^2 + 1$ needs to be perfect square, because $(n + 1)^2$ and m^2 are perfect squares.

From $n^2 + 1 = x^2$ we get $n = 0$ and $x = + - 1$, from there $m = + - 1$

And second solution would be for $m = 0$, then we have $n = -1$.

□ For a math contest there is a shortlist with 46 problems, of which 10 are geometry problems. The difficulty of every two problems is different (so there are no two problems with the same difficulty). Let N be the number of ways the selection committee can select 3 problems, such that - Problem 1 is easier than problem 2, - Problem 2 is easier than problem 3, - There is at least one geometry problem in the test. Calculate $\frac{N}{4}$.

Solution

Given an arbitrary selection of three problems, there is only one way to order them such that they are in ascending order of difficulty. Therefore, there are $\binom{46}{3} = 15180$ possible tests. However, we must compute the number of tests with no geometry problems. This is $\binom{36}{3} = 7140$. $N = \frac{15180-7140}{4} = \boxed{2010}$.

□ Show, using the binomial expansion, that $(1 + \sqrt{2})^5 < 99$. Show also that $\sqrt{2} > 1.4$. Deduce that $2^{\sqrt{2}} > 1 + \sqrt{2}$.

Solution

first we will prove that $\sqrt{2} > 1.4$. Squaring that we get that $2 > 1.96$ which is true and we 'll prove that $1.5 > \sqrt{2}$, which is also trivial when we square it.

Now $(1 + \sqrt{2})^5 < 99$. −> $(1 + \sqrt{2})^5 < (1 + 1.5)^5 = 97.65625 < 99$

$2^{\sqrt{2}} > 1 + \sqrt{2}$ is trivial by Bernoulli's inequality...Rewrite number 2 from left side of inequality in form $(1 + 1)$

□ Prove that: p is prime, $p \geq 3$, the equation $x^2 + 1 \equiv \pmod{p}$ have solution if $p = 4k + 1$

Solution

Assume $p = 4k + 3$, then obviously p does not divide x so

$$x^2 \equiv -1 \implies 1 \equiv x^{p-1} \equiv x^{2 \cdot \frac{p-1}{2}} = (-1)^{\frac{p-1}{2}} = -1 \pmod{p}.$$

□ $\Sigma x_i \leq \Sigma x_i^2$ for $x_i > 0$ prove that

Solution

$\Sigma x_i^p \leq \Sigma x_i^{p+1}$ for $p > 1$, $p \in R$

$\Sigma x_i \leq \Sigma x_i^2 \implies \Sigma x_i^2 - x_i \geq 0 \implies \Sigma x_i(x_i - 1) \geq 0$

So it is only natural to divide the terms depending on whether or not they are positive or negative, i.e.:

$\sum_{i:x_i>1} x_i(x_i-1) + \sum_{i:x_i<1} x_i(x_i-1) \geq 0$

Clearly all the terms in the first summand on LHS are positive, whereas all the terms in the second one are negative.

Since $x_i > 1 \implies x_i^{p-1} > 1$ we have, $\sum_{i:x_i>1} x_i^p(x_i-1) \geq \sum_{i:x_i>1} x_i(x_i-1)$

Similarly, $x_i < 1 \implies x_i^{p-1} < 1 \implies \sum_{i:x_i<1} x_i^p(x_i-1) \geq \sum_{i:x_i<1} x_i(x_i-1)$ (recall that both sides are negative)

Adding the two inequalities, we get: $\sum_{i:x_i>1} x_i^p(x_i-1) + \sum_{i:x_i<1} x_i^p(x_i-1) \geq \sum_{i:x_i>1} x_i(x_i-1) + \sum_{i:x_i<1} x_i(x_i-1) \geq 0 \implies \sum_{i:x_i>1} x_i^p(x_i-1) + \sum_{i:x_i<1} x_i^p(x_i-1) = \sum x_i^p(x_i-1) \geq 0 \implies \sum x_i^p \leq \sum x_i^{p+1}$ as desired

□ Let $a_1, a_2, ..., a_n$ be postive real numbers. Prove: $(a_1 + ... + a_n)^2 \leq \frac{\pi^2}{6}(1^2a_1^2 + 2^2a_2^2 + ... + n^2a_n^2)$

Solution

From Cauchy-Schwarz inequality,

$$\frac{\pi^2}{6}\left(\sum_{i=1}^{n} i^2a_i^2\right) = \left(\sum_{i=1}^{\infty} \frac{1}{i^2}\right)\left(\sum_{i=1}^{n} i^2a_i^2\right) \geq \left(\sum_{i=1}^{n} \frac{1}{i^2}\right)\left(\sum_{i=1}^{n} i^2a_i^2\right) \geq \left(\sum_{i=1}^{n} a_i\right)^2.$$

□ Find all pairs of integers (m,n) such that the numbers $A = n^2 + 2mn + 3m^2 + 2,\ B = 2n^2 + 3mn + m^2 + 2,\ C = 3n^2 + mn + 2m^2 + 1$ have a common divisor greater than 1.

Solution

suppose p is prime and $p|A, B, C$.

$A - B = 2m^2 - mn - n^2 = (m-n)(2m+n) \qquad (1)$

$C - B = m^2 - 2mn + n^2 - 1 = (m-n)^2 - 1 \qquad (2)$

From (1), $p|(m-n)$ or $p|(2m+n)$ but clearly $p \not|(m-n)$ because of (2)

replacing $n \equiv -2m \mod p$ in A and C gives $3m^2 + 2 \equiv 12m^2 + 1 \mod p$

but $\gcd(3m^2+2, 12m^2+1) = \gcd(3m^2+2, 7)$ so the greatest common denominator is at most 7

so $3m^2 + 1 \equiv 0 \mod 7 \Longrightarrow m \equiv 2, 5 \mod 7 \Longrightarrow n \equiv 3, 4 \mod 7$

hence $(m,n) = (7k_1 + 2, 7k_2 + 3) or (7k_1 + 5, 7k_2 + 4)$

□ 100 lines lie in the plane. Is it possible for them to have exactly 2010 points of intersection?

Solution

Let $(a, b, c, d, e..)$ be the parallel line sets and numbers of lines parallel. (suppose there are 7 line, Parallel sets are (1,2,3) (4,5) (6) (7), then the code will be (3,2,1,1)) It is easy to see that the intersections are in form $\frac{a(100-a)+b(100-b)+c(100-c)...}{2} = 2010$ where $a + b + c + ... = 100$

$100(a + b + c + ...) - a^2 + b^2 + c^2... = 4020\ 5980 = a^2 + b^2 + c^2...$

Then using trial and error, I obtained a set (77,4,2,2,2,2,2,2,2,2,1,1,1) so it is possible

□ Let f,g: R >R be functions like that so f(g(x))=g(f(x))=-x for any x is element of R a) prove that f and g are odd functions b) Make an example of these two functions f isn't equal to g

Solution

a) : $g(f(g(x))) = g(u)$ where $u = f(g(x)) = -x$ and so $g(f(g(x))) = g(-x)$ $g(f(g(x))) = g(f(v)) = -v$ where $v = g(x)$ and so $g(f(g(x))) = -g(x)$ So $g(-x) = -g(x)$ and $g(x)$ is an odd function.

Same computation with $f(g(f(x)))$ shows that $f(x)$ is an odd function.

b) Choose $f(x) = 2x$ and $g(x) = -\frac{x}{2}$

□ If a+b+c=1, a,b,c>0, prove that
$\frac{ab+\sqrt{a^3c}+\sqrt{b^3c}}{a+b} + \frac{bc+\sqrt{b^3a}+\sqrt{c^3a}}{b+c} + \frac{ca+\sqrt{a^3b}+\sqrt{c^3b}}{c+a} \leq \frac{3}{2}$

Solution

By AM-GM , $\sqrt{a^3c} \leq \frac{a^2+ac}{2}$ and, $\sqrt{b^3c} \leq \frac{b^2+bc}{2}$, therefore - $\sum_{cyclic} \frac{ab+\sqrt{a^3c}+\sqrt{b^3c}}{a+b} \leq \sum_{cyclic} \frac{2ab+a^2+b^2+c(a+b)}{2(a+b)} = \sum_{cyclic} \frac{(a+b)(a+b+c)}{2(a+b)} = \sum_{cyclic} \frac{a+b+c}{2} = \frac{3}{2}$ equality for $a = b = c = \frac{1}{3}$ $Q.E.D$

□ Solve for x, y such that $2x > y > x$,if $2(2x-y)^2 = (y-x)$

Solution

Let $z = y - x$, so $0 < z < x$ and $2(x-z)^2 = z$. Solving for z using the quadratic formula gives:

$$z = \frac{4x + 1 \pm \sqrt{8x+1}}{4}$$

The positive sign gives $z > x$, so take the negative sign. For z to be an integer, $8x + 1 = (4k+1)^2$ for some k. Solving for x gives $x = 2k^2 + k$ for some k, so $z = 2k^2$, so $\boxed{(x, y) = (2k^2 + k, 4k^2 + k)}$ for $k \in \mathbb{N}$

□ Find the sum $\sum_{k=1}^{89} \tan^2 k$

Solution

Let's find a polynomial such that this 89 numbers are the roots of it, then the coefficients will give the sum. We have $(cos(x)+i\cdot sin(x))^n = cos(nx)+i\cdot sin(nx) \implies (1+i\cdot tan(x))^n = \frac{1}{cos(x)^n}(cos(nx)+i\cdot sin(nx))$. Write $z := tan(x)$. Thus, $\sum_{k=0}^{n} \binom{n}{k} i^k z^k = \frac{1}{cos(x)^n}(cos(nx) + i \cdot sin(nx))$. Now let $n = 180$ and let x having 'integer-valued degree', so $\sum_{k=0}^{180} \binom{180}{k} i^k z^k = \frac{1}{cos(x)^n}(cos(nx) + i \cdot sin(nx)) = \frac{(-1)^x}{cos(x)^n}$. Now look at the imaginary part, giving: $z \sum_{k=0}^{89} \binom{180}{2k+1} (-1)^k (z^2)^k = 0$. But this is the polynomial we wanted, since its roots are $tan(k^\circ)^2$ (we also counted $tan(0) = 0$, which can be neglected). So $\sum_{k=1}^{89} tan(k^\circ)^2 = \frac{\binom{180}{177}}{\binom{180}{179}} = \frac{15931}{3}$.

□ Find positive integers a, b, c, d such that $a + b + c + d - 3 = ab = cd$.

Solution

Without loss of generality, $1 \leq a \leq b \leq c \leq d$ so we have $a + b + c + d - 3 \leq 4d - 3$. We also have $a + b + c + d - 3 = cd \leq 4d - 3 \implies 3 \leq (4 - c)d$. The product on the RHS must be positive and it follows that each factor must be positive because d must be a positive integer. Therefore, we have $1 \leq c \leq 3$. From here, we have 3 cases.

Case 1: $c = 1$ If $c = 1$, we must have $a = b = 1$ from our inequality chain. The equality chain becomes $d = 1 = d$ so the solution for this case is $a = b = c = d = 1$. Substituting values, we find that this solution works.

Case 2: $c = 2$ If $c = 2$, we have $a + b + d - 1 = 2d \implies a + b - 1 = d$. Note that $a + b \leq 4 \implies a + b - 1 = d \leq 3$. Now suppose that $d = 3$. Then we have $a + b = 4$ which is only satisfied by $a = b = 2$. Quickly checking, we find that this does not work. If $d = 2$, then we have $ab = 4$, which again is satisfied by $a = b = 2$, so there are no solutions for this case.

Case 3: $c = 3$ If $c = 3$, we have $a + b + d = 3d \implies a + b = 2d$. Note that $a + b \leq 6$ so that $d \leq 3$. Using the equation $ab = cd$ and checking $d = 3$, we find that no a, b exist. Thus, there are no solutions for this case.

The only solution is $\boxed{(a, b, c, d) = (1, 1, 1, 1)}$.

□ The age of the father is 5.5 times as that of the second daughter. Mom got married at 20; at that time grandfather was 57. The first son was born when mom was 22. At present, the first daughter is 19; her age differs from the second son by 5 and from the second daughter by 9. The last year, age of the third son was half of the first son. The sum of the age of the second daughetr and th ethird son equal the age of the second son. What is the the age of the first son?

Solution

Let the first son be x years old.

We know that the second daughter must be 10 years old and the third son's is $\frac{x+1}{2}$ years old.

Also, $10+\frac{x+1}{2}=14$ or 24 since the second son is 5 years older or younger than the first daughter. If $\frac{x+1}{2}=14, x=27$ and if $\frac{x+1}{2}=4, x=7$. Since the first son must be older than the second, then $\boxed{27}$.

□ Solve the following inequality:

$$yzt\sqrt{x-4}+xzt\sqrt{y-4}+xyt\sqrt{z-4}+xyz\sqrt{t-4}\geq xyzt$$

Solution

Because of the surds, we have $x,y,z,t\geqslant 4$, hence

$\sum_{\text{cyc}}\frac{\sqrt{x-4}}{x}\geqslant 1$

By the trivial inequality, $(\sqrt{x-4}-2)^2\geqslant 0 \iff x-4\sqrt{x-4}\geqslant 0 \iff \frac{\sqrt{x-4}}{x}\leqslant\frac{1}{4}$, hence we must have $\frac{\sqrt{\xi-4}}{\xi}=\frac{1}{4}\iff \xi=8$, where ξ is an arbitrary member of the set $\{x,y,z,t\}$

Therefore the only solution is $x=y=z=t=8$.

□ The cells of a $n\times m$ array are filled with real numbers of absolute value at most 1, in such a way that the sum of the entries of any 2×2 square subarray is zero. Find the maximum value of the sum of all entries of the array.

Solution

Let those entries be $|a_{i,j}|\leq 1$, $1\leq i\leq n$, $1\leq j\leq m$. If any of n,m is equal to 1, clearly $\max\sum\limits_{i,j}a_{i,j}=nm$, by taking all entries equal to 1, since the condition on the 2×2 squares is empty of content.

If both n,m are even, the array partitions in $nm/4$ 2×2 squares, so $\sum\limits_{i,j}a_{i,j}=0$ for any such array, therefore the maximum is also 0.

If only one of n,m is even, say n, the array partitions in $n(m-1)/4$ 2×2 squares, plus a column of n entries, therefore the maximum is at most n. On the other hand, a model made by having alternating columns of all 1, then all -1 entries, clearly yields the value n for the sum of all entries.

We are left with the case of both n,m odd. We claim that $\max\sum\limits_{i,j}a_{i,j}=\max\{n,m\}$. The proof goes by induction on $n+m$, the case of any of n,m being equal to 1 having been proved in the above. We cover the $n\times m$ array with a $(n-2)\times(m-2)$ array in the top left corner (of maximal sum $\max\{n-2,m-2\}$, by the induction hypothesis), one horizontal $2\times(m-1)$ array in the bottom left corner (of sum 0), one vertical $(n-1)\times 2$ array in the top right corner (of sum 0), and the entry $a_{n,m}$. Then $\sum\limits_{i,j}a_{i,j}\leq\max\{n-2,m-2\}+0+0-a_{n-1,m-1}+a_{n,m}\leq\max\{n,m\}$, since the entry $a_{n-1,m-1}$ is present in both strips of height/width 2, and all entries are of absolute value at most 1. A model for this maximum value is made by alternating columns of all 1, then all -1 entries if $n>m$, or alternating rows of all 1, then all -1 entries if $n\leq m$. Therefore, in the case of a square 1987×1987 array, the maximal value for the sum is 1987.

□ Prove that if x is real, the minimum value of $\dfrac{(a+x)(b+x)}{(c+x)}(x>-c)$, for $a>c, b>c$ is $(\sqrt{(a-c)}+\sqrt{(b-c)})^2$.

Solution

The function can be written as

$f(x)=x+a+b-c+\frac{(a-c)(b-c)}{x+c}=x+c+\frac{(a-c)(b-c)}{x+c}+a+b-2c$

Applying AM-GM on the first two terms, we get

$f(x) \geqslant 2\sqrt{(a-c)(b-c)} + (a-c) + (b-c)$ and the result follows.

The equality is attained for $x + c = \frac{(a-c)(b-c)}{x+c} \iff x = -c + \sqrt{(a-c)(b-c)}$

$\square$ Solve the following inequation $\frac{1}{1-x^2} + 1 > \frac{3x}{\sqrt{1-x^2}}$

Solution

Since $|x| < 1$, we can substitute $x = \sin\phi$ where $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, hence $\cos\phi > 0$. The inequality becomes:

$\frac{1}{\cos^2\phi} + 1 > \frac{3\sin\phi}{\cos\phi} \iff \tan^2\phi - 3\tan\phi + 2 > 0 \iff (\tan\phi - 1)(\tan\phi - 2) > 0$

Hence $\tan\phi < 1$ or $\tan\phi > 2$, which yields $\sin\phi < \frac{1}{\sqrt{1+1^2}} = \frac{1}{\sqrt{2}}$ or $\sin\phi > \frac{2}{\sqrt{1+2^2}} = \frac{2}{\sqrt{5}}$

Thus the solution is $x \in \left(-1, \frac{1}{\sqrt{2}}\right) \cup \left(\frac{2}{\sqrt{5}}, 1\right)$

$\square$ $x, y \in R$ $x + y = 3(\sqrt{x-2} + \sqrt{y+1} - 1)$ Find maximum of xy

Solution

By Schwarz inequality, $\sqrt{x-2} + \sqrt{y+1} \leq \sqrt{2}\sqrt{x+y-1}$ $(x \geq 2,\ y \geq -1)$

$3(\sqrt{x-2} + \sqrt{y+1} - 1) \leq 3(\sqrt{2}\sqrt{x+y-1} - 1)$

$\iff x + y \leq 3(\sqrt{2}\sqrt{x+y-1} - 1)$, let $t = \sqrt{x+y-1}$ $(x+y \geq 1)$, from $x + y = t^2 + 1$,

we have $t^2 + 1 \leq 3(\sqrt{2}t - 1)$, yielding $\sqrt{2} \leq t \leq 2\sqrt{2}$, which satisfies $t \geq 0$.

$\therefore 4xy \leq (x+y)^2 = (t^2+1)^2 \leq 9^2$, yielding $xy \leq \frac{81}{4}$.

The equality holds when $t^2 = 8 \iff t = 2\sqrt{2}$ $(\sqrt{2} \leq t \leq 2\sqrt{2}) \iff x + y = 9$ *and* $xy = \frac{81}{4}$

$\iff x = y = \frac{9}{2}$, which satisfies $x \geq 2,\ y \geq -1$.

The desired maximum value is $\frac{81}{4}$.

$\square$ Let A be a given positive number and a be the largest integer that is less than or equal to A. Show that the minimum value of non negative integer n such that $n + (-1)^n > A$ is given by $a + 1 - \frac{1}{2}\{1 + (-1)^a\}$.

Solution

Let $n(a)$ denote the smallest n such that $n + (-1)^n > A$ for a given a.

If $2 \mid a$, then $a + (-1)^a = a + 1 > A$ and $(a-1) + (-1)^{a-1} = a - 2 < A$, hence $n(a) = a$

If $2 \nmid a$, then $a + (-1)^a = a - 1 < A$ and $(a+1) + (-1)^{a+1} = a + 2 > A$, hence $n(a) = a + 1$

Thus $n(a) = \begin{cases} a & 2 \mid a \\ a+1 & 2 \nmid a \end{cases}$

On the other hand, $a + 1 - \frac{1+(-1)^a}{2} = \begin{cases} a & 2 \mid a \\ a+1 & 2 \nmid a \end{cases}$

Therefore the claim stands.

$\square$ A set of positive integers has the properties that Every number in the set, apart from 1, is divisible by at least one of 2, 3 or 5 If the set contains 2n 3n or 5n for some integer n, then it contains all three and n as well. The set contains between 300 and 400 integers. How many does it contain?

Solution

All the members of the set are of the form $2^p3^q5^r$ where p, q, r are non-negative integers.

If $2^p3^q5^r$ where $p, q, r > 0$ is an element of the set, then the set also contains the following numbers:

$2^{p-1}3^{q+1}5^r$

$2^{p-1}3^q5^{r+1}$

$2^{p+1}3^{q-1}5^r$

$2^{p+1}3^q5^{r-1}$

$2^p3^{q-1}5^{r+1}$

$2^p3^{q+1}5^{r-1}$

$2^{p-1}3^q5^r$

$2^p3^{q-1}5^r$

$2^p3^q5^{r-1}$

Hence inductively we conclude that:

(i) If some number $2^p3^q5^r$ such that $p+q+r=k$ is contained in the set, then ALL the numbers $2^p3^q5^r$ such that $p+q+r=k$ are contained in the set;

(ii) If the numbers $2^p3^q5^r$ such that $p+q+r=k>0$ are contained in the set, then the numbers $2^p3^q5^r$ such that $p+q+r=k-1$ are contained in the set.

By balls and urns, we get that there are $\binom{k+2}{2}=\frac{(k+1)(k+2)}{2}$ ordered solutions to $p+q+r=k$ if $p,q,r\geqslant 0$. Hence we must find n such that

$300\leqslant\sum_{k=0}^{n}\frac{k^2+3k+2}{2}\leqslant 400$

Using known formulas for the sum of the first and second powers of the first few natural numbers, we have

$300\leqslant\frac{n(n^2+6n+11)}{6}\leqslant 400$

With some trial and error, we get $n=11$, yielding 363 elements in the set.

$\square$

Find a six-digit number whose product by $2,3,4,5,$ and 6 contains the same digits as did the original number (in different order, of course).

Solution

Let $n=\overline{abcdef}$ be the desired number.

Since $6n$ must be a six-digit number, we have $n\leqslant 166666$. Therefore $a=1$.

We also note that the digit 0 can't appear in the number.

Since the units digit of $5n$ can't be zero, the units digit of n must be odd, and it can't be 1, as 1 is already taken as the rightmost digit. It also can't be 5, since that would yield a zero units digit for $2n,4n,6n$. Therefore $n=\overline{1pqrs3}\vee n=\overline{1pqrs7}\vee n=\overline{1pqrs9}$, where p,q,r,s are some digits.

Let's examine $n=\overline{1pqrs3}$. Multiplying it $2,3,4,5,6$, we obtain the following string of units digits: $6,9,2,5,8$. Therefore, with the addition of 1 and 3 already there, we would require a total of 7 digits, and that's impossible.

Let's examine $n=\overline{1pqrs9}$. Multiplication by $2,3,4,5,6$ yields the string of units digits $8,7,6,5,4$, hence the argument is the same as in the previous case.

So we're left with $n=\overline{1pqrs7}$. The string of the units digits is $4,1,8,5,2$, hence the complete set is $\{1,2,4,5,7,8\}$

Assume $p=2$. Then $3n$ would start with 3, which is impossible. If $p=5$, then $2n$ would start with 3. If $p=8$, then $n>166666$. Therefore $p=4$

So $n=\overline{14qrs7}$. If $2n$ would require a carryover from $2q$ to the next digit to the right, we would get either 9 or 0, and that's impossible. Hence $q<5\implies q=2$.

So $n=\overline{142rs7}$. If $s=8$, then $3n$ ends in 61, which is impossible. Hence $s=5\implies r=8$.

Finally we get $\boxed{n=142857}$. It's easy to check that the number fits all the requirements.

$\square$ find all polynominal P with integer coefficients such that for all integers a ,b and n with $a>n\geq 1$ and $b>0$,we have : $P(a)+P(b)=P(a-n)+P(b+n)$

Solution

So since P is a polynomial, let it have a constant term c. Then, let $Q(x)=P(x)-c$.

We still have

$Q(a) + Q(b) = Q(a-n) + Q(b+n)$

Consider $\lim_{n\to a} RHS = Q(0) + Q(a+b)$

As $Q(0) = 0$, we have $Q(a) + Q(b) = Q(a+b) \implies Q(x) = nx$ for some constant n.

Thus, $P(x) = nx + c$, i.e. all polynomials of degree 0 or 1. It can easily be checked that this works.

☐ A broken line inside a cube of side length 1 has a total length of 300. Prove that there exists a plane parallel to one of the faces of the cube that intersects the set of lines at least 100 times.

(A broken line just means a bunch of line segments connected together to form a longer curve that does not intersect itself)

Solution

Suppose the broken line is made up of $N-1$ line segments between N vertecies, $v_1, v_2, ...v_N$ inside a unit cube in the cartesian plane, for convenience let the verticies be $(0,0,0), (0,0,1), ...(1,1,1)$. In general $v_n = (i_n, j_n, k_n)$, and we are given that $\sum_{n=1}^{N-1} |v_n v_{n+1}| = 300$.

Consider a segment $\overline{v_\ell v_{\ell+1}}$. We can project that line onto the x, y and z axis, and from the triangle inequality we have

$$|i_\ell - i_{\ell+1}| + |j_\ell - j_{\ell+1}| + |k_\ell - k_{\ell+1}| \geq |v_\ell v_{\ell+1}|$$

Hence $\sum_{n=1}^{N-1} |i_n - i_{n+1}| + |j_n - j_{n+1}| + |k_n - k_{n+1}| \geq 300$

So there must be one axis, on which lies a projected line of length at least 100. Now since the line lies on an interval $[0,1]$, by the box principle there is some point on that interval that is crossed 100 times by the projected line, call that point P. Now if we take the plane that passes through P and is perpendicular to the axis on which P lies, then that plane will intersect the line in the cube at least 100 times as well.

☐ Find all n for which there are n consecutive integers whose sum of squares is a prime.

Solution

Let the numbers be $k+1, k+2, \ldots, k+n$. Then the sum of their squares is

$S = \frac{(k+n)(k+n+1)(2k+2n+1) - k(k+1)(2k+1)}{6}$ which after simplification yields

$S = kn(k+n+1) + \frac{n(n+1)(2n+1)}{6}$

If n had a prime factor other than 2 and 3 - i.e. a prime factor which couldn't be canceled with the denominator of the fractional term - then the two n's existing in the two terms would allow us to extract a common factor and S couldn't be prime.

Hence $n = 2^p 3^q$, yielding

$S = 2^p 3^q k(2^p 3^q + k + 1) + 2^{p-1} 3^{q-1} (2^p 3^q + 1)(2^{p+1} 3^q + 1)$

Similarly, if $p > 1$ or $q > 1$, then S couldn't be prime.

Therefore we need to check $n = 2, 3, 6$. All of them work as

$2^2 + 3^2 = 13$

$2^2 + 3^2 + 4^2 = 29$

$2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 139$

So our answer is $n \in \{2, 3, 6\}$.

☐ The diagonal of a convex quadrilateral $ABCD$ intersect at O. Let M_1 and M_2 be the centroids of ΔAOB and ΔCOD respectively. Let H_1 and H_2 be the orthocenters of ΔBOC and ΔDOA respectively. Prove that $M_1 M_2 \perp H_1 H_2$.

Solution

Denote M, N the midpoints of AB, CD. Let U, V be the orthogonal projections of A, C on DB and E, F the orthogonal projections of B, D on AC. Then $H_1 \equiv CV \cap BE$ and $H_2 \equiv AU \cap DF$. Moreover, U, E lie on the circumference (M) with diameter AB and F, V lie on the circumference

(N) with diameter CD. Obviously, the right triangles $\triangle BVH_1$ and $\triangle CEH_1$ are similar, then it follows that $H_1B \cdot H_1E = H_1C \cdot H_1V$. Analogously, we have $H_2A \cdot H_2U = H_2D \cdot H_2F \Longrightarrow H_1, H_2$ have equal powers with respect to the circles $(M),(N)$. Thus, H_1H_2 is the radical axis of $(M),(N)$ $\Longrightarrow H_1H_2 \perp MN$ $(\star)$. On the other hand, we have $OM_1 = 2MM_1$ and $OM_2 = 2NM_2$. By Thales theorem we get $M_1M_2 \parallel MN$. Together with $(\star)$, we conclude that $H_1H_2 \perp M_1M_2$.

□ Find all positive integers a, b such that $\frac{a}{b} + \frac{21b}{25a}$ is an integer.

Solution

$\frac{25a^2+21b^2}{25ab} = n \implies 5 \mid b$. Put $b = 5k$ to get $\frac{a^2+21k^2}{5ak} = n \iff a^2 - 5ank + 21k^2 = 0$, hence $a = \frac{k(5n \pm \sqrt{25n^2-84})}{2}$.

Solving $25n^2 - 84 = m^2$ we get $n = 2 \implies a = (5 \pm 2)k$.

Hence $(a, b) = (7k, 5k)$ or $(a, b) = (3k, 5k)$ where $k \in \mathbb{N}$

□ N is a natural number greater than 1. Prove the implication: $n^k + 1 | n^l + 1 \rightarrow k | l$

Solution

Clearly we need $\ell \geq k$. Write $\ell = qk + r$, with $0 \leq r < k$. Now $n^\ell + 1 = (n^k)^q n^r + 1 \equiv (-1)^q n^r + 1 \equiv 0 \pmod{n^k + 1}$ is only possible if q is odd and $r = 0$, hence $k \mid \ell$.

□ Prove that $\frac{2}{201} < \log \frac{101}{100}$

Solution

Note that $\ln(1+x) > \frac{2x}{x+2}$ $(x > 0)$ So we are done Another way: Let l is a tangent line to $f(x) = \frac{1}{x}$ for $x = 100.5$. Let $A(100, 0)$, B is a intersection point of l and $x = 100$, C is a intersection point of l and $x = 101$ and $D(101, 0)$. Hence, $S_{ABCD} = \frac{\frac{2}{201}+t+\frac{2}{201}-t}{2} \cdot 1 = \frac{2}{201}$, where $t = \frac{0.5}{100.5^2}$. Thus, $\ln \frac{101}{100} = \int\limits_{100}^{101} \frac{1}{x} dx > S_{ABCD} = \frac{2}{201}$.

□ Solve $\lfloor x + \lfloor 2x \rfloor \rfloor < 3$

Solution

If $x = n + a$ where $n = [x]$ and $a = \{x\}$, then

$[n + a + 2n + [2a]] < 3$

$3n + [2a] + [a] < 3$

As $[a] = 0$, we get

$3n + [2a] < 3$

Case 1. $0 \leqslant a < \frac{1}{2}$

Then $3n < 3 \iff n < 1 \implies n \leqslant 0$, hence the solution set is $S_1 = \bigcup_{n=-\infty}^{0} \left[n, n + \frac{1}{2}\right)$

Case 2. $\frac{1}{2} \leqslant a < 1$

Then $3n + 1 < 3 \iff n < \frac{2}{3} \implies n \leqslant 0$, hence the solution set is $S_2 = \bigcup_{n=-\infty}^{0} \left[n + \frac{1}{2}, n + 1\right)$

The union of the two sets yields $x < 1$.

□ Prove $\lfloor \sqrt{n} + \frac{1}{2} \rfloor = \lfloor \sqrt{n + \lfloor \sqrt{n} \rfloor} \rfloor$

Solution

Let$\lfloor n \rfloor$be$p$, and we would divided it into 2 case, case I ,$p + \frac{1}{2} \leq \sqrt{n} < p + 1$ since $(p + \frac{1}{2})^2 = p^2 + p + \frac{1}{2}$, so the least integer is $p^2 + p + 1$, and , $L.H.S. = \lfloor \sqrt{n} + \frac{1}{2} \rfloor = p + 1$ $p + 1 \leq \sqrt{(p^2 + p + 1) + p} \leq R.H.S. = \lfloor \sqrt{n + \lfloor \sqrt{n} \rfloor} \rfloor \leq \sqrt{(p + 1)^2 + p} < p + 2$

case II, $p \leq \sqrt{n} < p + \frac{1}{2}$ since $(p + \frac{1}{2})^2 = p^2 + p + \frac{1}{2}$, so the greatest integer is $p^2 + p$, and , $L.H.S. = \lfloor \sqrt{n} + \frac{1}{2} \rfloor = p$ $p \leq \sqrt{(p^2 + p) + p} \leq R.H.S. = \lfloor \sqrt{n + \lfloor \sqrt{n} \rfloor} \rfloor \leq \sqrt{(p)^2 + 2p} < p + 1$

□ Arrange numbers 1, 2, 3, 4, 5 in a line. Any arrangements are equiprobable. Find the probability such that the sum of the numbers for the first, second and third equal to the sum of that of the third, fourth and fifth. Note that in each arrangement each number are used one time without overlapping.

Solution

Total number of ways of keeping $= 5!$ Let the order be a, b, c, d, e We need $a+b+c = c+d+e = x$, say. So, $2x = 15 + c \implies c = 1, 3, 5$ Case 1: $c = 1$ $a, b, 1, d, e$ and $a+b = c+d = 7 \implies (a, b), (c, d) = (2, 5), (3, 4)$ and its seven more permutations So, number of sequences $= 8$ Case 2: $c = 3$ $a, b, 3, d, e$ and $a+b = c+d = 6$ There are 8 sequences similarly Case 3: $c = 5$ $a, b, 5, d, e$ and $a+b = c+d = 5$ There are 8 sequences. So, probability is $\frac{3\times 8}{5!} = \frac{1}{5}$

☐ a and d are non-negative. b and c are positive. Let $b+c \geq a+d$ Find the Min value of $\frac{b}{c+d} + \frac{c}{a+b}$

Solution

First of all, if a and d are both 0, then the least it can be is 2 because it is reduced to $\frac{b}{c} + \frac{c}{b}$. Dr. Graubner's solution is less than this, so we assume that $a + d > 0$.

First of all, if $b + c \neq a + d$, then replacing a with $a \cdot \left(\frac{b+c}{a+d}\right)$ and d with $d \cdot \left(\frac{b+c}{a+d}\right)$ decreases each of the fractions, and decreases the sum. Thus we can assume that $b + c = a + d$.

We can assume that $c \geq d$, because if not, we can switch a and b with d and c, respectively. Since their sum is the same, c is now greater than d. Also, this means that $b \leq a$. We can let $a = e + k$, $b = e - k$, $c = f + k$, meaning that $d = f - k$. Also, we know that $k \geq 0$, $k < e$, $k \leq f$, and $e, f > 0$. We now know that the sum we're looking for is equal to

$\frac{e-k}{2f} + \frac{f+k}{2e} = \frac{e^2-ek+f^2+fk}{2ef}$

$= \frac{e^2-2ef+f^2-ek+fk}{2ef} + 1 = \frac{(f-e+k)(f-e)}{2ef} + 1$

We need to minimize this. Note that it is a linear function in k if e and f are kept constant, and hence takes its min and max at the endpoints, which are 0 and $\min(e, f)$. First, suppose $f > e$. Then both terms of the product are positive, meaning that the sum is greater than 1, and Dr. Graubner's solution again is less than 1. So we assume that $min(e, f) = f$. Then the slope of the linear function, $f - e$, is nonpositive so it is least at the upper endpoint, when $k = f$. We now assume $k = f$.

We are trying to minimize

$1 + \frac{(2f-e)(f-e)}{2ef} = 1 - \frac{(2f-e)(e-f)}{2ef} = 1 - \left(1 - \frac{e}{2f}\right)\left(1 - \frac{f}{e}\right)$

$= 1 + \frac{f}{e} - \frac{1}{2} - 1 + \frac{e}{2f} = -\frac{1}{2} + \frac{\frac{e}{\sqrt{2}f} + \frac{\sqrt{2}f}{e}}{\sqrt{2}}$

$\geq \frac{2}{\sqrt{2}} - \frac{1}{2} = \sqrt{2} - \frac{1}{2}.$

Equality if $\frac{e}{\sqrt{2}f} = 1$, and $e = \sqrt{2}f$, and $k = f$, and $a = (\sqrt{2}+1)f$, $b = (\sqrt{2}-1)f$, $c = 2f$, $d = 0$. In Dr. Graubner's case, $f = \frac{1}{2}$.

☐ let: $a, b, c \geq 1$, $x, y, z \geq 0$ such that $a^x + b^y + c^z = 4$

$xa^x + yb^y + zc^z = 6$

$x^2a^x + y^2b^y + z^2c^z = 9$ Find the hightest value of C

Solution

From the Cauchy-Schwarz inequality we have

$$(x^2a^x + y^2b^y + z^2c^z)(a^x + b^x + c^x) \geq (xa^x + yb^y + zc^z)^2,$$

But if we put the values of the given expressions, $4 \cdot 9 = 6^2$; so that equality must occur in our application. So $x = y = z$ is forced, leading to

$$a^x + b^x + c^x = 4; x[a^x + b^x + c^x] = 6; x^2[a^x + b^x + c^x] = 9.$$

So we have $x = \frac{3}{2} \implies a^{\frac{3}{2}} + b^{\frac{3}{2}} + c^{\frac{3}{2}} = 4$ Therefore $c^{\frac{3}{2}} \leq 4 \implies c^3 \leq 16 \implies c \leq 2\sqrt[3]{2}$. A possible solution set is $(a, b, c) = \left(0, 0, 2\sqrt[3]{2}\right)$; and $(x, y, z) = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$. Hence we are done.

$\square$ Prove that: $\frac{1}{1005}+\frac{3}{1006}+\frac{5}{1007}+\ldots+\frac{2007}{2008}=\frac{2007}{2}-\frac{2006}{3}+\frac{2005}{4}\ldots-\frac{2}{2007}+\frac{1}{2008}$

Solution

Adding $-2\cdot\frac{2007}{2}-2\cdot\frac{2005}{4}-2\cdot\frac{2003}{6}-\cdots-2\cdot\frac{1}{2008}$ to the both sides, and adding the term $2008-\frac{2008}{1}$ at the beginning of the RHS, we get

$\sum_{k=1}^{2008}\frac{2k-2009}{k}=2008+\sum_{k=1}^{2008}\frac{k-2009}{k}$

This is equivalent to $\sum_{k=1}^{2008}2=2008+\sum_{k=1}^{2008}1$, and that's obviously true.

$\square$ Let define a number sequence $a_{n\,n=0}^{\infty}$ as follows : $a_0=13, a_{n+1}=4(a_0a_1a_2\cdots a_n)^2+6a_0a_1a_2\cdots a_n+3$. Find general term a_n

Solution

Define $b_n:=a_0a_1\cdots a_n$. Then $b_0=13$ and $b_{n+1}=4b_n^3+6b_n^2+3b_n \iff 2b_{n+1}+1=8b_n^3+12b_n^2+6b_n+1=(2b_n+1)^3$

Hence $2b_n+1=(2b_0+1)^{3^n}=27^{3^n}=3^{3^{n+1}}$, thus $b_n=\frac{3^{3^{n+1}}-1}{2}$

So $a_n=\frac{b_n}{b_{n-1}}=\frac{3^{3^{n+1}}-1}{3^{3^n}-1}=3^{2\cdot 3^n}+3^{3^n}+1, n\geqslant 0$

$\square$ Coefficient x^4 of $(1+x+x^2+x^3)^{11}$ is ... .

Solution

$$(1+x+x^2+x^3)^{11}=\sum_{\alpha+\beta+\gamma+\delta=11}\frac{11!}{\alpha!\beta!\gamma!\delta!}\cdot x^{\beta+2\gamma+3\delta}$$

$i)$ $\alpha=9,\ \beta=0,\ \gamma=2\wedge\delta=0:\dfrac{11!}{9!\cdot 2!\cdot 0!\cdot 0!}=55$

$ii)$ $\alpha=9,\ \beta=1,\ \gamma=0\wedge\delta=1:\dfrac{11!}{9!\cdot 1!\cdot 1!\cdot 0!}=110$

$iii)$ $\alpha=8,\ \beta=2,\ \gamma=1\wedge\delta=0:\dfrac{11!}{8!\cdot 2!\cdot 1!\cdot 0!}=495$

$iv)$ $\alpha=7,\ \beta=4,\ \gamma=0\wedge\delta=0:\dfrac{11!}{7!\cdot 4!\cdot 0!\cdot 0!}=330$

Hence, our result is: $55+110+495+330=990$.

$\square$ Let f be a real function such that $\forall x;a\in R;f(x+a)=\frac{1}{2}+\sqrt{f(x)-[f(x)]^2}$. Show that f is periodic.

Solution

$f(x+2a)=\frac{1}{2}+\sqrt{f(x+a)-f^2(x+a)}$

$f(x+2a)=\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{f(x)-f^2(x)}-\frac{1}{4}-\sqrt{f(x)-f^2(x)}-f(x)+f^2(x)}$

$f(x+2a)=\frac{1}{2}+\left|\frac{1}{2}-f(x)\right|$

$f(x+4a)=\frac{1}{2}+\left|\frac{1}{2}-f(x+2a)\right|=\frac{1}{2}+\left|\left(-\left|\frac{1}{2}-f(x)\right|\right)\right|$

$f(x+4a)=\frac{1}{2}+\left|\frac{1}{2}-f(x)\right|=f(x+2a)$. QED

$\square$ Let a , b , c be the affixes of an acute-angled triangle having its circumcenter in the origin of complex plane .

Prove that : $\left|\frac{a-b}{a+b}\right|+\left|\frac{b-c}{b+c}\right|+\left|\frac{c-a}{c+a}\right|=\left|\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}\right|$.

Solution

Let $|a|=|b|=R$. Then, if $a=Re^{i\phi}, b=Re^{i\theta}$

$\frac{a-b}{a+b}=\frac{(\cos\phi-\cos\theta)+i(\sin\phi-\sin\theta)}{(\cos\phi+\cos\theta)+i(\sin\phi+\sin\theta)}$

$\frac{a-b}{a+b}=\frac{-2\sin\frac{\phi+\theta}{2}\sin\frac{\phi-\theta}{2}+2i\cos\frac{\phi+\theta}{2}\sin\frac{\phi-\theta}{2}}{2\cos\frac{\phi+\theta}{2}\cos\frac{\phi-\theta}{2}+2i\sin\frac{\phi+\theta}{2}\cos\frac{\phi-\theta}{2}}$

$\frac{a-b}{a+b}=\frac{2i\sin\frac{\phi-\theta}{2}\left(\cos\frac{\phi+\theta}{2}+i\sin\frac{\phi+\theta}{2}\right)}{2\cos\frac{\phi-\theta}{2}\left(\cos\frac{\phi+\theta}{2}+i\sin\frac{\phi+\theta}{2}\right)}$

$\frac{a-b}{a+b}=i\tan\frac{\phi-\theta}{2}$

But $\phi-\theta=-2\gamma$, hence $\frac{a-b}{a+b}=-i\tan\gamma$

Since $0<\alpha,\beta,\gamma<\frac{\pi}{2}$, the equality reduces to

$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha+\tan\beta+\tan\gamma$

which is obviously true.

☐ In any triangle ABC there is an interesting and useful identity $(b+c)^2\cos A + a^2\cos B\cos C = bc(1+\cos A)^2$.

Solution

$\boxed{(b+c)^2\cos A + a^2\cos B\cos C = bc(1+\cos A)^2} \iff$

$\iff \quad (b+c)^2\cdot\frac{b^2+c^2-a^2}{2bc} + a^2\cdot\frac{c^2+a^2-b^2}{2ca}\cdot\frac{a^2+b^2-c^2}{2ab} = bc\left[1+\frac{b^2+c^2-a^2}{2bc}\right]^2 \iff$

$\iff \quad 2(b+c)^2(b^2+c^2-a^2) + \underline{(c^2+a^2-b^2)(a^2+b^2-c^2)} = [(b+c)^2-a^2]^2 \iff$

$\iff \quad 2(b+c)^2(b^2+c^2-a^2) + \underline{a^4-(b^2-c^2)^2} = (b+c)^4+a^4-2a^2(b+c)^2 \iff$

$\iff \quad 2(b+c)^2(b^2+c^2) = (b+c)^4+(b^2-c^2)^2 \iff$

$\iff \quad (b+c)^2\cdot[2b^2+2c^2-(b+c)^2] = (b^2-c^2)^2 \iff$

$\iff \quad (b+c)^2\cdot(b-c)^2 = (b^2-c^2)^2$ O.K. A nice identity ! .

Another way: I'll use the well-known identity $a = b\cdot\cos C + c\cdot\cos B$ a.s.o. Proof. $(b+c)^2\cos A + a^2\cos B\cos C = bc(1+\cos A)^2 \iff$

$(b^2+c^2)\cos A + (a\cdot\cos B)(a\cdot\cos C) = bc\,(1+\cos^2 A) \iff$

$(b^2+c^2)\cos A + (c-b\cdot\cos A)(b-c\cdot\cos A) = bc\,(1+\cos^2 A)$ O.K.

☐ Prove that

$$\boxed{\boxed{\triangle\ ABC \implies \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} + \frac{1}{2r}\left(\frac{a^2}{r_a} + \frac{b^2}{r_b} + \frac{c^2}{r_c}\right) \geq 9}}$$

Solution

Since $\boxed{\begin{array}{l} \blacktriangleright\ r_a = \frac{S}{s-a} \\ \blacktriangleright\ h_a = \frac{2S}{a} \end{array}}$ we have : $\boxed{\frac{r_a}{h_a} = \frac{a}{2(s-a)}}$ a.s.o. Thereby, $\sum \frac{r_a}{h_a} = \frac{1}{2}\cdot\sum \frac{a}{s-a} = \frac{1}{2}\cdot\frac{\sum a(s-b)(s-c)}{(s-a)(s-b)(s-c)}$

Using the well-known identities : $\boxed{\begin{array}{l} \blacktriangleright\ \prod(s-a) = sr^2 \ ;\ abc = 4Rrs \\ \blacktriangleright\ ab+bc+ca = s^2+r^2+4Rr \\ \blacktriangleright\ a^3+b^3+c^3 = 2s(s^2-6Rr-3r^2) \end{array}}$, the last sum becomes, after some computations : $\boxed{\boxed{\sum \frac{r_a}{h_a} = \frac{2R-r}{r}}}$.

But $\sum \frac{a^2}{r_a} = \sum \frac{a^2(s-a)}{rs} = \frac{s\sum a^2 - \sum a^3}{rs} \implies \boxed{\boxed{\sum \frac{a^2}{r_a} = 4(R+r)}}$, where I also used the relation $\boxed{\sum a^2 = 2(s^2-4Rr-r^2)}$.

Therefore, the proposed inequality is equivalent to : $\frac{2R-r}{r} + \frac{2(R+r)}{r} \geq 9 \iff R \geq 2r \iff$ Euler's inequality , so we are done .

☐ $x \in N$ Find x such that $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right]$

Solution

$n \leqslant \frac{x}{99} < n+1 \wedge n \leqslant \frac{x}{101} < n+1$

Hence $99n \leqslant x < 99n+99 \wedge 101n \leqslant x < 101n+101$

For there to be a common solution, the intervals must be either interlocked or enclosed.

1. $99n \leqslant 101n+101 \leqslant 99n+99 \implies -101 \leqslant 2n \leqslant -2 \implies -50 \leqslant n \leqslant -1$

In this case $x \in [99n, 101n + 101)$

2. $99n \leqslant 101n \leqslant 99n + 99 \implies 0 \leqslant 2n \leqslant 99 \implies 0 \leqslant n \leqslant 49$

In this case $x \in [101n, 99n + 99)$

3. Because the interval lengths are 99 and 101 respectively, enclosing is possible only thus: $101n \leqslant 99n < 99n + 99 \leqslant 101n + 101 \implies n \leqslant 0 \wedge n \geqslant -1 \implies n \in \{-1, 0\}$. The interval $[-99, 0)$ is already covered in Case 1, and the interval $[0, 99)$ is already covered in Case 2.

Hence $x \in \bigcup_{n=-50}^{-1}[99n, 101n + 101) \cup \bigcup_{n=0}^{49}[101n, 99n + 99)$

□ $x, y \in R^+$ $x^3 + y^3 = 4x^2$ Find the Max of $\quad x + y$

Solution

Let $x+y = k$. Hence, the equation $k(x^2 - x(k-x) + (k-x)^2) = 4x^2$ has real root. But $k(x^2 - x(k-x) + (k-x)^2) = 4x^2 \Leftrightarrow (3k-4)x^2 - 3k^2x + k^3 = 0$. If $k = \frac{4}{3}$ so $x = \frac{4}{9}$ and $y = \frac{8}{9}$. Let $k \neq \frac{4}{3}$. Hence, $(3k^2)^2 - 4(3k-4)k^3 \geq 0$, which gives $0 \leq k \leq \frac{16}{3}$. For $k = \frac{16}{3}$ we obtain: $x = \frac{32}{9}$ and $y = \frac{16}{9}$. Hence, $\max_{x^3+y^3=4x^2}(x+y) = \frac{16}{3}$. Since $\frac{32}{9} > 0$ and $\frac{16}{9} > 0$, the answer is $\frac{16}{3}$. □ For $x, y, p > 1$, prove that

$\sqrt[p]{\frac{x^p+y^p}{2}} \leq \frac{x+y}{2} + \frac{p-1}{8}(x-y)^2,$

the inequality sign is reversed for $0 \neq p < 1$.

Solution

I just prove when p>1,wlog $y \geq x$,let $y = kx, k \geq 1$ $\sqrt[p]{\frac{x^p+y^p}{2}} \leq \frac{x+y}{2} + \frac{p-1}{8}(x-y)^2, \Longleftrightarrow \sqrt[p]{\frac{1+k^p}{2}}x \leq \frac{1+k}{2}x + \frac{p-1}{8}(k-1)^2 x^2$ we only need to prove $\sqrt[p]{\frac{1+k^p}{2}} \leq \frac{1+k}{2} + \frac{p-1}{8}(k-1)^2$ let $f(x) = \sqrt[p]{\frac{1+k^p}{2}}$ Hance $f(x) = f(1) + f'(1)(k-1) + f''(1)(k-1)^2 + f^3(\xi)(k-1)^3$

$$= \frac{1+k}{2} + \frac{p-1}{8}(k-1)^2 - \frac{(1/2\,\xi^p + 1/2)^{p^{-1}}\xi^p(p-1)(\xi^p p - p + 2 + \xi^p)}{\xi^3(\xi^p+1)^3}(k-1)^3 \leq \frac{1+k}{2} + \frac{p-1}{8}(k-1)^2$$

where $1 < \xi < k$ it's easy to know the inequality sign is reversed for $0 \neq p < 1$.

□ Let three functions f, u, v: R–>R such that $f(x + \frac{1}{x}) = f(x) + \frac{1}{f(x)}$ for all non-zero x and $[u(x)]^2 + [v(x)]^2 = 1$ for all x. We know that $x_0 \in R$ such that $u(x_0).v(x_0) \neq 0$ and $f(\frac{1}{u(x_0)} \cdot \frac{1}{v(x_0)}) = 2$; find $f(\frac{u(x_0)}{v(x_0)})$.

Solution

$f(\frac{u(x_0)}{v(x_0)}) + \frac{1}{f(\frac{u(x_0)}{v(x_0)})} = f(\frac{u(x_0)}{v(x_0)} + \frac{v(x_0)}{u(x_0)}) = f(\frac{1}{u(x_0)*v(x_0)}) = 2.$

Then, let $f(\frac{u(x_0)}{v(x_0)}) = x \Leftrightarrow x + \frac{1}{x} = 2 \Leftrightarrow x = 1.$

□ $x_1 = 1$ $x_{n+1} = \frac{x_n^2}{\sqrt{3x_n^4 + 6x_n^2 + 2}}$

Find $\quad x_n$

Solution

Substitute $x_n^2 = \frac{1}{a_n}$ to get

$a_{n+1} = 2a_n^2 + 6a_n + 3$ with $a_1 = 1$

The above equation yields $2a_{n+1} + 3 = 4a_n^2 + 12a_n + 9 = (2a_n + 3)^2$

Thus $2a_n + 3 = (2a_1 + 3)^{2^{n-1}} = 5^{2^{n-1}} \iff a_n = \frac{5^{2^{n-1}} - 3}{2}$

So finally $x_n = \frac{1}{\sqrt{a_n}} = \sqrt{\frac{2}{5^{2^{n-1}} - 3}}$

□ Let s be the perimeter of an acute triangle ABC (not equilateral) with its circumcenter O, incenter I. P is a variable point inside $\triangle ABC$. D,E,F are projections of P on BC,CA,AB. Prove that $2(AF + BD + CE) = s$ if and only if P is on OI.

Solution

Let $(x : y : z)$ be the barycentric coordinates of P with respect to $\triangle ABC$. Therefore, coordinates of its projections D, E, F onto BC, CA, AB, in Conway's notation, are $D(0 : a^2y + xS_C : a^2z + xS_B)$, $E(b^2x + yS_C : 0 : b^2z + yS_A)$ and $F(c^2x + zS_B : c^2y + zS_A : 0)$. From these, we deduce that

$\overline{AF} = \frac{c^2y+zS_A}{c(x+y+z)}$, $\overline{BD} = \frac{a^2z+xS_B}{a(x+y+z)}$, $\overline{CE} = \frac{b^2x+yS_C}{b(x+y+z)}$

For any fixed k such that $\overline{AF} + \overline{BD} + \overline{CE} = k$, locus $f(x,y,z) = 0$ is linear. Indeed

$k(x+y+z) = \frac{c^2y+zS_A}{c} + \frac{a^2z+xS_B}{a} + \frac{b^2x+yS_C}{b}$

Therefore, locus of points P is a single line f with the above barycentric equation.

Particularly, the locus f for $k = \frac{1}{2}(a+b+c)$ contains the circumcenter O and incenter I. Indeed, if M, N, L are the midpoints of BC, CA, AB and X, Y, Z the tangency points of the incircle (I) with BC, CA, AB, we have

$\overline{AL} + \overline{BM} + \overline{CN} = \frac{1}{2}(a+b+c) \Longrightarrow O \in f$

$\overline{AZ} + \overline{BX} + \overline{CY} = (s-a) + (s-b) + (s-c) = \frac{1}{2}(a+b+c) \Longrightarrow I \in f.$

□ $p_1 = 2, p_2 = 3, p_3 = 5, \dots p_n$ nth prime.$s_n = \sum_{i=1}^{n} p_i$. Prove that there exists perfect square in $[s_n, s_{n+1}]$ interval.

Solution

It is true for $n = 1, 2, 3, 4$ [4,9,16,25]

For $n > 4$,

oh yes, just a litle easy thing/claim:

If $a \in \mathbb{R}+$, there is a perfect square in $[a^2, (a+1)^2]$ proof: assume I don't say the true: there exist $n \in N$ so that $n^2 < a^2 < (a+1)^2 < (n+1)^2$, but then we see that $n < a$, but $2a+1 < 2n+1 \Rightarrow a < n$, contradiction.

We say now that $\sqrt{s_n} < \frac{p_n+1}{2}$

Proof: $2+3+5+7+11 = 28 < 5.5^2$

For further primes, $p_{n+1} > p_n + 2$ (the further primes are odd and $\in N$)

So $s_n \leq p_n + (p_n - 2) + ... + 11 + 7 + 5 + 3 + 2 < p_n + (p_n - 2) + ... + 11 + 9 + 7 + 5 + 3 + 1 = \frac{(p_n+1)^2}{4}$.

$s_{n+1} - s_n \geq p_n + 2 = 2\frac{p_n+1}{2} + 1$, so $[s_n, s_n + 2\sqrt{s_n} + 1] \subset [s_n, s_{n+1}]$ and with the claim we know there is a perfect square in that interval.

□ A, B, C are 3 collinear points and let P be a point not on the line joining them. Prove that the circumcentres of triangles - ABP, BCP, ACP and the point P lie on a circle.

Solution

Let O_1, O_2, O_3 be the circumcenters of $\triangle PAC$, $\triangle PAB$, $\triangle PBC$. We use oriented angles (mod 180). Since $\angle PO_2A = 2\angle PBC = \angle PO_3C$, then the isosceles $\triangle PO_2A$ and $\triangle PO_3C$ are similar $\Longrightarrow$ $\angle APO_2 = \angle CPO_3$, which implies that $\angle O_2PO_3 = \angle APC$. Since $O_1O_2 \perp PA$ and $O_1O_3 \perp PC$, it follows that $\angle APC = \angle O_2O_1O_3$. Hence, $\angle O_2O_1O_3 = \angle O_2PO_3 \Longrightarrow P \in \odot(O_1O_2O_3)$.

□ Show that $(1+x)^n \geq (1-x)^n + 2nx(1-x^2)^{(n-1)/2}$ for all $0 \leq x \leq 1$ and all positive integers n.

Solution

Let $a = 1+x, b = 1-x$ so that $a, b > 0$ and we have to prove

$$a^n \geq b^n + n(a-b)(ab)^{\frac{n-1}{2}};$$

Which can be rewritten as (since $a > b$; this is obvious for $a = b \implies x = 0$)

$$\frac{a^n - b^n}{a-b} = a^{n-1} + a^{n-2}b + \cdots + b^{n-1} \geq n(ab)^{\frac{n-1}{2}};$$

Which is perfectly true on using the AM-GM inequality:

$$\begin{aligned} a^{n-1} + a^{n-2}b + \cdots + b^{n-1} &\geq n \cdot \sqrt[n]{(ab)^{(n-1)+\cdots+1}} \\ &= n \cdot (ab)^{\frac{1}{n} \cdot \frac{n(n-1)}{2}} = n(ab)^{\frac{n-1}{2}}. \end{aligned}$$

☐ Solve it: $x! = xy + x + y$ and $x, y \in \mathbb{N}$.

Solution

Using Simon's Favorite Factoring Trick, add 1 to both sides and factor:

$x! + 1 = (x+1)(y+1)$

First let's get the trivial cases out of the way. If $x = 1$, $y = 0$. If $x = 0$, $y = 1$. If $x = 2$, $y = 0$ as well. (These work only if you consider $\mathbb{N}$ to contain 0.)

Now we may assume $x \geq 3$. Obviously, if $x+1$ is composite, we can factor it into $p, q \leq x$, so that $x + 1 = pq \mid x!$. So the original equation cannot hold. Otherwise, if $x+1$ is prime, by Wilson's Theorem, $x! \equiv -1 \bmod x+1$, so $x+1 \mid x!+1$. Also, obviously $x! > x$, so that $\frac{x!+1}{x+1} > 1$. Therefore, $y = \frac{x!+1}{x+1} - 1$ is a positive integer.

We now have our solutions: for any prime $p > 2$, $x = p-1, y = \frac{(p-1)!+1}{p} - 1$, plus the three special cases above if you consider $\mathbb{N}$ to contain 0.

☐ Find the value $\sum_{k=1}^{n} \left(\sin\frac{k\pi}{n}\right)^4$.

Solution

let $S_n = \sum_{k=1}^{n} (\sin\frac{k\pi}{n})^4$

for $n = 1, S_1 = 0$ $n = 2, S_2 = 1$

we suppose that $n > 2$

We have $sin^4(x) = \frac{1}{8}(3 - 4cos(2x) + cos(4x))$

then $\sum_{k=1}^{n} (\sin\frac{k\pi}{n})^4 = \sum_{k=1}^{n} \frac{1}{8}(3 - 4cos(2\frac{k\pi}{n}) + cos(4\frac{k\pi}{n}))$

$= \frac{3}{8}n - \frac{1}{2}\sum_{k=1}^{n} Re(e^{\frac{i2k\pi}{n}}) + \frac{1}{8}\sum_{k=1}^{n} Re(e^{\frac{i4k\pi}{n}})$

$= \frac{3}{8}n - \frac{1}{2}Re(\sum_{k=1}^{n}(e^{\frac{i2k\pi}{n}})) + \frac{1}{8}Re(\sum_{k=1}^{n}(e^{\frac{i4k\pi}{n}}))$

$= \frac{3n}{8}$

☐ Let m, n be positive integers, $m > n$. Prove that

$$\operatorname{lcm}(m,n) + \operatorname{lcm}(m+1, n+1) > \frac{2mn}{\sqrt{m-n}}$$

Solution

Let $gcd(m-n, n) = gcd(m, n) = a$ and $gcd(m-n, n+1) = gcd(m+1, n+1) = b$ a and b are coprime and divide $m-n$ so $ab \leq m-n$ so $\frac{1}{\sqrt{ab}} \geq \frac{1}{\sqrt{m-n}}$ $\operatorname{lcm}(m,n) + \operatorname{lcm}(m+1,n+1) = \frac{mn}{a} + \frac{(m+1)(m+1)}{b} > mn\left(\frac{1}{a} + \frac{1}{b}\right)$ So $\operatorname{lcm}(m,n) + \operatorname{lcm}(m+1,n+1) \geq \frac{2mn}{\sqrt{ab}} \geq \frac{2mn}{\sqrt{m-n}}$

☐ r and s are distinct, nonreal complex numbers such that $r + \frac{1}{s} \in \mathbb{R}$ and $s + \frac{1}{r} \in \mathbb{R}$.

Evaluate $|r \cdot s|$.

Solution

Let $r = a + bi$ and $s = m + ni$. We have $a + bi + \frac{1}{m+ni}$, or $a + bi + \frac{m}{m^2+n^2} - \frac{ni}{m^2+n^2}$, belongs to reals. Thus, the coefficient of the i terms must be 0, so $bi - \frac{ni}{m^2+n^2} = 0 \implies b = \frac{n}{m^2+n^2}$. This becomes $m^2 + n^2 = \frac{n}{b}$.

We are also given that $s + \frac{1}{r}$, or $m + ni + \frac{a-bi}{a^2+b^2}$, is real. Hence, $ni - \frac{bi}{a^2+b^2} = 0$, so $n = \frac{b}{a^2+b^2}$. This becomes $a^2 + b^2 = \frac{b}{n}$.

What we wish to find is $|rs|$, or $|(a+bi)(m+ni)|$, or $|am-bn+ani+bmi|$. This is $\sqrt{(am-bn)^2 + (an+bm)^2}$ $\sqrt{a^2m^2 - 2abmn + b^2n^2 + a^2n^2 + b^2m^2 + 2abmn} = \sqrt{(a^2+b^2)(m^2+n^2)}$.

HEY! $a^2 + b^2 = \frac{n}{b}$ from earlier and $m^2 + n^2 = \frac{b}{n}$ from earlier. Thus, multiplying them will yield 1, and $\sqrt{1} = \boxed{1}$.

☐ Determine all the natural numbers x ,$y \geq 1$, such that $2^x - 3^y = 7$

Solution

Looking mod 3, we must have $2^x \equiv 1 \bmod 3$ which implies $x = 2a$

Now we have:

$2^{2a}-3^y=7\Rightarrow 4^a-3^y=7$

Now looking mod 4, we must have $3^y\equiv 1 \mod 4$ which implies $y=2b$

So:

$2^{2a}-3^{2b}=7\Rightarrow(2^a+3^b)(2^a-3^b)=7$

But this means $2^a-3^b=1$ and $2^a+3^b=7$

Adding the two equations gives $2^{a+1}=8$ so $a=2$ and $x=2a=4$

And then this means that $3^b=3$ so $b=1$ and $y=2b=2$

So the only solution is $(x,y)=(4,2)$ and this does satisfy the equation.

$\square$ Prove that $\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}=\frac{1}{n+1}$

Solution

It is equivalent to

$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}=1$

We know that $0=(1-1)^{n+1}=\sum_{k=0}^{n+1}1^{n+1-k}(-1)^k\binom{n+1}{k}=1^{n+1}\binom{n+1}{0}+\sum_{k=0}^{n}(-1)^{k+1}\binom{n+1}{k+1}=1-\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}$ by the binomium of Newton.

So, we can conclude that $\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}=1$.

Another way:

Note that $\frac{1}{k+1}\binom{n}{k}=\dfrac{n!}{(k+1)!(n-k)!}=\frac{1}{n+1}\binom{n+1}{k+1}$

So $LHS=\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\cdots$

$=\frac{1}{n+1}\left(\binom{n+1}{1}-\binom{n+1}{2}+\binom{n+1}{3}-\binom{n+1}{4}+\cdots\right)$

$=\frac{1}{n+1}\cdot-\left((1-1)^{n+1}-\binom{n+1}{0}\right)$

$=\frac{1}{n+1}$

$\square$ Let $a,b,c\geq 0$ be reals such that $a+b+c=1$. Prove that

$$(ab+bc+ca)\left(\frac{a}{b^2+b}+\frac{b}{c^2+c}+\frac{c}{a^2+a}\right)\geq\frac{3}{4}$$

Solution

Let $f(x)=\frac{1}{x(x+1)}$, then we get $f''(x)>0$. Let $ab+bc+ca=p$, we have

$$af(b)+bf(c)+cf(a)\geq f\left(\frac{ab+bc+ca}{a+b+c}\right)=\frac{1}{p(p+1)}$$

It's enough to prove that $\frac{1}{p+1}\geq\frac{3}{4}$. That's true $\iff 1\geq 3(ab+bc+ca)$. This follows from $(a+b+c)^2\geq 3(ab+bc+ca)$ $\square$ Let $f(m,n)=3m+n+(m+n)^2$. Calculate the value of $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}2^{-f(m,n)}$.

Solution

We need to prove that $f(m,n):\mathbb{N}_0\times\mathbb{N}_0\to 2\mathbb{N}_0$ is bijective. In other words, a distinct m and n map to a distinct even number, and every even number is mapped to.

Suppose we write $f(m,n)$ as $(m+n)(m+n+1)+2m=2\left(\binom{m+n+1}{2}+m\right)$. Then obviously it's even. To show it's surjective, suppose we want $f(m,n)=2k$. Then there is some number l so that $\binom{l+1}{2}\leq k<\binom{l+2}{2}$. Then $k-\binom{l+1}{2}=m$ and $l-m=n$. We now need to show that neither m nor n is negative. Obviously m is nonnegative by construction. We need to show that $l\geq m$, which is true because

$m=k-\binom{l+1}{2}\leq\left(\binom{l+2}{2}-1\right)-\binom{l+1}{2}=l+1-1=l.$

So $f(m,n)$ is surjective to the even numbers. Now suppose $f(m,n)=f(p,q)$ with $(m,n)\neq(p,q)$. Then

$\binom{m+n+1}{2} + m = \binom{p+q+1}{2} + p$

Either $m+n = p+q$, in which case $m = p$ and thus $n = q$, or WLOG $m+n \geq p+q+1$. Then we have

$\binom{m+n+1}{2} = \binom{p+q+1}{2} + p - m \leq \binom{m+n}{2} + p - m$

$= \binom{m+n+1}{2} - (m+n) + p - m$

$0 \leq p - (m+n) - m \leq p + q - m - n$

or $p+q \geq m+n$, impossible. Hence it is injective as well.

So the image of $f(m,n)$ is every even number, exactly once. So $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2^{-f(m,n)} = \sum_{k=0}^{\infty} 2^{-2k} = \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{4}{3}$.

$\square$ Find a formula for $\prod_{k=1}^{N}\left(xe^{\frac{i2\pi k}{N}} + y\right)$ in terms of x, y, and N.

Solution

Consider the polynomial $P(\xi) = (\xi - y)^N - x^N$. Its roots are $\xi_k = x\omega_k + y, k = \overline{1, N}$ where ω_k are all Nth roots of unity - which is obvious as $(\xi - y)^N - x^N = 0 \iff \left(\frac{\xi - y}{x}\right)^N = 1$.

The required product is the product of all ξ_k, which is obtained by Vieta:

$\prod_{k=1}^{N} \xi_k = (-1)^N P(0) = (-1)^N[(-y)^N - x^N] = y^N - (-x)^N$

$\square$ Let $a_0, a_1, \ldots, a_{2n}$ real numbers such that $\forall k \in \{1, 2\ldots, 2n-1\} : a_k \geq \frac{a_{k-1}+a_{k+1}}{2}$. Prove that $\frac{a_1+a_3+\cdots+a_{2n-1}}{n} \geq \frac{a_0+a_2+\cdots+a_{2n}}{n+1}$ and find equality condition.

Solution

Let us working out:

$(n+1)(a_1 + a_3 + \cdots + a_{2n-1}) \geq n(a_0 + a_2 + \cdots + a_{2n})$

and we know that $(a_1 + a_3 + \cdots + a_{2n-1}) \geq \frac{a_0+a_{2n}}{2} + (a_2 + a_4 + \cdots + a_{2n-2})$ by $a_k \geq \frac{a_{k-1}+a_{k+1}}{2}$.

So we have to prove $a_2 + a_4 + \cdots + a_{2n-2} \geq (n-1)\frac{a_0+a_{2n}}{2}$

and this follows by $a_{n-x} + a_{n+x} \geq a_0 + a_{2n}$ for $0 \leq x \leq n$

This thing can we prove by induction:

$2a_n \geq a_{n-1} + a_{n+1}$ is already known,

IH:$2a_n \geq a_{n-1} + a_{n+1} \geq \cdots \geq a_{n-k} + a_{n+k}$ for $0 < k < n$

We know also that $2a_{n-k} \geq a_{n-k+1} + a_{n-k-1}$ and similar $2a_{n+k} \geq a_{n+k+1} + a_{n+k-1}$ add this gives $2(a_n - k + a_{n+k}) \geq a_{n-k+1} + a_{n-k-1} + a_{n+k+1} + a_{n+k-1}$ and by (IH) we know $a_{n-k-1} + a_{n+k+1} \leq 2(a_n - k + a_{n+k}) - a_{n+k-1} - a_{n-k+1} \leq a_{n-k} + a_{n+k}$

There holds only equality if $a_0, a_1, \ldots, a_{2n}$ is an arithmetic sequence.

$\square$ Let a,b be positive rational numbers such that $a \neq b$ and $a^(1/3) + b^(1/3)$ is a rational number. Show that $a^(1/3)$ is a rational number.

Solution

By the identity $(\sqrt[3]{a} + \sqrt[3]{b})^3 = a + b + 3\sqrt[3]{ab}(\sqrt[3]{a} + \sqrt[3]{b})$, we can see that $\sqrt[3]{ab}$ is also rational. WLOG $a > b$. Then by the quadratic formula, $\sqrt[3]{a} = r + \sqrt{s}$ and $\sqrt[3]{b} = r - \sqrt{s}$, where r and s are rational.

Then $a = (r + \sqrt{s})^3 = r^3 + 3rs + \sqrt{s}(3r^2 + s)$. In order for this to be rational, either s is a perfect square, or $s = -3r^2 \leq 0$, impossible. Thus s is a perfect rational square, say q^2, and $\sqrt[3]{a} = r + q \in \mathbb{Q}$.

$\square$ Find all primes of the form $a^n + 1$ where a and n are natural numbers and n is not a power of 2.

Solution

Clearly n should be even. Suppose n isn't a power of 2. We can write n as $2^m t$ where t is an odd positive integer. Let $t = 2k+1$, we have

$$a^n + 1 = a^{2^m(2k+1)} = (a^{2m})^{2k+1} = x^{2k+1} + 1.$$

But then

$$x^{2k+1}+1=(x+1)(x^{2k}-x^{2k-1}+\cdots-a+1)$$

Which can't be a prime. Contradiction, so n is a perfect power of 2.

$\square$ Sum to n terms:

$$\frac{1}{4}+\frac{1\times 3}{4\times 6}+\frac{1\times 3\times 5}{4\times 6\times 8}+\frac{1\times 3\times 5\times 7}{4\times 6\times 8\times 10}+\cdots$$

Solution

For the limit, you can also use the generalized binomial formula (generally useful when double factorials are involved).

$S=\sum_{n=2}^{\infty}\frac{2(2n-3)!!}{(2n)!!}=\sum_{n=2}^{\infty}\frac{2(2n-3)!!}{2^n n!}$

$S=\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(-1)(-3)\cdots[-2(n-1)+1]}{2^{n-1}n!}$

$S=-2\sum_{n=2}^{\infty}\frac{(1)(-1)(-3)\cdots[-2(n-1)+1]}{2^n n!}(-1)^n$

$S=-2\sum_{n=2}^{\infty}\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!}(-1)^n$

$S=-2\sum_{n=2}^{\infty}\binom{1/2}{n}(-1)^n$

$S=-2\left((1-1)^{1/2}-\left(1-\frac{1}{2}\right)\right)=1$

$\square$ Find all positive integers n such that $17|3^n-n$.

Solution

3^n is periodic of period 16 modulo 17; n is periodic of period 17 modulo 17. Thus we only need consider the remainders modulo $16\cdot 17$ to find those that check. The first one is 5.

$\square$ For given $a>0$, $b>0$ find minimum value of $y>a$ so that is truly the implication $|x-y|\le a$, $x\ne 0 \implies \left|\frac{1}{x}-\frac{1}{y}\right|\le\frac{1}{b}$.

Solution

From the given condition , we deduce

$$\frac{1}{y+a}\le\frac{1}{x}\le\frac{1}{y-a}$$

Thus, it is enough to solve for the following quadratic inequation

$$y^2-ay-ab\ge 0$$

This yields

$$y\ge\frac{a+\sqrt{a^2+4ab}}{2}$$

Therefore, $y_{min}=\frac{a+\sqrt{a^2+4ab}}{2}$

$\square$ For a tetrahedron $ABCD$, let O be on the inside. $AO\cap\triangle BCD=A_1$, $BO\cap\triangle CDA=B_1$ $CO\cap\triangle DAB=C_1$, $DO\cap\triangle ABC=D_1$

What is the Min of $\sum_{cyclic}\frac{AA_1}{A_1O}$?

Solution

Let S,S_A,S_B,S_C,S_D be the volume of the tetrahedra ABCD,OBCD,OACD,OADB,OABC.

$\frac{AA_1}{OA_1}+\frac{BB_1}{OB_1}+\frac{CC_1}{OC_1}+\frac{DD_1}{OD_1}=\frac{S}{S_A}+\frac{S}{S_B}+\frac{S}{S_C}+\frac{S}{S_D}$

$\frac{AA_1}{OA_1}+\frac{BB_1}{OB_1}+\frac{CC_1}{OC_1}+\frac{DD_1}{OD_1}=S\left(\frac{1}{S_A}+\frac{1}{S_B}+\frac{1}{S_C}+\frac{1}{S_D}\right)$

By AM-HM on the positive numbers S_A,S_B,S_C,S_D, we get

$\frac{4}{\frac{1}{S_A}+\frac{1}{S_B}+\frac{1}{S_C}+\frac{1}{S_D}}\le\frac{S_A+S_B+S_C+S_D}{4}=\frac{S}{4}$

$\Longrightarrow \frac{AA_1}{OA_1} + \frac{BB_1}{OB_1} + \frac{CC_1}{OC_1} + \frac{DD_1}{OD_1} \geq 16$

Therefore, minimum value occurs when O coincides with the centroid of $ABCD$.

$\square$ Let P and Q be points on the side AB of the triangle $\triangle ABC$ (with P between A and Q) such that $\angle ACP = \angle PCQ = \angle QCB$,and let AD be the angle bisector of $\angle BAC$. Line AD meets lines CP and CQ at M and N respectively. Given that $PN = CD$ and $3\angle BAC = 2\angle BCA$, prove that triangles $\triangle CQD$ and $\triangle QNB$ have the same area.

Solution

First of all, let us denote $\angle BAD = \angle DAC = \angle ACP = \angle PCQ = \angle QCB = x$ for the sake of convenience.

By simple angle chasing, we have that the area of $\triangle CQD$ is $\frac{QC \cdot CD \sin x}{2}$ and that of $\triangle QNB$ is $\frac{QN \cdot QB \sin 4x}{2}$.

Consider $\triangle AQC$; as $\angle QAC = \angle QCA$, it is isosceles. Hence, $PN \parallel AC$ and we conclude that $\triangle QPN \sim \triangle QAC$. This gives $\frac{QN}{PN} = \frac{QC}{AC} \iff QN \cdot AC = CD \cdot QC$.

Hence, it suffices to prove that $AC \sin x = QB \sin 4x \iff \frac{AC}{\sin 4x} = \frac{QB}{\sin x} = \frac{BC}{\sin 4x}$; that is, $AC = BC$.

In $\triangle PNC$, as $PN \parallel AC$, we have $\angle NPC = \angle NCP = x \iff NP = NC = CD$, so $\angle CND = \angle CDN \iff 7x = 180$.

However, since $\angle CAB = 2x$ and $\angle CBA = 180 - 5x$, we indeed have $AC = BC$, and we are done. $\blacksquare$

$\square$

Calculate $\sum_{k=0}^{16} \cos^2\left(\frac{2k\pi}{17}\right)$

Solution

Denote by $\zeta = \cos\dfrac{2\pi}{17} + \mathrm{i}\sin\dfrac{2\pi}{17}$ the principal primitive root of order 17 of the unity. Then $\zeta^k = \cos\dfrac{2k\pi}{17} + \mathrm{i}\sin\dfrac{2k\pi}{17}$. On the other hand $(\zeta^k)^2 = 2\cos^2\dfrac{2k\pi}{17} - 1 + 2\mathrm{i}\sin\dfrac{2k\pi}{17}\cos\dfrac{2k\pi}{17}$.

Now, $\sum_{k=0}^{16}(\zeta^k)^2 = \sum_{k=0}^{16}(\zeta^2)^k = \dfrac{1-(\zeta^2)^{17}}{1-\zeta^2} = 0$, so looking at its real part, $\sum_{k=0}^{16}\left(2\cos^2\dfrac{2k\pi}{17} - 1\right) = 0$, whence $\sum_{k=0}^{16}\cos^2\dfrac{2k\pi}{17} = \dfrac{17}{2}$.

Another approach: if you combine a geometric argument:

$\cos^2\frac{2k\pi}{17} = \frac{1+\cos\frac{4k\pi}{17}}{2}$, so the sum is $\frac{17}{2} + \frac{1}{2}\sum_{k=0}^{16}\cos\frac{4k\pi}{17}$. The last sum is 17 times the x-coordinate of the centroid of the regular 17-gon inscribed in the unit circle centered at the origin, thus equals to zero and the result follows.

$\square$

$N = 1 + 10 + 10^2 + \cdots + 10^{1997}$. Determine the 1000^{th} digit after the decimal point of $\sqrt{N}$ in base 10.

Solution

$N = 1 + 10 + 10^2 + \cdots + 10^{2n-1} = \frac{10^{2n}-1}{9}$. $\sqrt{N} = \frac{\sqrt{10^{2n}-1}}{3}$

Let $\sqrt{10^{2n}-1} = 10^n - x$, then we can calculate that $\frac{5}{10^{n+1}} < x < \frac{6}{10^{n+1}}$.

So we see that $\sqrt{N} = 333\cdots 33, 333...331...$, where there are n 3's for the decimal point, n 3's after the decimal point and then a 1 (= the $n+1$-digit).

So here, were $n = 999$, the 1000^{th} digit after the point is a 1.

$\square$ Prove that $(mn!)^2$ is divisible by $(m!)^{n+1}(n!)^{m+1}$ for all positive integers m, n

Solution

The product of K consecutive natural numbers is divisible by $K!$, which follows from $\binom{M+K-1}{K} = \frac{M(M+1)(M+2)\cdots(M+K-1)}{K!}$ being an integer.

Thus $km(km-1)(km-2)\cdots(km-m+1)$ is divisible by $m!$, and moreover, we can write the quotient as $\frac{km(km-1)(km-2)\cdots(km-m+1)}{m!} = k\frac{(km-1)(km-2)\cdots(km-m+1)}{(m-1)!}$. Since the product of $m-1$ consecutive numbers is divisible by $(m-1)!$, we conclude that the quotient is the product of k and an integer number.

For shortness, denote $(K)_M = K(K-1)(K-2)\cdots(K-M+1)$. Then
$\frac{(mn)!}{(m!)^n} = \frac{(m)_m}{m!}\frac{(2m)_m}{m!}\frac{(3m)_m}{m!}\cdots\frac{(mn)_m}{m!}$.

By the previous argument, we can write this as $(1\cdot Q_1)(2Q_2)(3Q_3)\cdots(nQ_n)$, where Q_i are some integers. Therefore the expression is divisible by $n!$.

Thus $(m!)^n n! \mid (mn)!$. Similarly, $(n!)^m m! \mid (mn)!$ and the claim follows.

Another way:

Suppose we have mn people we wish to divide into m teams of n. We do this by lining them up in a row, and let the first n people form a team, the second n people form a team, etc.

We can line them up in $(mn)!$ ways, we divide by $(n!)^m$ to account for re-arrangements of people within their teams and we divide by $m!$ to account for re-arrangements of the teams within the row.

Therefore there are $\dfrac{(mn)!}{(n!)^m(m!)}$ ways of dividing the people into teams, and because of our interpretation this must be an integer.

Similarly, $\dfrac{(mn)!}{(m!)^n(n!)}$ is an integer, and the result follows by multiplying the two together.

☐ If $f(x)$ is a real valued polynomial and $f(x)=0$ has real and distinct roots, show that the equation $(f'(x))^2 - f(x)f''(x) = 0$ cannot have real roots.

Solution

Let $f(x) = (x-a)(x-b)...$ Then we have $\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + ...$ and differentiate both side, $\frac{-f''(x)f(x)+f'(x)^2}{f(x)^2} = \frac{1}{(x-a)^2} + \frac{1}{(x-b)^2} + ...$ so $f''(x)f(x) - f'(x)^2 = (x-b)^2(x-c)^2... + (x-a)^2(x-c)^2... + ...$ which is greater than zero for all real x.

☐ Given that n is a natural number (positive integer) prove that $1+n^{19}+n^{47}$ is prime if and only if $1+n^{17}+n^{76}$ is prime.

Solution

Let $\omega \neq 1$, such that $\omega^3 = 1$. Note that $P(\omega) = Q(\omega) = 0$, and $P(\omega^2) = Q(\omega^2) = 0$, hence $P(x)$ and $Q(x)$ are both divisible by $(x-\omega)(x-\omega^2) = x^2+x+1$. This implies that n^2+n+1 divides $n^{47}+n^{19}+1$ and $n^{76}+n^{17}+1$ for all $n \in \mathbb{N}$. If $n=1$, then $P(1)$ and $Q(1)$ are both primes and the statement is true. For the other hand, if $n>1$, then $n^2+n+1>1$ and $P(n), Q(n)$ are both greater than n^2+n+1. Hence $P(n)$ and $Q(n)$ are both composite numbers when $n>1$, and the statement is valid.

☐ Find the value of x such that
$\frac{(x+\alpha)^n-(x+\beta)^n}{(\alpha-\beta)} = \frac{sinn\theta}{sin^n\theta}$
where α and β are the roots of $t^2-2t+2=0$ and n is a natural number.

Solution

Just put $n=2$ to get $x = \cot\theta - 1$.

As $\alpha, \beta = 1 \pm i$, it's easy to check $\frac{(\cot\theta+i)^n-(\cot\theta-i)^n}{2i} = \frac{\sin n\theta}{\sin^n\theta} \iff \frac{e^{in\theta}-e^{-in\theta}}{2i} = \sin n\theta$

☐ $a,b,c,d \in N$ $a+b=c^2d$ $a+b+c=42$

Find all the possible values of c

Solution

We substitute $a+b$ with c^2d into the second equation. So we have $c^2d+c=42$. Factoring this yields $c(cd+1)=42$.

We'll now list out the ordered pairs of numbers that multiply to 42. These are $(1,42)$, $(2,21)$, $(3,14)$, $(6,7)$, (7 and $(42,1)$.

When $c=1$ and $cd+1=42$, we have $d=41$. Evidently this works. When $c=2$ and $cd+1=21$, we have $cd=20$ and $d=10$. This works too. When $c=3$ and $cd+1=14$, we have $cd=13$. However, if $c=3$, then d won't be natural. When $c=6$ and $cd+1=7$, we have $cd=6$ and $d=1$. This works.

None of the others will work since $cd+1>c$ for c,d natural. So there are three values of c that work: $\boxed{1, 2, \text{ and } 6}$.

□ Let $a_1, a_2, \dots, a_n$ be positive integers such that

$$\frac{a_1+1}{a_2}+\frac{a_2+1}{a_3}+\cdots+\frac{a_n+1}{a_1}$$

is also an integer. Show that

$$\gcd(a_1,a_2,\dots,a_n)\le\sqrt[n]{a_1a_2\cdots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)}.$$

Solution

Let $d=\gcd(a_1,a_2,\dots,a_n)$, and write $a_i=d\cdot x_i$ where $x_i\in\mathbb{Z}^+$. If we put $a_{n+1}=a_1$, we obtain that: $\sum_{i=1}^n\frac{a_i+1}{a_{i+1}}=\sum_{i=1}^n\frac{dx_i+1}{dx_{i+1}}=\frac{dS_1+S_2}{dx_1x_2\cdots x_n}$

Where $S_1=x_1^2x_3...x_n+x_1x_2^2x_4...x_n+....+x_1x_2...x_{n-1}^2+x_2x_3...x_n^2$ and $S_2=x_1x_2...x_n\sum_{i=1}^n\frac{1}{x_i}$. Clearly $d|dS_1+S_2$ (because by assumption $\frac{dS_1+S_2}{dx_1x_2\cdots x_n}$ is an integer), an this implies that $d|S_2$, further $d\le x_1x_2...x_n\sum_{i=1}^n\frac{1}{x_i}$. Hence $d^n\le a_1a_2...a_n\sum_{i=1}^n\frac{1}{a_i}$, as desired.

□ Solve in natural the system

$$\begin{cases} x^2+7^x=y^3\\ x^2+3=2^y\end{cases}$$

Solution

Subtracting, we get $7^x-3=y^3-2^y$. Since $x\in\mathbb{N}$, we have $7^x\ge7$

$\implies y^3-2^y=7^x-3\ge4>0\implies y^3>2^y$

But for $y>10$ we have $y^3<2^y$. This means $y\le10$.

[list] [*]$y=1$ no solution.

[*]$\boxed{y=2}\implies\boxed{x=1}$

[*]$y=3$ no solution.

[*]$y=4$ no solution.

[*]$y=5$ no solution.

[*]$y=6$ no solution.

[*]$y=7$ no solution.

[*]$y=8$ no solution.

[*]$y=9$ no solution.

[*]$y=10$ no solution.[/list]

Hence the unique solution $\boxed{(x,y)=(1,2)}$.

Another way : We find $(1,2)$ as only solution:

For $Y\ge3$:

$x^2 \equiv 5 \pmod 8$ and this hasn't solutions, $y = 1$ can't ($x^2 = -1$ isn't solvable in N) $y = 2$ had only $x = 1$ as solution ($7^x = 2^3 + 3 - 4 = 7$)

☐ Let a, b, c be compleks numbers for which $a + b + c = 0$ Prove that $max(|a|, |b|, |c|) \leq \frac{\sqrt{3}}{2}\sqrt{|a|^2 + |b|^2 + |c|^2}$

Solution

Let be $max(|a|, |b|, |c|) = |a|$ So we get to show that
$|a| \leq \frac{\sqrt{3}}{2}\sqrt{|a|^2 + |b|^2 + |c|^2}$ when we have
$|a|^2 \leq 3(|b|^2 + |c|^2)$
from $a + b + c = 0$ we have
$|a|^2 = |-b-c|^2 = |b+c|^2 = ||b| + |c||^2$ so we have
$|b|^2 + 2|b||c| + |c|^2 \leq 3|b|^2 + 3|c|^2$
$2(|b|^2 + |c|^2) + (|b| - |c|)^2 \geq 0$
so we are done

☐ You have an even number of N players. You want to form N/2 matches. How many different matches are possible ?

Solution

There are $(N-1)(N-3)(N-5)\dots(3)(1)$ possible matchings. If we consider one person at a time, then the first person has $(N-1)$ possible different people to choose from. There are two less people, so the next person then has $(N-2-1)$ people to choose from. We continue until we get to 1.

☐ Solve $x_1^2 + x_2^2 + .. + x_{2010}^2 = 2010x_1x_2..x_{2010}$ in N

Solution

If you have a solution $(x_1, x_2, \dots x_{2010})$, so that $x_2^2 + x_3^2 + \dots + x_{2010}^2 = b$ and $2010x_2x_3 \cdots x_{2010} = a$, the equation reduces to $x_1^2 - ax_1 + b = 0$. That means the two roots of the quadratic $x^2 - ax + b = 0$ are x_1 and $a - x_1$. Using this idea you can generate infinite families of solutions looking like $(1, 1, \dots 1)$ $(2009, 1, \dots 1)$ $(2010 \cdot 2009 - 1, 2009, 1, \dots 1)$ $(2010^2 \cdot 2009 - 2011, 2010 \cdot 2009 - 1, 2009, 1, \dots 1)$ $\vdots$

Because there's multiple variables that can be root flipped in this way (I was just doing it to 1s above), it seems unlikely that there will be any concise way to describe all solutions.

☐ Find all pairwise distinct primes a, b, c such that $a + 5b + 10c = abc$.

Solution

If either a or b is even, then the other has to be even also and both need to be 2, contradiction. So both a and b are odd primes. But then the LHS is even, so the RHS needs to be even. Thus $c = 2$.

The equation becomes $a + 5b + 20 = 2ab$. Transform this into $4ab - 10b - 2a + 5 = (2a-5)(2b-1) = 45$. The factors of 45 are $(1, 45), (3, 15), (5, 9), (9, 5), (15, 3), (45, 1)$, leading to

$(a, b) = (3, 23), (4, 8), (5, 5), (7, 3), (10, 2), (25, 1)$, of which the only 2 pairs that work are $(a, b, c) = (3, 23, 2)$ and $(a, b, c) = (7, 3, 2)$. ☐ Find the coefficient of $\frac{1}{n+i}$ when $\frac{1}{(n-k)(n-k+1)\dots(n-1)(n)(n+1)\dots(n+k-1)(n+k)}$ is expressed as a linear combination of $\frac{1}{n+i}, i \in \{-k, -k+1, ..., -1, 0, 1, ..., k-1, k\}$.

For example, $\frac{1}{(n-1)n(n+1)} = (1/2)\frac{1}{n-1} + (-1)\frac{1}{n} + (1/2)\frac{1}{n+1}$.

Solution

Call the fraction α, and let the coefficient of $\frac{1}{n+i}$ be c_i for all $i \in \{-k, -k+1, \cdots, k-1, k\}$. Therefore $\alpha = \frac{1}{(n-k)(n-k+1)\cdots(n+k-1)(n+k)} = \sum_{a=-k}^{k} \frac{c_a}{n+a}$. Multiplying both sides by the denominator of α gives $1 = \sum_{a=-k}^{k} c_a p_a(n)$, where $p_a(n)$ is the polynomial $\prod_{-k \leq b \leq k, b \neq a}(n+b)$. Note that if $i \in \{-k, -k+1, \cdots, k-1, k\}$, then $p_a(i) = 0$ if $a \neq i$, so $1 = c_i p_i(i)$ for all i in the relevant range. Therefore $c_i = \frac{1}{p_i(i)} = \prod_{-k \leq b \leq k, b \neq i}(i+b)$.

□ Anumber of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a single and that between a boy and a girl was called a mixed single. The total number of boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most how many schools were represented by an odd number of players?

Solution

Let there be n schools. Suppose the i^{th} school sends B_i boys and G_i girls. Let $B = \sum B_i$ and $G = \sum G_i$. We are given that $|B - G| = 1$.

The number of same sex matches is $1/2 \sum B_i(B - B_i) + 1/2 \sum G_i(G - G_i) = (B^2 - \sum B_i^2 + G^2 - \sum G_i^2)$. The number of opposite sex matches is $\sum B_i(G - G_i) = BG - \sum B_iG_i$. Thus we are given that $B^2 - \sum B_i^2 + G^2 - \sum G_i^2 - 2BG + 2\sum B_iG_i = 0$ or ± 2. Hence $(B - G)^2 - \sum(B_i - G_i)^2 = 0$ or ± 2. But $(B - G)^2 = 1$, so $\sum(B_i - G_i)^2 = -1, 1$ or 3. It cannot be negative, so it must be 1 or 3. Hence $B_i = G_i$ except for 1 or 3 values of i, where $|B_i - G_i| = 1$. Thus the largest number of schools that can have $B_i + G_i$ odd is 3.

This solution uses a slightly differently worded problem, one that says the number of boys and girls differed by 1 (not at most 1). But it doesn't make a difference (for difference 0, the largest value is 2).

□ Let $x > 0$ be a real number. Prove that

$$\frac{x(x+1)(x+2)\cdots(x+m-1)}{m!} \geq x^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}} \quad \forall m \in \mathbb{N}$$

Solution

Induction works. For $m = 1$ the inequality is an equality. Suppose that the inequality is true for $m = n$. Then, for $m = n + 1$

$$\frac{1}{(n+1)!}\prod_{k=1}^{n+1}(x+k-1) = \frac{x+n}{n+1} \cdot \frac{1}{n!}\prod_{k=1}^{n}(x+k-1) \geq \frac{x+n}{n+1} \cdot x^{H_n}$$

Where $H_n = \sum_{k=1}^{n} \frac{1}{k}$. But

$$\frac{x+n}{n+1}x^{H_n} = \frac{x^{1+H_n} + x^{H_n} + \ldots + x^{H_n}}{n+1} \geq \sqrt[n+1]{x^{1+(n+1)H_n}} = x^{H_{n+1}}$$

And we are done

Another way:

Weighted AM-GM has $\frac{x+(i-1)}{i} \geq x^{\frac{1}{i}}$ we take product $LHS = \prod_{i=1}^{m} \frac{x+(i-1)}{i} \geq \prod_{i=1}^{m} x^{\frac{1}{i}} = RHS$ which we wanted to proof

□ Determine all real-valued functions f that satisfy

$2f(xy + xz) + 2f(xy - xz) \geq 4f(x)f(y^2 - z^2) + 1$ for all real numbers x,y,z

We fill in: $(x, y, z) = (0, 0, 0)$ and $(1, 1, 0) : 0 \geq (2f(0) - 1)^2$ and $0 \geq (2f(1) - 1)^2$, so we find $f(0) = f(1) = 0.5$

Now we do $y = 1, z = 0 : 4f(x) \geq 4f(x)f(1) + 1 = 2f(x) + 1 \to f(x) \geq 0.5forallx \in R$. [1]

When we fill $y = z$ in;

$2f(2xy) + 2f(0) \geq 4f(0)f(x) + 1$ or $f(2xy) \geq f(x)$, now we choose $y = \frac{1}{2x}$, so we get $0.5 \geq f(x)$ for all $x \in R\backslash 0$. [2]

With [1] and [2] we know that $f(x) = 0.5$ for all $x \in R$

$\square$

Prove that $\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m}$, $m \leq n$

Solution

We need to prove:

$\sum_{k=m}^{n} \frac{\binom{n}{k}\binom{k}{m}}{\binom{n}{m}} = 2^{n-m}$

Expanding the binomial coefficients we must show:

$\sum_{k=m}^{n} \frac{(n-m)!}{(n-k)!(k-m)!} = 2^{n-m}$

But this sum expands as:

$1 + (n-m) + \frac{(n-m)(n-m-1)}{2!} + \cdots + \frac{(n-m)\cdots(n-m-(n-m-1))}{(n-m)!}$

But this is the expansion for $(1+1)^{n-m}$

So $LHS = 2^{n-m}$ as required.

Another way:

Combinatorial argument: There are n objects, and we choose k objects out of the n objects, and then choose another m objects out of the k objects. This is accounted by the LHS. To accomplish the same task, we can choose the m objects out of the n objects straightaway (which is $\binom{n}{m}$). But there are 2^{n-m} subsets of the objects left when m objects are taken from the n objects.

$\square$ $a, b, c, d \in N$ $a \leq b \leq c \leq d$

Find (a,b,c,d) such that $ab + cd = a + b + c + d + 3$

Solution

$ab + cd - a - b - c - d = 3 \iff (a-1)(b-1) + (c-1)(d-1) = 5$

Due to the given condition, we have $((a-1)(b-1), (c-1)(d-1)) \in \{(0,5), (1,4), (2,3)\}$, and now the casework is easy.

$\square$

Let P be an interior point of $\triangle$ ABC . Denote R_a , R_b , R_c the circumradii of the triangles PBC , PCA and PAB respectively. Prove that : $R_a + R_b + R_c \geq PA + PB + PC$.

Solution

Let M, N, L be the midpoints of PA, PB, PC. Perpendicular lines to PA, PB, PC through M, N, L pairwise meet at the circumcenters X, Y, Z of $\triangle PBC$, $\triangle PCA$ and $\triangle PAB$. By Erdős-Mordell inequality for $\triangle XYZ \cup P$ we get

$PX + PY + PZ = R_a + R_b + R_c \geq 2(PM + PN + PL) = PA + PB + PC$

$\square$ Let $m,\ n \in \mathbb{N}^*,\ m \leq 2n$ and $a,\ b,\ c > 0$. Prove that the following inequality holds:

$$\frac{a^m}{b^n + c^n} + \frac{b^m}{c^n + a^n} + \frac{c^m}{a^n + b^n} \geq \frac{3}{2}\sqrt{\frac{a^m + b^m + c^m}{a^{2n-m} + b^{2n-m} + c^{2n-m}}}.$$

Solution

► $LHS \overset{CEB}{\geq} \frac{1}{3} \cdot (a^m + b^m + c^m)\left(\frac{1}{a^n+b^n} + \frac{1}{b^n+c^n} + \frac{1}{c^n+a^n}\right) \overset{C.S.}{\geq} \frac{3}{2} \cdot \frac{a^m+b^m+c^m}{a^n+b^n+c^n}$.

► Thus, it remains to prove that : $\frac{a^m+b^m+c^m}{a^n+b^n+c^n} \geq \sqrt{\frac{a^m+b^m+c^m}{a^{2n-m}+b^{2n-m}+c^{2n-m}}}$ which rewrites as :

$(a^m + b^m + c^m)(a^{2n-m} + b^{2n-m} + c^{2n-m}) \overset{C.S.}{\geq} (a^n + b^n + c^n)^2 = \left(\sum a^{\frac{2n-m}{2}} \cdot a^{\frac{m}{2}}\right)^2$

$\square$

Find all possible digits x, y, z such that the number $\overline{13xy45z}$ is divisible by 792.

Solution

$\square$

There are forty weights: $1, 2, \cdots, 40$ grams. Ten weights with even masses were put on the left pan of a balance. Ten weights with odd masses were put on the right pan of the balance. The left and the right pans are balanced. Prove that one pan contains two weights whose masses di ffer by exactly 20 grams.

Solution

Assume for contradiction that neither pan contains two weights whose masses differ by exactly 20 grams. Split up the even weights into the sets $\{2, 22\}, \{4, 24\}, \cdots, \{20, 40\}$. There are ten sets, and at most one weight from each set may be picked, so we must pick exactly one weight from each set. Similarly, we must also pick exactly one weight from each of $\{1, 21\}, \{3, 23\}, \cdots, \{19, 39\}$.

Now, consider the sum of each side mod 20. The left side has sum $2(2+4+6+8+10) \equiv 0 \pmod{20}$. The right side has sum $2(1+3+5+7+9) = 10 \pmod{20}$. As 0 and 10 are not equal, we have reached a contradiction and we are done.

$\square$ $\triangle DEF$ is the tangential triangle of $\triangle ABC$. On the sides of $\triangle DEF$, take two equal segments AG, BH (A-G-E,B-H-F) Circle($\triangle ACG$) meet Circle($\triangle ABH$) at Q. Circle mean circumcircle. Show that A,Q,D are collinear.

Solution

Let M be the second intersection of $\odot(ACG)$ with line DE. Since $\triangle EAC$ is isosceles with apex E, it follows that $AGMC$ is an isosceles trapezoid with $GM \parallel AC \implies AG = CM$. Since $DC = DB$, then we deduce that $DM = DH$. Therefore, $\overline{DC} \cdot \overline{DM} = \overline{DB} \cdot \overline{DH} \implies D$ has equal power with respect to circles $\odot(ACG)$ and $\odot(ABH) \implies D$ lies on the radical axis AQ of $\odot(ACG)$ and $\odot(ABH)$.

$\square$

The function f has the property that, for each real number x,

$$f(x) + f(x-1) = x^2.$$

If $f(19) = 94$, what is the remainder when $f(94)$ is divided by 1000?

Solution

There must be a pattern with $f(94)$. $f(94) + f(93) = 94^2$, $f(93) + f(92) = 93^2$ $f(92) + f(91) = 92^2$, and so on. Hence, $f(94) = 94^2 - (93^2 - (92^2 - ...)))..)$ so this is $94^2 - 93^2 + 92^2 - 91^2 + ... + 20^2 - 94$.

We see that this is a sum of arithmetic series. Simplifying gives us $187+183+179+...+43+20^2-94$. We take out $187+183+...+43$. This is $43 \cdot 37 + 144 + 140 + ... + 0$ (we have 37 terms in the sequence and we take 43 away from all 37), which is $43 \cdot 37 + 4 \cdot (36+35+...+1) = 43 \cdot 37 + 2 \cdot 36 \cdot 37 = 115 \cdot 37 = 4255$. Adding 400 and subtracting 94 gives us 4561. Hence the remainder is $\boxed{561}$.

$\square$

Calculate:

$$\sum_{n=0}^{\infty} \left\lfloor \frac{10000 + 2^n}{2^{n+1}} \right\rfloor$$

Solution

First of all, notice that the smallest k such that $2^k > 10000$ is $k = 14$, since $2^{14} = 16384$. Also, since $\lfloor x + \frac{1}{2} \rfloor =< x >$, where $< x >$ is the closest integer to x, we can rewrite the sum as the following:

$\sum_{n=0}^{13} \left\lfloor \frac{10000+2^n}{2^{n+1}} \right\rfloor = \sum_{n=0}^{13} \left\lfloor \frac{10000}{2^{n+1}} + \frac{1}{2} \right\rfloor$

$= \sum_{n=0}^{13} < \frac{10000}{2^{n+1}} >$

$=< 5000 > + < 2500 > + < 1250 > + < 625 > + < 312.5 > + < 156.25 > + < 78.125 > + < 39.0625 > + < 19.53.. > + < 9.76.. > + < 4.88.. > + < 2.44 > + < 1.22.. > + < 0.61.. >$

$= 5000 + 2500 + 1250 + 625 + 313 + 156 + 78 + 39 + 20 + 10 + 5 + 2 + 1 + 1$

$= \boxed{10000}$.

$\square$

Let $x, y, x \in (0, \pi)$ and $x + y + z = \pi$. Prove that (without Jensen's inequality) $\sin x + \sin y + \sin z \le \frac{3\sqrt{3}}{2}$. Proof 1 (geometric). Let ABC be a triangle. Apply A.M. $\ge$ G.M. : $\frac{1}{3} \cdot \sum(s - a) \ge \sqrt[3]{(s-a)(a-b)(a-c)} \iff s^3 \ge 27\prod(s-a) \iff s^3 \ge 27sr^2 \iff \boxed{s \ge 3r\sqrt{3}}$ (1) . From well-known inequality $3 \cdot \sum r_b r_c \le \left(\sum r_a\right)^2$ obtain $3s^2 \le (4R + r)^2 \iff \boxed{s\sqrt{3} \le 4R + r}$ (2) . I used the well-known relations $\boxed{r_a r_b + r_b r_c + r_c r_a = s^2}$ and $\boxed{r_a + r_b + r_c = 4R + r}$. Using (1) , (2) obtain $\left\| \begin{array}{ccc} s\sqrt{3} & \le & 4R + r \\ r & \le & \frac{s}{3\sqrt{3}} \end{array} \right\| \bigoplus \implies s\sqrt{3} - \frac{s}{3\sqrt{3}} \le 4R \iff \boxed{s \le \frac{3R\sqrt{3}}{2}}$. In conclusion, $\boxed{3r\sqrt{3} \le s \le \frac{3R\sqrt{3}}{2}}$ (3) $\implies \sum \sin A = \frac{a+b+c}{2R} = \frac{s}{R} \le \frac{3\sqrt{3}}{2} \iff \boxed{\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}}$.

Proof 2 (trigonometric). Observe that $\sin y + \sin z = 2\sin\frac{y+z}{2}\cos\frac{y-z}{2} \le 2\cos\frac{x}{2}$ because $\frac{y+z}{2} = 90° - \frac{x}{2}$ and $\cos\frac{y-z}{2} \le 1$. Therefore, $\sum \sin x \le \sin x + 2\cos\frac{x}{2} = 2\cos\frac{x}{2}\left(1 + \sin\frac{x}{2}\right)$, i.e. $\boxed{\sum \sin x \le 2\cos\frac{x}{2}\left(1 + \sin\frac{x}{2}\right.}$ NowWe'llprove that $u > 0$, $v > 0$, $u^2 + v^2 = 1 \implies u(1 + v) \le \frac{3\sqrt{3}}{4}$. Indeed, observe that $\{u, v\} \subset (0, 1)$ and $u(1 + v) - \max \iff u^2(1 + v)^2 - \max \iff (1 - v)(1 + v)^3 - \max \iff E(u, v) \equiv (1 - v)\left(\frac{1+v}{3}\right)^3 - \max$. Observe that $(1 - v) + 3 \cdot \frac{1+v}{3} = 2$ (constant). Therefore $E(u, v)$ is maximum iff $1 - v = \frac{1+v}{3} = \frac{2}{4}$, i.e. $v = \frac{1}{2}$. Thus $u = \frac{\sqrt{3}}{2}$ and $\boxed{u(1 + v) \le \frac{3\sqrt{3}}{4}}$. For $u := \cos\frac{x}{2}$ and $v = \sin\frac{x}{2}$ and the relation (4) obtain $\sum \sin x \le 2u(1 + v) \le \frac{3\sqrt{3}}{2}$.

$\square$

Eliminate θ from the following.

$x^2 + y^2 = \frac{x\cos 3\theta + y\sin 3\theta}{\cos^3\theta} = \frac{y\cos 3\theta - x\sin 3\theta}{\sin^3\theta}$

Solution

$(x^2 + y^2)^2\cos^6\theta = (x\cos 3\theta + y\sin 3\theta)^2 = x^2\cos^2 3\theta + 2xy\sin 3\theta\cos 3\theta + y^2\sin^2 3\theta$ $(x^2 + y^2)^2\sin^6\theta = (y\cos 3\theta - x\sin 3\theta)^2 = y^2\cos^2 3\theta - 2xy\sin 3\theta\cos 3\theta + x^2\sin^2 3\theta$

so by summing $(x^2 + y^2)^2(\cos^6\theta + \sin^6\theta) = x^2 + y^2$. But $\cos^6\theta + \sin^6\theta = (\cos^2\theta + \sin^2\theta)(\cos^4\theta - \cos^2\theta\sin^2\theta + \sin^4\theta) = 1 - 3\cos^2\theta\sin^2\theta = 1 - \dfrac{3}{4}\sin^2 2\theta$, so $\sin^2 2\theta = \dfrac{4(x^2 + y^2 - 1)}{3(x^2 + y^2)}$.

$\square$ For a parallelogram $ABCD$, a line through A meet BC,CD at X,Y. Let K,L be the excenters of $\triangle ABX$, $\triangle AYD$. Show that $\angle KCL$ is constant.

Solution

Let us consider the configuration where X lies on $\overrightarrow{BC}$ and $Y \in \overline{CD}$, the remaining cases are treated analogously. Let I be incenter of $\triangle ABX$. Since B, I, X, K are concyclic and $XI \parallel AL$, it follows that $\angle AKB = \angle IXB = \angle DAL$. But since $\angle ADL = \angle KBA$, then $\triangle ADL \sim \triangle KBA$. Hence $\frac{DL}{AB} = \frac{AD}{BK} \implies \frac{DL}{BC} = \frac{BC}{BK}$.

Since $\angle LDC = \angle CBK = 90° - \frac{1}{2}\angle ADC$, we deduce that $\triangle DLC \sim \triangle BCK$. Then $\angle BCK = \angle DLC$ implies

$\angle KCL = 360° - \angle BCK - \angle LCD - \angle DCB = \angle LDC + \angle ADC = \angle ADL$

$\implies \angle KCL = 90° + \frac{1}{2}\angle ADC = \text{const}.$

$\square$ For which integers, n, is $\frac{n^2-71}{7n+55}$ a positive integer?

Solution

$\frac{n^2-71}{7n+55} \in \mathbb{Z} \implies 7n+55|n^2-71 \quad (1).$

$7n+55 = 7(n+8)-1 \implies 7n+55|7(n-8)(n+8)-7(n-8)$

$\implies 7n+55|7(n^2-64)-(n-8) \quad (2).$

$(1),(2) \implies 7n+55|7(n^2-64)-(n-8)-7(n^2-71)$

$\implies 7n+55|7(71-64)-(n-8) \implies 7n+55|57-n$

$\implies 7n+55|7(57-n) \implies 7n+55|399-7n \quad (4)$

$\implies 7n+55|(399-7n)+(7n+55) \implies 7n+55|454$

Note that divisors of 454 are $1, 2, 227, 454$ and their negatives.

So the solutions :

$7n+55 = 1 \implies$ impossible

$7n+55 = 2 \implies$ impossible

$7n+55 = 227 \implies$ impossible

$7n+55 = 454 \implies n = 57$

$7n+55 = -1 \implies n = -8$

$7n+55 = -2 \implies$ impossible

$7n+55 = -227 \implies$ impossible

$7n+55 = -454 \implies$ impossible

Hence the solutions are $\boxed{n \in \{-8, 57\}}$.

□ Let $\{a_1, a_2, \cdots\}$ be a sequence of non-negative numbers such that $a_{n+m} \leq a_n + a_m$ for all n and m Show that for all $n \geq m$, $\quad a_n \leq m a_1 + \left(\frac{n}{m} - 1\right) a_m$

Solution

It's a direct consequence of the condition. $(m-k)\cdot m\cdot a_1 \geq (m-k)\cdot a_m \Leftrightarrow m^2\cdot a_1 + k\cdot a_m \geq m\cdot a_m + km\cdot a_m$ dividing through m gives $m \cdot a_1 + \frac{k}{m} a_m \geq a_m + k \cdot a_m \geq a_{k+m}$. let $n = m+k$ and we're done.

□ The line l is tangent to the circle S at the point A. B and C are two points on l on opposite sides of A. The other tangents from B, C to S intersect at a point P. B, C move along l in such a way that $|AB| \cdot |AC|$ is constant. Find the locus of P.

Solution

Let us rename the point $A \equiv P$ and vice-versa, in order to use the common ABC-triangle notation. Thus $PB \cdot PC = (s-a)(s-b) = k^2$. Let r be the radius of S. Then

$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \Longrightarrow r^2 = \frac{(s-a)k^2}{s} \Longrightarrow \frac{s}{a} = \frac{k^2}{k^2-r^2}$

Let h_a be the length of the altitude issuing from vertex A. Then we have

$a \cdot h_a = 2r \cdot s \Longrightarrow h_a = \frac{2r\cdot s}{a} = \frac{2r\cdot k^2}{k^2-r^2} = \text{const.}$

Locus of A is a parallel line ℓ' to ℓ in the half-plane of S such that $\text{dist}(\ell, \ell') = \frac{2r\cdot k^2}{k^2-r^2}$

□ Consider the polynomial $P(x)$ from the seventh grade. Knowing that $P(x)+1$ is divisible by $(x-1)^4$ and $P(x)-1$ is divisible by $(x+1)^4$, determine $P(x)$.

Solution

Let $P(x) = (ax^3+bx^2+cx+d)(x-1)^4 - 1$ and $P(x) = (Ax^3+Bx^2+Cx+D)(x+1)^4 + 1$

Then $P(x) = ax^7 + (b-4a)x^6 + (6a-4b+c)x^5 + (d-4c+6b-4a)x^4 + (a-4b+6c-4d)x^3 + (6d-4c+b)x^2 + (c-4d)x + (d-1)$ Also $P(x) = Ax^7 + (4A+B)x^6 + (6A+4B+C)x^5 + (4A+6B+4C+D)x^4 + (A+4B+6C+4D)x^3 + (B+4C+6D)x^2 + (C+4D)x + (D+1)$

Then $A = a$ $B = (b-4a) - 4A = b - 8a$ $C = (6a-4b+c) - (6A+4B) = c - 8b + 32a$ $D = (d-4c+6b-4a) - (4A+6B+4C) = d - 8c + 32b - 88a$

Then $4d-26c+84b-191a=A+4B+6C+4D=a-4b+6c-4d \implies 24a+4c=11b+d$ Also $6d-44c+161b-408a=B+4C+6D=6d-4c+b \implies 51a+5c=20b$ Also $4d-31c+120b-320a=C+4D=c-4d \implies 40a+4c=15b+d$ Also $d-8c+32b-88a+1=D+1=d-1 \implies 44a+4c=16b+1$

Solving: $a=\frac{5}{16}$, $b=\frac{5}{4}$, $c=\frac{29}{16}$ and $d=1$

So $P(x)=\left(\frac{5}{16}x^3+\frac{5}{4}x^2+\frac{29}{16}x+1\right)(x-1)^4-1=\frac{5}{16}x^7-\frac{21}{16}x^5+\frac{35}{16}x^3-\frac{35}{16}x$

□ Find the rest of the division $x^{1959}-1$ by $(x^2+1)\cdot(x^2+x+1)$

Solution

$x^{12}-1=(x^6+1)(x^6-1)=(x^2+1)(x^4-x^2+1)(x^2+x+1)(x-1)(x^3+1)$ So $(x^2+1)(x^2+x+1)|x^{12n}-1$ $x^{1956}-1=x^{12\cdot 163}-1\equiv 0 \mod (x^2+1)(x^2+x+1)$ So $x^{1956}\equiv 1 \mod (x^2+1)(x^2+x+1)$ So $x^{1959}\equiv x^3 \mod (x^2+1)(x^2+x+1)$ So $x^{1959}-1\equiv x^3-1 \mod (x^2+1)(x^2+x+1)$

□ Consider the polynominal $\boxed{p=X^4+X^3-1}$ with the roots $\{a,b,c,d\}$. Ascertain the monic polynominal with the roots $\{ab,ac,ad,bc,bd,cd\}$.

Solution

Denote $\begin{array}{l} ab+cd=m \\ ac+bd=n \\ ad+bc=p \end{array}$. Observe that $\sum a^2=1$ and $\boxed{m+n+p=0}$.

Prove easily that $mn+np+pm=\sum abc(a+b+c)=\sum abc(-1-d)=$
$-\sum abc-4abcd \implies \boxed{mn+np+pm}=4$. Remain to ascertain mnp . Therefore,
$mnp=\sum a^2b^2c^2+abcd\cdot\sum a^2=\sum\frac{1}{a^2}-\sum a^2$. The polynominal which has the roots $\left\{\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}\right\}$ is X^4-X-1 from where obtain $\sum\frac{1}{a^2}=0$. Thus,
$\boxed{mnp=-1}$. Therefore, the required polynominal is $q=\prod\left[(X^2-1)-mX\right]=$
$\left(X^2-1\right)^3-X\left(X^2-1\right)^2\cdot\sum m+X^2\left(X^2-1\right)\cdot\sum mn-X^3\cdot mnp=$
$\left(X^2-1\right)^3+4X^2\left(X^2-1\right)+X^3$. In conclusion, $\boxed{q=X^6+X^4+X^3-X^2-1}$.

□ For the system ¡n $\mathbb{R}$:

$$\begin{cases} (x^2-x+1)(48y^2-24y+67)=48 \\ x+y+z=1 \end{cases}$$

If you consider x_0, y_0, z_0 as the solution of the system, find the value of $E=2x_0+3y_0+z_0$

Solution

$x^2-x+1=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\geq\frac{3}{4}$

and $48y^2-24y+67\geq 48.\frac{4}{3}$

$\Rightarrow (x^2-x+1)(48y^2-24y+67)\geq 48$ for all $x,y\in R$ Hence $x=\frac{1}{2}$ and $y=\frac{1}{4}$.

□ Find the largest n with no zeroes in its representation such that $2^{s(n)}=s(n^2)$. Here, $s(n)$ is the sum of the digits of n.

Solution

Let n have k digits. Then $s(n)\geq k$; on the other hand, n^2 has at most $2k$ digits, so $s(n^2)\leq 18k$.

But then $18k\geq s(n^2)=2^{s(n)}\geq 2^k$, whence $k\leq 6$. Thus $s(n^2)\leq 108$, so we need $s(n)\leq 6$. Now, it is known (and easy to prove) that $s(N)\equiv N \pmod 9$. For $s(n)=6$ it follows $n\equiv 6 \pmod 9$, so $s(n^2)\equiv n^2\equiv 36\equiv 0 \pmod 9$, but then $9\mid s(n^2)=2^6$, absurd. For $s(n)=5$ it follows $n\equiv 5 \pmod 9$, so $s(n^2)\equiv n^2\equiv 25\equiv 7 \pmod 9$, but $2^5=32\equiv 5 \pmod 9$. For $s(n)=4$ the largest is $n=1111$, with $n^2=1234321$, $s(n)=4$, $s(n^2)=16=2^4=2^{s(n)}$.

□ If three distinct integers are chosen at random show that there will exist two among them, say a and b such that 30 divides (a^3b-b^3a).

Solution

No matter whatever a,b you choose $ab(a^2-b^2)$ is divisible by 6(Indeed by 2 and if both of a,b

are not divisible by 3, then $3|a^2-b^2$). Now, among the three distinct integers, if one of them is divisible by 5, we have $5|ab(a^2-b^2)$. If none of them are divisibly by 5, then since there are two non quadratic non zero residues modulo 5, by pigeonhole principle, we have $5|ab(a^2-b^2)$ and hence, $[6,5]=30|ab(a^2-b^2)=a^3b-ab^3$ as required.

☐ Let $ABCDEFG$ be a regular heptagon and let lengths $AB=a, AC=b, AD=c$. Then find $\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}$.

Solution

$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=4\cos^2\frac{\pi}{7}+4\cos^2\frac{2\pi}{7}+\frac{cos^2\frac{\pi}{7}}{cos^2\frac{4\pi}{7}}=4\cos^2\frac{\pi}{7}+4\cos^2\frac{2\pi}{7}+4\cos^2\frac{3\pi}{7}$

We calculate this sum by noting that $\cos\frac{\pi}{7},\cos\frac{3\pi}{7}$,and $\cos\frac{5\pi}{7}$

satisfy the polynomial $8x^4+4x^3-8x^2-3x+1$. From here we find that the sum is just 5

Another way: By Ptolemy's theorem for quadrilaterals $ABDC, ABDE, ABDF, ABCE$ we have

$a^2+a\cdot c=b^2 \implies \frac{b^2}{a^2}=1+\frac{c}{a}$ (1)

$b\cdot c+a^2=c^2 \implies \frac{a^2}{c^2}=1-\frac{b}{c}$ (2)

$a\cdot b+b^2=c^2 \implies \frac{c^2}{b^2}=\frac{a}{b}+1$ (3)

$a\cdot b+a\cdot c=b\cdot c \implies a=\frac{b\cdot c}{b+c}$ (4)

Adding the expressions $(1),(2),(3)$ together and then combining with (4) yields

$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=3+\frac{c}{a}+\frac{a}{b}-\frac{b}{c}=4+\frac{c}{b}+\frac{c}{b+c}-\frac{b}{c}$

But $(3)\cap(4)$ yields $:\frac{c}{b}-\frac{b}{c}=1-\frac{c}{b+c}\implies \frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=5.$

☐ For $a,b,c>0$ Prove that

$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}}>\sqrt{a+b+c}$

Solution

Let $f(x)=\frac{1}{\sqrt{x}}$ then $f''(x)>0$.

Hence

$$a\cdot\frac{1}{\sqrt{a+b}}+b\cdot\frac{1}{\sqrt{b+c}}+c\cdot\frac{1}{\sqrt{c+a}}\ge(a+b+c)\frac{\sqrt{a+b+c}}{\sqrt{a^2+b^2+c^2+ab+bc+ca}}$$
$$>\sqrt{a+b+c}$$

Done !

Another appr$\hat{o}$ach: It might be helpful to mention that Jensen's inequality

$\lambda_1\phi(x_1)+\lambda_2\phi(x_2)+...+\lambda_n\phi(x_n)\ge\phi(\lambda_1x_1+\lambda_2x_2+...+\lambda_nx_n)$

is being used here with

$\lambda_1=\frac{a}{a+b+c},\lambda_2=\frac{b}{a+b+c},\lambda_3=\frac{c}{a+b+c};x_1=a+b,x_2=b+c,x_3=c+a$ and $\phi(x)=\frac{1}{\sqrt{x}}$.

☐ Prove that in any triangle the following equality holds: $ab+bc+ca=p^2+r^2+4Rr$ where a,b,c are the sides of the triangle ,p is half of the perimeter, r is the inradius and R is the circumradius.

Solution

From the area formulae we obtain

$$\frac{abc}{4R}=\Delta=rs=\sqrt{s(s-a)(s-b)(s-c)};\ \therefore\ (s-a)(s-b)(s-c)=r^2s.$$

Expanding our last inequality and applying $abc=4Rrs$ we get,

$$s^3-s^2(a+b+c)+s(ab+bc+ca)-4Rrs=r^2s;$$

Which, on applying $a+b+c=2s$, leads to

$$ab+bc+ca=\frac{1}{s}\left(r^2s+4Rrs+s^3\right)=s^2+r^2+4Rr.$$

$\square$ Find all polynomials satisfying $f(x^2) = \{f(x)\}^2$ for all real numbers x

Solution

Notice that $f(x) = \{f(\sqrt{x})\}^2 \in [0, 1)$ for all nonnegative real x. However, if $f(x)$ is a non-constant polynomial, then $\lim_{x\to+\infty} |f(x)| = \infty$, but here $f(x)$ is bounded as $x \to \infty$, so $f(x)$ is constant.

Hence, $f(x) = c$ for all real x and $f(x^2) = \{f(x)\}^2 \implies c = \{c\}^2$ which has $c = 0$ as its only real solution. Therefore, $\boxed{f(x) = 0, \forall x \in \mathbb{R}}$.

$\square$ Solve equation

$$(2+\sqrt{2})^{\sin^2 x} - (2+\sqrt{2})^{\cos^2 x} + (2-\sqrt{2})^{\cos 2x} = (1+\frac{\sqrt{2}}{2})^{\cos 2x}$$

Solution

We can rewrite the equation as

$(2+\sqrt{2})^{\sin^2 x} + (2-\sqrt{2})^{\cos 2x} = (2+\sqrt{2})^{\cos^2 x} + (2-\sqrt{2})^{-\cos 2x}$

So if $f(x) = (2+\sqrt{2})^{\sin^2 x} + (2-\sqrt{2})^{\cos 2x}$ we are looking for the solutions to $f(x) = f(\frac{\pi}{2} - x)$.

Now $f(x)$ is strictly increasing on intervals $(n\pi, (n+\frac{1}{2})\pi)$ and strictly decreasing on the intervals $((n+\frac{1}{2})\pi, (n+1)\pi)$, $n \in \mathbb{Z}$ Explain

1) $\sin^2(x)$ increases on $(n\pi, (n+\frac{1}{2})\pi)$ and decreases on $((n+\frac{1}{2})\pi, (n+1)\pi)$. Since $2+\sqrt{2} > 1$ we have $(2+\sqrt{2})^{\sin^2 x}$ increasing and decreasing on the same domains.

2) $\cos(2x)$ decreases on $(n\pi, (n+\frac{1}{2})\pi)$ and increases on $((n+\frac{1}{2})\pi, (n+1)\pi)$. Since $2-\sqrt{2} < 1$ we have $(2-\sqrt{2})^{\cos 2x}$ increasing and decreasing on the respective domains.

3) f is the sum of those two increasing/decreasing functions and is therefore also increasing/decreasing on the respective domains

Now we have that $f(x)$ and $f(\frac{\pi}{2} - x)$ are increasing/decreasing in different domains so they can meet only once in the domain $(n\pi, (n+\frac{1}{2})\pi)$ for all $n \in \frac{1}{2}\mathbb{Z}$

Since we can see the obvious solutions $x = \frac{(2k+1)\pi}{4}$, $k \in \mathbb{Z}$ and one of these fall in each of the domains specified above these are the only solutions.

$\square$ For a parallelogram $ABCD$, a line through A meet BC,CD at X,Y. Let K,L be the excenters of $\triangle ABX$, $\triangle AYD$. Show that $\angle KCL$ is constant.

Solution

Let us consider the configuration where X lies on $\overrightarrow{BC}$ and $Y \in \overline{CD}$, the remaining cases are treated analogously. Let I be incenter of $\triangle ABX$. Since B, I, X, K are concyclic and $XI \parallel AL$, it follows that $\angle AKB = \angle IXB = \angle DAL$. But since $\angle ADL = \angle KBA$, then $\triangle ADL \sim \triangle KBA$. Hence $\frac{DL}{AB} = \frac{AD}{BK}$ $\implies \frac{DL}{BC} = \frac{BC}{BK}$.

Since $\angle LDC = \angle CBK = 90^\circ - \frac{1}{2}\angle ADC$, we deduce that $\triangle DLC \sim \triangle BCK$. Then $\angle BCK = \angle DLC$ implies

$\angle KCL = 360^\circ - \angle BCK - \angle LCD - \angle DCB = \angle LDC + \angle ADC = \angle ADL$

$\implies \angle KCL = 90^\circ + \frac{1}{2}\angle ADC = \text{const.}$

$\square$ Prove that there aren't any positive odd numbers a and b satisfying the equation $x^2 = y^3 + 4$

Solution

Since x, y are odd numbers, then $gcd(x, y) = d \to d|4 \to d = 1$. So $x^2 - 4 = (x-2)(x+2) = y^3$ and again $d = gcd(x-2, x+2) \to d|4$, since both $x-2$ and $x+2$ are odd numbers, hence $d = 1$. Therefore, the product of two relatively prime numbers is a cube of an odd number, so $x - 2 = z^3$ and $x + 2 = t^3$ where $zt = y$ and $t^3 - z^3 = 4 \to (t-z)(t^2 + tz + z^2) = 4$. Now, we are left to examine that whether it holds for two odd numbers or not. Since $t, z \geq 1 \to t^2 + tz + z^2 \geq 3$, and hence we should have $t - z = 1$ and $t^2 + tz + z^2 = 4$. Put $t = z + 1$ in the second equation then we get $(z+1)^2 + (z+1)z + z^2 = 3z^2 + 3z + 1 = 4 \to 3z^2 + 3z - 3 = 0$ and the roots are $\frac{1}{2}(-3+3\sqrt{5}), \frac{1}{2}(-3-3\sqrt{5})$ which are not integers. Hence, contradiction.

$\square$ F is defined on R, where $F(x+F(y)) = F(x) + 2xy^2 + y^2F(y)$ Find $F(x)$

Solution

Let $P(x,y)$ be the assertion that $F(x+F(y)) = F(x) + 2xy^2 + y^2F(y)$ Then $P(x,0) \implies F(x+F(0)) = F(x)$ Then $P(x+F(0),1) \implies F(x+F(1)) = F(x+F(0)+F(1)) = F(x+F(0)) + 2(x+F(0)) + F(1) = F(x) + 2(x+F(0)) + F(1)$ But $P(x,1) \implies F(x+F(1)) = F(x) + 2x + F(1)$ Then $F(x) + 2(x+F(0)) + F(1) = F(x+F(1)) = F(x) + 2x + F(1) \implies 2F(0) = 0 \implies F(0) = 0$ $P(0,y) \implies F(F(y)) = y^2F(y)$ Then $P(F(x),y) \implies F(F(x)+F(y)) = F(F(x)) + 2F(x)y^2 + y^2F(y) = x^2F(x) + 2F(x)y^2 + y^2F(y)$

Let $Q(x,y)$ be the assertion that $F(F(x)+F(y)) = x^2F(x)+2F(x)y^2+y^2F(y)$ Then $Q(x,1) \implies F(F(x)+F(1) = x^2F(x)+2F(x)+F(1)$ Also $Q(1,x) \implies F(F(1)+F(x)) = F(1)+2F(1)x^2+x^2F(x)$ Then $x^2F(x)+2F(x)+F(1) = F(1)+2F(1)x^2+x^2F(x) \implies F(x) = F(1)x^2$

$P(0,1) \implies F(1)^3 = F(0+F(1)) = F(0)+F(1) = F(1) \implies F(1) =$ -1 or 0 or 1 $P(1,1) \implies F(1)(1+F(1))^2 = F(1+F(1)) = F(1)+2+F(1) \implies (F(1)-1)(F(1)+1)(F(1)+2) = 0 \implies F(1) =$ -2 or -1 or 1 Hence $F(1) =$ -1 or 1 So $F(x) = -x^2, \forall x \in \mathbb{R}$ or $F(x) = x^2, \forall x \in \mathbb{R}$

$\square$ Prove that,for real $x_1, x_2, ...x_n$ $(sinx_1 + sinx_2 + ... + sinx_n)^2 + (cosx_1 + cosx_2 + ... + cosx_n)^2 \leq n^2$ for $n \geq 1$

Solution

Let $z_i = \cos x_i + \iota \sin x_i$ for $i = 1, 2, 3, \cdots, n$

Now as we now that $\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|$

$\implies (\sin x_1 + \sin x_2 + \cdots + \sin x_n)^2 + (\cos x_1 + \cos x_2 + \cdots + \cos x_n)^2 \leq n^2$

$\square$ A set S_0 containing subsets of $\{1, 2, \ldots, n\}$ has the property that for all $1 \leq a, b \leq n$, $a \neq b$, there is some set $A \in S_0$ with $a \in A$ and $b \notin A$. A series of sets is defined recursively with S_{i+1} consisting of all sets in S_i, plus all pairwise intersections and unions of sets in S_i. Prove that for some k, S_k contains all subsets of $\{1, 2, \ldots, n\}$.

Solution

There is a set containing a but not b, call it P_b. Similarly, there is a set containing a but not c, which is P_c.

We have $P_b \cap P_c \cap P_d \cap \cdots \cap P_n = \{a\}$. Therefore, after $n-1$ iterations, we will have all unitary subsets of $\{1, 2, \ldots, n\}$. From there, it's easy to see that any other subset with elements $a_1, a_2, \ldots$ will be formed by the union of the sets containing the individual elements of the subset.

$\square$

Given a convex hexagon $ABCDEF$. The point Y lies inside the hexagon. Points K, L, M, N, P, Q are the midpoints of sides AB, BC, CD, DE, EF, FA . Prove that the sum of the squares of fields $QAKY, LCMY, NEPY$ does not depend on the choice point Y.

Solution

Lemma. If P is an arbitrary point on the plane of $\triangle ABC$, whose centroid is G, then one of the triangles $\triangle PAG$, $\triangle PBG$, $\triangle PCG$ is equivalent to the sum of the other two.

WLOG assume that line PG separates segment BC from vertex A. Let M be the midpoint of BC and let X, Y, Z, U be the orthogonal projections of A, B, C, M onto PG. Then UM is the median of the right trapezoid $BYZC$ and $\triangle MUG \sim \triangle AZG$ are similar with similarity coefficient $\frac{GM}{AG} = \frac{1}{2}$. Therefore

$BY + CZ = 2 \cdot MU = AX \Longrightarrow PG \cdot BY + PG \cdot CZ = PG \cdot AX$
$\Longrightarrow [\triangle PBG] + [\triangle PCG] = [\triangle PAG]$

• By similar reasoning, it's easy to show that the distance from G to an abitrary line ℓ in the plane ABC equals the arithmetic mean of the directed distances from A, B, C to ℓ.

Back to the problem, since $[\triangle AQK] = \frac{1}{4}[\triangle AFB]$, $[\triangle CLM] = \frac{1}{4}[\triangle CBD]$ and $[\triangle ENP] = \frac{1}{4}[\triangle EDF]$ are constant, then it's enough to show that the sum of areas $[\triangle YLM] + [\triangle YNP] + [\triangle YQK]$ is constant. Let G, G' be the centroids of $\triangle KMP$ and $\triangle LNQ$. Notation $\delta(P)$ stands for the distance from a point P to the line AF. Then we have

$3 \cdot \delta(G) = \delta(K) + \delta(M) + \delta(P) = \frac{1}{2}[\delta(B) + \delta(C) + \delta(D) + \delta(E)]$
$\Longrightarrow 3 \cdot \delta(G) = \delta(L) + \delta(N) = 3 \cdot \delta(G')$

Since, the same relation occurs with respect to the remaining sides of the hexagon, then we deduce that G and G' coincide. In other words, $\triangle KMP$ and $\triangle LNQ$ share the same centroid G. Now, WLOG assume that G lies inside $\triangle MYN$ and that line YG separates K, L, M from N, P, Q. Using the previous lemma in $\triangle KMP$ and LNQ, we get

$[\triangle YQG] + [\triangle YNG] = [\triangle YLG] \;\; (1)\;,\;\; [\triangle YMG] + [\triangle YKG] = [\triangle YPG] \;\; (2)$

On the other hand, by adding areas we obtain

$[\triangle YLM] = [\triangle YLG] + [\triangle LGM] - [\triangle YMG] \;\; (3)$
$[\triangle YNP] = [\triangle YPG] + [\triangle NGP] - [\triangle YNG] \;\; (4)$
$[\triangle YQK] = [\triangle QGK] - [\triangle YQG] - [\triangle YKG] \;\; (5)$

Adding the expressions $(1), (2), (3), (4), (5)$ properly gives

$[\triangle YLM] + [\triangle YNP] + [\triangle YQK] = [\triangle LGM] + [\triangle NGP] + [\triangle QGK] = \text{const.}$

□ Solve $x = \sqrt{3-x}\sqrt{4-x} + \sqrt{5-x}\sqrt{4-x} + \sqrt{5-x}\sqrt{3-x}$

Solution

Let $a = 4 - x$, we get

$$\sqrt{a^2-a} + \sqrt{a^2+a} + \sqrt{a^2-1} = 4 - a$$

Now, play with this equation :

$\sqrt{a^2-a} + \sqrt{a^2+a} + \sqrt{a^2-1} = 4 - a$
$\Longrightarrow \sqrt{a^2-a} + \sqrt{a^2+a} = 4 - a - \sqrt{a^2-1}$
$\overset{\text{square sides}}{\Longrightarrow} a^2 - a + a^2 + a + 2\sqrt{a^4-a^2} = (4-a)^2 + (a^2-1) - 2(4-a)\sqrt{a^2-1}$
$\Longrightarrow 2a^2 + 2\sqrt{a^4-a^2} = 2a^2 - 8a + 15 - 2(4-a)\sqrt{a^2-1}$
$\Longrightarrow 2\sqrt{a^2(a^2-1)} = 2\sqrt{a^2-1}(a-4) - 8a + 15$
$\Longrightarrow 8\left(\sqrt{a^2-1} + a\right) = 15$
$\Longrightarrow \sqrt{a^2-1} = \frac{15}{8} - a$
$\Longrightarrow a^2 - 1 = a^2 - \frac{15a}{4} + \frac{225}{64}$
$\Longrightarrow -1 = -\frac{15a}{4} + \frac{225}{64}$
$\Longrightarrow a = \frac{289}{240}$

And so $x = 4 - a = \boxed{\dfrac{671}{240}}$.

□ Can you explain it a bit better? I am not sure of what the question is asking for, better yet, can you give an example?

Solution

PLUS the very important piece of information that the 25 guests have each a DIFFERENT number of acquaintances. Without that, the trivial answer would be that nobody may know A (for example when the guests each know each other).

With that, it is an easy play on the easy to prove, classical result that in a finite graph all vertex degrees cannot be all distinct (I will leave you to find the proof). Assume then there are n guests at A's party, each knowing at least a person at the party, all having distinct number of acquaintances. The only possibility is they know respectively $n, n-1, \ldots, 2, 1$ people. Denote them respectively by $A_n, A_{n-1}, \ldots, A_2, A_1$. Therefore A_n must know A, and all the other guests A_k, $1 \leq k \leq n-1$

Assume A_n leaves the party. Now each A_k, $1 \leq k \leq n-1$, knows $k-1$ people at the party. Since A_{n-1} does not know A_1, it follows he knows A, and all the other guests A_k, $2 \leq k \leq n-2$. Continue this reasoning, with $A_{n-1}, \ldots, A_{m+1}$ leaving the party, each having known A. Now each A_k, $1 \leq k \leq m$, knows $\max\{0, k-n+m\}$ people at the party. But then A_m knows $\max\{0, 2m-n\}$ people. If $2m-n > 0$, since A_m does not know $A_1, A_2, \ldots, A_{n-m}$, it follows he knows A, and all the other guests A_k, $n-m+1 \leq k \leq m-1$, and we may still continue. The reasoning stops when $2m-n \leq 0$, when all guests still at the party know no more people. It means A was known by $A_n, A_{n-1}, \ldots, A_{m+1}$, i.e. by $n-m$ people, where m is the largest value such that $2m-n \leq 0$, i.e. $m = \lfloor n/2 \rfloor$.

It means the number of guests that know A is exactly $n - \lfloor n/2 \rfloor$, no more, no less, if the conditions of the problem are to be obeyed. For $n = 25$ this gives that 13 guests were acquainted with A.

Note. I think in the original post, there was no condition of each guest to know at least one other person (but still that they know different number of people). This relaxes the condition on the degrees of the vertices of this graph. Can you find the answer under these more relaxed conditions?

□ Let $a, b, c, d, e, f > 0$. Prove that
$\frac{ab}{(a+b)^2} + \frac{cd}{(c+d)^2} + \frac{ef}{(e+f)^2} \leq \frac{5}{8} + \frac{8abcdef}{(a+b)^2(c+d)^2(e+f)^2}$

Solution

let $x = \frac{ab}{(a+b)^2}, y = \frac{cd}{(c+d)^2}, z = \frac{ef}{(e+f)^2}$, then $x, y, z \in \left(0, \frac{1}{4}\right]$, and inequality becomes
$f(x,y,z) = \frac{5}{8} + 8xyz - (x+y+z) \geq 0$
but $f(x,y,z)$ is a linear function for each variable, and it's symmetric, so we get
$f(x,y,z) \geq \min\left\{f(0,0,0), f\left(0,0,\frac{1}{4}\right), f\left(0,\frac{1}{4},\frac{1}{4}\right), f\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right\} = 0.$

□ Let a_0 be a positive integer and $a_1, a_2, \cdots, a_n$ distinct integers. Then prove the polynomial $f(x) = a_0(x-a_1)(x-a_2)\cdots(x-a_n) - 1$ is irreducible in $\mathbb{Q}$.

Solution

By Gauss' lemma, it is enough to prove $f(x)$ irreducible over $\mathbb{Z}$. Assume then $f(x) = g(x)h(x)$, with $g(x), h(x) \in \mathbb{Z}[x]$ and $0 < \min\{\deg g, \deg h\} \leq \max\{\deg g, \deg h\} < \deg f = n$ (and of course $\deg g + \deg h = n$). Then for any $1 \leq i \leq n$ we have $g(a_i)h(a_i) = f(a_i) = -1$, therefore $\{g(a_i), h(a_i)\} = \{-1, 1\}$, and so $(g+h)(a_i) = 0$. Since $\deg(g+h) \leq \max\{\deg g, \deg h\} < n$, it follows $g+h$ is identically null (having n distinct roots), hence $h(x) = -g(x)$, and $f(x) = -g(x)^2$. But this is a contradiction, since $a_0 > 0$, while the leading coefficient of $-g(x)^2$ is negative. Notice that the condition $a_0 > 0$ is necessary; a simple example is enough: $f(x) = -(x-1)(x+1) - 1 = -x^2$.

□ IF $x = \frac{2\pi}{7}$,then find $tanx.tan2x + tan2x.tan4x + tan4x.tanx$

Solution

$\tan\frac{k\pi}{7}; 1 \leq k \leq 6$ are roots of $\tan 7\theta = 0$ which gives the polynomial $y^6 - \binom{7}{2}y^4 + \binom{7}{4}y^2 - 7 = 0$ with the substitution $y = \tan\theta$

This also yields that $\tan^2\frac{k\pi}{7}; k = 2, 4, 8$ are roots to the polynomial $y^3 - \binom{7}{2}y^2 + \binom{7}{4}y - 7 =$ so that from Vieta's formula $\tan^2\frac{2\pi}{7} + \tan^2\frac{4\pi}{7} + \tan^2\frac{8\pi}{7} = 21$

Further we have $\tan\frac{2\pi}{7} + \tan\frac{4\pi}{7} + \tan\frac{8\pi}{7} = \tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{8\pi}{7}$

But from Vieta, $\tan^2\frac{2\pi}{7}\tan^2\frac{4\pi}{7}\tan^2\frac{8\pi}{7}=7$ and hence $\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{8\pi}{7}=-\sqrt{7}$

Using the identity $2\sum ab=(a+b+c)^2-(a^2+b^2+c^2)$ we obtain the required sum as -7

□ If $a,b,c\in[1,2]$ and $a+b+c=4$, then :

$a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b}\le 4\sqrt{3}$

Solution

Dividing by 4 ,we have that $\frac{a}{4}+\frac{b}{4}+\frac{c}{4}=1$

Define $f(x)=\sqrt{x}$,which is concave since $f''(x)<0$

By Jensen's inequality we obtain

$LHS\le f\left(\frac{\frac{a}{4}(b+c)+\frac{b}{4}(c+a)+\frac{c}{4}(a+b)}{\frac{a}{4}+\frac{b}{4}+\frac{c}{4}}\right)\le\sqrt{3}\iff a(b+c)+b(c+a)+c(a+b)\le 12\iff a(4-a)+b(4-b)+c(4-c)\le 12\iff a^2+b^2+c^2\ge 4$

Since $2(a^2+b^2+c^2)\ge 2(ab+bc+ca)\implies a^2+b^2+c^2\ge\frac{16}{3}>4$

□ If a,b,c are positive real numbers and $a+b+c=1$ prove that

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\ge\sqrt{\frac{1}{2\left(ab+bc+ca\right)}}$$

Solution

Let $b+c=x, c+a=y, b+c=z$, and $f(t)=\frac{1}{\sqrt{t}}$. Observe that $f''(t)>0$. Hence we have $af(x)+bf(y)+cf(z)\ge f\left(\frac{ax+by+cz}{a+b+c}\right)$. The last one equal to $f(2(ab+bc+ca))=\frac{1}{\sqrt{2(ab+bc+ca)}}$.

□ Find all triples (x,y,z) of real numbers which satisfy the simultaneous equations

$$x=y^3+y-8$$

$$y=z^3+z-8$$

$$z=x^3+x-8.$$

Solution

Let $f(t)=t^3+t-8$, then $f'(t)=3t^2+1>0$ for all real t, so $f(t)$ is strictly increasing for all real t.

Notice that $f(f(f(x)))=x$, but since $f(t)$ is strictly increasing, this implies that $f(x)=x$ (*). So $x^3+x-8=x\implies x^3=8\implies x=2$. Similarly, for y and z, we have that $y=z=2$. Therefore, $(x,y,z)=\boxed{(2,2,2)}$ is the only real solution to the system.

Lemma: If $g(t)$ is a strictly increasing function then $g(g(t))=t\iff g(t)=t$. Proof: If $g(g(t))=t$ then we have that 3 cases:

Case 1: $g(t)=t$ This case is trivial.

Case 2: $g(t)>t$ Since g is increasing, we have that

$t=g(g(t))>g(t)>t;$

contradiction.

Case 3: $g(t)<t$

Since g is increasing, we have that

$t>g(t)>g(g(t))=t;$

contradiction.

Hence, $g(g(t))=t\iff g(t)=t$.

The lemma can then be extended by induction to $g^k(t)=t\iff g(t)=t$, when g is strictly increasing.

Remark: Lemma. If f , g are strict increasing $\nearrow$ dynamic function, i.e. $f,g:I\to I$, where I is an interval, then $\left\{\begin{array}{l} f(x)=g(y) \\ \\ f(y)=g(z) \\ \\ f(z)=g(x) \end{array}\right\| \implies x=y=z$. Indeed,

► $\underline{x<y} \iff f(x)<f(y) \iff g(y)<g(z) \iff \underline{y<z} \iff$

$f(y)<f(z) \iff g(z)<g(x) \iff \underline{z<x} \iff$ absurd .

► $\underline{x>y} \iff f(x)>f(y) \iff g(y)>g(z) \iff \underline{y>z} \iff$

$f(y)>f(z) \iff g(z)>g(x) \iff \underline{z>x} \iff$ absurd .

For $f,g:\mathrm{R}\to\mathrm{R}$, where $\left\{\begin{array}{lcll} f(x) & = & x & \nearrow \\ \\ g(x) & = & x^3+x-8 & \nearrow \end{array}\right\|$ obtain the proposed problem.

□ Let a,b,c such that:

$$a^2+c^2=1$$

$$b^2+2b(a+c)=6$$

Prove that: $b(c-a)\le 4$

Solution

$\left\{\begin{array}{l} a^2+c^2=1 \\ b^2+2b\left(a+c\right)=6 \end{array}\right.$ let $t=ac\in\left[-\frac{a^2+c^2}{2},\frac{a^2+c^2}{2}\right]=\left[-\frac{1}{2},\frac{1}{2}\right]$, then $1+2t=(a+c)^2=\left(\frac{6-b^2}{2b}\right)^2 \implies$

$t=\frac{\left(b^2-6\right)^2-4b^2}{8b^2}\in\left[-\frac{1}{2},\frac{1}{2}\right] \implies 2\le b^2\le 18$ so we have

$$b^2\left(c-a\right)^2=b^2\left(1-2t\right)=b^2\left(1-\frac{\left(b^2-6\right)^2-4b^2}{4b^2}\right)=$$

$$\le\frac{1}{4}\left(\frac{(18-b^2)+(b^2-2)}{2}\right)^2=16$$

$\implies b\left(c-a\right)\le 4$

□ Solve the equation $13x^4-19x^2-21+(6x^2+28)\sqrt{x^2-1}=0$

Solution

let $x^2=t^2+1,\ t\ge 0$

$\implies 13(t^2+1)^2-19(t^2+1)-21+(6t^2+34)t=0$

$\implies 13t^4+6t^3+7t^2+34t-27=0$

$\implies (t^2+t-1)(13t^2-7t+27)=0 \implies t^2+t-1=0 \implies t=\frac{-1\pm\sqrt{5}}{2} \implies t=\frac{-1+\sqrt{5}}{2}$

$\implies x^2=\frac{5-\sqrt{5}}{2} \implies x=\pm\sqrt{\frac{5-\sqrt{5}}{2}}$

□ Find x $\arcsin(1-x)-2\arcsin x=\frac{\pi}{2}$

Solution

$\arcsin(1-x)=\frac{\pi}{2}+2\arcsin x$

$\implies 1-x=\sin\left(\frac{\pi}{2}+2\arcsin x\right)$

$\implies 1-x=\cos\left(2\arcsin x\right)=1-2\sin^2(\arcsin x)=1-2x^2$

$\implies x=0,\frac{1}{2}$

For $x=0$, LHS$=\arcsin(1-0)-2\arcsin 0=\arcsin 1=\frac{\pi}{2} \implies x=0$ is a solution.

For $x=\frac{1}{2}$, LHS$=\arcsin\left(1-\frac{1}{2}\right)-2\arcsin\left(\frac{1}{2}\right)=\frac{\pi}{6}-\frac{\pi}{3}\neq\frac{\pi}{2} \implies x\neq\frac{1}{2}$

$\square$ Let $ABCD$ be a trapezoid, where $AD \parallel BC$ and $BC < AD$. For a point $M \in (AB)$ denote $N \in (CD)$ for which $MN \parallel AD$, $I \in MC \cap NB$ and $F \in AB$ for which $FI \parallel AD$. Prove that $MF = MA \iff BN \parallel FD$

Solution

Let $G \equiv FI \cap DC$. From $\triangle MIF \sim \triangle MCB$ and $\triangle NIG \sim \triangle NBC$, we obtain $\frac{FI}{BC} = \frac{MI}{MC}$ and $\frac{GI}{BC} = \frac{NI}{NB}$. But $\frac{MI}{MC} = \frac{NI}{NB}$, due to $\triangle MIN \sim \triangle CIB$. Therefore, $FI = GI$, i.e. I is the midpoint of FG. Assume that M is the midpoint of AF. Then $L \equiv FD \cap MN$ is the midpoint of $FD \Longrightarrow IL$ is the F-midline of $\triangle DFG \Longrightarrow IL \parallel CN$ and since IF is parallel to CB, it follows that $\triangle LIF$ and $\triangle NCB$ are homothetic through $M \Longrightarrow FLD$ is parallel to BN. The converse is proved analogously.

$\square$ p is prime. Find p such that $p^4 - 5p^2 + 9$ is prime.

Solution

Looking mod 3 tells us that if $p \equiv 0 \bmod 3$, then $p^4 - 5p^2 + 9$ is divisible by 9, and hence is not prime. If $p \not\equiv 0 \bmod 3$, then $p^4 - 5p^2 + 9 \equiv p^4 + p^2 \equiv 1 + 1 \equiv 2 \bmod 3$, which tells us nothing about the primality of the expression since numbers equivalent to 2 mod 3 may be prime or not prime.

But checking mod 5 tells us that $p^4 - 5p^2 + 9 \equiv p^4 + 4 \bmod 5$. And by Fermat's Little Theorem we know that $p^4 \equiv 1 \bmod 5$ for any p relatively prime to 5. Therefore, $p^4 - 5p^2 + 9 \equiv 0 \bmod 5$ for any prime p not equal to 5.

So the only way that $p^4 - 5p^2 + 9$ could be prime is if it equals 5 or if p=5. $p^4 - 5p^2 + 9 = 5 \Longrightarrow (p^2 - 1)(p^2 - 4) = 0 \Longrightarrow p = \pm 1$ or $p = \pm 2$. So the prime $p = 2$ makes the expression prime.

And if $p = 5$, then $p^4 - 5p^2 + 9 = 509$, which is prime. So $p = 5$ also makes the expression prime.

$\square$ Prove that

$$\sum_{i=1}^{n} (-1)^{n+i} \binom{n}{i} \binom{ni}{n} = n^n$$

Solution

$\sum_{i=1}^{n} (-1)^{n+i} \binom{n}{i} \binom{ni}{n} = (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{ni}{n}$

$= (-1)^n$coefficient of x^n in $\sum_{i=0}^{n} (-1)^i \binom{n}{i} (1+x)^{ni}$

$= (-1)^n$coefficient of x^n in $(1 - (1+x)^n)^n$

$= (-1)^n$coefficient of x^n in $\left(-nx - \binom{n}{2}x^2 - \binom{n}{3}x^3 - \cdots\right)^n$

$= (-1)^n (-n)^n$

$= n^n$

$\square$ Find the smallest natural $n > 11$ such that exists a polynomial $p(x)$ with degree n that verifies:

i) $p(k) = k^n$, for $k = 1, 2, \ldots, n$.

ii) $p(0) \in \mathbb{Z}$.

iii) $p(-1) = 2003$.

Solution

$P(x) = \lambda \prod_{r=1}^{n} (x - r) + x^n$

$\Longrightarrow P(0) = \lambda (-1)^n n! \Longrightarrow \lambda \in \mathbb{Q}$

And $P(-1) = \lambda (-1)^n (n+1)! + (-1)^n = 2003$

For $n \in$ even

$\lambda (n+1)! = 2002 = 2 \times 7 \times 11 \times 13$

$\Longrightarrow \min n = 12, \ \lambda = \frac{2002}{13!}$

For $n \in$ odd

$-\lambda(n+1)! = 2004 = 4\times 3\times 167$
$\implies \min n = 333,\ \lambda = -\frac{2004}{334!}$
Hence smallest $n=12$ and $P(x) = \frac{2002}{13!}\prod_{r=1}^{12}(x-r) + x^{12}$

☐ If $S_n = \sum_{i=1}^{n}\frac{1}{i}$; $n \geq 3$. Then prove that:
$n(n+1)^{\frac{1}{n}} - n < S_n < n-(n-1)n^{-\frac{1}{(n-1)}}$

Solution

$S_n = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} = (1+1)+(\frac{1}{2}+1)+(\frac{1}{3}+1)+...+(\frac{1}{n}+1)-n = 2+\frac{3}{2}+\frac{4}{3}+...+\frac{n+1}{n}-n \geq n\sqrt[n]{2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot...\cdot\frac{n+1}{n}} - n = n\sqrt[n]{n+1}-n$ $S_n = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} = n+(\frac{1}{2}-1)+(\frac{1}{3}-1)+...+(\frac{1}{n}-1) = n-(\frac{1}{2}+\frac{2}{3}+...+\frac{n-1}{n}) \leq n-(n-1)\sqrt[n-1]{2\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot...\cdot\frac{n-1}{n}} = n-(n-1)\sqrt[n-1]{\frac{1}{n}}$

☐ 1 ▶ For any $n\in\mathbb{N}^*$, $a_n > 0$ and $\lim_{n\to\infty}\frac{a_n}{n} = \infty$, prove that $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+a_k} = 0$.

2 ▶ It is well-known that for the sequence $a_n = \frac{\sqrt[n]{n!}}{n}$, $\lim_{n\to\infty} a_n = \frac{1}{\mathrm{e}}$. Prove that : $\lim_{n\to\infty}\frac{n}{\ln n}\cdot\left(a_n-\frac{1}{\mathrm{e}}\right) = \frac{1}{2\mathrm{e}}$.

Solution to problem (1) Using the simple inequality $x+y \geq 2\sqrt{xy}$, $x,y>0$ we obtain : $\sum_{k=1}^{n}\frac{1}{n+a_k} < \frac{1}{2\sqrt{n}}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{a_k}}$. On the other hand,

we have : $\lim_{n\to\infty}\frac{\sum_{k=1}^{n+1}\frac{1}{\sqrt{a_k}}-\sum_{k=1}^{n}\frac{1}{\sqrt{a_k}}}{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{a_{n+1}}} = \lim_{n\to\infty}\left[\sqrt{\frac{n+1}{a_{n+1}}}\cdot\left(1+\sqrt{\frac{n}{n+1}}\right)\right] = 0$
.

Thus, by the Cesaro-Stolz theorem we get : $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{a_k}} = 0$. Then we also have :
$\boxed{\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+a_k} = 0}$.

Solution to problem (2) $\frac{n}{\ln n}\cdot\left(a_n-\frac{1}{\mathrm{e}}\right) = \frac{1}{\mathrm{e}}\cdot\frac{n}{\ln n}\cdot(\mathrm{e}\cdot a_n - 1) = \frac{1}{\mathrm{e}}\cdot\frac{n}{\ln n}\cdot b_n\cdot\frac{\mathrm{e}^{b_n}-1}{b_n}$, where $b_n = \ln(\mathrm{e}\cdot a_n) = 1+\ln a_n \to 0$.

Therefore, $\lim_{n\to\infty}\frac{n}{\ln n}\cdot\left(a_n-\frac{1}{\mathrm{e}}\right) = \frac{1}{\mathrm{e}}\cdot\lim_{n\to\infty}\frac{n\cdot b_n}{\ln n}$. On the other hand, $\lim_{n\to\infty}\frac{(n+1)\cdot b_{n+1}-n\cdot b_n}{\ln(n+1)-\ln n} =$
$= \lim_{n\to\infty}\frac{1-n\cdot\ln\frac{n+1}{n}}{\ln(n+1)-\ln n} = \lim_{n\to\infty}\frac{n\cdot\left(1-n\ln\frac{n+1}{n}\right)}{\ln\left(1+\frac{1}{n}\right)^n} = \frac{1}{2}$, because $\lim_{n\to\infty} n\cdot\left(1-n\ln\frac{n+1}{n}\right) = \frac{1}{2}$
(one can easily prove it by l'Hospital's rule) . Consequently, $\lim_{n\to\infty}\frac{n\cdot b_n}{\ln n} = \frac{1}{2}$ (Stolz-Cesaro) and our conclusion follows.

☐ Solve in natural the equation $5^5-5^4+5^n = m^2$

Solution

Suppose $n\geq 4$. Then, $5^4(5-1+5^{n-4}) = m^2 \implies 4+5^{n-4} = k^2$, for some $k\in\mathbb{Z}$. Thus, $5^{n-4} = k^2-4 = (k-2)(k+2) \implies k=3 \implies n=5, m=75$.

For $n\in\{1,2,3\}$ it is easy to check that no solution exists.

Hence, the only solution is $(m,n)\in\{(75,5)\}$.

☐ Prove that for positive integer n,

$$\left(\sum_{k=1}^{n}\sqrt{\frac{k-\sqrt{k^2-1}}{\sqrt{k(k+1)}}}\right)^2 \leq n\sqrt{\frac{n}{n+1}}.$$

Solution

Use Cauchy-Schawz's ineq ,we have: $\left(\sum_{k=1}^{n}\sqrt{\frac{k-\sqrt{k^2-1}}{\sqrt{k(k+1)}}}\right)^2 \leq \left(\sum_{k=1}^{n}(k-\sqrt{k^2-1})\right)\left(\sum_{k=1}^{n}\frac{1}{\sqrt{k(k+1)}}\right)$
Because $\left(\sum_{k=1}^{n}\frac{1}{\sqrt{k(k+1)}}\right) \leq \sqrt{n\left(\sum_{k=1}^{n}\frac{1}{k(k+1)}\right)} = \frac{n}{\sqrt{n(n+1)}}$ $\left(\sum_{k=1}^{n}(k-\sqrt{k^2-1})\right) \leq \sqrt{n}$ We have done.

$\square$ Find all functions $f : \mathbb{Q}^+ \longrightarrow \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$, (i) $f(x+1) = f(x)+1$; and (ii) $f(x^3) = [f(x)]^3$.

Solution

By mathematical induction we have :

$f(x+n) = n + f(x)$ with $\forall n \in \mathbb{N}^+$

so if $m, n \in \mathbb{N}^+$ then we have :

$\left(\frac{m}{n} + n^2\right)^3 = \left(\frac{m}{n}\right)^3 + 3m^2 + 3mn^3 + n^6$

so :

$f\left[\left(\frac{m}{n} + n^2\right)^3\right] = f\left[\left(\frac{m}{n}\right)^3\right] + 3m^2 + 3mn^3 + n^6 = f^3\left(\frac{m}{n}\right) + 3m^2 + 3mn^3 + n^6$

But

$f\left[\left(\frac{m}{n} + n^2\right)^3\right] = \left[f\left(\frac{m}{n} + n^2\right)\right]^3 = \left[f\left(\frac{m}{n}\right) + n^2\right]^3$

Hence:

$\left[f\left(\frac{m}{n}\right) + n^2\right]^3 = f^3\left(\frac{m}{n}\right) + 3m^2 + 3mn^3 + n^6$

Solve this quadratic equation with $f\left(\frac{m}{n}\right)$ is variable we have root :

$f\left(\frac{m}{n}\right) = \frac{m}{n}$

Thus :$m, n \in \mathbb{N}^+$ so $\frac{m}{n} \in \mathbb{Q}^+$ and we have $f(x) = x$

$\square$ Determine the smallest integer which is half of a perfect square, one-third full cube and fifth complete fifth grade.

Solution

Answer: $2^{15}3^{20}5^{24}$

Let this smallest integer be x. Observe that 2, 3 and 5 are all primes. Then the smallest x possible must only have 2, 3 and 5 as its only prime factors. We let $x = 2^a3^b5^c$. Condition 1. x is half of a perfect square. $2x = 2^{a+1}3^b5^c$. This requires $2|a+1, b, c$. Condition 2. x is one-third of a perfect cube. $3x = 2^a3^{b+1}5^c$. This requires $3|a, b+1, c$. Condition 3. x is one-fifth of a fifth power. $5x = 2^a3^b5^{c+1}$. This requires $5|a, b, c+1$. We will examine each variable (a, b and c) one by one. For a, we have $2|a+1$ which implies a is odd. Then $3, 5|a$ implies that $15|a$. It follows that $min(a) = 15$. For b, we have $2, 5|b$ implies that $10|b$. $3|b+1$ means $b = 2(mod 3)$. The minimum b that fulfills this requirement is 20. For c, we have $2, 3|c$ implies that $6|c$. $5|b+1$ means $c = 4(mod 5)$. The minimum c that fulfills this requirement is 24. Hence the desired integer is $2^{15}3^{20}5^{24}$. QED.

$\square$ Find the minimum value of

$(u-v)^2 + \left(\sqrt{2-u^2} - \frac{9}{v}\right)^2$

for $0 < u < \sqrt{2}$ and $v > 0$

Solution

We can rearrange the thing to be minimized as follows:

$(u-v)^2 + \left(\sqrt{2-u^2} - \frac{9}{v}\right)^2$ Line 1

$= \left(\left(\frac{u+\sqrt{2-u^2}}{2} + \frac{u-\sqrt{2-u^2}}{2}\right) - \left(\frac{v+\frac{9}{v}}{2} + \frac{v-\frac{9}{v}}{2}\right)\right)^2 + \left(\left(\frac{u+\sqrt{2-u^2}}{2} - \frac{u-\sqrt{2-u^2}}{2}\right) - \left(\frac{v+\frac{9}{v}}{2} - \frac{v-\frac{9}{v}}{2}\right)\right)^2$ Line 2

$= \left(\left(\frac{u+\sqrt{2-u^2}}{2} - \frac{v+\frac{9}{v}}{2}\right) + \left(\frac{u-\sqrt{2-u^2}}{2} - \frac{v-\frac{9}{v}}{2}\right)\right)^2 + \left(\left(\frac{u+\sqrt{2-u^2}}{2} - \frac{v+\frac{9}{v}}{2}\right) - \left(\frac{u-\sqrt{2-u^2}}{2} - \frac{v-\frac{9}{v}}{2}\right)\right)^2$ Line 3

$= 2\left(\frac{u+\sqrt{2-u^2}}{2} - \frac{v+\frac{9}{v}}{2}\right)^2 + 2\left(\frac{u-\sqrt{2-u^2}}{2} - \frac{v-\frac{9}{v}}{2}\right)^2$ Line 4

$\geq 2\left(\frac{u+\sqrt{2-u^2}}{2} - \frac{v+\frac{9}{v}}{2}\right)^2$ Line 5

$= 2\left(\frac{v+\frac{9}{v}}{2} - \frac{u+\sqrt{2-u^2}}{2}\right)^2$ Line 6

$\geq 2\left(\frac{v-6+\frac{9}{v}+6}{2} - \frac{u+\sqrt{2-u^2+2(u-1)^2}}{2}\right)^2$ Line 7

$= 2\left(\frac{v^2-6v+9}{2v} + 3 - \frac{u+\sqrt{4-4u+u^2}}{2}\right)^2$ Line 8

$= 2\left(\frac{(v-3)^2}{2v} + 3 - \frac{u+2-u}{2}\right)^2$ Line 9

$= 2\left(\frac{(v-3)^2}{2v} + 3 - 1\right)^2$ Line 10

≥ 8 Line 11

Equality happens if $u = 1$ and $v = 3$.

In going from line 3 to line 4, we use the fact that $(x+y)^2 + (x-y)^2 = 2x^2 + 2y^2$.

In going from line 8 to line 9 we use the fact that $u \leq \sqrt{2}$ so $2 - u > 0$.

In going from line 10 to line 11, we use the fact that $v > 0$ because that means that $\frac{(v-3)^2}{2v} \geq 0$ and $\frac{(v-3)^2}{2v} + 2 \geq 2$.

□ Find all the triples of positeve integers (a, b, c) such as $(a+1)(b+1)(c+1) = 2abc$

Solution

The equation is equivalent to $abc = ab + bc + ca + a + b + c + 1 \Leftrightarrow 1 = \frac{1}{c} + \frac{1}{a} + \frac{1}{b} + \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} + \frac{1}{abc}$ Now, if a, b, c are all greater or equal than 4, then we get that: $1 = \frac{1}{c} + \frac{1}{a} + \frac{1}{b} + \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} + \frac{1}{abc}$ $\leq \frac{3}{4} + \frac{3}{16} + \frac{1}{64} = \frac{61}{64} < 1$ Contradiction, therefore at least one of them must be less than 4.

If at least one of them equals 1(WLOG a) then: $(b+1)(c+1) = bc$, an absurd.

If at least one of them equals 2(WLOG a) then: $3b+3c+3 = bc \Leftrightarrow c = \frac{3b+3}{b-3} \Leftrightarrow b-3|3b+3-3(b-3)$ $\Leftrightarrow b-3|12$ which leads to $b = \{2, 4, 5, 6, 7, 9, 15\}$ which leads to the triples $\{(2, 4, 15), (2, 6, 7), (2, 5, 9)\}$ up to permutations. (They must be checked)

If at least one of them equals 3(WLOG a) then: $2b+2c+2 = bc \Leftrightarrow \frac{2b+2}{b-2} = c \Leftrightarrow b-2|2b+2-2(b-2)$ $\Leftrightarrow b-2|6 \Leftrightarrow b = \{3, 4, 5, 8\}$ leading to the triples $\{(3, 3, 8), (3, 4, 5)\}$ up to permutations.

Checking out all the possible triples we get that they are $\{(2, 4, 15), (2, 6, 7), (2, 5, 9), (3, 3, 8), (3, 4, 5)\}$ up to permutations of course.

QED.

□ Consider addition $\oplus$ and multiplication $\otimes$ modulo 7 of the numbers in $S = \{0, 1, 2, 3, 4, 5, 6\}$. This means that

$$m \oplus n = \text{remainder when } m + n \text{ is divided by } 7$$

$$m \otimes n = \text{remainder when } m \times n \text{ is divided by } 7$$

Then 1 is the multiplicative identity and each element $a \in S$ has a multiplicative inverse $1/a$. Find the value of $\frac{1}{4} \oplus \left(2 \otimes \frac{1}{3}\right)$.

Solution

$1 \equiv 8 mod 7 \Rightarrow \frac{1}{4} \equiv \frac{8}{4} \equiv 2 mod 7$ $1 \equiv 15 mod 7 \Rightarrow \frac{1}{3} \equiv \frac{15}{3} \equiv 5 mod 7$

Thus we have:

$\frac{1}{4} \oplus \left(2 \otimes \frac{1}{3}\right) \equiv 2 \oplus 2 \otimes 5 \equiv 2 \oplus 10 \equiv 5 mod 7$

□ $a = \log_{150} 72$,$b = \log_{45} 180$. Find $\log_{200} 75$ in terms of a,b.

Solution

Let $\log 2 = x$, $\log 3 = y$, $\log 5 = z$. Then

$$a = \log_{200} 75 = \frac{\log 75}{\log 200} = \frac{\log 3 \cdot 5^2}{\log 2^3 \cdot 5^2} = \frac{y + 2z}{3x + 2z},$$

$$b = \log_{150} 72 = \frac{\log 72}{\log 150} = \frac{\log 2^3 \cdot 3^2}{\log 2 \cdot 3 \cdot 5^2} = \frac{3x + 2y}{x + y + 2z},$$

and

$$c=\log_{45}180=\frac{\log 180}{\log 45}=\frac{\log 2^2\cdot 3^2\cdot 5}{\log 3^2\cdot 5}=\frac{2x+2y+z}{2y+z}.$$

Solving for y,z in terms of x,a,b, we find

$$y=\frac{(2ab+3a+b-3)x}{ab-2a+2},\quad z=\frac{(-3ab+6a-b+3)x}{2(ab-2a+2)}.$$

Substituting the result gives

$$c=\frac{2x}{2y+z}+1=\frac{9ab+10a+3b-1}{5ab+18a+3b-9}.$$

□ Show that $\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7}=\sqrt{7}$.

Solution

The equation $\tan 7\theta=0$ has roots $\theta=\frac{\pi}{7},\frac{2\pi}{7},\frac{3\pi}{7},\ldots$.

$\frac{\sin 7\theta}{\cos 7\theta}=0$

$\frac{\Im(\cos 7\theta+i\sin 7\theta)}{\Re(\cos 7\theta+i\sin 7\theta)}=0$

$\frac{7c^6s-35c^4s^3+21c^2s^5-s^7}{c^7-21c^5s^2+35c^3s^4-7cs^6}=0$,

using De Moivre's theorem, $(\cos\theta+i\sin\theta)^7=(\cos 7\theta+i\sin 7\theta)$, and expanding using the binomial theorem, where $c=\cos\theta$ and $s=\sin\theta$.

$\frac{7t-35t^3+21t^5-t^7}{1-21t^2+35t^4-7t^6}=0$, on dividing top and bottom by $\cos^7\theta$.

$t^6-21t^4+35t^2-7=0$

with $t^2=x$, this is $x^3-21x^2+35x-7=0$, with roots $x=\tan^2(\frac{\pi}{7}),\tan^2(\frac{2\pi}{7})$ and $x=\tan^2(\frac{3\pi}{7})$.

Then by Viete's formulas, the product of the roots is 7 and so

$\tan(\frac{\pi}{7})\tan(\frac{2\pi}{7})\tan(\frac{3\pi}{7})=\sqrt{7}$.

□ Let $\triangle ABC$ with orthocenter H and circumcircle $C(O,R)$. Show $|OH|=R\sqrt{1-8\cos A\cdot\cos B\cdot\cos C}$.

Solution

$OH=3\cdot OG$ and $R^2-OG^2=\frac{1}{9}\cdot(a^2+b^2+c^2)\implies OH^2=9R^2-\sum a^2=9R^2-4R^2\cdot\sum\sin^2 A=9R^2-2R^2\cdot\sum(1-\cos 2A)=R^2\left(3+2\cdot\sum\cos 2A\right)\implies\boxed{OH^2=R^2\left(3+2\cdot\sum\cos 2A\right)}$ (1) . Observe that $\sum\cos 2A=\cos 2A+2\cos(B+C)\cos(B-C)=2\cos^2 A-1-2\cos A\cos(B-C)=-1-2\cos A[(\cos(B+C)+\cos(B-C)]=-1-4\cos A\cos B\cos C$, i.e. $\boxed{\sum\cos 2A=-1-4\cdot\prod\cos A}$. In conclusion, the relation (1) becomes $OH^2=R^2\cdot(1-8\cdot\cos A\cos B\cos C)$, i.e. $OH=R\cdot\sqrt{1-8\prod\cos A}$. On other hand, $1-8\cdot\prod\cos A=1-4\cos A[\cos(B+C)+\cos(B-C)]=1-4\cos A\cos(B-C)+4\cos^2 A=[2\cos A-\cos(B-C)]^2+\sin^2(B-C)$. Thus, $1-8\cdot\prod\cos A\ge 0$, i.e. $\cos A\cos B\cos C\le\frac{1}{8}$. We have the equality $\iff A=B=C\iff O\equiv H$.

□ Solve $|x^2-12|x|+20|\le 9$.

Solution

$x^2=|x|^2\implies x^2-12|x|+20=(|x|-2)(|x|-10)$ .We'lluse the substitution $\boxed{|x|-2=t\ge -2}$ $(*)$. Therefore, our

inequality becomes $|t(t-8)|\le 9\iff(t^2-8t-9)(t^2-8t+9)\le 0\iff t\in\left[-1,4-\sqrt{7}\right]\cup\left[4+\sqrt{7},9\right]\overset{(*)}{\iff}$

$|x|\in\left[1,6-\sqrt{7}\right]\cup\left[6+\sqrt{7},11\right]\iff\boxed{\boxed{x\in\left[-11,-6-\sqrt{7}\right]\cup\left[-6+\sqrt{7},-1\right]\cup\left[1,6-\sqrt{7}\right]\cup\left[6+\sqrt{7},1\right.}}$

□ The last digit of the number x^2+xy+y^2 is zero (where x, y are positive integers). Prove that two last digits of this number are zeros.

Solution

This means $x^3 \equiv y^3 \pmod{10}$. Let us show then $x \equiv y \pmod{10}$; it will follow $3x^2 \equiv 0 \pmod{10}$, hence $x \equiv 0 \pmod{10}$, therefore $x^2+xy+y^2 \equiv 0 \pmod{100}$.

But $x^3 \equiv y^3 \pmod{10}$ implies $x^3 \equiv y^3 \pmod 2$, and so $x \equiv y \pmod 2$. Also $x^3 \equiv y^3 \pmod{10}$ implies $x^3 \equiv y^3 \pmod 5$, and so $x \equiv y \pmod 5$. Together, they yield $x \equiv y \pmod{10}$, as claimed.

□ Find all a such that

$$\begin{cases} |x+1|a = y+\cos x \\ \sin^2 x + y^2 = 1 \end{cases}$$

have only real solution

Solution

Rearranging the second equation, we have

$y^2 = 1-\sin^2 x = \cos^2 x \implies y = \cos x,\ y = -\cos x.$

For every value of x that solves the system, there will be two corresponding values of y, which means there will be multiple solutions. The only case in which there will be one corresponding value of y is when $\cos x = 0$ (consequence of $y = \cos x = -\cos x$) or $x = \frac{\pi(2k+1)}{2}$ for some $k \in \mathbb{Z}$. Thus, we have $|x+1|a = y+\cos x = 2\cos x = 0$. So we must have $a = 0$. Or $x = -1 \implies -1 = \frac{\pi(2k+1)}{2}$. Solving, we obtain a non-integer value of k so we know that $|x+1| \neq 0$. Thus, $a = 0$ is the only value that will yield the desired condition.

□ A caravan of 7 horse-pulled wagons travels across the country. The journey lasts several days, and the horse riders are getting are getting tired of looking at the wagon ahead of him. In how many ways is it possible to permute the wagons so that each wagon is preceded by wagon different from the original one?

Solution

Call T_i is the number of permutations in which seven cars have i cars do not change position. We have $|T_i| = (7-i)!$ And $|T_{j_1} \bigcap ... \bigcap T_{j_k}| = (7-k)!$ $(j_1, j_2, .., j_k \in (1, 2, ..., 7)$ $(1 \le k \le 7) \Rightarrow$ Inverted several ways to satisfy the assignment is: $7! - |T_1 \bigcup T_2 \bigcup ... \bigcup T_7| = 7! - \sum_{k=1}^{7} (-1)^{k+1} \binom{7}{k} |T_{j_1} \bigcap ... \bigcap T_{j_k}| = 7! - \sum_{k=1}^{7} (-1)^{k+1} \binom{7}{k} (7-k)!$

□ Prove that $N = \sqrt{1+\sqrt{3+\sqrt{5+\sqrt{\cdots+\sqrt{2n-1}}}}} < 2$ for all $n > 1$

Solution

We use the inequality $(2(n-k)-1)+(n-k+2) < (n-k+1)^2$, equivalent to $(n-k)(n-k-1) > 0$ for $0 \le k \le n-2$. Start, for $k = 0$, with $2n-1 < (2n-1)+(n+2) < (n+1)^2$, hence $2n-3+\sqrt{2n-1} < (2(n-1)-1)+(n+1) < n^2$ (for $k = 1$). Then $2n-5+\sqrt{2n-3+\sqrt{2n-1}} < (2(n-2)-1)+(n) < (n-1)^2$ (for $k = 2$). And so on, until $1+\sqrt{3+\cdots+\sqrt{2n-1}} < 1+3 = 4$. A different approach. First look at $a_n = \sqrt{1+\sqrt{1+\cdots+\sqrt{1}}}$, with n nested radical signs. The sequence $(a_n)_{n\ge1}$ is clearly increasing, starting with $a_1 = 1$. For $n > 1$ we have $a_n^2 = 1+a_{n-1} < 1+a_n$, thus a_n must be less than the positive root of $\lambda^2-\lambda-1$, which is $\phi = \dfrac{1+\sqrt{5}}{2} < \sqrt{3}$. (In fact it is easy to see that $\lim_{n\to\infty} a_n = \phi$.)

Now define $x_n = \sqrt{1+\sqrt{3+\cdots+\sqrt{2n-1}}}$. Then $\sqrt{3+\sqrt{5+\cdots+\sqrt{2n-1}}} = \sqrt{3}\sqrt{1+\sqrt{5/9+\cdots+}}$ $\sqrt{3}a_{n-1} < \sqrt{3}\phi < 3$, since all fractions appearing are subunitary. This yields $x_n < \sqrt{1+3} =$

2 (in fact a stronger bound is found, and if we start later, e.g. $\sqrt{5+\sqrt{7+\cdots+\sqrt{2n-1}}} = \sqrt{5}\sqrt{1+\sqrt{7/25+\cdots+\sqrt{(2n-1)/5^{n-2}}}} < \sqrt{5}a_{n-2} < \sqrt{15}$, we will get better and better bounds).

$\square$ $(1),(4,7,10),(13,16,19,22,25),(28,31,34,37,40,43,46)$....... in the above sequence of brackets 2nd, 4th, 6th brackets are removed. in the new sequence of brackets formed (a) does 2011 appear in some bracket? if so in which bracket it appears. (b) Find the sum of the numbers in n th bracket.

Solution

The general term for all nos. appearing is $3k+1$. So 2011 appears in some bracket. The first term in the k^{th} bracket is $3(k-1)^2+1$ So let $3(k-1)^2+1 \leq 2011 < 3k^2+1 \Rightarrow (k-1)^2 \leq 670 < k^2 \Rightarrow k = 26$ So 2011 appears in the 26^{th} bracket.

Also k^{th} bracket has $2k-1$ numbers. So nos. in k^{th} bracket form an A.P. with $a = 3(k-1)^2+1$ and $d = 3$ and $n = 2k-1$ so $l = a+(n-1)d = 3k^2-6k+3+1+6k-3 = 3k^2+1$ $a+l = 6k^2-6k+5$

So sum $= \frac{n}{2}[a+l] = \frac{2k-1}{6k^2-6k+5} = \frac{12k^3-18k^2+16k-5}{2}$

$\square$ Find the value of $S_n = \arctan\frac{1}{2}+\arctan\frac{1}{8}+\arctan\frac{1}{18}+\cdots+\arctan\frac{1}{2n^2}$. Also find $\lim_{n\to\infty} S_n$.

Solution

here $T_r = tan^{-1}\frac{1}{2r^2} = tan^{-1}(2r+1) - tan^{-1}(2r-1)$

so telescopic series

$S_n = tan^{-1}(2n+1) - tan^{-1}(1)$

$\lim_{n\to\infty} S_n = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$\square$ What is n that makes the following numbers integers: $\frac{n+1}{5}, \frac{n+2}{7}, \frac{n+3}{9}$

Solution

Answer $= n = 159+315t, t \in \mathbb{N} \cup t = 0$

Condition 1. $\frac{n+1}{5}$ being an integer implies that $n \equiv 4 \pmod 5$. Condition 2. $\frac{n+2}{7}$ being an integer implies that $n \equiv 5 \pmod 7$. Condition 3. $\frac{n+3}{9}$ being an integer implies that $n \equiv 6 \pmod 9$.

Let us combine the first 2 conditions. Since $n \equiv 5 \pmod 7$, write $n = 5+7k$ for some integer k. Then: $5+7k \equiv 4 \pmod 5$ $2k \equiv 4 \pmod 5$ $k \equiv 2 \pmod 5$ $gcd(5,7) = 1$. By Chinese Remainder Theorem, we should have solution in $\mod 35$. $7(2)+5 = 19$. The solution is hence $\equiv 19 \pmod{35}$.

Now let us combine this with the last condition. In a similar manner, we have n = 19 + 35m for some integer m. Then: $19+35k \equiv 6 \pmod 9$ $8k \equiv -13 \pmod 9$ $8k \equiv 32 \pmod 9$ $k \equiv 4 \pmod 9$ $gcd(35,9) = 1$. Again, by Chinese Remainder Theorem, we should have solution in $\mod 315$. $35(4)+19 = 159$. The solution is hence $\equiv 159 \pmod{315}$.

Hence the solution is $n = 159+315t, t \in \mathbb{N} \cup t = 0$. QED.

$\square$ If f a two times differentiable function and $f(0) = f(2) = 0$ then prove that there is at least one $x_0 \in (0,2)$ such as $|f''(x_0)| \geq |f(1)|$

Solution

By mean value theorem, there exists $c \in (0,1)$ such that $f'(c) = f(1)$. Similarly there exists $d \in (1,2)$ such that $f'(d) = -f(1)$. Then there exists $x_0 \in (c,d)$ such that $f''(x_0) = \frac{-2f(1)}{d-c}$. $d-c < 2$, so the result follows.

$\square$ Let m and n be positive integers. Suppose that $\gcd(11k-1,m) = \gcd(11k-1,n)$ holds for all $k \in \mathbb{N}$. Assume that $m > n$. Prove that $\frac{m}{n}$ is a power of 11.

Solution

Let $p \neq 11$ be a prime. Then $p^\alpha \mid n$ if and only if $p^\alpha \mid m$. This follows because one can find k such that $p^\alpha \mid 11k-1$ (since 11 is inversible modulo p^α). Therefore $n = 11^\nu z$ and $m = 11^\mu z$ for some positive integer z with $11 \nmid z$. Since $m > n$, it follows $\mu > \nu$, and so $\dfrac{m}{n} = 11^{\mu-\nu}$.

$\square$ Show that
$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\le\frac{1}{\sqrt{3n}}$
without induction.

Solution

It is elementary to prove that $\dfrac{2k-1}{2k}\le\dfrac{\sqrt{3(k-1)+1}}{\sqrt{3k+1}}$ for $k\ge 1$ (in fact with strict inequality for $k>1$), since indeed, by squaring, it is equivalent to $1\le k$.

Then by telescoping $\prod_{k=1}^{n}\dfrac{2k-1}{2k}\le\prod_{k=1}^{n}\dfrac{\sqrt{3(k-1)+1}}{\sqrt{3k+1}}=\dfrac{1}{\sqrt{3n+1}}<\dfrac{1}{\sqrt{3n}}$.

$\square$ Show that the only solution to $5^x-3^y=2$ where $x,y\in\mathbb{N}$ is $x=y=1$

Solution

$5^x-3^y=2=5-3$, so $5(5^{x-1}-1)=3(3^{y-1}-1)$. Assume $x,y>1$. So $5\mid 3^{y-1}-1$, implying $y-1=4b$. Then $3\cdot 80=3(3^4-1)\mid 3(3^{4b}-1)$, therefore $48\mid 5^{x-1}-1$, implying $x-1=4a$. Then $13\cdot 48=624=5^4-1\mid 5^{4a}-1$, therefore $13\mid 3^{4b}-1$. This in turn implies $b=3c$, so $2^4\cdot5\cdot7\cdot13\cdot73=81^3-1\mid 3^{4b}-1$, hence $5\mid 5^{4a}-1$, absurd, since $x>1$ implies $4a>0$.

$\square$ Pairwise distinct real numbers a,b,c satisfies the equality

$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}.$$

Find all possible values of abc.

Solution

We have $a-b=\frac{1}{c}-\frac{1}{b}=\frac{b-c}{bc}$, $b-c=\frac{1}{a}-\frac{1}{c}=\frac{c-a}{ac}$, and $c-a=\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}$. Multiplying these equations together yields $(a-b)(b-c)(c-a)=\frac{(b-c)(c-a)(a-b)}{(abc)^2}$. Since a,b,c are pairwise distinct, $(a-b)(b-c)(c-a)\neq 0$, so $(abc)^2=1$, so $abc=\pm1$. $(a,b,c)=(1,-\frac{1}{2},-2)$ and $(a,b,c)=(-1,\frac{1}{2},2)$ yield solutions to the given equations satisfying $abc=1$ and $abc=-1$, respectively, so the set of all possible values of abc is $\{-1,1\}$.

$\square$ Let MN not perpendicular d, M lie on d. The circle ω variable touching d at M. NH, NK touches ω at H,K. Prove HK passes through a fixed point

Solution

Let (U) be the fixed circle centered at d and passing through M,N. Perpendicular d' to NM at M cuts (U) again at the fixed D. Variable circle ω cuts lines NM and d' again at P,R. Let $Q\equiv RP\cap DN$. Then the circles ω, (U) and $\odot(PNQ)$ concur at the Miquel point E of $\triangle DMN\cup\overline{PQR}$. But P is ortocenter of $\triangle DNR$, due to $\angle PRD=\angle PND\Longrightarrow N,E,R$ are collinear. Let $F\equiv EM\cap PR$. Since line pencil $N(M,R,F,D)$ is harmonic, it follows that NF is the polar of D WRT ω. Therefore, the polar HK of N WRT ω pass through the fixed point D. (thiếu hình vẽ đi kèm)

$\square$ In the triangle ABC given that $\angle ABC=120°$. The bisector of $\angle B$ meet AC at M and external bisector of $\angle BCA$ meet AB at P. Segments MP and BC intersects at K. Prove that $\angle AKM=\angle KPC$.

Solution

Lemma. In $\triangle ABC$, internal angle bisector of $\angle ABC$ and external angle bisectors of $\angle BCA$ and $\angle CAB$ are concurrent. Proof. It suffices to prove that B-excenter I_B lies on the angle bisector of $\angle B$. Let X and Y be the projections of I_B onto BA and BC, respectively. In $\triangle BXI_B$ and $\triangle BYI_B$, $BX=BY$ and $I_BX=I_BY$, implying that $\triangle BXI_B\cong\triangle BYI_B$. Hence, I_B lies on the angle bisector of $\angle B$, and the lemma is proven. $\square$

According to above lemma, P is the M-excenter of $\triangle MBC$, and therefore K lies on the angle bisector of $\angle BMC$.

By the lemma again, since K is A-excenter of $\triangle ABM$, we have that K lies on the angle bisector of $\angle BAC$.

From here on, easy angle chasing shows that $\angle AKM = \angle KPC = 30°$. We are done. ■

□ The base of pyramid is an equilateral triangle of side 'a'. The lateral sides are 'b' each. Find the largest volume of the sphere that can be inscribed in this pyramid.

Solution

Let I be the incenter of the inscribed sphere, $\triangle ABC$ be equilateral with side length a, and D be the apex of the pyramid such that $AD = BD = CD = b$. Then by symmetry, the sphere's point of tangency D' in plane ABC is the center of the equilateral triangle, and D, I, D' are collinear.

Now consider the plane $DD'A$, which intersects BC at M. Then again by symmetry, we must have the sphere's point of tangency A' be in this plane, and furthermore, D, A', M are collinear. Thus it suffices to consider right $\triangle DMD'$, in which a semicircle with center I is inscribed. We now proceed to find the side lengths of this triangle.

First, it is easy to see that $D'M = \frac{a}{2\sqrt{3}}$, since $\triangle D'MB$ is 30-60-90 with right angle at M. Next, by the Pythagorean theorem, $DM^2 = DB^2 - MB^2 = b^2 - (a/2)^2$, so $DM = \sqrt{b^2 - (a/2)^2}$. Hence $(DD')^2 = DM^2 - (D'M)^2 = b^2 - \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2\sqrt{3}}\right)^2 = b^2 - \frac{a^2}{3}$.

Now we observe that $ID' = IA' = r$, and so $ID = DD' - ID' = \sqrt{b^2 - a^2/3} - r$. Since $\triangle IA'D \sim \triangle MD'D$, it follows that $\frac{MD}{MD'} = \frac{ID}{IA'}$, or

$$\frac{\sqrt{b^2 - a^2/4}}{a/\sqrt{12}} = \frac{\sqrt{b^2 - a^2/3} - r}{r}.$$

Solving this equation for r yields

$$r = \frac{\sqrt{3(4k-1)} - 1}{4\sqrt{3(3k-1)}}a,$$

where $k = (b/a)^2 > 1/3$, since $a > b\sqrt{3} > 0$ for the pyramid to be non-degenerate.

□ n (≥ 2) is a natural number. Show that $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}$

Solution

let $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, then $\omega, \omega^2, \cdots, \omega^{n-1}$ are the roots of $x^{n-1} + x^{n-2} + \cdots + x + 1 = 0$

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = (x - \omega)(x - \omega^2) \cdots (x - \omega^{n-1})$$

Plugging $x = 1$

$$n = (1 - \omega)(1 - \omega^2) \cdots (1 - \omega^{n-1}) \Rightarrow n = |1 - \omega||1 - \omega^2| \cdots |1 - \omega^{n-1}|$$

and we have

$$|1 - \omega^k| = |1 - \cos \frac{2k\pi}{n} - i \sin \frac{2k\pi}{n}| = 2 \sin \frac{k\pi}{n} (0 \leq k \leq n-1)$$

so

$$2^{n-1} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = n \Rightarrow \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}$$

□ In a triangle ABC, bisector of $\angle BAC$, AU is drawn. From B and C perpendiculars BE and CF on AU are drawn. If AD is the altitude of $\triangle ABC$, then prove that $AE \cdot AF \geq AD^2$

Solution

Thiếu hình vẽ [hide="Diagram"](http://oi51.tinypic.com/21dk7zl.jpg)[img]http://oi51.tinypic.com/21dk7zl.j

Let M be the midpoint of EF and G be the point such that $EBFG$ is a parallelogram.

If $AB = AC$ then U, E, F, D coincide and equality occurs. Now suppose $AB > AC$, as in the diagram. Then $\angle C > \angle B$.

Suppose for a contradiction that $90° - \frac{A}{2} \geq C$.

Then $90° + \frac{C}{2} > 90° + \frac{B}{2} \geq \left(C + \frac{A}{2}\right) + \frac{B}{2} = 90° + \frac{C}{2}$, contradiction.

Then $90° - \frac{A}{2} \leq C$ which means $\angle ACE < C \implies E \in (AU)$. In a similar fashion we can prove that F lies on AU extended beyond U.

Now by the angle bisector theorem and similar triangles,

$$\frac{AF - AU}{AU - AE} = \frac{BU}{CU} = \frac{AB}{AC} = \frac{AF}{AE}$$

So $AE(AF - AU) = AF(AU - AE) \implies AE \cdot AF = AU \cdot \frac{AE+AF}{2}$. So the inequality is equivalent to proving $AU \cdot \frac{AE+AF}{2} \geq AD^2$.

In fact we of course have $AU \geq AD$ so it suffices to prove $\frac{1}{2}(\overrightarrow{AE} + \overrightarrow{AF}) \geq \overrightarrow{AU} \iff \overrightarrow{AM} \geq \overrightarrow{AU} \iff M \in (UF)$.

Since M is the midpoint of the diagonal EF of the parallelogram BG, it is also the midpoint of diagonal BG. It is easy to prove $CE > BF$, since for example $CE = AC \sin \frac{A}{2} < AB \sin \frac{A}{2} = BF$. Then $EG = BF > EC$, so G lies on EC extended beyond C which means that the intersection of BG with the line AU is further down that the intersection with BC. The inequality follows.

□ $x, y, z \geq 0;\ x + y + z = 4$ Find the minimum value of $P = \sqrt{2x+1} + \sqrt{3y+1} + \sqrt{4z+1}$

Solution

Use this lemma:

$$\begin{aligned} &a \geq 0, b \geq 0 \\ &\sqrt{a+1} + \sqrt{b+1} \geq \sqrt{a+b+1} + 1 \end{aligned}$$

The proof of this is easy Then we have

$$\begin{aligned} &\sqrt{2x+1} + \sqrt{3y+1} + \sqrt{4z+1} \geq \sqrt{2x+3y+1} + 1 + \sqrt{4z+1} \\ &\geq \sqrt{2x+3y+4z+1} + 2 \geq \sqrt{2(x+y+z)+1} + 2 = 5 \end{aligned}$$

equality is held when $x = 4, y = z = 0$

□ For the 3 positive real numbers a, b, c satisfy $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 16$, find the maximum and minimum value of $\frac{a^2+2b^2}{ab}$

Solution

By AM-GM,$\frac{a^2+2b^2}{ab} \geq 2\sqrt{2}$ Equality is held when $a = \sqrt{2}, b = 1, c = \frac{29-16\sqrt{2}+\sqrt{1353-932\sqrt{2}}}{2}$ On the other hand,By Cauchy,$16 = (a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \geq [\sqrt{(a+b)(\frac{1}{a} + \frac{1}{b})} + 1]^2$ So $(a+b)(\frac{1}{a} + \frac{1}{b}) \leq 9$ Let,$\frac{b}{a} = x$,x>=1,or we can change a and b $x + \frac{1}{x} \leq 7x \leq \frac{7+3\sqrt{5}}{2}$ $f(x) = 2x + \frac{1}{x} \leq \max\left\{f(1), f(\frac{7+3\sqrt{5}}{2})\right\} = \frac{21+3\sqrt{5}}{2}$ Equality is held when $a = 1, b = \frac{7+3\sqrt{5}}{2}, c = \frac{3+\sqrt{5}}{2}$

□ Find all positive integer solutions of equation $n^3 - 2 = k!$.

Solution

For $k > 3$ we have $4 \mid k!$, so $4 \mid n^3 - 2$, hence $2 \mid n^3$, therefore $8 \mid n^3$, leading to $4 \mid n^3 - k! = 2$, absurd. Thus the only solution is $2^3 - 2 = 3!$.

□ Given $\triangle ABC$, find the location of P such that its pedal triangle is an isosceles right-angled triangle.

Solution

Let P_1, P_2, P_3 be the orthogonal projections of P on the sidelines BC, CA, AB. Assume that $\triangle P_1P_2P_3$ is isosceles right with apex P_1. By generalized Simson theorem, $\triangle P_1P_2P_3$ is similar to the triangle

$\triangle A'B'C'$ formed by the inverses A', B', C' of A, B, C under any inversion with center P and arbitrary power k^2. Thus, $P_1P_2 = P_1P_3 \iff A'B' = A'C'$. By inversion properties, we get
$\frac{A'B'}{AB} = \frac{k^2}{PA \cdot PB}$, $\frac{A'C'}{AC} = \frac{k^2}{PA \cdot PC} \Longrightarrow \frac{AB}{AC} = \frac{PB}{PC}$

Hence, P lies on the A-Apollonian circle of $\triangle ABC$. On the other hand, we have that $\angle P_2P_1P_3 = 90°$. Thus, from the cyclic quadrilaterals PP_1BP_3 and PP_1CP_2 we deduce that $\angle PBA + \angle PCA = 90°$ $\Longrightarrow \angle BPC = 90° + \angle BAC \pmod{\pi}$. In other words, if the perpendicular to AC through C cuts AB at D, then P lies on the circle $\odot(BCD)$. Therefore, A-Apollonian circle of $\triangle ABC$ and $\odot(BCD)$ intersect at two points whose pedal triangles are isosceles right with apex on BC. Repeating the same construction for CA, AB yields at most 6 distinct points whose pedal triangles with respect to $\triangle ABC$ are isosceles right.

$\square$ Proof (without the use of pigeonhole principle) that a simple graph has at least two vertices of the same degree. Is this possible?

Give a counter example to show that the result is not true for a graph which is not a simple graph.

Solution

The pigeonhole principle is such a basic one, that it is likely that any proof will contain a hidden equivalent of it. Typically, if all degrees are distinct, and since any degree d obeys $0 \le d \le |G| - 1$ in a simple graph, it means the set of the values of the degrees is $\{0, 1, \ldots, |G| - 1\}$. But the vertex having degree $|G| - 1$ is therefore connected to all other, in contradiction with the fact that one of the vertices had degree 0, unless $|G| = 1$, where the only vertex has degree 0, and there are not enough vertices to have a degree equality.

The minimal counterexample for a not-simple graph is $|G| = 2$, with the only edge a loop.

$\square$ $\boxed{\frac{m}{1+m+mn} + \frac{n}{1+n+np} + \frac{p}{1+p+pm} + \frac{(mnp-1)^2}{(1+m+mn)(1+n+np)(1+p+pm)} = 1}$.

Remark. $\boxed{\{m, n, p\} \subset \mathbb{R}_+^* \implies \frac{m}{1+m+mn} + \frac{n}{1+n+np} + \frac{p}{1+p+pm} \le 1}$ with equality iff $mnp = 1$.

A geometrical interpretation. Let $\triangle ABC$ with the area $S = [ABC] = 1$. For the points $M \in (BC)$, $N \in (CA)$, $P \in (AB)$

define $X \in BN \cap CP$, $Y \in CP \cap AM$, $Z \in AM \cap BN$. Denote $\frac{MB}{MC} = m$, $\frac{NC}{NA} = n$, $\frac{PA}{PB} = p$. Observe that

$[ABZ] + [BCX] + [CAY] + [XYZ] = 1$ and prove easily that $[ABZ] = \frac{m}{1+m+mn}$, $[BCX] = \frac{n}{1+n+np}$,

$[CAY] = \frac{p}{1+p+pm}$. Then the area of the triangle XYZ is $[XYZ] = \frac{(1-mnp)^2}{(1+m+mn)(1+n+np)(1+p+pm)}$.

Particular case. $m = n = p \implies [XYZ] = \frac{(m-1)^2}{m^2+m+1} \cdot [ABC]$.

$\square$ A 3-digit number is divisible by 11 , and the quotient is the sum of all digits' square . Find the 3-digit number.

Solution

Let the number be $100a + 10b + c$

Now a+c-b=0 ,11

Let $100a + 10b + c = 11(a^2 + b^2 + c^2)$

Case 1: $b = a + c \implies 10a + c = 2(a^2 + ac + c^2) \implies c = 2c_1$

$\implies 5a + c_1 = a^2 + 2ac_1 + 4c_1^2 \implies c_1 =$ Even

$c_1 = 0 \implies a = 5, b = 5;\ c_1 = 2, 4 \implies$ No solution

$\implies$ required number is 550

$550 = 11 \times 50 = 11 \times (5^2 + 5^2 + 0^2)$

Case 2: $b = a + c - 11 \implies c \geq 2$ and $10a + c - 10 = (2a^2 + 2ac + 2c^2 + 121 - 22a - 22c)$

$\implies 2a^2 + 2ac + 2c^2 + 131 - 32a - 23c = 0$

$\implies c = \text{odd}$

$c = 3 \implies a^2 - 13a + 40 = 0 \implies a = 8, 5$

$\implies a = 8,\ b = 0$

$\implies$ required number is 803

$803 = 11 \times 73 = 11 \times (8^2 + 0^2 + 3^2)$

$c = 5, 7, 9 \implies$ No solution

Hence Required Number is $550, 803$

□ Let p is a prime number. Prove that $p^{p+1} + (p+1)^p$ is not a perfect square.

Solution

Of-course $p > 2$,so odd,let $p = 2k+1$ and then $p \not| a$ $(p+1)^p = a^2 - p^{p+1} = (a + p^{k+1})(a - p^{k+1})$ $gcd(a+p^{k+1}, a-p^{k+1}) = gcd(a-p^{k+1}, 2a) = gcd(a-p^{k+1}, 2) = 2$ So let $a-p^{k+1} = 2x^m, a+p^{k+1} = 2y^m$ with $gcd(x,y) = 1$ Then $y^m - x^m = p^{k+1}$ But from Tricky lemma,$m = p$ and $k = 1$.Which gives us $p = 3$ yielding $y^3 - x^3 = 9$ which has no solution.

□ $0 \leq \alpha < \beta < \gamma \leq 2\pi$ $\sin\alpha + \sin\beta + \sin\gamma = 0$ $\cos\alpha + \cos\beta + \cos\gamma = 0$ Find the value of $\beta - \alpha$

Solution

We have:

$\sin\alpha + \sin\beta = -\sin\gamma$ and $\cos\alpha + \cos\beta = -\cos\gamma$. This implies that:

$\sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta + \cos^2\alpha + 2\cos\alpha\cos\beta + \cos^2\beta = 1$

$\cos(\beta - \alpha) = -\frac{1}{2}$

Since α and β lie in [0,2pi] and $\beta > \alpha$, $\beta - \alpha$ must also be on [0,2pi]. We have two possibilities: $\beta - \alpha = \frac{2\pi}{3}$ and $\beta - \alpha = \frac{4\pi}{3}$

Substituting the first solution in the sine equation, we get $\sin(\alpha + \frac{\pi}{3}) = \sin -\gamma$, which implies that $\gamma = -\alpha - \frac{\pi}{3}$ (absurd, since for positive alpha, gamma would be negative) and $\gamma = \alpha - \frac{2\pi}{3}$ (also absurd, because gamma is larger than alpha).

Thus, the only solution is $\frac{4\pi}{3}$

□ n is a natural number, where $n > 1$ Find the value of n satisfying $\frac{3^n+1}{n^2} \in N$

Solution

Lemma: If $n > 1$ odd,$n \not| 3^n + 1$ Proof: Let p be the smallest prime factor of n Then $3^{2n} \equiv 1 \mod p, 3^{p-1} \equiv 1 \implies 3^{gcd(p-1,2n)} \equiv 1 \mod p$ Since $p - 1 < p$ and even,$gcd(p-1, 2n) = 2$ and therefore $p | 3^2 - 1$,contradiction. So $n = 1$

Now back to original problem,n even so $2||3^n + 1 \equiv 2 \mod 2$ but $4|n^2|3^n + 1$,contradiction. Thus $n = 1$ is the only solution.

□ n is a natural number. Show that $1 \cdot 3 \cdot 5 \cdots (2n-1) < 2n^{n-1}$ with no use of induction.

Solution

By AM $\geq$ GM $\implies \frac{2r+1+(2n-(2r+1))}{2} \geq \sqrt{(2r+1)(2n-(2r+1))}$

$\implies (2r+1)(2n-(2r+1)) \leq n^2$

$\implies \prod_{r=1}^{n-2}(2r+1)(2n-(2r+1)) \leq \prod_{r=1}^{n-2} n^2$

$\implies \prod_{r=1}^{n-2}(2r+1)\prod_{r=1}^{n-2}(2n-(2r+1)) \leq (n^2)^{n-2}$

$\implies \prod_{r=1}^{n-2}(2r+1)\prod_{r=1}^{n-2}(2r+1) \leq n^{2n-4}$

$\implies \prod_{r=1}^{n-2}(2r+1) \leq n^{n-2}$

Now $2n-1 < 2n \implies 1\cdot 3\cdot 5 \cdots (2n-1) < 2n^{n-1}$

$\square$ Let l be a line bisecting both of perimeter, area of triangle ABC Let O, H, I, G be its circumcenter, orthocenter, incenter, centroid. Does l pass through one of O, H, I, G certainly ?

Solution

If ℓ cuts AB, AC at M, N, then $AM+AN = BM+CN+BC$ (1). Let (I, r) be the incircle of $\triangle ABC$ and without loss of generality assume that I is inside $\triangle AMN$. From $[\triangle AMN] = [\square BMNC]$, we get

$[\triangle IAM] + [\triangle IAN] + [\triangle MIN] = [\triangle IBM] + [\triangle ICN] + [\triangle IBC] - [\triangle MIN]$

$2[\triangle MIN] + \frac{1}{2}r(AM+AN) = \frac{1}{2}r(BM+CN+BC)$ (2)

From (1) and (2), it follows that $[\triangle MIN] = 0 \implies I \in \ell$.

$\square$

a, b, c, d are natural numbers, where $a<b<c<d$ Show that there don't exist a, b, c, d between two consecutive perfect square numbers such that $ad = bc$

Solution

Let $(a,b) = x$ and $\frac{a}{x} = y$ and $\frac{b}{x} = w$, so that $(w,y) = 1$. The equation becomes $yd = wc$. So w must be a factor of the left hand side but it is relatively prime with y. Thus $d = zw$ for some z, and the equation finally becomes $yzw = wc$ or $c = yz$. Thus we can find integers w, x, y, z so that $a = xy, b = wx, c = yz, d = zw$.

Now because $d > b, c$ we know that $z > x$ and $w > y$. Because they're integers, we know that $z \geq x+1$ and $w \geq y+1$.

Let $a = k^2 + m$ where $0 \geq m < 2k+1$. Then $k^2 \leq xy\left(\frac{x-y}{2}\right)^2 + xy = \left(\frac{x+y}{2}\right)^2$ whence $x+y \geq 2k$. Adding this to $xy+1 \geq k^2+1$, we get $(x+1)(y+1) \geq (k+1)^2$. But $d = zw \geq (x+1)(y+1) \geq (k+1)^2$, so $d \geq (k+1)^2 > a$ which is a contradiction.

$\square$ Prove that : $\tan\alpha + \tan\beta \geq 2tan\sqrt{\alpha\beta}$ for each $\alpha, \beta \in [0, \frac{\pi}{2}]$

Solution

It is noticed that $f(x) = \tan(x)$ is convex on $x \in \left[0, \frac{\pi}{2}\right]$. Hence, by Jensen inequality, we have

$$\tan\alpha + \tan\beta \geq 2\tan\left(\frac{\alpha+\beta}{2}\right)$$

The last line is just an observation that $\tan(x)$ is increasing and $\alpha+\beta \geq 2\sqrt{\alpha\beta}$ So we are done for now.

$\square$ Let n is a positive integer. Prove that: $\left\lfloor \sqrt{n-\frac{3}{4}} + \frac{1}{2} \right\rfloor + \lfloor \sqrt{n-1} \rfloor = \lfloor \sqrt{4n-3} \rfloor$

Solution

Let x be the unique integer such that $x^2 - x + 1 \leq n < x^2 + x + 1$ $[\sqrt{n-\frac{3}{4}} + \frac{1}{2}] + [\sqrt{n-1}] = [\frac{\sqrt{4x^2-4x+1}}{2} + \frac{1}{2}] + [\sqrt{x^2-x}] = x + x - 1 = 2x-1$ And $[\sqrt{4n-3}] = [\sqrt{4x^2-4x+1}] = 2x-1$

$\square$ Let $n \in \mathbb{N}$. Prove $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \frac{3}{4}$.

Solution

Denote $A_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$ and apply the Chebyshev's inequality for two decreasing sentencies :

$A_n \cdot \left(A_n + \frac{1}{2n}\right) = A_n \cdot \left(A_n + \frac{1}{n} - \frac{1}{2n}\right) = \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) \cdot \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}\right) <$
$n \cdot \left[\frac{1}{n\cdot(n+1)} + \frac{1}{(n+1)\cdot(n+2)} + \cdots \frac{1}{(2n-1)\cdot 2n}\right] = n \cdot \sum_{k=0}^{n-1}\left(\frac{1}{n+k} - \frac{1}{n+k+1}\right) = \frac{1}{2}$.

In conclusion, $A_n \cdot \left(A_n + \frac{1}{2n}\right) < \frac{1}{2} \implies 2n \cdot A_n^2 + A_n - n < 0 \iff \boxed{A_n < \frac{-1+\sqrt{8n^2+1}}{4n}}$.

Remark. $A_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \frac{-1+\sqrt{8n^2+1}}{4n} < \frac{3}{4} - \frac{1}{4n} < \frac{3}{4}$.

□ Given an A- right triangle ABC with $b \le c$, where h_a , w_a , m_a are its altitude, bisector and median from vertex A respectively. Calculate $\lim_{b\to c} \frac{m_a - h_a}{w_a - h_a}$.

Solution

From the relations $2m_a = a$, $ah_a = bc$ and $w_a = \frac{2bc\cdot\cos\frac{A}{2}}{b+c} = \frac{bc\sqrt{2}}{b+c}$ obtain that $\frac{m_a-h_a}{w_a-h_a} = \frac{2am_a-2ah_a}{2aw_a-2ah_a} = \frac{a^2-2bc}{\frac{2abc\sqrt{2}}{b+c}-2bc} =$

$$\frac{(b^2+c^2-2bc)(b+c)}{2bc\cdot\left[\sqrt{2(b^2+c^2)}-(b+c)\right]} = \frac{(b-c)^2(b+c)\left[\sqrt{2(b^2+c^2)}+(b+c)\right]}{2bc(b-c)^2} \overset{(b\neq c)}{=} \frac{(b+c)\left[\sqrt{2(b^2+c^2)}+(b+c)\right]}{2bc} \implies$$

$\lim_{b\to c} \frac{m_a-h_a}{w_a-h_a} \overset{(t=\frac{b}{c})}{=} \lim_{t\nearrow 1} \frac{(t+1)\left[\sqrt{2(t^2+1)}+(t+1)\right]}{2t} = 4$.

□ Given a natural number n, such that $2n+1$ and $3n+1$ are both squares. Can $5n+3$ be a prime?

Solution

Take $2n+1 = a^2$, $3n+1 = b^2$. Then $1 = 3a^2 - 2b^2$, so $5n+3 = 1+a^2+b^2 = 4a^2-b^2 = (2a-b)(2a+b)$. So we need $2a-b = 1$, but then $4(a-1)^2+(2b-1)^2 = 1$. However, this has as only solution $a = b = 1$, and so $n = 0$, with $5n+3 = 3$ a prime. In most countries, 0 is a natural number, so it is the only solution.

□ Let ABC be a triangle with semiperimeter s , circumradius R and inradius r for which denote $Q = \sum_{\text{cyc}} \cos\frac{A}{2}$. Prove that : $\boxed{s = 2Q\cdot\left(\sqrt{(RQ)^2 - Rr} - 2R\right)}$.

Solution

Using the identities : $\prod_{cyc} cos\frac{\hat{A}}{2} = \frac{s}{4R}$ $\prod_{cyc} sin\frac{\hat{A}}{2} = \frac{r}{4R}$ It is equivalent to show that: $2\prod_{cyc} cos\frac{\hat{A}}{2} = (\sum_{cyc} cos\frac{\hat{A}}{2})(\sqrt{(\sum_{cyc} cos\frac{\hat{A}}{2})^2 - 4\prod_{cyc} sin\frac{\hat{A}}{2}}-2)$ We use the substitution : $\begin{cases} \hat{X} = \frac{\pi-\hat{A}}{2} \\ \hat{Y} = \frac{\pi-\hat{B}}{2} \\ \hat{Z} = \frac{\pi-\hat{C}}{2} \end{cases}$ So it is equivalent to show that : $2\prod_{cyc} sin\hat{X} = (\sum_{cyc} sin\hat{X})(\sqrt{(\sum_{cyc} sin\hat{X})^2 - 4\prod_{cyc} cos\hat{X}} - 2)$ Which is true because : $\prod_{cyc} sin\hat{X} = \frac{s'r'}{2R'^2}$ $\sum_{cyc} sin\hat{X} = \frac{s'}{R'}$ $\prod_{cyc} cos\hat{X} = \frac{s'^2-(2R'+r')^2}{4R'^2}$ where s' , R' , r' are the semi-perimeter, the circumradius and the inradius of $\triangle XYZ$ respectively.

□ For any non - empty set X of numbers, denote by a_X the sum of the largest and the smallest elements in X. What is the average value of a_X if X ranges over all non - empty subsets of $\{1, 2, \ldots\ldots\ldots\ldots\ldots\ldots 1000\}$?

Solution

Let $P_n = \{1,\ 2,\ 3,\ \cdots,\ n\}$

Let a_n be the required average over all non empty subsets of P_n

We can easily derive that $(2^{n+1}-1)a_{n+1} = (2^n-1)a_n + 2^n(n+3) - 1$

Now by telescopic sum from $(2^{n+1}-1)a_{n+1} - (2^n-1)a_n = 2^n(n+3) - 1 \implies (2^n-1)a_n = (n+1)(2^n-1)$

$\implies \boxed{\boxed{a_n = n+1}}$

For the given problem $n = 1000$ gives average value 1001

□ Show that for every positive integer $n \ge 4$:

$$lcm(1, 3, \ldots, 2n-1) > (2n+1)^2$$

Solution

We have

$$[1,3,\cdots,2n-1] > [2n-1,2n-3,2n-5] = (2n-1)(2n-3)(2n-5) > (2n+1)^2$$

as $2n-1, 2n-3, 2n-5$ are pairwise coprime.

□ Find all functions, $f(x)$, if they exist ($f:\mathbb{R}\to\mathbb{R}$), such that $f(f(x))+xf(x)=1\forall x$ that is an element of $\mathbb{R}$.

Solution

Let $P(x)$ denote the statement $f(f(x))+xf(x)=1$.

By $P(0)$ we have $f(f(0))=1$. By $P(f(0))$ we have $f(1)+f(0)=1$.

By $P(1)$ we have $f(f(1))=1-f(1)$. By $P(f(1))$ we have $f(f(f(1)))+f(1)f(f(1))=1$. Plugging in the above we have $f(1-f(1))+f(1)-f(1)^2=1$. Since $f(1)+f(0)=1$, $f(1-f(1))=f(f(0))=1$, so $f(1)=f(1)^2$. Thus $f(1)$ must be either 0 or 1.

If $f(1)=0$ then $f(0)=1$, but then $f(f(0))=f(1)=0$. If $f(1)=1$ then $f(0)=0$, but then $f(f(0))=f(0)=0$.

Thus no such functions exist.

□ For all natural numbers n (>1) , show that $\left(\frac{1+(n+1)^{n+1}}{n+2}\right)^{n-1} > \left(\frac{1+n^n}{n+1}\right)^n$

Solution

It can probably be done by induction, but I'll leave that to more pro inductors.

$\left(\frac{1+(n+1)^{n+1}}{n+2}\right)^{n-1} > \left(\frac{1+n^n}{n+1}\right)^n \iff \left(\frac{1+(n+1)^{n+1}}{(n+1)+1}\right)^{\frac{1}{(n+1)-1}} > \left(\frac{1+n^n}{n+1}\right)^{\frac{1}{n-1}}$.

Let $f(x)=\left(\frac{1+x^x}{x+1}\right)^{\frac{1}{x-1}}$. Then $f'(x)=\left(\frac{1}{x-1}\right)\left(\frac{1+x^x}{x+1}\right)^{\frac{2-x}{x-1}}\left(\frac{(x+1)(\ln x+1)x^x-(1+x^x)}{(x+1)^2}\right)$. The only factor that is not immediately obviously positive for $\forall x>1$ is $(x+1)(\ln x+1)x^x-(1+x^x)$. It must be shown that $(x-1)(\ln x+1)x^x > 1+x^x$ for $\forall x>1 \iff ((x+1)\ln x+x)x^x>1$, which is obvious.

Since $f'(x)>0$ for $\forall x>1$, $f(x)$ is increasing in that domain, which implies the given result from the stronger result: For $\forall x,y\in R$ such that $x>y>1$, it is true that $f(x)>f(y)$.

□ $x\geq 1$ Which of $\sqrt{[\sqrt{x}]}$ and $[\sqrt{\sqrt{x}}]$ is greater ?

Solution

Let $x=(a^2+b+c)^2$ for natural a, whole b and real c such that $0\leq b\leq 2a, 0\leq c<1$. Then, $\sqrt{[\sqrt{x}]}\geq\left[\sqrt{\sqrt{x}}\right]$ with equality only if $b=0$.

□ Prove that for each $n\in\mathbb{N}$, $(n!)!$ is multiple of $n!^{(n-1)!}$

Solution

Remember that $\dfrac{(a_1+a_2+\cdots+a_k)!}{a_1!a_2!\cdots a_k!}$, a multinomial coefficient, is a positive integer. So $\dfrac{(ka)!}{(a!)^k}$ is a positive integer. In our case, $k=(n-1)!$, and $a=n$, so $(ka)!=((n-1)!n)!=(n!)!$, and all falls into place.

□ Let $F(x),P(x),Q(x),R(x),S(x)$ are polynomial,with

$$F(x)=x^4+x^3+x^2+x+1$$

and

$$P(x^5)+xQ(x^5)+x^2R(x^5)=F(x)S(x)$$

prove that: $(x-1)$ is a common factor of $P(x),Q(x),R(x),S(x)$

Solution

Let $\omega=e^{2\pi i/5}$. Then, plugging in $x=1$ we have $P(1)+Q(1)+R(1)=5S(1)$. In addition, plugging in $x = \omega^k$ for $k = 1, 2, 3, 4$ yields $P(1) + \omega^k Q(1) + \omega^{2k} R(1) = 0$. Therefore,

$$\sum_{k=0}^{4} \left(P(1) + \omega^k Q(1) + \omega^{2k} R(1)\right) = 5P(1) = 5S(1),$$

so $P(1) = S(1)$. Similarly,

$$\sum_{k=0}^{4} \omega^{-k} \left(P(1) + \omega^k Q(1) + \omega^{2k} R(1)\right) = 5Q(1) = 5S(1),$$

so $Q(1) = S(1)$. Finally,

$$\sum_{k=0}^{4} \omega^{-2k} \left(P(1) + \omega^k Q(1) + \omega^{2k} R(1)\right) = 5R(1) = 5S(1),$$

so $R(1) = S(1)$. Therefore,

$$P(1) + Q(1) + R(1) = 3S(1) = 5S(1),$$

so $S(1) = P(1) = Q(1) = R(1) = 0$, so $x - 1$ is a common factor of P, Q, R, S.

□ Show that if the equation $x^4 + ax^3 + 2x^2 + bx + 1 = 0$ has at least a real root, then $a^2 + b^2 \geq 8$.

Solution

Consider the equation $x^3 \cdot a + x \cdot b + \left(x^2+1\right)^2 = 0$ of the line d in the analytical coordinate system aOb , where $x \in \mathbb{R}$. The distance $\sqrt{a^2+b^2}$ from the origin O to $M(a, b) \in d$ is equally to the distance $\delta = \frac{\left(x^2+1\right)^2}{\sqrt{x^2(x^4+1)}}$ from the origin to the line d . Thus, $\sqrt{a^2+b^2} = \delta \geq 2\sqrt{2}$. In conclusion, $a^2 + b^2 \geq 8$
Another way: Let u be a root to the equation. Clearly $u \neq 0$. Then $u^2 + au + 2 + \frac{b}{u} + \frac{1}{u^2} = 0$. Rewrite this as $(u + \frac{a}{2})^2 + (\frac{1}{u} + \frac{b}{2})^2 + 2 = \frac{a^2+b^2}{4}$ and we are done.

□ A soccer ball is tiled of hexagons (6-gons) and pentagons (5-gons). Each pentagon is surrounded by 5 hexagons, i.e., each edge of a pentagon is an edge of an hexagon. Each hexagon is surrounded by 3 hexagons and 3 pentagons (alternating), i.e., for any pair of edges of an hexagon with a common vertex one edge is also an edge of another hexagon and the other edge is also an edge of a pentagon. How many pentagons and hexagons are there on the soccer ball?

Solution

Denote by f_5 the number of pentagons and by f_6 the number of hexagons. From the givens, by a little bit of double counting, we get $5f_5 = 3f_6$. Denote by v_3 the total number of vertices (we used v_3 since at each vertex meet three polygons), and denote by e the total number of edges (sides of the polygons). By double counting, $3v_3 = 2e$ and $5f_5 + 6f_6 = 2e$. Finally, use Euler's formula for a map on the sphere $v + f = e + 2$, where in our case $v = v_3$ and $f = f_5 + f_6$. Solving this system of equations yields $2e = 5f_5 + 6f_6 = 3f_6 + 6f_6 = 9f_6 = 3v_3$, so $v_3 = 3f_6$. Plugging in Euler's formula, we get $3f_6 + (3/5)f_6 + f_6 = (9/2)f_6 + 2$, whence $f_6 = 20$ and $f_5 = 12$.

□ How many subsets $\{a_1, a_2, a_3\}$ of $\{1, 2, ...14\}$ satisfy $a_2 - a_1 \geq 3$ and $a_3 - a_2 \geq 3$?

Solution

$x_1 \rightarrow a_1 \leftarrow x_2 \rightarrow a_2 \leftarrow x_2 \rightarrow a_3 \leftarrow x_2$

Now $x_1 + x_2 + x_3 + x_4 = n - 3,\ x_1 \geq 0,\ x_2 \geq 2,\ x_3 \geq 2,\ x_4 \geq 0$

Number of solution $= \binom{n-4}{3}$

for $n = 14$, Answer$= \binom{10}{3} = 120$

☐ Prove that there can not exist an odd number of different integers k_i such that $|k_1 - k_2| = |k_2 - k_3| = \cdots = |k_p - k_1|$, where k is an odd integer.

Solution

Except for $p = 2$, when $|k_1 - k_2| = |k_2 - k_1|$, such numbers k_i (being integer is irrelevant) do not exist ($p \geq 3$ being odd is irrelevant).

Assume $k_1 < k_2$; then we need $k_2 < k_3$, otherwise $k_1 = k_3$, and of course $k_2 - k_1 = k_3 - k_2$. Then, similarly, $k_3 < k_4 < \cdots < k_p$, but then $k_p - k_1 > k_p - k_{p-1}$.

Assume $k_1 > k_2$; then we need $k_2 > k_3$, otherwise $k_1 = k_3$, and of course $k_1 - k_2 = k_2 - k_3$. Then, similarly, $k_3 > k_4 > \cdots > k_p$, but then $k_1 - k_p > k_{p-1} - k_p$.

☐ For two real numbers x and y, a function $f(x)$ satisfy that $f(xf(x) + f(y)) = x^2 + y$ Find $f(x)$

Solution

Let $P(x, y)$ denote the statement $f(xf(x) + f(y)) = x^2 + y$

From $P(0, y)$ we have $f(f(y)) = y$. f is therefore bijective and its own inverse.

Now, $xf(x) = f(x)f(f(x))$ so comparing $P(x, y)$ and $P(f(x), y)$ we find $f(x)^2 + y = x^2 + y$. Thus $|f(x)| = |x|$ for all x, and particularly $f(0) = 0$. Since f has to be bijective we have $f(-x) = -f(x)$, and f is odd.

From $P(x, 0)$ we have $f(xf(x)) = x^2$, so $xf(x) = f(x^2)$.

Applying f to both sides of $P(x, y)$ we find $xf(x) + f(y) = f(x^2 + y)$, or $f(x^2) + f(y) = f(x^2 + y)$. Since x^2 can be made to take any nonnegative value and f is odd, this is effectively Cauchy's functional equation.

Now $f(x) = x$ and $f(x) = -x$ are both solutions. If there are other solutions, that is $f(a) = a$ and $f(b) = -b$ for some $a, b \neq 0$, then $f(a + b) = a - b$, which can't be true from $|f(x)| = |x|$.

☐ Prove that in any triangle ABC exists the identity $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4 \cdot \sin\frac{\pi+A}{4} \cdot \sin\frac{\pi+B}{4} \cdot \sin\frac{\pi+C}{4}$.

Solution

We can obtain the proposed identity by the substitutions $\left\{\begin{array}{c} x := \frac{\pi-A}{2} \\ \\ y := \frac{\pi-B}{2} \\ \\ z := \frac{\pi-C}{2} \end{array}\right\|$ in the well-known conditioned

trigonometrical identity $\boxed{x + y + z = \pi \implies \sum \sin x = 4 \cdot \prod \cos\frac{x}{2}}$. Indeed, $x + y + z = \sum \frac{\pi-A}{2} = \pi$

and $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = \sum \sin\frac{\pi-A}{2} = 4 \cdot \prod \cos\frac{\pi-A}{4} = 4 \cdot \sin\frac{\pi+A}{4} \cdot \sin\frac{\pi+B}{4} \cdot \sin\frac{\pi+C}{4}$.

☐ Let a, b, c be positive real numbers such that $abc = 1$. Prove that $\sqrt{\frac{a^{11}}{b+c}} + \sqrt{\frac{b^{11}}{c+a}} + \sqrt{\frac{c^{11}}{a+b}} \geq \frac{3\sqrt{2}}{2}$

Solution

Rephrase this problem as

$$\sum_{cyc} \frac{a^6}{\sqrt{2a(b+c)}} \geq \frac{3}{2}.$$

Note that using the AM-GM inequality, we have $2\sqrt{2a(b+c)} \leq 2a + b + c$; so that it is sufficient to check that

$$\sum_{cyc} \frac{a^6}{2a + b + c} \geq \frac{3}{4}.$$

Using the Cauchy-Schwarz inequality, we have

$$\sum_{cyc} \frac{a^6}{2a+b+c} \geq \frac{(a^3+b^3+c^3)^2}{4(a+b+c)};$$

So that it will suffice to check that

$$(a^3+b^3+c^3)^2 \geq 3(a+b+c);$$

Which is obvious from the AM-GM inequality in accordance with the Power-mean inequality, or maybe, CS alone: $(a^3+b^3+c^3)(a+b+c) \geq (a^2+b^2+c^2)^2 \geq \frac{1}{9}(a+b+c)^4$; So that $(a^3+b^3+c^3)^2 \geq \frac{1}{81}(a+b+c)^6 \geq 3(a+b+c).\square$

$\square$ Prove the identity

$$(z+a)^n = z^n + a\sum_{k=1}^{n} \binom{n}{k}(a-kb)^{k-1}(z+kb)^{n-k}$$

Solution

Let's prove it by induction on n. It is clearly true for n=0. Suppose it is true for $0 \leq m \leq n-1$.

$z^n + a\sum_{k=1}^{n}\binom{n}{k}(a-kb)^{k-1}(z+kb)^{n-k}$
$= z^n + a\sum_{k=1}^{n}\left(\binom{n}{k}(a-kb)^{k-1}\sum_{i=0}^{n-k}\binom{n-k}{i}z^i(kb)^{n-k-i}\right)$
$= z^n + \sum_{k=1}^{n}\sum_{i=0}^{n-k} a\binom{n}{k}(a-kb)^{k-1}\binom{n-k}{i}z^i(kb)^{n-k-i}$
$= z^n + \sum_{i=0}^{n-1}\sum_{k=1}^{n-i} a\binom{n}{k}(a-kb)^{k-1}\binom{n-k}{i}z^i(kb)^{n-k-i}$
$= z^n + \sum_{i=0}^{n-1}\sum_{k=1}^{n-i} az^i\binom{n}{n-k}\binom{n-k}{i}(a-kb)^{k-1}(kb)^{n-i-k}$
$= z^n + \sum_{i=0}^{n-1}\sum_{k=1}^{n-i} az^i\binom{n}{i}\binom{n-i}{k}(a-kb)^{k-1}(kb)^{n-i-k}$
$= z^n + \sum_{i=0}^{n-1}\binom{n}{i}z^i a\sum_{k=1}^{n-i}\binom{n-i}{k}(a-kb)^{k-1}(kb)^{n-i-k}$
$= z^n + \sum_{i=0}^{n-1}\binom{n}{i}z^i a(0+a)^{n-i}$ by the induction hypotesis
$= z^n + \sum_{i=0}^{n-1}\binom{n}{i}z^i a^{n-i}$
$= (z+a)^n$

Therefore, it is true for every $n \geq 0$ by induction.

$\square$ During June (30 days), Anton play chess at least once a day. During that month, the game is not more than 45 times. Show that there are periods where Anton do the game exactly 14 times.

Solution

let s_n be the number of games that played from the first day until n-th day

therefor we have :

$1 \leq s_1 < s_2 < ... < s_{30} \leq 45$

$15 \leq s_1 + 14 < s_2 + 14 < ... < s_{30} + 14 \leq 59$

now we have 60 numbers between 1 and 59 therefor we have at least two number with same value and we now that for any two numbers i, j we have $s_i \neq s_j$ ... therefor we have two numbers i, j such that $s_i = s_j + 14 \Rightarrow s_i - s_j = 14$

$\square$ Solve the following simultaneous equations:

$2 - bc = 2ad$

$2 - ac = 2bd$

$2 - ab = 2cd$

Solution

(assuming a, b, c, d are reals)

Clearly a, b, c are symmetric.

First, check the case where one of a, b, c is zero. It's straightforward to see the only solutions are of the form $(0, \pm\sqrt{2}, \pm\sqrt{2}, \pm\frac{1}{\sqrt{2}})$. Now we assume a, b, c are nonzero.

Set $p = abcd$. Then ab, ac, bc are all roots of the equation $2 - x = 2p/x$ (and we know that x is nonzero). This is the quadratic $x^2 - 2x + 2p = 0$, except when $p = 0$ when it becomes linear. Either way it has at most two roots, so ab, ac, bc are not all distinct, that is a, b, c are not all distinct. WLOG set $a = b$.

If $a = b = c$ we have $2 - a^2 = 2ad$, yielding the solutions $(a, a, a, \frac{2-a^2}{2a})$ for $a \in \mathbb{R} \setminus \{0\}$.

Otherwise a^2 and ac are different two roots of the quadratic. Vieta's tells us $a^2 + ac = 2$, so $c = \frac{2-a^2}{a}$, and $a^3c = 2p = a^2cd$ or $a = 2d$. This yields the solutions $(a, a, \frac{2-a^2}{a}, \frac{a}{2})$ for $a \in \mathbb{R} \setminus \{0\}$, which are all indeed solutions of the original equation. (Also we get our case where one of a, b, c is zero back by setting $a = \pm\sqrt{2}$.)

□ n is a natural number. Show that the number of the divisors of n can't exceed $2\sqrt{n}$

Solution

For each divisor $d \mid n$, there is an associated divisor $n/d \mid n$. Consider the pairs $\{d, n/d\}$ (when n is a perfect square, one of these pairs is a singleton - that containing twice $\sqrt{n}$).

But $(\min\{d, n/d\})^2 \le d(n/d) = n$, so there are at most $\lfloor\sqrt{n}\rfloor$ such pairs, thus at most $2\lfloor\sqrt{n}\rfloor \le 2\sqrt{n}$ divisors. Notice that the maximum $2\lfloor\sqrt{n}\rfloor$ for the number of divisors may be reached, for example for $n = 2, 3, 6, 12, 24$. For this to happen we need $\operatorname{lcm}(1, 2, 3, \ldots, \lfloor\sqrt{n}\rfloor) \mid n$; it seems that there are only finitely many such cases.

Proof. Already $\operatorname{lcm}(1, 2, 3, 4, 5, 6, 7) = 420 > 196 = (2 \cdot 7)^2$. Assume that for a prime $p \le \lfloor\sqrt{n}\rfloor$ for which $\operatorname{lcm}(1, 2, \ldots, p) \mid n$ we have $\operatorname{lcm}(1, 2, \ldots, p) \ge (2p)^2$; that implies $n \ge (2p)^2$. But then $\lfloor\sqrt{n}\rfloor \ge 2p$, so there exists another prime $p < q < 2p \le \lfloor\sqrt{n}\rfloor$ (by Bertrand's postulate, now Tchebysheff's theorem). Since we now need $\operatorname{lcm}(1, 2, \ldots, p)q \mid \operatorname{lcm}(1, 2, \ldots, p, \ldots, q) \mid n$, it follows $n \ge \operatorname{lcm}(1, 2, \ldots, p)q \ge (2p)^2q > (2q)^2$, since $p^2 > 2p > q$. This process continues ad nauseam, so there exists no more eligible n.

For the small cases, $\operatorname{lcm}(1, 2, 3, 4, 5) = 60 > 49 = 7^2$ leads to the above proof. Finally, the only values are those listed above.

□ Prove that If $a_i \in \{1, -1\}, i = 1, 2, 3, \cdots, n$ such that $\sum_{i=1}^{n} a_i a_{i+1} = 0, a_{n+1} = a_1$, than $4|n$.

Solution

Clearly n is even. Now we divide all the numbers a_i into groups of two such that in no group there exist two adjacent numbers, i.e. no group contains numbers with cyclically consecutive indexes. If we change the sign of any two numbers in a group, then four numbers in the sum $\sum a_i a_{i+1}$ change their sign. However, there is no change modulo 4; the sum remains invariant modulo 4. We keep performing this step till every number a_i becomes 1. In that case the sum has not changed modulo 4, and it actually is $\sum a_i a_{i+1} = 1 + 1 + ... + 1 = n$. Since in the beginning the sum was 0, i.e. divisible by 4, and through this process it has not changed modulo 4, then n must be divisible by 4.

□ Given that $x, y, z, a, b, c > 0$, prove that $\frac{(x+y+z)^{a+b+c}}{x^a y^b z^c} \ge \frac{(a+b+c)^{a+b+c}}{a^a b^b c^c}$.

Solution

Rewrite this into the following form:

$$\frac{x+y+z}{a+b+c} \ge \sqrt[a+b+c]{\left(\frac{x}{a}\right)^a \left(\frac{y}{b}\right)^b \left(\frac{z}{c}\right)^c};$$

Which follows from the weighted AM-GM inequality as follows:

$$a \cdot \frac{x}{a} + b \cdot \frac{y}{b} + c \cdot \frac{z}{c} \geq (a+b+c) \sqrt[a+b+c]{\left(\frac{x}{a}\right)^a \left(\frac{y}{b}\right)^b \left(\frac{z}{c}\right)^c}.$$

We are done. $\square$

$\square$ Let a, b be two positive integers satisfying $0 < b \leq a$. Let p be any prime number. Show that

$$\binom{pa}{pb} \equiv \binom{a}{b} \mod p^3.$$

Solution

Lemma 1: For any two positive integers satisfying $0 < b \leq a$, we have that

$$\binom{pa}{pb} = \binom{a}{b} \frac{\prod_{k=a-b+1}^{a-1}(kp+1)(kp+2)\cdots(kp+p-1)}{\prod_{k=0}^{b-1}(kp+1)(kp+2)\cdots(kp+p-1)}$$

Proof is easy, and is omitted.

Lemma 2: Let p be any prime number. Then,

$$\sum_{i=1}^{p-1} \frac{1}{i} = 0, \quad \sum_{1 \leq i < j \leq p-1} \frac{1}{ij} = 0$$

when viewed in $\mathbb{Z}/p^2\mathbb{Z}$. Proof: Well, for the first one, just note that $\frac{1}{i} + \frac{1}{p-i} = \frac{p}{i(p-1)}$. Therefore, it suffices to prove that $\sum_{i=1}^{(p-1)/2} \frac{1}{i(p-i)} = 0$ in $\mathbb{Z}/p\mathbb{Z}$. But in $\mathbb{Z}/p\mathbb{Z}$, $\frac{1}{i(p-1)} = \frac{1}{i^2}$, and since the inverses of the set of quadratic residues $\pmod p$ is the same as the set of quadratic residues $\pmod p$, therefore, our required sum is nothing but $\sum_{i=1}^{(p-1)/2} i^2 = p(p-1)(p+1)/24$ and which is 0 in $\mathbb{Z}/p\mathbb{Z}$.

For the second one, we have $\frac{1}{ij} + \frac{1}{(p-i)j} + \frac{1}{i(p-j)} + \frac{1}{(p-i)(p-j)} = \frac{p^2}{ij(p-i)(p-j)}$ which is 0 in $\mathbb{Z}/p^2\mathbb{Z}$.

Lemma 3: Let p be a prime, k be any non-negative integer. Then,

$$(kp+1)(kp+2)\cdots(kp+p-1) \equiv (p-1)! \pmod{p^3}$$

Proof: Well, expanding the LHS, and taking only the powers of kp which are less than 3, as the others are cancelled out, we have that,

$$(kp+1)(kp+2)\cdots(kp+p-1) \equiv (p-1)! + (kp)(p-1)! \sum_{i=1}^{p-1} \frac{1}{i} + (kp)^2(p-1)! \sum_{1 \leq i < j \leq p-1} \frac{1}{ij} \pmod{p^3}$$

And this is easily seen to be congruent to $(p-1)!$ as the second and third terms are 0 due to Lemma 2.

Coming to the main proof, we have,

$$\binom{pa}{pb} = \binom{a}{b} \frac{\prod_{k=a-b+1}^{a-1}(kp+1)(kp+2)\cdots(kp+p-1)}{\prod_{k=0}^{b-1}(kp+1)(kp+2)\cdots(kp+p-1)}$$

by lemma 1. But, $(kp+1)(kp+2)\cdots(kp+p-1) \equiv (p-1)! \pmod{p^3}$ for every k, and therefore, since the numerator and denominator in our fraction both become $(p-1)!^b$, so, we can cancel it out, and finish the question.

$\square$ $a, b, c > 0$ $a+b+c = 1$ $a > bc, b > ac, c > ab$

Prove: $\sqrt{a-bc} + \sqrt{b-ca} + \sqrt{c-ab} \leq \sqrt{2}$

Solution

By C-S: $\left(\sum_{cyc}\sqrt{a-bc}\right)^2 \leq \sum_{cyc}\frac{a-bc}{2a+b+c}\cdot\sum_{cyc}(2a+b+c)$. Thus, it remains to prove that: $\sum_{cyc}\frac{a-bc}{2a+b+c}\cdot\sum_{cyc}(2a+b+c) \leq 2$. But $\sum_{cyc}\frac{a-bc}{2a+b+c}\cdot\sum_{cyc}(2a+b+c) \leq 2 \Leftrightarrow \Leftrightarrow 2\sum_{cyc}\frac{a-bc}{a+1} \leq 1$. We obtain: $1-2\sum_{cyc}\frac{a-bc}{a+1} = \sum_{cyc}\left(a-\frac{2a^2+2ab+2ac-2bc}{2a+b+c}\right) = = \sum_{cyc}\frac{2bc-ab-ac}{2a+b+c} = \sum_{cyc}\frac{b(c-a)-c(a-b)}{2a+b+c} = = \sum_{cyc}(a-b)\left(\frac{c}{2b+a+c}-\frac{c}{2a+b+c}\right) = \sum_{cyc}\frac{c(a-b)^2}{(2a+b+c)(2b+a+c)} \geq 0$ More way (by Sasha2): $\sum_{cyc}\sqrt{a-bc} = \frac{1}{2}\sum_{cyc}\left(\sqrt{a-bc}+\sqrt{b-ac}\right) \leq \leq \frac{1}{2}\sum_{cyc}\sqrt{2(a-bc+b-ac)} = \frac{1}{2}\sum_{cyc}\sqrt{2(a+b)(1-c)} = \frac{\sqrt{2}}{2}\sum_{cyc}(a+b) = \sqrt{2}$

□ Let ABC be an equilateral triangle. Let $M\in(BC)$, $N\in(CA)$, $P\in(AB)$ so that $\frac{MB}{MC}=\frac{NC}{NA}=\frac{PA}{PB}=x$.

Prove that the area of the triangle formed by the lines AM , BN , CP is equally to $\frac{(x-1)^2}{x^2+x+1}\cdot S$, where $S=[ABC]$.

Solution

Denote $X\in BN\cap CP$, $Y\in CP\cap AM$, $Z\in AM\cap BN$. Observe that $\triangle XYZ$ is equilateral.

Suppose w.l.o.g. $AB=1$ and using the generalized Pythagoras' theorem in $\triangle ABM$ obtain easily that $AM=\frac{\sqrt{x^2+x+1}}{x+1}$ $(*)$.

Apply the Menelaus's theorem to the transversals : $\left\{\begin{array}{ccccc} \overline{BZN}/\triangle AMC & : & \Longrightarrow & \frac{ZA}{x+1}=\frac{ZM}{x^2}=\frac{AM}{x^2+x+1} \\ \\ \overline{CYP}/\triangle ABM & : & \Longrightarrow & \frac{YA}{x^2+x}=\frac{YM}{1}=\frac{AM}{x^2+x+1}\end{array}\right\|$

. In conclusion,

$\frac{YZ}{x^2-1}=\frac{ZA}{x+1}=\frac{YM}{1}=\frac{AM}{x^2+x+1}$ and $YZ\overset{(*)}{=}\frac{x-1}{\sqrt{x^2+x+1}}$, $\frac{[XYZ]}{[ABC]}=YZ^2 \Longrightarrow [XYZ]=\frac{(x-1)^2}{x^2+x+1}\cdot S$.

□ Prove that $\sum_{k=1}^{\infty}\frac{k^{2005}}{2005^k}$ is rational

Solution

Denote $S_n=\sum_{k=1}^{\infty}\frac{k^n}{2005^k}$. For $n=0$ we have $S_0=\dfrac{2005}{2004}\in\mathbb{Q}$. Assume, by induction hypothesis, that $S_j\in\mathbb{Q}$ for all $0\leq j\leq n$.

But $2004S_{n+1} = 2005S_{n+1}-S_{n+1} = 1+\sum_{k=1}^{\infty}\frac{(k+1)^{n+1}-k^{n+1}}{2005^k} = 1+\sum_{k=1}^{\infty}\sum_{j=0}^{n}\binom{n+1}{j}\frac{k^j}{2005^k} = 1+\sum_{j=0}^{n}\binom{n+1}{j}\sum_{k=1}^{\infty}\frac{k^j}{2005^k} = 1+\sum_{j=0}^{n}\binom{n+1}{j}S_j$, and by the induction hypothesis all elements are rational, hence S_{n+1} will also be rational.

□ Find a closed-form expression equivalent to $\displaystyle\sum_{j=0}^{n}\frac{\binom{n}{j}}{n^j(j+1)}$

Solution

$S=\sum_{j=0}^{n}\frac{\binom{n}{j}}{n^j(j+1)} = \frac{1}{n+1}\sum_{j=0}^{n}\frac{\binom{n+1}{j+1}}{n^j}$

Then, $\frac{S}{n}=\frac{1}{n+1}\sum_{j=0}^{n}\frac{\binom{n+1}{j+1}}{n^{j+1}} = \frac{1}{n+1}\sum_{j=1}^{n+1}\frac{\binom{n+1}{j}}{n^j} = \frac{1}{n+1}\left(\left(1+\frac{1}{n}\right)^{n+1}-1\right)$

And we have $S=\frac{n}{n+1}\left(\left(1+\frac{1}{n}\right)^{n+1}-1\right)$

□ Let sequence $\{x_k\}$ is defined by: $x_k=\frac{1}{2!}+\frac{2}{3!}+...+\frac{k}{(k+1)!}$

Calculate $\lim_{n\to+\infty}\sqrt[n]{x_1^n+x_2^n+...+x_{1999}^n}$

Solution

$x_k=\sum_{r=1}^{k}\frac{r}{(r+1)!}=\sum_{r=1}^{k}(\frac{1}{r!}-\frac{1}{(r+1)!})=1-\frac{1}{(k+1)!}$

from that $x_1<x_2<x_3<.....<x_{1999}$

$\lim_{n\to+\infty}\sqrt[n]{x_1^n+x_2^n+...+x_{1999}^n}$

$=\lim_{n\to+\infty}x_{1999}(1+(\frac{x_{1998}}{x_{1999}})^n+(\frac{x_{1997}}{x_{1999}})^n+....+(\frac{x_1}{x_{1999}})^n)^{\frac{1}{n}}$

$=x_{1999}=1-\frac{1}{2000!}$

□ Let ΔABC is isosceles triangle in A, $\widehat{A}=\frac{\pi}{7}$, $AB=b, BC=a$. Prove: $a^5-4a^3b^2+3ab^4-b^5=0$

Solution

(Hình vẽ đi kèm) Locate the points P,Q on AC,AB such that $CB=BP=PQ$. By easy angle chase we get that $\angle PQB = \frac{2\pi}{7} \implies \triangle QPA$ is Q-isosceles. Thus, $BP = PQ = QA = a$. Parallels from P, Q to BC cut AB, AC at S, T, respectively. Then $\triangle CBP$ and $\triangle QAT$ are congruent $\implies$ $PC = QT = x$, but $\triangle BCP$ and $\triangle ABC$ are similar

$\implies \frac{PC}{BC} = \frac{BC}{b} \implies x = \frac{a^2}{b}$ (1)

It's easy to see that $QTPS$ is an isosceles trapezoid with $PS = QS = y$. Then

$\frac{SP}{BC} = \frac{AS}{AB} \implies \frac{y}{a} = \frac{y+a}{b} \implies y = \frac{a^2}{b-a}$ (2)

$QS = TP = y \implies TP + PC = AC - AT \implies y + x = b - a$ (3)

Combining $(1), (2), (3) \implies \frac{a^2}{b-a} + \frac{a^2}{b} = b - a \implies b^3 + a^3 - a^2b - 2ab^2 = 0$

$\implies (b^3 + a^3 - a^2b - 2ab^2)(a^2 + ab - b^2) = 0 \implies a^5 - b^5 + 3ab^4 - 4a^3b^2 = 0$

$\square$ Let ABC be a triangle with circumradius R , inradius r and semiperimeter s .

Denote $K \equiv \sum \sin\frac{A}{2}$. Prove that : $\boxed{s^2 = 4R \cdot (K-1)^2 \cdot \left[R(K+1)^2 + r\right]}$.

Solution

It's easy to prove this identity, but it's much more difficult to find it. Notice that : $\sum_{cyc} sin\hat{A} = \frac{s}{R}$ $\prod_{cyc} sin\frac{\hat{A}}{2} = \frac{r}{4R}$ So the identity is equivalent to : $(\sum_{cyc} sin\hat{A})^2 = 4(\sum_{cyc} sin\frac{\hat{A}}{2} - 1)^2((\sum_{cyc} sin\frac{\hat{A}}{2} + 1)^2 + 4\prod_{cyc} sin\frac{\hat{A}}{2})$ Use the following substitution : $\begin{cases} \hat{X} = \frac{\pi-\hat{A}}{2} \\ \hat{Y} = \frac{\pi-\hat{B}}{2} \\ \hat{Z} = \frac{\pi-\hat{C}}{2} \end{cases}$ Let s' , R' , r' be the semi-perimeter, the circumradius and the inradius of $\triangle XYZ$ respectively. So the identity is equivalent to : $(\sum_{cyc} sin2\hat{X})^2 = 4(\sum_{cyc} cos\hat{X} - 1)^2((\sum_{cyc} cos\hat{X} + 1)^2 + 4\prod_{cyc} cos\hat{X})$ Which is true since : $\sum_{cyc} sin2\hat{X} = \frac{2s'r'}{R'^2}$ $\sum_{cyc} cos\hat{X} = 1 + \frac{r'}{R'}$ $\prod_{cyc} cos\hat{X} = \frac{s'^2-(2R'+r')^2}{4R'^2}$

$\square$ Find the number of ordered triples of sets (A, B, C) such that $A \cup B \cup C = \{1, 2, ..., 2003\}$ and $A \cap B \cap C = \phi$

Solution

Find the number of ordered triples of sets (A, B, C) such that $A \cup B \cup C = \{1, 2, \ldots, n\}$ and $A \cap B \cap C = \emptyset$.

The six sets $A \setminus (B \cup C), B \setminus (C \cup A), C \setminus (A \cup B), A \cap B, B \cap C, C \cap A$ will therefore make up a partition of the set $\{1, 2, \ldots, n\}$ (a Venn diagram makes things obvious). Since any element can equally belong to any of these six sets, the required number is 6^n.

$\square$ Let $a, b\ and\ c$ be real numbers such that $a \geq b \geq c > 0$ and $a + b + c = 1$.

Show that $a\sqrt{\frac{b}{c}} + b\sqrt{\frac{c}{a}} + c\sqrt{\frac{a}{b}}$ is in $[1, +\infty)$.

Solution

We need $a\sqrt{\frac{b}{c}} + b\sqrt{\frac{c}{a}} + c\sqrt{\frac{a}{b}} \geq 1 = a + b + c$. Put $a = x^2, b = y^2, c = z^2$ so that $x \geq y \geq z > 0$ and the inequality becomes

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2$$

$$\left(\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y}\right) - \left(\frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}\right)$$

$$= \frac{1}{xyz}(x-y)(x-z)(y-z)(xy+yz+zx) \geq 0$$

and so we have

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq \frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}$$

and

$$(\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y})^2 \geq \left(\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y}\right)\left(\frac{zx^2}{y} + \frac{xy^2}{z} + \frac{yz^2}{x}\right)$$

$$\geq (x^2+y^2+z^2)^2$$

by C-S - taking the square root gives the result.

$\square$ ABC is a triangle, O is the midpoint of its side $[BC]$ and $A=\frac{4\pi}{7}$, $C=\frac{2\pi}{7}$. Calculate $m(\angle AOC)$.

Solution

Denote $m(\angle AOC)=x$. From the well-known property $1=\frac{OB}{OC}=\frac{AB}{AC}\cdot\frac{\sin\widehat{OAB}}{\sin\widehat{OAC}}=\frac{\sin C}{\sin B}\cdot\frac{\sin(x-B)}{\sin(A+B-x)}\iff$ $\sin B\sin(C+x)=\sin C\sin(x-B)\iff\cos(C-B+x)-\cos(B+C+x)=\cos(B+C-x)-$ $\cos(C+x-B)\iff 2\cos(C-B+x)=\cos(B+C+x)+\cos(B+C-x)\iff\cos(C-B+x)=$ $\cos(B+C)\cos x\iff\cos(C-B+x)=-\cos A\cos x\iff\cos(C-B)-\sin(C-B)\tan x=$ $-\cos A\iff\tan x=\frac{\cos(C-B)-\cos(B+C)}{\sin(C-B)}\iff\boxed{\tan x=\frac{2\sin B\sin C}{\sin(C-B)}=\frac{2\tan B\tan C}{\tan C-\tan B}}$. Our case : $\tan\widehat{BOC}=2\sin\frac{2\pi}{7}$.

$\square$ Suppose p is an odd prime and $4p+1$ is also prime. Prove that $4^p\equiv -1 mod(4p+1)$.

Solution

Let $4^p\equiv k\pmod{4p+1}$ where $0<k<4p+1$

Since $4^{4p}\equiv 1\pmod{4p+1}$, we have $k^4\equiv 1\pmod{4p+1}$

Now, $(2^p)^2\equiv k\pmod{4p+1}\implies k$ is a quadratic residue of $4p+1\implies k^{\frac{4p+1-1}{2}}\equiv 1\pmod{4p+1}$ $\implies k^{2p}\equiv 1\pmod{4p+1}$ Let m be the smallest positive integer such that $k^m\equiv 1\pmod{4p+1}$ $\implies m|gcd(2p,4)=2\implies k^2\equiv 1\pmod{4p+1}\implies 4p+1|(k+1)(k-1)$ So $4p+1$ must divide one of $k+1,k-1$ Since $k<4p+1$ We must have $4p+1|k+1\implies k=4p\implies 4^p\equiv -1\pmod{4p+1}$

$\square$ Find all primes m and n such that $2(m+n)$ is the difference of two integer squares.

Solution

If $m=n=2$, then $2(m+n)=3^2-1^2$ If $m=2,n\neq 2$, then we need $2n+4=x^2-y^2$ Note that $(a-k)^2-a^2$ is odd if k is odd and is divisible by 4 when k is even So there is no solution

Now if $m,n>2$ we need $2(m+n)=x^2-y^2$ Consider $(k+2)^2-k^2=4k+4=2(2k+2)$ As we vary k throughout the integers, we get every even number Since $m,n>2$, $m+n$ is even So we can find k such that $(k+2)^2-k^2=2(2k+2)=2(m+n)$ So the only primes which does not work is $m=2,n\neq 2$ and vice versa.

$\square$ Prove that if $p|m^2+9$ then there exists a x such that $p|x^2+1$, where p is a prime number

Solution

Let a^{-1} be the multiplicative inverse of a in modulo p

By Fermat's Little Theorem, we have $a^{p-1}\equiv 1\pmod p$ => $a^{p-2}\equiv a^{-1}\pmod p$

Since $p|m^2+9$ we have $m^2\equiv -9\pmod p$ => $9^{-1}m^2\equiv(-9)9^{-1}\equiv(-1)(9)(9^{-1})\equiv -1\pmod p$ And $9^{-1}\equiv 9^{p-2}\equiv(3^{p-2})^2\pmod p$ => $(3^{p-2}m)^2\equiv -1\pmod p$ => $p|(3^{p-2}m)^2+1$

$\square$ Show that $\binom{-\frac{1}{2}}{k}=(-1)^k\binom{2k}{k}\frac{1}{2^{2k}}$.

Solution

$$\binom{-\frac{1}{2}}{k}=\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})...(-\frac{(2k-1))}{2})}{k!}=\frac{(-1)^k(1.3.5...(2k-1))}{2^k.k!}$$

$$1.3.5...(2k-1)=\frac{1.2.3.4.5.6...(2k-2)(2k-1)}{2.4.6...(2k-2)}=\frac{(2k-1)!}{2^{k-1}(k-1)!}$$

$$\binom{-\frac{1}{2}}{k}=\frac{(-1)^k(2k-1)!}{2^k.2^{k-1}(k)!(k-1)!}.\frac{2k}{2k}$$

$$=\frac{(-1)^k(2k)!}{2^{2k}(k)!(k)!}=(-1)^k\binom{2k}{k}\frac{1}{2^{2k}}$$

$\square$ What is the coefficient of x^2 from :
$((((...((((x-2)^2-2)^2-2)^2-2)^2-........2)^2$

Solution

$(x-2)^2 = x^2-4x+4$, coefficients are 1,-4 and 4

$((x-2)^2-2)^2 = ... + 20x^2-16x+4$, coefficients are 20,-16 and 4

$(((x-2)^2-2)^2-2)^2 = ... + 336x^2-64x+4$, coefficients are 336,-64 and 4

It's easy to see that if for n brackets coefficients are (a,b,4), then for n+1 brackets coefficients are $(4a+b^2, 4b, 4)$. Indeed,

$((x^3P(x)+ax^2+bx+4)-2)^2 = (x^3P(x)+ax^2+bx+2)^2 =$
$x^3\left[x^3P(x)^2+2ax^2P(x)+x(a^2+2bP(x))+4P(x)+2ab\right]+$
$+x^2(4a+b^2)+4bx+4=$
$=x^3R(x)+x^2(4a+b^2)+4bx+4.$

Now we just continue a sequence: $(1,-4,4) \to (4+4^2,-4^2,4) \to (4^2+4^3+4^4,-4^3,4) \to (4^3+4^4+4^5+4^6,-4^4,4) \to ... \to (4^{n-1}+4^n+...+4^{2n-2},-4^n,4)$

So the coefficient of x^2 is equal to $4^{n-1}+4^n+...+4^{2n-2} = \boxed{\dfrac{4^{2n-1}-4^{n-1}}{3}}$ $\square$ Find the sum where n is positive integer: $\sum_{k=0}^{\infty}\left[\frac{n+2^k}{2^{k+1}}\right]$

[x] means floor of x

Solution

Firstly we will prove that $[a+\frac{1}{2}] = [2a]-[a]$ for any a.

1) If $a = k+r$ (k- integer , $0 \le r < 1/2$) then
$[a+\frac{1}{2}] = [k+r+\frac{1}{2}] = k$
$[2a] = [2k+2r] = 2k$
$[a] = k$ and we have identity $k = 2k-k$

2) If $a = k+r$ (k - integer, $1/2 \le r < 1$) then
$[a+\frac{1}{2}] = [k+r+\frac{1}{2}] = k+1$
$[2a] = [2k+2r] = 2k+1$
$[a] = k$ and we have identity $k+1 = 2k+1-k$

$[\frac{n}{2}+\frac{1}{2}]+[\frac{n}{4}+\frac{1}{2}]+... = ([n]-[\frac{n}{2}])+([\frac{n}{2}]-[\frac{n}{4}])+... = [n] = n$

Hence the answer is n

$\square$ Prove that $\sqrt[n]{a^n+b^n} \ge \sqrt[n+1]{a^{n+1}+b^{n+1}}$ for $n \in \mathbb{N}^*$ and $a,b \in \mathbb{R}_+^*$ with $a \ne b$?

Solution

Suppose w.l.o.g. that $0 < a < b$ and denote $c = \frac{a}{b} < 1$. Observe that $\left\{\begin{array}{c} \frac{1}{n} > \frac{1}{n+1} > 0 \\ \ln(1+c^n) > \ln(1+c^{n+1}) > 0 \end{array}\right\| \odot =$

$\frac{1}{n}\cdot\ln(1+c^n) > \frac{1}{n+1}\cdot\ln(1+c^{n+1}) \iff \sqrt[n]{1+c^n} > \sqrt[n+1]{1+c^{n+1}} \iff b\cdot\sqrt[n]{1+c^n} > b\cdot\sqrt[n+1]{1+c^{n+1}} \iff$

$\sqrt[n]{a^n+b^n} \ge \sqrt[n+1]{a^{n+1}+b^{n+1}}$

$\square$ Find all solutions to $7^x-3^y=4$

Solution

Modulo 4 we must have $7^x \equiv 3^y \pmod 4$, i.e. $(-1)^x \equiv (-1)^y \pmod 4$, hence $x \equiv y \pmod 2$.

1. If both x, y are even, say $x = 2a$, $y = 2b$, then $(7^a-2)(7^a+2) = 3^{2b}$, thus $7^a-2 = 3^u$ and $7^a+2 = 3^v$, therefore $4 = 3^v-3^u = 3^u(3^{v-u}-1)$. This implies $u = 0$ and $3^v-1 = 4$, impossible.

2. If both x, y are odd, say $x = 2a + 1$, $y = 2b + 1$, clearly $a = b = 0$ is a solution: $7 - 3 = 4$. Otherwise we will have $7 \cdot 49^a - 3 \cdot 9^b = 4$, thus $7(49^a - 1) = 3(9^b - 1)$. We need $7 \mid 9^b - 1$, thus $3 \mid b$. But then $7 \cdot 8 \cdot 13 = 9^3 - 1 \mid 9^b - 1$, so we need $13 \mid 49^a - 1$. This in turn requires $6 \mid a$, but then $9 \mid 49^a - 1$, and so $3 \mid 9^b - 1$, absurd for $b \geq 1$.

Therefore the only solution is $x = y = 1$.

☐ Let n be an integer such that $2n^2$ has exactly 28 distinct positive divisors and $3n^2$ has exactly 24 distinct positive divisors. How many distinct positive divisors does $6n^2$ have?

Solution

If $p_1, p_2, p_3, \ldots$ are distinct primes then the number of divisors of an integer $x = p_1^{k_1} p_2^{k_2} p_3^{k_3} \ldots$ is $(k_1 + 1)(k_2 + 1)(k_3 + 1)\ldots$ Suppose $n^2 = 2^{2a}.3^{2b}.p^{2u}.q^{2v}\ldots$, where $p, q, \ldots$ are further distinct primes and $a, b, u, v, \ldots$ are non-negative integers.. Then the number of divisors of $2n^2$ is $28 = (2a + 2)(2b + 1)(2u + 1)(2v + 1)\ldots$ and the number of divisors of $3n^2$ is $24 = (2a + 1)(2b + 2)(2u + 1)(2v + 1)\ldots$ Dividing these gives

$$\frac{(2a + 2)(2b + 1)}{(2a + 1)(2b + 2)} = \frac{28}{24} = \frac{7}{6} \Rightarrow b = \frac{8a + 1}{5 - 2a}$$

The only solution of this in non-negative integers which satisfies the original equations is $a = 1, b = 3$ (with $u, v, \ldots = 0$), i.e. $n^2 = 2^2.3^6$ $6n^2 = 2^3.3^7$ and this has $(3 + 1)(7 + 1) = 32$ divisors.

☐ For even n, prove that $\sum_{i=1}^{n} \left((-1)^{i+1} \cdot \frac{1}{i}\right) = 2 \sum_{i=1}^{n/2} \frac{1}{n+2i}$.

Solution

Induction on n works. Easy to verify the base cases. Assume the result for some natural n. $\sum_{i=1}^{n+2} \left((-1)^{i+1} \cdot \frac{1}{i}\right) = \frac{1}{n+1} - \frac{1}{n+2} + \sum_{i=1}^{n} \left((-1)^{i+1} \cdot \frac{1}{i}\right)$ and by the assumption, this will be equal to $\frac{1}{n+1} - \frac{1}{n+2} + \frac{2}{n+2} + 2\left(\frac{1}{n+4} + \cdots + \frac{1}{n+n}\right)$ and since $\frac{1}{n+1} + \frac{1}{n+2} = 2\left(\frac{1}{(n+2)+n} + \frac{1}{(n+2)+(n+2)}\right)$ and the sum becomes $2\sum_{i=1}^{\frac{n+2}{2}} \frac{1}{n+2i}$ which proves the problem.

☐ Let $\alpha + \beta + \gamma = \pi$. Prove that $\sum_{cyc} \sin 2\alpha = 2 \cdot \left(\sum_{cyc} \sin \alpha\right) \cdot \left(\sum_{cyc} \cos \alpha\right) - 2\sum_{cyc} \sin \alpha$.

Solution

$\sum_{cyc} \sin 2\alpha = 4 \sin \alpha \sin \beta \sin \gamma$ and $2 \cdot \left(\sum_{cyc} \sin \alpha\right) \cdot \left(\sum_{cyc} \cos \alpha\right) - 2\sum_{cyc} \sin \alpha = 8(\sum \sin \alpha) \prod \sin \frac{\alpha}{2}$ as $\sum \cos \alpha = 1 + 4 \prod \sin \frac{\alpha}{2}$ and it becomes $32 \prod(\sin \frac{\alpha}{2} \cos \frac{\alpha}{2}) = 4 \sin \alpha \sin \beta \sin \gamma$ by using $\sum \sin \alpha = 4 \prod \cos \frac{\alpha}{2}$ and so, they are equal.

☐ Ascertain $l(x) = \lim_{n \to \infty} a_n(x)$ for $x > 0$, where $a_n = \sqrt{x + 2\sqrt{x + 2\sqrt{x + \ldots + 2\sqrt{x + 2\sqrt{3x}}}}}$ (n radicals).

Solution

The expression $3x$ of the innermost radical is a red herring. Replace it with some positive constant A. Then

• for $x + 2\sqrt{A} = A$, the sequence $(a_n)_{n \geq 1}$ is seen to be constant equal to $\sqrt{A}$; • for $x + 2\sqrt{A} < A$, the sequence $(a_n)_{n \geq 1}$ is seen to be decreasing, and since being lower bounded by 0, convergent; • for $x + 2\sqrt{A} > A$, the sequence $(a_n)_{n \geq 1}$ is seen to be increasing. But take some value $B > A$ such that $x + 2\sqrt{B} \leq B$; then $a_n < b_n$, where the sequence $(b_n)_{n \geq 1}$ is obtained by replacing A with B, and then by the previous remark $a_n < b_n \leq b_1$. Thus the sequence $(a_n)_{n \geq 1}$ is upper bounded, hence convergent.

Now that we have established the existence of a finite limit $\ell(x)$, we can pass to the limit in the relation $a_{n+1}^2 = x + 2a_n$, so as to get $\ell(x)^2 = x + 2\ell(x)$, whence $\ell(x) = 1 + \sqrt{x + 1}$. Notice how this checks with the original problem where $A = 3x$ and $x = 3$, for which $\ell(3) = 1 + \sqrt{3 + 1} = 3$.

☐ Solve the following inequality in real numbers $x + \frac{x}{\sqrt{x^2 - 1}} > \frac{35}{12}$.

Solution

Observe that $x < -1 \implies x \in \emptyset$. Therefore, $x > 1$ and $(\forall)$ $x > 1$ there is uniquely $\phi \in \left(0, \frac{\pi}{2}\right)$ so that

$\boxed{x = \dfrac{1}{\sin\phi} > 1}$. Our inequation becomes $\frac{1}{\sin\phi} + \frac{1}{\cos\phi} > \frac{35}{12}$ $(*)$. Denote $\boxed{\sin\phi + \cos\phi = t \in \left(1, \sqrt{2}\right]}$
.

The equation $(*)$ becomes $\frac{2t}{t^2-1} > \frac{35}{12}$, i.e. $t \in \left(-\frac{5}{7}, \frac{7}{5}\right) \cap \left(1, \sqrt{2}\right] \implies t \in \left(1, \frac{7}{5}\right)$. Now you can return

easily to the initial variable x , i.e. exists $\theta \in \left(0, \frac{\pi}{4}\right)$, where $\sin\theta + \cos\theta = \frac{7}{5}$ and $\phi \in (0, \theta) \cup \left(\frac{\pi}{2} - \theta, \frac{\pi}{2}\right)$.

In conclusion, $\sin\theta \in \left\{\frac{3}{5}, \frac{4}{5}\right\}$ and $\sin\phi \in \left(0, \frac{3}{5}\right) \cup \left(\frac{4}{5}, 1\right) \implies \boxed{x \in \left(1, \dfrac{5}{4}\right) \cup \left(\dfrac{5}{3}, \infty\right)}$.

Remark. Prove analogously that $(\forall)$ $x > 1$, $x + \frac{x}{\sqrt{x^2-1}} \ge 2\sqrt{2} \iff$
$(\forall)$ $\phi \in \left(0, \frac{\pi}{2}\right)$, $\frac{1}{\sin\phi} + \frac{1}{\cos\phi} \ge 2\sqrt{2} \iff (\forall)$ $t \in \left(1, \sqrt{2}\right]$, $\frac{t}{t^2-1} \ge \sqrt{2}$.

☐ Let a, b, c, d be the complex numbers satisfying $a+b+c+d = a^3+b^3+c^3+d^3 = 0$ Prove that a pair of the a, b, c, d must add up to 0.

Solution

Assume $a + b = -(c + d) \neq 0$. Then $a^3 + b^3 = -(c^3 + d^3)$ writes as $(a + b)(a^2 - ab + b^2) = -(c+d)(c^2 - cd + d^2)$. We can cancel $a + b = -(c + d)$, to obtain $a^2 - ab + b^2 = c^2 - cd + d^2$.

But we then also have $a^2 + 2ab + b^2 = c^2 + 2cd + d^2$, so $ab = cd$. It follows (a, b) and $(-c, -d)$ are roots of the same quadratic polynomial, hence $a = -c$ or $a = -d$.

☐ Let $\mathbb{N}$ be the set of positive integers. Define $a_1 = 2$ and for $n = 2, 3, ...,$ $a_{n+1} = min\{\lambda | \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} +$
Show that $a_{n+1} = a_n^2 - a_n + 1$ for $n = 1, 2, ...$

Solution

Since $a_{n+1} = \frac{1}{1 - \frac{1}{a_1} - \frac{1}{a_2} - \dots - \frac{1}{a_n}} + 1$, we have

$a_{n+1} = \frac{1}{\frac{1}{a_n - 1} - \frac{1}{a_n}} + 1 = a_n(a_n - 1) + 1 = a_n^2 - a_n + 1$. QED

☐ Solve the equation

$\sqrt{x-1} + \sqrt{3-x} + 4x\sqrt{2x} \le x^3 + 10$

Solution

From AM-QM, we have $\sqrt{x-1} + \sqrt{3-x} \le 2\sqrt{\frac{x-1+3-x}{2}} = 2$.

From the Trivial Inequality, we have $(x\sqrt{x} - 2\sqrt{2})^2 \ge 0 \to x^3 - 4x\sqrt{2x} + 8 \ge 0$.

Thus, $x^3 + 10 \ge 4x\sqrt{2x} + 2 \ge 4x\sqrt{2x} + \sqrt{x-1} + \sqrt{3-x}$ as desired. □ ☐ n is a natural number, where $n \ge 50$ Show that there is no n such that is divided by all the natural numbers m, where $m \le \sqrt{n}$

Solution

Let $2 = p_1 < p_2 < \cdots < p_k < \cdots$ be the sequence of the prime numbers. We have $11^2 = 121 < 210 = 2 \cdot 3 \cdot 5 \cdot 7$, hence $p_m^2 < \prod_{k=1}^{m-1} p_k$ for $m = 5$. But $p_{m+1} < 2p_m$ by Tchebyshev's theorem (Bertrand's postulate), so $p_{m+1}^2 < 4p_m^2 < 4\prod_{k=1}^{m-1} p_k < \prod_{k=1}^{m} p_k$, thus the inequality holds for all $m \ge 5$, by simple induction.

Now (for $m \ge 5$), if $\prod_{k=1}^{m-1} p_k \mid n$, it means $n \ge \prod_{k=1}^{m-1} p_k > p_m^2$, hence $\sqrt{n} > p_m$, and, in order to have such an n as required, we will need also $p_m \mid n$, and the process continues indefinitely, so no such n exists.

Since $\sqrt{50} > 7 = p_4$, it follows that for $n \ge 50$ we need have $\prod_{k=1}^{m-1} p_k \mid n$ for $m = 5$, so no such required n does exist. In fact, the largest n with that property is $n = 24$.

☐ $1 \le x \le y \le z \le 4$ Find the Min of $(x-1)^2 + \left(\frac{y}{x} - 1\right)^2 + \left(\frac{z}{y} - 1\right)^2 + \left(\frac{4}{z} - 1\right)^2$

Solution

By QM-AM, we have that $\sqrt{(x-1)^2+\left(\frac{y}{x}-1\right)^2+\left(\frac{z}{y}-1\right)^2+\left(\frac{4}{z}-1\right)^2} \geq \frac{x+\frac{y}{x}+\frac{z}{y}+\frac{4}{z}-4}{2}$.

By AM-GM, we have that $\frac{x+\frac{y}{x}+\frac{z}{y}+\frac{4}{z}}{4} \geq \sqrt{2}$, so $\frac{x+\frac{y}{x}+\frac{z}{y}+\frac{4}{z}-4}{2} \geq 2\sqrt{2}-2$.

So our minimum occurs at $(2\sqrt{2}-2)^2 = 12-8\sqrt{2}$. Equality occurs when $x = \frac{y}{z} = \frac{z}{y} = \frac{4}{z} \Rightarrow x = \sqrt{2}, y = 2, z = 2\sqrt{2}$.

□ Prove if $r \geq s \geq t \geq u \geq v$ then $r^2 - s^2 + t^2 - u^2 + v^2 \geq (r-s+t-u+v)^2$

Solution

Note that (a) if $x \geq y \geq 0$, then $2xy \geq 2y^2, x^2-y^2 \geq x^2-2xy+y^2 \Rightarrow \sqrt{x^2-y^2} \geq x-y$ (b) if $z \leq y \leq 0$ then $-z \geq -y \geq 0, z^2-y^2 \geq 0$ and the positive $\sqrt{z^2-y^2} \geq z-y$ (≤ 0). (c) if $y \leq 0 \Rightarrow \sqrt{y^2} \geq y$

If there are no positive pairs of the numbers, (i.e. $s \leq 0$) , then by C-S,

$$r^2+(t^2-s^2)+(v^2-u^2) \geq (\sqrt{r^2}+\sqrt{t^2-s^2}+\sqrt{v^2-u^2})^2 \geq (r-s+t-u+v)^2$$

If exactly one positive pair, (i.e. $v \leq u \leq 0$), then

$$(r^2-s^2)+t^2+(v^2-u^2) \geq (\sqrt{r^2-s^2}+\sqrt{t^2}+\sqrt{v^2-u^2})^2 \geq (r-s+t-u+v)^2$$

If two positive pairs

$$(r^2-s^2)+(t^2-u^2)+v^2 \geq (\sqrt{r^2-s^2}+\sqrt{t^2-u^2}+\sqrt{v^2})^2 \geq (r-s+t-u+v)^2$$

□ Find all functions $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$ such that

$$f(x)+f\left(\frac{1}{1-x}\right) = 1+\frac{1}{x(1-x)}.$$

Solution

Define the set $D \equiv R-\{0.1\}$. Then the function $\phi \ : \ D \to D$, $\phi(x) = \frac{1}{1-x}$ is a bijection and $\phi\circ\phi\circ\phi = 1_D$. We observe that $\phi^{-1} = \phi\circ\phi$. Denote the function $\psi \ : \ D \to D$, where $\psi(x) = 1+\frac{1}{x(1-x)}$. Therefore, functional equation becomes the following system :

$f+f\circ\phi = \psi$; $f\circ\phi+f\circ\phi\circ\phi = \psi\circ\phi$; $f\circ\phi\circ\phi+f = \psi\circ\phi\circ\phi$.

The its solution is $f = \frac{1}{2}(\psi-\psi\circ\phi+\psi\circ\phi\circ\phi) \Longrightarrow f(x) = x+\frac{1}{x}$.

Another examples :

1. ► $2f(x)-f\left(\frac{1-x}{1+x}\right) = x-1$, $x \notin \{-1,0,1\}$; Indication. $\phi(x) = \frac{1-x}{1+x}$; $\phi\circ\phi = 1_D$. 2. ► $f(x)+f\left(-\frac{1}{x}\right)+f\left(\frac{x-1}{x+1}\right) = x$, $x \notin \{-1,0,1\}$; [hide="Indication."]$\phi(x) = \frac{x-1}{x+1}$; $\phi\circ\phi\circ\phi\circ\phi = 1_D$. 3. ► $f(x)+f\left(\frac{x-1}{x}\right) = \frac{1}{x}-x+1$, $x \notin \{0,1\}$. Indication.

$\phi(x) = \frac{x-1}{x}$; $\phi\circ\phi\circ\phi = 1_D$.

□ Let $A_1A_2\ldots A_7$ be a regular heptagon, and let A_1A_3 and A_2A_5 intersect at X. Compute $\angle A_1XA_7$.

Solution

See the attached diagram (Thiếu Hình vẽ)

$\angle A_1A_2A_3 = \frac{5\pi}{7} \implies \angle A_3A_1A_2 = \angle A_1A_3A_2 = \frac{\pi}{7} \implies \angle A_7A_1X = \frac{4\pi}{7}$

From the trapezoid $A_2A_3A_4A_5$ we have $\angle A_3A_2A_5 = \frac{2\pi}{7} \implies \angle XA_2A_1 = \frac{3\pi}{7}$. Thus from the $\triangle XA_1A_2$ we get $\angle A_1XA_2 = \frac{3\pi}{7}$. Hence it's isosceles, therefore $XA_1 = A_1A_2 = A_1A_7$, thus $\triangle XA_1A_7$ is also isosceles.

So finally $\angle A_1XA_7 = \frac{\pi-\frac{4\pi}{7}}{2} = \frac{3\pi}{14}$

□ If AD, BE, CF are the bisectors of a triangle ABC of semiperimetre s prove that $DE^2+EF^2+FD^2 \leq \frac{s^2}{3}$

Solution

The points D , E , F are not the tangent points of the incircle with the sides of $\triangle ABC$. Therefore, $EF \neq 2r \cdot \cos\frac{A}{2}$ a.s.o. and $AE = \frac{bc}{a+c}$, $AF = \frac{bc}{a+b}$ a.s.o.

If the incircle of $\triangle ABC$ touches its sides in the points $X \in (BC)$, $Y \in (CA)$ and $Z \in (AB)$, then

$XY^2+YZ^2+ZX^2 \equiv \sum YZ^2 = \sum\left(2r\cdot\cos\frac{A}{2}\right)^2 = \sum 4r^2\cdot\frac{s(s-a)}{bc} =$
$\frac{4r^2s}{abc}\cdot\sum a(s-a) = \frac{2r^2}{R}\cdot(4R+r) \leq \frac{2}{R}\cdot\frac{s^2}{27}\cdot(4R+r) = \frac{s^2}{3}\cdot\frac{2(4R+r)}{9R} \leq \frac{s^2}{3}$.

I used the well-known relations $\sum a(s-a) = 2r(4R+r)$ and $3r\sqrt{3} \leq s$, $2r \leq R$.

□ Find all functions $f : \mathbb{R}-(0,1) \to \mathbb{R}$ such that :

$(\forall x \in \mathbb{R}-(0,1)) f(\frac{x-1}{x}) + f(x) = \frac{1}{x} - x + 1$

Note: $R-(0,1) = (-\infty, 0] \cup [0,1] \cup [1, +\infty]$

Solution

The key observation here is that $(\tau\circ\tau\circ\tau)(x) = x$, where $\tau(x) = \frac{x-1}{x}$. Iterating this function, we have

$$f(\tau(x)) + f(x) = \frac{1}{x} - x + 1,$$

$$f((\tau\circ\tau)(x)) + f(\tau(x)) = \frac{1}{\tau(x)} - \tau(x) + 1,$$

$$f(x) + f((\tau\circ\tau)(x)) = \frac{1}{(\tau\circ\tau)(x)} - (\tau\circ\tau)(x) + 1.$$

(We also note that $\tau(x) \neq 0,1$ when $x \neq 0,1$.) Now we just have a linear system of equations. The unique solution is

$$f(x) = \frac{3}{2} - x - \frac{1+x}{2(1-x)}.$$

■

□ If $sin\ \alpha + sin\ \beta + sin\ \gamma = cos\ \alpha + cos\ \beta + cos\ \gamma = 0$

Then prove that $cos\ (\alpha+\beta) + cos\ (\beta+\gamma) + cos\ (\gamma+\alpha) = 0$

Solution

$\begin{cases} u = \cos\alpha + i\cdot\sin\alpha \\ v = \cos\beta + i\cdot\sin\beta \\ w = \cos\gamma + i\cdot\sin\gamma \end{cases} \Longrightarrow |u| = |v| = |w| = 1$ and $\overline{u+v+w} = \overline{u}+\overline{v}+\overline{w} = \frac{1}{u}+\frac{1}{v}+\frac{1}{w} =$

$\frac{uv+vw+wu}{uvw}$ $(*)$. Therefore, $\left\{\begin{array}{c} cos\alpha + cos\beta + cos\gamma = 0 \\ sin\alpha + sin\beta + sin\gamma = 0 \end{array}\right\| \iff u+v+w = 0 \iff$

$\overline{u+v+w} = 0 \overset{(*)}{\iff} uv+vw+wu = 0 \iff \left\{\begin{array}{c} cos\ (\alpha+\beta) + cos\ (\beta+\gamma) + cos\ (\gamma+\alpha) = 0 \\ sin\ (\alpha+\beta) + sin\ (\beta+\gamma) + sin\ (\gamma+\alpha) = 0 \end{array}\right\|$

.

□ Let n be a positive integer. If $4^n + 2^n + 1$ is a prime, prove that n is a power of three.

Solution

I will use the known and easy to establish fact that $2^M - 1 \mid 2^N - 1$ if and only if $M \mid N$. Let $n = 3^a m$, with $\gcd(3,m) = 1$. Since $2^{3n} - 1 = (2^n-1)(4^n+2^n+1)$, and also $2^{3^{a+1}} - 1 \mid 2^{3n} - 1$ (according with the above), but $2^{3^{a+1}} - 1 \nmid 2^n - 1$ (again according with the above), it follows that $\frac{2^{3n}-1}{2^{3^{a+1}}-1} \mid \frac{2^{3n}-1}{2^n-1} = 4^n + 2^n + 1$. If $m \neq 1$, this will be a proper divisor, so in order for 4^n+2^n+1 to be a prime we need $m = 1$, and so $n = 3^a$.

□ Let $ABCDEF$ be a convex hexagon in which diagonals AD, BE, CF are concurrent at O. Suppose $[OAF]$ is geometric mean of $[OAB]$ and $[OEF]$ and $[OBC]$ is geometric mean of $[OAB]$ and $[OCD]$. Prove that $[OED]$ is the geometric mean of $[OCD]$ and $[OEF]$. (Here $[XYZ]$ denotes are of $\triangle XYZ$)

Solution

Let $OA = a, OB = b, OC = c, OD = d, OE = e, OF = f$ and $\angle AOB = \angle EOD = \alpha$, $\angle BOC = \angle FOE = \beta$ and $\angle COD = \angle AOF = 180 - \alpha - \beta$

Note that $[OAB][OCD][OEF] = [OBC][ODE][OAF]$

iff $ab\sin\alpha \cdot cd\sin(\alpha+\beta) \cdot ef\sin B = bc\sin\beta \cdot de\sin\alpha \cdot af\sin(\alpha+\beta)$ which is obviously true.

Hence

$[ODE] = \frac{[OAB][OCD][OEF]}{[OBC][OAF]}$

$= \frac{[OAB][OCD][OEF]}{\sqrt{[OAB][OCD][OAB][OEF]}} = \sqrt{[OCD][OEF]}$

□ Prove that among numbers $\lfloor 2^{\frac{1}{2}+k} \rfloor$, ($k$ is natural number) there are infinity even numbers.

Solution

Consider the binary representation $\sqrt{2} = \overline{1.01\ldots}_{(2)}$. Since $\sqrt{2}$ is irrational, the sequence is not ultimately periodic, hence there are infinitely many 0's in it, and infinitely many 1's. Then for $\sqrt{2} = \overline{1.01\ldots0\ldots}_{(2)}$, where the 0 after the $\ldots$ is on k-th position, we have $\lfloor 2^k\sqrt{2} \rfloor$ even (since multiplying with 2^k is tantamount to shifting the . with k positions to the right, thus the integer part ends in 0).

We can use instead of $\sqrt{2}$ any other irrational number, but also any rational $\frac{p}{q}$ with q different from a power of 2, since it also generates a representation with infinitely many 0's in it, and infinitely many 1's.

□ Find the coefficient of x^{48} in the product of $(x-1)(x-2)(x-3)\ldots\ldots\ldots\ldots(x-49)(x-50)$

Solution

By Vieta

$a_{48} = \sum_{k=1}^{49} \left(k \sum_{j=k+1}^{50} j \right)$

$a_{48} = \sum_{k=1}^{49} k \frac{51+k}{2}(50-k)$

$a_{48} = \frac{1}{2} \sum_{k=1}^{49} 2550k - k^2 - k^3$

$a_{48} = \frac{1}{2} \left(2550 \cdot \frac{49 \cdot 50}{2} - \frac{49 \cdot 50 \cdot 99}{6} - \left(\frac{49 \cdot 50}{2} \right)^2 \right)$

$a_{48} = \frac{49 \cdot 50}{4}(2550 - 33 - 49 \cdot 25)$

$a_{48} = \frac{49 \cdot 25}{2} \cdot 1292$

$a_{48} = 791350$

□ Let b and c be two elements from $[-1, 1]$. And consider the equation $x^2 + bx + c = 0$

Determin the set of values of the solutions for the given equation .

Solution

We know that $-1 \le b, c \le 1$, so

$$-1 \le -b \le 1b^2 - 4c \le 1 - 4(-1) = 5 \implies \sqrt{b^2 - 4c} \le \sqrt{5} \text{ and } -\sqrt{b^2 - 4c} \ge -\sqrt{5}$$

.

Using this, we can find bounds on the two solutions $x_1 = \frac{-b+\sqrt{b^2-4c}}{2}$ and $x_2 = \frac{-b-\sqrt{b^2-4c}}{2}$.

$x_1 = \frac{-b+\sqrt{b^2-4c}}{2} \le \frac{1+\sqrt{5}}{2}$

$x_2 = \frac{-b-\sqrt{b^2-4c}}{2} \ge \frac{-1-\sqrt{5}}{2}$

$x_2 \le x_1$

$\therefore \frac{-1-\sqrt{5}}{2} \le x_2 \le x_1 \le \frac{1+\sqrt{5}}{2}$

Hence, the solutions to $x^2 + bx + c = 0$ must be in the interval $\boxed{[\dfrac{-1-\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}]}$.

Note: $\phi = \frac{1+\sqrt{5}}{2}$ is known as the golden ratio, so the interval may be written as $[-\phi, \phi]$.

☐ Solve the following system of equations: $\begin{cases} ab(a+b) = 6 \\ bc(b+c) = 30 \\ ac(a+c) = 12 \end{cases}$

Solution

Obviously none of the variables can be zero. Put $S = a+b+c, P = abc$. Then the equations become

$\frac{P}{c}(S-c) = 6 \iff c = \frac{PS}{P+6}$

$\frac{P}{a}(S-a) = 30 \iff a = \frac{PS}{P+30}$

$\frac{P}{b}(S-b) = 12 \iff b = \frac{PS}{P+12}$

Thus $\left(\frac{P}{P+6} + \frac{P}{P+12} + \frac{P}{P+30}\right) S = S$

If $S = 0$, then $a + b = -c$, hence the first equation becomes $abc = -6$, but then the other two become $abc = -30 \wedge abc = -12$ which can't hold.

Therefore $\frac{P}{P+6} + \frac{P}{P+12} + \frac{P}{P+30} = 1$

After clearing the denominators and simplifying, we get

$P^3 + 24P^2 - 1080 = 0 \iff (P-6)(P^2 + 30P + 180) = 0$

For $P = 6$ we get $a = \frac{S}{6} \wedge b = \frac{S}{3} \wedge c = \frac{S}{2} \implies ab(a+b) = \frac{S^3}{36} = 6 \implies S = 6$, hence $\boxed{(a,b,c) = (1,2,3)}$

Then $P^2 + 30P + 180 = 0 \implies P_{1,2} = -15 \pm 3\sqrt{5}$

For $P = -15 + 3\sqrt{5}$ we get

$a = \frac{-15+3\sqrt{5}}{15+3\sqrt{5}} S = \frac{\sqrt{5}-3}{2} S$

$b = \frac{-15+3\sqrt{5}}{-3+3\sqrt{5}} S = -\sqrt{5} S$

$c = \frac{-15+3\sqrt{5}}{-9+3\sqrt{5}} S = \frac{5+\sqrt{5}}{2} S$

Thus $ab(a+b) = 6 \iff \frac{\sqrt{5}-3}{2} \cdot (-\sqrt{5}) \cdot \frac{-\sqrt{5}-3}{2} S^3 = 6$

$-\sqrt{5} S^3 = 6 \iff S = -\sqrt[6]{\frac{36}{5}}$

Hence

$$\boxed{(a,b,c) = \left(\frac{3-\sqrt{5}}{2}\sqrt[6]{\frac{36}{5}}, \sqrt{5}\sqrt[6]{\frac{36}{5}}, -\frac{5+\sqrt{5}}{2}\sqrt[6]{\frac{36}{5}}\right)}$$

For $P = -15 - 3\sqrt{5}$, by using similar technique, we find

$$\boxed{(a,b,c) = \left(-\frac{3+\sqrt{5}}{2}\sqrt[6]{\frac{36}{5}}, \sqrt{5}\sqrt[6]{\frac{36}{5}}, \frac{5-\sqrt{5}}{2}\sqrt[6]{\frac{36}{5}}\right)}$$

☐ Simplify $\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \ , \ 0 < c < a$.

Solution

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\left[\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}\right]\left[\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}\right]=2a\left[\sqrt{(x+c)^2+y^2}-\sqrt{(x-c$$

$$(x+c)^2-(x-c)^2=2a\left[\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}\right]$$

$$\left\{\begin{array}{l}\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\frac{2cx}{a}\\ \\ \sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a\end{array}\right\| \oplus$$

$$\sqrt{(x+c)^2+y^2}=\frac{cx}{a}+a$$

$$(x+c)^2+y^2=\left(\frac{cx}{a}\right)^2+2cx+a^2$$

$$x^2+c^2+y^2=\frac{c^2x^2}{a^2}+a^2$$

$$\left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right)$$

$$\boxed{a^2-c^2=b^2} \implies b^2x^2+a^2y^2=a^2b^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

.

□ If there are 8 teams that play in a tournament, 2 teams per game, in how many ways can the tournament be organized if each team is to participate in exactly 2 games against different opponents? (Two tournaments that have the same teams playing in each game, but have the games ordered differently, are considered to be organized the same way. Also, individual games are symmetrical and there is no home-field advantage.)

Solution

In theoretical terms, you ask for the maximal number of 2-regular (undirected) simple labeled graphs. However, it is well-known that such graphs may be described as disjoint unions of cycles spanning the whole graph; this way, since the smallest cycle is a triangle, by counting vertices we see that all the possible configurations are (C_5, C_3), (C_4, C_4) and (C_8). In the first case, there are $\binom{8}{3}=56$ ways to choose vertices for the C_3; in the second, $\frac{1}{2}\binom{8}{4}=35$ ways (two 4-subsets whose union is the whole set of vertices lead to the same partition); in the third, the C_8 spans the whole graph.

This generates $56\times 12=672$ tournaments for the first case, $35\times 3^2=315$ for the second and 2520 for the last one, for a total of 3507. I used the well-known fact that k vertices may be arranged in a k-cycle in $\frac{(k-1)!}{2}$ ways - to see that, just consider the $k!$ strings $a_1a_2\cdots a_k$ indicating that $(a_1,a_2),\ldots,(a_{k-1},a_k),(a_k,a_1)$ are edges and note that the $2k$ different cyclings $a_j\ldots a_ka_1\cdots a_{j-1}$, $a_j\ldots a_1a_k\ldots a_{j+1}$ for all j yield the same configuration.

The answer is thus 3507.

□ Define in $\mathbb{R}$ the following equivalence relation : for any $\{x,y\}\subset\mathbb{R}$ we"ll mean $\boxed{x\ .s.s.\ y \iff x=y=0\ \vee\ xy>0}$, i.e. x and y have "same sign".

Some examples. Consider $0<a,b\neq 1$, $\{x,y\}\subset\mathbb{R}$.

▶ $\frac{x}{y}\ .s.s.\ xy$, where $y\neq 0$; $|x|\ .s.s.\ x^2$; $|x|-|y|\ .s.s.\ x^2-y^2$.

► $(a^x - a^y)$ .s.s. $(a-1)(x-y)$; $a^x - b^x$ .s.s. $x(a-b)$.

► $\log_a x - \log_a y$ .s.s. $(a-1)(x-y)$; $\log_a x$ .s.s. $(a-1)(x-1)$, where $x > 0$ and $y > 0$.

► $\log_a x - \log_b x$ .s.s. $(a-1)(b-1)(x-1)(b-a)$, where $x > 0$.

Exercises.

1.⊙ Prove that for any $0 < a \neq 1$, $0 < b \neq 1$ have the relation $\left(a^b - a\right)(b^a - b) > 0$.

2.⊙ Solve the following inequations :

2.1 $(2x-1)(|x-2| - |x|) \lg |x-1| \geq 0$.

2.2 $\max\left\{a^x, a^{-x}\right\} \geq a^{\max\left\{x, \frac{1}{x}\right\}}$, where $0 < a \neq 1$.

Problem: Show that $\log_{\frac{1}{2}} x > \log_{\frac{1}{3}} x$ only when $0 < x < 1$.

Solution

$E \equiv \log_{\frac{1}{2}} x - \log_{\frac{1}{3}} x$ .s.s. $\left(\frac{1}{2} - 1\right)\left(\frac{1}{3} - 1\right)(x-1)\left(\frac{1}{3} - \frac{1}{2}\right)$ .s.s. $-(x-1)$.

Therefore, $\log_{\frac{1}{2}} x > \log_{\frac{1}{3}} x \iff E > 0 \iff x - 1 < 0 \iff 0 < x < 1$.

□ Find all positive integers x, y, z satisfying the equation $3^x + 4^y = 5^z$

Solution

Taking the original equation mod4, we find that $(-1)^x \equiv 1 \pmod 4$, so x is even. Let $x = 2x_1$. By similar reasoning, taking the original equation mod5 gives that z is even, allowing us to let $z = 2z_1$. Therefore, we have

$4^y = 2^{2y} = (5^{z_1} - 3^{x_1})(5^{z_1} + 3^{x_1})$

implying that we can let $5^{z_1} - 3^{x_1} = 2^a$ and $5_1^z + 3_1^x = 2^b$, where $a + b = 2y$. Adding the resulting system gives

$2^a + 2^b = 2(5^{z_1}) \implies 5^{z_1} = 2^{a-1} + 2^{b-1}$.

Therefore, since 5^{z_1} is odd and $a < b$, we have that $a - 1 = 0 \implies a = 1$.

Subtracting the first equation from the second equation in the new system therefore gives

$3^{x_1} = 2^{b-1} - 1$

Taking this equation mod3, we find that $(-1)^{b-1} \equiv 1 \pmod 3$, implying that $b-1$ is even. Let $b - 1 = 2c$. Therefore,

$3^{x_1} = (2^c - 1)(2^c + 1)$, so let $2^c - 1 = 3^u$ and $2^c + 1 = 3^v$, where $u + v = x_1$. Then, $3^v - 3^u = 2 \implies u = 0 \implies v = 1 \implies x_1 = z_1 = 1 \implies x = y = z = 2$.

Therefore, all solutions are given by $(x, y, z) \in \{(2, 2, 2)\}$.

□ Let there be a system of $2n-1$ equations, where $n \in \mathbb{N}$. The i^{th} equation is $x_i \cdot x_{i+1} = a_i$, for real variables x_i and real constants a_i for which $\prod_{i=1}^{2n-1} a_i > 0$. Note that in the $(2n-1)^{\text{th}}$ equation, $x_{(2n-1)+1} = x_{2n} = x_1$.

Solution

By multiplying all odd numbered equations and dividing by all even numbered equations, we get $\frac{\prod_{i=1}^{n}(x_{2i-1} \cdot x_{2i})}{\prod_{i=1}^{n-1}(x_{2i} \cdot x_{2i+1})} = \frac{\prod_{i=1}^{n} a_{2i-1}}{\prod_{i=1}^{n-1} a_{2i}} \implies x_1^2 = \frac{\prod_{i=1}^{n} a_{2i-1}}{\prod_{i=1}^{n-1} a_{2i}}$

$\implies \boxed{x_1 = \pm\sqrt{\frac{\prod_{i=1}^{n} a_{2i-1}}{\prod_{i=1}^{n-1} a_{2i}}}}$

Since $\boxed{x_i \cdot x_{i+1} = a_i}$ $\forall i \in \mathbb{N}, 1 \leq i \leq 2n-1$, all x_i with $2 \leq i$ can be recursively defined as $\boxed{x_i = \frac{a_i}{x_{i-1}}}$.

□ Prove that amongst six people in a room there are at least three who know one another or at least three who do not know one another.

Solution

Treat the six people as six vertices in a graph and name the vertices A, B, C, D, E, F. Pick a random vertex, say A. Color an edge black if the people represented by the vertices joined know each other and white if otherwise. The problem is solved when we obtain either a black triangle or a white triangle. By pigeonhole principle, A will be joined by at least 3 edges of the same color, let's say black. WLOG, assume that edges AB, AC, AD are black. If any of the edges BC, BD, CD is colored black, a black triangle is formed. If none, a white triangle BCD is formed. QED.

$\square$ In triangle $\triangle ABC$, $\angle A > \angle B$. Prove that $BC > AC$.

Solution

$\angle A > \angle B$ implies $\sin A > \sin B$. From the sine formula $\frac{BC}{\sin A} = \frac{AC}{\sin B}$, we get $BC > AC$. Even if $\angle A > \frac{\pi}{2}$, it won't reach the case when $\sin A < \sin B$ because for this case to occur, $\angle A + \angle B > \pi$ which is impossible.

$\square$ Let $A =1, 2, 3, 4, ..., n$, $n > 4$. Prove that we always can divide A into the two disjoint sets, S and P, such that the sum of elements of S is equal to the product of elements of P.

Solution

When n is even, choose $P = \{1, n, \frac{n-2}{2}\}$ and S the rest. When n is odd, choose $P = \{1, n-1, \frac{n+1}{2}\}$. It is easy to check that they work.

$\square$ Following equation:

$$21x - 25 + 2\sqrt{x-2} = 19\sqrt{x^2 - x + 2} + \sqrt{x+1}$$

Solution

Condition: $x \geq 2$ We have $21x - 25 + 2\sqrt{x-2} = 19\sqrt{x^2 - x - 2} + \sqrt{x+1}$

$\Leftrightarrow 21(x-2) + 2\sqrt{x-2} + 17 = 19\sqrt{(x+1)(x-2)} + \sqrt{x+1} \quad (1)$

We put: $\begin{cases} U = \sqrt{x-2} \geq 0 \\ V = \sqrt{x+1} \geq \sqrt{3} \end{cases}$

$\Rightarrow U^2 - V^2 + 3 = 0 \quad (*)$

Then: $(1) \Leftrightarrow 21U^2 + 2U + 17 = 19UV + V \Rightarrow V = \dfrac{21U^2 + 2U + 17}{19U + 1}$

Instead $(*)$ we obtain: $40U^4 + 23U^3 - 183U^2 - 23U + 143 = 0$

$\Leftrightarrow (U-1)(U+1)(5U+11)(8U-13) = 0$ By itself it is then.OK But the original problem solution like? anyone know?

$\square$ Let $mn + 1$ different real numbers be given. Prove that there is either an increasing sequence with at least $n + 1$ members or a decreasing sequence with at least $m + 1$ members.

Solution

The precise result is that if we are given a $mn + 1$-length sequence of distinct elements of a linearly ordered set we can find either a $m + 1$-length increasing or $n + 1$-length decreasing subsequence. As you state it (choose "members"...) we can simply re-order the numbers as we want!

This has been posted a lot of times before; just consider for the number in the k-th position the longest increasing subsequence starting with it, of length $\alpha(k)$. If there is some $\alpha(k) \geq m+1$, we are done. Otherwise, all $\alpha(k) \in \{1, \ldots, m\}$, so, by pigeonhole principle, there are $n + 1$ indices k with the same $\alpha(k)$-value, which must necessarily be in decreasing order.

$\square$ Prove that there is no function $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy

$$f(x + f(y)) = f(x) - y$$

for all $x, y \in \mathbb{Z}$.

Solution

We have $f^4(x+f(y)) = f^3(-y+f(x)) = f^2(-x+f(-y)) = f(y+f(-x)) = x+f(y)$, so $f^4 = \mathrm{id}_{\mathbb{Z}}$, since any integer z can be written as $z = x + f(y)$ (just take $x = z - f(0)$ and $y = 0$).

Therefore f is injective, so for $y = 0$, from $f(x+f(0)) = f(x) - 0 = f(x)$ follows $x + f(0) = x$, so $f(0) = 0$. Now, for $x = 0$, from $f(f(y)) = f(0+f(y)) = f(0) - y = -y$ follows $f^2 = -\mathrm{id}_{\mathbb{Z}}$. Therefore from $-x - f(y) = f^2(x+f(y)) = -x + f(-y)$ follows $f(-y) = -f(y)$. Take now $y = f(-z)$, so $f(x+z) = f(x+f(f(-z))) = f(x) - f(-z) = f(x) + f(z)$. This is the Cauchy equation, which on $\mathbb{Z}$ has as only solution $f(t) = f(1)t$. But then $-1 = f^2(1) = f(1)^2$, absurd.

Another way: $f(x+f(y)) = f(x) - y$

$f(f(x+f(y))+z) = f(f(x)-y+z)$

$f(z) - x - f(y) = f(z-y) - x$

$f(z) - f(y) = f(z-y)$

By putting $z = x + y$ we get

$f(x+y) = f(x) + f(y)$, which is Cauchy equation with solution $f(x) = cx$.

But now we have $c(x+cy) = cx - y \implies c^2 y = -y$ which is a contradiction.

□ Let ABC be a triangle. Prove that $\angle A = 60^\circ \iff s = \sqrt{3}(R+r)$ (s-semiperimeter, R-radius of the circumcircle, r-radius of the incircle).

Remarks.

▶ The equivalence $\boxed{A = 60^\circ \iff s = (R+r)\sqrt{3}}$ is false because the implication $\boxed{A = 60^\circ \implies s = (R+r)\sqrt{3}}$ is true and $\boxed{s = (R+r)\sqrt{3} \implies A = 60^\circ}$ is false.

▶ This equivalence $\boxed{60^\circ \in \{A, B, C\} \iff s = (R+r)\sqrt{3}}$ is true. Indeed, prove easily that

$$\left\{\begin{array}{c} \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{4R+r}{s} \\ \\ \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1 \\ \\ \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{r}{s} \end{array}\right\| \implies s \cdot \prod\left(1 - \sqrt{3}\cdot\tan\frac{A}{2}\right) = 4\left[s - (R+r)\sqrt{3}\right]$$

.

Other method. Prove easily that $\boxed{x+y+z=0 \implies \sum \sin x = -4\prod \sin\frac{x}{2}}$ $(*)$. Therefore, $s = (R+r)\sqrt{3} \iff$

$$\frac{s}{R} = \sqrt{3}\cdot\left(1+\frac{r}{R}\right) \iff \sum \sin A = \sqrt{3}\cdot\sum\cos A \iff \sum \sin\left(A - 60^\circ\right) = 0 \text{ . For } \left\{\begin{array}{c} x := A - 60^\circ \\ y := B - 60^\circ \\ z := C - 60^\circ \end{array}\right\|$$

, where

$x+y+z = 0$ apply the identity $(*)$. In conclusion, $s = (R+r)\sqrt{3} \iff \prod \sin\left(\frac{A}{2} - 30^\circ\right) = 0 \iff$ $60^\circ \in \{A, B, C\}$.

□ Let $x, y \geq 0, x + y = 1$. Find min,max $A = \sqrt{1+x^{2009}} + \sqrt{1+y^{2009}}$

Solution

Lets find extrema of the function $f = \sqrt{1+x^{2009}} + \sqrt{1+(1-x)^{2009}}$.

$f'(x) = \frac{2009x^{2008}}{2\sqrt{1+x^{2009}}} - \frac{2009(1-x)^{2008}}{2\sqrt{1+(1-x)^{2009}}} = 0 \implies$

$\implies \frac{\sqrt{1+x^{2009}}}{x^{2008}} = \frac{\sqrt{1+y^{2009}}}{y^{2008}} \implies \frac{1}{x^{4016}} + \frac{1}{x^{2007}} = \frac{1}{y^{4016}} + \frac{1}{y^{2007}}$.

Let $g(x) = x^{-4016} + x^{-2007}$.

$g'(x) = -4016x^{-4017} - 2007x^{-2008} < 0$ (remind that $x > 0$), so if $a > b$ we get $g(a) < g(b)$.

Hence $x^{-4016} + x^{-2007} = y^{-4016} + y^{-2007}$ only if $x = y$.

We have found the only extremum $x = 1 - x \implies x = \frac{1}{2}$.

Now it's easy to see that in $x = y = \frac{1}{2}$ function f has minimum and in $x = 0, y = 1$ it has maximum.

Answer: $f_{min} = 2\sqrt{1 + 2^{-2009}}$, $f_{max} = 1 + \sqrt{2}$.

$\square$ Solve the equation

$x + sinx = \pi$

Solution

$x + \sin x = \pi \Leftrightarrow \sin(\pi - x) = \pi - x \overset{(\pi - x) = t \in [-1;1]}{\rightarrow} \Rightarrow \sin t = t \Rightarrow \sin t - t = 0$; Let: $f(t) = \sin t - t, \forall t \in [-1;1]$; $f'(t) = \cos t - 1 \leq 0, \forall t \in [-1;1] \Rightarrow only : t = 0 \Rightarrow \boxed{x = \pi}$

$\square$ Find all positive integers c such that $a^3 + b^3 = c! + 4$ has solutions in integers.

Solution

Note that if $gcd(n, 9) = 1, n^6 \equiv 1 \mod 9 \implies n^3 \equiv \pm 1 \mod 9$ So if $c > 5, c! + 4 \equiv 4 \mod 9$.But $a^3 + b^3 \equiv 0 + 0, 0 + 1, 1 + 1, -1 - 1, -1 + 1 \mod 9$ So try with $c = 1, 2, 3, 4, 5$ $c = 1, c! + 4 = 5 \mod 9$,impossible. $c = 2, c! + 4 = 6 \equiv -3 \mod 9$,impossible. $c = 3, c! + 4 = 10$,no solution. $c = 4, c! + 4 = 28 = 3^3 + 1^3$ $c = 5, c! + 4 = 124 = 5^3 - 1^3$ Hence $\boxed{c \in \{4, 5\}}$

$\square$ Show that if the points of the plane are coloured with three colours, there will always exist two points of the same colour which are one unit apart.

Solution

Assume that we had a map $\mathbf{c} : \mathbb{R}^2 \to \{1, 2, 3\}$ such that for any segment AB of length 1, $\mathbf{c}(A) \neq \mathbf{c}(B)$.

Pick some point A at random and consider a point B such that $AB = \sqrt{3}$. Then there are points C, D such that ACD and BCD are equilateral, of side 1. Thus, $\mathbf{c}(A)$, $\mathbf{c}(C)$ and $\mathbf{c}(D)$ are all distinct, as are $\mathbf{c}(B)$, $\mathbf{c}(C)$ and $\mathbf{c}(D)$. We conclude that $\mathbf{c}(A) = \mathbf{c}(B)$, whence the circle $\mathcal{C}(A, \sqrt{3})$ is monochromatic - a contradiction since we can then choose a chord of $\mathcal{C}$ of length 1 whose endpoints will be of the same color.

$\square$ Solve the following equation : $x^x = x$ such that x is an integer

Solution

If $x > 0$ then $x \ln x = \ln x \iff (x - 1) \ln x = 0 \iff x = 1$

If $x < 0$ then $x = -m \implies \frac{1}{(-m)^m} = -m \implies m = 2k + 1, k \geqslant 0$ (both sides must have the same sign)

$-(2k + 1) \ln(2k + 1) = \ln(2k + 1) \iff 2(k + 1) \ln(2k + 1) = 0$, thus $k = 0 \iff x = -1$

Therefore the solutions are $x \in \{-1, 1\}$

$\square$ Calculate $\begin{cases} 3(x + \frac{1}{x}) = 4(y + \frac{1}{y}) = 5(z + \frac{1}{z}) \\ xy + yz + zx = 1 \end{cases}$

Solution

Substituting $x = \tan \frac{\alpha}{2}, y = \tan \frac{\beta}{2}, z = \tan \frac{\gamma}{2}$ we get $\alpha + \beta + \gamma = \pm \pi$ and $\frac{6}{\sin \alpha} = \frac{8}{\sin \beta} = \frac{10}{\sin \gamma}$, thus α, β, γ are the angles of an $6 - 8 - 10$ triangle (OR their negative counterparts), yielding $\cos \alpha = \frac{4}{5} \implies x = \tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \pm \frac{1}{3}, y = \pm \frac{1}{2}, z = \pm 1$

As all the variables must obviously have the same sign, the solutions are $(x, y, z) \in \left\{ \left(\frac{1}{3}, \frac{1}{2}, 1 \right), \left(-\frac{1}{3}, -\frac{1}{2}, -1 \right)\right.$

$\square$ Let $a^2 + b^2 < 1$ and $c^2 + d^2 < 1$. Prove that: $(a - c)^2 + (b - d)^2 \geq (ad - bc)^2$

Solution

However, initial statement is true as well.

Let $a = c + \psi_1, b = d + \psi_2$, and we don't mind whether $\psi_1, \psi_2 > 0$ or not.

Our statement rewrites as follows:

$\psi_1^2+\psi_2^2 \geq (d\psi_1 - c\psi_2)^2$

$\psi_1^2+\psi_2^2 \geq d^2\psi_1^2 + c^2\psi_2^2 - 2cd\psi_1\psi_2$

$\psi_1^2(1-d^2)+\psi_2^2(1-c^2) \geq -2cd\psi_1\psi_2$

$\psi_1^2 c^2+\psi_2^2 d^2 \geq -2cd\psi_1\psi_2$

$(\psi_1 c+\psi_2 d)^2 \geq 0$

□ When $\sum_{i=1}^{x}(2i-1)=n$ prove or disprove that $\sum_{i=1}^{n}\lfloor\sqrt{i}\rfloor=\sum_{i=1}^{x}(2i-1)i$ assuming i is a variable.

Solution

We know that :

$\sum_{i=1}^{x}(2i-1)=n$

And we know that : $\sum_{i=1}^{x}(2i-1)=\sum_{i=1}^{x}(2i)-\sum_{i=1}^{x}(1)=2(\frac{x(x+1)}{2})-x=x^2+x-x=x^2$. Such that $n=x^2$

Afterthat : $\sum_{i=1}^{n}\lfloor\sqrt{i}\rfloor=\sum_{i=1}^{x^2}\lfloor\sqrt{i}\rfloor=\lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+\ldots+\lfloor\sqrt{x^2}\rfloor$

It is equivalent, with your problem on Knockout Tournament. So we obtain :

$\lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+\ldots+\lfloor\sqrt{x^2}\rfloor$

$=1(3)+2(5)+\ldots+(x-1)(2x-1)+x=\sum_{i=1}^{x-1}i(2i+1)+x$

$=\sum_{i=1}^{x-1}(2i^2)+\sum_{i=1}^{x-1}(i)+x$

$=\frac{(x-1)(x)(2x-3)}{3}+\frac{(x-1)(x)}{2}+x$

In the other hand, we obtain :

$\sum_{i=1}^{x}(2i-1)i=\sum_{i=1}^{x}(2i^2)-\sum_{i=1}^{x}(i)$

$\frac{(x)(x+1)(2x-1)}{3}-\frac{(x)(x+1)}{2}$

And, we can solve the rest.

□ Prove that in $\triangle ABC$ there is the identity $\boxed{a\cdot\tan\frac{A}{2}+b\cdot\tan\frac{B}{2}+c\cdot\tan\frac{C}{2}=4R-2r}$.

Solution

Method 1. $\left\|\begin{array}{cc} r_a=s\cdot\tan\frac{A}{2} & (1) \\ \\ ar_a=s\left(r_a-r\right) & (2) \\ \\ r_a+r_b+r_c=4R+r & (3) \end{array}\right\| \Longrightarrow \sum a\cdot\tan\frac{A}{2} \overset{(1)}{=} \sum\frac{ar_a}{s} \overset{(2)}{=} \sum\left(r_a-r\right) \overset{(3)}{=} 2(2R-r)$

.

Method 2. $\left\|\begin{array}{cc} S=s(s-a)\tan\frac{A}{2} & (4) \\ \\ S=sr=(s-a)r_a & (5) \end{array}\right\| \Longrightarrow \sum a\cdot\tan\frac{A}{2} \overset{(4)}{=} S\cdot\sum\frac{a}{s(s-a)}=\sum\left(\frac{S}{s-a}-\frac{S}{s}\right) \overset{(5)}{=} \sum\left(r_a-r\right)=2(2R-r)$.

Applications. $\sum\frac{a}{r_a}\geq\frac{2s}{2R-r}$. Define $S_n=\sum a^n\cdot\tan\frac{A}{2}$, $n\in\mathrm{N}$. Then $\boxed{S_{n+1}=s\cdot S_n-r\cdot\sum a^n\ ,\ n\in\mathrm{N}}$

.

□

$$\begin{cases} x-2\sqrt{y+1}=3 \\ x^3-4x^2\sqrt{y+1}-9x-8y=-52-4xy \end{cases}$$

Solution

The second equation can be rewritten thus:

$x\left(x^2-4x\sqrt{y+1}+4(y+1)\right)-13x-8y=-52$

Hence by using the first equation we get

$9x-13x-8y=-52 \iff x+2y=13$

So
$13-2y=2\sqrt{y+1}+3 \implies \sqrt{y+1}=5-y$
Squaring, we get $y^2-11y+24=0 \implies y\in\{3,8\}$.
However $y=8$ doesn't work, hence the only solution is $y=3 \implies x=7$, ie. $(x,y)=(7,3)$
$\square$ Find the minimum value of the expression

$$P=\sqrt{2x^2+2y^2-2x+2y+1}+\sqrt{2x^2+2y^2+2x-2y+1}+\sqrt{2x^2+2y^2+4x+4y+4}$$

Solution

The expression rewrites as
$P=\sqrt{2}\left(\sqrt{\left(x-\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2}+\sqrt{(x+1)^2+(y+1)^2}\right)$
Thus the sum in the outermost parentheses will be minimized if (x,y) is the Fermat point of the triangle $\left(\frac{1}{2},-\frac{1}{2}\right),\left(-\frac{1}{2},\frac{1}{2}\right),(-1,-1)$

Since the triangle is isosceles, the Fermat point F lies on its symmetrial axis $y=x$, and since it makes the angle of 120° with the triangle basis, it's easy to calculate its position.

If I'm not mistaken, its coordinates are $F\left(-\frac{\sqrt{3}}{6},-\frac{\sqrt{3}}{6}\right)$, and $P_{\min}=2+\sqrt{3}$

$\square$ The equation $f_a(x)\equiv x^2+(a+2)x+a^2-a+2=0$ $(*)$ has real roots, where $a\in\mathbb{R}^*$. Find the range of these roots.

Solution

r is a root of $(*) \iff f_a(r)=\underline{r}^2+(a+2)\underline{r}+a^2-a+2=0$, i.e. $\underline{a}^2+(r-1)\cdot\underline{a}+(r^2+2r+2)=0$.

► $r\in\mathbb{R} \iff \Delta'_r(a)\equiv(a+2)^2-4\left(a^2-a+2\right)\geq 0 \iff 3a^2-8a+4\leq 0 \iff a\in\left[\frac{2}{3},2\right]$.

► $a\in\mathbb{R}^* \iff \Delta'_a(r)\equiv(r-1)^2-4\left(r^2+2r+2\right)\geq 0 \iff 3r^2+10r+7\leq 0 \iff$ $\boxed{r\in\left[-\frac{7}{3},-1\right]}$.

$\square$ Let P be an interior point of the square $ABCD$ so that $PA=1$, $PB=2$, $PC=3$. Find the length of $[AB]$.

Solution

Proof 1. Let $l=AB$ and $\phi=m\left(\widehat{ABP}\right)$. Thus, $m\left(\widehat{CBP}\right)=90^\circ-\phi$ and $l\sqrt{2}>3$, $l<1+2$, i.e. $l\in\left(\frac{3\sqrt{2}}{2},3\right)$ $(*)$.

Apply the generalized Pytagoras' theorem to : $\left\{\begin{array}{lcl} PA/\triangle ABP & \implies & 4l\cdot\cos\phi=l^2+3 \\ PC/\triangle CPB & \implies & 4l\cdot\sin\phi=l^2-5\end{array}\right\| \implies$

$\left(l^2+3\right)^2+\left(l^2-5\right)^2=16l^2 \iff l^4-10l^2+17=0 \iff l^2\in\{5\pm 2\sqrt{2}\} \overset{(*)}{\implies} \boxed{l=\sqrt{5*2\sqrt{2}}}$.

$\square$ Prove that for any natural number n, $a_n=\frac{240\cdot(4n+3)!}{n!\cdot(n+1)!\cdot(n+3)!\cdot(n+5)!}$ is an integer.

Solution

Consider the multinomial coefficient $C=\binom{N}{a_1,\ldots,a_k}$ (so $a_1+\cdots+a_k=N$); take $d=\gcd(a_1,\ldots,a_k)$; then $\dfrac{d}{N}C\in\mathbb{N}$.

Proof. There exist integers $u_1,\ldots,u_k$ such that $d=\sum_{j=1}^k u_ja_j$ (by Bézout's relation). Then $\dfrac{d}{N}C=\sum_{j=1}^k u_j\dfrac{a_j}{N}C=\sum_{j=1}^k u_j\binom{N-1}{a_1,\ldots,a_{j-1},a_j-1,a_{j+1},\ldots,a_k}$.

By repeatedly applying this to $240\binom{4n+9}{n,n+1,n+3,n+5}$, having $240=2^4\cdot 3\cdot 5$, and analyzing what greatest common divisor may the four elements at the denominators have, the thesis follows.

□ Prove that, if $n \geq 2$, for all natural n, $(n+1)\cos\left(\frac{\pi}{n+1}\right) - n\cos\left(\frac{\pi}{n}\right) \geq 1$

Solution

Your relation holds for $n = 1$ as well; anyway, assume now that $n \geq 2$.

Consider the Taylor series $\sum_{k\geq 0}(-1)^k \frac{x^{2k}}{(2k)!}$ of the cosine function. Trivial computations reduce your relation to $\sum f_k(y) \geq \sum f_k(x)$ where $y = \pi/n > x = \pi/(n+1)$ and

$f_k(t) = \frac{t^{2k-1}}{(2k)!} - \frac{t^{2k+1}}{(2k+2)!}$,

and all we need to see is that the derivatives of all f_k are non-negative on the interval $[0, 2]$.

□ Find all x, y such that: $\begin{cases} x^2 + y^2 - xy + 4y + 1 = 0 \\ y[7 - (x-y)^2] = 2(x^2+1) \end{cases}$

Solution

From the first equation, $x^2 + 1 = xy - y^2 - 4y$

Plugging that into the second equation, we get

$y[7 - (x-y)^2] = 2y(x - y - 4)$

If $y = 0$, then the first equation becomes $x^2 + 1 = 0$, which has no solution.

Thus $y \neq 0 \implies 7 - (x-y)^2 = 2(x-y) - 8 \iff (x-y)^2 + 2(x-y) - 15 = 0$

Hence $x - y \in \{-5, 3\}$

1. Plugging $y = x + 5$ into the second equation we get $-9(x+5) = x^2 + 1 \iff x^2 + 9x + 46 = 0$, which has no real solutions.

2. Plugging $y = x - 3$ into the second equation we get $-2(x-3) = 2(x^2+1) \iff x^2 + x - 2 = 0 \iff x \in \{-2, 1\}$

Hence the solutions are $(x, y) \in \{(-2, -5), (1, -2)\}$

□ $\left.\begin{cases} a_1,\ a_2, \ldots,\ a_n \in (0,\ +\infty) \\ S = \sum_{k=1}^{n} a_k \\ a_{n+1} = a_1 \end{cases}\right| \Longrightarrow \sum_{k=1}^{n} \sqrt{\frac{a_k + a_{k+1}}{2S - a_k - a_{k+1}}} \geq 2.$

Solution

$\sqrt{\frac{2S - a_k - a_{k+1}}{a_k + a_{k+1}}} \cdot 1 \leq \frac{\frac{2S - a_k - a_{k+1}}{a_k + a_{k+1}} + 1}{2} = \frac{S}{a_k + a_{k+1}}$

$\Rightarrow \sum \sqrt{\frac{a_k + a_{k+1}}{2S - a_k - a_{k+1}}} \geq \sum \frac{a_k + a_{k+1}}{S} = 2$

□ If $(1 + x + x^2)^n = k_0 + k_1.x + k_2.x^2 + k_3.x^3 + ... + k_{2n}.x^{2n}$, what is the value of : $k_0.k_1 - k_1.k_2 + k_2.k_3 - ....$???

Solution

The coefficients k_i are symmetric, ie. $k_i = k_{2n-i}$, which is obvious from

$\left(\frac{1}{x} + 1 + x\right)^n = \frac{k_0}{x^n} + \frac{k_1}{x^{n-1}} + \cdots + k_{2n-1}x^{n-1} + k_{2n}x^n$

Now just substitute $\frac{1}{x}$ for x to obtain the claim.

Therefore

$S = \sum_{i=0}^{2n-1}(-1)^i k_i k_{i+1} = \sum_{i=0}^{2n-1}(-1)^i k_{2n-i} i_{2n-1-i}$

Put $j := 2n - 1 - i \iff i = 2n - 1 - j$ to get

$S = \sum_{j=0}^{2n-1}(-1)^{2n-1}(-1)^{-j} k_j k_{j+1} = -\sum_{j=0}^{2n-1}(-1)^j k_j k_{j+1} = -S$

Hence $S = 0$

□ Let $S_1, S_2, ...S_{2011}$ be nonempty sets of consecutive integers such that any 2 of them have a common element. Prove that there is a positive integer that belongs to every $S_i, i = 1, ..., 2011$ (For example, $2, 3, 4, 5$ is a set of consecutive integers while $2, 3, 5$ is not.)

Solution

Let M be the minimum attained by $\max(S_k)$, m the maximum attained by $\min(S_k)$. Then $m \leq M$, so any $a \in [m, M]$ belongs to all S_k.

$\square$ Find the real values of m such that the following system of equations have solution:
$\begin{cases} x^2+\frac{4x^2}{(x+2)^2}\geq 5 \\ x^4+8x^2+16mx+16m^2+32m+16=0 \end{cases}$

Solution

Simplifying the first inequality we get

$x^4+4x^3+3x^2-20x-20\geqslant 0 \iff (x+1)(x-2)(x^2+5x+10)\geqslant 0$

Hence $x\leqslant -1 \vee x\geqslant 2$

Regarding the second equation as a quadratic in m, we must have a non-negative discriminant:

$-x^4-4x^2+16x\geqslant 0 \iff x(x-2)(x^2+2x+8)\leqslant 0$

Hence $0\leqslant x\leqslant 2$

Therefore the only possible common solution is $x=2$. Then the second equation becomes $(m+2)^2=0 \iff m=-2$

$\square$ In the triangle ABC prove that the perpendiculars from the Gergonne's point N to the interior bisectors are intersecting the sides of the triangle in 6 points that are situated on the same circle, concentric with $C(ABC)$.

Solution

Incircle (I) of $\triangle ABC$ touches BC, CA, AB at D, E, F. N is symmedian point of $\triangle DEF$. Perpendiculars to AI through N cut AB, AC at Z_a, Y_a, perpendiculars to BI through N cut BC, BA at X_b, Z_b, perpendiculars to CI through N cut CA, CB at $Y_c, X_c \Longrightarrow EY_a=FZ_a, FZ_b=DX_b, DX_c=EY_c$. BC is antiparallel of EF WRT $\angle FDE \Longrightarrow \triangle NX_bX_c\sim\triangle DEF$. DN is symmedian of $\triangle DEF \Longrightarrow ND$ is median of $\triangle NX_bX_c$. Similarly, NE, NF are medians of $\triangle NY_cY_a, \triangle NZ_aZ_b$ $\Longrightarrow X_b, X_c, Y_c, Y_a, Z_a, Z_b$ are on circle concentric with (I).

$\square$ Proof. $1.000\ldots001^{10^m}=\left[\left(1+\frac{1}{10^{p+1}}\right)^{10^{p+1}}\right]^{10^{m-p-1}} \overset{\text{(Bernoulli)}}{\geq} \left(1+10^{p+1}\cdot\frac{1}{10^{p+1}}\right)^{10^{m-p-1}}=$
$2^{10^{m-p-1}}=\left(2^{10}\right)^{10^{m-p-2}}=1024^{10^{m-p-2}}>1000^{10^{m-p-2}}$.

Solution

Let $\{p,m\}\subset\mathbb{N}$ so that $m\geq p+2$. Prove that $1.000\ldots001^{10^m}>1000^{10^{m-p-2}}$,

where the base from the left side has p zeroes after point, i.e. $1.000\ldots001=1+\frac{1}{10^{p+1}}$.

Particular case. $p=1$ and $m=3 \implies 1.01^{1000}>1000$ or more generally $\left(1+\frac{1}{10^p}\right)^{10^{p+1}}>1000$ for any $p\in\mathbb{N}$. Another way: Use the known inequality $\left(1+\dfrac{1}{n}\right)^{n+1}>\mathrm{e}$ for all integer $n\geq 1$. Then, since $\left(1+\frac{1}{10^{p+1}}\right)^{10^m}>\mathrm{e}^{10^m/(10^{p+1}+1)}$, it is enough to prove $\mathrm{e}^{10^m}>10^{3\cdot 10^{m-1}+3\cdot 10^{m-p-2}}$.

We will first prove by induction on p that $\mathrm{e}^{10^{p+2}}>10^{3\cdot 10^{p+1}+3}$. For $p=0$ it's $\mathrm{e}^{100}>10^{33}$, which is true. And $\mathrm{e}^{10^{p+3}}>\left(10^{3\cdot 10^{p+1}+3}\right)^{10}=10^{3\cdot 10^{p+2}+30}>10^{3\cdot 10^{(p+1)+1}+3}$.

But this is the base case for $m=p+2$ for the main inequality. We will prove that by induction on m now. $\mathrm{e}^{10^{m+1}}>\left(10^{3\cdot 10^{m-1}+3\cdot 10^{m-p-2}}\right)^{10}=10^{3\cdot 10^{(m+1)-1}+3\cdot 10^{(m+1)-p-2}}$.

$\square$ Find $x,y\in\mathbb{R}$ such that: $\begin{cases} 6x^2y+2y^3+35=0 \\ 5(x^2+y^2+x)+2xy+13y=0 \end{cases}$

Solution

Substitute $x=\frac{a+b}{2}, y=\frac{a-b}{2}$. Then the equations, after simplification, become
$\begin{cases} a^3-b^3=-35 \\ 3a^2+9a+2b^2-4b=0 \end{cases}$

The second equation yields $9a^2+27a=-6b^2+12b$, hence

$(a+3)^3-(b-2)^3=(a^3-b^3)+[(9a^2+27a)-(-6b^2+12b)]+27+8=-35+0+35=0$

Therefore $a=b-5$. Plugging that into the first equation we get

$b^2-5b+6=0 \iff b\in\{3,2\}$

Hence

$b=3 \implies a=-2 \implies (x,y)=\left(\frac{1}{2},-\frac{5}{2}\right)$

$b=2 \implies a=-3 \implies (x,y)=\left(-\frac{1}{2},-\frac{5}{2}\right)$

Both solutions satisfy the given system.

$\square$ Find the value of:

$cos\left(\frac{2\pi}{13}\right)+cos\left(\frac{6\pi}{13}\right)+cos\left(\frac{8\pi}{13}\right)$

Solution

Let $z_k, k=0,...,12$ be complex roots of $z^{13}-1=(z-1)(z^{12}+z^{11}+...+1)=0 \implies z_0=1$ and $z_k=\cos\frac{2\pi k}{13}+\mathrm{i}\cdot\sin\frac{2\pi k}{13}=\overline{z}_{13-k} \implies \sum_{k=1}^{12} z_k=\sum_{k=1}^{6}(z_k+\overline{z}_k)=2\sum_{k=1}^{6}\cos\frac{2\pi k}{13}=-1.$

Let $X=(z_1+\overline{z}_1)+(z_3+\overline{z}_3)+(z_4+\overline{z}_4)=2\left(\cos\frac{1\cdot 2\pi}{13}+\cos\frac{3\cdot 2\pi}{13}+\cos\frac{4\cdot 2\pi}{13}\right)$ and $Y=(z_2+\overline{z}_2)+(z_5+\overline{z}_5)+(z_6+\overline{z}_6)=2\left(\cos\frac{2\cdot 2\pi}{13}+\cos\frac{5\cdot 2\pi}{13}+\cos\frac{6\cdot 2\pi}{13}\right) \implies Y<0<X$ and $X+Y=-1.$

$X\cdot Y=$

$=(z_1+\overline{z}_1)(z_2+\overline{z}_2)+(z_1+\overline{z}_1)(z_5+\overline{z}_5)+(z_1+\overline{z}_1)(z_6+\overline{z}_6)+$

$+(z_3+\overline{z}_3)(z_2+\overline{z}_2)+(z_3+\overline{z}_3)(z_5+\overline{z}_5)+(z_3+\overline{z}_3)(z_6+\overline{z}_6)+$

$+(z_4+\overline{z}_4)(z_2+\overline{z}_2)+(z_4+\overline{z}_4)(z_5+\overline{z}_5)+(z_4+\overline{z}_4)(z_6+\overline{z}_6)=$

$=(z_1+\overline{z}_1)+(z_3+\overline{z}_3)+(z_4+\overline{z}_4)+(z_6+\overline{z}_6)+(z_5+\overline{z}_5)+(z_7+\overline{z}_7)+$

$+(z_1+\overline{z}_1)+(z_5+\overline{z}_5)+(z_2+\overline{z}_2)+(z_8+\overline{z}_8)+(z_3+\overline{z}_3)+(z_9+\overline{z}_9)+$

$+(z_2+\overline{z}_2)+(z_6+\overline{z}_6)+(z_1+\overline{z}_1)+(z_9+\overline{z}_9)+(z_2+\overline{z}_2)+(z_{10}+\overline{z}_{10})=$

$=(z_1+\overline{z}_1)+(z_3+\overline{z}_3)+(z_4+\overline{z}_4)+(z_6+\overline{z}_6)+(z_5+\overline{z}_5)+(z_6+\overline{z}_6)+$

$+(z_1+\overline{z}_1)+(z_5+\overline{z}_5)+(z_2+\overline{z}_2)+(z_5+\overline{z}_5)+(z_3+\overline{z}_3)+(z_4+\overline{z}_4)+$

$+(z_2+\overline{z}_2)+(z_6+\overline{z}_6)+(z_1+\overline{z}_1)+(z_4+\overline{z}_4)+(z_2+\overline{z}_2)+(z_3+\overline{z}_3)=$

$=3\sum_{k=1}^{6}(z_k+\overline{z}_k)=-3$

As a result, X,Y are roots of $\xi^2+\xi-3=0 \implies X=\frac{-1+\sqrt{13}}{2}>Y=\frac{-1-\sqrt{13}}{2}$.

$\square$

Let $a\le b\le c$ be real numbers such that :

$a+b+c=2$, And

$ab+bc+ca=1$.

Prove that $0\le a\le\frac{1}{3}\le b\le 1\le c\le\frac{4}{3}$

Solution

From $b+c=2-a\wedge bc=1-a(b+c)=1-a(2-a)=(a-1)^2$, we get that b,c are the solutions to the equation $t^2+(a-2)t+(a-1)^2=0$, whose discriminant must be non-negative. Hence $(a-2)^2-4(a-1)^2\geqslant 0 \iff a(3a-4)\leqslant 0 \iff 0\leqslant a\leqslant\frac{4}{3}$.

Thus a,b,c are all non-negative, so is their product.

From $a+b+c=2\wedge a\leqslant b\leqslant c$ we get $3a\leqslant 2 \iff a\leqslant\frac{2}{3}$, therefore $1-a$ is non-negative:

$p:=abc=a(bc)=a(1-a)^2=4a\left(\frac{1-a}{2}\right)^2 \overset{\text{AM-GM}}{\leqslant} 4\left(\frac{a+\frac{1-a}{2}+\frac{1-a}{2}}{3}\right)^3=\frac{4}{27}$, with the equality for $a=\frac{1-a}{2} \iff a=\frac{1}{3} \implies (b,c)=\left(\frac{1}{3},\frac{4}{3}\right)$

Consider the polynomial $Q(x)=x^3-2x^2+x-p$. We've already established $0\leqslant p\leqslant\frac{4}{27}$. Now we have

$Q(0)=-p\leqslant 0, Q\left(\frac{1}{3}\right)=\frac{4}{27}-p\geqslant 0, Q(1)=-p\leqslant 0, Q\left(\frac{4}{3}\right)=\frac{4}{27}-p\geqslant 0$

Therefore a,b,c, which are real roots of $Q(x)$, lie in the segments $\left[0,\frac{1}{3}\right],\left[\frac{1}{3},1\right],\left[1,\frac{4}{3}\right]$ respectively and the claim is thus proven.

□ Find the point P on BC , side of the $\triangle ABC$ such that $\frac{AB}{PD}+\frac{AC}{PE}$ is minimum where PD and PE are perpendiculars on AB and AC.

Solution

By Cauchy,

$(AB\cdot PD+AC\cdot PE)\left(\frac{AB}{PD}+\frac{AC}{PE}\right)\geqslant(AB+AC)^2$

but $AB\cdot PD+AC\cdot PE=2[ABC]$, hence

$\frac{AB}{PD}+\frac{AC}{PE}\geqslant\frac{(AB+AC)^2}{2[ABC]}$

The equality is attained iff $\frac{AB\cdot PD}{\frac{AB}{PD}}=\frac{AB\cdot PE}{\frac{AB}{PE}}\iff PD=PE$, ie. iff AP is the bisector of $\angle A$.

□ Show that the cube root of 3 cannot be the root of a quadratic equation with integer coefficients.

Solution

If $ax^2+bx+c=0$ is a quadratic with integer coefficients, then its roots are of the form $x_{1,2}=p\pm\sqrt{q}$ where p,q are rational numbers.

Thus $p\pm\sqrt{q}=\sqrt[3]{3}\iff p^3+3pq\pm(3p^2+q)\sqrt{q}=3$

Therefore either (i) $\sqrt{q}$ must be rational, but then $p\pm\sqrt{q}$ is rational too, which is impossible as it's equal to $\sqrt[3]{3}$ by the assumption, or (ii) $3p^2+q=0$ which (having in mind that q is radicand) can hold iff $p=q=0$, which doesn't satisfy the assumption. QED

□ It seems that for all natural n, $\prod_{k=1}^{n}\cos\frac{k\pi}{2n+1}=\frac{1}{2^n}$.

Solution

The identity is indeed true. Below is a fairly simple proof that does not require complex numbers or roots of unity.

Let $\cos\dfrac{\pi}{2n+1}\cos\dfrac{2\pi}{2n+1}...\cos\dfrac{n\pi}{2n+1}=x$. Then,

$$x=\cos\frac{\pi}{2n+1}\cos\frac{2\pi}{2n+1}...\cos\frac{n\pi}{2n+1}$$

$$x\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}...\sin\frac{n\pi}{2n+1}=\frac{1}{2^n}\sin\frac{2\pi}{2n+1}\sin\frac{4\pi}{2n+1}...\sin\frac{2n\pi}{2n+1}$$

where we have multiplied both sides of the equation by $\sin\dfrac{\pi}{2n+1}\sin\dfrac{2\pi}{2n+1}...\sin\dfrac{n\pi}{2n+1}$ and then used the fact that $\sin x\cos x=\dfrac{1}{2}\sin 2x$. We now apply $\sin x=\sin(\pi-x)$:

$$\sin\frac{2n\pi}{2n+1}=\sin\frac{(2n+1)\pi-2n\pi}{2n+1}=\sin\frac{\pi}{2n+1}$$

$$\sin\frac{2(n-1)\pi}{2n+1}=\sin\frac{(2n+1)\pi-2(n-1)\pi}{2n+1}=\sin\frac{3\pi}{2n+1}$$

This will repeat for all terms above $\dfrac{n\pi}{2n+1}$, and all the sine functions on both sides of the equation will cancel, leaving

$$x=\frac{1}{2^n}$$

. Therefore,

$$\prod_{k=1}^{n}\cos\frac{k\pi}{2n+1}=\frac{1}{2^n}\boxed{QED}$$

□ Find all positive n such that $n^5+n^4+n^3+n^2+n+1$ is a perfect square.

Solution

$n^5+n^4+...+1=(n^3+1)(n^2+n+1)$, by eulidean algorithm, $(n^3+1,n^2+n+1)=(n^2+n-1,n^2+n+1)=$

1, since $n(n+1)$ is always even. Thus, both of these term are squares. Since $n^2 < n^2+n+1 < (n+1)^2$ there are no solutions. □

□ $\begin{cases} (x+\sqrt{x^2+1})(y+\sqrt{y^2+1}) = 1 \\ y + \frac{y}{\sqrt{x^2-1}} + \frac{35}{12} = 0 \end{cases}$

Solution

For $f(y) = y + \sqrt{1+y^2}$ is monotone increasing , $f(y) = \frac{1}{x+\sqrt{1+x^2}} = -x + \sqrt{1+x^2} = f(-x)$ So $y = -x$ then $y + \frac{y}{\sqrt{y^2-1}} + \frac{35}{12} = 0$ Let $y = \sec x$ and the equation turns to $\sec x + \csc x = -\frac{35}{12}$ Let $\sin x + \cos x = t$ and the equation turn to be $24t = -35(t^2-1) \to t = \frac{5}{7}, -\frac{7}{5}$ Note that $\sin x \cos x > 0$ because $\frac{y^2}{\sqrt{y^2-1}} > 0$ So $t = -7/5$ and that turns to be $\sin x = -0.6, -0.8$ $y = -\frac{5}{3}, -\frac{5}{4}$

□ Simplify this expression

$S = \frac{n+\sqrt{n^2-1}}{\sqrt{n}+\sqrt{n+\sqrt{n^2-1}}} + \frac{n-\sqrt{n^2-1}}{\sqrt{n}-\sqrt{n-\sqrt{n^2-1}}}.$

Solution

$S = \frac{2n\sqrt{n}+n\left(\sqrt{n+\sqrt{n^2-1}}-\sqrt{n-\sqrt{n^2-1}}\right)-\sqrt{n^2-1}\left(\sqrt{n-\sqrt{n^2-1}}+\sqrt{n+\sqrt{n^2-1}}\right)}{n-1} = \frac{2n\sqrt{n}+n\left(\sqrt{\frac{n+1}{2}}+\sqrt{\frac{n-1}{2}}-\sqrt{\frac{n+1}{2}}+\sqrt{\frac{n-1}{2}}\right)-\sqrt{n^2-1}\Big(}{n-1}$

$= \frac{2n\sqrt{n}+2n\sqrt{\frac{n-1}{2}}-2(n+1)\sqrt{\frac{n-1}{2}}}{n-1} = \frac{2n\sqrt{n}-\sqrt{2}\sqrt{n-1}}{n-1}$

□ Calculate

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i^2 j}{5^i(j5^i+i5^j)}$$

Solution

Rewrite the sum as

$S = \sum_i \sum_j \frac{1}{\frac{5^i}{i}\left(\frac{5^i}{i}+\frac{5^i}{j}\right)}$

By exchanging the indices, we get that $S = \sum_i \sum_j \frac{1}{\frac{5^j}{j}\left(\frac{5^i}{i}+\frac{5^i}{j}\right)}$, hence

$2S = \sum_i \sum_j \left(\frac{1}{\frac{5^i}{i}\left(\frac{5^i}{i}+\frac{5^i}{j}\right)} + \frac{1}{\frac{5^j}{j}\left(\frac{5^i}{i}+\frac{5^i}{j}\right)}\right) = \sum_i \sum_j \frac{ij}{5^i 5^j} = \left(\sum_j \frac{j}{5^j}\right)^2$

Let $T = \sum_{j=1}^{\infty} \frac{j}{5^j}$. Then $T = \frac{1}{5} + \sum_{j=2}^{\infty} \frac{j}{5^j} = \frac{1}{5} + \sum_{j=1}^{\infty} \frac{j+1}{5^{j+1}}$

$T = \frac{1}{5} + \frac{1}{5}\sum_{j=1}^{\infty}\left(\frac{j}{5^j} + \frac{1}{5^j}\right)$

$T = \frac{1}{5} + \frac{1}{5}\left(T + \frac{\frac{1}{5}}{1-\frac{1}{5}}\right)$

$\frac{4T}{5} = \frac{1}{4}$

$T = \frac{5}{16}$

Now $S = \frac{T^2}{2} = \frac{25}{512}$

□ $\left[\frac{8x+1}{6}\right] + \left[\frac{4x-1}{3}\right] = \frac{16x-7}{9}$

Solution

Since $\frac{16x-7}{9}$ is an integer, put $\frac{16x-7}{9} = k, k \in \mathbb{Z}$. Then $x = \frac{9k+7}{16}$, hence the equation becomes

$\left[\frac{3k+3}{4}\right] + \left[\frac{3k+1}{4}\right] = k$

Using $\left[t+\frac{1}{2}\right] = [2t] - [t]$ on $\left[\frac{3k+1}{4}+\frac{1}{2}\right]$, this becomes

$\left[\frac{3k+1}{2}\right] = k$

$k \leqslant \frac{3k+1}{2} < k+1$

$-1 \leqslant k < 1$

Hence $k \in \{-1, 0\} \iff x \in \left\{-\frac{1}{8}, \frac{7}{16}\right\}$

□ Let ABC be a nonisosceles and acute triangle with $a \le b \le c$. Denote its circumcirle $C(O,R)$ and its orthocenter H .

Prove easily that this hexagon is inscribed in the Euler's circle $w = C\left(E, \frac{R}{2}\right)$, where E is the midpoint of $[OH]$.

If X , Y , Z are the midpoints of $[HA]$, $[HB]$, $[HC]$ respectively, then $\{X,Y,Z\} \subset w$, $E \in M_aX \cap OH$, i.e.

XOM_aH is parallelogram and $OA \perp F_bF_c$, where $F_bF_c = a \cdot \cos A$. For $a \le b \le c$ the hexagon $M_aF_aF_bM_bM_cF_c$

is convex and $\left\{\begin{array}{ccc} M_aF_a = \frac{c^2-b^2}{2a} & ; & F_aF_b = c \cdot \cos C \\ F_bM_b = \frac{c^2-a^2}{2b} & ; & M_bM_c = \frac{a}{2} \\ M_cF_c = \frac{b^2-a^2}{2c} & ; & F_cM_a = \frac{a}{2} \end{array}\right\|$. From here show that this hexagon isn't regularly. 2 ► $\left\{\begin{array}{l} \text{Method 1 :} \quad M_aF_a = \left|\frac{a}{2} - c \cdot \cos B\right| = \frac{\left|a^2-2ac\cdot\cos B\right|}{2a} = \frac{\left|a^2-\left(a^2+c^2-b^2\right)\right|}{2a} \\ \text{Method 2 :} \quad M_aF_a = \frac{1}{2} \cdot |F_aB - F_aC| = \frac{\left|F_aB^2-F_aC^2\right|}{2a} = \frac{\left|\left(c^2-h_a^2\right)-\left(b^2-h_a^2\right)\right|}{2a} \end{array}\right\| \Longrightarrow$

$\boxed{M_aF_a = \dfrac{|b^2-c^2|}{2a}}$. Proposed problem. Prove that $\phi = m\left(\widehat{AM_aF_a}\right) \Longrightarrow 4S = |b^2-c^2| \cdot \tan\phi$.

Observe that $F_aF_b = M_bM_c \iff A = 2C$ or $B = C$ and $M_aF_a = F_bM_b \iff A = 120^\circ$ or $A = B$.

□ Let ABCD be a cyclic quadrilateral inscribed in a circle O with diagonals AC and BD perpendicular at X. Denote P,Q,R,S as the projections of X onto the sides of the quadrilateral. Denote J,K,L,M as the midpoints of the sides of the quadrilateral.

Show that P,Q,R,S,J,K,L, and M lie on a circle with the center as the midpoint of OX.

Solution

Prove easily that $JKLM$ is a rectangle inscribed in the circle with diameters JL and MK . Since $\widehat{PXA} \equiv \widehat{ABD} \equiv \widehat{ACD} \equiv \widehat{CXL}$ obtain that $\widehat{PXA} \equiv \widehat{CXL}$, i.e. $X \in PL$. Show analogously $X \in JR \cap QM \cap KS$, i.e. the points $\{P,Q,R,S\}$ belong to the circle with diameters JL and MK .Since $OJXL$, $OKXM$ are two parallelograms with common diagonal $[OX]$ obtain easily required conclusion. □

Let us agree to say that a non-negative integer is "scattered" if its binary expansion has no occurence of two ones in a row. For example, 37 is scattered but 43 is not, since the binary expansion of 37 is 100101 in which the ones are all separated by at least one zero, while the binary expansion of 43 is 101101 which has two ones in successive places. For an integer $n \ge 0$, how many scattered non-negative integers are there less than 2^n?

Solution

Let a_n be the number of such binary strings with n digits. Among them, let 0_n be the number of those which end in 0 and 1_n the number of those which end in 1. Then

$a_n = 0_n + 1_n$ $0_{n+1} = 1_n + 0_n$ [zero can appear after any digit] $1_{n+1} = 0_n$ [one can appear only after zero]

Thus $0_{n+1} = a_n \implies 0_n = a_{n-1} \implies 1_{n+1} = a_{n-1}$

$a_{n+1} = 0_{n+1} + 1_{n+1} = a_n + a_{n-1}$, and with $a_1 = 2, a_2 = 3$, we see that $a_n = F_{n+2}$, where F_n is Fibonacci sequence ($F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n$).

□ Find all the natural numbers N such that N is one unit greater than the sum of squares of its

Solution

Let k be the number of digits of the desired number $N = \overline{a_1a_2\ldots a_k}$. Then $10^{k-1} \leqslant N = a_1^2 + \cdots + a_k^2 + 1 \leqslant 81k + 1$. It is easily shown that this can't be satisfied for $k \geqslant 4$.

Thus $k = 1, 2, 3$

(i) $k = 1$. The equation $a = a^2 + 1$ has no integer solution.

(ii) $k = 2$. If $a^2 + b^2 + 1 = 10a + b$, then $(2a-10)^2 + (2b-1)^2 = 97$. Checking all the even squares (because of $(2a-10)^2$) less than 100, we find that only 16 works. Thus $2b - 1 = 9 \iff b = 5$ and $|2a - 10| = 4 \iff a = 5 \pm 2$. Therefore the solutions are $N = 35$ and $N = 75$.

(iii) $k = 3$. If $a^2 + b^2 + c^2 + 1 = 100a + 10b + c$, then $(100 - 2a)^2 + (2b - 10)^2 + (2c - 1)^2 = 10097$. Assume $a \geqslant 1$. Then $(100 - 2a)^2 \leqslant 98^2 = 9604 \implies (2b - 10)^2 + (2c - 1)^2 \geqslant 493$, but that can't be achieved as $(2b - 10)^2 + (2c - 1)^2 \leqslant 10^2 + 17^2 = 389$. Thus $a = 0$ and there are no three-digit solutions.

Therefore all the solutions are $N = 35$ and $N = 75$.

□ Prove that $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = 0$, where a, b, c are the roots to $x^3 - 3x^2 - 5x - 1 = 0$

Solution

$x^3 - 3x^2 - 5x - 1 = 0$ $x^3 - 3x^2 - 5x - 1 + (8x) = 8x$ $x^3 - 3x^2 + 3x - 1 = 8x$ $(x-1)^3 = 8x$ $(x-1) = 2\sqrt[3]{x}$

If a,b,c are the roots of the equation:

$(a-1) = 2\sqrt[3]{a}$ $(b-1) = 2\sqrt[3]{b}$ $(c-1) = 2\sqrt[3]{c}$

$(a-1) + (b-1) + (c-1) = 2(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c})$

$(a+b+c) - 3 = 2(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c})$ $3 - 3 = 2(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c})$

Then,

$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = 0$ Another way: We apply the identity mentioned by mavroperevna. It states for x, y, z, we have the following relation:

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$.

Hence if $x + y + z = 0$, then $x^3 + y^3 + z^3 - 3xyz$ is zero, and the converse is also true. Let $x = \sqrt[3]{a}, y = \sqrt[3]{b}, z = \sqrt[3]{c}$. By Vieta's, $x^3 + y^3 + z^3 = a + b + c = 3$ and $xyz = \sqrt[3]{abc} = \sqrt[3]{1} = 1$. Thus $x^3 + y^3 + z^3 - 3xyz = 3 - 3(1) = 0$. Hence $x + y + z = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = 0$.

□ Find all polynomials in complex coefficients, such that $P(x^2) = P(x)^2$.

Solution

We will first prove that such a polynomial, if it is not constant, must be monic. Proof: Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + ...a_1 x + a_0$ $P(x^2) = a_n x^{2n} + stuff$ and $P(x)^2 = (a_n)^2 x^{2n} + stuff$ For this to be an identity; to work for all x in the domain of P, the coefficients must match so $a_n = (a_n)^2$ The only two solutions are $a_n = 0, 1$, but $a_n \neq 0$ because then the degree of P would be $n - 1$. Thus, $a_n = 1$ and P is monic. Now look at $P(x^2)$. There is no term of degree $2n - 1$, but in $P(x)^2$, there is, and it is given by $2(1)(a_{n-1})x^{2n-1}$. For the condition to be an identity, the coefficient of this term must be 0 to match the other side, so $a_{n-1} = 0$. In this manner, we can easily use strong induction to prove that $a_{n-1}, a_{n-3}, a_{n-5}...$ are all 0. Proof: From the coefficient of $x^{2n-1} = 0$ in $P(x)^2$, we have that $a_{n-1} = 0$, because $a_n = 1$. This is the base case. Since $a_{n-1} = 0$, we have that a_{n-3}=0 from the coefficient of x^{2n-3} in $P(x)^2$, once again, because a_n cannot be 0. Suppose we have proven that $a_{n-1}, a_{n-3}, ...a_{n-(2k-1)}$ are all 0. Now, the coefficient of $x^{2n-(2k+1)}$=$2(a_n)(a_{n-(2k+1)})$+$2(a_{n-1})(a_{n-2k})$+$2(a_{n-2})(a_{n-(2k-1)})$+...+$2(a_{n-k})(a_{n-k-1})$=0, because the coefficient of this term on the other side is 0. By parity, one of the terms in each product is of the form $a_{2k-(2m-1)}$, where m is an integer at least 1. Thus, all terms but the first are obviously 0 by the inductive hypothesis.

Then, since $a_n = 1$, $(a_{n-(2k+1)})$=0. Thus, we are done, and $a_{n-1}, a_{n-3}, ...a_1 = 0$.

We can do a similar thing for the evens. We claim that $a_{n-2}, a_{n-4}, ...a_2 = 0$ Proof: We use strong induction. Base case: $a_{n-1}x^{2n-2} = [(a_{n-1}^2 + 2(a_n)(a_{n-2})]x^{2n-2}$. Since $a_{n-1} = 0$ and a_n cannot be 0,

$a_{n-2} = 0$.

Assume that we have proved that $a_{n-2}, a_{n-4}, ...a_{n-2k} = 0$. Then, $a_{n-(2k-1)}x^{2n-4k+2}$=$[a^2_{n-(2k-1)}$+$2(a_n)(a_{n-(2k$

Then, all the terms $a_{n-(2m-1)}$ disappear since they are 0, as above. By the inductive hypothesis, all we have left is that $2(a_n)(a_{n-(2k+2)}) = 0$, or $a_{n-(2k+2)}$=0 since a_n cannot be 0, as desired. Thus, $P(x)$, if not constant, is of the form $x^n + c$ for some constant c. One can easily verify that this type of form can not satisfy the equation unless $c = 0$.

Proof: Assume this works and c is not 0. Then, we have $x^2n + c = x^2n + 2cx^n + c^2$. Comparing coefficients, $c = c^2$, and $2c = 0$, and c must be 0, which is a contradiction. Thus, polynomials x^n for any positive integers n, work. For constant functions, plugging in $x = 0$, $P(0) = P(0)^2$, so $P(0) = 0, 1$. Therefore, our only options for a constant P are 0 and 1, which both work. $P = 0, 1, x^n$ Q.E.D

□ Given a positive integer n, and the set $M = \{1, 2, 3, ...50\}$, we choose 35 elements of M. Among these 35 elements of M, there always exists 2 distinct numbers a, b such that $a + b = n$ or $a - b = n$. Find all possible values of n.

Solution

Take $1, 2, ..., 35$ to see we can't reach 69.

Now we prove 1 tot 69 is possible.

$a - b = x$

$x \in \{1, ..., 15\}$ is possible by looking to subsets $\{(1, 1+x), \cdots, (x-1, 2x-1), (2x, 3x), \cdots, (3x-1, 4x-1)\cdots\}$ with $x \in \{16, .., 34\}$ is possible by looking to subsets $\{(1, 1+x), \cdots, (16, x+16)\}$

It is easy to find we have in each time constructed at least 16 distint subsets with each numbers less than 51 and hence because we have only 15 numbers we don't take, there are two of that set.

At the same way: $a + b = x$ take $\{(1, x-1)\cdots, (17, x-17)\}$ for $35 \leq x \leq 51$ and $\{50, x-50, \cdots, 35, x-35\}$ for $x \in [52, 69]$

and use again PHP to see the result.

□ prove that the expression

$$n + \left[\sqrt[3]{n - \frac{1}{27}} + \frac{1}{3}\right]^2$$

is not the cube of any integer with n is an integer

Solution

Supposing that If $n > 0$ $t \leq \sqrt[3]{n - \frac{1}{27}} + \frac{1}{3} < t + 1 \rightarrow t^3 - t^2 + \frac{t}{3} < n < t^3 + 2t^2 + \frac{4t}{3} + \frac{1}{3}$ $\rightarrow t^3 + \frac{t}{3} < n + t^2 < t^3 + 3t^2 + \frac{4t}{3} + \frac{1}{3}$ If $n + t^2$ is a cube then $n + t^2 \geq t^3 + \frac{t}{3} > t^3$ $n + t^2 < t^3 + 3t^2 + \frac{4t}{3} + \frac{1}{3} < t^3 + 3t^2 + 3t + 1 = (t + 1)^3$ So $t^3 < n + t^2 < (t + 1)^3$ cannot be a cube. If $n \leq 0$ the statement is not true because when $n = t^3 - t^2 (t < 0) \rightarrow LHS = t^3$ Q.E.D

□ Solve the system equations :

$\frac{a}{x} + \frac{b}{y} = (3x^2 + y^2)(x^2 + 3y^2)$

$\frac{a}{x} - \frac{b}{y} = 2(y^4 - x^4)$

Solution

$$\begin{cases} \frac{a}{x} + \frac{b}{y} = 3x^4 + 10x^2y^2 + 3y^4 \\ \frac{a}{x} - \frac{b}{y} = -2x^4 + 2y^4 \end{cases}$$

Adding the equations up and multiplying by x we get

$2a = x^5 + 10x^3y^2 + 5xy^4 \quad (1)$

Subtracting the second equation from the first and multiplying by y we get

$2b = 5x^4y + 10x^2y^3 + y^5 \quad (2)$

Adding (1) and (2) we get

$2(a+b)=(x+y)^5 \iff x+y=\sqrt[5]{2(a+b)}$

Subtracting (2) from (1) we get

$2(a-b)=(x-y)^5 \iff x-y=\sqrt[5]{2(a-b)}$

Therefore $(x,y)=\left(\frac{\sqrt[5]{2(a+b)}+\sqrt[5]{2(a-b)}}{2},\frac{\sqrt[5]{2(a+b)}-\sqrt[5]{2(a-b)}}{2}\right)$

Since x,y can't be zero, the necessary condition is $ab\neq 0$.

□ $\begin{cases} x^2+y^2=\dfrac{1}{5} & (1)\\ 4x^2+3x-\dfrac{57}{25}=-y\left(3x+1\right) & (2)\end{cases}$

Solution

Multiply the second equation by 2 and rearrange:

$8x^2+6xy+6x+2y=\frac{114}{25}$

$(3x+y)^2-x^2-y^2+2(3x+y)=\frac{114}{25}$

Now use the first equation:

$(3x+y)^2+2(3x+y)=\frac{114}{25}+\frac{1}{5}$

Put $t=3x+y$ to get

$t^2+2t-\frac{119}{25}=0$

$t_{1,2}=\frac{-2\pm\sqrt{\frac{576}{25}}}{2}=\frac{-2\pm\frac{24}{5}}{2}$

$t_1=\frac{7}{5},t_2=-\frac{17}{5}$

(i) $3x+y=\frac{7}{5} \implies y=\frac{7}{5}-3x \implies x^2+\frac{49}{25}-\frac{42}{5}x+9x^2=\frac{1}{5}$

$5x^2-\frac{21}{5}x+\frac{22}{25}=0$

$x_1=\frac{11}{25} \implies y_1=\frac{2}{25}$

$x_2=\frac{2}{5} \implies y_2=\frac{1}{5}$

(ii) $3x+y=-\frac{17}{5} \implies y=-\frac{17}{5}-3x \implies x^2+\frac{289}{25}+\frac{102}{5}x+9x^2=\frac{1}{5}$

$5x^2+\frac{51}{5}x+\frac{142}{25}=0$

The discriminant of this equation is negative, hence the solutions are complex.

Thus all the real solutions are $(x,y)=\left(\frac{11}{25},\frac{2}{25}\right)$ and $(x,y)=\left(\frac{2}{5},\frac{1}{5}\right)$

(Though for the sake of completeness, the complex solutions are $(x,y)_{1,2}=\left(\frac{-51\pm i\sqrt{239}}{50},\frac{-17\mp 3i\sqrt{239}}{50}\right)$)

□ Find all polynomials that satisfy the equation $(x+1)P(x)=(x-10)P(x+1)$

Solution

If this is for all $x\in\mathbb{R}$ then you can put x = 10 to obtain $11P(10)=0$ so 10 is a root;

Set x = -1 to get $-11P(0)=0$ so 0 is also a root;

Set x = 0 to get $-10P(1)=P(0)=0$, so 1 is also a root;

Set x = 1 to get that 2 is also a root, and so on, so all integers between 0 and 10 inclusive are roots.

So you can let $P(x)=x(x-1)(x-2)...(x-10)\cdot Q(x)$ for some polynomial $Q(x)$.

Substitute in the original equation to get that $Q(x)=Q(x+1)$ so $Q(x)$ is a constant, let it be c.

Therefore the solutions are $P(x)=c\cdot x(x-1)(x-2)...(x-10)$ for some real costant c.

□ If $gcd(a,b)=1$, prove that:

(a) $gcd(a-b,a+b)\leq 2$, (b) $gcd(a-b,a+b,ab)=1$, (c) $gcd(a^2-ab+b^2,a+b)\leq 3$.

Solution

(a) $\gcd(a-b,a+b)=\gcd(2a,a+b)\leq\gcd(2a,2(a+b))=\gcd(2a,2b)=2$ $(b)A=\gcd(a-b,a+b,ab)\leq\gcd(a+b,a-b)\leq 2$ since (a) is true.If $A=2$, $2\mid a,b$ which contradicts a and b are relatively prime. (c) If $A=\gcd(a^2-ab+b^2,a+b)>3$ $a=-b(\mod A) \implies a^3=-b^3(\mod A)$

$0 = a^2 - ab + b^2 = (a+b)(a^2 - ab + b^2) = a^3 - b^3 (\mod A) \implies a^3 = b^3 (\mod A)$. So $2a^3, 2b^3$ numbers are divisible by A . So we can say that $1 = \gcd(a,b) > 1$ since $A > 3$. Done!

□ Demonstrate the inequality:
$\frac{2}{2!} + \frac{7}{3!} + \frac{14}{4!} + ... + \frac{k^2-2}{k!} + ... + \frac{9998}{100!} < 3$

Solution

$\frac{k^2}{k!} = \frac{k}{(k-1)!} = \frac{k-1+1}{(k-1)!} = \frac{1}{(k-2)!} + \frac{1}{(k-1)!}$ and so $\frac{k^2-2}{k!} = \frac{1}{(k-2)!} + \frac{1}{(k-1)!} - \frac{2}{k!}$.

Let S be the infinite series $\frac{2}{2!} + \frac{7}{3!} + \frac{14}{4!} + ... + \frac{k^2-2}{k!} + ...$ Then

$$S = \frac{2}{2!} + \left(\frac{1}{1!} + \frac{1}{2!} - \frac{2}{3!}\right) + \left(\frac{1}{2!} + \frac{1}{3!} - \frac{2}{4!}\right) + \left(\frac{1}{3!} + \frac{1}{4!} - \frac{2}{5!}\right) + ... =$$

$$= \frac{2}{2!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{2!} = 3$$

with the rest of the terms cancelling. Since all the terms are positive, the given series

$$\frac{2}{2!} + \frac{7}{3!} + \frac{14}{4!} + ... + \frac{k^2-2}{k!} + ... + \frac{9998}{100!} < S = 3$$

as required.

□ When $x^n + x^{n-1} + ... + x + 1$ is divided by $x^2 + x + 1$, find the quotient and remainder with proof.

Solution

If $P_n(x) = x^n + x^{n-1} + \cdots + 1$ and $R_n(x)$ is the desired remainder, then $P_{n+1}(x) = xP_n(x) + 1 \implies R_{n+1}(x) \equiv xR_n(x) + 1 \pmod{x^2+x+1}$

Hence $R_0(x) = 1, R_1(x) = x+1, R_2(x) = 0, R_3(x) = 1 = R_0(x)$ etc. with period 3.

Thus if k is a non-negative integer, then $R_n(x) = \begin{cases} 1 & n = 3k \\ x+1 & n = 3k+1 \\ 0 & n = 3k+2 \end{cases}$

Therefore the quotient $Q_n(x)$ is $Q_n(x) = \begin{cases} x^{n-2} + x^{n-5} + \cdots + x & n = 3k \\ x^{n-2} + x^{n-5} + \cdots + x^2 & n = 3k+1 \\ x^{n-2} + x^{n-5} + \cdots + 1 & n = 3k+2 \end{cases}$

□ In a quadrilateral $ABCD$ let K be a point inside the triangle ABD such that $\triangle ABD \sim \triangle KCD$. Prove that $\triangle BCD \sim \triangle AKD$ as well.

Solution

$$\triangle ABD \sim \triangle KCD \overset{(S.A.S)}{\Longleftrightarrow} \left| \begin{array}{c} \frac{BD}{CD} = \frac{DA}{DK} \\ \\ \widehat{ADB} \equiv \widehat{KDC} \end{array} \right| \Longleftrightarrow \left| \begin{array}{c} \frac{BD}{AD} = \frac{CD}{KD} \\ \\ \widehat{BDC} \equiv \widehat{ADK} \end{array} \right| \overset{(S.A.S)}{\Longleftrightarrow} \triangle BCD \sim \triangle AKD \ .$$

Example. The proof of the Ptolemy's theorem : $ef = ac + bd \iff ABCD$ is cyclically.

Proof. Let $ABCD$ be a convex quadrilateral. Denote $\left| \begin{array}{ccc} AB = a & BC = b & CD = c \\ DA = d & AC = e & BD = f \end{array} \right|$. Construct $\triangle DEC \sim \triangle ABC$

so that the line CD doesn't separate E and A . Thus, $\frac{AB}{DE} = \frac{CA}{CD} \implies \boxed{DE = \frac{ac}{e}}$. Observe that $\widehat{BCE} \equiv \widehat{ACD}$

□ Suppose $\binom{n_1}{3} + \binom{n_2}{3} + \binom{n_3}{3} = 2012$. What is the maximum value of $n_1 + n_2 + n_3$?

Solution

$\binom{n}{3} = 0 \mod 5$ if $n = 0, 1, 2 \mod 5$ $\binom{n}{3} = 1 \mod 5$ if $n = 3 \mod 5$ $\binom{n}{3} = -1 \mod 5$ if $n = 4$

mod 5. $2012 = 2 \mod 5$, therefore 2 of $n_i = 3 \mod 5$. $\binom{3}{3} = 1, \binom{8}{3} = 56, \binom{13}{3} = 286, \binom{18}{3} = 816, \binom{23}{3} = 1771$. Let $n_1 \le n_2, n_1 \equiv n_2 = 3 \mod 5$. It is easy to chek, that $n_3 \le 22$ and $n_3 = 22, 21$ are not solution. If $n_3 = 20$ we get solution $n_1 = 8, n_2 = 18, n_3 = 20$. $n_3 = 15, 16, 17$ are not solution. If $n_3 \le 12$, then $n_2 = 23$ and $\binom{n_1}{3} + \binom{n_3}{3} = 241$. Therefore $n_1 = 8, \binom{n_3}{3} = 185$ - not solution. $(n_1 = 8, n_2 = 18, n_3 = 20$ is unique solution.

□ Prove that if $n \mid \underbrace{111...1}_{n \text{ ori}}$ then $3 \mid n$

Solution

Clearly $n > 1$ with that property must be odd (the OP forgot to restrict to $n > 1$). Let p be the least prime dividing n (so $p \ge 3$). Now, the repunit divided by n is equal to $\frac{10^n - 1}{10-1}$, hence $9n \mid 10^n - 1$, and so a fortiori $p \mid n \mid 10^n - 1$. Let ν be the order of 10 modulo p; then we must have both $\nu \mid n$ and $\nu \mid p - 1$ (by Fermat's little). However, any prime in the factorization of $p - 1$ must be less than p, and so cannot divide n, hence $\gcd(n, p-1) = 1$, and so $\nu = 1$. This means $10^1 - 1 = 9 \equiv 0 \pmod{p}$, forcing $p = 3$, and so $3 \mid n$.

□ Solve the system of equations $\begin{cases} x^3 = 6z^2 - 12z + 8 \\ y^3 = 6x^2 - 12x + 8 \\ z^3 = 6y^2 - 12y + 8 \end{cases}$.

Solution

Observe that $x^3 = 2\left(3z^2 - 6z + 4\right) \ge 2 \implies x \ge \sqrt[3]{2}$. Prove analogously that $\{x, y, z\} \subset I = \left[\sqrt[3]{2}, \infty\right)$.

Consider the functions $f(x) = x^3\ (\nearrow)$ and $g(x) = 6x^2 - 12x + 8\ (\nearrow)$, where $x \in I$. Observe that these functions

are strict increasing and our system becomes $\begin{cases} f(x) = g(z) \\ f(y) = g(x) \\ f(z) = g(y) \end{cases}$. Thus, $\underline{x \le y} \iff f(x) \le f(y) \iff g(z) \le g(x) \iff$

$\underline{z \le x} \iff f(z) \le f(x) \iff g(y) \le g(z) \iff \underline{y \le z}$. In conclusion, $z \le x \le y \le z$, i.e. $\boxed{x = y = z = 2} \ge \sqrt[3]{2}$.

and $\frac{BC}{CE} = \frac{AC}{CD}$, what means $\triangle BCE \sim \triangle ACD \implies \frac{BC}{AC} = \frac{BE}{AD} \implies \boxed{BE = \frac{bd}{e}}$. In conclusion, $ABCD$ is

cyclically $\iff B + D = 180^\circ \iff E \in BD \iff BD = BE + ED \iff f = \frac{bd}{e} + \frac{ac}{e} \iff$ $\boxed{\boxed{ef = ac + bd}}$.

Remark. We can construct the point E outside of quadrilateral $ABCD$ and the proof in this case is likewise.

□ Prove that

$C \equiv \sum_{k=0}^{n} C_n^k \cos(k+1)x = C_n^0 \cos x + C_n^1 \cos 2x + C_n^2 \cos 3x + \ldots + C_n^n \cos(n+1)x = 2^n \cos^n \frac{x}{2} \cos \frac{(n+2)x}{2}$

Solution

$C + i \cdot S \equiv \sum_{k=0}^{n} C_n^k \cos(k+1)x + i \cdot \sum_{k=0}^{n} C_n^k \sin(k+1)x = \sum_{k=0}^{n} C_n^k \left[\cos(k+1)x + i \cdot \sin(k+1)x\right] = \sum C_n^k z^{k+1} =$

$z \cdot \sum_{k=0}^{n} C_n^k z^k = z(z+1)^n$, where $z = \cos x + i \cdot \sin x$ and $z + 1 = (1 + \cos x) + i \cdot \sin x = 2 \cos \frac{x}{2} \cdot \left(\cos \frac{x}{2} + i \cdot \sin \frac{x}{2}\right)$.

Therefore, $\boxed{C \equiv \sum_{k=0}^{n} C_n^k \cos(k+1)x = 2^n \cos^n \frac{x}{2} \cos \frac{(n+2)x}{2}} \wedge \boxed{S \equiv \sum_{k=0}^{n} C_n^k \sin(k+1)x = 2^n \cos^n \frac{x}{2} \sin \frac{(n}{}}$

. $\square$ $x^2 - 2\lfloor x \rfloor + \{x\} = 0$

Solution

i set $x = n + r$ with $n \in Z$ and $0 \le r < 1$ now, we have $x^2 - 2\lfloor x \rfloor + \{x\} = (n+r)^2 - 2n + r = r^2 + (2n+1)r + n^2 - 2n = 0$ we have a polynomial degree 2 with variable is r equation want solution must satisfying $\delta = (2n+1)^2 - 4(n^2 - 2n) = 12n + 1 \ge 0 \to n \ge 0$ and $r_1 = \frac{-(2n+1)+\sqrt{12n+1}}{2}$ $r_2 = \frac{-(2n+1)-\sqrt{12n+1}}{2}$ we can see $r_2 < 0$ with $n \ge 0$ now, $r_1 \ge 0 \leftrightarrow 0 \le n \le 2$ and $r_1 < 1$ with all $n \ge 0$ let $n = 0 \to r = 0$ therefor $x = 0$ let $n = 1 \to r = \frac{\sqrt{13}-3}{2}$ therefor $x = \frac{\sqrt{13}-1}{2}$ let $n = 2 \to r = 0$ therefor $x = 2$ $\square$

In $\triangle ABC$, cevians $\overline{AA'}, \overline{BB'}, \overline{CC'}$ concur at P. Prove that

$$\frac{PA \cdot PB \cdot PC}{PA' \cdot PB' \cdot PC'} \ge 8.$$

Solution

For any interior point P w.r.t. $\triangle ABC$ exist $\{x, y, z\} \subset (0, \infty)$ so that $\left\{\begin{array}{c} \frac{A'B}{A'C} = \frac{z}{y} \\ \\ \frac{B'C}{B'A} = \frac{x}{z} \\ \\ \frac{C'A}{C'B} = \frac{y}{x} \end{array}\right|$. Using the van Aubel's relation

obtain that $\left\{\begin{array}{c} \frac{PA}{PA'} = \frac{B'A}{B'C} + \frac{C'A}{C'B} = \frac{y+z}{x} \\ \\ \frac{PB}{PB'} = \frac{C'B}{C'A} + \frac{A'B}{A'C} = \frac{z+x}{y} \\ \\ \frac{PC}{PC'} = \frac{A'C}{A'B} + \frac{B'C}{B'A} = \frac{x+y}{z} \end{array}\right| \Longrightarrow \frac{PA}{PA'} \cdot \frac{PA}{PA'} \cdot \frac{PC}{PC'} = \frac{(y+z)(z+x)(x+y)}{xyz} \ge 8$.

Remark. For example, can choose $\frac{x}{[BPC]} = \frac{y}{[CPA]} = \frac{z}{[APB]} = \frac{1}{[ABC]}$ (normalized barycentrical coordinates w.r.t. $\triangle ABC$). $\square$ Prove that :

$(p-1)(p-2)\dots(p-k) \equiv (-1)^k.k! \pmod p$. Where, p is prime number and $1 \le k \le p-1$, k is integer

Solution

First note that, $(p-m) \equiv -m \pmod p$ for any positive integer m. Applying this fact, we have

$$(p-1)...(p-k) \equiv (-1)...(-k) \equiv k!(-1)^k \pmod p$$

as desired.

Remark: The requirement that p is prime, and $k \le p-1$, just makes it impossible for the RHS of the expression to be $\equiv 0 \pmod p$. Otherwise, we can still have the same equivilance, but one has to be careful whether the RHS is $\equiv 0 \pmod p$. $\square$ In the convex pentagon ABCDE, the sides BC, CD and DE are equal and each diagonal is parallel to a side. Prove that ABCDE is a regular pentagon.

Solution

Consider side AB of pentagon $ABCDE$. Diagonal CE is the only diagonal that does not contain either A or B. Thus, for any side, only one diagonal could possibly be parallel to it.

Now, consider the three given sides, $BC = CD = DE$. Without loss of generality, let them all be 1. Since $BE \| CD$, let $\angle BCD = \angle EDC = \alpha$. Let k be the line through D such that $k \| BC$. Let ℓ

be the line through C such that $\ell \| DE$. Let m be the line through E such that $m \| BD$. Let n be the line through B such that $n \| CE$. We must prove that $k \cap \ell = m \cap n$.

Since $BCDE$ is an isosceles trapezoid, we have that $BE = 2\sin\left(\alpha - \frac{\pi}{2}\right) + 1$ and that the distance between BE and CD (let this value be x) is $\cos\left(\alpha - \frac{\pi}{2}\right)$. Look at the altitude from A to CD, we have the equation

$$\frac{1}{2}BE \cdot \tan\left(\frac{\pi-\alpha}{2}\right) + x = \frac{1}{2} \cdot \tan(\pi - \alpha)$$

Plugging in the values of BE and x, we find that $x = 108^\circ$, which implies that the given pentagon is regular.

$\square$ In a meeting, there are 2011 scientists attending. We know that, every scientist know at least 1509 other ones. Prove that a group of five scientists can be formed so that each one in this group knows 4 people in his group.

Solution

From Caro-Wei theorem.

Consider the graph $G = (V, E)$, with $|V| = 2011$ being the scientists, and E being the acquaintance relationships, thus $\deg_G v \geq 1509$ for all $v \in V$. The complementar graph $\overline{G}$ will thus have $\deg_{\overline{G}} v \leq 501$ for all $v \in V$. Then, by the Caro-Wei theorem,

$$\omega(G) = \alpha(\overline{G}) \geq \sum_{v \in V} \frac{1}{\deg_{\overline{G}} v + 1} \geq \sum_{v \in V} \frac{1}{501+1} = \frac{2011}{502} = 4 + \frac{3}{502},$$

therefore $\omega(G) \geq 5$, where $\omega(G)$ is the cliquomatic number of G, thus G contains a K_5.

$\square$ Let $p(x) = x^2 + x + 1$. Find the fourth smallest prime q such that p(x) has a root mod q.

Solution

If $q \mid x^2 + x + 1$ then $x^3 \equiv 1 \pmod{q}$. If the order of x modulo q is 1, then $x^2 + x + 1 \equiv 3 \pmod{q}$, so this happens only for $q = 3$. Otherwise the order of x modulo q is 3, so we need $3 \mid \varphi(q) = q - 1$. Then, using the fact that $\left(\mathbb{Z}_q^*, \cdot\right)$ is cyclic, using a generator g of it we obtain $\left(g^{(q-1)/3}\right)^3 - 1 \equiv 0 \pmod{q}$, and since $g^{(q-1)/3} \not\equiv 1 \pmod{q}$, it means all is well. Thus, in the sequence $3, 7, 13, 19, 31, \ldots$ of such primes, the fourth term is $q = 19$.

$\square$ Find the limit of the sequence $(x_n)_n$ given by

$$x_n = ac + (a+ab)c^2 + (a+ab+ab^2)c^3 + \cdots + (a+ab+\cdots+ab^n)c^{n+1},$$

where a, b, c are real numbers with $-1 < c, bc < 1$ and $b \neq 1$.

Solution

We have $x_n = \sum_{k=0}^{n}\left(ac^{k+1}\sum_{j=0}^{k} b^j\right) = \frac{a}{b-1}\sum_{k=0}^{n}\left((bc)^{k+1} - c^{k+1}\right) = \frac{a}{b-1}\left(\frac{(bc)^{n+2}-1}{bc-1} - \frac{c^{n+2}-1}{c-1}\right)$.

According to the givens, we thus have $\lim_{n\to\infty} x_n = \frac{a}{b-1}\left(\frac{-1}{bc-1} - \frac{-1}{c-1}\right) = \frac{ac}{(c-1)(bc-1)}$.

$\square$ $\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2} \geq 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, where A, B, C are the angles of the the triangle ABC .

Solution

$\left|\prod\cos\frac{B-C}{2} \geq 8\prod\sin\frac{A}{2}\right| \odot \prod\left(2\cos\frac{A}{2}\right) > 0 \iff \prod\left(2\cos\frac{B-C}{2}\sin\frac{B+C}{2}\right) \geq$
$8\prod\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right) \iff \prod(\sin B + \sin C) \geq 8\prod\sin A \iff \prod(b+c) \geq 8abc$,
what is well-known. I used the simple relation $\cos\frac{A}{2} = \sin\frac{B+C}{2}$.

$\square$ For all odd number u ,Prove that there exist n that $u \mid 10^n - 1$

Solution

If $\gcd(u, 10) = 1$, then there exists a repunit $\dfrac{10^n - 1}{9} = \overline{11\ldots1}$ divisible by u. Consider the numbers

$a_k = \overline{11\dots 1}$ with k digits 1, for $1 \le k \le u$. If there exists such a k so that $u \mid a_k$, we are done. Otherwise, there must exist $1 \le i < j \le u$ so that $u \mid a_j - a_i = 10^i a_{j-i}$; but $\gcd(u, 10^i) = 1$, thus $u \mid a_{j-i}$ (so in fact this latter case cannot occur).

□ Prove that the prime divisors of $\frac{x^7-1}{x-1}$ are always in the form $7k, 7k+1$.

Solution

Generalization

We claim that all prime divisors of

$$x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1$$

are either p or congruent to 1 modulo p.

Let q be a prime divisors of the expression.

$$x^{p-1} + \cdots + x^2 + x + 1 \equiv 0 \pmod q \Longrightarrow (x-1)(x^{p-1} + \cdots + x^2 + x + 1) \equiv 0 \pmod q$$

Then $x^p \equiv 1 \pmod q \Longrightarrow ord_q(x)|p$.

Thus $ord_q(x)$ is 1 or p.

Case 1: $ord_q(x) = 1$

This means that $x \equiv 1 \pmod q$. Then

$$x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1 \equiv 1 + 1 + 1 + \cdots + 1 \equiv p \equiv 0 \pmod q$$

Since p, q are both primes, this implies that $p = q$.

Case 2: $ord_q(x) = p$

This implies that by Fermat's little theorem, that $p|q-1$, as p is the order and $x^{q-1} \equiv 1 \pmod q$. Then $q - 1 \equiv 0 \pmod p \Longrightarrow q \equiv 1 \pmod p$.

Therefore all prime divisors are in the form p or $pk+1$.

□ Solve equation in integer numbers $x^2 + y^2 + z^2 = y^2.x^2$

Solution

Notice that quadratic residues are $0, 1$ mod 4. Trying all 8 possible cases for x,y, and z, only $x^2 \equiv y^2 \equiv z^2 \equiv 0$ mod 4. Thus, we can let $x = 2a$, $y = 2b$,$z = 2c$. Substituting this back into the original equation, we have

$4(a^2 + b^2 + c^2) = 16a^2b^2$

$a^2 + b^2 + c^2 = 4a^2b^2$.

From this, we know that $a^2 \equiv b^2 \equiv c^2 \equiv 0$ mod 4. Then, let $a = 2x_1$, $b = 2y_1$,$c = 2z_1$

$16(x_1^2 + y_1^2 + z_1^2) = 256x_1^2y_1^2$

$x_1^2 + y_1^2 + z_1^2 = 16x_1^2y_1^2$

By these methods, the equation will eventually become

$x_\infty^2 + y_\infty^2 + z_\infty^2 = 2^\infty x_\infty^2 y_\infty^2$.

By infinite descent, the only solution is $x = y = z = 0$.

□ Prove that $a > 0$, $b > 0 \implies \frac{1}{a^2} + \frac{1}{b^2} + \frac{4}{a^2+b^2} \ge \frac{32(a^2+b^2)}{(a+b)^4}$ (in my opinion, the level of this problem is "easier").

Proof. On the one hand $\frac{1}{a^2} + \frac{1}{b^2} + \frac{4}{a^2+b^2} = \frac{a^2+b^2}{a^2b^2} + \frac{4}{a^2+b^2} \ge 2 \cdot \sqrt{\frac{a^2+b^2}{a^2b^2} \cdot \frac{4}{a^2+b^2}} \implies$ $\boxed{\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{4}{a^2+b^2} \ge \dfrac{4}{ab}}$ (1) and on the other hands $\boxed{\dfrac{4}{ab} \ge \dfrac{32(a^2+b^2)}{(a+b)^4}}$ (2) $\iff (a+b)^4 \ge 8ab\left(a^2+b^2\right) \iff$

$(a^2+2ab+b^2)^2 \ge 8ab\left(a^2+b^2\right) \iff \left(\frac{a}{b}+\frac{b}{a}+2\right)^2 \ge 8\left(\frac{a}{b}+\frac{b}{a}\right) \iff (y+2)^2 \ge 8y$, where $y=\frac{a}{b}+\frac{b}{a} \iff$

$(y-2)^2 \ge 0$, what is truly. In conclusion, the inequality (2) is truly and so from the relations (1) and (2) obtain the required inequality.

Remark. $\left\{\begin{array}{c} (a+b)^2 \boxed{\ge} 4ab \\ \\ (a+b)^2 \boxed{\le} 2\left(a^2+b^2\right) \end{array}\right|$. But $(a+b)^4 \boxed{\ge} 8ab\left(a^2+b^2\right)$. Indeed, $(a+b)^4=\left[\left(a^2+b^2\right)+2ab\right]^2=$

$\left[\left(a^2+b^2\right)^2+4a^2b^2\right]+4ab\left(a^2+b^2\right) \ge 2\sqrt{\left(a^2+b^2\right)^2\cdot 4a^2b^2}+4ab\left(a^2+b^2\right)=8ab\left(a^2+b^2\right)$.

$\square$ Let ABC be a triangle. The its angled bisectors meet again its circumcircle $C(O,R)$ in the points A' , B' , C'

respectively, i.e. its incenter $I \in AA' \cap BB' \cap CC'$. Prove that $\frac{1}{[BA'C]}+\frac{1}{[CB'A]}+\frac{1}{[AC'B]} \ge \frac{9}{[ABC]}$.

Solution

We'll use the well-known (or you prove easily them) relations $\left\{\begin{array}{lr} [BA'C]=\frac{ra^2}{4(s-a)} & (1) \\ \\ \sum a^2(s-a)=4sr(R+r) & (2) \\ \\ \frac{s^2}{r} \ge 16R-5r \ge 9(R+r) & (3) \end{array}\right|$.

Therefore,

$\sum \frac{1}{[BA'C]} \overset{(1)}{=} \frac{4}{r}\cdot\sum\frac{s-a}{a^2} = \frac{4}{r}\cdot\sum\frac{(s-a)^2}{a^2(s-a)} \overset{\text{(C.B.S)}}{\ge} \frac{4}{r}\cdot\frac{s^2}{\sum a^2(s-a)} \overset{(2)}{=} \frac{4}{r}\cdot\frac{s^2}{4sr(R+r)} =$

$\frac{s}{r^2(R+r)} = \frac{s^2}{9r(R+r)}\cdot\frac{9}{S} \overset{(3)}{\ge} \frac{16R-5r}{9(R+r)}\cdot\frac{9}{S} = \left(1+\frac{7}{9}\cdot\frac{R-2r}{R+r}\right)\cdot\frac{9}{S} \ge \frac{9}{S}$, where S is the area

$[ABC]$ of $\triangle ABC$. In conclusion, $\boxed{\sum\frac{1}{[BA'C]} \ge \left(1+\frac{7}{9}\cdot\frac{R-2r}{R+r}\right)\cdot\frac{9}{[ABC]} \ge \frac{9}{[ABC]}}$.

$\square$ Let

$$\left\{\begin{array}{l} 0 \le a \le 1; 0 \le b \le 1; 0 \le c \le 1 \\ a+b+c=\frac{3}{2} \end{array}\right.$$

Prove the inequality:

$$a^{10}+b^{10}+c^{10} \le \frac{1025}{1024}$$

Solution

Let $a \ge b \ge c$ $a+b=m \ge 1$ we can easily get $a=1, b=u-1$ is the greatest because $f(x)=x^{10}+(u-x)^{10}$ $f'(x)=10x^9-10(u-x)^9$ where $x \ge \frac{u}{2}$,the function is increasing so $a^{10}+b^{10} \le 1+(u-1)^10$ Let $b+c=0.5$ similarly we can have $b=0.5, c=0$ is the greatest So $a^{10}+b^{10}+c^{10} \le \frac{1025}{1024}$ the quation holds where $a=1, b=0.5, c=0$

$\square$ Let ABC be a triangle, a line d so that $d \parallel BC$, $A \notin d$ and a mobile point $M \in d$. Denote $N \in AC$ for which $NB \parallel MA$. Prove that the area of the triangle CMN is constant.

Solution

Let $P \equiv d \cap AB$, $Q \equiv d \cap AC$ and $D \equiv AM \cap BC$. Then we have

$\frac{[CMN]}{[CMA]}=\frac{CN}{CA}$, $\frac{[CMA]}{[CPA]}=\frac{QM}{QP} \Longrightarrow [CMN]=[CPA]\cdot\frac{QM}{QP}\cdot\frac{CN}{CA}$

But $d \parallel BC, DA \parallel NB$ yield $\frac{QM}{QP}=\frac{CD}{CB}=\frac{CA}{CN} \Longrightarrow [CMN]=[CPA]=\text{const}$

$\square$ Evaluate: $4\cos 18^\circ - 3\sec 18^\circ - 2\tan 18^\circ$

Solution

$4\cos 18^\circ - 3\sec 18^\circ - 2\tan 18^\circ = \frac{4\cos^2 18^\circ - 3 - 2\sin 18^\circ}{\cos 18^\circ}$

$= \frac{4\cos^3 18^\circ - 3\cos 18^\circ - 2\sin 18^\circ \cos 18^\circ}{\cos^2 18^\circ}$
$= \frac{\cos 3(18^\circ) - \sin 2(18^\circ)}{\cos^2 18^\circ}$
$= \frac{\cos 54^\circ - \sin 36^\circ}{\cos^2 18^\circ}$
$= \frac{\sin 36^\circ - \sin 36^\circ}{\cos^2 18^\circ}$
$= 0$

$\square$ Find all integers, with proof, $n \geq 2$ that satisfies $\sqrt[n]{3^n + 4^n + 5^n + 8^n + 10^n}$ and that the expression is an integer.

Solution

The expression is equivalent to $k^n = 3^n + 4^n + 5^n + 8^n + 10^n$. By trying, we get $12^3 = 3^3 + 4^3 + 5^3 + 8^3 + 10^3$. If we divide by 10^n, we get $(\frac{k}{10})^n = (\frac{3}{10})^n + (\frac{4}{10})^n + (\frac{5}{10})^n + (\frac{8}{10})^n + 1^n$. It's obvious that $k > 10$, so as the LHS increases, the RHS tends to decrease, this means, when n is bigger than 3, we get $k^n > 3^n + 4^n + 5^n + 8^n + 10^n$. So, $n = 3$.

$\square$ Each member of the sequence $112002, 11210, 1121, 117, 46, 34, \ldots$ is obtained by adding five times the rightmost digit to the number formed by omitting that digit. Determine the billionth (10^9th) member of this sequence.

Solution

Elementary solution: the sequence goes 112002,11210,1121, 117, 46, 34, 23, 17, 36, 33, 18, 41, 9, 45, 29, 47, 39, 48, 44,24, 22, 12, 11, 6, 30, 3, 15, 26, 32, 13, 16, 31, 8, 40, 4, 20, 2, 10, 1, 5, 25, 27, 37, 38, 43, 19, 46, and hence repeats per 42 and we only need to investigate the 10^9 mod 42th element. Since there are 42 integers appearing in the sequence, and the relation is recursive with 1 parent, we can take the index mod 42.

More interesting solution: Consider the set $A = \{i \in \mathbb{N} | i > 0, i < 50, 7 \not| i\}$ and denote $f(x)$ the number after x in the sequence. We can check that $f(x) \in A \Leftrightarrow \forall x \in A$, thus f is bijective. We now know that if $G = \{f^k, k \in N\}$, then $(G, \circ)$ is a group. As the relation was bijective, and our group is cyclic, $|G|$ divides 42, thus $f^{42} = e$, thus we can take the index mod 42.

$\square$ Prove that

$(\forall)\ k \in \overline{1,n}\ ,\ x_k \in (0,1] \implies \sum_{k=1}^n x_k + \frac{1}{x_1 x_2 \ldots x_{n-1} x_n} \geq \sum_{k=1}^n \frac{1}{x_k} + x_1 x_2 \ldots x_{n-1} x_n$.

Solution

Denote $f(x) = \frac{1}{x} - x$, where $x \in (0,1]$. Prove easily that $f(ab) \geq f(a) + f(b) \iff (1-a)(1-b)(1-ab) \geq 0$.

In conclusion, $\left\{\begin{array}{c} f(x_1x_2) \geq f(x_1) + f(x_2) \\ \\ f(x_1x_2x_3) \geq f(x_1x_2) + f(x_3) \\ \\ f(x_1x_2x_3x_4) \geq f(x_1x_2x_3) + f(x_4) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f(x_1x_2\ldots x_{n-1}x_n) \geq f(x_1x_2\ldots x_{n-1}) + f(x_n) \end{array}\right| \bigoplus \implies f(x_1\, x_2 \ldots x_{n-1} x_n) \geq \sum_{k=1}^n f(x_k) \iff$

$\frac{1}{x_1 x_2 \ldots x_n} - x_1 x_2 \ldots x_n \geq \sum_{k=1} \left(\frac{1}{x_k} - x_k\right) \iff \sum x_k + \frac{1}{x_1 x_2 \ldots x_n} \geq \sum \frac{1}{x_k} + x_1 x_2 \ldots x_n$.

$\square$ (a) Find all primes p such that $\frac{p-1}{2}$ and $\frac{p^2-1}{2}$ are perfect squares. (b) Find all primes p such that $\frac{p+1}{2}$ and $\frac{p^2+1}{2}$ are perfect squares.

Solution

Letting $x^2 = \frac{p^2+1}{2}$ and $y^2 = \frac{p+1}{2}$ we subtract to get

$$(x+y)(x-y) = p \cdot \frac{p-1}{2}$$

since p and $\frac{p-1}{2}$ are relatively prime we have the following system of equations

$$\begin{cases} x+y = p & (1) \\ \\ x-y = \frac{p-1}{2} & (2) \end{cases}$$

Adding (1) and (2) we get $x = \frac{3p-1}{4}$ and therefore $y = \frac{p+1}{4}$ therefore we have the following equation:

$$\left(\frac{p+1}{4}\right)^2 = \frac{p+1}{2}$$

solving the only solution is $\boxed{p = 7}$

□ Let p be a prime. If $p \neq 3$, then show that there exists an integer r such that $3r \equiv 1 \pmod{p}$.

Solution

Since $p \neq 3$ and obviously $p \nmid r$, then we have that the set $\{1 \cdot 3, 2 \cdot 3, \cdots, p \cdot 3\}$ is a complete set of residue classes. Therefore there must exist some $r, 1 \leq r \leq p-1$ such that $3r \equiv 1 \mod p$, as there must be some element in this set where its congruence is 1 (otherwise it contradicts the fact that the set is a complete residue set).

I used the fact that if positive integer m and integer a satisfy $\gcd(a, m) = 1$, then the set $\{1 \cdot a, 2 \cdot a, \cdots, m \cdot a\}$ is a complete set of residue classes (in fact, it's not too difficult to prove).

■

□ Prove that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that:

$$f(f(n)) = 3n$$

Solution

Denote X by the set of the numbers which are not divisible by 3. The numbers in X is put increased, so: $X = x_1, x_2, ..., x_k, ...$ which satisfies $x_i < x_{i+1} (x_i \epsilon X)$ Every number n which is not in X can be illustrated by this expression: $n = x_k.3^i$ which n is not zero. Let $a_{i,k} = x_k.3^i$ f is determined by this rule: $f(0) = 0$, $f(a_{i,k}) = a_{i,k+1}$ (if k is odd) and $f(a_{i,k}) = a_{i+1,k-1}$ (if k is even) We can easily prove that if k is even or k is odd, $f(a_{i,k}) = a_{i+1,k} = 3.a_{i,k}$ satisfying the condition of the problem.

□ Let p(x) be a polynomial of degree n not necessarily with integer coefficients. For how many consecutive integer values of x must p(x) be an integer in order to guarantee that P(x) is an integer for all integers x?

Solution

Take $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$. Write the given $n+1$ consecutive integers as $s, s+1, s+2, \ldots, s+n$.

Now, let $P(x+1) - P(x) = P_1(x)$. Note that $P_1(x)$ has degree $n-1$, and $P_1(x)$ is an integer for $x = s, s+1, s+2, \ldots, s+(n-1)$.

We can repeat this process until we reach $P_n(x)$ with degree 0, which is an integer for $x = s$. But $P_n(x) = c$ for some c, and then using the relation $P_{n-1}(s+n+1) - P_{n-1}(s+n) = c$ we can recursively solve for the value of $P(s+n+1)$, which will be an integer since all of the values we are dealing with are integers. If we repeat this process we can show that $P(s+n+2), P(s+n+3), \ldots$ are all integers. Similarly, we can also solve for $P(s-1), P(s-2), \ldots$.

☐ Find the real-numbered solution to the equation below and demonstrate that it is unique.

$$\frac{36}{\sqrt{x}} + \frac{9}{\sqrt{y}} = 42 - 9\sqrt{x} - \sqrt{y}$$

Solution

We rearrange to $\dfrac{36}{\sqrt{x}} + \dfrac{9}{\sqrt{y}} + 9\sqrt{x} + \sqrt{y} = 42$. By AM-GM, $\dfrac{36}{\sqrt{x}} + 9\sqrt{x} \geq 36$ and $\dfrac{9}{\sqrt{y}} + \sqrt{y} \geq 6$, so $\dfrac{36}{\sqrt{x}} + \dfrac{9}{\sqrt{y}} + 9\sqrt{x} + \sqrt{y} \geq 42$, with equality if and only if $\dfrac{36}{\sqrt{x}} = 9\sqrt{x}$ and $\dfrac{9}{\sqrt{y}} = \sqrt{y}$. This gives the solution $x = 4, y = 9$. We can check that this works easily.

☐ Prove that for all g, function $f(m) = m$ is the solution of $f(g(m)) = g(f(m))$.

Solution

Say $f\colon M \to M$ (for $M \neq \emptyset$) is such that, for all $g\colon M \to M$ and for all $m \in M$, we have $f(g(m)) = g(f(m))$. For any particular $c \in M$, consider $g_c\colon M \to M$ defined by $g_c(m) = c$ for all $m \in M$. Then $c = g_c(f(c)) = f(g_c(c)) = f(c)$. Thus the only such f is $f(m) = m$ for all $m \in M$, i.e. $f = \mathrm{id}_M$, which clearly satisfies.

☐ If S is the sum of positive real numbers $x_1, x_2, ...x_n$, prove that: $(1+x_1)(1+x_2)...(1+x_n) \leq 1 + S + \frac{S^2}{2!} + ... + \frac{S^n}{n!}$

Solution

By the Lagrange multipliers' method, the system of equations $\frac{\partial L}{\partial x_i} = 1+S+\cdots+\frac{S^{n-1}}{(n-1)!} - \frac{1}{x_i+1}\prod_{j=1}^n (1+x_j) = 0$ yields as unique critical points (interior to the domain) those with $x_1 = \cdots = x_n = \frac{S}{n}$. Any critical points on the border of the domain (where some of the x_i's are 0) correspond in fact to lower values of n.

Now, the function $\phi_n\colon [0,\infty) \to \mathbb{R}$ given by $\phi_n(x) = 1+x+\cdots+\frac{x^n}{n!} - \left(1+\frac{x}{n}\right)^n$ has as derivative $\phi_n'(x) = 1+x+\cdots+\frac{x^{n-1}}{(n-1)!} - \left(1+\frac{x}{n}\right)^{n-1} = \phi_{n-1}(x) + \left(1+\frac{x}{n-1}\right)^{n-1} - \left(1+\frac{x}{n}\right)^{n-1}$. Assuming by induction that $\phi_{n-1}(x) \geq 0$, this yields $\phi_n'(x) \geq \left(1+\frac{x}{n-1}\right)^{n-1} - \left(1+\frac{x}{n}\right)^{n-1} \geq 0$, so $\phi_n(x)$ is increasing, and since $\phi_n(0) = 0$, the claim follows.

☐ Solve equation in integer numbers with $n \geq 2$. $[\sqrt{n}] + [\sqrt[3]{n}] + [\sqrt[4]{n}] + ... + [\sqrt[n]{n}] = [\log_2 n] + [\log_3 n] + ... + [\log_n n]$

Solution

LHS and RHS are the number of lattice points satisfying $x^y \leq n$ $(2 \leq x, y \leq n)$ fix y and count we get LHS,fix x we get RHS so the equality holds for every positive integer n which is greater than 2

☐ Show that the GCD of three consecutive triangular numbers is 1.

Solution

Let the three triangular numbers be $\frac{(n-1)n}{2}, \frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}$.

$\gcd\left(\frac{(n-1)n}{2}, \frac{n(n+1)}{2}\right) = \frac{n}{2}$ or n. $\gcd\left(\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}\right) = \frac{n+1}{2}$ or $n+1$. Exactly one of n and $n+1$ is even so therefore the gcd must be $\frac{n}{2}$ and $n+1$ or n and $\frac{n+1}{2}$. Then seeing if any of those two pairs of gcds have any common factors, we can determine if all three triangular numbers have any common factors. Since $\gcd(n, n+1) = 1$, none of those have common factors. Therefore, $\gcd\left(\frac{(n-1)n}{2}, \frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}\right) = \boxed{1}$. ■

☐ Find, with proof, a positive integer n such that

$$\frac{(n+1)(n+2)\cdots(n+500)}{500!}$$

is an integer with no prime factors less than 500.

Solution

Thinking that the 500! will take out all the prime factors less than 500, let us find an integer that when added to any number, still has the divisor of all the integers less than or equal to 500. 500! is such an integer. Thus, we have such an expression:

$$\frac{(500!+1)(500!+2)\cdots(500!+500)}{500!}=\left(\frac{500!}{1}+1\right)\left(\frac{500!}{2}+1\right)\cdots\left(\frac{500!}{500}+1\right)$$

I've taken out all the prime factors less than 500 in the factors in the numerator. Also, adding 1 to a number n ensures that the new number has no factors in common with n. However, that still doesn't account for larger primes. So squaring 500! works because all the primes are still present in the factors and the addition of 1 makes it have no prime factor less than 500. Therefore, $n=\boxed{(500!)^2}$.

□ Prove that if $0<x<\pi/2$, then $\sec^6 x+\csc^6 x+\sec^6 x\csc^6 x\geq 80$.

Solution

$\sec^6 x+\csc^6 x+\sec^6 x\csc^6 x\geq 80 \iff 1+\sin^6 x+\cos^6 x\geq 80\sin^6 x\cos^6 x \iff$

$1+\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x\geq 80\sin^6 x\cos^6 x \iff 2-3\sin^2 x\cos^2 x\geq 80\sin^6 x\cos^6 x \iff$ $80\sin^6 x\cos^6 x+3\sin^2 x\cos^2 x-2\leq 0$. Denote $\sin^2 x\cos^2 x=t$, where $0\leq t\leq\frac{1}{4}$.

In conclusion, our inequality is equivalently with $80t^3+3t-2\leq 0$ for any $t\in\left[0,\frac{1}{4}\right]$, what is truly

$$\text{because } 0\leq t\leq\frac{1}{4}\implies\left\{\begin{array}{c} 80t^3\leq\frac{5}{4} \\ \\ 3t-2\leq-\frac{5}{4}\end{array}\right|\bigoplus\implies 80t^3+3t-2\leq 0\ .$$

□ Let ABC be a triangle with the circumcircle w . The A-symmedian of $\triangle ABC$ meet again the circle w at D . Denote the midpoint E of $[AD]$. Prove that $m\left(\widehat{BEC}\right)=2A$.

Solution

Denote the midpoints M , N , P of $[BC]$, $[CA]$, $[AB]$ respectively and the intersection $S\in AD\cap BC$.

$$\text{Thus, }\left\{\begin{array}{ccc} \triangle ABD\sim\triangle AMC & \implies & m(\angle BED)=m(\angle MNC)=A \\ \\ \triangle ACD\sim\triangle AMB & \implies & m(\angle CED)=m(\angle MPB)=A\end{array}\right|\implies m(\angle BEC)=2A\ .$$

An easy extension. Let ABC be a triangle with the circumcircle w . Consider two points $\{M,S\}\subset(BC)$ so that $S\in(BM)$ and $\widehat{SAB}\equiv\widehat{MAC}$. Denote $AS\cap w=\{A,D\}$ and the points $\{E,F\}\subset(AS)$ so that $\frac{EA}{ED}=\frac{MB}{MC}=\frac{FD}{FA}$. Denote $K\in BE\cap CF$. Prove that $KE=KF$ and $m(\angle BKC)=2A$.

□ Prove that
$(1+\frac{1}{n-1})^{n-1}<e<(1+\frac{1}{n})^{n+1}$.

Solution

Apply AM-GM for $n\in\mathbb{N}^*$ to $x_k:=1+\frac{1}{n}$, $k\in\overline{1,n}$ and $x_{n+1}:=1$. Thus,

$\frac{n\cdot\left(1+\frac{1}{n}\right)+1}{n+1}>\sqrt[n+1]{\left(1+\frac{1}{n}\right)^n}\iff\left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^n$.

Apply again AM-GM for $n\in\mathbb{N}$, $n\geq 2$ to $x_k:=1-\frac{1}{n}$, $k\in\overline{1,n}$ and $x_{n+1}:=1$. Thus,

$\frac{n\cdot\left(1-\frac{1}{n}\right)+1}{n+1}>\sqrt[n+1]{\left(1-\frac{1}{n}\right)^n}\iff\left(1+\frac{1}{n}\right)^{n+1}>\left(1+\frac{1}{n-1}\right)^n$. In conclusion,

$2<\ldots<\left(1+\frac{1}{n-1}\right)^{n-1}<\left(1+\frac{1}{n}\right)^n<\ldots<\left(1+\frac{1}{n}\right)^{n+1}<\left(1+\frac{1}{n+1}\right)^{n+2}<\ldots<3$, $(\forall)\,n\geq 5$.

□

Prove that $m!n!(m+n)!$ divides $(2m)!(2n)!$

Solution

It is enough to prove that, for any positive integer N, we have $\left\lfloor \frac{2m}{N} \right\rfloor + \left\lfloor \frac{2n}{N} \right\rfloor \geq \left\lfloor \frac{m}{N} \right\rfloor + \left\lfloor \frac{n}{N} \right\rfloor + \left\lfloor \frac{m+n}{N} \right\rfloor$, which is true since equivalent to $\left\lfloor 2\left\{\frac{m}{N}\right\} \right\rfloor + \left\lfloor 2\left\{\frac{n}{N}\right\} \right\rfloor \geq \left\lfloor \left\{\frac{m}{N}\right\} + \left\{\frac{n}{N}\right\} \right\rfloor$, which is trivial.

Now Legendre's formula $\sum_{j=1}^{\infty} \left\lfloor \frac{k}{p^j} \right\rfloor$ for the exponent of a prime p dividing the factorial $k!$, applied to $2m, 2n, m, n$ and $m+n$, yields the claim.

□ Prove that there aren't exist integer a,b bot not zero such that for any prime p, q $>$ 1000, p difference q , ap+bq is a prime, too.

Solution

Dirichlet's theorem kills this. Fix a prime p, and consider a prime $r > p$, not dividing a nor b. Then there exists an integer c such that $bc \equiv -1 \pmod{r}$. The arithmetic progression $cap + mr$, for $m = 1, 2, \ldots$ contains infinitely many primes; take one such $q = cap + mr$. We have $ap + bq = ap + bcap + bmr \equiv ap - ap = 0 \pmod{r}$, i.e. $r \mid ap + bq$, and $r \neq ap + bq$, $q \neq p$, for q taken large enough.

□ Solve $\arctan \frac{1}{7} + 2\arctan \frac{1}{3}$.

Solution

Denote $\left\{\begin{array}{ccc} \arctan \frac{1}{7} = x & \iff & x \in \left(0, \frac{\pi}{2}\right) \ , \ \tan x = \frac{1}{7} \\ & & \\ \arctan \frac{1}{3} = y & \iff & x \in \left(0, \frac{\pi}{2}\right) \ , \ \tan y = \frac{1}{3} \end{array}\right|$. Prove easily that $\left\{\begin{array}{c} \tan \frac{\pi}{12} = 2 - \sqrt{3} \\ \\ \tan \frac{\pi}{8} = \sqrt{2} - 1 \end{array}\right|$. Observe that

$0 < \frac{1}{7} < 2-\sqrt{3} < \frac{1}{3} < \sqrt{2}-1 \implies 0 < \arctan \frac{1}{7} < \frac{\pi}{12} < \arctan \frac{1}{3} < \frac{\pi}{8} < \frac{\pi}{6}$. Since $(x+y) \in \left(0, \frac{5\pi}{24}\right)$ and

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{1}{7}+\frac{1}{3}}{1-\frac{1}{7}\cdot\frac{1}{3}} = \frac{1}{2} \implies x+y = \arctan \frac{1}{2}$. Denote $x+y = z$. Since $y+z \in \left(0, \frac{\pi}{3}\right)$ and $\tan(y+z) = \frac{\tan y + \tan z}{1 - \tan y \tan z} = \frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}} = 1 \implies x+2y = y+z = \frac{\pi}{4} \iff \arctan \frac{1}{7} + 2\arctan \frac{1}{3} = \frac{\pi}{4}$

.

□ The incircle of triangle ABC is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let I_A, I_B, I_C be the incenters of triangles AEF, BDF, CDE, respectively. Prove that I_AD, I_BE, I_CF are concurrent.

Solution

Let ω be the incircle of $\triangle ABC$ and let $X = AI \cap \omega$. Then arc EX =arc XF since AI is the perpendicular bisetor of EF. Therefore $\angle AFX = \angle XEF = \angle XFE$, i.e. FX is the bisector of $\angle AFE$. So $I_A = AI \cap EF$, $I_B = BI \cap DF$, $I_C = CI \cap ED$. So, I_AD, I_BE, I_CF, are the bisectors of $\triangle DEF$, so they are concurrent.

□ Let S be a set of 10 distinct positive real numbers. Show that there exist $x, y \in S$ such that

$$0 < x - y < \frac{(1+x)(1+y)}{9}.$$

Solution

Divide the set of positive reals into the 9 sets of the form $\left[\frac{k-1}{10-k}, \frac{k}{9-k}\right)$ for $k = 1, 2, \ldots 9$. If $k = 9$, then the right side should just be ∞, and if $k = 1$, then the left boundary should be open. Note that all sets are disjoint and all positive reals are covered. By the pigeonhole principle, one of these sets contains 2 or more numbers from set S. Let x be the greater and y be the smaller. Then $x - y > 0$. Now we need to show that $\frac{(x+1)(y+1)}{9} > x - y$.

Transform the inequality into

$$\frac{(x+1)(y+1)}{9} > (x+1)-(y+1) \Rightarrow \frac{1}{9} > \frac{1}{y+1} - \frac{1}{x+1}.$$

We know that $x < \frac{k}{9-k}$ and $y \geq \frac{k-1}{10-k}$ for the same k. Then $x+1 < \frac{9}{9-k}$ and $y+1 \geq \frac{9}{10-k}$, so $\frac{1}{x+1} > \frac{9-k}{9}$ and $\frac{1}{y+1} \leq \frac{10-k}{9}$. Hence

$$\frac{1}{y+1} - \frac{1}{x+1} < \frac{10-k}{9} - \frac{9-k}{9} = \frac{1}{9}$$

and it is proven.

□ Find all pairs of positive integers $(a,\ b)$ such that $\frac{b-1}{a}$, $\frac{a+4}{b}$ are positive integers. Let $\frac{b-1}{a} = k_1$, $\frac{a+4}{b} = k_2$. Then, $b = ak_1 + 1$. Therefore, $a + 4 = k_2 b = k_2(ak_1 + 1) = ak_1k_2 + k_2$. So $a = \frac{k_2-4}{1-k_1k_2}$. Since a is positive and $1-k_1k_2 < 0$, then, $0 < k_2 < 4$. So $k_2 = 1, 2, 3$. Next we need to check the cases $k_2 = 1, 2, 3$. For $k_2 = 1$, we have $\frac{b+4}{a} = 1$. Then, $a = b + 4$. So $\frac{b-1}{a} = 1 + \frac{3}{a}$ $(a, b) = (1, 5), (3, 7)$ are satisfyed. For the case $k_2 = 2$, $(a, b) = (2, 3)$ For the case $k_2 = 3$, there are no solutions. Therefore, $(a, b) = (2, 3), (3, 7), (1, 5)$ are three groups of pairs of positive integers.

□ Let $\triangle ABC$ be a triangle with incenter I such that $\angle A = 120$, consider the points D, E and F such that :

$D = (AI) \cap (BC)$ and $E = (BI) \cap (AC)$ and $F = (CI) \cap (AB)$

Show that D lies on the circle with diameter $[EF]$.

Solution

$\frac{DA}{DB} = \frac{\sin \widehat{ABD}}{\sin \widehat{BAD}} = \frac{\sin \widehat{B}}{\sin 60} = \frac{\sin \widehat{B}}{\sin 120} = \frac{\sin \widehat{B}}{\sin \widehat{A}} = \frac{AC}{BC} = \frac{AF}{FB}$. So, DF is the bisector of $\widehat{ADB}$ and similarly DE is the bisector of $\widehat{ADC}$. Therefore $\widehat{FDE} = 90$ and the result follows

□ Find all integers n for which both n+27 and 8n+27 are perfect cubes.

Solution

If $n+27$ is perfect cube, then $8(n+27)$ is also perfect cube. Then difference between two cubes($(8n+8\cdot 27)$ and $(8n+27)$) is equal to $7 \cdot 27 = 189$. Then we write first some cubes: 1, 8, 27, 64, 125, 216, 343, 512, 729, ... We see that differnce between adjacent cubes(after 512) is greater than 189, that's why both cubes are least or equal to 512. Searching in first 8 cubes pairs with such difference, we find pairs $(27, 216)$. So $n = 0$

□ Let ABC be a non-obtuse triangle and let m_a be the length of the median issued from vertex A .

Prove that the following inequality holds: $\boxed{m_a \leq \sqrt{\frac{b^2+c^2}{2}} \cdot \cos\frac{A}{2}}$.

Consequence. In any non-obtuse $\triangle\, ABC$ the following inequality holds:

$$\boxed{\left(\frac{m_a}{\cos\frac{A}{2}}\right)^2 + \left(\frac{m_b}{\cos\frac{B}{2}}\right)^2 + \left(\frac{m_c}{\cos\frac{C}{2}}\right)^2 \leq a^2+b^2+c^2}\ .$$

Solution

$m_a \leq \sqrt{\frac{b^2+c^2}{2}} \cdot \cos\frac{A}{2} \iff 4m_a^2 \leq 2(b^2+c^2)\cdot\cos^2\frac{A}{2} \iff 2(b^2+c^2)-a^2 \leq 2(b^2+c^2)\cdot\cos^2\frac{A}{2} \iff$
$b^2+c^2-a^2 \leq (b^2+c^2)\cdot\left(2\cos^2\frac{A}{2}-1\right) \iff 2bc\cdot\cos A \leq (b^2+c^2)\cos A \iff (b-c)^2\cos A \geq 0$.

In conclusion, $\boxed{\frac{b+c}{2}\cdot\cos\frac{A}{2} \leq m_a \leq \sqrt{\frac{b^2+c^2}{2}}\cdot\cos\frac{A}{2}}$ in any non-obtuse triangle ABC .

□ Let $ax^2+bx+c=0$ be a equation with the roots x_1 , x_2 . Find the relation $f(a,b,c)=0$ so that $x_1^2 = x_2 \ \vee\ x_2^2 = x_1$.

Solution

Denote $\left\{\begin{array}{lllll} x_1+x_2 & = & S & = & -\frac{b}{a} \\ \\ x_1x_2 & = & P & = & \frac{c}{a}\end{array}\right\|$. Therefore, $x_1^2=x_2 \ \vee \ x_2^2=x_1 \iff (x_1^2-x_2)(x_2^2-x_1)= 0 \iff$

$P^2-(S^3-3PS)+P=0 \iff S^3=P(3S+P+1) \iff \boxed{b^3+ac(a+c)=3abc}$.

□ Find the values of x that satisfy the equation: $\sqrt{\pi^2-4x^2}=arcsin(cosx)$

Solution

By trig identity $arcsin(cosx)=arcsin(sin(\frac{\pi}{2}-x))=\frac{\pi}{2}-x$, so

$$(\pi+2x)(\pi-2x)=\left(\frac{\pi}{2}-x\right)^2$$

and by cancellation we get $2(\pi+2x)=\frac{\pi}{2}-x$, so $x=\frac{-3\pi}{10}$ and $x=\frac{\pi}{2}$)

□ Let $n>2$ be a composite number. Prove that not all of the terms in the sequence

$$\binom{n}{1},\binom{n}{2},\binom{n}{2},\ldots,\binom{n}{n-1}$$

Solution

When n is composite, there exists a prime $p\mid n$, $p<n$. Then in $\binom{n}{p}=\frac{n(n-1)\cdots(n-(p-1))}{p!}$ the factors $n-j$, $1\le j\le p-1$, are co-prime with p, hence the power of p dividing $\binom{n}{p}$ is one less than that dividing n, therefore $n\nmid\binom{n}{p}$.

□ prove that $cos(\frac{2\pi}{2n+1})+cos(\frac{4\pi}{2n+1})+---cos(\frac{2n\pi}{2n+1})=\frac{-1}{2}$ n is natural number

Solution

One has $\cos\frac{2k\pi}{2n+1}=\cos\frac{2(2n-k+1)\pi}{2n+1}$ for all $1\le k\le n$, since $\cos\theta=\cos(2\pi-\theta)$. On the other hand, $0=\sum_{k=0}^{2n}\cos\frac{2k\pi}{2n+1}=1+\sum_{k=1}^{n}\cos\frac{2k\pi}{2n+1}+\sum_{k=n+1}^{2n}\cos\frac{2k\pi}{2n+1}=1+\sum_{k=1}^{n}\cos\frac{2k\pi}{2n+1}+\sum_{k=1}^{n}\cos\frac{2(2n-k+1)\pi}{2n+1}=1+2\sum_{k=1}^{n}\cos\frac{2k\pi}{2n+1}$, since these are the real parts of the roots of $z^{2n+1}-1=0$.

□ Find the largest positive integer k such that $\phi(\sigma(2^k))=2^k$. ($\phi(n)$ denotes the number of positive integers that are smaller than n and relatively prime to n, and $\sigma(n)$ denotes the sum of divisors of n). As a hint, you are given that $641|2^{32}+1$.

Solution

The hint makes it fairly obvious that the correct answer is $k=31$.

$\sigma(2^k)=\sum_{i=0}^{k}2^i=2^{k+1}-1$, so $\phi(2^{k+1}-1)=2^k$. Suppose $2^{k+1}-1=\prod_{i=1}^{m}p_i^{e_i}$; then $2^k=\phi(2^{k+1}-1)=(2^{k+1}-1)\prod_{i=1}^{m}\left(\frac{p_i-1}{p_i}\right)$. Since 2^k has no odd factors, it follows that $\prod_{i=1}^{m}p_i\ge 2^{k+1}-1$, but by definition, $\prod_{i=1}^{m}p_i\le 2^{k+1}-1$; so $\prod_{i=1}^{m}p_i=2^{k+1}-1$. Then $2^k=\prod_{i=1}^{m}(p_i-1)$, so it follows that $p_i=2^{j_i}+1$ for integers j_i, and

$$\prod_{i=1}^{m}(2^{j_i}+1)=2^{k+1}-1.$$

Because 2^{j_i+1} is prime, j_i cannot have any odd factors, so $j_i=2^{l_i}$ for integers l_i. Multiplying both sides by $\prod_{i=1}^{m}(2^{2^{l_i}}-1)$ gives

$$\prod_{i=1}^{m}(2^{2^{l_i+1}}-1)=(2^{k+1}-1)\prod_{i=1}^{m}(2^{2^{l_i}}-1).$$

Thus $\gcd\left(2^{2^{l_i}}-1,2^{k+1}-1\right)\ne 1$ for some i. By the Euclidean Algorithm, this is true iff k is a multiple of 2^{l_i} or vice versa, and $k+1$ has no odd prime factors (otherwise it divides out; if l is odd and $l|k+1$ then $2^l-1|2^{k+1}-1$, but by Euclidean Algorithm $\gcd\left(2^{2^{l_i}}-1, 2^l-1\right)=1$ for all i, giving us an extraneous prime factor), so $k=2^r-1$ for some r.

If $k+1=2^r$, then $2^{2^i}+1|2^{k+1}-1$ and $\gcd\left(2^{2^i}+1, 2^{2^j}+1\right)=1$ for $0\le i,j\le r-1$, and expanding the last expression and using a telescoping difference of squares product gives

$$\left[\frac{1}{2^1-1}\right]\cdot(2^{k+1}-1)\prod_{i=0}^{r-1}\left(\frac{2^{2^i}}{2^{2^i}+1}\right)=2^{2^r-1}=2^k,$$

which holds true iff $2^{2^i}+1$ is prime for all $1\le i\le r-1$, so $r\le 5$ and $k\le \boxed{31}$, which works.

□ Prove that the equation $x=\frac{1}{x+1}+\frac{1}{x+2}+...+\frac{1}{x+2010}$ has exactly 2011 solutions.

Solution

Let $f(x)=\frac{1}{x+1}+\frac{1}{x+2}+...+\frac{1}{x+2010}$ and $g(x)=x$. Since $f'(x)<0$ and for all $i\in\{1,2,...,2010\}$: $\lim_{x\to -i^+}f(x)=+\infty$, $\lim_{x\to -i^-}f(x)=-\infty$, $\lim_{x\to\pm\infty}f(x)=0$ and g is increasing function, we see that the equation has exactly one real root on all following interval: $(-\infty,-2010)$, $(-2010,-2009)$,..., $(-1,+\infty)$. Done!

□ Let $\{u,v\}\subset C^*$ and $L_{u,v}=\{\ z\in C\mid\ z+u\overline{z}+v=0\ \}$. What are the necessary and sufficient conditions for $L_{u,v}$ is a line ?

Proof. $z+u\overline{z}+v=0 \iff \overline{z}+\overline{u}z+\overline{v}=0$. Eliminate $\overline{z}$ between the equivalent equations $\left\{\begin{array}{l} z+u\cdot\overline{z}+v=0\\ \overline{u}\cdot z+\overline{z}+\overline{v}=0\end{array}\right|\implies$

$(1-|u|^2)\cdot z+(v-u\overline{v})=0$, a relation what is verified by an infinitude of points $z\in L_{u,v}$. In conclusion, $|u|=1\ \wedge\ u\overline{v}=v$.

Otherwise. The equivalent equations $\left\{\begin{array}{l} z+u\cdot\overline{z}+v=0\\ \overline{u}\cdot z+\overline{z}+\overline{v}=0\end{array}\right| \iff \frac{1}{\overline{u}}=\frac{u}{1}=\frac{v}{\overline{v}} \iff |u|=1\ \wedge\ u\overline{v}=v \iff u\overline{v}=v$

because $|u\overline{v}|=|v|$ and $v\ne 0 \implies |u|=1$.

Another way: The general form of a line is: $Ax+By+C=0$ $(A,B,C$ are real; A,B are not both zero) Let $z=x+iy$, then $x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}$, so $A\ \frac{z+\bar{z}}{2}+B\ \frac{z-\bar{z}}{2i}+C=0$, $z+\frac{a^2}{|a|^2}\bar{z}+\frac{ba}{|a|^2}=0$ $(a=A+iB\ne 0,\ b=2C$ is real) Compare coefficients of $L_{u,v}$, $u=\frac{a^2}{|a|^2}$, $v=\frac{ba}{|a|^2}$; $|u|=1$, $u\overline{v}=v$ ■

□ $f(x)=x^{x^{x^x}}$. Example: $f(2)=2^{2^{2^2}}=2^{2^4}=2^16$. Find the last 2 digits of: $f(17)+f(18)+f(19)+f(20)$.

Solution

$17^2=289=-11 \mod 100$

$17^4=(-11)^2=121=21 \mod 100$

$17^8=21^2=441=41 \mod 100$

$17^{16}=41^2=1681=81 \mod 100$

$17^{20}=17^{16}17^4=81\times 21=1701=1 \mod 100$

$17^4=(-3)^4=81=1 \mod 20$

$17=1 \mod 4$

$17^{17}=1 \mod 4$

$17^{17^{17}}=17 \mod 20$

$17^{17^{17^{17}}}=17^{17}=17^{16}17=81\times 17=1377=77 \mod 100$

$18^2=324=24 \mod 100$

$18^4=24^2=576=76 \mod 100$

$18^8=76^2=5776=76 \mod 100$

$18^{4n} = 76 \mod 100$ for all $n \geq 1$

$18 = 2 \mod 4$

$18^{18} = 2^{18} = 4^9 = 0 \mod 4$

$18^{18^{18}} = 2^{18^{18}} = 0 \mod 4$

$18^{18^{18^{18}}} = 76 \mod 100$

$19^2 = 361 = 61 \mod 100$

$19^4 = 61^2 = 3721 = 21 \mod 100$

$19^5 = 19^4 19 = 21 \times 19 = 399 = -1 \mod 100$

$19^{10} = (-1)^2 = 1 \mod 100$ $19^2 = 1 \mod 10$ $19^{19^{19}} = 19 = 9 \mod 10$ $19^{19^{19^{19}}} = 19^9 = 19^5 19^4 = -1 \times 21 = -21 = 79 \mod 100$

$20^{20^{20^{20}}} = 0 \mod 100$

Last 2 digits of $f(17) + f(18) + f(19) + f(20) = 32$

$\square$ Prove that if x, y are integers such that $x^2 + y^2 + 3 \geq 6xy$, then $x^2 + y^2 \geq 6xy$

Solution

Suppose that $x^2 + y^2 < 6xy$. We have $3 \geq 6xy - x^2 - y^2$ and $0 < 6xy - x^2 - y^2$. So, $\exists\ x, y \in \mathbb{Z}$ such that $6xy - x^2 - y^2 \in \{1, 2, 3\}$. But $\forall\ k \in \{1, 2, 3\}$, $k = 6xy - x^2 - y^2 \Leftrightarrow (x + y)^2 + k = 8xy \Rightarrow (x + y)^2 \equiv 5, 6, 7 \pmod 8$, contradiction because $\forall a \in \mathbb{Z}\ a^2 \equiv 0, 1, 4 \pmod 8$.

$\square$ Let ABC be a triangle with the centroid G and let M be an arbitrary interior point. The line MG cut AB , BC , CA in Z , X , Y . Prove that $\frac{\overline{XM}}{\overline{XG}} + \frac{\overline{YM}}{\overline{YG}} + \frac{\overline{ZM}}{\overline{ZG}} = 3$.

Solution

Denote the midpoints D , E , F of the sides $[BC]$, $[CA]$, $[AB]$ respectively and $U \in AM \cap BC$, $V \in BM \cap CA$,

$W \in CM \cap AB$. Prove easily that $\frac{\overline{UM}}{\overline{UA}} + \frac{\overline{VM}}{\overline{VB}} + \frac{\overline{WM}}{\overline{WC}} = \sum \frac{[BMC]}{[BAC]} = 1 \quad (*)$. Apply the Menelaus' theorem to

$$\text{transversals}: \left\{\begin{array}{lcl} \overline{XDU}/\triangle AGM & : & \frac{\overline{XM}}{\overline{XG}} \cdot \frac{\overline{DG}}{\overline{DA}} \cdot \frac{\overline{UA}}{\overline{UM}} = 1 \quad \Longrightarrow \quad \frac{\overline{XM}}{\overline{XG}} = 3 \cdot \frac{\overline{UM}}{\overline{UA}} \\ \\ \overline{YEV}/\triangle BGM & : & \frac{\overline{YM}}{\overline{YG}} \cdot \frac{\overline{EG}}{\overline{EB}} \cdot \frac{\overline{VB}}{\overline{VM}} = 1 \quad \Longrightarrow \quad \frac{\overline{YM}}{\overline{YG}} = 3 \cdot \frac{\overline{VM}}{\overline{VB}} \\ \\ \overline{ZFW}/\triangle CGM & : & \frac{\overline{ZM}}{\overline{ZG}} \cdot \frac{\overline{FG}}{\overline{FC}} \cdot \frac{\overline{WC}}{\overline{WM}} = 1 \quad \Longrightarrow \quad \frac{\overline{ZM}}{\overline{ZG}} = 3 \cdot \frac{\overline{WM}}{\overline{WC}} \end{array}\right| \stackrel{(*)}{\Longrightarrow} \sum \frac{\overline{XM}}{\overline{XG}} = 3$$

.

$\square$ 25 points are given on the plane. Among any three of them, one can choose two less than one inch apart. Prove that there are 13 points among them which lie in a circle of radius 1

Solution

Consider the graph G whose vertices are the 25 points, with edges between points no less than 1 inch apart. The given condition means G contains no triangle K_3, therefore its number of edges is at most that of a complete bipartite $K_{12,13}$ graph, by Turán's theorem. Thus the number of pairs of points at pairwise distance less than 1 inch is at least $\binom{12}{2} + \binom{13}{2} = 144$, and these are the edges $E(\overline{G})$ of the complementary graph $\overline{G}$.

But then $\sum_{v \in V(\overline{G})} \deg v = 2E(\overline{G}) \geq 288$, hence for at least one vertex v we have $\deg v \geq \lceil 288/25 \rceil = 12$. Acircle of radius 1 drawn having v as its center thus contains at least 12 other points, so at least 13 in all.

Alternatively, considering the graph G whose edges are between points less than 1 inch apart, the given condition means the independence number $\alpha(G)$ is at most 2. By the Caro-Wei theorem we have $2 \geq \alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg v + 1}$, so for at least one vertex v we have $\frac{1}{\deg v + 1} \leq \frac{2}{25}$, i.e. $\deg v + 1 \geq \frac{25}{2}$, so $\deg v \geq 12$.

□ find the sum to n terms of the series, $\frac{1}{1.3}+\frac{2}{1.3.5}+\frac{3}{(1.3.5.7}+\ldots\ldots\ldots\ldots\ldots$

Solution

Denote $a_n=\frac{1}{1\cdot 3}+\frac{2}{1\cdot 3\cdot 5}+\cdots+\frac{n}{1\cdot 3\cdot 5\cdots(2n+1)}=\frac{b_n}{1\cdot 3\cdot 5\cdots(2n+1)}$. Then $a_n=\frac{b_{n-1}}{1\cdot 3\cdot 5\cdots(2n-1)}+\frac{n}{1\cdot 3\cdot 5\cdots(2n+1)}=\frac{(2n+1)b_{n-1}+n}{1\cdot 3\cdot 5\cdots(2n+1)}$, leading to $b_n=(2n+1)b_{n-1}+n$.

This also writes $2b_n+1=2(2n+1)b_{n-1}+2n+1=(2n+1)(2b_{n-1}+1)$. By iterating, $2b_n+1=(2n+1)(2n-1)\cdots 3\cdot 1=(2n+1)!!$, so $b_n=\frac{1}{2}((2n+1)!!-1)$, thus $a_n=\frac{(2n+1)!!-1}{2\cdot(2n+1)!!}=\frac{1}{2}-\frac{1}{2\cdot(2n+1)!!}$.

This also suggests an alternative solution. Compute $a_n+\frac{1}{2\cdot(2n+1)!!}$; the sum of its last two terms is $\frac{n}{(2n+1)!!}+\frac{1}{2\cdot(2n+1)!!}=\frac{2n+1}{2\cdot(2n+1)!!}=\frac{1}{2\cdot(2n-1)!!}$, and it all telescopes to $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 3!!}=\frac{1}{2}$. □ The cubic equation $x^3+2x-1=0$ has exactly one real root r. Note that $0.4<r<0.5$.

(a) Find, with proof, an increasing sequence of positive integers $a_1<a_2<a_3<\cdots$ such that

$$\frac{1}{2}=r^{a_1}+r^{a_2}+r^{a_3}+\cdots.$$

(b) Prove that the sequence that you found in part (a) is the unique increasing sequence with the above property.

Solution

a)Since $r^3+2r-1=0$, we have $1-r^3=2r$ or $\frac{1}{2}=\frac{r}{1-r^3}$. Writing this as a geometric series, we get $\frac{1}{2}=\frac{r}{1-r^3}=r^1+r^4+r^7+\cdots$. So $a_k=3k-2$ for positive integers k suffices. b) If $1\leq b_1<b_2<b_3<\cdots$ would be another such sequence, then there will exist a first index k such that $3k-2=a_k\neq b_k$.

1. If $b_k<a_k$, thus $b_k\in\{3k-3,3k-4\}$, then, from the index k on, the sum for the "b"-sequence is larger than r^{b_k}, while the sum for the "a"-sequence is equal to $\frac{r^{3k-2}}{1-r^3}=\frac{r^{3k-3}}{2}<r^{b_k}$, contradiction. 1. If $b_k>a_k$, thus $b_k\geq 3k-1$, then, from the index k on, the sum for the "b"-sequence is at most $\frac{r^{3k-1}}{1-r}$, while the sum for the "a"-sequence is still equal to $\frac{r^{3k-3}}{2}$. We thus need $\frac{r^{3k-1}}{1-r}\geq\frac{r^{3k-3}}{2}$, that is $2r^2\geq 1-r$, or $(2r-1)(r+1)\geq 0$, impossible, since $0.4<r<0.5$, thus $0<2r<1$. □ $\sum_{k=0}^{1998}\frac{k+3}{(k+1)!+(k+2)!+(k+3)!}+\frac{1}{2001!}$

Find the value $2008.k$

Solution

We rewrite the expression inside the summation as $\frac{1}{(k+1)!}\cdot\frac{k+3}{1+(k+2)+(k+2)(k+3)}=\frac{1}{(k+3)(k+1)!}=\frac{k+2}{(k+3)!}$. After experimenting a little, we find

$$\sum_{k=0}^{n}\frac{k+2}{(k+3)!}=\frac{1}{2}-\frac{1}{(n+3)!}$$

which we prove by induction. The base case is trivial. For the induction step, assume the result for some $n=t$ to find $\sum_{k=0}^{t+1}\frac{k+2}{(k+3)!}=\frac{1}{2}-\frac{1}{(n+3)!}+\frac{t+3}{(t+4)!}=\frac{1}{2}-\frac{1}{(t+4)!}$ as desired. Now we just use $n=1998$ to find $\sum_{k=0}^{1998}\frac{k+3}{(k+1)!+(k+2)!+(k+3)!}=\sum_{k=0}^{1998}\frac{k+2}{(k+3)!}=\frac{1}{2}-\frac{1}{2001!}$. Adding this to $\frac{1}{2001!}$ conveniently leaves $\boxed{\frac{1}{2}}$.

□ Let $m,n\in\mathbb{N}-\{0,1\}$ such that $\sqrt{6}-\frac{m}{n}>0$. Prove that $\sqrt{6}-\frac{m}{n}>\frac{1}{2mn}$.

Solution

Otherwise $\sqrt{6}n<\frac{1}{2m}+m \implies 6n^2<\frac{1}{4m^2}+1+m^2$

Combining this with the given condition $0<(6n^2-m^2)<1+\frac{1}{4m^2}$

So, $6n^2-m^2=1 \implies m^2\equiv -1 \pmod 6$

But this is impossible. So we arrive at a contradiction.

So $\sqrt{6}-\frac{m}{n}>\frac{1}{2mn}$

□ If $a \neq b$ and $a, b \in \mathbb{R}_+$, then find biggest k that $(\sqrt{a}-\sqrt{b})^2 \geq k\sqrt{ab}$ is true for all a, b.

Solution

$(\sqrt{a}-\sqrt{b})^2 \geq k\sqrt{ab} \Leftrightarrow a+b \geq k\sqrt{ab}+2\sqrt{ab}$. By $AM-GM$ we have $a+b \geq 2\sqrt{ab}$. For $k=0$ the inequality is true. Suppose that $k>0$. $a+b \geq k\sqrt{ab}+2\sqrt{ab} \Leftrightarrow \frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{b}}{\sqrt{a}} \geq k+2$. Denote $a=x$ and let $b=1$. The inequality becomes $x+\frac{1}{x} \geq k+2$. But $k>0$, then $\exists\ y>0$ such that $k+y>2$ and $y<2$. If we find $x>0$ $(x \neq 1)$ such that $x+\frac{1}{x}=k+y$, we'll obtain a contradiction. We have $x^2-x(k+y)+1=0$. Then, we obtain $x=\frac{k+y+\sqrt{(k+y)^2-4}}{2}>0$. So, for $(a,b)=(\sqrt{\frac{k+y+\sqrt{(k+y)^2-4}}{2}},1)$ we have a contradiction! Then $\boxed{k=0}$.

□ Let ABC be a triangle with the incenter I. Denote $D \in AI \cap BC$, $E \in BI \cap CA$, $F \in CI \cap AB$ and $M \in BE \cap DF$, $N \in CN \cap DE$. Prove that $\widehat{IAM} \equiv \widehat{IAN}$.

Solution

Using an well-known relation obtain that $\left\{\begin{array}{ccc} \frac{MF}{MD}=\frac{EA}{EC}\cdot\frac{BF}{BD}\cdot\frac{BC}{BA} & \Longrightarrow & \frac{MF}{MD}=\frac{b+c}{a+b} \\ & & \\ \frac{NE}{ND}=\frac{FA}{FB}\cdot\frac{CE}{CD}\cdot\frac{CB}{CA} & \Longrightarrow & \frac{NE}{ND}=\frac{b+c}{a+c}\end{array}\right|$.

Denote $X \in AM \cap BC$ and $Y \in AN \cap BC$. Using Menelaus' theorem for the transversals in the mentioned

triangles $\left\{\begin{array}{ccccc} \overline{AMX}/\triangle BDF & : & XB=\frac{ac}{2b+c} & \Longrightarrow & \frac{XB}{XC}=\frac{c}{2b} \\ & & & & \\ \overline{ANY}/\triangle CDE & : & YC=\frac{ab}{b+2c} & \Longrightarrow & \frac{YB}{YC}=\frac{2c}{b}\end{array}\right| \Longrightarrow \frac{XB}{XC}\cdot\frac{YB}{YC}=\left(\frac{AB}{AC}\right)^2$.

From the Steiner's theorem obtain that $\widehat{DAX} \equiv \widehat{DAY}$, i.e. $\widehat{IAM} \equiv \widehat{IAN}$.

□ ► $n=10 \Rightarrow 2^{10}=1024>1000=10^3$. So the given claim is true for $n:=10$. ► Suppose that for some $m \geq 10$, $2^m>m^3$ $(*)$. Then $2^{m+1}=2\cdot 2^m>2m^3=m^3+m\cdot m^2>m^3+(3+3+1)\cdot m^2=m^3+3m^2+3m+1=(m+1)^3$. Now we proved that $2^n>n^3, \forall\ n>9$.

Remark. Suppose that for a given $p \in \mathbb{N}^*$ exists $s \in \mathbb{N}^*$ so that $s \geq 2^p-1$ and $2^s>s^p$.

Then $(\forall)\ n \in \mathbb{N}^*$, $n \geq s$ we have $2^n>n^p$. Indeed :

► $n=s \Rightarrow 2^s>s^p$. So the given claim is true for $n:=s$.

► Suppose that for some $m \geq s>2^p-1$, $2^m>m^p$ $(*)$. Then $2^{m+1}=2\cdot 2^m>2\cdot m^p=m^p+m\cdot m^{p-1}>m^p+(2^p-1)\cdot m^{p-1}=m^p+\left(\sum_{k=1}^p C_p^k\right)\cdot m^{p-1}>m^p+\sum_{k=1}^p C_p^k m^{p-k}=\sum_{k=0}^p C_p^k m^{p-k}=(m+1)^p$. Now we proved that $2^n>n^p$, $\forall\ n>s$.

Particular case. For $p=3$ exists $s=10 \geq 2^3-1=7$ so that $2^{10}>10^3$. Then $2^n>n^3$ for any $n \geq 10$.

□ Lemma. Consider the real numbers $r \neq 0$, x_1 and $x_{k+1}=x_k+r$, where $k \in \mathbb{N}^*$.

Then $\boxed{C \equiv \sum_{k=1}^n \cos x_k=\frac{\cos\frac{x_1+x_n}{2}\sin\frac{nr}{2}}{\sin\frac{r}{2}}}$ and $\boxed{S \equiv \sum_{k=1}^n \sin x_k=\frac{\sin\frac{x_1+x_n}{2}\sin\frac{nr}{2}}{\sin\frac{r}{2}}}$.

Proof. $2\sin\frac{r}{2}\cdot C=\sum_{k=1}^n 2\sin\frac{r}{2}\cos x_k=\sum_{k=1}^n\left[\sin\left(x_k+\frac{r}{2}\right)-\sin\left(x_k-\frac{r}{2}\right)\right]=\sin\left(x_n+\frac{r}{2}\right)-\sin\left(x_1-\frac{r}{2}\right)$, because $x_{k+1}-\frac{r}{2}=x_k+\frac{r}{2}$. In conclusion, $2\sin\frac{r}{2}\cdot C=$

$2\sin\frac{x_n-x_1+r}{2}\cos\frac{x_1+x_n}{2}=2\sin\frac{(n-1)r+r}{2}\cos\frac{x_1+x_n}{2} \Longrightarrow \boxed{C=\frac{\cos\frac{x_1+x_n}{2}\sin\frac{nr}{2}}{\sin\frac{r}{2}}}$.

$\boxed{2\sin\frac{r}{2}\cdot S}=\sum_{k=1}^n 2\sin\frac{r}{2}\sin x_k=\sum_{k=1}^n\left[\cos\left(x_k-\frac{r}{2}\right)-\cos\left(x_k+\frac{r}{2}\right)\right]=$

$\cos\left(x_1-\frac{r}{2}\right)-\cos\left(x_n+\frac{r}{2}\right)$, because $x_{k+1}-\frac{r}{2}=x_k+\frac{r}{2}$. In conclusion, $2\sin\frac{r}{2}\cdot S=$

$2\sin\frac{x_n-x_1+r}{2}\sin\frac{x_1+x_n}{2}=2\sin\frac{(n-1)r+r}{2}\sin\frac{x_1+x_n}{2} \implies \boxed{S=\frac{\sin\frac{x_1+x_n}{2}\sin\frac{nr}{2}}{\sin\frac{r}{2}}}$.

Problem: Denote $E\equiv\sum_{k=1}^n\sin^2\frac{k\pi m}{n}$, where $m,n\in\mathbb{Z}$ and $0<m<n$. Find E as a function of n .

Solution

$2E=\sum_{k=1}^n\left(1-\cos\frac{2k\pi m}{n}\right)=n-F$, where $F\equiv\sum_{k=1}^n\cos\frac{2k\pi m}{n}$. Apply upper lemma for $x_1=r=\frac{2m\pi}{n}$ and

obtain that $F=\frac{\cos\frac{x_1+x_n}{2}\sin\frac{nr}{2}}{\sin\frac{r}{2}}=\frac{\cos\frac{x_1+x_n}{2}\sin\left(\frac{n}{2}\cdot\frac{2m\pi}{n}\right)}{\sin\frac{r}{2}}=0 \implies F=0$. In conclusion, $2E=n \implies E=\frac{n}{2}$.

□ Prove that $\sin x=k\cdot\sin(a-x) \iff \tan\left(x-\frac{a}{2}\right)=\frac{k-1}{k+1}\cdot\tan\frac{a}{2}$.

Solution

$\sin x=k\cdot\sin(a-x) \iff \frac{k}{1}=\frac{\sin x}{\sin(a-x)} \iff \frac{k-1}{k+1}=\frac{\sin x-\sin(a-x)}{\sin x+\sin(a-x)} \iff$

$\frac{k-1}{k+1}=\frac{2\cdot\sin\left(x-\frac{a}{2}\right)\cos\frac{a}{2}}{2\cdot\sin\frac{a}{2}\cos\left(x-\frac{a}{2}\right)} \iff \frac{k-1}{k+1}=\frac{\tan\left(x-\frac{a}{2}\right)}{\tan\frac{a}{2}} \iff \tan\left(x-\frac{a}{2}\right)=\frac{k-1}{k+1}\cdot\tan\frac{a}{2}$.

Remark. For $x:=B$, $a:=-A$, $k:=-k$ obtain the Mollweide's identity in $\triangle ABC$: $\sin B=k\cdot\sin C \iff$

$\tan\frac{B-C}{2}=\frac{k-1}{k+1}\cdot\cot\frac{A}{2} \iff \tan\frac{B-C}{2}=\frac{b-c}{b+c}\cdot\cot\frac{A}{2}$ because $k=\frac{\sin B}{\sin C}=\frac{b}{c}$.

□ Let $z=\cos(1)+i\sin(1)$, $\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n=\frac{1}{1-\frac{z}{2}}=\frac{4-2\cos(1)+2i\sin(1)}{5-4\cos(1)}$ Compare real parts, $\sum_{n=0}^{\infty}\frac{\cos(n)}{2^n}=\frac{4-2\cos(1)}{5-4\cos(1)}$ ■ – You can find those general results

$$\sum_{k=0}^{\infty}x^k cos(ak+b)=\frac{cos(b)-xcos(b-a)}{x^2-2xcosa+1}$$

$$\sum_{k=0}^{\infty}x^k sen(ak+b)=\frac{sen(b)-xsen(b-a)}{x^2-2xcosa+1}$$

or the indefinite summations

$$\sum_k x^k cos(ak+b)=\frac{x^{k+1}cos[a(k-1)+b]-x^k cos[ak+b]}{x^2-2xcosa+1}$$

$$\sum_k x^k sen(ak+b)=\frac{x^{k+1}sen[a(k-1)+b]-x^k sen[ak+b]}{x^2-2xcosa+1}$$

□ If $tanx=ntany$, then the maximum value of $sec^2(x-y)$ is ???

Solution

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}=\frac{(n-1)\tan y}{1+n\tan^2 y}$$

. Using $1+\tan^2 A=\sec^2 A$ this gives

$$\sec^2(x-y)=1+\left(\frac{(n-1)Y}{1+nY^2}\right)^2$$

where $Y=\tan y$

Differentiating this expression with respect to Y gives

$$\frac{\mathrm{d}[\sec^2(x-y)]}{\mathrm{d}Y}=\frac{2Y(n-1)^2(1-nY^2)}{(1+nY^2)^3}$$

For turning points, the numerator is zero, i.e. $2Y(n-1)^2(1-nY^2)=0$.

For either $n = 1$ or $Y = 0$ we have $\sec^2(x-y) = 1$, a minimum point. If $1 - nY^2 = 0$ we have $\sec^2(x-y) = 1 + \left(\frac{n-1}{2\sqrt{n}}\right)^2 = \frac{(n+1)^2}{4n}$

Examining the sign of the derivative, we see that at just less the turning points when $Y^2 = \frac{1}{n}$, it is positive for both positive and negative Y and at just more than it is negative. We conclude that the maximum value of $\sec^2(x-y)$ is $\frac{(n+1)^2}{4n}$

□ Show that $P(z) = 0 \Rightarrow |z| \le 2$ if $P(z) = z^7 + 7z^4 + 4z + 1$

Solution

Clearly $|z^7+7z^4+4z+1| \ge |z^7| - |7z^4| - |4z| - |1|$, which can be proved by showing $|a+b| \ge |a| - |b|$. Then if $|z| > 2$, $P(z) \ge |z^7| - |7z^4| - |4z| - |1| \ge |\frac{z^7}{8}| - |4z| - |1| \ge |\frac{z^7}{16}| - |1| \ge 8 - 1 = 7 > 0$, therefore it cannot be that $|z| > 2$, and therefore $|z| \le 2$.

□ Let $ABCD$ be a rhombus. Let $P \in (BC)$ and $Q \in (CD)$ such that $BP = CQ$. Prove that the centroid of $\triangle APQ$ lies on (BD) .

Solution

Proof 1 (synthetic). Denote $R \in (AD)$ so that $AR = BP = CQ$, $S \in QR \cap BD$, the midpoint T of $[AQ]$ and $G \in BD \cap PT$. Observe that $ACQR$ is an isosceles trapezoid, TS is the Q-middle line

in $\triangle AQR$ and $TS \parallel BP$ with $\frac{GP}{GT} = \frac{BP}{TS} = \frac{AR}{TS} = 2$, i.e. G is the centroid of the triangle APQ .

Proof 2 (with vectors). Denote $M \in (BD)$ so that $CPMQ$ is a parallelogram. Observe that $BP = PM = CQ$. Thus,

$\overrightarrow{AP} - \overrightarrow{AM} = \overrightarrow{MP} = \overrightarrow{QC} = \overrightarrow{AC} - \overrightarrow{AQ}$, i.e. the triangles APQ and AMC have a common A-median AS , where

$S \in PQ \cap CM$. Hence these triangles have and a common centroid G , where $G \in AS \cap MD$, i.e. $G \in BD$.

Proof 3 (analytic). Suppose w.l.o.g. $AB = 1$, $m\left(\widehat{BAD}\right) = \phi < 90^\circ$ and $A(0,0)$, $B(\cos\phi, \sin\phi)$, $C(1+\cos\phi, \sin\phi)$

and $D(1,0)$. For $BP = CQ = r < 1$ obtain easily that $P(1 - r + \cos\phi, \sin\phi)$ and $Q(1 + r\cos\phi, r\sin\phi)$ .Therefore,

the centroid $G_{APQ}\left(\frac{2-r+(1+r)\cos\phi}{3}, \frac{(1+r)\sin\phi}{3}\right) \in (BD) \iff \begin{vmatrix} 1 & 0 & 1 \\ \cos\phi & \sin\phi & 1 \\ 2-r+(1+r)\cos\phi & (1+r)\sin\phi & 3 \end{vmatrix} = 0$

$$\iff \begin{vmatrix} 1 & 0 & 1 \\ \cos\phi & 1 & 1 \\ 2-r+(1+r)\cos\phi & (1+r) & 3 \end{vmatrix} = 0 \iff \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2-r & 1+r & 3 \end{vmatrix} + \cos\phi \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1+r & 1+r & 3 \end{vmatrix} = 0 ,$$

what is truly because we have in the first determinant $C_3 = C_1 + C_2$ and in the second determinant $C_1 = C_2$.

Otherwise, prove easily that $G \in BD \iff y_G = \frac{\sin\phi}{\cos\phi - 1} \cdot (x_G - 1)$, i.e. $(1+r)\sin\phi = \frac{\sin\phi}{\cos\phi-1} \cdot (\cos\phi - 1)(1+r)$

□ Prove that $a_n < (\frac{1+\sqrt{5}}{2})^n, \forall\, n \ge 2, a_n$ nth term of Fibonacci.

Solution

We use induction. Check the base case for $n=2$ and $n=3$ Now let the statement be true for $n=k$ and $n=k-1$ $a_{k+1}=a_k+a_{k-1}<(\frac{1+\sqrt{5}}{2})^k+(\frac{1+\sqrt{5}}{2})^{k-1}=(\frac{1+\sqrt{5}}{2})^{k-1}\cdot(1+\frac{1+\sqrt{5}}{2})=$

$(\frac{1+\sqrt{5}}{2})^{k-1}\cdot(\frac{5+1+2\sqrt{5}}{4})=(\frac{1+\sqrt{5}}{2})^{k-1}\cdot(\frac{1+\sqrt{5}}{2})^2=(\frac{1+\sqrt{5}}{2})^{k+1}$ So this statement is true for $n=k+1$ too. this completes our induction step. So this statement is true for all $n\geq 2$

☐ Let $ABCD$ be a trapezoid with $AB\parallel CD$ and $AC\perp BD$. Denote $O\in AC\cap BD$. Prove that $AB\cdot CD=AO\cdot OC+BO\cdot OD$.

Solution

Method 1 (trigonometric). Denote $\left\{\begin{array}{c} m(\angle OAB)=m(\angle OCD)=x \\ \\ m(\angle OBA)=m(\angle ODC)=y \end{array}\right|$. Thus, $x+y=90^\circ$

and $AB\cdot CD=AO\cdot OC+BO\cdot OD \iff 1=\frac{AO}{AB}\cdot\frac{OC}{CD}+\frac{BO}{AB}\cdot\frac{OD}{CD} \iff$

$1=\sin y\cos x+\sin x\cos y \iff 1=\sin(y+x) \iff x+y=90^\circ$, what is truly.

Method 2 (sax - metric). Denote $\left\{\begin{array}{c} OA=x\ ;\ OB=y \\ \\ OC=z\ ;\ OD=t \end{array}\right|$. Thus, $AB\cdot CD=AO\cdot OC+BO\cdot OD \iff$

$AB^2\cdot CD^2=(AO\cdot OC+BO\cdot OD)^2 \iff (x^2+y^2)\cdot(z^2+t^2)=(xz+yt)^2 \iff x^2t^2+y^2z^2=(xz+yt)^2 \iff x^2t^2+y^2z^2=2xyzt \iff (xt-yz)^2=0 \iff xt=yz$, what is truly (well-known) because

$[ACD]=[BCD] \iff [ACD]-[COD]=[BCD]-[COD] \iff [AOD]=[BOC] \iff xt=yz$.

An easy extension. Let $ABCD$ be a convex quadrilateral. Denote $O\in AC\cap BD$ and the area $[AOD]=S$,

$m(\angle AOD)=\phi$. Prove that $AB^2\cdot CD^2=\left[(xz+yt)+(xt+yz)\cdot\cos\phi\right]^2+(xt-yz)^2\cdot\sin^2\phi$.

Therefore, $AB\cdot CD\geq|(xz+yt)+(xt+yz)\cdot\cos\phi|$, with equality where $\phi=90^\circ$.

☐ Let $a,b,c\in\mathbb{Z}$ such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ and $\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$ are integers. Prove that $|a|=|b|=|c|$.

Solution

Note $x=\dfrac{a}{b};y=\dfrac{b}{c};z=\dfrac{c}{a};m=\sum\dfrac{a}{b}$ and $n=\sum\dfrac{b}{a}$. We have $\sum x=m;\sum\dfrac{1}{x}=n$ and $xyz=1$, so $x^3-mx^2+nx-1=0$. Be $x=\dfrac{p}{q}$, with $p,q\in\mathbb{N};(p;q)=1 => q|p => a=\pm b$ and analogues.

☐ Find the value m $\sum_{p=0}^m\binom{m}{p}2^p=729$

Solution

We can see from the binomial expansion of $(1+2)^m$ that $\sum_{p=0}^m\binom{m}{p}2^p=3^m$.

Thus, $3^m=729=3^6$, and $m=6$.

Doing the math to be sure, we have on the left side when $m=6$:

$\sum_{p=0}^6\binom{6}{p}2^p$

$\binom{6}{0}2^0+\binom{6}{1}2^1+\binom{6}{2}2^2+\binom{6}{3}2^3+\binom{6}{4}2^4+\binom{6}{5}2^5+\binom{6}{6}2^6$

$1+12+60+160+240+192+64$

729

☐ A walk consists of a sequence of steps of length 1 taken in the directions north, south, east, or west. A walk is self-avoiding if it never passes through the same point twice. Let $f(n)$ be the number of n-step self-avoiding walks which begin at the origin. Compute $f(1)$, $f(2)$, $f(3)$, $f(4)$, and show that

$$2^n < f(n) \leq 4 \cdot 3^{n-1}.$$

Solution

We have $f(1) = 4$, obviously, $f(2) = 12$ (4 ways for the first, 3 for the second), $f(3) = 36$ (12 ways, and then 3 ways for the 3rd step).

For $f(4)$, we should have 108 ways; the only way this fails is if we make a square. For each first step, there are obviously two ways to make a square, so $f(4) = 108 - 8 = 100$

For the upper bound of our bounds, this is because at best, we can have 4 choices for our first move, and 3 for each one after (as we can't double back on the prior move), so $f(n) \leq 4 \cdot 3^{n-1}$

As for the lower bound: If all we do is go up or right at each turn, the path will clearly never intersect itself. This gives 2^n possibilities. Also, we can just go straight left n times in a row, so $f(n) \geq 2^n + 1 > 2^n$

□ Given: (i) a, $b > 0$; (ii) a, A_1, A_2, b is an arithmetic progression; (iii) a, G_1, G_2, b is a geometric progression. Show that

$$A_1 A_2 \geq G_1 G_2.$$

Solution

Because a, h, k, d are in GP, we know that $ad = hk$. Also, since a, b, c, d are in arithmetic progression, we know $a + d = b + c$. Hence

$(b + c)^2 = 4bc + (b - c)^2 = 4ad + (a - d)^2$

and thus $0 < (a - d)^2 - (b - c)^2 = 4(bc - ad)$ and finally $bc > ad = hk$.

□

If x is a root of the equation $x^2 + px + q = 0$, $p, q \in \mathbb{C}$ then show that: if $|p| + |q| < 1$, then $|x| < 1$.

Solution

We have $|p| + |q| < 1$. Assume $|x| \geq 1$. Then we see that:

$|x^2 + px + q| \geq |x^2| - |px| - |q| \geq |x^2| - |px| - (1 - |p|) = |x|^2 - |p| \cdot |x| - 1 + |p| = |x|^2 - |p|(|x| - 1) - 1$

As $|x| \geq 1$, this equation is decreasing as p increases. As $|p| < 1$, we set $|p| = 1. > |x|^2 - |x| \geq 0$ since $|x| \geq 1$. Thus $|x^2 + px + q| > 0$ for $|x| \geq 1$, and thus $x^2 + px + q \neq 0$ whenever $|x| \geq 1$, therefore if $x^2 + px + q = 0$ then $|x| < 1$.

□

Let a, b, c the roots of $x^3 - 9x^2 + 11x - 1 = 0$ and $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$.Find numeric value of $s^4 - 18s^2 - 8s$.

Solution

From the equation, $a + b + c = 9, ab + bc + ca = 11, abc = 1$

$$11 = ab + bc + ca = (\sqrt{ab} + \sqrt{bc} + \sqrt{ca})^2 - 2\sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}) =$$

$$= (\sqrt{ab} + \sqrt{bc} + \sqrt{ca})^2 - 2s \Rightarrow (\sqrt{ab} + \sqrt{bc} + \sqrt{ca})^2 = 11 + 2s$$

$$s^2 = (\sqrt{a} + \sqrt{b} + \sqrt{c})^2 = a + b + c + 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}) = 9 + 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$$

$$\Rightarrow s^2 - 9 = 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$$

Squaring both sides of this,

$$(s^2 - 9)^2 = s^4 - 18s^2 + 81 = 4(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})^2 = 44 + 8s$$

Rearranging,

$$s^4 - 18s^2 - 8s = 44 - 81 = -37$$

$\square$

If S is a sequence of positive integers let $p(S)$ be the product of the members of S. Let $m(S)$ be the arithmetic mean of $p(T)$ for all non-empty subsets T of S. Suppose that S' is formed from S by appending an additional positive integer. If $m(S) = 13$ and $m(S') = 49$, find S'.

Solution

Let $S = \{a_1, a_2, ..., a_n\}$. Then clearly $m(S) = \frac{\prod_{i=1}^{n}(1+a_i)-1}{2^n-1}$ Thus if a_{n+1} is appended to form S', then: $13 \cdot (2^n - 1) = \prod_{i=1}^{n}(1+a_i) - 1$ $49 \cdot (2^{n+1} - 1) = \prod_{i=1}^{n+1}(1+a_i) - 1 = (13 \cdot (2^n - 1) + 1)(1 + a_{n+1}) - 1$ $= 13 \cdot (2^n - 1) + 13 \cdot (2^n - 1) \cdot a_{n+1} + a_{n+1}$ Expanding, $98 \cdot 2^n - 49 = 13 \cdot 2^n - 13 + 13a_{n+1}2^n - 12a_{n+1}$ $\Longrightarrow 85 \cdot 2^n - 36 = 13a_{n+1}2^n - 12a_{n+1}$ By plugging in various values of a_{n+1} and solving for n, we find $n = 3, a_{n+1} = 7$ is a solution. Thus we must find $a_1, a_2, a_3 \in \mathbb{Z}$ such that $92 = (1+a_1)(1+a_2)(1+a_3)$. We easily see $a_1 = 1, a_2 = 1, a_3 = 22$ is a solution and indeed $m(\{1, 1, 22\}) = 13$ and $m(\{1, 1, 22, 7\}) = 15$, thus a possible solution of $S' = \boxed{\{1, 1, 7, 22\}}$ (Note there are multiple solutions, but they can easily be found by application of $92 = (1+a_1)(1+a_2)(1+a_3)$ giving all the solutions. You can show it is impossible for a solution to have $n = 1$, thus $n \geq 2$. This would show a solution for a_{n+1} only exists for $a_{n+1} \leq 7$, and then we can easily verify it must be $n = 3$ and $a_4 = 7$.)

$\square$

For every positive integer n show that

$$[\sqrt{4n+1}] = [\sqrt{4n+2}] = [\sqrt{4n+3}] = [\sqrt{n} + \sqrt{n+1}]$$

where $[x]$ is the greatest integer less than or equal to x (for example $[2.3] = 2$, $[\pi] = 3$, $[5] = 5$).

Solution

Trivially $[\sqrt{4n+1}] = [\sqrt{4n+2}] = [\sqrt{4n+3}]$ as there are no perfect squares $2, 3 \pmod 4$. Thus we need to show $[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}]$. Let $k \leq [\sqrt{n} + \sqrt{n+1}] < k+1$. Then: $k^2 \leq 2n + 1 + 2\sqrt{n}\sqrt{n+1} < (k+1)^2$ Then clearly: $2n + 1 + 2\sqrt{n}\sqrt{n+1} > 2n + 1 + 2\sqrt{n}\sqrt{n} > 4n + 1$, $2n+1+2\sqrt{n}\sqrt{n+1} < 2n+1+2\sqrt{n+1}\sqrt{n+1} < 4n+3$ Thus $4n+1 < 2n+1+2\sqrt{n}\sqrt{n+1} < 4n+3$. This would mean $[\sqrt{4n+1}] \leq [\sqrt{n} + \sqrt{n+1}] \leq [\sqrt{4n+3}]$. But $[\sqrt{4n+1}] = [\sqrt{4n+3}]$, thus we have universal equality and we are done.

$\square$

Suppose that $0 \leq x_i \leq 1$ for $1 \leq i \leq n$. Prove that $2^{n-1}\left(1 + \prod_{k=1}^{n} x_k\right) \geq \prod_{k=1}^{n}(1 + x_k)$ with equality iff at least $n-1$ of the $x_i's$ are equal to 1 .

Solution

Extension. Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in \mathbb{M}_{mn}$ be a matrix so that for any $i \in \overline{1, m}$ have $0 \leq a_{i1} \leq a_{i2} \leq \ \dots \ \leq a_{in}$. Then exists the inequality $\prod_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \leq n^{m-1} \cdot \sum_{j=1}^{n} \prod_{i=1}^{m} a_{ij}$

.

Proof.Here is a generalization of the Cebasev-inequality"] I note $X.s.s.Y \Longleftrightarrow XY > 0 \ \vee \ X = Y = 0$, i.t. the real numbers X, Y have same sign. Prove that exists the following inequality (a generalization over R_+^* of the Cebasev-s inequality):

For the matrix $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, where $\{m, n\} \subset N^*$ and $(\forall)\, \{i, l\} \subset \overline{1, m},\ (\forall)\, \{j, k\} \subset \overline{1, n},\ a_{ij} > 0,\ (a_{ij} - a_{ik}).s.s.(a_{lj} - a_{lk}) \Longrightarrow$

$$\prod_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \leq n^{m-1} \sum_{j=1}^{n} \prod_{i=1}^{m} a_{ij}.$$

A particular case: $A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n \end{pmatrix} \in M_{mn}\left(R_+^*\right)$, i.e.

$(\forall)\, i \in \overline{1, m},\ (\forall)\, j \in \overline{1, n},\ a_{ij} = a_j > 0 \Longrightarrow \left(\frac{1}{n} \cdot \sum\limits_{j=1}^{n} a_j\right)^m \leq \frac{1}{n} \cdot \sum\limits_{j=1}^{n} a_j^m.$

Particular case. $A = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \dots & \dots \\ x_n & 1 \end{pmatrix} \in \mathbb{M}_{n2} \quad \Longrightarrow \quad \prod_{k=1}^{n} (1 + x_k) \leq 2^{n-1} \cdot \left(1 + \prod_{k=1}^{n} x_k\right).$

□

Let $S = \sum_{k=0}^{1998} \frac{k+3}{(k+1)! + (k+2)! + (k+3)!} + \frac{1}{2001!}$. Find the numeric value of $2008S$

Solution

We have $(k+1)! + (k+2)! + (k+3)! = (k+1)!(1 + (k+2) + (k+2)(k+3)) = (k+1)!(k+3)^2$. So $\frac{k+3}{(k+1)!+(k+2)!+(k+3)!} = \frac{1}{(k+1)!(k+3)} = \frac{k+2}{(k+3)!} = \frac{1}{(k+2)!} - \frac{1}{(k+3)!}$. Therefore the sum $\sum_{k=0}^{m} \frac{k+3}{(k+1)!+(k+2)!+(k+3)!}$ telescopes to $\frac{1}{2!} - \frac{1}{(m+3)!}$. So the answer is $2008 \cdot \frac{1}{2} = 1004$.

□ Solve the system , $x, y, z \in \left[0, \frac{\pi}{2}\right)$ $\begin{cases} tgx + siny + sinz = 3x \\ sinx + tgy + sinz = 3y \\ sinx + siny + tgz = 3z \end{cases}$

Solution

The function $f \colon [0, \pi/2) \to [0, \infty)$ given by $f(x) = \tan t + 2 \sin t - 3t$ is increasing, since $f'(t) = \frac{1}{\cos^2 t} + 2 \cos t - 3 = \frac{(1-\cos t)^2 (2 \cos t + 1)}{\cos^2 t} \geq 0$. Thus $f(t) \geq f(0) = 0$, with equality for $t = 0$ only.

Adding the three equations yields $f(x) + f(y) + f(z) = 0$, therefore the only solution is $x = y = z = 0$.

□

Let P_1 and P_2 be regular polygons of 1985 sides and perimeters x and y respectively. Each side of P_1 is tangent to a given circle of circumference c and this circle passes through each vertex of P_2. Prove $x + y \geq 2c$. (You may assume that $\tan \theta \geq \theta$ for $0 \leq \theta < \frac{\pi}{2}$.)

Solution

For inscribed and circumscribed regular n-gons ($n \geq 3$), the inequality boils down to proving $\sin \frac{\pi}{n} +$

$\tan\frac{\pi}{n} > 2\frac{\pi}{n}$. Denote $\alpha = \frac{\pi}{2n}$, and $t = \tan\alpha$. By known formulae $\sin 2\alpha = \frac{2t}{1+t^2}$ and $\tan 2\alpha = \frac{2t}{1-t^2}$. Thus we need prove $\frac{t}{1+t^2} + \frac{t}{1-t^2} > 2\alpha$. But $\frac{t}{1+t^2} + \frac{t}{1-t^2} = \frac{2t}{1-t^4} > 2t > 2\alpha$, since $0 < \alpha \leq \frac{\pi}{6} < \frac{\pi}{4}$, hence $0 < t < 1$, while $t = \tan\alpha > \alpha$, and so we are done.

$\square$

Given numbers $a_1, a_2, ...a_n$, find the number x which the sum

$(x - a_1)^2 + (x - a_2)^2 + ... + (x - a_n)^2$

is a minimum and compute this minimum.

Solution

For a random variable A defined on a finite probability field $\Omega = \{1, 2, \ldots, n\}$ with probability $P(k) = \frac{1}{n}$ for all $1 \leq k \leq n$, and taking values $A(k) = a_k$, we have $\mu(A) = \int_\Omega A \mathrm{d}P = \frac{1}{n}\sum_{k=1}^n a_k$, and $\sigma^2(A) = \int_\Omega (A - \mu)^2 \mathrm{d}P = \mu(A^2) - \mu(A)^2$.

Now compute $\mu((A-x)^2) = \int_\Omega (A-x)^2 \mathrm{d}P = \mu(A^2) - 2x\mu(A) + x^2 = \sigma^2(A) + \mu(A)^2 - 2x\mu(A) + x^2 = \sigma^2(A) + (\mu(A) - x)^2 \geq \sigma^2(A)$, with equality for $x = \mu(A)$. Therefore $\sum_{k=1}^n (a_k - x)^2 \geq n\sigma^2(A)$, with equality for $x = \frac{1}{n}\sum_{k=1}^n a_k$. $\square$ Factorize over Z: $x^{2n} + x^n + 1$

Solution

$x^{2n}+x^n+1 = \frac{x^{3n}-1}{x^n-1} = \frac{\prod_{d|3n}\Phi_d(x)}{\prod_{d|n}\Phi_d(x)}$, where $\Phi_n(x)$ is the n^{th} cyclotomic polynomial. Let k be the maximum integer k such that $3^k|n$. Then: $\frac{\prod_{d|3n}\Phi_d(x)}{\prod_{d|n}\Phi_d(x)} = \prod_{d|\frac{n}{3^k}} \Phi_{d\cdot 3^{k+1}}(x)$ As the n^{th} cyclotomic polynomial is irreducible in $\mathbb{Z}[x]$, and $\mathbb{Z}[x]$ is a UFD this is the fully factorized form and we are done.

$\square$ Find the last non-zero digit in the number 2011!.

Solution

Clearly there are $[\frac{2011}{5}] + [\frac{2011}{5^2}] + [\frac{2011}{5^3}] + [\frac{2011}{5^4}] = 501$ zeroes, so we need to find what $\frac{2011!}{10^{501}}$ is modulo 10. As clearly $2|\frac{2011!}{10^{501}}$, we need only find this modulo 5. Expanded, we see: $\frac{2011!}{10^{501}} \equiv (1 \cdot 2 \cdot 3 \cdot 4)^{402} \cdot 1 \cdot (1 \cdot 2 \cdot 3 \cdot 4)^{80} \cdot (1 \cdot 2 \cdot 3 \cdot 4)^{20} \cdot (1 \cdot 2 \cdot 3 \cdot 4)^3 \cdot (1 \cdot 2 \cdot 3) \pmod 5$ (We get this from considering the maximal power of 5 which divides each term, and then splitting into cases) But by Wilson's Theorem, $4! \equiv -1 \pmod 5$, thus: $\frac{2011!}{10^{501}} \equiv (-1)^{80} \cdot (-1)^{20} \cdot (-1)^3 \cdot 1 \equiv 4 \pmod 5$. Thus as $\frac{2011!}{10^{501}} \equiv 4 \pmod 5$ and $\frac{2011!}{10^{501}} \equiv 0 \pmod 2$, $\frac{2011!}{10^{501}} \equiv 4 \pmod{10}$, and thus the last non-zero digit is $\boxed{4}$.

$\square$ The number 1987 can be written as a three digit number xyz in some base b. If $x + y + z = 1 + 9 + 8 + 7$, determine all possible values of x, y, z, b.

Solution

Let $p(x) = ax^2 + cx + d$ where a, c, d are the digits of 1987 in base b. Then $p(b) = 1987$, and $p(1) = 25$.

Thus $(b - 1)|p(b) - p(1) = 1987 - 25 = 1962 = 2 \cdot 3^2 \cdot 109$ because it's an integer coefficient polynomial.

We know that $b^2 \leq 1987$ and $b^3 > 1987$ because it's a 3 digit number. Therefore, by taking square/cube roots, we get $12 < b < 45$, so $11 < b - 1 < 44$.

The only number in those bounds that divides 1962 is 18. Thus $b - 1 = 18$ and $b = 19$. Then 1987 is written as $5 \cdot 19^2 + 9 \cdot 19 + 11$ so its base 19 expansion is $59b$ where b is the numeral in base 19 for 11 in base 10. The sum of these digits is $b + 9 + 5 = 11 + 9 + 5 = 25$. Thus base 19 works, and it is the only base that works.

$\square$ Show that: $(\forall n \in \mathbb{N}_{\cup 0})$: $\left[\frac{n+2-[\frac{n}{25}]}{3}\right] = \left[\frac{8n+24}{25}\right]$

Solution

Let $n = 25q + r$, then: $\left[\frac{n+2-[\frac{n}{25}]}{3}\right] = \left[\frac{25q+r+2-q}{3}\right] = \left[\frac{24q+r+2}{3}\right] = 8q + \left[\frac{r+2}{3}\right]$ Similarly, $\left[\frac{8n+24}{25}\right] = \left[\frac{200q+8r+24}{25}\right] = 8q + \left[\frac{8r+24}{25}\right]$ By looking at the 25 cases, it can be shown that $\left[\frac{r+2}{3}\right] = \left[\frac{8r+24}{25}\right]$, and thus we are done. Note: We can save analyzing that many cases by using the property $\left\lfloor\frac{\lfloor x\rfloor}{m}\right\rfloor = \left\lfloor\frac{x}{m}\right\rfloor$

for all $x \in \mathbb{R}$ and $m \in \mathbb{N}^*$

$\square$ Show that :

1)$-$ $(\forall n \in \mathbb{N}_0)\,(\exists!\,(p_n, q_n) \in \mathbb{N} \times \mathbb{N}_{\cup 0})$: $\begin{cases} (2+\sqrt{3})^n = p_n + q_n\sqrt{3} \\ \\ 3q_n^2 = p_n^2 - 1 \end{cases}$

2)$-$ Show that $(\forall n \in \mathbb{N}_{\cup 0})$: $\left[p_n + q_n\sqrt{3}\right]$ is an odd number .

Solution

For 1, let $(2+\sqrt{3})^n = p_n + q_n\sqrt{3}$. Then we use norms in the ring $\mathbb{Z}[\sqrt{3}]$ ($N(a+b\sqrt{3}) = a^2 - 3b^2$) to know that $N(ab) = N(a)N(b)$ for any $a, b \in \mathbb{Z}[\sqrt{3}]$. Clearly then $N(p_n + q_n\sqrt{3}) = (2^2 - 3\cdot 1^2)^n = 1$. Thus $p_n^2 - 3q_n^2 = 1 \Longrightarrow p_n^2 - 1 = 3q_n^2$.

For part 2, we know for sufficiently large n, $p_n + q_n\sqrt{3} \approx (2+\sqrt{3})^n + (2-\sqrt{3})^n$. It can then be easily shown $[p_n + q_n\sqrt{3}] = (2+\sqrt{3})^n + (2-\sqrt{3})^n - 1$. Expanding using the binomial theorem, it is clear this must always be odd.

$\square$ A competition involving $n \geq 2$ players was held over k days. In each day, the players received scores of $1, 2, 3, \ldots, n$ points with no players receiving the same score. At the end of the k days, it was found that each player had exactly 26 points in total. Determine all pairs (n, k) for which this is possible.

Solution

Clearly $\frac{n(n+1)}{2}$ points are handed out each day, therefore after k days there are $k\frac{n(n+1)}{2} = 26n$ points handed out. Then clearly $k(n+1) = 52$, so $k = 1, 2, 4, 13, 26, 52$. Clearly we can throw out $k = 52$ immediately. But $k = 26$ results in $n = 1$, which clearly violates $n \geq 2$. The $k = 1$ case has $n = 51$, does not work. For the case $k = 2$, we have $n = 25$, and this works out because if a player receives x points on the first day, give them $26 - x$ points on the second. The case $k = 4$ and $n = 12$ works out in the same way. For $k = 13$, $n = 3$, we find some difficulty in constructing a strategy to attain 26. However, after some guesswork we find the strategy: 1. On the first four days give the first player 2 points, the second 1 point and the third 3 points. 2. For the next three days give the first player 2 points, the second 3 points and the third 1 point. 3. On the eight day give the first player 1 points, the second player 3 points and the third player 2 points. 4. For the next two days give the first player 1 point, the second player 2 points and the third 3 points. 5. On the last three days give the first player 3 points, the second player 2 points and the third 1 points. (This strategy can be derived as well as many others by letting Player 1 get a $1's$ and a $3's$, Player 2 b $1's$ and b $3's$, etc.) Thus the possible pairs are $\boxed{(25,2),(12,4),(3,13)}$

$\square$ Prove that for any real x the following inequality holds: $1^x + 2^x + 6^x + 12^x \geq 4^x + 8^x + 9^x$ Find all x for which an equality holds.

Solution

Let $2^x = m$, $3^x = n$. Then we seek to show for all $x \in \mathbb{R}$: $1 + m + mn + m^2n \geq m^2 + m^3 + n^2$ $\Longrightarrow m^2(n-1-m) + mn + m + 1 - n^2 \geq 0 \Longrightarrow m^2(n-1-m) - n(n-1-m) + m + 1 - n \geq 0$ $\Longrightarrow (m^2 - n - 1)(n - 1 - m) \geq 0$.

Thus we must show $(m^2-n-1)(n-1-m) \geq 0$ for all x. We see this is true when $(m^2-n-1) \geq 0$ and $(n-1-m) \geq 0$ or $(m^2-n-1) \leq 0$ and $(n-1-m) \leq 0$.

Case 1: $x \geq 1$ Then clearly $4^x - 3^x - 1 \geq 0$ because this is an increasing function, and equality holds for $x = 1$, thus for all $x \geq 1$ this will hold as well. But also $3^x - 2^x - 1 \geq 0$ by the same reasoning above; this is increasing and equality holds for $x = 1$.

Case 2: $x < 1$ As $4^x - 3^x - 1$ is increasing and it equals 0 at $x = 1$, we find $4^x - 3^x - 1 \leq 0$ for all $x < 1$. Similarly $3^x - 2^x - 1 \leq 0$ by the same logic of increasing and equality.

Thus in all cases we find $(m^2 - n - 1)(n - 1 - m) \geq 0$, and therefore we are done.

$\square$ For five integers a, b, c, d, e we lnow that the sums $a + b + c + d + e$ and $a^2 + b^2 + c^2 + d^2 + e^2$ are divisible by an odd number n. Prove that the expression $a^5 + b^5 + c^5 + d^5 + e^5 - 5abcde$ is also divisible by n.

Solution

Let a, b, c, d, e be the roots of the monic quintic polynomial $x^5 - \sigma_1 x^4 + \sigma_2 x^3 - \sigma_3 x^2 + \sigma_4 x - \sigma_5$, where σ_i is the i-th symmetric sum.

Then by Newton's sums we get that

$$s_5 - \sigma_1 s_4 + \sigma_2 s_3 - \sigma_3 s_2 + \sigma_4 s_1 - 5\sigma_5 = 0$$

Note that we want to show $s_5 - 5\sigma_5 \equiv 0 \mod n$. Therefore it suffices to show $\sigma_1 s_4 - \sigma_2 s_4 + \sigma_3 s_2 - \sigma_4 s_1 \equiv 0 \mod n$. Note that $\sigma_1 \equiv s_1 \equiv s_2 \equiv 0 \mod n$.

So we get $\sigma_1 s_4 + \sigma_2 s_4 - \sigma_3 s_2 + \sigma_4 s_1 \equiv \sigma_2 s_4 \mod n$. It suffices to show that $\sigma_2 s_4 \equiv 0 \mod n$.

Note that we have $\sigma_1^2 = s_2 + 2\sigma_2$. Note that $0 \equiv \sigma_1^2 \equiv s_2 + 2\sigma_2 \mod n \implies 2\sigma_2 \equiv 0 \mod n$. Since n is odd we have that $\sigma_2 \equiv 0 \mod n$, and we are done.

$\square$ LEt $a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 be positive real numbers such that$a_1 = a_7 = 0$

Show that : $(\exists i \in \{2, 3, 4, 5, 6\}) : \ a_{i+1} + a_{i-1} \leq \sqrt{3} a_i$

Solution

Assume all inequalities are reversed.

Then $2\sqrt{3}a_4 > (3a_3 - \sqrt{3}a_2) + (3a_5 - \sqrt{3}a_6) = 2(a_3 + a_5) + (a_3 - \sqrt{3}a_2) + (a_5 - \sqrt{3}a_6) > 2(a_3 + a_5)$, absurd, since we assumed $a_3 + a_5 > \sqrt{3}a_4$.

Alternatively, square those (reversed) inequalities and add them up; then conveniently group the terms, in order to get $(a_3 - a_5)^2 + ((a_2 + a_6) - a_4)^2 + (a_2 - a_6)^2 < 0$, impossible. However, the equality to 0 case occurs if and only if $a_3 = a_5$, $a_2 = a_6 = a_4/2$, and further on $a_3 = \sqrt{3}a_2$, hence for the unique type of sequence $0, x, x\sqrt{3}, 2x, x\sqrt{3}, x, 0$, when all inequalities mentioned in the statement of the problem turn into equalities.

$\square$ Let $a, b,$ and c denote three distinct integers, and let P denote a polynomial having integer coefficients. Show that it is impossible that $P(a) = b, P(b) = c$, and $P(c) = a$.

Solution

Suppose for the sake of contradiction that all of a, b, and c are distinct. $P(a) - P(b) = b - c$, $P(b) - P(c) = c - a$, and $P(c) - P(a) = a - b$. Multiplying these all together yields $(P(a) - P(b))(P(b) - P(c))(P(c) - P(a)) = (b - c)(c - a)(a - b)$.

Rearrange to get $\left(\frac{P(a)-P(b)}{a-b}\right)\left(\frac{P(b)-P(c)}{b-c}\right)\left(\frac{P(c)-P(a)}{c-a}\right) = 1$ (the division is valid as none of $a - b$, $b - c$, or $c - a$ are zero.) Since P is a polynomial with integer coefficients, each of $\frac{P(a)-P(b)}{a-b}$, $\frac{P(b)-P(c)}{b-c}$, and $\frac{P(c)-P(a)}{c-a}$ are integers. But since they are integers that multiply to one, they must all have absolute value one.

If some one of $\frac{P(a)-P(b)}{a-b}$, $\frac{P(b)-P(c)}{b-c}$, and $\frac{P(c)-P(a)}{c-a}$ is -1 (without loss of generality, let us suppose that $\frac{P(a)-P(b)}{a-b} = -1$), then $P(a) - P(b) = b - a$. But $P(a) = b$ and $P(c) = c$, so $b - c = b - a$, yielding $a = c$, contradicting our assumption that all the variables were distinct.

It follows that each of $\frac{P(a)-P(b)}{a-b}$, $\frac{P(b)-P(c)}{b-c}$, and $\frac{P(c)-P(a)}{c-a}$ is equal to 1, so we have $P(a) - P(b) = a - b$, $P(b) - P(c) = b - c$, and $P(c) - P(a) = c - a$. Substituting, $b - c = a - b$, $c - a = b - c$, and $a - b = c - a$. Rearrange to get $2b = a + c$, $2a = b + c$, and $2c = a + b$. This equation quickly yields $a = b = c$, which again contradicts our assumption that all of a, b, and c are distinct.

Hence, we may conclude that not all of a, b, and c are distinct, which completes our proof. Another way

Suppose that $a < b < c$. Then $|P(a) - P(c)| = |b - a| < |a - c|$, a contradiction. Next, suppose that $b < a < c$. Then $|P(b) - P(c)| = |c - a| < |c - b|$, also a contradiction.

Note that these are both contradictions because $|a - c|$ divides $|P(a) - P(c)|$ and $|c - b|$ divides $|P(b) - P(c)|$.

☐ Let $S(n, n-1) = 1$ and we also know that $(n-1) \cdot (k-1) \cdot S(n, k) = (n-k) \cdot S(n, k-1)$. Show that $S(n, k) = \binom{n-2}{k-1}(n-1)^{n-k-1}$.

Solution

Obviously you use induction here. Note that here you induct on $n-k$ rather than k or induction fails. Base case of $k = n-1$ is trivial, $S(n, n-1) = \binom{n-2}{n-2}(n-1)^{n-n+1-1}$ obviously. Now we proceed to the inductive step. Let this be true for all values of $n - k \leq m$. Consider $m + 1$, or $S(n, k-1)$. $S(n, k-1) = \frac{(n-1)(k-1)S(n,k)}{n-k}$ Now by the inductive hypothesis:

$$S(n, k-1) = \frac{(n-1)(k-1)\binom{n-2}{k-1}(n-1)^{n-k-1}}{n-k} = \frac{(k-1)\frac{(n-2)!}{(k-1)!(n-k-1)!}(n-1)^{n-k}}{n-k} = \frac{(n-2)!}{(k-2)!(n-k)!}(n-1)^{n-k} =$$ $\binom{n-2}{(k-1)-1}(n-1)^{n-(k-1)}$ And thus we are done.

☐ Find x in Z $\left[\frac{x}{1!}\right] + \left[\frac{x}{2!}\right] + \ldots + \left[\frac{x}{10!}\right] = 1001$.

Solution

Well clearly we can drop out all the terms such with factorials greater than 5!, because if $x \geq 720$ it clearly doesn't equal 1001. Thus we must find $x \in \mathbb{Z}$, such that: $\left[\frac{x}{1}\right] + \left[\frac{x}{2}\right] + \left[\frac{x}{6}\right] + \left[\frac{x}{24}\right] + \left[\frac{x}{120}\right] = 1001$ Let $x = 120q + r$. Then: $120q + r + 60q + [r/2] + 20q + [r/6] + 5q + [r/24] + q = 1001$ $206q + r + [r/2] + [r/6] + [r/24] = 1001$ From this we see $q = 4$. Then: $r + [r/2] + [r/6] + [r/24] = 177$ We estimate if we drop the floors that $r \approx 104$. Plugging in 104, we find it equals 177. Thus $x = 120 \cdot 4 + 104 = \boxed{584}$.

☐ find all functions f(x):R->R such that $f(kx) = f(lx)$ where k, l are constant and $k \neq l$

Solution

If k or ℓ is null, the only possibility is $f(x) = f(0)$, an arbitrary constant. If both k and ℓ are not null, define $x \sim y$ if there exists $n \in \mathbb{Z}$ such that $y = (k/\ell)^n x$. This is clearly an equivalence relation; the class $\hat{0}$ of 0 is $\{0\}$, and the other classes $\hat{x}$ of $x \neq 0$ are countable. Clearly if $x \sim y$ we have $f(x) = f(y)$. conversely, for $f(x) = C_{\hat{x}}$, with $C_{\hat{x}}$ arbitrary constants, f fulfills (the functions are given by arbitrary projections $\hat{f} \colon \mathbb{R}/_{\sim} \to \mathbb{R}$.

☐ Find the solution of the equation

$$\sqrt{\frac{x-7}{3}} + \sqrt{\frac{x-6}{4}} + \sqrt{\frac{x-8}{2}} = \sqrt{\frac{x-3}{7}} + \sqrt{\frac{x-4}{6}} + \sqrt{\frac{x-2}{8}}$$

.

Solution

$\sqrt{\frac{x-7}{3}} + \sqrt{\frac{x-6}{4}} + \sqrt{\frac{x-8}{2}} = \sqrt{\frac{x-3}{7}} + \sqrt{\frac{x-4}{6}} + \sqrt{\frac{x-2}{8}} \Leftrightarrow \Leftrightarrow (x-10)\left(\frac{\frac{4}{21}}{\sqrt{\frac{x-7}{3}}+\sqrt{\frac{x-3}{7}}} + \frac{\frac{1}{12}}{\sqrt{\frac{x-6}{4}}+\sqrt{\frac{x-4}{6}}} + \frac{\frac{3}{8}}{\sqrt{\frac{x-8}{2}}+\sqrt{\frac{x-2}{8}}}\right) = 0 \Leftrightarrow \Leftrightarrow x = 10.$

☐ For positive reals a, b, c with $a + b + c = 1$, show that

$$(a + \frac{c}{2})^n (b + \frac{c}{2}) \leq \frac{n^n}{(n+1)^{n+1}}$$

Solution

Let $a+\frac{c}{2}=r, b+\frac{c}{2}=s$, then $r+s=1$. From weighted AM-GM, we get

$$1=n(\frac{r}{n})+s\geq(n+1)\sqrt[n+1]{\frac{r^ns}{n^n}}$$

Which is equivalent to the desired $r^ns\leq\frac{n^n}{(n+1)^{n+1}}$.

$\square$ Suppose that n is the smallest number satisfying

$a^m=1(mod b)$ where a,b are given number $(a,b)=1$

Prove that $n|\phi(m)$

Solution

Let $\phi(m)=nq+r$, where $n,r\in\mathbb{Z}$ and $0\leq r<n$. Then: $a^{\phi(m)}\equiv 1\pmod{b}$ by Euler's Theorem.

However, we also have $a^{\phi(m)}\equiv a^{nq+r}\equiv(a^n)^q\cdot a^r\equiv 1^q\cdot a^r\equiv a^r\pmod{b}$ Now if then clearly $a^r\equiv 1\pmod{b}$. But as n is the least such positive number that this is satisfied, we have either $r\geq n$ or $r\notin\mathbb{Z}^+$. Clearly the first is false, so r is not positive and thus $r=0$. Then $\phi(m)=nq$, and thus $n|\phi(m)$.

$\square$ Give triangle ABC inscribed a circle (O) with center O, AJ is the angle bisector of $\angle BAC$. JE, JF is perpendicular with CA, BA, at E,F. AO cut JE at Q, AO cut JF at N. CF cut BE at M, FN cut BQ at P, CN cut EQ at S. Prove that M,S,P are collinear

Solution

Let D be the foot of the A-altitude of $\triangle ABC$. $AQJB$ is cyclic, due to $\angle BAO=\angle CJE=90^\circ-\angle ACB$. Thus, $\angle JBQ=\angle JAQ=\angle JAD\Longrightarrow BQ$ is B-altitude of $\triangle BAJ\Longrightarrow P$ is orthocenter of $\triangle BAJ$, i.e. $P\in AD$. By similar reasoning, $S\in AD$. $AEJDF$ is clearly cyclic and A is the midpoint of the arc EF of its circumcircle $\Longrightarrow DA,BC$ bisect $\angle EDF\Longrightarrow$ Pencil $D(E,F,A,B)$ is harmonic $\Longrightarrow M\equiv BE\cap CF\cap AD$. So, M,S,P lie on AD.

$\square$ Solve equation $\sqrt[5]{x^3-6x^2+9x}=\sqrt[3]{x^5+6x^2-9x}$.

Solution

If we have $y^5=x^3-6x^2+9x$ and $y^3=x^5+6x^2-9x$ then $y^5+y^3=x^5+x^3$ hence $y=x$. (If $y>x$ then $LHS>RHS$ and vice versa.) Now we have $x^5-x^3+6x^2-9x=0$ and observe that $x^5-x^3+6x^2-9x=x(x^4-(x-3)^2)$. We are done. All real solutions are $0, \frac{-1+\sqrt{13}}{2}, \frac{-1-\sqrt{13}}{2}$.

$\square$ The real numbers x,y satisfy $x^3-3x^2+5x-17=0$, $y^3-3y^2+5y+11=0$. Find $x+y$.

Solution

Let $x-1=a$ and $y-1=b$. Hence, $a^3+2a-14=0$ and $b^3+2b+14=0$, which gives $a^3+2a+b^3+2b=0$, which is $(a+b)(a^2-ab+b^2+2)=0$. Id est, $a+b=0$ and $x+y=2$.

$\square$ The diagonals of a convex quadrilateral ABCD are mutually perpendicular. Perpendicular lines from the midpoints of sides AB and AD are dropped to their opposite sides CD and CB, respectively. Prove that these two lines and line AC have a common point.

Solution

Let M,N be the midpoints of AB,AD, respectively. Let $K\equiv AC\cap BD$

Let S and T be the feet of the perpendiculars from M and N to CD and CB, respectively.

Let $E\equiv MS\cap BD$ and $F\equiv NT\cap BC$.

Let $P\equiv MS\cap AC$ and $P'\equiv NT\cap AC$.

Using menelaus theorem on triangle AKB with line MPE we get

$$\frac{AP}{PK}\cdot\frac{KE}{BE}\cdot\frac{BM}{AM}=1=>\frac{AP}{PK}=\frac{BE}{KE}(*)$$

Similarly, in triangle ADK with line $NP'F$ we have

$$\frac{AP'}{P'K} \cdot \frac{KF}{DF} \cdot \frac{DN}{AN} = 1 => \frac{AP'}{PK'} = \frac{DF}{KF}(**)$$

So by $(*), (**)$ we have $P \equiv P'$ iff $\frac{BE}{KE} = \frac{DF}{KF} \leftrightarrow \frac{BK}{KE} = \frac{DK}{KF}(***)$
Now since $PKTB$ is cyclic we have $\angle KPF = \angle KBC$ so we have $PKF \sim BKC$ and we get

$$\frac{PK}{BK} = \frac{KF}{KC} => BK \cdot KF = PK \cdot CK$$

Similarly, we see that $KPE \sim KDC$ and therefore

$$PK \cdot CK = DK \cdot KE$$

hence we get $BK \cdot KF = PK \cdot CK = DK \cdot KE$ which implies $(***)$ and we are done.

$\square$ Show that $\sum_{k=0}^{n} \binom{n+k}{n} \frac{1}{2^k} = 2^n$.

Solution

Indeed, denote $f(n) = \sum_{k=0}^{n} \binom{n+k}{n} \frac{1}{2^k}$. Notice that $\binom{n+1+n+1}{n+1} \frac{1}{2^{n+1}} = \binom{n+1+n}{n+1} \frac{1}{2^n}$. Then $f(n+1)-f(n) = 1-1+\sum_{k=1}^{n} \left(\binom{n+1+k}{n+1} - \binom{n+k}{n} \right) \frac{1}{2^k} + \binom{n+1+n+1}{n+1} \frac{1}{2^{n+1}} = \frac{1}{2} \left(\sum_{k=1}^{n} \binom{n+1+(k-}{n+1}\right.$
$\frac{1}{2} f(n+1)$. Once we have $f(n+1) = 2f(n)$, and clearly $f(1) = 2$, the claim is immediate by iteration.

$\square$ ABC is acute-angled. What point P on the segment BC gives the minimal area for the intersection of the circumcircles of ABP and ACP?

Solution

The point P that satisfies is the foot of the altitude from A to BC. Let R_1 and R_2 be the circumradii of the circumcircles of ABP and ACP respectively.

We can express the area of the intersection of the circumcircles as

$$\frac{\angle B}{360} \cdot R_1^2 \cdot \pi - (ABP) + \frac{\angle C}{360} \cdot R_2^2 \cdot \pi - (ACP) = \frac{\pi}{360}(\angle B \cdot R_1^2 + \angle C \cdot R_2^2) - (ABC)$$

Since (ABC) is constant it is enough to find the minimum value of $\angle B \cdot R_1^2 + \angle C \cdot R_2^2$
From law of sines in ABP and ACP we have $R_1^2 = \frac{AB^2}{4\sin^2 \angle BPA}$ and $R_2^2 = \frac{AC^2}{4\sin^2 \angle CPA} = \frac{AC^2}{4\sin^2 \angle BPA}$
Hence we have to minimize

$$\frac{AB^2 \cdot \angle B}{4\sin^2 \angle BPA} + \frac{AC^2 \cdot \angle C}{4\sin^2 \angle BPA} = \frac{1}{\sin^2 \angle BPA}(\frac{AB^2 \cdot \angle B + AC^2 \cdot \angle C}{4})$$

But $\frac{AB^2 \cdot \angle B + AC^2 \cdot \angle C}{4}$ is constant and therefore it is enough to find the minimum of $\frac{1}{\sin^2 \angle BPA}$ which is obviously 1 when $\angle BPA = 90$ and we are done.

$\square$
Solve for x:
$\lfloor x + \lfloor x + \lfloor x+1 \rfloor + 1 \rfloor + 1 \rfloor = 117$

Solution

Let $n = \lfloor x \rfloor \in \mathbb{Z}$. Then $\lfloor x+1 \rfloor = n+1$, and $\lfloor x + \lfloor x+1 \rfloor + 1 \rfloor = \lfloor x+n+2 \rfloor = \lfloor x \rfloor + n + 2 = 2n+2$. Continuing, we have $\lfloor x + \lfloor x + \lfloor x+1 \rfloor + 1 \rfloor + 1 \rfloor = \lfloor x + 2n + 3 \rfloor = 3n+3$. Since this equals 117, we solve to obtain $\lfloor x \rfloor = n = 38$, from which it follows $x \in [38, 39)$. $\square$

Calculate the value of:

$$\sum_{k=0}^{100} \frac{5^k}{k+1} \binom{100}{k}$$

Solution

$$
\begin{aligned}
S(n,z) = \sum_{k=0}^{n} \frac{z^k}{k+1}\binom{n}{k} &= \sum_{k=0}^{n} \frac{n!}{(k+1)!(n-k)!} z^k \\
&= \frac{1}{n+1}\sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} z^k \\
&= \frac{1}{(n+1)z}\sum_{k=0}^{n} \binom{n+1}{k+1} z^{k+1} \\
&= \frac{1}{(n+1)z}\sum_{k=1}^{n+1} \binom{n+1}{k} z^{k} \\
&= \frac{1}{(n+1)z}\left(-1+\sum_{k=0}^{n+1} \binom{n+1}{k} z^{k}\right) \\
&= \frac{(1+z)^{n+1}-1}{(n+1)z}.
\end{aligned}
$$

Therefore, $S(100,5) = \frac{6^{101}-1}{505}$. □

Solve the equation: $2^{x-1} + 2^{-x-1} = cos(x^3 + x)$

Solution

Multiplying by 2: $2^x + 2^{-x} = 2\cos(x^3 + x)$ Note that there is a solution at $x = 0$. Note that $f(x) = 2^x + 2^{-x}$ is increasing for all $x > 0$, much faster than $2\cos(x^3 + x)$ ever increases, thus there are no solutions for $x > 0$. However, both sides of this equation are symmetric in that x is a solution iff $-x$ is a solution. Thus $\boxed{x = 0}$ is the only solution. □

Solve the equation

$3^{\frac{2}{x}} + (11 \cdot 3^x - 1)^{\frac{1}{x}} \cdot 3^{x+1} = 11 \cdot 3^{x+\frac{2}{x}}$

Solution

Do the obvious manipulations to reduce it to $(11 \cdot 3^x - 1)^{x-1} = 3^{(x-1)(x+2)}$. □

Solve the equation

$(1 + x^2)(y^2 + 2y\sqrt[4]{2} + 2\sqrt{2}) = 1 + 2x - x^2$

Solution

The original equation equivalents to $(y^2 + 2y\sqrt[4]{2} + \sqrt{2}) + \frac{x^2-2x-1}{1+x^2} + \sqrt{2} = 0 \Leftrightarrow (y + \sqrt[4]{2})^2 + \frac{(\sqrt{2}+1)x^2-2x+\sqrt{2}-1}{1+x^2} = 0 \Leftrightarrow (y + \sqrt[4]{2})^2 + \frac{(\sqrt{2}+1)(x-(\sqrt{2}-1))^2}{1+x^2} = 0$

Now the correct answer is $\boxed{x = \sqrt{2} - 1, y = -\sqrt[4]{2}}$

□

Solve the system equation

$max(x + 2y, 2x - 3y) = 4, min(-2x + 4y, 10y - 3x) = 4$

Solution

Just solve 4 systems of equations

$$(1)\ \begin{cases} x + 2y = 4 \\ -3x + 10y = 4 \end{cases} \Leftrightarrow (x, y) = (2, 1)$$

$$(2)\ \begin{cases} x + 2y = 4 \\ -2x + 4y = 4 \end{cases} \Leftrightarrow (x, y) = (1, \tfrac{3}{2})$$

(3) $\begin{cases} 2x-3y=4 \\ -2x+4y=4 \end{cases} \Leftrightarrow (x,y)=(14,8)$

(4) $\begin{cases} 2x-3y=4 \\ -3x+10y=4 \end{cases} \Leftrightarrow (x,y)=(\frac{52}{11},\frac{20}{11})$

Now it is easy to check that the answer that we need to find is

$$\boxed{(x,y)\in\{(2,1),(1,\frac{3}{2})\}}$$

$\square$

A magician and his assistant appeared to the public with lots of people. In the scenary, there is a board 4 x 4. The magician close his eyes, and then, the assistant invites people of the public to write the numbers 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16 on the squares of the board to complete the 16 numbers. After that, the assistant covers two adjacent houses, which he chooses, with a black patch and leaves the scene. In the end, the magician open his eyes and has to guess the number in each house that the assistant hid. Explain how the trick works.

Solution

The assistant (either in his head or on paper) creates a 4 x 4 box with 1 through 16 in their respective boxes, ordered from left to right, then from top to bottom. In this way, for example, 4 is in the upper right corner, and 10 is in the third row, second box from the left. He then (again, either in his head or on paper) starts with 1 and sees what number is in the box on the public board that is the same as the box on his new board that 1 is in, suppose it's a. He then writes a after 1. He then checks to see which number is in the box in the public square that corresponds to the box that a is in on his new square, call it b. Then he writes b after a. He continues this way until he reaches 1. If there are other numbers that have not been written down, he picks one and starts over. He repeats this until all numbers are used.

For example: if we created this square,

11	7	2	13
6	15	4	14
5	1	12	8
10	9	3	16

then the assistant would mentally create this square,

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

and create the sequence 1, 11, 12, 8, 14, 9, 5, 6, 15, 3, 2, 7, 4, 13, 10 and the sequence 16. The assistant counts the total number of sequences he has written down, and if it's even he blacks out boxes 1 and 2, and if it's odd he blacks out boxes 1 and 3.

The magician knows which two numbers are blacked out trivially, but not which number is in which box. He knows what parity the number of sequences are, however. So the magician takes one of the numbers that is blacked out and starts a chain as above with it. He stops the chain either when he gets to 1 or he gets to the other box that is blacked out. At this point, he starts with the other number that is blacked out and creates a second chain with it, again stopping at either 1 or (2 or 3). Now if any number is not part of either of these chains, he creates a sequence starting with this number. It's obvious that this sequence repeats itself. He does the same thing again, and again, until he's used all numbers.

So the magician has two chains and a number of cycles. He knows the parity of the total number of cycles, and he knows that in each chain, the blacked out box that the chain stopped on can either contain the starting number for that chain or the other chain. If it's the first one, then the two chains are each their own separate cycles, and if it's the second case, the two chains form one long sequence. So the magician picks whichever choice gives the correct parity of the number of cycles. In this way he discovers which box contains which number.

In our case, here's what the magician would see:

X	X	2	13
6	15	4	14
5	1	12	8
10	9	3	16

and he realizes that 7 and 11 are the crossed out ones. He then creates the chains 7, 4, 13, 10, 1 and 11, 12, 8, 14, 9, 5, 6, 15, 3, 2 and the sequence 16. Then he sees that the crossed out numbers are in spaces 1 and 2, so there are an even number of cycles. This leads to the knowledge that the two chains must combine to create one long cycle, so in box 1 must be 11 and in box 2 must be 7. This completes the solution.

Sidenote: There are $\binom{16}{2} = 120$ ways for the assistant to pick which two squares are marked off, but for every position the magician gets there are only two different scenarios that he must choose between, so there is lots of room for improvement on this scheme. If the assistant marks off k squares, then the number of ways he can do this is $\binom{16}{k}$. The number of scenarios that the magician must guess between is $k!$ so in order for the trick to still work we need $\binom{16}{k} > k!$ which happens for all $k \leq 7$. If $k > 7$ then there are too many possibilities for the assistant to be able to encode each one for the magician. However, with $k \leq 7$, theoretically the magician and assistant could come up with a code that works for all arrangements. $\square$

Fin $n \in \mathbb{N}$ such that: $cos\varphi < \dfrac{1}{\sqrt[8]{1+nsin^4\varphi}}; \forall\varphi \in (0;\frac{\pi}{2}]$

Solution

solve for $n < \frac{(1+\cos^2 x)(1+\cos^4 x)}{(1-\cos^2 x)\cos^8 x}\ \forall\ x \in \left(0, \frac{\pi}{2}\right]$

let $u = \cos^2 x$ to get $n < f(u) = \frac{(1+u)(1+u^2)}{u^4(1-u)}\ \forall\ u \in [0, 1)$

but it's easy to see that $f(u) > 1$ in that interval, so $n = 1$ works. $\square$

Wich function is periodic, and determin their periods:

$f(x) = 3x - \lfloor 3x \rfloor$ $f(x) = \lfloor x \rfloor^2 - 2(x-1)\lfloor x \rfloor + (x-1)^2$ $f(x) = \lfloor x \rfloor + \lfloor x+0.5 \rfloor - \lfloor 2x \rfloor$ $f(x) = x - \lfloor 3x \rfloor$

Solution

A function on the real line f is periodic with period p if $p > 0$ is the smallest value such that

$$f(x+p) = f(x)$$

for all x. Thus, we compute

$$\begin{aligned}0 = f_1(x+p) - f_1(x) &= 3(x+p) - \lfloor 3(x+p) \rfloor - 3x + \lfloor 3x \rfloor \\ &= 3p - \lfloor 3x + 3p \rfloor + \lfloor 3x \rfloor.\end{aligned}$$

Since this must equal 0 for all x, we choose $x = 0$ and observe this implies $3p = \lfloor 3p \rfloor$, or $3p \in \mathbb{Z}$. The smallest positive p for which this is true is $p = 1/3$, and a quick substitution shows that this indeed gives us $f_1(x+1/3) = f_1(x)$ for all x. The subsequent examples are treated similarly:

$$\begin{aligned} 0 = f_2(x+p) - f_2(x) &= (\lfloor x+p \rfloor - x - p + 1)^2 - (\lfloor x \rfloor - x + 1)^2 \\ &= (\lfloor x+p \rfloor - \lfloor x \rfloor - p)(\lfloor x+p \rfloor + \lfloor x \rfloor - 2x - p + 2), \end{aligned}$$

and with $x = 0$, this gives $0 = (\lfloor p \rfloor - p)(\lfloor p \rfloor - p + 2)$. The second factor obviously can never be zero, so the first factor gives us $p \in \mathbb{Z}$, of which the smallest positive value is $p = 1$. Again, it is easy to check that $f_2(x+1) = f_2(x)$. For the third example, we can actually prove that the function is identically zero. Note that since

$$\begin{aligned} f_3(x+1/2) &= \lfloor x+1/2 \rfloor + \lfloor x+1 \rfloor - \lfloor 2(x+1/2) \rfloor \\ &= \lfloor x+1/2 \rfloor + \lfloor x \rfloor + 1 - \lfloor 2x \rfloor - 1 \\ &= f_3(x), \end{aligned}$$

we need only consider $0 \le x < 1/2$. But in this case $f_3(x) = 0 + 0 - 0 = 0$, so $f_3(x) = 0$ everywhere. Thus $f_3(x)$ is periodic with any period p, but there is no least period as defined above. Finally, in the last example,

$$\begin{aligned} 0 = f_4(x+p) - f_4(x) &= x + p - \lfloor 3(x+p) \rfloor - x + \lfloor 3x \rfloor \\ &= p - \lfloor 3x + 3p \rfloor + \lfloor 3x \rfloor, \end{aligned}$$

and with the choice $x = 0$ we have $p = \lfloor 3p \rfloor$, which is true only if $p = 0$; hence f_4 is not periodic. $\square$

Given the quadrilateral $ABCD$, the inscribed circle (I), $A = 90^0$. BI intersects AD at M, DI intersects AB at N. Prove that : AC is perpendicular to MN

Solution

(I, r) is the quadrilateral incircle, x-y coordinate origin is at I and right angle vertex A is at $(-r, -r)$. Let $P = (r\cos\psi, r\sin\psi)$, $Q = (r\cos\vartheta, r\sin\vartheta)$ be tangency points of BC, CD with (I). Equations of $AB \parallel x$ and $DA \parallel y$ are $y = -r$ and $x = -r$, respectively. Equations of $BC \perp PI$ and $CD \perp QI$ are $y - r\sin\psi = -\frac{\cos\psi}{\sin\psi}(x - r\cos\psi)$ and $y - r\sin\vartheta = -\frac{\cos\vartheta}{\sin\vartheta}(x - r\cos\vartheta)$, respectively. Solving proper equation pairs yields coordinates of $B \equiv AB \cap BC$, $C \equiv BC \cap CD$ and $D \equiv CD \cap DA$:

$B = \left(r\frac{1+\sin\psi}{\cos\psi}, -r\right)$, $C = \left(r\frac{\sin\psi - \sin\vartheta}{\sin(\psi-\vartheta)}, r\frac{\cos\vartheta - \cos\psi}{\sin(\psi-\vartheta)}\right)$, $D = \left(-r, r\frac{1+\cos\vartheta}{\sin\vartheta}\right)$.

Bisectors BI, DI of $\angle B, \angle C$ have equations $y = -\frac{\cos\psi}{1+\sin\psi}x$, $y = -\frac{1+\cos\vartheta}{\sin\vartheta}x$, respectively. Solving proper equation pairs yields coordinates of $M \equiv BI \cap DA$ and $N \equiv DI \cap AB$:

$M = \left(-r, r\frac{\cos\psi}{1+\sin\psi}\right)$, $N = \left(r\frac{\sin\vartheta}{1+\cos\vartheta}, -r\right)$.

Using formulas $\sin\phi = \frac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}$, $\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}$, slopes a, m of AC, MN are equal to

$a = \frac{\cos\vartheta - \cos\psi + \sin(\psi-\vartheta)}{\sin\psi - \sin\vartheta + \sin(\psi-\vartheta)} = \frac{\cos\vartheta(1+\sin\psi) - \cos\psi(1+\sin\vartheta)}{\sin\psi(1+\cos\vartheta) - \sin\vartheta(1+\cos\psi)} = \frac{(1-\tan^2\frac{\vartheta}{2})(1+\tan\frac{\psi}{2})^2 - (1-\tan^2\frac{\psi}{2})(1+\tan\frac{\vartheta}{2})^2}{4(\tan\frac{\psi}{2} - \tan\frac{\vartheta}{2})} =$

$= \frac{2(\tan\frac{\psi}{2} - \tan\frac{\vartheta}{2})(1+\tan\frac{\psi}{2})(1+\tan\frac{\vartheta}{2})}{4(\tan\frac{\psi}{2} - \tan\frac{\vartheta}{2})} = \frac{(1+\tan\frac{\psi}{2})(1+\tan\frac{\vartheta}{2})}{2}$

$m = -\frac{1+\cos\vartheta}{1+\sin\psi} \cdot \frac{1+\sin\psi+\cos\psi}{1+\sin\vartheta+\cos\vartheta} = -\frac{2}{(1+\tan\frac{\psi}{2})^2} \cdot \frac{2(1+\tan\frac{\psi}{2})}{2(1+\tan\frac{\vartheta}{2})} = -\frac{2}{(1+\tan\frac{\psi}{2})(1+\tan\frac{\vartheta}{2})}$

Since $am = -1 \Longrightarrow AC \perp MN$. $\square$

Prove that integers $n \ge 2 \wedge k \ge 0$ satisfy inequality $\frac{1}{n^n} > \sum_{i=0}^{k}\left(\sum_{j=1}^{n+i} j^j\right)^{-1}$.

Solution

Let $S = \sum_{j=1}^{n} j^j$.

Then the RHS equivalent to

$$\frac{1}{S} + \frac{1}{S + (n+1)^{n+1}} + \cdots + \frac{1}{S + (n+k)^{n+k}}$$

Since all of the terms are positive, it suffices only to prove that $\frac{1}{n^n} > \frac{1}{S} \implies 1^1 + 2^2 + \cdots n^n > n^n$, which is obviously true when $n \geq 2$, so we're done. $\square$

Consider the small sets S of the set $1, 2, 3, ..., 15$, which has the property: the product of any three elements of S is not a square number. k is the greatest number so that set S has k elements satisfying the conditions above. Find k

Solution

Consider one subset of the given set:

$$S_0 = \{1, 3, 5, 6, 7, 9, 10, 11, 13, 14\}$$

It's obvious that the mentioned set is satisfied our condition. In this case, $k = 10$.

Assume there exist a set S such that $|S| > 10$ also satisfy our condition (which means at most 4 numbers of $1, 2, 3, ..., 15$ don't belong to S).

If 1 belongs to S, consider 3 following pairs:

$$(2, 8), (3, 12), (4, 9)$$

The product of each pair is a square, if we multiply it with 1 then we also have a square. Hence, at least 3 numbers of 3 pairs above mustn't belong to S.

If 2 or 8 belongs to S, then in 2 pairs $(5, 10), (7, 14)$, there's at least 2 numbers mustn't belong to S. Sum up with the 3 numbers which don't belong to S above, there're a least 5 numbers don't belong to S (contradiction).

So, $1 \notin S$.

If 2 belongs to S, then in 4 pairs which each product is a square: $(4, 8), (3, 6), (5, 10), (7, 14)$. Hence, at least 4 mentioned numbers don't belong to S. Sum up with number 1, there's at least 5 numbers don't belong to S (contradiction).

So, $2 \notin S$.

Similarly, if 8 belongs to S, then after considering 3 pairs $(3, 6), (5, 10), (7, 14)$, we can conclude at least 3 mentioned numbers don't belong to S. Sum up with number 1 and number 2, there's at least 5 numbers don't belong to S (contradiction).

So, $8 \notin S$.

If 15 belongs to S, then at least 2 numbers of 2 pairs $(3, 5), (6, 10)$ don't belong to S. Sum up with $1, 2, 8$, there's at least 5 numbers don't belong to S (contradiction).

So, $15 \notin$.

If 3 belongs to S, then at least 1 numbers of the pair $(4, 12)$ doesn't belong to S. Sum up with $1, 2, 8, 15$, there's at least 5 numbers don't belong to S (contradiction).

So, $3 \notin$.

But in this case, then at least 5 numbers $1, 2, 3, 8, 15$ don't belong to S, contradiction.

Hence, no existence of a set S such that $|S| > 10$ also satisfy our condition. Which means $k_{\max} = 10$.

$\square$

Solve in $\mathbb{N}^2$ equation : $\frac{n(n+1)}{2} + n! = 2.6^m$

Solution

Multiplying both sides by 2, we get $n(n+1) + 2n! = 4 \cdot 6^m$ Case 1: $n+1$ is composite. Then for $n > 3$ we know $(n+1)|n!$. It follows that $n+1$'s only prime factors are $2, 3$. However, n divides both sides as well. Thus n's only prime factors are $2, 3$. But this is a contradiction as $\gcd(n, n+1) = 1$. For $n \leq 3$, we find the solution of $(n, m) = (3, 1)$

Case 2: $n+1$ is prime. Notice that for all $p>3$, we need $n(n+1)+2n! \not\equiv 0 \pmod p$. Notice however we need for $n>3$ that 6 divides the LHS. This would imply n is multiple of 6. Let 6^a be the highest power of 6 dividing n, so let $n=6^a k$. Now observe that $6^k a(6^k a+1)+2(6^k a)! = 2\cdot 6^m$ Observe that then $m \geq k$. If $m>k$, then the powers of 3 dividing each side won't match up. Thus $m=k$. However, then $2(6^k a)! > 2\cdot 6^k$, a contradiction. Thus the unique solution of $\boxed{(n,m)=(3,1)}$.

□

Find $x,y\in\mathbb{Q}$, if $\dfrac{x^2-y^2}{(x^2+y^2)^2}=-11x+2y$ and $\dfrac{2xy}{(x^2+y^2)^2}=2x+11y$.

Solution

According to the second equation: $x=0 \Leftrightarrow y=0$ but this is not a solution. So we can divide the first equation by second one. Introduce substitution $\frac{x}{y}=a$. Then: $\frac{x^2-y^2}{2xy}=\frac{-11x+2y}{2x+11y} \Leftrightarrow \frac{a-\frac{1}{a}}{2}=\frac{-11a+2}{2a+11} \Leftrightarrow$ $(a^2-1)(2a+11)=2a(2-11a) \Leftrightarrow 2a^3+33a^2-6a-11=0 \Leftrightarrow (2a+1)(a^2+16a-11)=0$ Since $a\in Q$ the only solution is $a=-\frac{1}{2} \Leftrightarrow y=-2x$ Using this result in the second equation we get: $\frac{-4x^2}{25x^4}=-20x \Leftrightarrow 125x^3=1$ and $\boxed{(x,y)=(\frac{1}{5},-\frac{2}{5})}$

□

Let (x_1,y_1) , (x_2,y_2) and (x_3,y_3) be three different real solutions to the system of equations $x^3-5xy^2=21$ and $y^3-5x^2y=28$. Find the value of

$(11-\dfrac{x_1}{y_1})(11-\dfrac{x_2}{y_2})(11-\dfrac{x_3}{y_3})$.

Solution

Observe that $y\neq 0$. Let $z=x/y$. Then

$$21=y^3(z^3-5z),$$
$$28=y^3(1-5z^2).$$

It follows that

$$\frac{28}{21}=\frac{4}{3}=\frac{1-5z^2}{z^3-5z},$$

or equivalently,

$$0=4z^3+15z^2-20z-3=f(z).$$

Note f has three real roots, since $f(z)<0$ for sufficiently small z, $f(-1)=28$, $f(0)=-3$ and $f(z)>0$ for sufficiently large z. Hence

$$\left(11-\frac{x_1}{y_1}\right)\left(11-\frac{x_2}{y_2}\right)\left(11-\frac{x_3}{y_3}\right)=\frac{f(11)}{4}=1729.$$

□

Consider the following sequence:

$u_0=2009$ and $u_{n+1}=\frac{u_n^2}{u_n+1}$. Show that for all $n\in(0,1,2,....1005):[u_n]=2009-n$.

Solution

$u_n-u_{n+1}=u_n-\frac{u_n^2}{u_n+1}=\frac{u_n}{u_n+1}>0$ (1) Easily to see that $u_n>0$ for every n then $u_n>u_{n+1}$. i.e. u_n is decreasing.

$u_n=u_0+(u_1-u_0)+(u_2-u_1)+...+(u_n-u_{n-1})$ (2) From (1) and (2) we get $u_n=2009-\frac{u_0}{u_0+1}-\frac{u_1}{u_1+1}-...-\frac{u_{n-1}}{u_{n-1}+1}$

$=2009-n+\frac{1}{u_0+1}+\frac{1}{u_1+1}+...+\frac{1}{u_{n-1}+1}$ (3)

Since $u_k>0$ for $k=1,2,...,n-1$ then (3) $\implies u_n>2009-n$ (4)

On the other hand, (u_n) is decreasing then

$\frac{1}{u_0+1}+\frac{1}{u_1+1}+...+\frac{1}{u_{n-1}+1}<\frac{n}{u_{n-1}+1}$ (5)

From (1) has $u_{n+1} = u_n - \frac{u_n}{u_n+1} > u_n - 1$ (6)

Apply (6) again and again we get $u_{n+1} > u_0 - (n+1)$ (7)

So that from (5) gives

$\frac{1}{u_0+1} + \frac{1}{u_1+1} + ... + \frac{1}{u_{n-1}+1} < \frac{n}{u_{n-1}+1} < \frac{n}{1+u_0-(n-1)}$

Substituting $u_0 = 2009$ yields

$\frac{1}{u_0+1} + \frac{1}{u_1+1} + ... + \frac{1}{u_{n-1}+1} < \frac{n}{2011-n} < 1$ (8) Because $n = 1, 2, 3, ..., 1005$

Plugging (8) into (3) gives

$u_n < (2019 - n) + 1$ (9)

Combinating (4) and (9) we obtain

$2009 - n < u_n < (2019 - n) + 1$

Now

$$\boxed{[u_n] = 2009 - n}$$

□

How many positive integers x which satisfies $x < 10^{2012}$ and $x^2 - x$ is divisible by 10^{2012}?

Solution

We need $x(x-1) \equiv 0 \pmod{2^{2012}}$ and $x(x-1) \equiv 0 \pmod{5^{2012}}$. Obviously $\gcd(x, x-1) = 1$. Thus $2^{2012}|x$ or $x-1$ and similarly for 5^{2012}. Therefore we get $x \equiv 0, 1 \pmod{2^{2012}}$ and $x \equiv 0, 1 \pmod{5^{2012}}$. We can always combine the solutions using CRT, so there are $2 \cdot 2 - 1 = \boxed{3}$ solutions (the -1 because $x < 10^{2012}$).

□

Solve equation

$$\sqrt{\frac{x+7}{x+1}} + 8 = 2x^2 + \sqrt{2x-1}$$

Solution

$\sqrt{\frac{x+7}{x+1}} + 8 = 2x^2 + \sqrt{2x-1} \Leftrightarrow \Leftrightarrow 2x^2 - 8 + \frac{\sqrt{2x^2+x-1} - \sqrt{x+7}}{\sqrt{x+1}} = 0 \Leftrightarrow \Leftrightarrow (2x^2-8)\left(1 + \frac{1}{(\sqrt{2x^2+x-1}+\sqrt{x+7})\sqrt{x+1}}\right) = 0 \Leftrightarrow x = 2.$

□ In A-isosceles $\triangle ABC$ denote the midpoint D of $[BC]$, the projection E of D on AC and the midpoint F of DE . Prove that $BE \perp AF$.

Solution

Proof 1 (synthetic). Denote $L \in AC$ for which $BL \parallel DE$. Show easily $\triangle ADE \sim \triangle BCL$. The median $[AF$ of $\triangle ADE$ and the median of $\triangle BCL$ are omologously. In conclusion, $\widehat{CBE} \equiv \widehat{DAF} \iff ABDX$ is cyclically $\iff \widehat{ADB} \equiv \widehat{AXB} \iff BE \perp AX$.

Proof 2 (metric). Denote $AB = AC = b$, $DB = DC = a$ and $AD = h$, where $a^2 + h^2 = b^2$. Denote $m(\angle DAF) = x$, $m(\angle EBD) = y$, $U \in AF \cap BC$ and $V \in BC$ for which $EV \perp BC$. Observe that $AE = \frac{h^2}{b}$ and $CE = \frac{a^2}{b}$. Apply the Menelaus' theorem to the transversal $\overline{AFU}$ and $\triangle CDE$: $\frac{AE}{AC} \cdot \frac{UC}{UD} \cdot \frac{FD}{FE} = 1 \iff \frac{h^2}{b^2} \cdot \frac{UC}{UD} = 1 \iff \frac{UC}{b^2} = \frac{UD}{h^2} = \frac{a}{b^2+h^2}$ from where $UD = \frac{ah^2}{b^2+h^2}$ and $\tan x = \frac{UD}{AD}$, i.e. $\boxed{\tan x = \frac{ah}{b^2+h^2}}$ (1) . Since $EV \parallel AD$ obtain that $\frac{CV}{CD} = \frac{EV}{AD} = \frac{CE}{CA} = \frac{\frac{a^2}{b}}{b}$ $\implies EV = \frac{a^2h}{b^2}$ and $CV = \frac{a^3}{b^2}$. Therefore, $BV = 2a - \frac{a^3}{b^2} \implies BV = \frac{a\left(b^2+h^2\right)}{b^2}$ and $\tan y = \frac{EV}{BV} = \frac{\frac{ha^2}{b^2}}{\frac{a\left(b^2+h^2\right)}{b^2}} \iff \boxed{\tan y = \frac{ah}{b^2+h^2}}$ (2) . From the relations (1) and (2) obtain that $\tan x = \tan y$, i.e. $x = y \iff ABDX$ is cyclically $\iff AF \perp BE$.

a.s.o.

☐ Solve the equation $2\sqrt{3x+4}+3\sqrt{5x+9}=x^2+6x+13$

Solution

$2\sqrt{3x+4}+3\sqrt{5x+9}=x^2+6x+13 \Leftrightarrow \Leftrightarrow 2\sqrt{3x+4}-4+3\sqrt{5x+9}-9=x^2+6x \Leftrightarrow \Leftrightarrow \frac{6x}{\sqrt{3x+4}+2}+\frac{15x}{\sqrt{5x+9}+3}=x^2+6x \Leftrightarrow xf(x)=0$, where $f(x)=x+6-\frac{6}{\sqrt{3x+4}+2}-\frac{15}{\sqrt{5x+9}+3}$. But f is an increasing function. Hence, the equation $f(x)=0$ has maximum one real root. $f(-1)=0$. Thus, we get the answer: $\{0,-1\}$.

☐ Prove that $1-2.3+4.5-6.7+8.9...+(n-1)n=1+2+3+4+5...+(n-1)+n$ provided that n is the second digit of the added multiplication part, i.e. 5 or 9 or 13.

Solution

For the equality to be true, $n=4k+1$ for some positive integer k.

The left side can be expressed like $1+(4*5-2*3)+(9*8-7*6)+...+(4k)(4k+1)=1+\sum_{x=1}^{k}4(4x-2)+6=1+\sum_{x=1}^{k}16x-2=1+\left(16\frac{k(k+1)}{2}\right)-2k=1+8k^2+8k-2k=8k^2+6k+1$

The right side is $\sum_{x=1}^{4k+1}x=\frac{(4k+1)(4k+2)}{2}=(4k+1)(2k+1)=8k^2+6k+1$

Therefore, the original equality is true for $n=4k+1$.

☐ Show that the equation $\sqrt{2-x^2}+\sqrt[3]{3-x^3}=0$ has no real roots.

Solution

$9>8 \Longrightarrow \sqrt[3]{3}>\sqrt{2}$ by taking 6th roots both sides. Now, given that $\sqrt{2-x^2}=\sqrt[3]{x^3-3}$. Now, $2-x^2\geq 0 \Longrightarrow x\leq\sqrt{2}$. But $x^3-3\geq 0$ due to LHS being non negative implies that $x\geq\sqrt[3]{3}>\sqrt{2}$ which is a contradiction.

Another way

$2-x^2\geq 0$ implies that $3-x^3\geq 0$ else we would have a contradiction in the domain(Check by assuming $3-x^2<0$). From here we see that the only way for two non-negative numbers to add up to zero is if both numbers are zero but that is also impossible. Therefore $\sqrt{2-x^2}+\sqrt[3]{3-x^3}=0$ has no real roots.

☐ We prove the generalization

$$S(m,n)=\sum_{k=1}^{n}\binom{k+m-1}{m}=\frac{n}{m+1}\binom{n+m}{m}.$$

Solution

We proceed by induction on n. The case $n=1$ is easily verified. So there exists a positive integer ν such that $S(m,\nu)=\frac{\nu}{m+1}\binom{\nu+m}{m}$. We then note

$$\begin{aligned}
S(m,\nu+1)&=S(m,\nu)+\binom{\nu+m}{m}\\
&=\frac{\nu}{m+1}\binom{\nu+m}{m}+\binom{\nu+m}{m}\\
&=\left(\frac{\nu}{m+1}+1\right)\binom{\nu+m}{m}\\
&=\frac{\nu+m+1}{m+1}\cdot\frac{(\nu+m)!}{\nu!\,m!}\\
&=\frac{\nu+1}{m+1}\cdot\frac{(\nu+m+1)!}{(\nu+1)!\,m!}\\
&=\frac{\nu+1}{m+1}\binom{(\nu+1)+m}{m},
\end{aligned}$$

thus proving that the claim is true for $n=\nu+1$ if it is true for $n=\nu$.

Next, we simply observe that

$$m!S(m,n)=\sum_{k=1}^{n}\frac{(k+m-1)!}{(k-1)!}=\sum_{k=1}^{n}\prod_{j=0}^{m-1}(k+j),$$

so with the choice $m=3$, we immediately obtain

$$\sum_{k=1}^{n}k(k+1)(k+2)=3!S(3,n)=\frac{1}{4}n(n+1)(n+2)(n+3).$$

Well, if you're going to change the question entirely, there's not much point in answering.

☐ If $f(x)$ is a polynomial satisfying $f(x)f(y)=f(x)+f(y)+f(xy)-2$ for all real x,y and $f(3)=10$, find $f(4)$.

Solution

This can be written $(f(x)-1)(f(y)-1)=f(xy)-1$. If f is a constant c, then we have $c^2-3c+2=0$, so $c=1$ or $c=2$. If not, let $f(y)=a_ny^n+\cdots+a_1y+a_0$, for some $n\geq 1$. It follows $(f(x)-1)a_ny^n=a_nx^ny^n$, thus $f(x)=x^n+1$, which indeed verifies. Asking for $f(3)=10$ forces $n=2$, hence $f(4)=17$.

☐ Find floor of the function $\sqrt{2}+(\frac{3}{2})^{\dfrac{1}{3}}+...+(\frac{n+1}{n})^{\dfrac{1}{n+1}}$

Solution

$S=\sqrt{2}+(\frac{3}{2})^{\frac{1}{3}}+...+(\frac{n+1}{n})^{\frac{1}{n+1}}$ Obviously , $n<S$. On the other hand applying Bernoulli's inequality we've for any $0<k<1$ $(1+\frac{1}{k})^{\frac{1}{k+1}}\geq 1+\frac{1}{k(k+1)}=1+\frac{1}{k}-\frac{1}{k+1}$. So $S<\sqrt{2}+\sum_{k=2}^{n}1+\frac{1}{k}-\frac{1}{k+1}=n-\frac{1}{2}-\frac{1}{n+1}+\sqrt{2}<n+1$. $\lfloor S\rfloor=n$.

☐ Given is a convex quadrilateral $ABCD$ and its diagonals intersecting at O with the angle $m\left(\widehat{AOB}\right)=90^o$. Let K,L,M,N be orthogonal projections of O on AB,BC,CD,DA respectively. Prove that $KLMN$ is cyclic.

Solution

An easy extension. Given is a convex quadrilateral $ABCD$ and its diagonals intersecting at O with the angle $\phi=m\left(\widehat{AOB}\right)$. Let K , L , M , N be orthogonal projections of O on AB , BC , CD , DA respectively. Prove that $m\left(\widehat{LKN}\right)+m\left(\widehat{LMN}\right)=2\phi$. Particular case. $OA\perp OB\implies\phi=90^\circ\implies m\left(\widehat{LKN}\right)+m\left(\widehat{LMN}\right)=180^\circ\implies KLMN$ is cyclically.

Proof. Observe that the quadrilaterals $OKAN$, $OLBK$, $OMCL$, $ONDM$ are cyclically. Therefore,

$$\left\{\begin{array}{c} m\left(\widehat{OKL}\right)=m\left(\widehat{OBL}\right)=m\left(\widehat{OBC}\right)\quad ;\quad m\left(\widehat{OKN}\right)=m\left(\widehat{OAN}\right)=m\left(\widehat{OAD}\right)\\ \\ m\left(\widehat{OML}\right)=m\left(\widehat{OCL}\right)=m\left(\widehat{OCB}\right)\quad ;\quad m\left(\widehat{OMN}\right)=m\left(\widehat{ODN}\right)=m\left(\widehat{ODA}\right)\end{array}\right\|\implies$$

$m\left(\widehat{LKN}\right)+m\left(\widehat{LMN}\right)=\left[m\left(\widehat{OKL}\right)+m\left(\widehat{OKN}\right)\right]+\left[m\left(\widehat{OML}\right)+m\left(\widehat{OMN}\right)\right]=$
$\left[m\left(\widehat{OBC}\right)+m\left(\widehat{OAD}\right)\right]+\left[m\left(\widehat{OCB}\right)+m\left(\widehat{ODA}\right)\right]=$
$\left[m\left(\widehat{OBC}\right)+m\left(\widehat{OCB}\right)\right]+\left[m\left(\widehat{OAD}\right)+m\left(\widehat{ODA}\right)\right]=$

$2\cdot m\left(\widehat{AOB}\right)=2\phi$. ☐ Lemma 1. Denote in $\triangle ABC$ the points $\left\{\begin{array}{c} D\in BC\ ;\quad AD\perp BC\\ E\in AC\ ;\quad DE\perp AC\\ F\in AB\ ;\quad DF\perp AB\end{array}\right\|$ and $\left\{\begin{array}{c} X\in BE\cap DF\\ Y\in CF\cap DE\end{array}\right\|$. Then $XY\parallel BC$.

Lemma 2. Denote in $\triangle ABC$ the orthocenter H and the points $\left\{\begin{array}{ccc} E\in AC & ; & BE\perp AC \\ F\in AB & ; & CF\perp AB \\ U\in AB\ ,\ V\in AC & ; & H\in UV \\ M\in BE & ; & UM\perp UV \\ N\in CF & ; & VN\perp VU \end{array}\right\|$. Then $MN\parallel BC$.

Solution

Proof of Lemma 1:

Let P,Q be second intersections of BE,CF respectively with the circle $\odot AFDE$. From $AF\cdot AB = AD^2 = AE\cdot AC$ we get $BCEF$ cyclic, i.e. $\widehat{FEB}=\widehat{FCB}$, or $arcPD = arcDQ$, so $\angle YFX = \angle QFD = \angle PED = \angle XEY$, or $FXYE$ is cyclic, done.

Proof of Lemma 2:

$\triangle HUM\sim\triangle HEV, \triangle HUF\sim\triangle HNV \implies \frac{HM}{HN}=\frac{FH}{HE}=\frac{HB}{HC}$, done.

□ Let $ABCD$ be a cyclic quadrilateral. (I_1) and (I_2) are the incircles of two triangles ADC and BCD. Prove that the common external tangent of $(I_1),(I_2)$, different from CD, is parallel to AB.

Solution

Let M,N be midpoints of the arcs CD,AB of the circle $\odot(ABCD)$, arcs which do not contain other vertex of the quad, and I,J the incenters of the two triangles. Well known: $MI=MJ=MC=MD$, so $IJ\perp MN$ (MN is the angle bisector of $\angle AMB$). The other common tangent will be, as reflection of CD about IJ, perpendicular to ON, i.e. parallel to AB, done (both tangents are perpendicular to two lines, symmetrical about IJ; as $AB\perp OM$, logically, the other one to be perpendicular to ON).

□ We are given a convex quadrilateral $ABCD$. Each of its sides is divided into N line segments of equal length. The corresponding division points of opposite sides are conected. This forms N^2 smaller quadrilaterals. Choose N of such that any two are in different "rows" and "columns". Prove that the sum of the areas of these chosen quadrilaterals is equal to the area of $ABCD$ divided by N.

Solution

For $N=2, E,F,G,H$ are mid-points of AB,BC,CD,DA respectively, I is the mid-point of EG. $S_{IHAE}+S_{IFCG}=S_{IHA}+S_{IAE}+S_{ICG}+S_{IFC}=S_{IDH}+S_{IEB}+S_{IGD}+S_{IBF}=S_{IEBF}+S_{IGDH}$. For $N=3$, suppose $ABCD$ is divided into $A_{ij}(i,j=1\text{ to }3)$. $3(A_{12}+A_{23}+A_{31})=2A_{12}+2A_{23}+A_{13}+A_{22}+3A_{31}=2A_{12}+2A_{23}+A_{13}+A_{21}+A_{32}+2A_{31}=A_{11}+A_{22}+A_{12}+2A_{23}+A_{13}+A_{21}+A_{32}+2A_{31}$ $=A_{11}+A_{12}+2A_{23}+A_{13}+A_{21}+2A_{32}+A_{21}+A_{31}=A_{11}+A_{22}+A_{12}+A_{33}+A_{23}+A_{13}+A_{21}+A_{32}+A_{21}+A_{31}$ Similarly for any N. ■

□ Find the remainder when $\tan^6 20°+\tan^6 40°+\tan^6 80°$ is divided by 1000.

Solution

$\tan 9\theta=(9t-84t^3+126t^5-36t^7+t^9)/(\ldots)$ $(t=\tan\theta)$ $x^4-36x^3+126x^2-84x+9=0$ has roots $\tan^2 20°,\tan^2 40°,\tan^2 60°=3,\tan^2 80°$ $(x=t^2)$ Using Newton's identities and Vieta's formulas, $p_1=e_1=36$ $p_2=e_1p_1-2e_2=36^2-2*126=1044$ $p_3=e_1p_2-e_2p_1+3e_3=36*1044-126*36+3*84=33300$ $\tan^6 20°+\tan^6 40°+\tan^6 80°=33273=273 \mod 1000$ ■

□ Find the length of internal and external bisector of angle A in triangle ABC in terms of the sides of the triangle. Let AD be the internal angle bisector of $\angle BAC$ and $\{A,E\}\in AD\cap\odot(ABC)$. from $\triangle ABD\sim\triangle AEC$ we get $AD\cdot AE=AB\cdot AC \implies AD(AD+DE)=AB\cdot AC \implies AD^2+BD\cdot DC=AB\cdot AC$, but from power of $D \implies AD\cdot DE=BD\cdot DC$. Next, calculate BD,CD from angle bisector theorem and, if a,b,c are the side lengths, we get $AD^2=\frac{b\cdot c\cdot[a^2-(b+c)^2]}{(b+c)^2}$.

For the external angle bisector, a similar process.

$\square$ Find $x, y, z (x, y, z \in \mathbb{Z})$ $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}$

Solution

The equation writes $z^2(x^2+y^2) = x^2y^2$. Take $d = \gcd(x, y)$, $x = da$, $y = db$, with $\gcd(a, b) = 1$. So $z^2(a^2+b^2) = d^2a^2b^2$. It follows $a^2b^2 \mid z^2$ (since $\gcd(a^2+b^2, a^2b^2) = 1$). Therefore $z = abc$, and so $c^2(a^2+b^2) = d^2$. This means $d = ce$, and so $a^2 + b^2 = e^2$.

Now use the parametrization of the primitive Pythagorean triples: $a = m^2 - n^2$, $b = 2mn$, $e = m^2 + n^2$, with $\gcd(m, n) = 1$, $|m| \neq |n|$. It follows $x = c(m^2+n^2)(m^2-n^2)$, $y = 2cmn(m^2+n^2)$, $z = 2cmn(m^2-n^2)$, with arbitrary integer not-null c (and, of course, also with reversed formulae for x, y, due to symmetry).

$\square$ Let I be the incenter and AD be a diameter of the circumcircle of a triangle ABC. If the point E on the ray BA and the point F on the ray CA satisfy the condition

$BE = CF = \frac{AB+BC+CA}{2}$,

show that the lines EF and DI are perpendicular.

Solution

Let I_A, I_B, I_C be the excenters opposite to A, B, C, respectively.

Let X and Y be the feet of the perpendiculars from I_C and I_B to AB and AC, respectively.

From the problem conditions we have that $AF = s - b = AX$. Also, $\angle I_CAX = 90 - \frac{\angle A}{2}$ and $\angle I_CAF = 180 - \angle A - (90 - \frac{\angle A}{2}) = 90 - \frac{\angle A}{2} = \angle I_CAX$ and therefore we obtain that AI_C is the perpendicular bisector of FX or equivalently, F is the reflection of X in side I_BI_C.

Similarly, E is the reflection of Y in side I_BI_C.

Let N be the circumcenter of ABC and O be the circumcenter of $I_AI_BI_C$.

Clearly N is midpoint of AD. Also, note that I and N are the orthocenter and nine-point center of $I_AI_BI_C$ and therefore N is midpoint of OI.

From this we obtain that $AIDO$ is a parallelogram and therefore $DI \parallel AO$. So it is enough to show that $AO \perp EF$.

This is basically just angle chase.

Note that ABC is the orthic triangle of $I_AI_BI_C$ and therefore, since $I_CO \perp AB$ and $I_BO \perp AC$ (this is well known) we have $I_C - X - O$ and $I_B - Y - O$.

Since $\angle FAX = \angle EAY$ and both AFX and AEY are A-isosceles we have that $\angle AXF = \angle AYE$ and therefore $FEYX$ is cyclic.

Let $S \equiv AO \cap EF$. We have

$$\angle SAF + \angle AFS = \angle YAO + \angle AFE = \angle YXO + \angle YXA = 90$$

and therefore $AO \perp EF$ and we are done.

$\square$ Consider the progression $u_0 = \frac{1}{2}$ $u_{n+1} = \frac{u_n}{3-2u_n}$

we will put $\forall n \in \mathbb{N} : w_n = \frac{u_n}{u_n+a}, a \in \mathbb{R}$

1.Find the value of a such that $\{w_n\}$ is a geometric progression

Solution

We can prove the following through induction: $u_n = \frac{1}{3^n+1}$.

First we see that it's true for $n = 0$: $u_0 = \frac{1}{2} = \frac{1}{1+3^0}$. Now assume $u_k = \frac{1}{3^k+1}$ for some integer $k \geq 0$. We prove that $u_{k+1} = \frac{1}{3^{k+1}+1}$, just using the definition of u_k. We get:

$u_{k+1} = \frac{u_k}{3-2u_k} = \frac{\frac{1}{3^k+1}}{3-2\cdot\frac{1}{3^k+1}} = \frac{\frac{1}{3^k+1}\cdot(3^k+1)}{(3-2\cdot\frac{1}{3^k+1})\cdot(3^k+1)} = \frac{1}{3(3^k+1)-2} = \frac{1}{3^{k+1}+1}$.

Therefore we know that $u_{k+1} = \frac{1}{3^{k+1}+1}$ if $u_k = \frac{1}{3^k+1}$, so since it's true for 0 it's true for $1, 2, 3\ldots$ and for all integers k.

Thus $w_n = \frac{u_n}{u_n+a} = \frac{\frac{1}{3^n+1}}{\frac{1}{3^n+1}+a} = \frac{\frac{1}{3^n+1}\cdot(3^n+1)}{(\frac{1}{3^n+1}+a)(3^n+1)} = \frac{1}{(1+a)+a3^n}$. So we calculate w_n^2 and $w_{n-1}w_{n+1}$, and see what values of a make these equal for all values of n.

$w_n^2 = \frac{1}{((1+a)+a3^n)^2} = w_{n-1}w_{n+1} = \frac{1}{(1+a)+a3^{n-1}}\frac{1}{(1+a)+a3^{n+1}}$ and taking reciprocals gives

$((1+a)+a3^n)^2 = (1+a)^2 + 2(1+a)a3^n + a^2 3^{2n} = ((1+a)+a3^{n-1})((1+a)+a3^{n+1}) = (1+a)^2 + (1+a)a(3^{n-1}+3^{n+1}) + a^2 3^{2n}$

$0 = (1+a)a(3^{n+1} - 2\cdot 3^n + 3^{n-1}) = (1+a)a3^{n-1}(9-6+1) = 4a(1+a)3^{n-1} = 0$ and so $a=0$ or $a=-1$. These both work: if $a=0$ then $w_n = \frac{u_n}{u_n} = 1$, and if $a=-1$ then $w_n = \frac{1}{-3^n} = -\left(\frac{1}{3}\right)^n$ both of which are geometric sequences. So the answers are $\boxed{a = 0 \text{ and } a = -1}$.

A much simpler solution is as follows:

If $\{w_n\}$ is geometric, then there exists a constant r such that

$$\begin{aligned} r = \frac{w_{n+1}}{w_n} &= \frac{u_{n+1}(u_n+a)}{(u_{n+1}+a)u_n} \\ &= \frac{u_n(u_n+a)}{(3-2u_n)(\frac{u_n}{3-2u_n}+a)u_n} \\ &= \frac{u_n+a}{u_n+a(3-2u_n)} \\ &= \frac{u_n+a}{(1-2a)u_n+3a}. \end{aligned}$$

Consequently, $r(1-2a)u_n + 3ar = u_n + a$ for all u_n, which implies

$$r(1-2a) = 1,$$
$$3ar = a.$$

This system yields two solutions: $(a,r) \in \{(0,1),(-1,\frac{1}{3})\}$. Since $a=0$ yields the trivial geometric sequence $w_n = 1$ which is satisfied regardless of u_n, this solution is valid. The second solution gives $w_n = -3^{-n}$. To find u_n, we let $v_n = 1/u_n$ and easily observe that the given recursion relation is equivalent to

$$v_{n+1} = \frac{1}{u_{n+1}} = \frac{3-2u_n}{u_n} = \frac{3}{u_n} - 2 = 3v_n - 2 = 3(v_n-1)+1.$$

Therefore, $v_{n+1} - 1 = 3(v_n - 1)$, and since $v_0 - 1 = 1$, we have $v_n - 1 = 3^n$. Thus $u_n = \frac{1}{3^n+1}$, which confirms our earlier result for w_n.

ABC is a triangle. BK is median, CL is bisector. If $(BK)\cap(CL) = P$ then prove this $\frac{PC}{PL} - \frac{AC}{BC} = 1$.

Solution

Areas Method

Denote $x := \frac{PC}{PL}$ and $S := [PCK]$, alongside with the usual $a = BC, b = CA, c = AB$. Then $[PAK] = S$, and $[APL] : [APC] = 1 : x \implies [APL] = \frac{2S}{x}$

Since $AL = \frac{bc}{a+b}$, we get $[ALC] : [ABC] = AL : c = \frac{b}{a+b} \iff 2S\left(1+\frac{1}{x}\right) = \frac{b[ABC]}{a+b} \quad (*)$

On the other hand, $[KPC] + [BPC] = \frac{[ABC]}{2}$ and $[KPC] : [BPC] = KP : BP = KC : BC$, hence $[KPC] : [BPC] = b2a)$

Therefore $[KPC] = S = \frac{b[ABC]}{2(2a+b)}$

Plugging that into $(*)$ we get

$\frac{1}{2a+b}\left(1+\frac{1}{x}\right) = \frac{1}{a+b} \iff \frac{1}{x} = \frac{a}{a+b} \iff x = 1 + \frac{b}{a}$. QED

Mass points Method

Assign mass μ to A. Then $AL : LB = b : a \implies m(B) = \frac{b}{a}\mu \implies m(L) = \frac{a+b}{a}\mu$. On the other hand, $AK = KC \implies m(C) = \mu$. Thus $PL \cdot m(L) = PC \cdot m(C) \implies \frac{PC}{PL} = 1 + \frac{b}{a}$. QED

Another ways: Method 1. Denote $M \in (BK)$ so that $LM \parallel AC$. Thus,
$\frac{PC}{PL} = \frac{CK}{ML} = \frac{AK}{ML} = \frac{AB}{BL} = 1 + \frac{LA}{LB} = 1 + \frac{CA}{CB}$.

Methos 2. Denote $S \in (BC)$ so that $LS \parallel AC$. Thus, $S \in AP$ and in the trapezoid $ACSL$ have $\frac{PC}{PL} = \frac{AC}{LS} = \frac{BA}{BL} = \frac{BL+LA}{BL} = 1 + \frac{LA}{LB} = 1 + \frac{CA}{CB}$.

□ Solve the equation
$2012^{x-2} + 2012^{\frac{4}{x}-2} = 2$

Solution

From the way that the equation is defined, we conclude that $x \neq 0$. If $x < 0$, we can easily see that

$$2012^{x-2} < 2012^{-2} < 1$$

and

$$2012^{\frac{4}{x}-2} < 2012^{-2} < 1$$

This shows that LHS $<$ RHS hence a contradiction is reached.

If $x > 0$, we proceed the problem as in previous post's.

□ Find all positive integers n such that for all odd integers a, if $a^2 \leq n$ then $a|n$.

Solution

$1^2 = 1 \leq n = 1, 2, 3, 4, 5, 6, 7, 8 < 9 = 3^2$ are all good; $3^2 = 9 \leq n = 9, 12, 15, 18, 21, 24 < 25 = 5^2$ are all good; $5^2 = 25 \leq n = 30, 45 < 49 = 7^2$ are all good; $7^2 = 49 \leq n = 105 < 121 = 11^2$ seems good, but is not, since $9^2 = 81 < 105$, while $9 \nmid 105$.

Others there are not. When $p_k \leq n < p_{k+1}^2$, where $(p_k)_{k\geq 1}$ is the prime numbers sequence, we need $p_2p_3 \cdots p_k \mid n$. But $3 \cdot 5 \cdot 7 > 4 \cdot 11$, and when $p_2p_3 \cdots p_{k-1} > 4p_k$ it follows, by Bertrand's postulate (Chebyshev's theorem) $p_2p_3 \cdots p_k > 4p_k^2 = (2p_k)^2 > p_{k+1}^2 > 4p_{k+1}$, by induction. But then $p_2p_3 \cdots p_k > p_{k+1}^2 > n$, so it is not possible to have $p_2p_3 \cdots p_k \mid n$. Surely there may be simpler results one could use (instead of Bertrand's), for example a variant of Bonse's theorem.

□ Solve the equation $\frac{9(\cos x+3\sin x)^2}{(\cos 2x+3\sin 2x)^2} = 3 + \cot x$.

Solution

$\frac{9(\cos x+3\sin x)^2}{(\cos 2x+3\sin 2x)^2} = 3 + \cot x \iff 9(\cos x + 3\sin x)^2 = \left(3 + \frac{\cos x}{\sin x}\right)(\cos 2x + 3\sin 2x)^2 \iff$

$9\sin x(\cos x + 3\sin x)^2 \left(\sin^2 x + \cos^2 x\right) = (3\sin x + \cos x)(\cos^2 x - \sin^2 x + +6\sin x\cos x)^2$, what is an

homogeneous equation. We'lluse the standard substitution $\tan x = t$, i.e. $\boxed{\frac{\sin x}{t} = \frac{\cos x}{1}}$.

Our equation becomes:

$9t(3t+1)^2\left(t^2+1\right) = (3t+1)(1-t^2+6t)^2 \overset{t\neq -\frac{1}{3}}{\iff} 9t(3t+1)\left(t^2+1\right) = (t^2-6t-1)^2 \iff$

$26t^4 + 21t^3 - 7t^2 - 3t - 1 = 0 \iff (t+1)(2t-1)\left(13t^2+4t+1\right) = 0$. Thus, $\boxed{t \in \left\{-\frac{1}{3}, -1, \frac{1}{2}\right\}}$

a.s.o.

□ Let $ABCD$ be a convex quadrilateral. Prove that for any $X \in (AB)$, $XA \cdot [BCD] + XB \cdot [ACD] = AB \cdot [XCD]$.

Proof. Denote $\{U, V, Y\} \subset CD$ so that $\left\{\begin{array}{c} AU \perp CD \\ BV \perp CD \\ XY \perp CD \end{array}\right|$ and $\left\{\begin{array}{c} K \in XY \ ; \ L \in BV \\ \\ A \in KL \parallel CD \end{array}\right|$. Thus,

$\frac{KX}{LB} = \frac{AX}{AB} \iff$

$\frac{XY-AU}{BV-AU} = \frac{AX}{AB} \iff AX(BV - AU) = AB(XY - AU) = \iff XA \cdot BV + XB \cdot AU = AB \cdot XY \iff$

$$XA\cdot(BV\cdot CD)+XB\cdot(AU\cdot CD)=AB\cdot(XY\cdot CD)\iff XA\cdot[BCD]+XB\cdot[ACD]=AB\cdot[XCD]$$
.

Problem: Let M , N be the midpoints of $[AB]$, $CD]$ respectively in the convex quadrilateral $ABCD$. Prove that there is the equivalence $[CMD]=[ANB]\iff AD\parallel BC$.

Solution

We'll use the upper lema. Denote $I\in AC\cap BD$ and $[IAB]=x$, $[IBC]=y$, $[ICD]=z$ and $[IDA]=t$.

Thus, $[CMD]=[ANB]\iff \frac{MA}{AB}\cdot[BCD]+\frac{MB}{AB}\cdot[ACD]=\frac{ND}{CD}\cdot[CAB]+\frac{NC}{CD}\cdot[DAB]\iff \boxed{[BCD]+[ACD]=[CAB]+[DAB]}\iff (y+z)+(z+t)=(x+y)+(x+t)\iff z=x\iff AD\parallel BC$.

An easy extension. Let $ABCD$ be a convex quadrilateral and let $M\in(AB)$, $N\in(CD)$ so that $\frac{MB}{AB}+\frac{NC}{CD}=1$. Prove that $[MCD]+[NAB]=[ABCD]$.

□ Solve the exponential equation $f(x)\equiv 2^x+2^{\sqrt{1-x^2}}=3$.

Solution

Observe that $\{0,1\}\subset\mathbb{S}$ - the set of the zeroes for given equation and $x\in\mathbb{S}\implies x\ge 0$. Suppose $x\in(0,1)$.

▶ $2^x+2^{\sqrt{1-x^2}}=2^{x-1}+2^{x-1}+2^{\sqrt{1-x^2}}>3\sqrt[3]{2^{2(x-1)+\sqrt{1-x^2}}}\ge 3\iff 2(x-1)+\sqrt{1-x^2}\ge 0\iff \sqrt{1+x}\ge 2\sqrt{1-x}\iff 1+x\ge 4(1-x)\iff 5x\ge 3\iff x\in\left[\frac{3}{5},1\right)$. Thus, $f(x)>3$, $(\forall)\ x\in\left[\frac{3}{5},1\right)$.

▶ $2^x+2^{\sqrt{1-x^2}}=2^x+2^{\sqrt{1-x^2}-1}+2^{\sqrt{1-x^2}-1}>3\sqrt[3]{2^{x-2+2\sqrt{1-x^2}}}\ge 3\iff (x-2)+2\sqrt{1-x^2}\ge 0\iff 2\sqrt{1-x^2}\ge 2-x\iff 4\left(1-x^2\right)\ge 4-4x+x^2\iff 5x^2\le 4x\iff x\in\left(0,\frac{4}{5}\right]$. Thus, $f(x)>3$, $(\forall)\ x\in\left(0,\frac{4}{5}\right]$.

Therefore, $f(x)>3$, $(\forall)\ x\in\left(0,\frac{4}{5}\right]\cup\left[\frac{3}{5},1\right)=(0,1)$. In conclusion, $\boxed{\mathbb{S}=\{0,1\}}$.

□ Let ABC be an A-isosceles triangle with the incenter I such that $IA=2\sqrt{3}$ and $IB=3$. Find the length c of the side $[AB]$.

Solution

Proof 1 (trigonometric). Let $m\left(\widehat{IBA}\right)=\frac{B}{2}=\phi$, i.e. $m\left(\widehat{IAB}\right)=\frac{A}{2}=90^\circ-2\phi$. Apply the theorem of **Sines** in $\triangle AIB$:

$\frac{IA}{\sin\widehat{IBA}}=\frac{IB}{\sin\widehat{IAB}}=\frac{AB}{\sin\widehat{AIB}}\iff\frac{2\sqrt{3}}{\sin\phi}=\frac{3}{\cos 2\phi}=\frac{c}{\cos\phi}=\sqrt{c^2+12}$. Since $\boxed{\tan\phi=\frac{2\sqrt{3}}{c}}$ obtain that $\boxed{\cos 2\phi=\frac{3}{\sqrt{c^2+12}}}=\frac{1-\tan^2\phi}{1+\tan^2\phi}=\frac{c^2-12}{c^2+12}\implies\boxed{c^2-12=3\sqrt{c^2+12}}\ (*)$. Denote $c^2+12=y^2$, where $y>0$, i.e. $c^2=y^2-12\implies y^2-24=3y\implies y^2-3y-24=0\implies y=\frac{3+\sqrt{105}}{2}\implies c^2=y^2-12=3y+24-12=$

$3(y+4)=\frac{3}{2}\cdot\left(11+\sqrt{105}\right)\implies c^2=\frac{3}{2}\cdot\left(11+\sqrt{105}\right)\overset{\left(11^2-105=4^2\right)}{\implies} c=\sqrt{\frac{3}{2}}\cdot\left(\sqrt{\frac{11+4}{2}}+\sqrt{\frac{11-4}{2}}\right)\implies c=\frac{\sqrt{3}\left(\sqrt{7}+\sqrt{15}\right)}{2}\implies\boxed{b=c=\frac{3\sqrt{5}+\sqrt{21}}{2}}$. Prove easily $\tan\phi=\frac{\sqrt{15}-\sqrt{7}}{2}$ and $\boxed{\tan B=\frac{3\sqrt{7}+\sqrt{15}}{6}}$
.

Proof 2 (metric). $\left\{\begin{array}{l} IA^2=\frac{bc(s-a)}{s}\implies\frac{b^2(2b-a)}{2b+a}=12\\ \\ IB^2=\frac{ca(s-b)}{s}\implies\frac{a^2b}{2b+a}=9\end{array}\right|\implies\frac{b(2b-a)}{a^2}=\frac{4}{3}\implies 4a^2+3ab-6b^2=0\implies$
$\left\{\begin{array}{l}\frac{a}{b}=\frac{-3+\sqrt{105}}{8}\\ \\ b^2=12\cdot\frac{2b+a}{2b-a}\end{array}\right|$. Therefore, $\frac{a}{-3+\sqrt{105}}=\frac{b}{8}=\frac{2b+a}{13+\sqrt{105}}=\frac{2b-a}{19-\sqrt{105}}\implies\frac{2b+a}{2b-a}=\frac{13+\sqrt{105}}{19-\sqrt{105}}\implies$

$b^2 = 12 \cdot \frac{13+\sqrt{105}}{19-\sqrt{105}} = 12 \cdot \frac{\left(13+\sqrt{105}\right)\left(19+\sqrt{105}\right)}{256} \implies b^2 = \frac{3}{2} \cdot \left(11 + \sqrt{105}\right) \implies$

$b = \sqrt{\frac{3}{2}} \cdot \left(\sqrt{\frac{11+4}{2}} + \sqrt{\frac{11-4}{2}}\right) = \frac{\sqrt{3}\left(\sqrt{15}+\sqrt{7}\right)}{2} \implies \boxed{b = c = \dfrac{3\sqrt{5}+\sqrt{21}}{2}}$.

$\square$ Let ABC be a triangle. Find the points $D \in (BC)$, $E \in (CA)$ and $F \in (AB)$ so that :

1 ▶ $DF \parallel AC$, $DE \parallel AB$ and EF is antiparallel to BC .

2 ▶ DF is antiparallel to AC , DE is antiparallel to AB and $EF \parallel BC$.

3 ▶ DF is parallel to AC , DE is parallel to AB and $EF \parallel BC$.

4 ▶ DF is antiparallel to AC , DE is antiparallel to AB and EF is antiparallel to BC

Solution

Denote $\frac{DB}{DC} = m$, i.e. $\frac{DB}{m} = \frac{DC}{1} = \frac{a}{m+1}$. Therefore:

1 ▶ $\left\{\begin{array}{ccccc} DF \parallel AC & \implies & \frac{FB}{FA} = \frac{DB}{DC} = m & \implies & \frac{FB}{m} = \frac{FA}{1} = \frac{c}{m+1} \\ & & & & \\ DE \parallel AB & \implies & \frac{EA}{EC} = \frac{DB}{DC} = m & \implies & \frac{EA}{m} = \frac{EC}{1} = \frac{b}{m+1} \end{array}\right|$.

In conclusion, EF is antiparallel to $BC \iff AF \cdot c = AE \cdot b \iff \frac{c^2}{m+1} = \frac{mb^2}{m+1} \iff$

$m = \frac{c^2}{b^2} \iff \frac{DB}{DC} = \left(\frac{c}{b}\right)^2$, i.e. the ray $[AD$ is the A-**symmedian** of $\triangle ABC$.

2 ▶ $\left\{\begin{array}{ccccccc} DF \text{ is antiparallel to } AC & \implies & \triangle BDF \sim \triangle BAC & \implies & \frac{BD}{c} = \frac{BF}{a} & \implies & BF = \frac{a^2m}{c(m+1)} \\ & & & & & & \\ DE \text{ is antiparallel to } AB & \implies & \triangle CDE \sim \triangle CAB & \implies & \frac{CD}{b} = \frac{CE}{a} & \implies & CE = \frac{a^2}{b(m+1)} \end{array}\right|$

.

In conclusion, $EF \parallel BC \iff \frac{BF}{BA} = \frac{CE}{CA} \iff \frac{a^2m}{c^2(m+1)} = \frac{a^2}{b^2(m+1)} \iff$

$m = \frac{c^2}{b^2} \iff \frac{DB}{DC} = \left(\frac{c}{b}\right)^2$, i.e. the ray $[AD$ is the A-**symmedian** of $\triangle ABC$.

3 ▶ $\left\{\begin{array}{ccccc} DF \parallel AC & \implies & \frac{FB}{FA} = \frac{DB}{DC} = m & \implies & \frac{FB}{m} = \frac{FA}{1} = \frac{c}{m+1} \\ & & & & \\ DE \parallel AB & \implies & \frac{EA}{EC} = \frac{DB}{DC} = m & \implies & \frac{EA}{m} = \frac{EC}{1} = \frac{b}{m+1} \end{array}\right|$.

In conclusion, EF is parallel to $BC \iff \frac{AF}{c} = \frac{AE}{b} \iff \frac{1}{m+1} = \frac{m}{m+1} \iff$

$m = 1 \iff DB = DC$, i.e. the ray $[AD$ is the A-**median** of $\triangle ABC$.

4 ▶ $\left\{\begin{array}{ccccccc} DF \text{ is antiparallel to } AC & \implies & \triangle BDF \sim \triangle BAC & \implies & \frac{BD}{c} = \frac{BF}{a} & \implies & BF = \frac{a^2m}{c(m+1)} \\ & & & & & & \\ DE \text{ is antiparallel to } AB & \implies & \triangle CDE \sim \triangle CAB & \implies & \frac{CD}{b} = \frac{CE}{a} & \implies & CE = \frac{a^2}{b(m+1)} \end{array}\right|$

.

In conclusion, EF is antiparallel to $BC \iff c \cdot AF = b \cdot AE \iff c^2 - \frac{a^2m}{m+1} = b^2 - \frac{a^2}{m+1} \iff$

$m = \frac{a^2+c^2-b^2}{a^2+b^2-c^2} \iff \frac{DB}{DC} = \frac{a^2+c^2-b^2}{a^2+b^2-c^2} \iff \frac{DB}{DC} = \frac{c\cos B}{b\cos C} \implies$ the ray $[AD$ is the A-**altitude** of $\triangle ABC$.

$\square$ Let ABC be a triangle with the incircle $w = C(I,r)$. Denote $\{D,E,F\} = ABC \cap w$. Prove that:

1 ▶ $A = 45^\circ \ \wedge \ \frac{AC}{AB} = \frac{2\sqrt{2}}{3} \implies \tan B = 2$.

2 ▶ $[DEF] = \frac{r}{2R} \cdot [ABC]$.

3 ▶ $\cos^2 \frac{B-C}{2} \geq \frac{2r}{R}$.

4.1 ▶ $a\cot A + b\cot B + c\cot C = 2(R+r)$.

4.2 ▶ $\frac{\sin A+\sin B+\sin C}{\cos A+\cos B+\cos C} = \frac{s}{R+r}$.

Solution

1 ▶ $\frac{b}{c} = \frac{2\sqrt{2}}{3} \iff \iff 3\sin B = 2\sqrt{2}\cos(B-45^\circ) \iff 3\tan B = 2\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\tan B\right)$ $\iff \tan B = 2$.

2 ▶ Suppose that $D \in BC$, $E \in CA$ and $F \in AB$. Thus, $\left\{\begin{array}{l} EF = 2(s-a)\sin\frac{A}{2} \\ \\ FD = 2(s-b)\sin\frac{B}{2} \\ \\ DE = 2(s-c)\sin\frac{C}{2} \end{array}\right| \implies$

$[DEF] = \frac{EF\cdot FD\cdot DE}{4r} =$
$\frac{2(s-a)(s-b)(s-c)}{r} \cdot \prod \sin\frac{A}{2} = 2sr \cdot \frac{(s-a)(s-b)(s-c)}{abc} = \frac{2rS^2}{4RS} \implies [DEF] = \frac{r}{2R} \cdot [ABC]$.

3 ▶ **Method 1.** Prove easily that $\boxed{(b+c)^2 \ge 4a(b+c-a)}$ $(*)$ and $\cos\frac{B-C}{2} = \frac{b+c}{4R\cos\frac{A}{2}}$ (1) . Therefore,

$\cos\frac{B-C}{2} = \frac{b+c}{4R} \cdot \sqrt{\frac{bc}{s(s-a)}} \implies \cos^2\frac{B-C}{2} = \frac{(b+c)^2bc}{16R^2s(s-a)} = \frac{(b+c)^2 4Rsr}{16R^2s(s-a)a} =$
$\frac{(b+c)^2r}{4R(s-a)a} = \frac{(b+c)^2}{a(b+c-a)} \cdot \frac{r}{2R} \overset{(*)}{\ge} 4 \cdot \frac{r}{2R} = \frac{2r}{R} \implies \cos^2\frac{B-C}{2} \ge \frac{2r}{R}$.

Method 2. Let $\{A, S\} = AI \cap w$ and diameter $[SN]$ of the circumcircle $C(O, R)$ of $\triangle ABC$. Thus, $m\left(\widehat{ASN}\right) = \frac{B-C}{2}$

and $\cos\widehat{ASN} = \frac{SA}{SN} \iff \cos\frac{B-C}{2} = \frac{IA+IS}{2R} \ge \frac{\sqrt{IA\cdot IS}}{R} = \frac{1}{R}\cdot\sqrt{2Rr} = \sqrt{\frac{2r}{R}} \implies \cos^2\frac{B-C}{2} \ge \frac{2r}{R}$.

Remark. $\boxed{\cos^2\frac{B-C}{2} = \frac{2r}{R} \iff b+c = 2a \iff IA = IS \iff IO \perp IA \iff IG \parallel BC}$.

4 ▶ Is well-known the remarkable identity $\boxed{\cos A + \cos B + \cos C = 1 + \frac{r}{R}}$ $(*)$.

⊙ 4.1 ▶ $\sum a\cot A = \sum \frac{a\cos A}{\sin A} = 2R\sum\cos A \overset{(*)}{=} 2R\left(1+\frac{r}{R}\right) = 2R + r$.

⊙ 4.2 ▶ $\frac{\sin A+\sin B+\sin C}{\cos A+\cos B+\cos C} = \frac{1}{2R} \cdot \frac{a+b+c}{\sum\cos A} \overset{(*)}{=} \frac{s}{R} \cdot \frac{1}{1+\frac{r}{R}} = \frac{s}{R+r}$.

□ Let $\in \left(0, \frac{\pi}{2}\right)$. Prove that $\frac{3\sqrt{3}}{\sin x} + \frac{1}{\cos x} \ge 8$.

Solution

Proof 1. Observe that $2\left(\frac{3\sqrt{3}}{\sin x} + \frac{1}{\cos x}\right) \ge \left(\sqrt{3}\sin x + \cos x\right)\left(\frac{3\sqrt{3}}{\sin x} + \frac{1}{\cos x}\right) =$
$\left(\frac{\sqrt{3}}{\sin x} + \frac{\sqrt{3}}{\sin x} + \frac{\sqrt{3}}{\sin x} + \frac{1}{\cos x}\right)\left(\frac{\sin x}{\sqrt{3}} + \frac{\sin x}{\sqrt{3}} + \frac{\sin x}{\sqrt{3}} + \cos x\right) \ge 4^2 = 16 \implies \frac{3\sqrt{3}}{\sin x} + \frac{1}{\cos x} \ge 8$.

Proof 2 (ugly). Observe that $\frac{3\sqrt{3}}{\sin x} + \frac{1}{\cos x} \ge 8 \iff \left(\sin^2 x + \cos^2 x\right)\left(3\sqrt{3}\cos x + \sin x\right)^2 \ge 64\sin^2 x\cos^2 x \overset{\tan x = t}{\iff}$
$\left(t^2+1\right)\left(t+3\sqrt{3}\right)^2 \ge 64t^2 \iff t^4+6\sqrt{3}t^3-36t^2+6\sqrt{3}t+27 \ge 0 \iff \left(t-\sqrt{3}\right)^2\left(t^2+8\sqrt{3}t+9\right) \ge 0$,

what is truly. We have the equality if and only if $t = \sqrt{3}$, i.e. $\tan x = \sqrt{3} \iff x = \frac{\pi}{3}$. **An easy extension (sqing).** Prove that for any $\{a, b, u, v\} \in \mathbb{R}_+^*$ there is the inequality $\boxed{\frac{a}{u} + \frac{b}{v} \ge \frac{(a+b)^2}{\sqrt{\left(a^2+b^2\right)\left(u^2+v^2\right)}}}$.

Proof. $\frac{a}{u} + \frac{b}{v} = \frac{a^2}{au} + \frac{b^2}{bv} \overset{C.B.S}{\ge} \frac{(a+b)^2}{au+bv} \overset{C.B.S}{\ge} \frac{(a+b)^2}{\sqrt{(a^2+b^2)(u^2+v^2)}}$.

Particular case. If $\{a, b, u, v\} \in \mathbb{R}_+^*$ and $x \in \left(0, \frac{\pi}{2}\right)$, then $\frac{a^2}{u\sin x} + \frac{b^2}{v\cos x} \ge \frac{(a+b)^2}{\sqrt{u^2+v^2}}$.

□ (hình học) Let ABC be a triangle. Denote the midpoint M of $[BC]$, the B-bisector $[BD]$, where $D \in (AC)$

and the projection P of A on $[BC]$. Prove that $BD = 2 \cdot AM \iff m\left(\widehat{MAP}\right) = \frac{|A-3C|}{2}$. – Let $\triangle ABC$ with $b \ge c$. Denote the midpoint D of $[BC]$ and $m\left(\widehat{ADB}\right) = \phi$. Prove that $\tan\frac{\phi}{2} \le \frac{b}{c} \le \cot\frac{\phi}{2}$. – For $\triangle ABC$ denote the **semiperimeter** p and lengths R, r, h_a, r_a of **circumradius**, **inradius**, A- **altitude**, A- **exinradius**. Prove that :

1 ▶ $OA \perp OI \iff \boxed{h_a = R + r} \iff (b+c)r = aR \iff \cos(B-C) = \cos B + \cos C$.

2 ▶ $IO \perp IA \iff b+c=2a \iff \boxed{r_a = h_a} \iff \sin\frac{A}{2} = \sqrt{\frac{r}{2R}} \iff \cos\frac{B-C}{2} = \sqrt{\frac{2r}{R}} \iff p^2+9r^2=18Rr$.

3 ▶ $\boxed{r_a = R + r} \iff ar = (p-a)R$. Find another nice equivalencies and a geometrical interpretation of this relation.

Find for each case partly a nonisosceles (therefore and nonequilateral) $\triangle ABC$ which verifies the respective relation.

□ Prove that for $\{a,b,c\} \subset (0,\infty)$, $\left\{\begin{array}{c} ax+by=(x-y)^2 \\ by+cz=(y-z)^2 \\ cz+ax=(z-x)^2 \end{array}\right\| \Longrightarrow x=y=z=0.$

Solution

We have $ax+by+cz = \sum x^2 - \sum yz \Rightarrow ax+(y-z)^2 = \sum x^2 - \sum yz \Rightarrow ax = x^2-xy-xz$ Similarly, $by = y^2-yz-yx$ Thus, $ax+by = x^2-xy-xz+y^2-yz-yx \Rightarrow (x-y)^2 = (x-y)^2 - z(x+y) \Rightarrow z(x+y)=0$

Similarly, we also have $x(y+z)=0$ and $y(z+x)=0$ And now we can easily to show that $x=y=z=0$

□ The lines AD , BE , CF are altitudes in $\triangle ABC$. For a mobile point $K \in [BC]$ define the point $\left\{\begin{array}{c} L \in EF \\ KL \parallel CF \end{array}\right.$. Prove that the circumcenter of $\triangle LDK$ belongs always to the line AC.

Solution

Denote the intersection $S \in EF \cap BC$. Observe that $KL \parallel CF \Longrightarrow \frac{LF}{KC} = \frac{SF}{SC} = \frac{\sin\widehat{SCF}}{\sin\widehat{SFC}} = \frac{\sin\widehat{BCF}}{\sin\widehat{CFE}}$ $\Longrightarrow \boxed{\frac{LF}{KC} = \frac{\cos B}{\cos C}}$ (1).

Denote the point $\left\{\begin{array}{c} N \in AD \\ LN \perp LK \end{array}\right.$. Observe that the quadrilateral $NLDK$ is inscribed in the circle with the dianeter $[KN]$.

Thus, $\widehat{LND} \equiv \widehat{LKD} \equiv \widehat{FCB} \equiv \widehat{BAD} \Longrightarrow \widehat{LND} \equiv \widehat{BAD} \Longrightarrow LN \parallel AB \Longrightarrow \frac{AN}{FL} = \frac{TA}{TF} = \frac{\sin\widehat{AFT}}{\sin\widehat{FAT}} \Longrightarrow \boxed{\frac{AN}{FL} = \frac{\sin C}{\cos B}}$ (2).

(1) $\wedge$ (2) $\Longrightarrow \frac{AN}{KC} = \tan C \Longrightarrow \frac{AN}{KC} = \frac{DA}{DC}$. Denote $M \in NK \cap AC$. Apply the **Menelaus' theorem** to the transversal $\overline{AMC}$ for $\triangle DNK$:

$\frac{CK}{CD} \cdot \frac{AD}{AN} \cdot \frac{MN}{MK} = 1 \Longrightarrow \frac{MN}{MK} = \frac{DC}{DA} \cdot \frac{AN}{KC} \Longrightarrow MN = MK$, i.e. the point M is the circumcenter of the triangle LDK and $M \in AC$.

□ Let ABC be a triangle with the circumradius R, the inradius r and the A- exinradius r_a. Prove that $\boxed{R^2+a^2 \ge 5rr_a}$. Establish when is equality.

Solution

The relations $abc = 4RS$ and $rr_a = (p-b)(p-c)$ are well-known. Therefore, $\left\{\begin{array}{ll} 1 ▶ & a^2 \ge a^2-(b-c)^2 \Longrightarrow \\ 2 ▶ & \left\| \begin{array}{c} bc \ge 2S \\ a \ge 2\sqrt{rr_a} \end{array} \right\| \Longrightarrow \end{array}\right.$

$\Longrightarrow \boxed{R^2+a^2 \ge 5rr_a}$.

We'll have equality if and only if $\left\| \begin{array}{c} bc = 2S \\ a^2 = 4(p-b)(p-c) \end{array} \right\| \iff A = 90^\circ$ and $b=c$.

Remark. $\left\{\begin{array}{ccc} a=(p-b)+(p-c) & \Longrightarrow & a \ge 2\sqrt{(p-b)(p-c)} = 2\sqrt{rr_a} \\ bc = p(p-a)+(p-b)(p-c) & \Longrightarrow & bc \ge 2\sqrt{p(p-a)(p-b)(p-c)} = 2S \end{array}\right\|$ a.s.o.

More precisely, for any positive numbers x, y, k we have $(x+y)(k^2+xy) \geq 4kxy$.

□ Let ABC be a triangle with $B = 30°$.Take the point $D \in [BC]$ such that $CD = AB$ and $m(\widehat{BAD}) = 15°$. Ascertain $m(\widehat{ACB})$.

Solution

Generally,We'llsuppose that $m(\widehat{BAD}) = \alpha$ is known. Apply the **Mr. Sinus' theorem** :

$$\left\{\begin{array}{lll} \triangle ABD & : & \frac{AB}{\sin(B+\alpha)} = \frac{AD}{\sin B} \\ & & \\ \triangle ACD & : & \frac{CD}{\sin(B+\alpha+C)} = \frac{AD}{\sin C} \end{array}\right\| \implies \sin B \cdot \sin(B+\alpha+C) = \sin C \cdot \sin(B+\alpha) \implies$$

$$\sin B \cdot [\sin(B+\alpha) + \cos(B+\alpha) \cdot \tan C] = \tan C \cdot \sin(B+\alpha) \implies \boxed{\tan C = \frac{\sin B \cdot \sin(B+\alpha)}{\sin(B+\alpha) - \sin B \cdot \cos(B+\alpha)}}.$$

Remark. $C = 45° \iff \tan(B+\alpha) = \frac{\sin B}{1-\sin B}$. In the particular case $\left\{\begin{array}{l} B = 30° \\ \alpha = 15° \end{array}\right\| \implies \tan C = 1$ $\implies C = 45°$.

In this case $A = 105°$, $B = 30°$, $C = 45°$ and $\frac{AB}{2} = \frac{BC}{1+\sqrt{3}} = \frac{CA}{\sqrt{2}}$. □ hình học □ hình □

$$\begin{array}{l} x > y > 0 \\ \qquad \implies \left(x^b - y^b\right)^a < (x^a - y^a)^b . \\ a > b > 0 \end{array}$$

Solution

Consider the function $f \ : \ (0, \infty) \to \mathbb{R}$, where $f(t) = (p^t - 1)^{\frac{1}{t}}$ and $p > 1$.

Define x .s.s. $y \iff x = y = 0$ or $xy > 0$. Prove easily that the function f is increasing.

Indeed, $f'(x)$ .s.s. $\left[\frac{tp^t \ln p}{p^t - 1} - \ln(p^t - 1)\right]$ .s.s. $[p^t \ln p^t - (p^t - 1)\ln(p^t - 1)] > 0$.

Therefore, for $p := \frac{x}{y} > 1$ obtain $f(b) < f(a) \implies \left[\left(\frac{x}{y}\right)^b - 1\right]^{\frac{1}{b}} < \left[\left(\frac{x}{y}\right)^a - 1\right]^{\frac{1}{a}} \implies$

$\left(x^b - y^b\right)^{\frac{1}{b}} < (x^a - y^a)^{\frac{1}{a}} \implies \boxed{\left(x^b - y^b\right)^a < (x^a - y^a)^b}$. Another way $(x^k - y^k)^n < (x^n - y^n)^k$

$\iff n\ln(x^k - y^k) < k\ln(x^n - y^n)$

$\iff n\ln x^k + n\ln(1 - (y/x)^k) < k\ln x^n + k\ln(1 - (y/x)^n)$

$\iff n\ln(1 - (y/x)^k) < k\ln(1 - (y/x)^n)$

$\iff (n-k)\ln(1 - (y/x)^k) < k[\ln(1 - (y/x)^n) - \ln(1 - (y/x)^k)]$

Now LHS<0 because $n - k > 0$ and the thing inside Ln is less than 1.

Also 0<RHS since $1 - (y/x)^n > 1 - (y/x)^k$ and Ln is an increasing function. So the last inequality holds hence so is the first.

□ Circles k_1 and k_2 intersect in the points A and B . Let $C \in k_1$ and $D \in k_2$ be two points for which the line CB intersects again the circle k_2 at E and the line DB intersects again the circle k_1 at F . Prove that $\frac{CE}{DF} = \frac{\sin \widehat{ABC}}{\sin \widehat{ABD}}$.

Solution

Denote the distance $\delta_{XY}(A)$ of the point A to the line XY . Observe that

$\triangle ACF \sim \triangle AED \implies \triangle ACE \sim \triangle AFD \implies \frac{CE}{FD} = \frac{\delta_{BC}(A)}{\delta_{BD}(A)} \implies \frac{CE}{DF} = \frac{\sin \widehat{ABC}}{\sin \widehat{ABD}}$.

Remark. Denote the point $G \in BF \cap ED$. Then $G \in AB \iff$ The quadrilateral $CDFE$ is cyclically.

□ Ascertain as **more simply** as possible $\int \frac{\sin 2x}{\sin x + \cos x}$ dx .

Solution

A proof for fun.We'lluse the easy identities $\boxed{\begin{array}{c} (\sin x + \cos x)^2 = 1 + \sin 2x \\ \\ (\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2 \end{array}}$

and the well-known formula $\boxed{\int \frac{1}{x^2-a^2}\ \mathrm{dx} = \frac{1}{2a}\cdot \ln\left|\frac{x-a}{x+a}\right| + \mathcal{C}\ ,\ a>0}$. Thus,
$\frac{\sin 2x}{\sin x+\cos x} = \frac{(\sin x+\cos x)^2-1}{\sin x+\cos x} = \sin x+\cos x - \frac{1}{\sin x+\cos x} =$
$\sin x+\cos x - \frac{\sin x+\cos x}{(\sin x+\cos x)^2} = \sin x+\cos x + \frac{(\sin x-\cos x)'}{(\sin x-\cos x)^2-2}$.
Therefore, $\int \frac{\sin 2x}{\sin x+\cos x}\ \mathrm{dx} = \int \left[\sin x+\cos x+\frac{(\sin x-\cos x)'}{(\sin x-\cos x)^2-2}\right]\mathrm{dx}$.

$$\boxed{\int \frac{\sin 2x}{\sin x+\cos x}\ \mathrm{dx} = \sin x-\cos x+\frac{1}{2\sqrt{2}}\cdot\ln\frac{\sqrt{2}+\cos x-\sin x}{\sqrt{2}+\sin x-\cos x}+\mathcal{C}}$$

without $|\bullet|$ because $\pm(\sin x-\cos x)\le|\sin x-\cos x|\le\sqrt{2}$.

For example, $\boxed{\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sin x+\cos x}\ \mathrm{dx} = 2+\frac{\sqrt{2}}{2}\cdot\ln\left(3-2\sqrt{2}\right)}$.

□ Does $(\exists)\ f:\mathcal{R}\to\mathcal{R}$ so that $f(1)=2$ and $(\forall)\ x\in\mathcal{R}$, $f(f(x))=x^2-2x+2$?

Solution

No. Put x=1 in the equation given to get $f(2)=1$. Since $f(1)=2$, by continuity there exists $1<y<2$ with $f(y)=y$. So $f(f(y))=y^2-2y+2$ implying $(y-1)(y-2)=0$. Contradiction.

□ Ascertain $\int_0^{\frac{n\pi}{4}} \frac{|\sin 2x|}{|\sin x|+|\cos x|}\ \mathrm{dx}$, where $n\in\mathbb{N}^*$.

Solution

Let $I_n=\int_0^{\frac{n\pi}{4}}\frac{|\sin 2x|}{|\sin x|+|\cos x|}dx$,

$I_{4(k+1)}-I_{4k}=\int_{k\pi}^{(k+1)\pi}\frac{|\sin 2x|}{|\sin x|+|\cos x|}\ dx$
$=\int_0^{\pi}\left(|\sin x|+|\cos x|-\frac{1}{|\sin x|+|\cos x|}\right)dx$
$=4-\int_0^{\pi}\frac{1}{|\sin x|+|\cos x|}\ dx$
$=4-2\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+\cos x}\ dx$
$=4-2\sqrt{2}\ln(1+\sqrt{2})$.

$\therefore I_{4k}=I_0+k(4-2\sqrt{2}\ln(1+\sqrt{2}))$, $I_0=0$, yielding $\boxed{I_n=\frac{1}{4}n(4-2\sqrt{2}\ln(1+\sqrt{2}))}$

□ hình □ hình Let ABC be a triangle with the circumcircle $w=C(O,R)$. The A-symmedian cut BC in D and meet again w in E. Denote the midpoint M of $[BC]$, $T\in AA\cap BC$, $L\in BB\cap CC$ and $AM=m_a$, $AD=s_a$. Prove that the following relations:

1 ▶ $s_a=\frac{2bc}{b^2+c^2}\cdot m_a$

2 ▶ $AE\cdot m_a=bc$ and $TA^2=\frac{abc}{b^2+c^2}$, $T\in EE$.

3 ▶ $\frac{EB}{c}=\frac{EC}{b}=\frac{a}{2m_a}$ (the quadrilateral $ABEC$ is *harmonically*) and $L\in\overline{ADE}$.

4 ▶ $\boxed{h_a+\left(m_a+\frac{a}{2}\right)\sin B\le\frac{3\sqrt{3}}{2}m_a}$. ***Particular case.*** If $A=90^\circ$, then $h_a+\max\{b,c\}\le\frac{3\sqrt{3}}{4}a$.

Remark. I used the notation XX - *the tangent line to the circle w at the point X* . □ hình

□ Let a B-rightangled $\triangle ABC$ with $AB=3$ and $BC=4$. For a mobile point $M\in[AC]$ define the point $P\in[BC]$ so that $m\left(\widehat{BMP}\right)=120^\circ$. Finf the range of BP .

Proof. Denote $BM=r$ and $m\left(\widehat{MBP}\right)=x$. Prove easily that $\frac{r\cos x}{4}+\frac{r\sin x}{3}=1\iff$ $\boxed{r=\frac{12}{3\cos x+4\sin x}}$ $(*)$.

Apply the *theorem of* ***sines*** in $\triangle BMP$: $\frac{BP}{\sin 120^\circ}=\frac{BM}{\sin(120^\circ+x)}\iff BP=\frac{r\sqrt{3}}{2\cos(30^\circ+x)}\overset{(*)}{\iff}$

$\boxed{BP=\frac{6\sqrt{3}}{\cos(30^\circ+x)(3\cos x+4\sin x)}}$ (1) . Thus, $\cos(30^\circ+x)(3\cos x+4\sin x)=\frac{3}{2}\left[\cos(2x+30^\circ)+\cos 3\right.$

$2\left[\sin(2x+30^\circ)-\sin 30^\circ\right] = \left(\frac{3\sqrt{3}}{4}-1\right)+\frac{1}{2}\left[3\cos(2x+30^\circ)+4\sin(2x+30^\circ)\right] \le \left(\frac{3\sqrt{3}}{4}-1\right)+\frac{5}{2} \Longrightarrow$

$\boxed{\cos(30^\circ+x)(3\cos x+4\sin x)\le \frac{3\sqrt{3}+6}{4}}$. Therefore, $BP \overset{(1)}{\ge} \frac{6\sqrt{3}}{\frac{3\sqrt{3}+6}{4}} = 8\left(2\sqrt{3}-3\right) \Longrightarrow 8\left(2\sqrt{3}-3\right)\le BP$

with equality iff $\tan(2x+30^\circ)=\frac{4}{3} \iff x_m=\frac{1}{2}\left(\arctan\frac{4}{3}-30^\circ\right)$. In conclusion, $\boxed{8\left(2\sqrt{3}-3\right)\le BP\le 4}$.

Remark. I used only the well-known inequality $|a\sin x+b\cos x|\le\sqrt{a^2+b^2}$ with equality iff $\tan x=\frac{a}{b}$.

An easy extension. Let a B-rightangled $\triangle ABC$ with $AB=a$ and $BC=b$. For a mobile point $M\in[AC]$ define the point $P\in[BC]$ so that $m\left(\widehat{BMP}\right)=\phi$, so that $\boxed{\frac{\pi}{2}\le\phi\le\pi-A}$ $(*)$. Find the range of BP .

Proof. Denote $m\left(\widehat{MBC}\right)=x$ and $AC=c\sqrt{a^2+b^2}$. Apply an well-known relation for $\triangle PMC$: $\frac{BP}{BC}=\frac{MP}{MC}\cdot\frac{\sin\widehat{BMP}}{\sin\widehat{BMC}} \iff \frac{BP}{b}=\frac{\sin\widehat{MCP}}{\sin\widehat{MPC}}\cdot\frac{\sin\phi}{\sin(C+x)} \iff BP=b\cdot\frac{\frac{a}{c}}{\sin(\phi+x)}\cdot\frac{\sin\phi}{\frac{a}{c}\cos x+\frac{b}{c}\sin x}=$ $\frac{ab\sin\phi}{(a\cos x+b\sin x)\sin(\phi+x)}=\frac{2ab\sin\phi}{b[\cos\phi-\cos(2x+\phi)]+a[\sin(2x+\phi)+\sin\phi]} \Longrightarrow$

$\boxed{BP=\frac{2ab\sin\phi}{(b\cos\phi+a\sin\phi)+[a\sin(2x+\phi)-b\cos(2x+\phi)]}}$. In conclusion, $\boxed{\frac{2ab\sin\phi}{(b\cos\phi+a\sin\phi)+c}\le BP\le}$

because $|a\sin(2x+\phi)-b\cos(2x+\phi)|\le\sqrt{a^2+b^2}=c$.

An interesting particular case. If $\phi=\frac{\pi}{2}$, then $BP\ge\frac{2ab}{a+c}$. If and $a+c=2b$, then $BP\ge a$.

Remark. See now why must the condition $(*)$. Indeed, $a\sin\phi+b\cos\phi\ge 0 \iff a\tan\phi+b\le 0 \iff \tan\phi\le-\frac{b}{a} \iff \tan(\pi-\phi)\ge\frac{b}{a}=\tan A \iff \pi-\phi\ge A \iff \phi\le\pi-A$.

□ ***Lemma.***Let $A(a)$, $X(x)$, $Y(y)$ be three points so that $A\notin XY$.

Choose $\phi\in[0,\pi)$. Then $\boxed{m(\widehat{XAY})=\phi \iff \frac{x-a}{y-a}\in\left\{\rho\omega\ ,\ \frac{\rho}{\omega}\right\}}$,

where $\rho=\frac{|x-a|}{|y-a|}$ and $\omega=\cos\phi+i\cdot\sin\phi$. ***Example.*** The triangle ABC is equilateral $\iff$

$$\left\|\begin{array}{c} AB=AC \\ \\ m(\widehat{BAC})=\frac{\pi}{3}\end{array}\right\| \iff$$

$$\frac{b-a}{c-a}\in\left\{\omega\ ,\ \frac{1}{\omega}\right\}\text{ , where }\left\|\begin{array}{c}\omega^3=-1 \\ \\ w^2+1=\omega \\ \\ \overline{\omega}=\frac{1}{\omega}=-\omega^2\end{array}\right\| \iff$$

$[(b-a)-\omega(c-a)][(c-a)-\omega(b-a)]=0 \iff$

$(b-a)(c-a)\left(\omega^2+1\right)=\omega\cdot\left[(b-a)^2+(c-a)^2\right] \iff$

$\boxed{a^2+b^2+c^2=ab+bc+ca}$ because $\omega^2+1=\omega\ne 0$.

Remark. We can choose A as the origin of the complex plane, i.e. $a=0$. In this case $b^2+c^2=bc \iff \omega\cdot\left(b^2+c^2\right)=\left(\omega^2+1\right)\cdot bc \iff (b-\omega\cdot c)(c-\omega\cdot b)=0$. □ Let ABC be an equilateral triangle, and let P and Q be the midpoints of sides AB and AC respectively. Let D be a **mobile** point on PQ. Extend the lines CD and BD so that they meet AB and AC at E and F respectively. **Ascertain the position of the point D so that the product $EB\cdot FC$ is minimum.**

Solution

Denote $\begin{cases} X \in CE & , & AX \parallel BC \\ Y \in BF & , & AY \parallel BC \end{cases}$. Observe that $\frac{EA}{EB} + \frac{FA}{FC} = \frac{XY}{BC}$ and $\frac{XY}{BC} = \frac{PQ}{BC-PQ}$.

Thus, $\frac{EA}{EB} + \frac{FA}{FC} = 1$, i.e. $\frac{AB}{EB} + \frac{AC}{FC} = 3$. In conclusion, $\frac{1}{BE} + \frac{1}{CF} = \frac{3}{BC}$.

Proof II (of my student). Denote $AB = a$. Apply the **Menelaus' theorem** to the transversals :

▶ $\overline{AFC}$ in $\triangle BDE$: $\frac{AE}{AB} \cdot \frac{FB}{FD} \cdot \frac{CD}{CE} = 1 \Longrightarrow \frac{AE}{AB} \cdot \frac{BC}{DQ} \cdot \frac{BP}{BE} = 1 \Longrightarrow$

$\frac{AE}{BE} = \frac{2 \cdot DQ}{a} \Longrightarrow \frac{a-BE}{BE} = \frac{2 \cdot DQ}{a} \Longrightarrow \boxed{\frac{1}{BE} = \frac{1}{a} + \frac{2 \cdot DQ}{a^2}}$.

▶ $\overline{AEB}$ in $\triangle CDF$: $\frac{AF}{AC} \cdot \frac{EC}{ED} \cdot \frac{BD}{BF} = 1 \Longrightarrow \frac{AF}{AC} \cdot \frac{BC}{DP} \cdot \frac{CQ}{CF} = 1 \Longrightarrow$

$\frac{AF}{CF} = \frac{2 \cdot DP}{a} \Longrightarrow \frac{a-CF}{CF} = \frac{2 \cdot DP}{a} \Longrightarrow \boxed{\frac{1}{CF} = \frac{1}{a} + \frac{2 \cdot DP}{a^2}}$.

In conclusion, $\frac{1}{BE} + \frac{1}{CF} = \frac{2}{a} + \frac{2 \cdot PQ}{a^2}$, i.e. $\boxed{\frac{1}{BE} + \frac{1}{CF} = \frac{3}{a}}$ (constant).

Thus, the product $BE \cdot CF$ is minimum $\Longleftrightarrow$ the product $\frac{1}{BE} \cdot \frac{1}{CF}$ is maximum $\Longleftrightarrow$ $\frac{1}{BE} = \frac{1}{CF}$, i.e. $BE = CF = \frac{2a}{3}$, what means the point D is the middle of the segment $[PQ]$.

Let ABC be a triangle and let $P \in (AB)$, $Q \in (AC)$ be two points so that $PQ \parallel BC$.

For a mobile point $D \in (PQ)$ denote $\begin{cases} E \in CD \cap AB \\ F \in BD \cap AC \end{cases}$. Prove that the sum $\frac{AC}{CF} + \frac{AB}{BE}$ is constant

and ascertain the position of the point D for which the area $[AEF]$ is maximum. ▭ hình ▭ Let ABC be a nonisosceles triangle with centroid G and incircle $C(I, r)$. Denote the midpoint M of the side $[BC]$

and the point $P \in (AB)$ which has the distance $2r$ to the line BC . The A - exincircle touches the side $[BC]$

in the point D and the sideline AB in the point T . Prove that $IG \perp AB \iff DA \perp DT \iff P \in MI$.

Solution

Denote the diameter XY of the incircle, where $X \in (BC)$. The relations $Y \in (AD)$, $MX = MD = \frac{1}{2} \cdot |b - c|$

and $MI \parallel \overline{AYD}$ are well-known. Observe that $\frac{PY}{BD} = \frac{h_a - 2r}{h_a} = \frac{ah_a - 2ar}{ah_a} = \frac{2pr - 2ar}{2pr} \Longrightarrow \boxed{PY = \frac{(p-a)(p-c)}{p}}$

.

▶ $\boxed{IG \perp AB} \iff IA^2 - IB^2 = GA^2 - GB^2 \iff \frac{bc(p-a)}{p} - \frac{ac(p-b)}{p} = \frac{4}{9} \cdot (m_a^2 - m_b^2) \iff$

$9c(b-a) = [2(b^2 + c^2) - a^2] - [2(a^2 + c^2) - b^2] \iff 9c(b-a) = 3(b^2 - a^2) \iff \boxed{a + b = 3c}$.

▶ $\boxed{P \in MI} \iff PY = MD \iff 2(p-a)(p-c) = p(b-c) \iff \boxed{a + b = 3c}$. Othewise (without calculus PY) .

$\boxed{P \in MI} \iff \frac{BP}{BA} = \frac{BM}{BD} \iff \frac{2r}{h_a} = \frac{a}{2(p-c)} \iff \frac{a}{p} = \frac{a}{2(p-c)} \iff p = 2c \iff \boxed{a + b = 3c}$.

▶ Since $BD = BT$ obtain $\boxed{DA \perp DT} \iff BD = BA \iff p - c = c \iff \boxed{a + b = 3c}$.

Remark. In the right trapezoid $BPYX$, $YX \perp BX$ the incircle with the diameter $[XY]$ is tangent to the side $[BP]$. Thus,

$IB \perp IP \iff IC^2 = CP \cdot CB \iff IC^2 = PY \cdot BX \iff PY = \frac{r^2}{p-b} \iff \boxed{PY = \frac{(p-a)(p-c)}{p}}$

.

▭ Let $ABCD$ be a parallelogram. Construct the isosceles triangles ABE , CBF , where $AB = AE$, $CB = CF$ so that

$\widehat{BAE} \equiv \widehat{BCF}$, the line AB doesn't separate the points E , C and the line BC separates the points F , A . Prove that :

1 ▶ $EF = 2 \cdot AC \cdot \sin\frac{\alpha}{2}$ and the value of the acute angle between the lines EF , AC is $\frac{\pi-\alpha}{2}$.

2 ▶ The points E , F , D are collinearly if and only if the quadrilateral $ABCD$ is a rhombus.

Solution

Here is a proof with the complex numbers. ***Proof.*** Denote $X(x)$ - the point X with the affix x and $\omega = \cos\alpha + i \cdot \sin\alpha$, where $m(\widehat{BAE}) = m(\widehat{BCF}) = \alpha$.

Observe that $\omega \cdot \overline{\omega} = 1$, $a + c = b + d$, i.e. $d = a + c - b$ and $\begin{cases} e = a + \omega(b-a) \\ f = c + \omega(b-c) \end{cases}$.

1 ▶ Thus, $e - f = (a-c)(1-\omega)$ and $1 - \omega = 2 \cdot \sin\frac{\alpha}{2}\left[\cos\frac{\pi+\alpha}{2} + i \cdot \sin\frac{\pi+\alpha}{2}\right]$, i.e. $EF = 2 \cdot AC \cdot \sin\frac{\alpha}{2}$ and he value of the acute angle between the lines EF , AC is $\pi - \frac{\pi+\alpha}{2} = \frac{\pi-\alpha}{2}$.

2 ▶ $D \in EF \iff r \equiv \frac{e-d}{f-d} \in \mathbb{R} \iff \frac{(b-c)+\omega\cdot(b-a)}{(b-a)+\omega\cdot(b-c)} \in \mathbb{R} \iff [(b-c) + \omega \cdot (b-a)] \cdot \left[\overline{b-a} + \overline{\omega} \cdot \overline{b-c}\right] \in \mathbb{R}$.

Observe that $(b-c) \cdot \overline{b-a} + (b-a) \cdot \overline{b-c} \in \mathbb{R}$. Therefore, $r \in \mathbb{R} \iff |b-a|^2 \cdot \omega + |b-c|^2 \cdot \overline{\omega} \in \mathbb{R} \iff$

$|b-a| = |b-c| \iff AB = BC$, i.e. $ABCD$ is rhombus.[/hide]

□ $\left\{\begin{array}{c} \triangle ABC\ ,\ L \in (BC) \\ \\ B \in (AM)\ ,\ m(\widehat{ALC}) = 2 \cdot m(\widehat{AMC}) \\ \\ C \in (AN)\ ,\ m(\widehat{ALB}) = 2 \cdot m(\widehat{ANB}) \end{array}\right\| \implies OL \perp BC$, where the point O is the circumcenter of $\triangle MAN$.

Solution

Denote the second intersections E , F of the line BC with the circumcircle $C(O)$ of $\triangle MAN$ and the its diameter AA'.

Therefore, $\left\{\begin{array}{ccc} \widehat{EAA'} \equiv \widehat{AMF} & \implies & AE \parallel FA' \\ \\ \widehat{FAA'} \equiv \widehat{ANE} & \implies & AF \parallel EA' \end{array}\right\| \implies$ the quadrilateral $AEA'F$ is a rectangle

$\implies \begin{cases} O \in (EF)\ ,\ OE = OF \\ \\ \widehat{AOE} \equiv \widehat{ALB} \end{cases}$.

Apply the **Pascal's theorem** to the cyclic hexagon $AAMFEN$: $\left\{\begin{array}{c} X \in AA \cap FE \\ \\ B \in AM \cap EN \\ \\ C \in MF \cap NA \end{array}\right\| \implies$

$X \in BC \implies$

$\left\{\begin{array}{c} O \in \overline{XEF} \\ \\ L \in \overline{XBC} \end{array}\right\| \implies \widehat{AOX} \equiv \widehat{ALX} \implies$ the quadrilateral $AOLX$ is cyclically $\implies OL \perp BC$.

□ Let $ABCD$ be a parallelogram with the area $[ABCD] = 1$. Denote the midpoint M of $[BC]$ and $Q \in AM \cap BD$. Find the area $[QMCD]$.

Solution

Proof 1. $[QMCD] = \frac{[QMCD]}{[ABCD]} = \frac{[BCD]-[MBQ]}{[BCD]} \cdot \frac{[BCD]}{[ABCD]} =$

$\left(1 - \frac{BM}{BC} \cdot \frac{BQ}{BD}\right) \cdot \frac{1}{2} = \frac{1}{2} \cdot \left(1 - \frac{1}{2} \cdot \frac{1}{3}\right) \implies \boxed{[QMCD] = \frac{5}{12}}$.

Proof 2. Denote $[BQM] = a$, $[AQD] = b$, $[ABQ] = [DMQ] = x$, Observe that $[MCD] = \frac{1}{4}$, $x^2 = ab$, $b = 4a \implies x = 2a$, $b = 4a$. Thus, $a + b + 2x = \frac{3}{4} \implies a + 4a + 4a = \frac{3}{4} \implies$ $a = \frac{1}{12} \implies [QMCD] = x + \frac{1}{4} = 2a + \frac{1}{4} = \frac{1}{6} + \frac{1}{4} \implies \boxed{[QMCD] = \frac{5}{12}}$

Hoàn chỉnh

In a triangle ABC prove that a^3
$cos(B - C) + b^3$
$cos(C - A) + c^3$
$cos(A - B) = 3abc$ – Show that $7^{2010} - 2^{2010}$ is a multiple of $3^3 \cdot 5^2 \cdot 11 \cdot 13 \cdot 31 \cdot 61 \cdot 67$. – Given that $f(x) = \cos x and g(x) = \sin x$. Find the value of x if $f[f[f[f(x)]]] = g[g[g[g(x)]]]$ – Solve the equation $\frac{11}{5}x - \sqrt{2x+1} = 3y - \sqrt{4y-1} + 2, \quad (x \geq 0; y \geq 1; x = 5k; x, y, k \in Z)$ – Hoàn chỉnh

□ Let s_1 be any positive integer.Define s_n to be so that $\sum_{k=1}^{n} s_k \equiv 0(\bmod n)$ with $0 \leq s_n \leq n-1$. Show $\exists N \in \mathbb{N}$ so that $\forall p, q \geq N$ we have $s_p = s_q$.

Solution

We have, for $n \geq 2$, $\sum_{k=1}^{n} s_k \leq s_1 + \sum_{k=2}^{n}(k-1) = s_1 + \frac{n(n-1)}{2}$ So, whatever could be s_1, it exists $m \in \mathbb{N}$ such that $s_1 + \frac{m(m-1)}{2} < m^2$ and then $\sum_{k=1}^{m} s_k \leq m^2$ Let then $u = \frac{1}{m}\sum_{k=1}^{m} s_k$. u is an integer (since $\sum_{k=1}^{m} s_k = 0 \pmod{m}$) and $u < m$

Then $s_{m+1} = -\sum_{k=1}^{m} s_k =_m u = u \pmod{m+1}$ and, since $0 \leq u < m < m+1$, we have $s_{m+1} = u$ And $\sum_{k=1}^{m+1} s_k = mu + u = (m+1)u$

And it is obvious (by induction for example) that $s_p = u \ \forall p \geq m$ (and $\sum_{k=1}^{p} s_k = \sum_{k=1}^{m} s_k + \sum_{k=m+1}^{p} s_k = mu + (p-m)u = pu = 0 \pmod{p}$)

□ If $a + b + c = 1, a, b, c > 0$, prove that $\frac{1+c(27ab-1)}{a+b} \leq 1 + (a+b)^2 + 3c + \frac{c^3}{a+b}$

Solution

Since $a + b = 1 - c$, that inequality is equivalent to $\dfrac{(1-c) + 27abc}{1-c} \leq 1 + (1-c)^2 + 3c + \dfrac{c^3}{1-c}$. Multiplying by $(1-c)$, we get $(1-c) + 27abc \leq (1-c) + (1-c)^3 + 3c(1-c) + c^3$, and combining like terms, we get $27abc \leq 1$. We homogenize this equation to $27abc \leq (a+b+c)^3$, which is equivalent to $\sqrt[3]{abc} \leq \dfrac{a+b+c}{3}$, AM-GM.

□

For a, b, c are all positive real,$a^2 + b^2 + c^2 = 3$, prove that
$a + b + c^2 \leq \frac{7}{2}$

Solution

Rewrite it as:

$$a^2 + b^2 + \frac{1}{2} \geq a + b$$

AM-GM gives:

$$a^2 + b^2 + \frac{1}{2} \geq \sqrt{2(a^2+b^2)} \geq a + b$$

so it's done. Equality occurs only when $a = b = \frac{1}{2}$ and $c = \sqrt{\frac{5}{2}}$.

□

Let $a = e^{(i2\pi)/n}$ Evaluate $\sum_{k=0,n-1}(1+a^k)^n$

Solution

$\sum_{k=0}^{n-1}(1+a^k)^n = \sum_{k=0}^{n-1}\sum_{i=0}^{n}\binom{n}{i}a^{ki} = \sum_{i=0}^{n}\binom{n}{i}\sum_{k=0}^{n-1}a^{ki} = 2n + \sum_{i=1}^{n-1}\binom{n}{i}\frac{a^{ni}-1}{a^i-1} = 2n$ since $a^n = 1$

□

What is the exact value of $\frac{\cos 1^o+\cos 2^o+...+\cos 43^o+\cos 44^o}{\sin 1^o+\sin 2^o+...+\sin 43^o+\sin 44^o}$?

Solution

We have to compute $A = \frac{\sum_{k=1}^{44}\cos(k\frac{\pi}{180})}{\sum_{k=1}^{44}\sin(k\frac{\pi}{180})} = \frac{N}{D}$

$N = \sum_{k=1}^{44}\cos(k\frac{\pi}{180}) = \sum_{k=1}^{22}(\cos(k\frac{\pi}{180})+\cos(\frac{\pi}{4}-k\frac{\pi}{180})) = \sum_{k=1}^{22}2\cos(\frac{\pi}{8})\cos(k\frac{\pi}{180}-\frac{\pi}{8}) = 2\cos(\frac{\pi}{8})\sum_{k=1}^{22}\cos(k\frac{\pi}{180}-\frac{\pi}{8})$

$D = \sum_{k=1}^{44}\sin(k\frac{\pi}{180}) = \sum_{k=1}^{22}(\sin(k\frac{\pi}{180})+\sin(\frac{\pi}{4}-k\frac{\pi}{180})) = \sum_{k=1}^{22}2\sin(\frac{\pi}{8})\cos(k\frac{\pi}{180}-\frac{\pi}{8}) = 2\sin(\frac{\pi}{8})\sum_{k=1}^{22}\cos(k\frac{\pi}{180}-\frac{\pi}{8})$

And so $S = \frac{N}{D} = \cot(\frac{\pi}{8}) = \sqrt{2}+1$

□

Let $f(x) = x^n + ... + x + 1$ and let $g(x) = f(x^{n+1})$. Find the remainder when $g(x)$ is divided by $f(x)$.

Solution

Let $z_k = e^{\frac{2ki\pi}{n+1}}$ for $k \in \{1, 2, \dots, n\}$. We have $z_k^{n+1} = 1$ and $f(z_k) = 0\ \forall k \in \{1, 2, \dots, n\}$

We have $g(x) = f(x)q(x)+r(x)$ with degree of $r(x) < n$. So $g(z_k) = f(z_k^{n+1}) = f(1) = n+1 = r(z_k)$

So $r(x)$ is a polynomial of degree $\leq n-1$ and taking the same value for n different values. So $r(x)$ is a constant.

And $r(x) = n+1$

□

$f(xy) = f(x)f(y)$ has $f(x) = x$ for x in Q, can we easy conclude that in R without give more info in question? Let $P(x, y)$ be the assertion $f(xy) = f(x)f(y)$

If $f(0) \neq 0 : P(x, 0) \implies f(x) = 1\ \forall x$ which indeed is a solution. So let us consider from now that $f(0) = 0$

If $\exists u \neq 0$ such that $f(u) = 0$, then $P(\frac{x}{u}, u) \implies f(x) = 0\ \forall x$ which indeed is a solution So let us consider from now that $f(x) = 0 \iff x = 0$

Let $x > 0 : P(x, x) \implies f(x^2) = f(x)^2 > 0$. Let us then consider $g(x)$ from $\mathbb{R} \to \mathbb{R}$ defined as $g(x) = \ln(f(e^x))$. The functional equation becomes $g(x+y) = g(x)+g(y)$

So it is easy to show that, for any $x > 0 : f(x) = e^{g(\ln x)}$ where $g(x)$ is any solution of Cauchy's equation.

Then $P(1, 1)$ implies $f(1) = 1$ and $P(-1, -1)$ implies $f(-1) = \pm 1$ and $P(x, -1)$ implies $f(-x) = f(-1)f(x)$

Hence the solutions :

S1 : $f(x) = 0\ \forall x$

S2 : $f(x) = 1\ \forall x$

S3 : Let $g(x)$ any solution of additive Cauchy equation

$f(0) = 0$

$\forall x > 0 : f(x) = e^{g(\ln x)}$

$\forall x < 0 : f(x) = e^{g(\ln -x)}$

S4 : Let $g(x)$ any solution of additive Cauchy equation

$f(0) = 0$

$\forall x > 0 : f(x) = e^{g(\ln x)}$

$\forall x < 0 : f(x) = -e^{g(\ln -x)}$

Some remarks : 1) So $f(x) = x$ for $x \in \mathbb{Q}$ is not enough to conclude $f(x) = x\ \forall x \in \mathbb{R}$

2) The general non constant continuous solutions are $|x|^c$ and $sign(x)|x|^c$ with $c > 0$

□

Solve the equation $x^3 - [x] = 3$,where $x \in \mathbb{R}$

Solution

So $x = \sqrt[3]{n+3}$ for any integer n such that $n+1 > \sqrt[3]{n+3} \geq n$

$\iff (n+1)^3 > n+3 \geq n^3$

$n+3 \geq n^3 \implies n \leq 1\ (n+1)^3 - n - 3 > 0 \implies n \geq 1$

So the unique solution $n = 1$ and $x = \sqrt[3]{4}$

□

Let $a_0, a_1, a_2, ..., a_n$ be a sequence of numbers such that $(3 - a_{n+1}) * (6 - a_n)$ =18 and $a_0 =$ 3 then FIND:

$\sum_{i=0}^{n} 1/a_i$

Solution

From the induction formula, we get $\frac{1}{a_{n+1}} - \frac{1}{9} = -2(\frac{1}{a_n} - \frac{1}{9})$ and so $\frac{1}{a_n} = \frac{1}{9} + \frac{2}{9}(-2)^n \implies \sum_{i=0}^{n} \frac{1}{a_i} = \frac{n+1}{9} + \frac{2}{9}\frac{(-2)^{n+1}-1}{-3}$

Hence the result : $\boxed{\sum_{i=0}^{n} \frac{1}{a_i} = \frac{3n+5+(-2)^{n+2}}{27}}$

□

Prove that $\cos(n \arctan 2\sqrt{2}) \in Q$, $n \in N$

Solution

$\cos(n \arctan 2\sqrt{2}) = \text{Re}[(\cos \arctan 2\sqrt{2} + i \sin \arctan 2\sqrt{2})^n]$ by DeMoivre.

Drawing a right triangle with legs 4 and $\sqrt{2}$, we find the hypotenuse to be $3\sqrt{2}$, and therefore $\cos\theta = \frac{1}{3}$ and $\sin\theta = \frac{4}{3\sqrt{2}}$

So we have $\text{Re}\left[\left(\frac{1}{3} + \frac{4i}{3\sqrt{2}}\right)^n\right] = \text{Re}\left[\left(\frac{\sqrt{2}+4i}{3\sqrt{2}}\right)^n\right] = \frac{1}{3^n} \cdot \text{Re}\left[(1+2\sqrt{2}i)^n\right]$.

$(1+2\sqrt{2}i)^n = \sum_{k=0}^{n} \binom{n}{k}(2\sqrt{2}i)^n$. These terms are only real when $n \equiv 0, 2 \bmod 4$. In both cases, n is even, which makes the square root integral, so the real part of this whole thing is some integer j.

Therefore, the answer is $\frac{j}{3^n}$ for some integer j, which is a rational number. □ Another way Let $a_n = \cos n\theta$ where $\theta = \tan^{-1} 2\sqrt{2}$

We have the recurrence $a_{n+2} = 2a_{n+1}a_1 - a_n$

$a_1 = \cos\theta = \frac{1}{3} \in \mathbb{Q}$ and $a_2 = \cos 2\theta = 2\cos^2\theta - 1 = -\frac{7}{9} \in \mathbb{Q}$

The above recurrence tells us that whenever a_n and a_{n+1} are rational so is a_{n+2}

Since a_1 and a_2 are rational, this result is true for all a_n

□

Let f,g: R >R be functions like that so **f(g(x))=g(f(x))=-x** for any x is element of R a) prove that f and g are odd functions b) Make an example of these two functions **f** isn't equal to **g** Please solve it

Solution

a) : $g(f(g(x))) = g(u)$ where $u = f(g(x)) = -x$ and so $g(f(g(x))) = g(-x)$ $g(f(g(x))) = g(f(v)) = -v$ where $v = g(x)$ and so $g(f(g(x))) = -g(x)$ So $g(-x) = -g(x)$ and $g(x)$ is an odd function.

Same computation with $f(g(f(x)))$ shows that $f(x)$ is an odd function.

b) Choose $f(x) = 2x$ and $g(x) = -\frac{x}{2}$

$\square$

Prove that the equation $x^2 - 16nx + 7^5 = 0$ equation $n \in N$ has no integer solutions.

Solution

So $7^5 = x(16n - x)$ and so 12 possibilities :

$x = 1$ and $16n - x = 7^5$ and so $n = \frac{7^5+1}{16} \notin N$

$x = 7$ and $16n - x = 7^4$ and so $n = \frac{7^4+7}{16} \notin N$

$x = 7^2$ and $16n - x = 7^3$ and so $n = \frac{7^3+7^2}{16} \notin N$

$x = 7^3$ and $16n - x = 7^2$ and so $n = \frac{7^2+7^3}{16} \notin N$

$x = 7^4$ and $16n - x = 7^1$ and so $n = \frac{7+7^4}{16} \notin N$

$x = 7^5$ and $16n - x = 1$ and so $n = \frac{1+7^5}{16} \notin N$

$x = -1$ and $16n - x = -7^5$ and so $n = -\frac{7^5+1}{16} \notin N$

$x = -7$ and $16n - x = -7^4$ and so $n = -\frac{7^4+7}{16} \notin N$

$x = -7^2$ and $16n - x = -7^3$ and so $n = -\frac{7^3+7^2}{16} \notin N$

$x = -7^3$ and $16n - x = -7^2$ and so $n = -\frac{7^2+7^3}{16} \notin N$

$x = 7^4$ and $16n - x = -7^1$ and so $n = -\frac{7+7^4}{16} \notin N$

$x = -7^5$ and $16n - x = -1$ and so $n = -\frac{1+7^5}{16} \notin N$

Hence the result

$\square$

Let $a, b, c \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\overline{abc} + \overline{bca} = 10abc$ (and not $\overline{abc}$), find $a + b + c$.

Solution

$100a + 10b + c + 100b + 10c + a = 10abc \iff 101a + 110b + 11c = 10abc$

So $a + c = 0 \pmod{10}$ Hence a first solution $a = c = 0$ and so $b = 0$

If $a \neq 0$, we get $c = 10 - a$ and the equation becomes $9a + 11b + 11 = ab(10 - a)$ or again $b = \frac{9a+11}{-a^2+10a-11}$

In order to have $-a^2 + 10a - 11 > 0$, we need to have $a \in \{2, 3, 4, 5, 6, 7, 8\}$, so just 7 tests in order to find either $a = 5$, either $a = 6$. So three solutions :

$a = 0,\ b = 0,\ c = 0 \implies 000 + 000 = 10 \times 0 \times 0 \times 0$ and $\boxed{a + b + c = 0}$

$a = 5,\ b = 4,\ c = 5 \implies 545 + 455 = 1000 = 10 \times 5 \times 4 \times 5$ and $\boxed{a + b + c = 14}$

$a = 6,\ b = 5,\ c = 4 \implies 654 + 546 = 1200 = 10 \times 6 \times 5 \times 4$ and $\boxed{a + b + c = 15}$

$\square$

After how many terms of the summation $4\sum_{n=0} \frac{(-1)^n}{2n+1}$ will the sum be within .01 of pi?

Solution

We have $\pi = 4\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = 4\sum_{p=0}^{+\infty}(\frac{1}{4p+1} - \frac{1}{4p+3}) = 8\sum_{p=0}^{+\infty} \frac{1}{(4p+1)(4p+3)}$.

Then we have $\int_{k+1}^{+\infty} \frac{8dx}{(4x+1)(4x+3)} < 8\sum_{p=k+1}^{+\infty} \frac{1}{(4p+1)(4p+3)} < \int_{k}^{+\infty} \frac{8dx}{(4x+1)(4x+3)}$

So $\ln(\frac{4k+7}{4k+5}) < 8\sum_{p=k+1}^{+\infty} \frac{1}{(4p+1)(4p+3)} < \ln(\frac{4k+3}{4k+1})$

We want to have $8\sum_{p=k+1}^{+\infty} \frac{1}{(4p+1)(4p+3)}$ about $\frac{1}{100}$, so $\ln(\frac{4k+3}{4k+1})$ about $\frac{1}{100}$.

So $1 + \frac{2}{4k+1}$ about $e^{\frac{1}{100}}$, which is about $1 + \frac{1}{100}$. So k is about 50

More precisely : $0.0097 < \ln(\frac{4\times 50+7}{4\times 50+5}) =< 8\sum_{p=51}^{+\infty} \frac{1}{(4p+1)(4p+3)} < \ln(\frac{4\times 50+3}{4\times 50+1}) < 0.00991$

And so $\pi - 0.00991 < 8\sum_{p=0}^{50} \frac{1}{(4p+1)(4p+3)} < \pi - 0.0097$

And so $\pi - 0.00991 < 4\sum_{p=0}^{50}(\frac{1}{4p+1} - \frac{1}{4p+3}) < \pi - 0.0097$

And so $\pi - 0.00991 < 4\sum_{n=0}^{101} \frac{(-1)^n}{2n+1} < \pi - 0.0097$

And also obviously $\pi - 0.00991 < 4\sum_{n=0}^{N} \frac{(-1)^n}{2n+1} < \pi\ \forall N > 100$

□

Let $0 < p < 1$. Prove that: $\frac{2}{e} \leq p^{\frac{p}{1-p}} + p^{\frac{1}{1-p}}$

Solution

$f(x) = x^{\frac{x}{1-x}} + x^{\frac{1}{1-x}}$ is a strictly decreasing function for $x \in [0, 1)$

Since $\lim_{x\to 1^-} \frac{1}{1-x}\ln(x) = -1$, we have $\lim_{x\to 1^-} x^{\frac{1}{1-x}} = \lim_{x\to 1^-} x^{\frac{x}{1-x}} = e^{-1}$

So $\lim_{x\to 1^-} f(x) = \frac{2}{e}$ and so the result.

□

Let x and y be positive real numbers such that x + 2y = 8. Determine the minimum value of x + y + (3/x) + (9/2y)

Solution

Setting $x = 8 - 2y$, we are looking for the minimum value of $f(y) = 8 - y + \frac{3}{8-2y} + \frac{9}{2y}$ when $y \in (0, 4)$

Just write then $f(y) = 8 + \frac{(y-3)^2(y+2)}{y(4-y)}$ and you get $f(y) \geq f(3) = 8$ $\forall y \in (0, 4)$

Hence the answer : $\boxed{8}$ reached when $(x, y) = (2, 3)$

□

Find all integers n such that $(7n - 12)/2^n + (2n - 14)/3^n + (24n)/6^n = 1$

Solution

If $n < 0$ implies $LHS < 0 < RHS$ and so no solution $n = 0$ is not a solution

If $n > 0$: $(7n - 12)3^n + (2n - 14)2^n + 24n = 6^n$ implies $7n - 12 \equiv 0 \pmod 2$ and $(3n - 14 \equiv 0 \pmod 3$ and so $n = 6p + 4$

For $n \geq 9$: $21n - 36 \leq 2^{n-1}$ and so $(7n - 12)3^n \leq 6^{n-1}$

For $n \geq 1$: $4n - 28 \leq 3^{n-1}$ and so $(2n - 14)2^n \leq 6^{n-1}$

For $n \geq 4$: $24n \leq 6^{n-1}$

So $(7n - 12)3^n + (2n - 14)2^n + 24n \leq 3 \times 6^{n-1} < 6^n$ $\forall n \geq 9$

And the only value possible for $0 < n = 6p + 4 < 9$ is $n = 4$ which indeed is a solution

Hence the answer : $\boxed{n = 4}$

□

Solve the equation: $[\frac{x-3}{2}] = [\frac{x-2}{3}]$.

Solution

Let $n \in \mathbb{Z}$ be the common value. We get : $n \leq \frac{x-3}{2} < n + 1$ and $n \leq \frac{x-2}{3} < n + 1$

$\iff$ $2n + 3 \leq x < 2n + 5$ and $3n + 2 \leq x < 3n + 5$

This implies $3n + 2 < 2n + 5$ and $2n + 3 < 3n + 5$ and so $-2 < n < 3$ and so $n \in \{-1, 0, 1, 2\}$

1) If $n = -1$ then the solution is $1 \leq x < 3$ and $-1 \leq x < 2$ and so $x \in [1, 2)$

2) If $n = 0$ then the solution is $3 \leq x < 5$ and $2 \leq x < 5$ and so $x \in [3, 5)$

3) If $n = 1$ then the solution is $5 \leq x < 7$ and $5 \leq x < 8$ and so $x \in [5, 7)$

4) If $n = 2$ then the solution is $7 \leq x < 9$ and $8 \leq x < 11$ and so $x \in [8, 9)$

Hence the answer : $\boxed{x \in [1, 2) \cup [3, 7) \cup [8, 9)}$

□ Let : $x_1, x_2, ..., x_{2011}$ are positive integer. such that $x_1 + x_2 + ... + x_{2011} = 2011^{2011}$ How many integer solution.

Solution

Let $S(m, n)$ (where $m \geq n$ are two natural numbers) be the number of ordered n-tuples of natural numbers whose sum is m,.

Then obviously $\boxed{S(n, m) = \binom{m-1}{n-1}}$ (just place $n-1$ integer distinct dots in the segment $(0, m)$)

$\square$ From the digits $1, 2, ..., 9$, we write all the numbers formed by these nine digits (the nine digits are all distinct), and we order them in increasing order as follows : 123456789, 123456798, ..., 987654321. What is the $100000th$ number ?

Solution

The first $8! = 40320$ such numbers are all these numbers beginning with 1 The next $8! = 40320$ such numbers are all these numbers beginning with 2

So the 100000^{th} such number is the $100000 - 2 \times 40320 = 19360^{th}$ such number beginning by 3

The first $7! = 5040$ such new numbers are all these numbers beginning with 31 The next $7! = 5040$ such new numbers are all these numbers beginning with 32 The next $7! = 5040$ such new numbers are all these numbers beginning with 34

So the 19360^{th} such new number is the $19360 - 3 \times 5040 = 4240^{th}$ such number beginning by 35

The first $6! = 720$ such new numbers are all these numbers beginning with 351 The next $6! = 720$ such new numbers are all these numbers beginning with 352 The next $6! = 720$ such new numbers are all these numbers beginning with 354 The next $6! = 720$ such new numbers are all these numbers beginning with 356 The next $6! = 720$ such new numbers are all these numbers beginning with 357

So the 4240^{th} such new number is the $4240 - 5 \times 720 = 640^{th}$ such number beginning by 358

The first $5! = 120$ such new numbers are all these numbers beginning with 3581 The next $5! = 120$ such new numbers are all these numbers beginning with 3582 The next $5! = 120$ such new numbers are all these numbers beginning with 3584 The next $5! = 120$ such new numbers are all these numbers beginning with 3586 The next $5! = 120$ such new numbers are all these numbers beginning with 3587

So the 640^{th} such new number is the $640 - 5 \times 120 = 40^{th}$ such number beginning by 3589

The first $4! = 24$ such new numbers are all these numbers beginning with 35891

So the 40^{th} such new number is the $40 - 1 \times 24 = 16^{th}$ such number beginning by 35892

The first $3! = 6$ such new numbers are all these numbers beginning with 358921 The next $3! = 6$ such new numbers are all these numbers beginning with 358924

So the 16^{th} such new number is the $16 - 2 \times 6 = 4^{th}$ such number beginning by 358926

And so $358926 - 147$, $358926 - 174$, $358926 - 417$, $358926 - 471$

Hence the answer : $\boxed{358926471}$

$\square$ Let $f(n)$ denotes the number of positive integral solutions of the equation $4x + 3y + 2z = n$. Find $f(2009) - f(2000)$ (x, y, z) solution of $4x + 3y + 2z = 2000 \implies (x + 1, y + 1, z + 1)$ solution of $4x + 3y + 2z = 2009$

So $f(2009) - f(2000)$ is the number of solutions of $4x + 3y + 2z = 2009$ where at least one of x, y, z is 1

1) $x = 1$: looking for number of solutions of $3y + 2z = 2005$ y must be odd $= 2p + 1$ and so $z = 1001 - 3p$ and so $p \in [0, 333]$ and so 334 such solutions

2) $x > 1$ and $y = 1$: looking for number of solutions of $4x + 2z = 2006 \iff z = 1003 - 2x$ And so $x \in [2, 501]$ and so 500 such solutions

3) $x > 1$ and $y > 1$ and $z = 1$: looking for number of solutions of $4x + 3y = 2007$ SO $y = 4p + 1$ and $x = 501 - 3p$ and so $p \in [1, 166]$ and so 166 such solutions.

$334 + 500 + 166 = 1000$

Hence the result $\boxed{f(2009) - f(2000) = 1000}$ Another solution :

(x, y, z) solution of $4x + 3y + 2z = n \implies (x, y + 1, z)$ solution of $4x + 3y + 2z = n + 3$

So $f(n + 3) - f(n)$ is the number of solutions of $4x + 3y + 2z = n + 3$ where $y = 1$ and so the number of solutions of $4x + 2z = n$ and so :

If n is odd : $f(n+3)-f(n)=0$ If n is even : $f(n+3)-f(n)=\lfloor\frac{n-2}{4}\rfloor$
$f(2009)-f(2006)=\lfloor\frac{2004}{4}\rfloor=501$ $f(2006)-f(2003)=0$ $f(2003)-f(2000)=\lfloor\frac{1998}{4}\rfloor=499$
And so $\boxed{f(2009)-f(2000)=1000}$

☐ $\sqrt{2^{2^{2^{2^2}}}}\geq 1+2^2+3^3+...+2004^{2004}$ or $1+2^2+3^3+...+2004^{2004}\geq\sqrt{2^{2^{2^{2^2}}}}$

Solution

$\sqrt{2^{2^{2^{2^2}}}}=\sqrt{2^{2^{65536}}}=2^{2^{65535}}>10^{\frac{3}{10}10^{\frac{3}{10}65535}}>10^{10^{10000}}$
$1+2^2+3^3+...+2004^{2004}<2004\times 2004^{2004}=2004^{2005}<10^{8010}$
Hence the result $\boxed{\sqrt{2^{2^{2^{2^2}}}}>1+2^2+3^3+...+2004^{2004}}$

☐ Find $f:(1,+\infty)\to R$ satisfy $f(x)-f(y)=(y-x)f(xy)$ for all $x,y>1$

Solution

Let $P(x,y)$ be the assertion $f(x)-f(y)=(y-x)f(xy)$ Let $a>2$ and $x\in(\frac{a^2}{4},4a^2)$
$P(\frac{2}{a}\sqrt{x},\frac{a}{2}\sqrt{x})\implies f(\frac{2}{a}\sqrt{x})-f(\frac{a}{2}\sqrt{x})=(\frac{a}{2}-\frac{2}{a})\sqrt{x}f(x)$
$P(\frac{a}{2}\sqrt{x},\frac{2a}{\sqrt{x}})\implies f(\frac{a}{2}\sqrt{x})-f(\frac{2a}{\sqrt{x}})=(\frac{2a}{x}-\frac{a}{2})\sqrt{x}f(a^2)$
$P(\frac{2a}{\sqrt{x}},\frac{2}{a}\sqrt{x})\implies f(\frac{2a}{\sqrt{x}})-f(\frac{2}{a}\sqrt{x})=(\frac{2}{a}-\frac{2a}{x})\sqrt{x}f(4)$
Adding these three lines, we get $(\frac{a}{2}-\frac{2}{a})f(x)+(\frac{2a}{x}-\frac{a}{2})f(a^2)+(\frac{2}{a}-\frac{2a}{x})f(4)=0$
And so $f(x)=\frac{\alpha(a)}{x}+\beta(a)$ $\forall x\in(\frac{a^2}{4},4a^2)$
So $\alpha(a)$ and $\beta(a)$ are constant and $f(x)=\frac{\alpha}{x}+\beta$
Plugging this in original equation, we get $\beta=0$ and so $\boxed{f(x)=\frac{\alpha}{x}}$

☐ A sequence is defined by $u_1=8$ and $u_{n+1}=u_n+3$.
Find the value of N such that $\sum_{n=1}^{2N}u_n-\sum_{n=1}^{N}u_n=1256$.

Solution

From the recursive equation $u_n=u_{n-1}+3$, we can find the closed equation $u_n=8+3(n-1)$. So,
$\sum_{n=1}^{2N}u_n-\sum_{n=1}^{N}u_n=(8+11+...+u_{2N})-(8+11+...+u_N)$
$=u_{N+1}+u_{N+2}+...+u_{2N}$
$=(8+3N)+(8+3(N+1))+...+(8+3(2N-1))$
$=8N+3\frac{N(3N-1)}{2}=1256$
$16N+9N^2-3N=2512$
$9N^2+13N-2512=0$
$(N-16)(9N+157)=0$
$\therefore\boxed{N=16}$

☐ Find a formula counting the number of all 2013-digits natural numbers which are multiple of 3 and all digits are taken from the se $X=3;5;7;9$

Solution

So we have exactly 3n digits ($n\in[0,671]$) in $\{5,7\}$ and so $\sum_{n=0}^{671}\binom{2013}{3n}2^{3n}2^{2013-3n}=2^{2013}\sum_{n=0}^{671}\binom{2013}{3n}$
$=2^{2013}\frac{2^{2013}+(1+j)^{2013}+(1+j^2)^{2013}}{3}$ where $j=e^{\frac{2i\pi}{3}}$
Hence the result : $\boxed{\frac{2^{2013}\left(2^{2013}-2\right)}{3}}$

☐ Solve the recurrence relation $a_n-2a_{n-1}+a_{n-2}=\binom{n+4}{4}$ for $n\geq 2$, $a_0=0$ and $a_1=5$.

Solution

Let $b_n=a_{n+1}-a_n$ and the relation is $b_{n+1}-b_n=\binom{n+6}{4}$ with $b_0=5$
So $b_n=5+\sum_{k=0}^{n-1}\binom{k+6}{4}=5+\sum_{k=1}^{n}\binom{k+5}{4}=5+\sum_{k=1}^{n}\frac{k^4+14k^3+71k^2+154k+120}{24}$
$b_n=5+\frac{1}{24}\sum_{k=1}^{n}k^4+\frac{7}{12}\sum_{k=1}^{n}k^3+\frac{71}{24}\sum_{k=1}^{n}k^2+\frac{77}{12}\sum_{k=1}^{n}k+5\sum_{k=1}^{n}1$

$b_n = \frac{1}{24}\frac{n(n+1)(6n^3+9n^2+n-1)}{30} + \frac{7}{12}\frac{n^2(n+1)^2}{4} + \frac{71}{24}\frac{n(n+1)(2n+1)}{6} + \frac{77}{12}\frac{n(n+1)}{2} + 5n + 5$
$b_n = \frac{n^5+20n^4+155n^3+580n^2+1044n+600}{120}$
Then $a_n = a_0 + \sum_{k=0}^{n-1} b_k = \sum_{k=0}^{n-1} \frac{k^5+20k^4+155k^3+580k^2+1044k+600}{120}$
$a_n = \frac{1}{120}\frac{n^2(n-1)^2(2n^2-2n-1)}{12} + \frac{1}{6}\frac{n(n-1)(6n^3-9n^2+n+1)}{30} + \frac{31}{24}\frac{n^2(n-1)^2}{4} + \frac{29}{6}\frac{n(n-1)(2n-1)}{6} + \frac{87}{10}\frac{n(n-1)}{2} + 5n$

$$\boxed{a_n = \frac{n^6 + 21n^5 + 175n^4 + 735n^3 + 1624n^2 + 1044n}{720}}$$

☐ Show that if there is no positive integers n such that $n^2 + n + 2010$ is a perfect square and n does not equal to 2009

Solution

$n^2 + n + 2010 = u^2$ with $u > n \geq 0 \iff (2u + 2n + 1)(2u - 2n - 1) = 8039$ with $2u + 2n + 1 > 2u - 2n - 1 \geq 1$

So, since 8039 is prime, $2u + 2n + 1 = 8039$ and $2u - 2n - 1 = 1$ and so the unique solution $(n, u) = (2009, 2010)$

Q.E.D.

☐ Solve equation $[x^2 + 1] = [2x]$.

Solution

Let $x^2 = m + y$ with $m \geq 0$ integer and $y \in [0, 1)$

Obviously, $x > 0$ and the equation becomes $m + 1 = \lfloor 2\sqrt{m+y} \rfloor$ and so $m + 2 > \lfloor 2\sqrt{m+y} \rfloor \geq m + 1 \geq 1 \iff$ (squaring) $m^2 + 4 > 4y \geq m^2 - 2m + 1$ So we need $m^2 - 2m + 1 < 4$ and so $m \in \{0, 1, 2\}$ $m = 0$ gives the solutions $x^2 \in [\frac{1}{4}, 1)$ and so $x \in [\frac{1}{2}, 1)$ $m = 1$ gives the solutions $x^2 \in [1, 2)$ and so $x \in [1, \sqrt{2})$ $m = 2$ gives the solutions $x^2 \in [\frac{9}{4}, 3)$ and so $x \in [\frac{3}{2}, \sqrt{3})$

Hence the answer : $\boxed{x \in [\frac{1}{2}, \sqrt{2}) \cup [\frac{3}{2}, \sqrt{3})}$

Another appraoch: The solution lies in the range $x \in [0,\ 2]$ since $x^2 + 1 > 2x + 1$ when $x < 0$ or $x > 2$. We split x into sections:

[list][*]$0 \leqslant x < \frac{1}{2}: \quad \lfloor x^2+1 \rfloor = 1,\ \lfloor 2x \rfloor = 0$ [*]$\frac{1}{2} \leqslant x < 1: \quad \lfloor x^2+1 \rfloor = 1,\ \lfloor 2x \rfloor = 1$ [*]$1 \leqslant x < \sqrt{2}: \quad \lfloor x^2+1 \rfloor = 2,\ \lfloor 2x \rfloor = 2$ [*]$\sqrt{2} \leqslant x < \frac{3}{2}: \quad \lfloor x^2+1 \rfloor = 3,\ \lfloor 2x \rfloor = 2$ [*]$\frac{3}{2} \leqslant x < \sqrt{3}: \quad \lfloor x^2+1 \rfloor = 3,\ \lfloor 2x \rfloor = 3$ [*]$\sqrt{3} \leqslant x < 2: \quad \lfloor x^2+1 \rfloor = 4,\ \lfloor 2x \rfloor = 3$ [*]$x = 2: \quad \lfloor x^2+1 \rfloor = 5,\ \lfloor 2x \rfloor = 4$[/list]

Hence the solution is $\frac{1}{2} \leqslant x < \sqrt{2}$ or $\frac{3}{2} \leqslant x < \sqrt{3}$.

Another way: $\lfloor x^2 + 1 \rfloor = \lfloor 2x \rfloor = z \in \mathbb{Z} \iff 0 \leq z \leq 2x \leq x^2 + 1 < z + 1$ and $(x^2 + 1) - 2x \leq 1$, i.e. $x \in [0, 2]$.

Thus, $\max\left\{\frac{(z-1)+1}{2}, \sqrt{z-1}\right\} = \boxed{\frac{z}{2} \leq x < \sqrt{z}} = \min\left\{\frac{z+1}{2}, \sqrt{z}\right\}$, where $\frac{z}{2} < \sqrt{z} \iff z \in \overline{1,3}$.

In conclusion, $x \in \bigcup_{z\in\overline{1,3}} \left[\frac{z}{2}, \sqrt{z}\right) = \left[\frac{1}{2}, 1\right) \cup \left[1, \sqrt{2}\right) \cup \left[\frac{3}{2}, \sqrt{3}\right) = \left[\frac{1}{2}, \sqrt{2}\right) \cup \left[\frac{3}{2}, \sqrt{3}\right)$.

☐ For a given positive integer n, how many n-digit natural numbers can be formed from five possible digits 1, 2, 3, 4, and 5 so that an odd numbers of 1 and even numbers of 2 are used?

Solution

Let a_n be the number of n-digits strings of $\{1, 2, 3, 4, 5\}$ with an even number of 1 and an even number of 2 Let b_n be the number of n-digits strings of $\{1, 2, 3, 4, 5\}$ with an even number of 1 and an odd number of 2 Let c_n be the number of n-digits strings of $\{1, 2, 3, 4, 5\}$ with an odd number of 1 and an even number of 2 Let d_n be the number of n-digits strings of $\{1, 2, 3, 4, 5\}$ with an odd number of 1 and an odd number of 2 :

$a_0 = 1$ and $b_0 = c_0 = d_0 = 0$ $a_{n+1} = 3a_n + b_n + c_n$ $b_{n+1} = a_n + 3b_n + d_n$ $c_{n+1} = a_n + 3c_n + d_n$ $d_{n+1} = b_n + c_n + 3d_n$

We obviously have $a_n+b_n+c_n+d_n=5^n$ and $b_n=c_n$ and we quickly get then $c_{n+1}=5^n+c_n$ and so $c_n=\boxed{\dfrac{5^n-1}{4}}$

☐ Solve the equation:$[x]^2+1=|2x|$

Solution

The equation shows that $2x\in\mathbb{Z}$ and so four cases :

$x=n\ge 0$: the equation becomes $n^2+1=2n$ and so $n=1$ and the solution $x=1$

$x=n<0$: the equation becomes $n^2+1=-2n$ and so $n=-1$ and the solution $x=-1$

$x=n+\frac12\ge 0$: the equation becomes $n^2+1=2n+1$ and so $n=0,2$ and the solutions $x=\frac12,\frac52$

$x=n+\frac12<0$: the equation becomes $n^2+1=-2n-1$ and so no solution

Hence the answer : $\boxed{x\in\left\{-1,\dfrac12,1,\dfrac52\right\}}$

☐ Solve for x:

$\frac{6}{2x+1}>\frac{1}{x}\left(1+\log_2\left(2+x\right)\right)$

Solution

Equation is equivalent to $0>\frac{A}{x}$ where $A=\frac{3}{2x+1}-\log_2(\frac{4}{x+2})$

For $x\le -2$, expression is not defined

For $-2<x<-\frac12$, we have $A<0$ and so $\frac{A}{x}>0$ and so no solution

For $x=-\frac12$, expression is not defined

For $x>-\frac12$: Let $A=\frac{3}{2x+1}-\log_2(\frac{4}{x+2})$

$\forall x>0$, we have $\ln x\le x-1$

So $\log_2(\frac{4}{x+2})=\frac{1}{\ln 2}\ln(\frac{4}{x+2})\le\frac{1}{\ln 2}(\frac{4}{x+2}-1)$ and so $A\ge\frac{3}{2x+1}-\frac{1}{\ln 2}(\frac{4}{x+2}-1)=\frac{2x^2+3x(\ln 2-1)+6\ln 2-2}{(x+2)(2x+1)\ln 2}$

It's easy to see that the quadratic (numerator) has no real root and so $A>0$ $\forall x>-\frac12$

And so $0>\frac{A}{x}$ and $x>-\frac12$ $\iff$ $\boxed{x\in\left(-\dfrac12,0\right)}$

☐ A positive integer n is called "FLIPPANT" if n does not end in 0 (when writtenFLIPPANT easy question..indecimal notation) and, moreover, n and the number obtained by reversing thedigits of n are both divisible by 7. How manyintegers are there between10 and 1000 ?

Solution

Writing the number $\overline{abc}$ with $c\ne 0$ and a,b not both zero, the problem is $2a+3b+c\equiv 2c+3b+a\equiv 0 \pmod 7$

And so $a\equiv -b\equiv c\pmod 7$

$a\equiv 0\pmod 7$ gives $a\in\{0,7\}$ and $b\in\{0,7\}$ and $c\in\{7\}$ and so 4 numbers less the one where $a=b=0$ and so 3 solutions $a\equiv 1\pmod 7$ gives $a\in\{1,8\}$ and $b\in\{6\}$ and $c\in\{1,8\}$ and so 4 solutions $a\equiv 2\pmod 7$ gives $a\in\{2,9\}$ and $b\in\{5\}$ and $c\in\{2,9\}$ and so 4 solutions $a\equiv 3\pmod 7$ gives $a\in\{3\}$ and $b\in\{4\}$ and $c\in\{3\}$ and so 1 solution $a\equiv 4\pmod 7$ gives $a\in\{4\}$ and $b\in\{3\}$ and $c\in\{4\}$ and so 1 solution $a\equiv 5\pmod 7$ gives $a\in\{5\}$ and $b\in\{2,9\}$ and $c\in\{5\}$ and so 2 solutions $a\equiv 6\pmod 7$ gives $a\in\{6\}$ and $b\in\{1,8\}$ and $c\in\{6\}$ and so 2 solutions

And so $3+4+4+1+1+2+2=\boxed{17}$ such numbers

☐ You are at a carnival and decide to play a game that can win you a beautiful stuffed teddy bear. For one dollar, you get to randomly pick two numbered balls out of a jar without replacement and without looking. The jar contains 50 numbered balls from 1 to 50. To win the bear, you must pick two numbered balls whose difference is ten or less. What is the probability that the difference between the two balls you select is 10 or less?

Solution

If the first ball is $n \in [1, 11]$, you get $n+9$ possibilities for the second If the first ball is $n \in [12, 39]$, you get 20 possibilities for the second If the first ball is $n \in [40, 50]$, you get $60-n$ possibilities for the second

So the required probability is $\sum_{n=1}^{11} \frac{n+9}{49\times 50} + \sum_{n=12}^{39} \frac{20}{49\times 50} + \sum_{n=40}^{50} \frac{60-n}{49\times 50}$

So $2\sum_{n=1}^{11} \frac{n+9}{49\times 50} + \sum_{n=12}^{39} \frac{20}{49\times 50}$

So $\frac{11\times 30+28\times 20}{49\times 50}$

So $\boxed{\frac{89}{245} \sim 36.33\%}$

☐ Let be the points $A(1,2)$ and $B(4,4)$ in the cartesian system xOy. Find C on Ox for the max angle BCA.

Solution

We get the max when the circle ABC is tangent to Ox

The center of this circle is then on the parabola $y^2 = (x-1)^2 + (y-2)^2$ of the points at same distance from A and $0x$ The center of this circle is also on the parabola $y^2 = (x-4)^2 + (y-4)^2$ of the points at same distance from B and $0x$

Eliminating y between these two equations gives the required result : $\boxed{C\left(\sqrt{26}-2, 0\right)}$ Another solution: Alternatively, the line extension of AB meets the x-axis at $P(-2,0)$. Since PC must be tangent to the circumcircle of ABC, the power of point P is

$$PC^2 = PA \cdot PB = \sqrt{13} \cdot 2\sqrt{13} = 26\,,$$

or $PC = \sqrt{26}$. Hence,

$$C \in \left\{(-2-\sqrt{26}, 0), (-2+\sqrt{26}, 0)\right\}.$$

It should be easy to you to verify which of them gives a bigger value of $\angle BCA$. (In fact, if angles are measured with direction, then $\angle BCA$ is smallest at $C = (-2-\sqrt{26}, 0)$ and is largest at $C = (-2+\sqrt{26}, 0)$.)

☐ Several pairs of positive integers (m,n) satisfy the equation 19m+90+8n = 1998. Of these, (100,1) is the pair with the smallest value for n. Find the pair with the smallest value for m

Solution

Since $19\times 100 + 8\times 1 = 1908$, all solutions are $19(100-8k) + 8(1+19k)$ and the required value is obtained with $k = \left\lfloor \frac{100}{8} \right\rfloor = 12$

Hence the answer : $\boxed{(m,n) = (4, 229)}$

Another solution It's a linear Diophantine equation, hence the solution is : $(m,n) = (8k + 100, -19k+1)$.

(m,n) are positive imply that : $-12 \geq k \geq 0$, obviously m take it smallest value when $k = -12$, just plug it to get : $(m,n) = (4, 229)$.

☐ Let n be positive integer and equation : $x + 2y + 5z = n$. (1) S_n is number of positive integer roots of (1) Prove that $S_n = (n-4)\left\lfloor \frac{n}{10} \right\rfloor + \left\lfloor \frac{n+2}{10} \right\rfloor - \left\lfloor \frac{n-1}{10} \right\rfloor - 5\left\lfloor \frac{n}{10} \right\rfloor^2$

Solution

Let $f(n) = (n-4)\left\lfloor \frac{n}{10} \right\rfloor + \left\lfloor \frac{n+2}{10} \right\rfloor - \left\lfloor \frac{n-1}{10} \right\rfloor - 5\left\lfloor \frac{n}{10} \right\rfloor^2$

The positive integer solutions of $x + 2y = m$ are $(m-2, 1), (m-4, 2), \ldots, (m - 2\left\lfloor \frac{m-1}{2} \right\rfloor, \left\lfloor \frac{m-1}{2} \right\rfloor)$

And so the number of positive integer solutions of $x + 2y = m$ is $T_m = \left\lfloor \frac{m-1}{2} \right\rfloor$ for any $m > 0$ and $T_m = 0\ \forall m \leq 0$

$S_n = T_{n-5} + T_{n-10} + T_{n-15} + \ldots = \sum_{\frac{n-1}{5} \geq k > 0} \left\lfloor \frac{n-5k-1}{2} \right\rfloor$

Writing $n-1 = 5u + r$, with $r \in \{0,1,2,3,4\}$, we get $S_n = \sum_{k=1}^{u} \left\lfloor \frac{5u-5k+r}{2} \right\rfloor$

And so $S_n=\sum_{k=0}^{u-1}\left\lfloor\frac{5k+r}{2}\right\rfloor$ with the convention $S_n=0$ if $u<1$

From there, finding directly the expression $f(n)$ from the sum S_n is possible but rather ugly and I suggest a shorter path :

It's immediate to check that $f(n)=S_n\ \forall n\in[0,9]$ It's immediate to check that $f(n+10)-f(n)=n+1$ It remains to check that $S_{n+10}-S_n=n+1$ in order to conclude the proof :

Let $n-1=5u+r$ and so $n+10-1=5(u+2)+r$: $S_{n+10}=\sum_{k=0}^{u+1}\left\lfloor\frac{5k+r}{2}\right\rfloor$

$S_{n+10}-S_n=\sum_{k=u}^{u+1}\left\lfloor\frac{5k+r}{2}\right\rfloor=\left\lfloor\frac{5u+r}{2}\right\rfloor+\left\lfloor\frac{5u+5+r}{2}\right\rfloor$

$S_{n+10}-S_n=\frac{5u+r}{2}+\frac{5u+5+r}{2}-\frac{1}{2}$ (exactly one of the two summands numerator is odd)

$S_{n+10}-S_n=5u+r+2=n+1$ And this concludes the proof.

☐ Which of the two numbers is greater: 100! or 10^{150}?

Solution

So let's take this: $(1\cdot 20)(21\cdot 40)(41\cdot 60)(61\cdot 80)(81\cdot 100)$

and that is definitely greater than $(1\cdot 20)(20\cdot 40)(40\cdot 60)(60\cdot 80)(80\cdot 100)=147456\cdot 10^{10}>10^{15}$

Now take these nine values...

$(2\cdot 19)(22\cdot 39)(42\cdot 59)(62\cdot 79)(82\cdot 99)\ \cdots\ (10\cdot 11)(30\cdot 31)(50\cdot 51)(70\cdot 71)(90\cdot 91)$

..all of which are greater than $(1\cdot 20)(21\cdot 40)(41\cdot 60)(61\cdot 80)(81\cdot 100)$.

Now notice that when you multiply these ten ugly expressions together, you get (whoa) 100!. Since each of these are greater than 10^{15}, it follows that $100!>10^{150}$.

Hand-made solution :

$100!=2^{97}3^{48}5^{24}7^{16}11^{9}13^{7}\ 17^{5}19^{5}23^{4}29^{3}31^{3}37^{2}\ 41^{2}43^{2}47^{2}53^{1}59^{1}61^{1}\ 67^{1}71^{1}73^{1}79^{1}83^{1}89^{1}97^{1}$

$3^{48}=9^{24}>8^{24}=2^{72}$ $7^{16}=2401^4>2000^4=2^{16}5^{12}$ $11^9>10^9=2^95^9$ $13^7>10^7=2^75^7$ $17^519^5=323^5>320^5=2^{30}5^5$ $23^4>20^4=2^85^4$ $29^331^3=899^3>800^3=2^{15}5^6$ $37^247^2=1739^2>1600^2=2^{12}5^4$ $41^243^2>1600^2=2^{12}5^4$ $53^197^1=5141^1>5000=2^35^4$ $59^189^1=5251^1>5000=2^35^4$ $61^183^1=5063^1>5000=2^35^4$ $67^179^1=5293^1>5000=2^35^4$ $71^173^1=5183^1>5000=2^35^4$

And so $100!>2^{97+72+16+9+7+30+8+15+12+12+3+3+3+3+3}\ 5^{24+12+9+7+5+4+6+4+4+4+4+4+4+4}=2^{293}5^{95}$

Then we have $2^7>5^3$ and so $100!>2^{293}5^{95}>2^{150}(2^7)^{20}5^{95}>2^{150}(5^3)^{20}5^{95}=2^{150}5^{155}>2^{150}5^{150}$ $=10^{150}$

☐ The variable x varies directly as the cube of y, and y varies directly as the square root of z. If x equals 1 when z equals 4, what is the value of z when x equals 27?

Solution

So $x=ay^3$ and $y=b\sqrt{z}$ and so $x=ab^3z\sqrt{z}$

$z=4$ and $x=1\implies ab^3=\frac{1}{8}$ and so $x=\frac{z\sqrt{z}}{8}$ and $z=4x^{\frac{2}{3}}$ And so $x=27\implies z=36$

☐ Find all continuous function $f(x)$ that $f:R^+\to R^+$ that:

$f(2x)=f(x)\forall x\in R^+$

Solution

One general solution is $h\left(\left\{\frac{\ln(x)}{\ln(2)}\right\}\right)$ for any fonction $h(x):[0,1)\to\mathbb{R}^+$ ($h(x)$ continuous on $[0,1)$ and $\lim_{x\to 1}h(x)=h(0)$) and where $\{u\}$ is the fractional part of u.

It's a general solution because : 1) All functions in this form are solutions 2) All solutions may be written in this form.

If you transform the problem in $f(x):\mathbb{R}_0^+\to$ anything, then the unique family of solutions is $f(x)=c$ (the key difference is "0 is in domain of $f(x)$ or not")

☐ Prove that for some natural number n, $n!$ begins with the digit sequence 2007.

Solution

Let S the beginning sequence and k its length (number of digits). Here $S = 2007$ and $k = 4$.

Let $p > k+3$, $N = 10^{2p} - 1$, $m =$the number of decimal digits of $N!$ and $n = 3 \times 10^p$. Let then the sequence $a_i = (N+i)!10^{-(m-k+2ip)}$ for $i \in \{0, ..., n\}$ $\lfloor a_0 \rfloor$ has exactly k digits.

$\frac{a_n}{a_0} = \frac{(N+n)!}{N!}10^{-2np} = \prod_{k=0}^{n-1}(1 + k10^{-2p})$ Using inequality $x - \frac{x^2}{2} < \ln(1+x) < x$, it's rather easy to show that $10 < \frac{a_n}{a_0} < 100$

Then, $\forall i \in \{1, ..., n\}$, $a_i - a_{i-1} = a_{i-1}(i-1)10^{-2p} < a_n n 10^{-2p} < 100 a_0 3 \times 10^p 10^{-2p} < 300 \times 10^{k-p} < 1$ and so the $\lfloor a_i \rfloor = \lfloor a_{i-1} \rfloor$ or $\lfloor a_i \rfloor = \lfloor a_{i-1} \rfloor + 1$.

So the sequence $\lfloor a_i \rfloor$ is a sequence of integers, each equal to the previous or to the previous+1, beginning with a number of k digits and ending with a number greater than 10 times the first one. Then one of these numbers, say $\lfloor a_q \rfloor$ must be S or $10S$.

Then the k first digits of $(N+q)!$ are the required sequence S.

☐ Find the maximum value of k such that $\binom{2008}{1000}$ is divisible by 21^k, where k is a natural number.

Solution

We have $ord_p(\binom{2008}{1000}) = ord_p(2008!) - ord_p(1000!) - ord_p(1008!) = \sum_{k=1}^{+\infty} \lfloor \frac{2008}{p^k} \rfloor - \sum_{k=1}^{+\infty} \lfloor \frac{1000}{p^k} \rfloor - \sum_{k=1}^{+\infty} \lfloor \frac{1008}{p^k} \rfloor$

So $ord_3(\binom{2008}{1000}) = \lfloor \frac{2008}{3} \rfloor + \lfloor \frac{2008}{9} \rfloor + \lfloor \frac{2008}{27} \rfloor + \lfloor \frac{2008}{81} \rfloor + \lfloor \frac{2008}{243} \rfloor + \lfloor \frac{2008}{729} \rfloor - \lfloor \frac{1008}{3} \rfloor - \lfloor \frac{1008}{9} \rfloor - \lfloor \frac{1008}{27} \rfloor - \lfloor \frac{1008}{81} \rfloor - \lfloor \frac{1008}{243} \rfloor - \lfloor \frac{1008}{729} \rfloor - \lfloor \frac{1000}{3} \rfloor - \lfloor \frac{1000}{9} \rfloor - \lfloor \frac{1000}{27} \rfloor - \lfloor \frac{1000}{81} \rfloor - \lfloor \frac{1000}{243} \rfloor - \lfloor \frac{1000}{729} \rfloor$

And $ord_3(\binom{2008}{1000}) = 669 + 223 + 74 + 24 + 8 + 2 - 336 - 112 - 37 - 12 - 4 - 1 - 333 - 111 - 37 - 12 - 4 - 1 = 0$

And so $k = 0$

☐ Suppose p is a prime gretaer than 3. Find all pairs (a, b) of integers satisfying the equation $a^2 + 3ab + 2p(a+b) + p^2 = 0$

Solution

The equation may be written $(a+b+p)^2 = b(b-a)$

So $b(b-a)$ is a perfect square and we have $b = ku^2$ and $a = ku^2 - kv^2$ fore some k, u, v.

Then $a + b + p = \epsilon_0 kuv$, which may be written $2ku^2 - kv^2 + p = \epsilon_0 kuv$ (where ϵ_0 is -1 or $+1$)

So $p = kv^2 - 2ku^2 + \epsilon_0 kuv = k(v - \epsilon_0 u)(v + 2\epsilon_0 u)$

Since p is prime, we have three cases : 1) $k = \epsilon_1$ $v - \epsilon_0 u = \epsilon_2$ $p = \epsilon_0\epsilon_1\epsilon_2(3u + \epsilon_0\epsilon_2)$ This is equivalent to : $k = \epsilon_1$ $v - \epsilon_0 u = \epsilon_0\epsilon_1$ $p = 3u + \epsilon_1$ And so : $p = 3u + \epsilon_1$ $a = -2u - \epsilon_1$ $b = \epsilon_1 u^2$

2) $k = \epsilon_1$ $v + 2\epsilon_0 u = \epsilon_2$ $p = -\epsilon_0\epsilon_1\epsilon_2(3u - \epsilon_0\epsilon_2)$ This is equivalent to : $k = \epsilon_1$ $v = 2\epsilon_1\epsilon_2 u + \epsilon_2$ $p = 3u + \epsilon_1$

And so : $p = 3u + \epsilon_1$ $a = -\epsilon_1(3u^2 + 1) - 4u$ $b = \epsilon_1 u^2$

3) $v - \epsilon_0 u = \epsilon_1$ $v + 2\epsilon_0 u = \epsilon_2$ $p = \epsilon_1\epsilon_2 k$

Which gives : $p = k$ $a = -k$ $b = 0$

****

As a conclusion

For any prime p we have the solution $(-p, 0)$

For any prime $p = 1 \pmod 3$, we also have the two solutions :

$(-\frac{2p+1}{3}, \frac{(p-1)^2}{9})$

$(-\frac{p(p+2)}{3}, \frac{(p-1)^2}{9})$

For any prime $p = 2 \pmod 3$, we also have the two solutions :

$(-\frac{2p-1}{3}, -\frac{(p+1)^2}{9})$

$\left(\frac{p(p-2)}{3}, -\frac{(p+1)^2}{9}\right)$

□ In any triangle ABC, if $BC + AC = 2AB$, show that $\cot(A/2) + \cot(B/2) = 2\cot(C/2)$

Solution

$a + b = 2c \iff 2s - (a + b) = 2s - 2c \iff s - a + s - b = 2(s - c)$

$\iff \frac{s(s-a)}{[ABC]} + \frac{s(s-b)}{[ABC]} = \frac{2s(s-c)}{[ABC]}$

Now using **<u>Heron</u>**<u>'s formula</u> $\implies \sqrt{\frac{s^2(s-a)^2}{s(s-a)(s-b)(s-c)}} + \sqrt{\frac{s^2(s-b)^2}{s(s-a)(s-b)(s-c)}} = 2\sqrt{\frac{s^2(s-c)^2}{s(s-a)(s-b)(s-c)}}$

$\iff \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = 2\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \iff \cot\frac{A}{2} + \cot\frac{B}{2} = 2\cot\frac{C}{2}$.

Other relations in triangles having **sides in aritmethic progression**: $\left\| \begin{array}{c} \frac{2}{h_a} = \frac{1}{h_b} + \frac{1}{h_c} \\ \\ 4\cos A + \cos(B - C) = 3 \\ \\ \sin\frac{A}{2} = \frac{1}{2}\cos\frac{B-C}{2} \end{array} \right\|$

and $\left\| \begin{array}{c} 3\tan\frac{B}{2}\tan\frac{C}{2} = 1 \\ \\ 3\sin^2\frac{A}{2} = \sin B \sin C \\ \\ 2\cos A + \cos B + \cos C = 2 \end{array} \right\|$.

□ Find $x \in R$ such: $[x+\frac{1}{2}]+[x-\frac{1}{2}] = [2x]$ Let $y = x-\frac{1}{2}$ and the equation is $\lfloor y+1\rfloor+\lfloor y\rfloor = \lfloor 2y+1\rfloor$ $\iff 2\lfloor y\rfloor = \lfloor 2y\rfloor \iff y \in [n, n+\frac{1}{2})$

Hence the answer : $\boxed{x \in \bigcup_{n\in\mathbb{Z}} [n + \frac{1}{2}, n + 1)}$

Another way: Using the well-known identity $[x] + \left[x + \frac{1}{2}\right] = [2x]$ obtain that $\left[x + \frac{1}{2}\right] + \left[x - \frac{1}{2}\right] = [2x] \iff [x] = \left[x - \frac{1}{2}\right] = z \in \mathrm{Z} \iff \left\{ \begin{array}{c} z \le x < z+1 \\ z \le x - \frac{1}{2} < z+1 \end{array} \right\| \iff \left\{ \begin{array}{c} z \le x < z+1 \\ z + \frac{1}{2} \le x < z + \frac{3}{2} \end{array} \right\| \iff$ $z + \frac{1}{2} \le x < z + 1$, $z \in \mathbb{Z}$. In conclusion, $x \in \bigcup_{z\in\mathbb{Z}} \left[z + \frac{1}{2}, z + 1\right)$.

<u>Remark</u>. $\left[x + \frac{1}{2}\right] + \left[x - \frac{1}{2}\right] = [2x] \iff x \in \bigcup_{z\in\mathbb{Z}} \left[z + \frac{1}{2}, z + 1\right) \iff [2 \cdot \{x\}] = 1$.

Note: We could apply Hermite's identity one more time as $\left\lfloor x - \dfrac{1}{2} \right\rfloor + \left\lfloor x - \dfrac{1}{2} + \dfrac{1}{2} \right\rfloor = \lfloor 2x - 1\rfloor$, in order to obtain $2\lfloor x\rfloor = \lfloor 2x - 1\rfloor = \lfloor 2\lfloor x\rfloor + 2\{x\} - 1\rfloor = 2\lfloor x\rfloor - 1 + \lfloor 2\{x\}\rfloor$, whence $\lfloor 2\{x\}\rfloor = 1$, and then the conclusion.

□ Let n be a positive integer. Find the number of $2n$-digit positive integers $a_1a_2 \ldots a_{2n}$ such that (i) none of the digits a_i is equal to 0, and (ii) the sum $a_1a_2 + a_3a_4 + \ldots + a_{2n-1}a_{2n}$ is even.

Followup: What if we mandate that $a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{2n-1}a_{2n}$ be even instead?

Solution

For a number $x = \overline{a_1a_2 \ldots a_{2n}}$, let us call $p(x) = a_1a_2 + a_3a_4 + \ldots + a_{2n-1}a_{2n}$.

If we call S_n the required number (count of numbers x without 0, with length $2n$ and such that $p(x)$ is even), we can say that $S_{2(n+1)}$ is : The count of numbers with length $2n$ and $p(x)$ even (S_n) multiplied by 56 (the number of possibilities for $a_{2n+1}a_{2n+2}$ even). Plus the count of numbers with length $2n$ and $p(x)$ odd ($9^{2n} - S_n$) multiplied by 25 (the number of possibilities for $a_{2n+1}a_{2n+2}$ odd).

And so $S_{n+1} = 56S_n + 25(81^n - S_n)$ and $S_1 = 56$

This formula is quite easy to solve (compute first $T_n = \frac{S_n}{81^n}$) and gives :

$\boxed{S_n = \dfrac{31^n + 81^n}{2}}$

$\square$ Find all pairs of integers (x, y) satisfying
$1 + x^2y = x^2 + 2xy + 2x + y$.

Solution

Writing this as $y = -\frac{x^2+2x-1}{x^2-2x-1} = -1 - \frac{4x}{x^2-2x-1}$, we must find all integer x such as the last fraction is integer.

First let's check the domain where the denominator is negative. Since the roots of $x^2 - 2x - 1$ are $1 \pm \sqrt{2}$, those values of x are $x \in \{0, 1, 2\}$. For all of them we get an integer y, so the first three pairs are $(0, 1), (1, -1), (2, -7)$

Now we check $x \geqslant 3 \vee x \leqslant -1$. We have two inequalities:

1. $\frac{4x}{x^2-2x-1} \geqslant 1 \iff 4x \geqslant x^2 - 2x - 1$ (note: the denominator is positive in the examined domain), and that yields $x^2 - 6x - 1 \leqslant 0$. The corresponding values for x are $x \in \{3, 4, 5, 6\}$. Checking shows that only $x = 3$ yields an integer y, so another solution is $(3, 7)$

2. $\frac{4x}{x^2-2x-1} \leqslant -1 \iff 4x \leqslant -x^2 + 2x + 1 \iff x^2 + 2x - 1 \leqslant 0$. The corresponding values for x are $x \in \{-2, -1\}$, but only $x = -1$ yields an integer y, hence another solution is $(-1, -1)$

Therefore the complete set of solutions is $\{(-1, -1), (0, 1), (1, -1), (2, -7), (3, 7)\}$

$\square$ Let ABCD is a prallelogram.Choose 2 point E and F on the side AB.($E \in [AF]$)DF and CE meet at P.2 circumcircles of triangles PAE and PFB meet at Q($Q \neq F$).Prove that PQ is parallel to AD

Solution

Let the radical axis PQ of $\odot(PAE)$ and $\odot(PFB)$ cut AB and DC at R and R', respectively. Thus, $RE \cdot RA = RF \cdot RB$. On the other hand, from the similar triangles $\triangle PEF \sim \triangle PCD$, we have $\frac{RE}{RF} = \frac{R'C}{R'D} \Longrightarrow \frac{R'C}{R'D} = \frac{RB}{RA}$. Since $AB = DC$, we conclude that $RA = R'D$ and $RB = R'C \Longrightarrow PQ \parallel AD \parallel BC$.

$\square$ Let $f(x) = a_nx^n + \ldots + a_1x + a_0$ have n solutions $x_1, x_2, ..., x_n$ Prove:

$$\frac{1}{f'(x_1)} + \ldots + \frac{1}{f'(x_n)} = 0$$

Solution

For the problem to make sense, $n > 1$. If $f(x) = a_n \prod_{k=1}^{n}(x - x_k)$, by the chain rule we obtain $f'(x_i) = a_n \prod_{k=1,k\neq i}^{n}(x_i - x_k)$. Then

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = \sum_{i=1}^{n} \left(\frac{1}{a_n} \cdot \prod_{k=1,k\neq i}^{n} \frac{1}{(x_i - x_k)} \right).$$

This is the coefficient of the x^{n-1} term of the Lagrange interpolation polynomial $L(x)$ through the n points $(x_i, 1/a_n)$, but clearly $L(x) = 1/a_n$ so it follows that the desired sum is equal to 0.

$\square$ Let P be the Fermat Point of ΔABC. Prove that the Euler lines of Δs PAB,PBC,PCA are concurrent and the point of concurrence is G, the centroid of ΔABC?

Solution

Let $\triangle A'BC, \triangle B'CA$ and $\triangle C'AB$ be three equilateral triangles erected outside $\triangle ABC$. Let X, Y, Z denote their circumcenters. Thus, $P \equiv (X) \cap (Y) \cap (Z)$ and $P \equiv AA' \cap BB' \cap CC'$. Let G_1, G_2, G_3 denote the centroids of $\triangle PBC$, $\triangle PCA$, $\triangle PAB \Longrightarrow XG_1, YG_2$ and ZG_3 are the Euler lines of $\triangle PBC, \triangle PCA, \triangle PAB$. If M is the midpoint of BC and G is the centroid of $\triangle ABC$, we get

$\frac{MX}{MA'} = \frac{MG_1}{MP} = \frac{MG}{MA} = \frac{1}{3}$

Therefore, X, G_1, G are collinear on a parallel line to AA'. Hence, Euler lines of $\triangle PBC$, $\triangle PCA$, $\triangle PAB$ concur at the centroid G of $\triangle ABC$.

☐ Find al positive integers m, n, where n is odd, that satisfy
$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}$.

Solution

Put $n = 2k-1$. Then
$\frac{1}{m} = \frac{1}{12} - \frac{4}{2k-1} = \frac{2k-49}{24k-12} \iff m = \frac{24k-12}{2k-49}$
Write this as $m = \frac{12(2k-49)+576}{2k-49} = 12 + \frac{576}{2k-49}$
Since $576 = 2^6 \cdot 3^2$ and $2k-49$ is odd, we get $2k-49 \in \{1, 3, 9\}$. Computing k, m, n in those cases, we get
$(m, n) \in \{(588, 49), (204, 51), (76, 57)\}$

☐ Determine $a, b \in \mathbb{R}$ such that the function $f : [0, 2] \to [-1, 3]$, $f(x) = ax + b$ is bijective.

Solution

If $f(x) = ax + b$ is bijective, then all the real numbers in the interval $[0, 2]$ must map to all the real numbers in the interval $[-1, 3]$. Note that the length of the first interval is 2 and that the length of the second interval is 4. Therefore, we let $a = 2$ so we have $[0, 2] \mapsto [0, 4]$. Now we let $b = -1$ so that we have $[0, 4] \mapsto [-1, 3]$. Thus, $f(x) = \boxed{2x-1}$.

☐ How many real solutions does the equation $x^3 3^{1/x^3} + \frac{1}{x^3} 3^{x^3} = 6$ have?
(A) 0 **(B)** 2 **(C)** 3 **(D)** Infinitely many **(E)** None

Solution

x must be positive, since otherwise LHS would be negative.
$LHS \geqslant 2\sqrt{3^{x^3+x^{-3}}} \geqslant 2\sqrt{3^2} = 6$
Hence $x^3 = x^{-3} \iff x = 1$. That's the unique solution, so the answer is E.

☐ 1. For how many nonnegative integers n does $x^3 + (n-1)x^2 + (n-n^2)x - n^3$ have all integer roots?

2. Consider the set of all equations $x^3 + a_2x^2 + a_1x + a_0 = 0$, where a_2, a_1, a_0 are real constants and $|a_i| \leq 2$ for $i = 0, 1, 2$. Let r be the largest positive real number which satisfies at least one of these equations. Find r.

Solution

Problem 1) We expand and factor: $x^3 + (n-1)x^2 + (n-n^2)x - n^3 = x^3 + nx^2 - x^2 + nx - n^2x - n^3$ $= (x-n)(x^2 + 2xn + n^2 - x) = (x-n)(x^2 + (2n-1)x + n^2)$.

For this to factor, the discriminant of the quadratic must be a perfect square. Then since the coefficient of x^2 is 1 and the other coefficients are integers, the roots will be integers.

The discriminant is $(2n-1)^2 - 4(n^2)(1) = 4n^2 - 4n + 1 - 4n^2 = 1 - 4n$. This is negative unless $n = 0$, and in this case it is in fact a square. Thus there is 1 nonnegative integer n so that that equation has all integer roots.

Problem 2) The greatest r is the root of $x^3 - 2x^2 - 2x - 2$. I couldn't find any useful closed form of it, and now we show that this is the greatest.

Let $s > r$ be the root of $x^3 + ax^2 + bx + c$ where $|a|, |b|, |c| \leq 2$. Then $0 = s^3 + as^2 + bs + c = s^3 - 2s^2 - 2s - 2 + (2+a)s^2 + (2+b)s + (2+c) > (2+a)s^2 + (2+b)s + (2+c) > 0$, impossible.

We know that $s^3 - 2s^2 - 2s - 2$ is positive, because after the root, it only increases. (shown by taking derivatives...) and we know that $(2+a)s^2 + (2+b)s + (2+c) > 0$ because s and s^2 are positive, and so are all of $2+a, 2+b, 2+c$. Hence the root of $x^3 - 2x^2 - 2x - 2$ is the greatest r possible.

☐ i) Solve in $\mathbb{R}$ the equation : $\sqrt[3]{x+6} + \sqrt{x-1} = x^2 - 1$

ii) Solve in $\mathbb{R}$ the equation : $2003x = 2004.2003^{\log_x 2004}$

Solution

Problem 1) $x = 2$ is a solution. We will show that it is the only solution. Moving everything to the LHS, $1 - x^2 + \sqrt[3]{x+6} + \sqrt{x-1} = 0$ and let $f(x) = 1 - x^2 + \sqrt[3]{x+6} + \sqrt{x-1}$. Since $1 - x^2$, $\sqrt[3]{x+6}$, and $\sqrt{x-1}$ are all concave functions, their sum $f(x)$ is concave throughout its domain. $f'(x)$ is decreasing. Assume for the sake of contradiction that the original equation has two or more solutions. Then, in $[1, \infty)$, the domain of f, there are two or more zeroes. Call the first two zeroes a and b, where $a < b$. Since $f(1) = \sqrt[3]{7} > f(a) = 0$ and $1 < a$, $f(x)$ must be decreasing at some value $c \in (1, a)$. Thus $f'(c) < 0$. Since $f'(x)$ is decreasing, for all $x > c$, $f'(x) < 0$, and $f(x)$ is decreasing. Since $b > a > c$, $f(b) < f(a) = 0$, contradiction

Another way: (no derivatives necessary)

First of all, $x \geqslant 1$ because of the second term.

Write the equation as

$\sqrt[3]{x+6} - 2 + \sqrt{x-1} - 1 = x^2 - 4$

$\frac{x-2}{\sqrt[3]{(x+6)^2} + 2\sqrt[3]{x+6} + 4} + \frac{x-2}{\sqrt{x-1}+1} = (x-2)(x+2)$

Obviously, $x = 2$ is a solution. Assume $x \neq 2$. Then

$\frac{1}{\sqrt[3]{(x+6)^2} + 2\sqrt[3]{x+6} + 4} + \frac{1}{\sqrt{x-1}+1} = x + 2$

The LHS is $\leqslant \frac{1}{4} + \frac{1}{1}$, and the RHS is $\geqslant 3$, so there can be no solution.

Therefore $x = 2$ is the only solution.

□ Find all positive integers n for which $\cos(\pi\sqrt{n^2+n}) \geq 0$

Solution

Put $\sqrt{n^2+n} = 2k + \delta$ where $k \in \mathbb{N}, \delta \in \mathbb{R}, 0 \leqslant \delta < 2$. Then by the given condition, $\delta \in \left[0, \frac{1}{2}\right] \cup \left[\frac{3}{2}, 2\right)$. That also can be written as

$\left\{\frac{\sqrt{n^2+n}}{2}\right\} \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right) \qquad (*)$

It is easily shown that $n - \frac{1}{2} < \sqrt{n^2+n} < n + \frac{1}{2}$ for natural n. Hence

$\frac{n}{2} - \frac{1}{4} < \frac{\sqrt{n^2+n}}{2} < \frac{n}{2} + \frac{1}{4}$

Case 1. $n = 2M, M \in \mathbb{N}$. Then we get $M - \frac{1}{4} < \frac{\sqrt{n^2+n}}{2} < M + \frac{1}{4}$. Obviously, the fractional part of the middle expression satisfies the condition $(*)$

Case 2. $n = 2M - 1, M \in \mathbb{N}$. Then we get $M - \frac{3}{4} < \frac{\sqrt{n^2+n}}{2} < M - \frac{1}{4}$. Obviously, the fractional part of the middle expression does not satisfy the condition $(*)$.

Hence we conclude that **all even** n satisfy the initial condition.

□ Solve in $\mathbb{R}$ the equation : $\log_2(\sin x) + \log_3(\tan x) = \log_4(\cos^2 x) + \log_5(\cot x)$.

Solution

$\frac{\log \sin x}{\log 2} + \frac{\log \tan x}{\log 3} - \frac{\log \cos x}{\log 2} + \frac{\log \tan x}{\log 5} = 0$

$\frac{\log \tan x}{\log 2} + \frac{\log \tan x}{\log 3} + \frac{\log \tan x}{\log 5} = \log \tan x \left(\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 5}\right) = 0$

$\tan x = 1 \Rightarrow x = \frac{\pi}{4} + k\pi$

□ Solve in R $\frac{1}{5}\frac{(x+1)(x-3)}{(x+2)(x-4)} + \frac{1}{9}\frac{(x+3)}{(x+4)}\frac{(x-5)}{(x-6)} - \frac{2}{13}\frac{(x+5)}{(x+6)}\frac{(x-7)}{(x-8)} = \frac{92}{585}$

Solution

Write this as

$\frac{1}{5}\frac{x^2-2x-3}{x^2-2x-8} + \frac{1}{9}\frac{x^2-2x-15}{x^2-2x-24} - \frac{2}{13}\frac{x^2-2x-35}{x^2-2x-48} = \frac{92}{585}$

$\frac{1}{5}\left(1 + \frac{5}{x^2-2x-8}\right) + \frac{1}{9}\left(1 + \frac{9}{x^2-2x-24}\right) - \frac{2}{13}\left(1 + \frac{13}{x^2-2x-48}\right) = \frac{92}{585}$

$\frac{1}{x^2-2x-8} + \frac{1}{x^2-2x-24} - \frac{2}{x^2-2x-48} = 0$

Put $u := x^2 - 2x - 8$. Then

$\frac{1}{u} + \frac{1}{u-16} - \frac{2}{u-40} = 0$

$u^2-56u+640+u^2-40u-2u^2+32u=0$

$-64u+640=0$

$u=10$

$x^2-2x-18=0$

$x_{1,2}=1\pm\sqrt{19}$

$\square$ Solve the equation: $\frac{x^2}{\sqrt{x+2}}+1=2x^2$

Solve the system of equations: $x^2(x^4+2)+y^3=\sqrt{xy(1-xy)}$
$2y^3(4x+1)+1\geq 4x^2+2\sqrt{1+(2x-y)^2}$

Solution

Problem 1) We can rewrite this as

$x^2+\sqrt{x+2}=2x^2\sqrt{x+2}$

$\implies x^2=\sqrt{x+2}(2x^2-1)$

$\implies \sqrt{x+2}=\frac{x^2}{2x^2-1}$

$\implies x+2=\frac{x^4}{4x^4-4x^2+1}$

$\implies 4x^5+8x^4-4x^3-9x^2+x+2=0$

By the Rational Root Theorem, $x=-1$ is a root of the equation. Thus, we may write

$4x^5+8x^4-4x^3-9x^2+x+2=(x+1)(4x^4+4x^3-8x^2-x+2)=0.$

However, by the Rational Root Theorem, $4x^4+4x^3-8x^2-x+2$ does not have any rational roots (it has more real roots, but not rational ones).

Thus, $x=-1$ is the only rational solution.

$\square$ Let a,b,c are positive number sastify that

$$a^2+b^2+c^2=12$$

Prove that:

$$\frac{1}{\sqrt{(1+ab^2)^3}}+\frac{1}{\sqrt{(1+bc^2)^3}}+\frac{1}{\sqrt{(1+ca^2)^3}}\geq\frac{1}{9}$$

Solution

Using Jensen for $f(x)=\frac{1}{\sqrt{x^3}}$ we come to the obvious by CS result $24\geq ab^2+bc^2+ca^2$

$\square$ Let ABC be a triangle , R the radius of circumcircle and S its area .

If $a^2+b^2+c^2=4$ then prove that $6R^2+S^2\geq 3$.

Solution

If you put $x=a^2+b^2-c^2$ and the similars it becomes:

$$\frac{xy+yz+zx}{4}\geq\frac{3xyz}{xy+yz+zx}$$

which is trivial.

$\square$ Let x_i be a set of reals such that $\sum^n x_i=n$. Prove that

$\sum^n(n+x_i+\frac{1}{x_i})(n+x_i+x_i^2)\geq n^3+4n^2+4n$

Solution

The LHS is equivalent to $n^3+2n^2+2n+\sum x_i^3+(n+1)\sum x_i^2+\sum x_i\sum\frac{1}{x_i}\geq n^3+4n^2+4n$.

so, it suffices to prove that $\sum x_i^3+(n+1)\sum x_i^2+\sum x_i\sum\frac{1}{x_i}\geq 2n^2+2n$.

But from Chebychev's inequality we get that $\sum x_i^3\geq\sum x_i^2$. So, we need to prove that $(n+2)\sum x_i^2+\sum x_i\sum\frac{1}{x_i}\geq n^2+n^2+2n$ which is obviously true from CS and AM-GM

$\square$ Let $d > c > b > a$. Prove that $a^b b^c c^d d^a \geq b^a c^b d^c a^d$

Solution

It is equivalent to $(\frac{c}{a})^{d-b} \geq (\frac{d}{b})^{c-a}$ Taking the ln of both sides, $\frac{\ln c - \ln a}{c-a} \geq \frac{\ln d - \ln b}{d-b}$ But note that these are the slopes of the lines on $f(x) = lnx$ between $(a, \ln a)$ and $(c, \ln c)$ and the line between $(b, \ln b)$ and $(d, \ln d)$. As $f(x) = \ln x$ is a concave function and $a < b < c < d$, this inequality must be true.

$\square$ Let x_n be a sequence. It is known that $(n+2)x_{n+2} - 6(n+1)x_{n+1} + 8nx_n = 0$ and $x_1 = \frac{1}{6}$,$x_2 = \frac{1}{20}$ .Find x_n.

Solution

Let $y_n = nx_n$ then we have $y_{n+2} - 6y_{n+1} + 8y_n = 0$ and $y_1 = \frac{1}{6}, y_2 = \frac{1}{10}$.It is well known that y_n can be written with constants a, b as $y_n = a2^{n-1} + b4^{n-1}$.By the case of $n = 1, 2$, we have $a + b = \frac{1}{6}$ and $2a + 4b = \frac{1}{10}$,or $a = \frac{17}{60}, b = -\frac{7}{60}$,hence $y_n = \frac{1}{60}(17 \cdot 2^{n-1} - 7 \cdot 4^{n-1})$.

Eventually we have $x_n = \frac{1}{60n}(17 \cdot 2^{n-1} - 7 \cdot 4^{n-1})$.

$\square$ Given any three non-collinear points in a plane, two of which are fixed and one variable, what is the locus of the center of the circle through these points?

Solution

By definition, the center of the circle must be equidistant (the distance is all the same) from all three points, and therefore from the two fixed points.

It is also well known that the locus of all points equidistant from two points is the points' segment's perpendicular bisector.

Finally, to generate a specific point on the perpendicular bisector, just draw the circle and pick a random point on it.

$\square$ Let a,b,c $>$0 and a+b+c=1, prove that
$\frac{1+a^2}{2bc+a} + \frac{1+b^2}{2ca+b} + \frac{1+c^2}{2ab+c} \geq 6$

Solution

$\sum \frac{1}{2bc+a} + \sum \frac{a^2}{2bc+a} \geq \frac{9}{1+2(ab+bc+ca)} + \frac{1}{1+2(ab+bc+ca)} = \frac{10}{1+2(ab+bc+ca)}$.

So we have to prove that:

$\frac{10}{1+2(ab+bc+ca)} \geq 6 \iff 1 \geq 3(ab+bc+ca)$ which is true.

$\square$ If $(a+b+c)^2 = a^2 + b^2 + c^2$, prove that:

$$\sum_{cyc} \frac{a^2}{a^2 + 2bc} = 1$$

Solution

Since the condition $ab + bc + ca = 0$. Hence:

$$\sum_{cyc} \frac{a^2}{a^2 + 2bc} = 1 \Leftrightarrow \sum_{cyc} \frac{a}{a - 2(b+c)} = 1 \Leftrightarrow$$

$$\Leftrightarrow 4(a^3 + b^3 + c^3) + 15abc = 4(a^3 + b^3 + c^3) - 6(ab(a+b) + bc(b+c) + ca(c+a)) - 3abc$$

which is true because

$$ab(a+b) + bc(b+c) + ca(c+a) = (a+b+c)(ab+bc+ca) - 3abc = -3abc$$

$\square$ Prove that of all triangles inscribed in a given triangle, the one with least perimeter connects the feet of the given triangle.

Solution

Let $\triangle ABC$ be an acute triangle with orthocenter H and circumcenter O. X, Y, Z are the feet of the altitudes on BC, CA, AB. $\triangle DEF$ is an arbitrary triangle such that D, E, F lie on BC, CA, AB. So we have to prove

$DE + EF + FD \geq XY + YZ + ZX.$

In the quadrangles $AEOF, BFOD$ and $CDOE$ we have the following inequalities:

$[AEOF] \leq \frac{R \cdot EF}{2}$, $[BFOD] \leq \frac{R \cdot FD}{2}$, $[CDOE] \leq \frac{R \cdot DE}{2}$

$\Longrightarrow [\triangle ABC] \leq \frac{R \cdot (DE+EF+FD)}{2}$

Since O, H are isogonal conjugates, we get $OA \perp YZ,\ OB \perp ZX,\ OC \perp XY$

$[AYHZ] = \frac{R \cdot YZ}{2}$, $[BZHX] = \frac{R \cdot ZX}{2}$, $[CXHY] = \frac{R \cdot XY}{2}$

$\Longrightarrow [\triangle ABC] = \frac{R \cdot (XY+YZ+ZX)}{2}$

Therefore, $DE + EF + FD \geq XY + YZ + ZX$ and the proof is completed.

□ Prove that in a convex cyclic quadrilateral $\overline{AB} \cdot \overline{AD} \cdot \overline{EC} = \overline{CB} \cdot \overline{CD} \cdot \overline{EA}$ (where A, B, C, D are the vertices, E is the intersection of the diagonals). Is the converse true?

Solution

We have $\triangle AEB \sim \triangle DEC$ and $\triangle AED \sim \triangle BEC$ (since all the angles are equal, this is how you prove power of a point). Thus, $\frac{AB}{AE} = \frac{CD}{ED}$ and $\frac{BC}{CE} = \frac{AD}{ED}$ Dividing them, we get the desired result.

□ If there exists a regular n-gon with its vertices at lattice points in a cartesian plane, prove that n=4

Solution

Suppose we had regular polygon with latice point vertices.

The area of a regular n-gon with side length s is given by $A = \frac{n}{4} \cdot s^2 \cdot \cot\left(\frac{180}{n}\right)$

Note that the side length squared, s^2, is an integer because $s = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ where (x_1, y_1), (x_2, y_2) are two adjacent vertices

Also $\cot\left(\frac{180}{n}\right)$ is irrational when $n \neq 4$

Thus we conclude that our polygon has an irrational area when $n \neq 4$

However, by the Shoelace Theorem, a polygon with lattice point vertices has a rational area. Contradiction ■

□ Prove that if $a + b + c = 0$, then

$$(ax + b)^4 + (bx + c)^4 + (cx + a)^4 = (bx + a)^4 + (cx + b)^4 + (ax + c)^4$$

for any complex number x

Solution

Consider the polynomial $P(x) = (ax+b)^4+(bx+c)^4+(cx+a)^4-(bx+a)^4-(cx+b)^4-(ax+c)^4$ I claim that 0,1,2,-1 and $\frac{1}{2}$ are all roots. 0,1, and -1 are obviously roots. Note that $2a+b = a+(a+b) = a-c$, and so on, thus 2 is a root. Also, $\frac{1}{2}a + b = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}b = \frac{1}{2}(a + b) + \frac{1}{2}b = \frac{1}{2}(b - c)$ and so on, thus $\frac{1}{2}$ is a root. Since $P(x)$ is a fourth degree polynomial with 5 roots, it must just be 0. Actually, if you expand $P(x)$ is only a cubic so we only needed four roots. In fact, for any a,b, and c, $P(x)$ will be divisible by $x^3 - x$.

□ Let ABC be a triangle, and let D, E, F be the feet of the altitudes from A, B, C respectively. Construct the incircles of triangles AEF, BDF and CDE; let the points of tangency with DE, EF and FD be C", A", and B" respectively. Prove that AA", BB", CC" concur.

Solution

Let A', B', C' be the tangency points of the incircle (I) of $\triangle ABC$ with BC, CA, AB. Sidelines

EF, FD, DE of the orthic triangle are respectively antiparallel to BC, CA, AB. Thus, there exists the product of the axial symmetry about the angle bisector of $\angle BAC$ and a positive homothety centered at A taking EF into BC and therefore carrying the incircle of $\triangle AEF$ into $(I) \Longrightarrow AA'$ and AA'' are isogonals with respect to $\angle CAB$. Similarly, BB', BB'' and CC', CC'' are isogonals with respect to $\angle ABC$ and $\angle BCA$. Since AA', BB', CC' concur at the Gergonne point of $\triangle ABC$, then AA'', BB'', CC'' concur at its isogonal conjugate with respect to $\triangle ABC$.

□ Solve in natural the equation $(x+y)(x+z) = xyz$

Solution

Let $g = gcd(x,y)$, $x = ga$, $y = gb$, then $(a+b)(ga+z) = gabz$. But $gcd(a,b) = 1$ so $a|ga+z$, so $a|z$. So $z = ka$. Thus, $(a+b)(g+k) = gbak$. Let $gcd(g,k) = n$. Then $g = nr$, $k = ns$ and $gcd(r,s) = 1$. Thus, $(a+b)(r+s) = nrsab$. So we must have $rs|a+b$, $ab|r+s$. Thus, $a+b \geq rs$ and $r+s \geq ab$. But if either r or s is at least 2 then $a+b \geq rs \geq r+s \geq ab$, so either a or b is at most 2. I have the solutions $(a,b,r,s) = (1,1,1,1)$,$(2,2,2,2)$,$(2,1,1,1)$,$(3,1,2,1)$ which yield the solutions $(x,y,z) = (4,4,4)$,$(6,3,6)$,$(6,6,3)$,$(12,4,6)$, and $(12,6,4)$. I think that's it, I may have left out some.

□ Prove that $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}$.

Solution

we know that $i^{3n+1} \exp\left(\frac{i\pi(n-1)}{2}\right) = 1$ now Consider the function $f(x) = x^{n-1} + x^{n-2} + ... + x + 1 = \frac{x^n - 1}{x-1} = \prod_{j=1}^{n-1} (x - w^j)$ note that $w^j = \exp\left(\frac{2\pi k i}{n}\right) = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right) = 1 - 2\sin^2\left(\frac{\pi k}{n}\right) + 2i \sin\left(\frac{\pi k}{n}\right)\cos\left(\frac{\pi k}{n}\right)$ then $1 - w^j = \left(\sin\left(\frac{\pi k}{n}\right) - i\cos\left(\frac{\pi k}{n}\right)\right) 2\sin\left(\frac{\pi k}{n}\right)$ now, we know that $n = 1+1+1+ ... + 1 + 1 = 1^{n-1} + 1^{n-2} + ... + 1 + 1 = f(1)$ $f(1) = \prod_{j=1}^{n-1} (1 - w^j) = \prod_{j=1}^{n-1} \left(\sin\left(\frac{\pi k}{n}\right) - i\cos\left(\frac{\pi k}{n}\right)\right) 2\sin\left(\frac{\pi k}{n}\right)$

$= \prod_{j=1}^{n-1} \left(\sin\left(\frac{\pi k}{n}\right) - i\cos\left(\frac{\pi k}{n}\right)\right) \prod_{j=1}^{n-1} 2 \prod_{j=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = 2^{n-1} \prod_{j=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) \cdot i^{3n+1} \exp\left(\frac{i\pi(n-1)}{2}\right) = n$

$2^{n-1} \prod_{j=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) \cdot 1 = n \Rightarrow \prod_{j=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = \frac{n}{2^{n-1}}$

□ Let $f: N \to N$ solve for $f(n)$ from system equation

$$f(n) + f(n+1) = f(n+2)f(n+3) - 1996$$

Solution

You can stack two equations to get $f(n+2) - f(n) = f(n+3)(f(n+4) - f(n+2))$. If $f(n+2) - f(n)$ is nonzero then we must have a steadily decreasing sequence or a steadily increasing sequence over the evens. Decreasing is impossible because we are using the natural numbers. Increasing is impossible or else we will need infinite divisors for the odd integers. Therefore $f(n+2) = f(n)$. Set $f(even) = a$ and $f(odd) = b$. Then we get $(a-1)(b-1) = 1997$, and thus our solutions are $f(odds) = 1998, f(evens) = 2$ or $f(odds) = 2, f(evens) = 1998$.

□ Let $x, y, z > 0$ so that $x + y + z = 1$. Prove that :

$\log_x(x^2+y^2+z^2) + \log_y(x^2+y^2+z^2) + \log_z(x^2+y^2+z^2) \leq x\log_x(xyz) + y\log_y(xyz) + z\log_z(xyz)$

Solution

Pick $a \in (0,1)$, it suffices to prove

$\log_a(x^2+y^2+z^2) \sum \frac{1}{\log_a(x)} \leq \log_a(xyz) \sum \frac{1}{\log_a(x)}$

Since $a \in (0,1)$, $\log_a(x)$ is a decreasing concave functions. Therefore, by Jensen

$\log_a(xyz) = \log_a(x) + \log_a(y) + \log_a(z) \geq 3\log_a\left(\frac{x+y+z}{3}\right) = 3\log_a(\frac{1}{3}) \qquad (1)$

Also, since $log_a(x)$ is decreasing, and $x^2+y^2+z^2 \geq \frac{1}{3}(x+y+z)^2 = \frac{1}{3}$ it follows that

$\log_a\left(\frac{1}{3}\right) \geq \log_a(x^2+y^2+z^2) \qquad (2)$

From (1) and (2), $\log_a(xyz) \geq 3\log_a(x^2+y^2+z^2)$

So it now suffices to prove that

$\sum \frac{1}{\log_a(x)} \leq \sum \frac{3x}{\log_a(x)} \qquad (3)$

But since $log_a(x)$ is decreasing, this implies that

(x,y,z) and $\left(\frac{1}{\log_a(x)}, \frac{1}{\log_a(y)}, \frac{1}{\log_a(z)}\right)$ are similarly sorted

Therefore (3) follows from Chebychev. ■

□ Let $a = \sqrt[2010]{2010}$,To compare $a^{a^{\cdots^{a}}}$ (2010) 2010 ?

Solution

$\sqrt[2010]{2010} = 2010^{\frac{1}{2010}} \Longrightarrow (a^{a^{a^{\cdots a}}})^{2010} = 2010$

□ x and y are real numbers. Prove that

$$|2x-y-1| + |x+y| + |y| \geq \frac{1}{3}$$

Find the minimal value of $|2x-y-1|+|x+y|+|y|$, where $\{x,y\} \subset \mathbb{C}$.

Solution

$3|2x-y-1|+3|x+y|+3|y| \geq |2x-y-1|+2|x+y|+3|y| \geq |(2x-y-1)-2(x+y)+3y| = |-1| = 1$

equality when $x = -y = \frac{1}{3}$

□ Find alll the natural number solutions such that $m > n > 1$

and $m^n = n^m$

Solution

For any prime p and any integer x, let $v_p(x)$ be the largest power of p dividing x. $m^n = n^m$ implies that $nv_p(m) = mv_p(n)$. Since $m > n$, $v_p(n) < v_p(m)$, or else $mv_p(n) > nv_p(m)$. This means that $n|m$, so we may set $m = kn$ for some integer k. Then $(kn)^n = n^{kn}$, giving $kn = n^k$, so $n^{k-1} = k$. If $n > 2$, then $n^{k-1} > 3^{k-1}$. It can easily be shown by induction that $3^{k-1} > k$ for all $k > 1$, so we must have that $n = 2$, trivially resulting in $k = 2$ as well. Hence, our only solution is $(2,4)$. Another way: You could take the ln of both sides to get $\frac{\ln m}{m} = \frac{\ln n}{n}$ and since $\frac{\ln x}{x}$ is decreasing for $x > e$, we only need to check $n = 2$. But $(2,4)$ is clearly a solution, and it can be the only as for $m > 4$, $\frac{\ln m}{m}$ would just be lower. This has been posted many times.

□ Find all the real positive numbers x, y knowing that $a = \frac{x+y}{2}, b = \sqrt{xy}, c = \frac{2xy}{x+y}, d = \sqrt{\frac{x^2+y^2}{2}}$ they are natural numbers which sum is 66

Solution

$a = \frac{x+y}{2}, b = \sqrt{xy}, c = \frac{2xy}{x+y}, d = \sqrt{\frac{x^2+y^2}{2}}$

We have $b^2 + d^2 = 2a^2$ where $a, b, d \in \mathbb{N}$

Hence for some coprime integers (m,n) we have

$b = k(m^2-n^2-2mn)$, $d = k(m^2-n^2+2mn)$, $a = k(m^2+n^2)$

In addition $c = \frac{b^2}{a}$, which implies that

$\frac{b^2}{a} = \frac{k(m^2-n^2-2mn)^2}{m^2+n^2} = k(m^2+n^2) - \frac{4kmn(m^2-n^2)}{m^2+n^2} \in \mathbb{N}$

so $m^2+n^2|4kmn(m^2-n^2)$ however since $\gcd(m,n) = 1$, suppose p is prime and $p|m^2+n^2$ then $p \not| mn$ otherwise p divides both m and n which is a contradiction.

Similarly if $p|m^2-n^2$ then $p|((m^2-n^2)+(m^2+n^2)) \Rightarrow p|2m^2$ hence $p|2$

So we can conclude that $m^2+n^2|8k$.

Now $a+b+c+d = 4km^2 - \frac{4mnk(m^2-n^2)}{m^2+n^2} = 66$

Since $\frac{4k(m^2-n^2)}{m^2+n^2}$ is an integer, we can factor out the m on the LHS to yield

$m\left(4km - \frac{4nk(m^2-n^2)}{m^2+n^2}\right) = 66$

so $m|66 \implies m = 1, 2, 3, 6, 11...$

Now $m < 6$ because

$m^2+n^2|8k$ implies $k \geq \frac{(m^2+n^2)}{8}$

And therefore

$66 = 4km^2 - \frac{4kmn(m^2-n^2)}{m^2+n^2} > 2k(m^2+n^2) \geq \frac{(m^2+n^2)^2}{4}$

Subbing in $m \geq 6$ leads to contradiction, therefore (after noting that $m > n \geq 1$) we are left with only three cases

$(m, n) = (2, 1), (3, 1), (3, 2)$

Plugging in shows that only the first case works and $k = 5$, yielding $(a, b, c, d) = (25, 5, 1, 35)$

Solving the system and checking that the solution works gives

$(x, y) = (25 + 5\sqrt{24}, 25 - 5\sqrt{24})$

□ $g(n) = (n^2-2n+1)^{\frac{1}{3}} + (n^2-1)^{\frac{1}{3}} + (n^2+2n+1)^{\frac{1}{3}}$. $\frac{1}{g(1)} + \frac{1}{g(3)} + \ldots + \frac{1}{g(999999)} = ?$

Solution

Factoring the function gives $(n-1)^{2/3} + [(n+1)(n-1)]^{1/3} + (n+1)^{2/3}$, which is in the form $x^2+xy+y^2 = \frac{x^3-y^3}{x-y}$, where $x = (n+1)^{1/3}$ and $y = (n-1)^{1/3}$ Substituting gives $\frac{2}{(n+1)^{1/3}-(n-1)^{1/3}}$, and so $\frac{1}{g(n)} = \frac{(n+1)^{1/3}-(n-1)^{1/3}}{2}$ The answer just telescopes: $\frac{\sqrt[3]{2}-\sqrt[3]{0}+\sqrt[3]{4}-\sqrt[3]{2}+\cdots+\sqrt[3]{1000000}-\sqrt[3]{999998}}{2} = \frac{\sqrt[3]{1000000}}{2} = \boxed{50}$

□ Find all positive solution of system of equation:

$\frac{xy}{2005y+2004x} + \frac{yz}{2004z+2003y} + \frac{zx}{2003x+2005z} = \frac{x^2+y^2+z^2}{2005^2+2004^2+2003^2}$

Solution

Let (x, y, z) be any positive reals such that $x+y+z = 1$, and let $k = \frac{\frac{xy}{2005y+2004x}+\frac{yz}{2004z+2003y}+\frac{zx}{2003x+2005z}}{\frac{x^2+y^2+z^2}{2005^2+2004^2+2003^2}}$. Then (kx, ky, kz) seems to satisfy the equation.

Conversely, for any solution (x, y, z) can be written as (ka, kb, kc), where $a+b+c = 1$ (simply let $k = (x+y+z), a = \frac{x}{k}, b = \frac{y}{k}, c = \frac{z}{k}$.)

Aside from that, setting $a = \frac{x}{2005}$, $b = \frac{y}{2004}$, and $c = \frac{z}{2003}$ seemed to make things a bit nicer...

□ Solve (a,b $>$ 0) $\sqrt[4]{a+x} + \sqrt[4]{a-x} = b$

Solution

Let $y = \sqrt[4]{a+x}$ and $z = \sqrt[4]{a-x}$. We have $y+z = b$ and $y^4+z^4 = 2a$. Let $xy = c$. Then $b^4 - 4b^2c + 2c^2 = 2a$ and $2(c-b^2)^2 = 2a+b^4$. Thus $c = b^2 \pm \sqrt{\dfrac{2a+b^4}{2}}$ and $y, z = b \pm \sqrt{b^2-4c}$ and $x = \max(y, z)^4 - a$.

□ How many three-digit numbers are there such that no two adjacent digits of the number are consecutive?

Solution

If the first digit is 9, then there are 9 choices (0-7, 9) for the second digit. But if the second digit is 0 or 9, then there are 9 choices for the third digit; otherwise, there are 8 choices. In total, this case accounts for $2 \cdot 9 + 7 \cdot 8 = 18 + 56 = 74$ possibilities.

If the first digit is 8 or 1, then there are 8 choices (0-6,8) for the second digit. If the second digit is 0 (in the case of first digit 8) or 9 (in the case of first digit 2) then there are 9 choices for the third digit; otherwise, there are 8 choices. This case accounts for $2(9 + 7 \cdot 8) = 2 \cdot 65 = 130$ possibilities.

In the rest of the cases, there are 8 choices for the second digit. If the second digit is 0 or 9, then there are 9 choices for the third digit; otherwise, there are 8 choices. This case accounts for

$6(2 \cdot 9 + 6 \cdot 8) = 6 \cdot 66 = 396$ possibilities.

The total is $74 + 130 + 396 = \boxed{600}$.

□ Let be $p \in \mathbb{N}, p \neq 0$. Prove that there is an increasing sequence of positives integers $(a_n)_{n\in\mathbb{N}^*}$ such that $a_{2n} + a_{2n-1} = pa_n$ for every $n \in \mathbb{N}$ only and only if $p \geq 4$.

Solution

First $a_2 + a_1 = pa_1 \Rightarrow a_2 = (p-1)a_1 \Rightarrow p - 1 > 1 \rightarrow p > 2$

Proof that $p \neq 3$

Assume there exists such a sequence $\{a_i\}_{i\in\mathbb{N}}$ with $p = 3$.

$\therefore a_{2n} + a_{2n-1} = 3a_n(*)$ and since $a_{2n-1} > a_n \Rightarrow a_{2n} < 2a_n$

which becomes $a_{2^n} < 2a_{2^{n-1}} < \cdots < 2^n a_1 \qquad (1)$

Similarly, $a_{2n} > a_{2n-1} \Rightarrow a_{2n-1} < \frac{3}{2}a_n$ (because of $(*)$)

when combined with (1) becomes: $a_{2^n-1} < \frac{3}{2}a_{2^{n-1}} < 3 \cdot 2^{n-2}a_1 \qquad (2)$

Now, from $(*)$ follows this identity: $a_{2^n} + a_{2^n-1} + 3a_{2^{n-1}-1} + 3^2 a_{2^{n-2}-1} + \cdots + 3^{n-1}a_1 = 3^n a_1 \qquad (3)$

Using (3) and pluggin in inequalities (1) and (2), then dividing through a_1 gives

$2^n + (3 \cdot 2^{n-1} + 3^2 \cdot 2^{n-2} + \cdots + 3^{n-1}) > 3^n$

$\therefore (3 \cdot 2^{n-1} + 3^2 \cdot 2^{n-2} + \cdots + 3^{n-1}) > 3^n - 2^n$

$\therefore 0 > 2^{n-1}$ contradiction!

Hence $p \neq 3$, ■

Sequence for $p \geq 4$

When p is even define a_i as follows

$a_1 = 2$, $a_{2n} = \frac{p}{2}a_n + 1$, $a_{2n-1} = \frac{p}{2}a_n - 1$

When p is odd:

$a_1 = 2$, $a_{2n} = \left\lfloor \frac{p}{2}a_n \right\rfloor + 1$, $a_{2n-1} = \left\lfloor \frac{p}{2}a_n \right\rfloor - 2\left(\frac{1}{2} - \frac{p}{2}a_n + \left\lfloor \frac{p}{2}a_n \right\rfloor\right)$

□ $17 + 187 + 1887 + ... + 188....87$, where the alst term contains exactly n 8's

Solution

We can see intuitively that the sum is $\sum_{k=0}^{n+1} 10^k + 8 \cdot \frac{10^k - 1}{9} - 1$.

Now we simply break this up...

$\sum 10^k = \frac{10^{n+2}-1}{9}$.

$\sum 8 \cdot \frac{10^k - 1}{9} = \frac{8}{9}\sum 10^k - 1 = \frac{8}{9}\left(\frac{10^{n+2}-1}{9} - (n+2)\right)$.

$\sum(-1) = -(n+2)$.

Summing, we get $\boxed{\dfrac{17}{9}\left(\dfrac{10^{n+2}-1}{9} - n - 2\right)}$.

Another way Define a function $f(n) = \underbrace{188\ldots87}_{n \text{ eights}}$.

Notice that $f(n+1) - f(n) = 17 \cdot 10^n$.

Therefore, $f(n+1) = 17 \cdot 10^n + f(n) = 17 \cdot 10^n + 17 \cdot 10^{n-1} + f(n-1) = \cdots = \sum_{k=0}^{n} 17 \cdot 10^k$.

Summing the geometric series, we find $f(n+1) = 17 \cdot \frac{10^{n+1}-1}{9}$.

Therefore, $\sum_{k=1}^{n} f(n) = \frac{17}{9}\sum_{k=1}^{n} 10^{k+1} - 1$, which gives the same answer as the above solution.

□ let $a, b, c > 0$ such that $abc = 1$ prove that:

$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq 1$

Solution

We can rewrite the inequality as:

$$\frac{1}{a + b + (abc)^{\frac{1}{3}}} + \frac{1}{b + c + (abc)^{\frac{1}{3}}} + \frac{1}{c + a + (abc)^{\frac{1}{3}}} \leq \frac{1}{(abc)^{\frac{1}{3}}}$$

By substituting $a = x^3, b = y^3, c = z^3$ where $x, y, z \geq 0, xyz = 1$; we get this to be equivalent to:

$$\frac{1}{x^3+y^3+xyz}+\frac{1}{y^3+z^3+xyz}+\frac{1}{z^3+x^3+xyz} \leq \frac{1}{xyz}$$

Using the inequality $x^3+y^3 = (x+y)(x^2-xy+y^2) \geq xy(x+y)$(AM-GM) we have:

$$\sum_{cyc}\frac{1}{x^3+y^3+xyz} \leq \sum_{cyc}\frac{1}{xy(x+y+z)} = \sum_{cyc}\frac{z}{x+y+z} = 1$$

Since $xyz = 1$. Equality holds iff $a = b = c = 1$.

$\square$ Solve in natural the equation :$\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}=\frac{1}{\sqrt{8}}$

Solution

Obviously, both x and y are non-squares. From

$\frac{1}{x}+\frac{1}{y}+\frac{2}{\sqrt{xy}}=\frac{1}{8}$

we conclude that xy is a perfect square, hence there exist $a, b, z \in \mathbb{N}$ such that $x = a^2z, y = b^2z$ and z is not a perfect square.

Then

$\sqrt{\frac{2}{z}}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{1}{2}$

Therefore $z = 2t^2$ for some $t \in \mathbb{N}$. Now

$\frac{2}{a}+\frac{2}{b}=t \iff b=\frac{2a}{at-2}$

Since this implies $2a \geqslant at-2$, we get $a \leqslant \frac{2}{t-2}$. From there, possible values for t are $t \in \{3, 4\}$ (since a can't be less than 1).

Case 1. $t = 3$. Then $a \leqslant 2$. For $a = 1$ we get $b = 2$, and for $a = 2$ we get $b = 1$. Corresponding values for (x, y) are $(x, y) = (2a^2t^2, 2b^2t^2) \in \{(18, 72), (72, 18)\}$

Case 2. $t = 4$. Then $a \leqslant 1$. For $a = 1$ we get $b = 1$. Corresponding values for (x, y) are $(x, y) = (2a^2t^2, 2b^2t^2) = (32, 32)$

NOTE: From $x = 2a^2t^2 \wedge y = 2b^2t^2$ we can simplify into $x = 2u^2, y = 2v^2$ and reduce the equation into a known and easy problem $\frac{1}{u}+\frac{1}{v}=\frac{1}{2}$, with solutions $(u, v) \in \{(3, 6), (6, 3), (4, 4)\}$

$\square$ Let R denote a non-negative rational number. Determine a fixed set of integers a, b, c, d, e, f, such that for *every* choice of R,

$$\left|\frac{aR^2+bR+c}{dR^2+eR+f}-\sqrt[3]{2}\right| < \left|R-\sqrt[3]{2}\right|.$$

Solution

We wish to determine fixed a, b, c, d, e, f to satisfy the inequality for all nonnegative rational R. As $R \to \sqrt[3]{2}$ through a sequence of rational numbers, the right hand side of this inequality approaches zero. Consequently, the left hand side must vanish if we set $R = \sqrt[3]{2}$. Hence,

$$a \cdot 2^{\frac{2}{3}} + b \cdot 2^{\frac{1}{3}} + c = 2d + e \cdot 2^{\frac{2}{3}} + f \cdot 2^{\frac{1}{3}}.$$

It follows that $a = e$, $b = f$, $c = 2d$. On substituting back into the inequality and factoring out the common factor $R - \sqrt[3]{2}$ from both sides, we obtain

$$\left|\frac{aR+b-d \cdot 2^{\frac{1}{3}}\left(R+2^{\frac{1}{3}}\right)}{dR^2+aR+b}\right| < 1.$$

For the last inequality to be satisfied, it suffices to let a, b, d be positive integers and make the numerator nonnegative, i.e., by letting $a > d \cdot 2^{\frac{1}{3}}$, $b > d \cdot 2^{\frac{2}{3}}$. A simple choice is $d = 1$, $a = b = 2$, leading to

$$\frac{2R^2 + 2R + 2}{R^2 + 2R + 2}.$$

☐ Prove that the function $n\varphi(n)$ is 1-1? Does it follow from unique factorization?

Solution

Suppose $m\phi(m) = n\phi(n)$

Let the canonical representations of m and n be

$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_a^{\alpha_a}$

$n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_b^{\beta_b}$

such that $p_1 < p_2 < \cdots < p_a$ and $q_1 < q_2 < \cdots < q_b$

We know that

$\phi(m) = p_1^{\alpha_1-1} p_2^{\alpha_2-1} \cdots p_a^{\alpha_a-1}(p_1-1)(p_2-1)\cdots(p_a-1)$

$\phi(n) = q_1^{\beta_1-1} q_2^{\beta_2-1} \cdots q_b^{\beta_b-1}(q_1-1)(q_2-1)\cdots(q_b-1)$

$m\phi(m) = n\phi(n)$

$\implies p_1^{2\alpha_1-1} p_2^{2\alpha_2-1} \cdots p_a^{2\alpha_a-1}(p_1-1)(p_2-1)\cdots(p_a-1) = q_1^{2\beta_1-1} q_2^{2\beta_2-1} \cdots q_b^{2\beta_b-1}(q_1-1)(q_2-1)\cdots(q_b-1) \quad \longrightarrow (1)$

Claim: $p_a = q_b$

Proof:

On the contrary, assume $p_a \neq q_b$

Without loss of generality, assume $p_a > q_b$

Then, all the primes composed in the canonical representation of RHS of (1) are less than p_a and hence, equality doesn't occur which is false.

So, $p_a = q_b$

If follows that $\alpha_a = \beta_b$ as $q_b \nmid q_1^{2\beta_1-1} q_2^{2\beta_2-1} \cdots q_{(b-1)}^{2\beta_{(b-1)}-1}(q_1-1)(q_2-1)\cdots(q_b-1)$

So, $p_a^{2\alpha_a-1}(p_a-1) = q_b^{2\beta_b-1}(q_b-1)$ and (1) reduces to

$p_1^{2\alpha_1-1} p_2^{2\alpha_2-1} \cdots p_{a-1}^{2\alpha_{(a-1)}-1}(p_1-1)(p_2-1)\cdots(p_{(a-1)}-1) = q_1^{2\beta_1-1} q_2^{2\beta_2-1} \cdots q_{(b-1)}^{2\beta_{(b-1)}-1}(q_1-1)(q_2-1)\cdots(q_{(b-1)}-1)$

Now, Without loss of generality, assume $a = b + x$ for $x \geq 0$

Also, we can similarly argue and claim that $p_{(a-1)} = q_{(b-1)}, \alpha_{(a-1)} = \beta_{(b-1)}$

And continuing in this manner, we get

$p_1^{2\alpha_1-1} p_2^{2\alpha_2-1} \cdots p_x^{2\alpha_x-1}(p_1-1)(p_2-1)\cdots(p_x-1) = 1$

$\implies x = 0 \implies a = b$ and also, $p_i = q_i, \alpha_i = \beta_i$ for $i = 1, 2, \cdots, a$

and hence, $m = n$

☐ Find the equations of the lines that pass through the origin and are inclined at 75^o to the line $x + y + (y - x)\sqrt{3} = a$

Solution

After rearranging and simplifying, the given equation becomes

$y = -(2-\sqrt{3})x + \frac{a}{2}(\sqrt{3}-1)$

If k is the slope of the desired line, then

$\frac{k+2-\sqrt{3}}{1-(2-\sqrt{3})k} = \pm \tan 75^\circ$

$\tan 75^\circ = \tan(30^\circ + 45^\circ) = \frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}$

Case 1.
$\frac{k+2-\sqrt{3}}{1-(2-\sqrt{3})k} = 2+\sqrt{3}$
$k+2-\sqrt{3} = 2+\sqrt{3}-k$
$2k = 2\sqrt{3}$
Hence the first line is $\boxed{y = x\sqrt{3}}$
Case 2.
$\frac{k+2-\sqrt{3}}{1-(2-\sqrt{3})k} = -(2+\sqrt{3})$
$k+2-\sqrt{3} = -2-\sqrt{3}+k$
This can be "satisfied" only for $k=\infty$, hence the second line is $\boxed{x=0}$

□ Let $a, b, c \geq 0$ and $a+b+c=5$. Determine the min value of:

$$A = \sqrt{a+1} + \sqrt{2b+1} + \sqrt{3c+1}$$

Solution

$1+\sqrt{x+y+1} \leq \sqrt{x+1} + \sqrt{2y+1}$ (1)
$1+\sqrt{x+y+1} \leq \sqrt{x+1} + \sqrt{3y+1}$ (2)
expanding/rearranging (1) and (2) shows that they are true
$\therefore A \geq 1+\sqrt{a+b+1} + \sqrt{3c+1} \geq 2+\sqrt{a+b+c+1} = 2+\sqrt{6}$
equality $(a,b,c) = (5,0,0)$

□ In a triangle ABC, choose an interior point P. Let AP meet BC at L, BP meet AC at M and CP meet AB at N.

(a) Prove that the value of the expression $\frac{AP}{AL} + \frac{BP}{BM} + \frac{CP}{CN}$ is independent of the choice of triangle or the choice of P: it is constant for every triangle and point P.

(b) Given a triangle ABC, find the point P such that $(\frac{AP}{AL})^2 + (\frac{BP}{BM})^2 + (\frac{CP}{CN})^2$ is minimised.

Solution

Denote by $[XYZ]$ the area of $\triangle XYZ$. For part a) $\frac{AP}{AL} = \frac{[ABP]}{[BLP]} = \frac{[APC]}{[PLC]} = \frac{[ABP]+[APC]}{[ABC]} = 1 - \frac{[BPC]}{[ABC]}$. So summing all of the terms give $3-1=2$.

For part b) just use Cauchy-Schwarz: $\frac{AP}{AL}^2 + \frac{BP}{BM}^2 + \frac{CP}{CN}^2 \geq \frac{(\frac{AP}{AL}+\frac{BP}{BM}+\frac{CP}{CN})^2}{3} = \frac{4}{3}$ with equality if and only $\frac{AP}{AL} = \frac{BP}{BM} = \frac{CP}{CN}$. From above, $\frac{AP}{AL} = 1 - \frac{[BPC]}{[ABC]}$ etc, so $\frac{[BPC]}{[ABC]} = \frac{1}{3}$ etc. This implies that P is the centroid.

□ Let $x, y, x \in \mathbb{C} - \mathbb{R}$ so that $\begin{cases} x^2 = y+z \\ y^2 = z+x \\ z^2 = x+y \end{cases}$. Prove that $|x|+|y|+|z| = 2+\sqrt{2}$.

Solution

The system gives $x^2+x = y^2+y = z^2+z = x+y+z$
Since a quadratic has only two complex solutions, two of x, y, z are equal. wlog $x=y$
Then $x^2 = x+z$ and $z^2 = 2x$ giving $(x^2-x)^2 = 2x \implies x(x-2)(x^2+1) = 0$
$x = y = 0, 2, \pm i \implies x^2 - x = z = -1 \pm i$ because $x \notin \mathbb{Z}$
$|x|+|y|+|z| = 2|\pm i| + |-1 \pm i| = 2+\sqrt{2}$

□ Find x such that: $1+a+a^2+...+a^x = (1+a)(1+a^2)(1+a^4)(1+a^8)$, where $a>0, a \neq 1$.

Solution

Check the coefficient of x^r in RHS
<u>Lemma:</u> The product of unique terms gives x^r for unique r

Proof: Note that $r = a^s + a^t + a^z + a^w$ is the only possible way because it has to be obtained through products of powers of a where s, t, z, w are whole numbers.

It is $base - a$ representation, there is a unique representation of r and hence coefficient of a^r is 1 for any r and $r \leq 16 \Longrightarrow \boxed{x = 16}$

Another way: By GP summation,

$LHS = \frac{a^{x+1}-1}{a-1} = RHS = (a+1)(a^2+1)(a^4+1)(a^8+1)$

$\Longrightarrow a^{x+1} - 1 = (a-1)(a+1)(a+1)(a^2+1)(a^4+1)(a^8+1)$

$\Longrightarrow a^{x+1} - 1 = (a^2-1)(a^2+1)(a^4+1)(a^8+1) = (a^4-1)(a^4+1)(a^8+1) = (a^8-1)(a^8+1) = a^{16}-1$

$\Longrightarrow x + 1 = 16 \Longrightarrow \boxed{x = 15}$

□ Let $f(x) = p(\sin x)^2 + q \sin x \cos x + r(\cos x)^2, p \neq r, q \neq 0$ .Find max and min value of $f(x)$ in the form of p, q, r.

Solution

The maximum of $a \sin x + b \cos x$ is $\sqrt{a^2 + b^2}$. The proof of this relies on the computation of $\sin \alpha + \beta$, by letting $\alpha = x$ and $\beta = \sin^{-1} \frac{b}{\sqrt{a^2+b^2}}$.

$f(x) = p \sin^2 x + q \sin x \cos x + r \cos^2 x$.

There are three cases: $p < r$, $p = r$, and $p > r$.

Case 1: $p = r$. Thus, $f(x) = p + q \sin x \cos x = p + \frac{q}{2} \sin 2x$. Max: $p + \frac{q}{2}$. Min: $p - \frac{q}{2}$.

Case 2: $p < r$. Thus, $f(x) = p + \frac{q}{2} \sin 2x + (r-p) \cos^2 x = p + \frac{q}{2} \sin 2x + (r-p) \cos^2 x + \frac{r-p}{2} - \frac{r-p}{2}$ $= p + \frac{q}{2} \sin 2x + \frac{r-p}{2} \cos 2x + \frac{r-p}{2}$.

Thus, the maximum is $p + \frac{r-p}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{r-p}{2}\right)^2}$, and the minimum is $p + \frac{r-p}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{r-p}{2}\right)^2}$

Case 3: $p > r$. Thus, $f(x) = r + \frac{q}{2} \sin 2x + (p-r) \sin^2 x = r + \frac{q}{2} \sin 2x + (p-r) \cos^2 x + \frac{p-r}{2} - \frac{p-r}{2}$ $= r + \frac{q}{2} \sin 2x - \frac{p-r}{2} \cos 2x + \frac{p-r}{2}$.

Thus, the maximum is $r + \frac{p-r}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p-r}{2}\right)^2}$, and the minimum is $r + \frac{p-r}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p-r}{2}\right)^2}$

□ prove that $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} < \frac{n\sqrt{n+1}-1}{n+1}$

Solution

$\Longleftrightarrow \sum_{k=2}^{n+1} \frac{1}{k} < \frac{n}{\sqrt{n+1}}$.

Proof:

$\sum_{k=2}^{n+1} \frac{1}{k} < \int_1^{n+1} \frac{1}{x} dx < \int_1^{n+1} \frac{1}{\sqrt{x}} dx = 2(\sqrt{n+1} - 1)$

$= \frac{2}{\sqrt{n+1}+1} < \frac{2}{\sqrt{n+1}} < \frac{n}{\sqrt{n+1}}$

□ In triangle ABC, $AB = AC = 1$ and $BC = \sqrt{2}$. Let O be the midpoint of BC and P be a point chosen at random on the interior of the triangle. If H is the foot of the altitude from P to AB, l is the perpendicular bisector of OH, and P' is the intersection of l and AB, compute the probability that $\angle POP'$ is acute.

Solution

Let the coordinates of A, B, C be $(0,0), (1,0), (0,1)$ respectively. Then $O\left(\frac{1}{2}, \frac{1}{2}\right)$. If $P(a, b)$ then $H(a, 0)$, where a, b are variables.

If M is the midpoint of OH, then $M\left(\frac{2a+1}{4}, \frac{1}{4}\right)$.

The slope of line OH is $k_{OH} = \frac{\frac{1}{2}}{\frac{1}{2}-a} = \frac{1}{1-2a}$. Hence the slope of l is $k_l = 2a - 1$, and its equation is $y - \frac{1}{4} = (2a-1)\left(x - \frac{2a+1}{4}\right)$. For $y = 0$ we get $x = \frac{2a+1}{4} - \frac{1}{4(2a-1)} = \frac{2a^2-1}{2(2a-1)}$, hence $P'\left(\frac{2a^2-1}{2(2a-1)}, 0\right)$

Angle POP' is acute iff $\overrightarrow{OP} \cdot \overrightarrow{OP'} > 0$

$\overrightarrow{OP} = \left\langle a - \frac{1}{2}, b - \frac{1}{2} \right\rangle = \left\langle \frac{2a-1}{2}, \frac{2b-1}{2} \right\rangle$

$\overrightarrow{OP'} = \left\langle \frac{2a^2-1}{2(2a-1)} - \frac{1}{2}, -\frac{1}{2} \right\rangle = \left\langle \frac{a^2-a}{2a-1}, -\frac{1}{2} \right\rangle$

$\overrightarrow{OP} \cdot \overrightarrow{OP'} = \frac{2a-1}{2} \cdot \frac{a^2-a}{2a-1} - \frac{2b-1}{4} = \frac{1}{2}\left(a^2 - a - b + \frac{1}{2}\right)$

For this to be positive, we must have $b < a^2 - a + \frac{1}{2}$

The equation of the line BC is $b = 1 - a$. The intersection of this line and the above parabola in the first quadrant is obtained from $1 - a = a^2 - a + \frac{1}{2} \wedge a > 0 \implies a = \frac{1}{\sqrt{2}}$.

Let $V\left(0, \frac{1}{2}\right), Q\left(\frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}\right), R\left(\frac{1}{\sqrt{2}}, 0\right)$.

The area of "parabolic trapezoid" $ARQV$ is

$S_1 = \int_0^{1/\sqrt{2}} \left(a^2 - a + \frac{1}{2}\right) \mathrm{d}a = \frac{a^3}{3} - \frac{a^2}{2} + \frac{a}{2}\Big|_0^{1/\sqrt{2}} = \frac{4\sqrt{2}-3}{12}$

The area of triangle RQB is $S_2 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)^2 = \frac{3-2\sqrt{2}}{4}$

The total favorable area is $S = S_1 + S_2 = \frac{3-\sqrt{2}}{6}$

Since the area of $\triangle ABC$ is $S_0 = \frac{1}{2}$, we get $p = \frac{S}{S_0} = \frac{3-\sqrt{2}}{3} = 1 - \frac{\sqrt{2}}{3}$

□ In how many ways can 5 persons be seated in a circle such that there are only three chairs where they can be seated? and also please generalise it

Solution

Suppose there are m people and n seats, with $m \geq n$. There are $\binom{m}{n}$ ways of choosing n people to seat, and within these n people there are $n!$ ways of seating them. However, for most 'circular table' problems, the sequence of people ABC is the same as BCA. In the sequence of people $p_1, p_2 ... p_n$ there are a total of n derangements sustaining the same order ($a_1, a_2, \dots a_n$ and $a_2, a_3, \dots a_n, a_1$ etc.). So in that case we divide $n!\binom{m}{n}$ by n to receive $(n-1)!\binom{m}{n} = \frac{m!}{(m-n)!n}$. In your example, $(n-1)!\binom{m}{n} = \frac{m!}{(m-n)!n} = 20$.

It may interest you to note that when $n > m$, the situation is symmetric with the first and there are $(m-1)!\binom{n}{m} = \frac{n!}{(n-m)!m}$ combinations.

□ Consider the point $A(5,1)$. Find the equations of the sides of the triangle ΔABC, knowing that this triangle has a median with the equation $y = 2x$, and an altitude with the equation $y = -x$.

Solution

Lines $m : y = 2x$ and $h : y = -x$ are given.

Line AB is perpendicular to h.

$AB : y - 1 = 1(x - 5) \iff AB : y = x - 4$

The vertex B is the intersection of AB and m.

$y_B = x_B - 4 \wedge y_B = 2x_B$

The solution to the system is $B(-4, -8)$

The vertex C must belong to h and the midpoint of AC must belong to m.

$y_C = -x_C \wedge \frac{y_C+1}{2} = 2\frac{x_C+5}{2}$

The solution to the system is $C(-3, 3)$

The line AC: $y - 1 = \frac{-2}{8}(x - 5) \iff y = -\frac{x-9}{4}$

The line BC: $y - 3 = \frac{11}{1}(x + 3) \iff y = 11x + 36$

□ Solve the equation $x \in \mathbb{C}$:

$$\left(\frac{x+1}{x+2}\right)^2 + \left(\frac{x+1}{x}\right)^2 = m(m-1)$$

Solution

$\frac{x^2(x+1)^2+(x+1)^2(x+2)^2}{(x^2+2x)^2} = m^2 - m$

$\frac{(x+1)^2(2x^2+4x+4)}{(x^2+2x)^2} = m^2 - m$

$\frac{2(x^2+2x+1)(x^2+2x+2)}{(x^2+2x)^2} = m^2 - m$

Put $u := x^2 + 2x$

$(m^2 - m)u^2 = 2(u + 1)(u + 2) = 2u^2 + 6u + 4$

$(m^2 - m - 2)u^2 - 6u - 4 = 0$

Case 1. $m \in \{-1, 2\}$. Then the equation becomes $3u + 2 = 0 \iff u = -\frac{2}{3} \iff x^2 + 2x + \frac{2}{3} = 0 \iff x_{1,2} = -1 \pm \frac{\sqrt{3}}{3}$

Case 2. $m \notin \{-1, 2\}$. Then $u_{1,2} = \frac{6 \pm \sqrt{16m^2 - 16m + 4}}{2(m^2 - m - 2)} = \frac{3 \pm (2m-1)}{m^2 - m - 2}$

$u_1 = \frac{2m+2}{m^2-m-2} \iff (x + 1)^2 = \frac{m^2+m}{m^2-m-2} = \frac{m}{m-2} \iff \iff x_{1,2} = -1 \pm \sqrt{\frac{m}{m-2}}$

$u_2 = \frac{4-2m}{m^2-m-2} \iff (x + 1)^2 = \frac{m^2-3m+2}{m^2-m-2} = \frac{m-1}{m+1} \iff \iff x_{3,4} = -1 \pm \sqrt{\frac{m-1}{m+1}}$

Neither of $x_{1,2,3,4}$ can take values of 0 or -2, hence there aren't any redundant solutions.

□ Prove that if $d|2n^2$ than $n^2 + d$ cannot be a perfect square.

Solution

Suppose $n^2 + \frac{2n^2}{k} = m^2$ where k is a positive integer. Then $n^2(k + 2) = km^2$, or $\frac{k+2}{k} = \frac{m^2}{n^2}$. Let g be the greatest common divisor of m and n so $m = gx$, $n = gy$, and $\frac{m^2}{n^2} = \frac{x^2}{y^2}$ and this fraction is irreducible. If k is odd then $gcd(k, k + 2) = 1$ so the fraction $\frac{k+2}{k}$ is also irreducible, so we must have $k + 2 = x^2$, $k = y^2$ which is impossible as no two squares differ by 2. If k is even then we can let $k = 2z$ and we have $\frac{k+2}{k} = \frac{z+1}{z}$ which is irreducible. Thus, $z + 1 = x^2$ and $z = y^2$, which is impossible as no two positive squares differ by 1.

□ Let function f be defined such that - $f(c) = (b - c)(a + c) + c^2$ where a,b,c are positive reals and a,b are fixed with $a \geq b$. Pove that the following inequality holds true - $f(a - b) + f(c) \leq a^2 + b^2$.

Solution

$$\begin{aligned} f(a-b) + f(c) =& (b - a + b)(a + a - b) + (a - b)^2 + (b - c)(a + c) + c^2 \\ &= (2b - a)(2a - b) + (a - b)^2 + ab - ca + bc - c^2 + c^2 \\ &= 4ab - 2(a^2 + b^2) + ab + (a - b)^2 + ab - ca + bc \\ &= 2[2ab - a^2 - b^2] + a^2 + b^2 - c(a - b) \\ &\leq 2[2ab - 2ab] + a^2 + b^2 - c(a - a) = a^2 + b^2; \end{aligned}$$

Hence proved.

□ Let c be a nonnegative integer, and define $a_n = n^2 + c$ (for $n \geq 1$). Define d_n as the greatest common divisor of a_n and a_{n+1}. (a) Suppose that $c = 0$. Show that $d_n = 1$, $\forall n \geq 1$. (b) Suppose that $c = 1$. Show that $d_n \in \{1, 5\}$, $\forall n \geq 1$. (c) Show that $d_n \leq 4c + 1$, $\forall n \geq 1$.

Solution

a) We have $a_n = n^2$ and $a_{n+1} = (n + 1)^2$, and since $(n, n + 1) = 1$, the result follows.

b) We have $a_n = n^2 + 1$ and $a_{n+1} = n^2 + 2n + 2$. Now use Euclidean Algorithm. If n is even, put $n = 2k$. So we have $d_n = (4k^2 + 4k + 2, 4k^2 + 1) = (4k^2 + 1, 4k + 1) = (4k + 1, k - 1) = (k - 1, 5)$, and the result follows. If n is odd, put $n = 2k - 1$. So we have $d_n = (4k^2 + 1, 4k^2 - 4k + 2) = (4k^2 + 1, 4k - 1) = (4k - 1, k + 1) = (k + 1, 5)$, and the result follows.

c) We have $a_n = n^2 + c$ and $a^2_{n+1} = n^2 + 2n + c + 1$. Now use Euclidean Algorithm again. If n is even, put $n = 2k$. So we have $d_n = (4k^2 + 4k + c + 1, 4k^2 + c) = (4k^2 + c, 4k + 1) = (4k + 1, k - c) = (k - c, 4c + 1)$, and the result follows. If n is odd, put $n = 2k - 1$. So we have $d_n = (4k^2 + c, 4k^2 - 4k + c + 1) = (4k^2 + c, 4k - 1) = (4k - 1, k + c) = (k + c, 4c + 1)$, and the result follows.

□ Find all natural number such as n that that $(2n)^2n+1$ and n^n+1 are prime number[/list][/code]

Solution

First note the trivial solution $n = 1$, and from now on assume $n > 1$

Suppose $a, n \in \mathbb{N}$ and $a^n + 1$ is prime, then n must be a power of 2

proof

let $n = r \cdot 2^k$ with $\gcd(r, 2) = 1$

then $a^{r \cdot 2^k} + 1 = \left(a^{2^k}\right)^r + 1 = \left(a^{2^k} + 1\right)\left(\left(a^{2k}\right)^{r-1} - \left(a^{2k}\right)^{r-2} + \cdots + 1\right)$ Hence not prime

So if $n = 2^k$ we have $\left(2^k\right)^{2^k} + 1 = 2^{k \cdot 2^k} + 1$ is prime

From our last proof it is clear that $k2^k$ must be a power of 2, hence k is also a power of 2.

Now if $(2n)^{2n} + 1$ is also a prime then $2^{(k+1)2^{k+1}} + 1$ is a prime. and as before $k + 1$ must be a power of two.

Ovbiously the only k such that both k and $k + 1$ are powers of 2 is $k = 1$

Hence our answer is $\boxed{n = 1, 2}$

□ Evaluate: $\sum\limits_{n=1}^{\infty} \arctan\left(\frac{1}{n^2-n+1}\right)$

Solution

Now $\tan(A - B) = \frac{tanA - tanB}{1 + tanAtanB}$ So when you sub. $X = tanA$ and $Y = tanB$ in the above then you get $arctanX - arctanY = \arctan\left(\frac{X-Y}{1+XY}\right)$ Sub. $X = n$ and $Y = n - 1$ in the above then you get $\tan^{-1}(n) - \tan^{-1}(n-1) = \arctan\left(\frac{1}{n^2-n+1}\right)$ So $\sum\limits_{n=1}^{n} \arctan\left(\frac{1}{n^2-n+1}\right) = [\tan^{-1}(n) - \tan^{-1}(n-1)] + ... + [\tan^{-1}(1) - \tan^{-1}(0)]$ Therefore $\sum\limits_{n=1}^{n} \arctan\left(\frac{1}{n^2-n+1}\right) = \tan^{-1}(n)$ As n tends to infinity, $\tan^{-1}(n)$ tends to $\frac{\pi}{2}$. Therefore the sum is equal to $\frac{\pi}{2}$.

□ Find all complex numbers a, b, c so that : $\begin{cases} a^3 + b^3 + c^3 = 24 \\ (a+b)(b+c)(c+a) = 64 \\ |a+b| = |b+c| = |c+a| \end{cases}$.

Solution

From $(a+b+c)^3 - a^3 - b^3 - c^3 = 3(a+b)(b+c)(c+a)$ we get

$(a+b+c)^3 = 216 \iff a+b+c = 6e^{i\phi}$ where $\phi \in \{0, \pm 2\pi/3\}$

If $a + b = \rho e^{i\gamma}, b + c = \rho e^{i\alpha}, c + a = \rho e^{i\beta}, \rho \in \mathbb{R}$, then

$\rho^3 e^{i(\alpha+\beta+\gamma)} = 64 \iff \rho = 4 \wedge \alpha + \beta + \gamma = 2k\pi, k \in \mathbb{Z}$

From $a + b = 4e^{i\gamma}, b + c = 4e^{i\alpha}, c + a = 4e^{i\beta}$ we get $a + b + c = 2(e^{i\alpha} + e^{i\beta} + e^{i\gamma})$, hence

$e^{i\alpha} + e^{i\beta} + e^{i\gamma} = 3e^{i\phi}$

Case 1. $\phi = 0$

Then $\cos\alpha + \cos\beta + \cos\gamma = 3 \wedge \sin\alpha + \sin\beta + \sin\gamma = 0$.

Obviously, the first equation can be satisfied only for $\alpha, \beta, \gamma \in 2\mathbb{Z}\pi$, hence $a + b = b + c = c + a = 4 \iff a = b = c = 2$

Case 2. $\phi = 2\pi/3$

Then $\cos\alpha + \cos\beta + \cos\gamma = -\frac{3}{2} \wedge \sin\alpha + \sin\beta + \sin\gamma = -\frac{3\sqrt{3}}{2}$

Squaring and adding up those two, we get

$3 + 2\left(\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha)\right) = 9$

which yields

$\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = 3$

Obviously, this can be satisfied only if $\alpha = \beta = \gamma$, and with $\alpha + \beta + \gamma = 2k\pi$ we get $\alpha = \beta = \gamma = 2k\pi/3$. From $a + b = b + c = c + a = 4e^{i2k\pi/3}$ we get

$\boxed{a = b = c = 2e^{i2k\pi/3}, k \in \mathbb{Z}}$

or

$\boxed{a = b = c = 2 \vee a = b = c = -1 \pm i\sqrt{3}}$

Case 3. $\phi = -2\pi/3$

Similar discussion as in the Case 2.

☐ Let ABC be a triangle for which $AB \neq AC$. Denote the its centroid G and the its incircle $C(I, r)$. Prove that $IB \cdot IC = r \cdot IA \iff IG \perp BC$.

Solution

Let M be the midpoint of BC, D the tangency point of the incircle (I) with BC and P the second intersection of AI with the circumcircle (O) of $\triangle ABC$. It's well-known that P is the circumcenter of $\triangle IBC$. Thus, IP, ID are isogonals with respect to $\angle BIC \Longrightarrow IB \cdot IC = 2ID \cdot IP = 2r \cdot IP$ $(\star)$. Obviously, $IG \perp BC \iff IG \parallel MP$. Hence by Thales theorem we claim that $IG \perp BC \iff \frac{AI}{IP} = \frac{AG}{GM} = 2$. Then, from the expression $(\star)$, it follows that $IG \perp BC \iff IB \cdot IC = r \cdot IA$.

☐ $ABCD$ is a rectangle, labelled anti-clockwise, with A at the bottom left-hand corner. E is a point on AB, closer to B than to A. F is a point on BC (roughly half-way between them). EC meets DF at G, AF meets EC at H and AF meets DE at J. Triangle CGF has an area of 1, the quadrilateral $BEHF$ has an area of 2 and triangle AEJ has an area of 3. What is the area of the quadrilateral $DJHG$?

Solution

Let $BF = a$, $FC = b$, so $DA = a + b$

Note that the area of CDE is $\frac{1}{2}CD(a + b)$

Also, $[CDF] = \frac{1}{2}CDb$ and $[ABF] = \frac{1}{2}CDa$

So $[CDE] - [CDF] - [ABF] = \frac{1}{2}CD(a + b) - \frac{1}{2}CDb - \frac{1}{2}CDa = 0$

But also,

$[CDE] - [CDF] - [ABF]$

$= [DJHG] + [CDG] + [JHE] - [CDG] - [CGF] - [AJE] - [JHE] - [HEBF]$

$= [DJHG] - [CGF] - [AJE] - [HEBF]$

$= [DJHG] - 6$

By transtitivity, $[DJHG] - 6 = 0$

So $[DJHG] = 6$.

☐ Let ABC be an isosceles triangle $(BA = BC)$. (O, R) is the circumcircle of $\triangle ABC$. It's known that : There exists a point D inside (O) such that $\triangle BCD$ is an equilateral triangle . AD intersects (O) at E . Prove that : $DE = R$

Solution

Since $BA = BD = BC$, it follows that $\triangle BAD$ is isosceles with apex B. Thus, $\angle BDA = \angle BAE = \pi - \angle BCE \Longrightarrow BCED$ is a kite $\Longrightarrow BE$ is the perpendicular bisector of DC. Consequently, $\angle COE = 2\angle CBE = 60^\circ \Longrightarrow \triangle OCE$ is equilateral with side lenght $R \Longrightarrow DE = EC = EO = R$.

☐ Find all positive-integer solutions (a, b, c, d) to the equation:

$$a + b + c + d = abcd$$

Solution

WLOG we can assume $a \leqslant b \leqslant c \leqslant d$, since the other possible solutions are mere permutations of those cases.

Then $abcd = a + b + c + d \leqslant 4d \iff abc \leqslant 4$

Therefore $(a, b, c) \in \{(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 2, 2)\}$

Solving all those cases for d, we find only two integer quadruplets: $(1,1,2,4)$ and $(1,1,4,2)$, which are essentially the same.

Hence there are 12 solutions, which are permutations of the basic quadruplet $(a,b,c,d)=(1,1,2,4)$

□ Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+f(y))=yf(x)+(2-x)f(y), \forall x,y\in\mathbb{R}$.

Solution

Put $y:=0$ and $x:=x-f(0)$ to get $f(x)=-f(0)x+(2+f(0))f(0)=ax+b$.

Substituing to the original equation we get $f(x)=0\ \forall x$ or $f(x)=x-1\ \forall x$

□ Let the medians of the triangle ABC intersect at point M. A line d through M intersects the circumcircle ABC at X and Y so that A and C lie on the same side of d. Prove that $BX\cdot BY=AX\cdot AY+CX\cdot CY$.

Solution

Let N be the midpoint of AC and A',B',C',N' the orthogonal projections of A,B,C,N on the line d. Segment NN' becomes the median of the trapezoid $ACC'A'\implies NN'=\frac{1}{2}(AA'+CC')$. But from $\triangle MBB'\sim\triangle MNN'$, we get the proportion $\frac{BB'}{NN'}=\frac{BM}{NM}=2$. Hence, it follows that $BB'=AA'+CC'$ $(\star)$.

On the other hand, if R denotes the circumradius of $\triangle ABC$, we have the relations

$BX\cdot BY=2R\cdot BB'$, $AX\cdot AY=2R\cdot AA'$, $CX\cdot CY=2R\cdot CC'$.

Combining these expressions with $(\star)$ yields

$\frac{BX\cdot BY}{2R}=\frac{AX\cdot AY}{2R}+\frac{CX\cdot CY}{2R}\implies BX\cdot BY=AX\cdot AY+CX\cdot CY.$

□ At a prize award five books are shared to three students. In how many ways can be shared the books, knowing that each student receives at least a book? But seven books to four students?

Solution

Suppose $H(k,n)$ is the number of ways to distribute k distinguishable books among n students such that each student gets at least one book.

Intuitively we have $H(k,1)=1$ and $H(k,2)=k-1$

Now suppose there are n students, the first student receives i books $i\in[1,k-n+1]$, and he can receive these i books in $\binom{k}{i}$ ways. This leaves $k-i$ books to distribute among $n-1$ students. So we get

$H(k,n)=\sum_{i=1}^{k-n+1}\binom{k}{i}H(k-i,n-1)$

Therefore,

$H(5,3)=\sum_{i=1}^{3}\binom{5}{i}H(k-i,2)=\sum_{i=1}^{3}\binom{5}{i}(4-i)=45$

$H(7,4)=\sum_{i=1}^{4}\binom{7}{i}H(k-i,3)=2576$

□ In acute triangle ABC, w is the circumcircle and O the circumcenter. w_1 is the circumcircle of triangle AOC, and OQ is the diameter of w_1. Let M,N be on AQ,AC respectively such that $AMBN$ is a parallelogram. Prove that MN,BQ intersect on w_1.

Solution

Let L be the midpoint of AB and P be the second intersection of ω_1 with BQ. Then $\angle APQ=\angle BNA=\angle ABC$. Thus if $R\equiv BQ\cap AC$ and $D\equiv BN\cap AP$, then $PDNR$ is cyclic. But notice that PQ bisects $\angle APC$ since Q is the midpoint of the arc AC of ω_1. As a resut, $\angle BPC=180^\circ-\angle ABC=\angle BNC\implies BPNC$ is cyclic $\implies\angle NPR=\angle BCA$, but since $PDNR$ is cyclic, we obtain $\angle NPR=\angle NDR=\angle BCA=\angle ABN\implies DR\parallel BA$. Therefore, the cevian NP of $\triangle BNA$ goes trough the midpoint L of $AB\implies P\equiv BQ\cap MN\in\omega_1$.

□ A deck of n playing cards, which contains three aces, is shuffled at random (it is assumed that all possible card distributions are equally likely). The cards are then turned up one by one from the top until the second ace appears. Prove that the expected (average) number of cards to be turned up is $(n+1)/2$.

Solution

The probability that the m^{th} card is the second ace is given by

$P(m) = (m-1)\left(\frac{3}{n}\cdot\frac{n-3}{n-1}\cdot\frac{n-4}{n-2}\cdots\frac{n-m}{n-m+2}\right)\frac{2}{n-m+1}$

$= 6(m-1)\frac{(n-m)!}{n!}\frac{(n-3)!}{(n-m-1)!} = \frac{(n-m)(m-1)}{\binom{n}{3}}$

Where $2 \le m \le n-1$

Therefore $E(x) = \sum_{k=2}^{n-1} kP(k)$

$= \frac{1}{\binom{n}{3}}\sum_{k=2}^{n-1} k(n-k)(k-1)$

For n odd

There is symmetry in the expression $p(m) = m(n-m)(m-1)$, that is $p(k)+p(n-k) = (n-2)(n-k)k$

$\therefore \sum_{k=2}^{n-1} k(n-k)(k-1) = (n-1)P(n-1)+(n-2)\sum_{k=2}^{\frac{n-1}{2}} k(n-k) = (n-2)\left((n-1)+\frac{(n-1)(n^2+n-12)}{12}\right) = \frac{(n-2)(n-1)n(n+1)}{12}$

For n even

The symetry is $p(k)+p(n-k+1) = (k-1)(n-k)(n+1)$

$\therefore \sum_{k=2}^{n-1} k(n-k)(k-1) = (n+1)\sum_{k=2}^{\frac{n}{2}}(k-1)(n-k)$

$= (n+1)\left(\frac{x(x-2)(x-1)}{12}\right) = \frac{(n-2)(n-1)n(n+1)}{12}$

$\therefore E(x) = \frac{1}{\binom{n}{3}}\frac{(n-2)(n-1)n(n+1)}{12} = \boxed{\frac{n+1}{2}}$

☐ If the inequality $1+\log_2\left(2x^2+2x+\frac{7}{2}\right) \ge \log_2\left(ax^2+a\right)$ has atleast one solution for a which belongs to $(0,\phi]$.Then find ϕ.

Solution

The inequation is equivalent to $4x^2+4x+7 \geqslant ax^2+a \iff (4-a)x^2+4x+7-a \geqslant 0$

Let's find the conditions for the last inequation not to have any solutions. This only can happen if

$4-a<0 \wedge 16-4(4-a)(7-a)<0$

After easy solving, we find that this is equivalent to

$(a>4)\wedge(a>8 \vee a<3) \iff a>8$

Hence we must have $a \leqslant 8$, so $\phi = 8$

☐ How many different ways are there to express $\frac{2}{15}$ in the form $\frac{1}{a}+\frac{1}{b}$, where a and b are positive integers with $a \le b$?

Solution

$\frac{1}{a}+\frac{1}{b} = \frac{2}{15} \iff (2a-15)(2b-15) = 225 = 3^2\cdot 5^2$

Hence $2a-15 \in \{1,3,5,9,15\}$, so there are 5 different ways:

$(a,b) \in \{(8,120),(9,45),(10,30),(12,20),(15,15)\}$

☐ Let a triangle ABC and I is its incenter. AI cuts the incircle (I) at D. Prove that the tangent of (I) at D and the external bisector of angle BIC meet on BC.

Solution

Denote X,Y,Z the tangency points of (I) with BC,CA,AB and let the internal angle bisector of $\angle BIC$ and the tangent of (I) at D cut BC at V,P, respectively. We shall prove that IP is the external bisector of $\angle BIC$. Midpoint M of the arc BC of the circumcircle $\odot(ABC)$ is circumcenter of $\triangle BIC$. Thus, IX and $IM \equiv IA$ are isogonals with respect to $\angle BIC \implies IV$ bisects $\angle MIX$. Analogously, if R is the projection of X on YZ, the rays XI, XR are isogonals with respect to

$\angle YXZ$. Lines IM, XR are parallel since they are both perpendicular to $YZ \implies$ angle bisectors of $\angle MIX$ and $\angle RXI$ are parallel $\implies XD \parallel IV$, but $IP \perp XD$. Then $IP \perp IV \implies IP$ is the external bisector of $\angle BIC$.

□ Prove that if the opposite sides of a skew (non-planar) quadrilateral are congruent, then the line joining the midpoints of the two diagonals is perpendicular to these diagonals, and conversely, if the line joining the midpoints of the two diagonals of a skew quadrilateral is perpendicular to these diagonals, then the opposite sides of the quadrilateral are congruent.

Solution

Label $ABCD$ the given quadrilateral. M, N denote the midpoints of the diagonals AC, BD, respectively.

• Assume that $AD = CB$ and $AB = CD$. Then $\triangle ADC \cong \triangle CBA$ by SSS criterion $\implies$ their medians DM and BM are congruent. Hence $\triangle DMB$ is isosceles with apex M. The median MN is identical to the altitude on $DB \implies MN \perp DB$. Likewise, $\triangle ADB \cong \triangle CBD$, then $\triangle ANC$ is isosceles with apex $N \implies NM \perp AC$.

• Conversely, if $MN \perp DB$ and $NM \perp AC$, the triangles $\triangle MDB$ and $\triangle NAC$ are isosceles with legs $MB = MD$ and $NA = NC$, respectively, which implies that

$CD^2 + CB^2 = AB^2 + AD^2$, $AD^2 + CD^2 = AB^2 + CB^2$.

Substracting and adding both expressions yields $AD = CB$ and $AB = CD$.

□ Find a set of integer solutions for the following equations. $17w + 13x = 3$ $13y + 17z = 7$ $w + x + y + z = 10$ If possible, find all integer solutions.

Solution

$x = \frac{3-17w}{13} = -w + \frac{3-4w}{13}$

$3 - 4w = 13a$

$w = \frac{3-13a}{4} = -3a + \frac{3-a}{4}$

$3 - a = 4b$

$a = 3 - 4b$

$w = -9 + 12b + b = 13b - 9$

$x = -13b + 9 + 3 - 4b = -17b + 12$

So, $x = -17b + 12, w = 13b - 9, b \in \mathbb{Z}$ (1)

$y = \frac{7-17z}{13} = -z + \frac{7-4z}{13}$

$7 - 4z = 13c$

$z = \frac{7-13c}{4} = -3c + \frac{7-c}{4}$

$7 - c = 4d$

$c = 7 - 4d$

$z = -21 + 12d + d = 13d - 21$

$y = -13d + 21 + 7 - 4d = -17d + 28$

So, $y = -17d + 28, z = 13d - 21, d \in \mathbb{Z}$ (2)

Plugging (1) and (2) into the third equation we get

$-4b + 3 - 4d + 7 = 10 \iff d = -b$

Therefore, the complete set of solutions is given by

$(x, y, z, w) = (-17n + 12, 17n + 28, -13n - 21, 13n - 9), n \in \mathbb{Z}$

□ Show that if $a_i \geq 1$, for $i \in \mathbb{N}^*$, then:

$(1 + a_1)(1 + a_2)(1 + a_3) \cdot \ldots \cdot (1 + a_n) \geq \frac{2^n}{n+1} \cdot (1 + a_1 + \cdot \ldots \cdot + a_n)$

Solution

Just use induction, let $M_n = \frac{2^n}{n+1}\left(1+\sum_{i=1}^{n} a_i\right)$

$M_{n+1} = \frac{2(n+1)}{n+2}M_n + \frac{2^{n+1}}{n+2}a_{n+1} \qquad (1)$

All you have to show is

$(1+a_{n+1})M_n \geq M_{n+1}$

Using (1) and the fact that $M_n \geq 2^n$ the result follows easily

$\square$ If $f(x)f(y) - f(xy) = x + y, \forall (x, y) \in \Re$ find $f(x)$.

Solution

$f(x)f(y) - f(xy) = x + y$

Setting $x = y = 0$, we get

$f^2(0) - f(0) = 0 \Longrightarrow f(0) = 0$ or 1

Setting $y = 0$, we get

$f(x)f(0) - f(0) = x \Longrightarrow f(0)(f(x) - 1) = x$

If $f(0) = 0$, we get $x = 0$ for all x, which is impossible. So we get $f(0) = 1$

Plugging this into the equation, we get

$f(x) - 1 = x \Longrightarrow f(x) = (x+1)$

Thus, the solution is $\boxed{f(x) = (x+1)}$

$\square$ Prove that the equation $x^9 + y^9 + z^9 = x + y + z + 2002^{2001}$ has no solution in $\mathbb{N}$.

Solution

We first note that $x^3 \equiv x \pmod 3$

This follows because $x^3 - x = (x-1)x(x+1)$ and one of the products on the right side must be a multiple of 3.

Thus, $x^9 = (x^3)^3 \equiv x^3 \equiv x \pmod 3$

Thus, $(x^9 + y^9 + z^9) \equiv (x + y + z) \pmod 3$

As 2002^{2001} is not a multiple of 3 (because 2002 is not a multiple of 3),

$x + y + z + 2002^{2001}$ is not congruent to $(x + y + z) \equiv (x^9 + y^9 + z^9) \pmod 3$

Thus, the given equation has no solutions in $\mathbb{N}$.

$\square$ Find the remainder when $41^{10^{41}}$ is divided by 251

Solution

$250 \mid 10^{41}$ because $250 = 2 \times 5^3$ and, thus, $250 \mid 10^m$ for all $m \geq 3$.

Let $10^{41} = 250k$ for some $k \in \mathbb{N}$.

$41^{10^{41}} = 41^{250k} = (41^{250})^k$

However, because 251 is prime, by Fermat's Little Theorem,

$a^{p-1} \equiv 1 \pmod p \Longrightarrow 41^{250} \equiv 1 \pmod{251}$

Thus, raising both sides to the kth power,

$(41^{250})^k \equiv 1^k \equiv 1 \pmod{251}$. Hence, we get

$\boxed{41^{10^{41}} \equiv 1 \pmod{251}}$

$\square$ Find all the integer positive solutions x, y of

$x^4 + 3x^2y^2 + 9y^4 = 12^{2005}$

Solution

$x^4 + 3x^2y^2 + 9y^4 = 12^{2005}$

We observe that $3 \mid x$. Replacing x by $3x_1$ and dividing the equation by 9, we obtain

1) $9x_1^4 + 3x_1^2y^2 + y^4 = 12^{2003} \cdot 16^1$

Now, $3 \mid y$, so let $y = 3y_1$ and divide the equation by 9 after substitution to get

2) $x_1^4 + 3x_1^2y_1^2 + 9y_1^4 = 12^{2001} \cdot 16^2$

We see that the process can be repeated a very large number of times, and in general, we have

$2i-1)$ $9x_i^4+3x_i^2y_{i-1}^2+y_{i-1}^4=12^{2007-4i}\cdot 16^{2i-1}$

$2i)$ $x_i^4+3x_i^2y_i^2+9y_i^4=12^{2005-4i}\cdot 16^{2i}$

where $x=3^ix_i$, $y=3^iy_i$ and $x_0=x$ and $y_0=y$

We can repeat the process until we reach a stage when the exponent of 12 is less than 2, so that the right hand side is no longer a multiple of 9. In particular, we have the 1002nd iteration of this process, for $i=501$,

1002) $x_{501}^4+3x_{501}^2y_{501}^2+9y_{501}^4=12\cdot 16^{1002}$

At this stage, we see that $3\mid x_{501}$ and replacing by $3x_{502}$, we see that 9 divides the left hand side, but not the right hand side.

Thus, we have no solution to the original equation.

□ Prove that

$$(a^n-1,a^m-1)=a^{(m,n)}-1$$

Solution

Let $(a^m-1,a^n-1)=d$

So, $a^m\equiv a^n\equiv 1\, mod\, d$

Let $k=ord_d a|m,n\Longrightarrow k|(m,n)$

So, $a^{(m,n)}\equiv 1\, mod\, d\Longrightarrow d|a^{(m,n)}-1$

Now, since $a^{(m,n)}-1|a^m-1,a^n-1$, we have $a^{(m,n)}-1|d$ but $d|a^{(m,n)}-1$

So, $d=a^{(m,n)}-1$

as desired.

□ Prove that the equation

$x^3+y^5=z^7$ has infinite solutions if x,y,z are integers.

Solution

We have : $2^{300}+2^{300}=2^{301}\iff 2^{300}\cdot 2^{105k}+2^{300}\cdot 2^{105k}=2^{301}\cdot 2^{105k}$

$\iff 2^{3(100+35k)}+2^{5(60+21k)}=2^{7(43+15k)}\iff (2^{100+35k})^3+(2^{60+21k})^5=(2^{43+15k})^7$.

Therefore, $(x,y,z)\in\{(2^{100+35k},2^{60+21k},2^{43+15k})\}$, where $k\in\mathbb{N}^*$.

□ If p is a prime greater than 3, then prove that p divides the sum of the quadratic residues between 0 and p

Solution

Let S denote the sum of the quadratic residues $\pmod p$. Suppose we square every element of $\{1,2,\ldots,p-1\}$. Then we obtain a list of the quadratic residues. Furthermore, each quadratic residue appears twice, once for x^2 and once for $(-x)^2$. (It is well known that there are $(p-1)/2$ quadratic residues $\pmod p$.) So we have

$$S\equiv 1^2+2^2+\cdots+(p-1)^2 \pmod p$$

which is

$$\frac{p(p+1)(2p+1)}{12}$$

which is divisible by p since p is a prime greater than 3. It follows that $p|S$, as desired.

□ Let ABCD be a quadrilateral where $\widehat{ABC}=\widehat{ADC}=90^0$ and $\widehat{BCD}<90^0$.Choose a point E on the opposite ray of AC such that DA is the angle-besector of BDE.Let M be the chosen arbitrarily between D and E.choose another point N on the opposite ray of BE such that $\widehat{NCB}=\widehat{MCD}$.Prove that MC is the angle-besector of DMN

Solution

Let $P \equiv AD \cap BC$ and $Q \equiv BA \cap DC$. Then A becomes the orthocenter of the acute $\triangle CPQ$ and $\triangle BED$ is its orthic triangle. Thus, EC bisects $\angle BED$ and C becomes the E-excenter of $\triangle EBD$. If $\angle MCD = \angle NCB$, it follows that $\angle MCN = \angle DCB \Longrightarrow \angle DCB = \angle MCN = \angle 90° - \frac{1}{2}\angle BED$, which implies that C is common E-excenter of $\triangle BED$ and $\triangle NEM \Longrightarrow MC$ bisects $\angle DMN$.

$\square$ In $\triangle ABC$, let I be the incenter, O be the circumcenter, H be the orthocenter, R be the circumradius, E be the midpoint of OH, r be the inradius, and s be the semiperimeter.

a) Find the distance IH in form of R, r, s b) Find the distance IE in form of R, r

Solution

Using Leibniz theorem for the circumcenter O, we obtain the relation

$OG^2 = \frac{1}{3}(OA^2 + OB^2 + OC^2) - \frac{1}{9}(a^2 + b^2 + c^2) = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$

Since $OG = \frac{1}{3}OH$, it follows that $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$.

Incircle (I) and 9-point circle (E) of $\triangle ABC$ are internally tangent $\Longrightarrow IE = \frac{1}{2}R - r$. Notice that IE becomes the I-median of $\triangle IOH$, therefore

$IE^2 = \frac{1}{2}(IO^2 + IH^2) - \frac{1}{4}OH^2 \Longrightarrow IH^2 = 2IE^2 + \frac{1}{2}OH^2 - IO^2$

$IH^2 = 2\left(\frac{R}{2} - r\right)^2 + \frac{9}{2}R^2 - \frac{1}{2}(a^2 + b^2 + c^2) - (R^2 - 2Rr)$

$IH^2 = 4R^2 + 2r^2 - \frac{1}{2}(a^2 + b^2 + c^2)$

Because of $a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr \Longrightarrow IH = \sqrt{4R^2 + 3r^2 + 4Rr - s^2}$.

$\square$ Find all reals solutions of

$x[x[x]] = 84$

where $[x]$ means the integer part of x

Solution

Obviously, x can't be non-positive, since then $[x] \leqslant 0 \implies x[x] \geqslant 0 \implies [x[x]] \geqslant 0 \implies x[x[x]] \leqslant 0$. So $x > 0$ - furthermore, $x > 1$, since $[x] = 0$ turns the whole LHS into zero.

Put $n := [x], a := \{x\}$. Then

$(n^2 + [an])(n + a) = 84 \implies [an] = \frac{84 - n^3 - an^2}{n + a}$

Since $[an]$ must be positive, we get $n \leqslant 4$, and from there it's just case-bashing.

$n = 1$:

$[a] = \frac{83 - a}{1 + a}$ doesn't have a solution since $0 \leqslant a < 1$ by definition, hence $[a] = 0$

$n = 2$

$[2a] = \frac{76 - 4a}{2 + a}$. Since $[2a] \in \{0, 1\}$, we solve the equations $\frac{76 - 4a}{2 + a} = 0$ and $\frac{76 - 4a}{2 + a} = 1$ and recheck if the obtained solutions satisfy $0 \leqslant a < 1$ and the chosen value of $[2a]$, but no solution in this case.

Going on like this, we find the only solution: $n = 4 \wedge a = \frac{2}{3} \implies x = \frac{14}{3}$

$\square$ We have n objects with weights $1, 2, 3, \cdots, n$ grams. We wish to choose two or more of these objects so that the total weight of the chosen objects is equal to the average weight of the remaining objects. Prove that, if the task is possible, then n is one less than a perfect square.

Solution

Let the number of chosen numbers be k, and let S be the sum of chosen numbers. Then if the task is possible we must have

$$(n - k) \cdot S = \frac{n(n + 1)}{2} - S,$$

or $(n - k + 1)S = \frac{n(n+1)}{2}$. Also, S is at least the sum of the first k numbers, so $S \geq \frac{k(k+1)}{2}$. Therefore, we must have

$$(n - k + 1)\frac{k(k + 1)}{2} \leq \frac{n(n + 1)}{2},$$

which can be written as

$$(n-k)(n+1-k^2) \geq 0,$$

which implies that $k^2 \leq n+1$ as $n > k$ by the problem statement. This means that $n-k+1 \geq k^2-k$. On the other hand,

$$\frac{n(n+1)}{n-k+1} = n+k+\frac{k^2-k}{n-k+1}$$

must be a integer, so $n-k+1$ must divide k^2-k, which means that $n-k+1 \leq k^2-k$. This is only possible if $n-k+1 = k^2-k$, which is the equality case. Therefore $n+1 = k^2$ so n is one less than a perfect square, as desired.

□ Prove without induction that if $0 < \alpha < \frac{\pi}{4(n-1)}$ where $n = 2, 3, ...$ then:

$$\tan n \cdot \alpha > n \cdot \tan \alpha$$

Solution

Note that the functions $\tan(n\alpha)$ and $\tan \alpha$ are both continuous on the given intervals. It suffices to show that the function $f(x) = \frac{\tan x}{x}$ is increasing in $\left(0, \frac{\pi}{2}\right)$, since then

$$\frac{\tan(n\alpha)}{n\alpha} > \frac{\tan \alpha}{\alpha} \implies \tan(n\alpha) > n \tan \alpha.$$

We have $f'(x) = \frac{x \sec^2 x - \tan x}{x^2}$, which can be written as $\frac{1}{2(x \cos x)^2}(2x - \sin 2x)$. This is greater than zero since $\frac{1}{2(x \cos x)^2}$ is obviously positive on $\left(0, \frac{\pi}{2}\right)$ and $2x > \sin 2x$ on $0 < x < \frac{\pi}{2}$ since $x > \sin x$ on $0 < x < \pi$. (Actually $x > \sin x$ for all $x > 0$.) Therefore $f(x)$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and we are done.

□ Find all couples (x, y) of real numbers such that

$\sqrt[15]{x} - \sqrt[15]{y} = \sqrt[5]{x} - \sqrt[5]{y} = \sqrt[3]{x} - \sqrt[3]{y}$

Solution

Note that $x = y$ is a solution. Now, assume $x \neq y$

Let $a = \sqrt[15]{x}; b = \sqrt[15]{y}$

$a - b = a^3 - b^3 \implies a^2 + ab + b^2 = 1 \implies (a^2 + ab + b^2)^2 = 1$

$a - b = a^5 - b^5 \implies a^4 + a^3b + a^2b^2 + ab^3 + b^4 = 1$

Now, $(a^2 + ab + b^2)^2 = a^4 + a^3b + a^2b^2 + ab^3 + b^4$

$\implies ab(a+b)^2 = 0$

$\implies x = 0$ or $y = 0$ or $x = -y$

So, the solution set is $(x, y) = (a, a), (a, -a), (0, a), (a, 0)$ where a is an integer.

□ Let $(a, b) \in \mathbb{R}^2$ such that the polynomial $P(x) = x^3 + \sqrt{3}(a-1)x^2 - 6ax + b$ has 3 reals solution . Show that : $|b| \leq |(a+1)^3|$

Solution

Let $P(x) = (x+p)(x+q)(x+r)$ Then $pqr = b$ Also $pq + qr + rp = -6a$ And $p + q + r = \sqrt{3}(a-1)$ Since $p^2 + q^2 + r^2 = (p+q+r)^2 - 2(pq+qr+rp)$ We get $p^2 + q^2 + r^2 = 3(a-1)^2 + 12a = 3(a+1)^2$ By AM-GM inequality: $p^2 + q^2 + r^2 \geq 3(pqr)^{\frac{2}{3}}$ So $(a+1)^2 \geq b^{\frac{2}{3}}$ Hence $(a+1)^6 \geq b^2$ Therefore $\mid (a+1)^3 \mid \geq \mid b \mid$ as required.

□ 1) Find the ratio of the radius of an escribed circle of a triangle to the radius of the circumscribing circle, in terms of the angles of the triangle.

2) Prove that the ratio of the radii of the two circles which touch the inscribed circle and the sides AB, AC of a triangle ABC is $\tan^4 \frac{1}{4}(B+C)$

Solution

1) Denote R and ϱ_a the circumradius and A-exradius of $\triangle ABC$, respectively. From the well-known identities

$[\triangle ABC] = 2R^2 \cdot \sin A \cdot \sin B \cdot \sin C$, $[\triangle ABC] = \varrho_a(s-a)$

$\implies \frac{R}{\varrho_a} = \frac{b+c-a}{4R\cdot\sin A\cdot\sin B\cdot\sin C} = \frac{\sin B+\sin C-\sin A}{2\cdot\sin A\cdot\sin B\cdot\sin C}$.

2) The radii R_i of a chain of circles (O_i, R_i) tangent to the sides of an angle $\angle(p,q)$ such that (O_i, R_i) is externally tangent to (O_{i-1}, R_{i-1}) and (O_{i+1}, R_{i+1}) form a decreasing geometric progression with ratio $\frac{1-\sin\theta}{1+\sin\theta}$, where θ stands for $\frac{1}{2}\angle(p,q)$.

Therefore, by denoting R_1, R_2 $(R_2 > R_1)$ the radii of the two circles externally tangent to the incircle (I, r) and tangent to the rays AB, AC, we obtain

$\frac{R_1}{r} = \frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}$, $\frac{r}{R_2} = \frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}$

$\implies \frac{R_1}{R_2} = \left(\frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}\right)^2 = \tan^4\left(\frac{\pi}{4} - \frac{A}{4}\right) = \tan^4\left(\frac{B+C}{4}\right)$.

□ Solve the equation $4x - 14[\sqrt{2x+19}] + 59 = 0$

Solution

Put $u := \sqrt{2x+19} \implies 2x = u^2 - 19$, hence the equation becomes

$2u^2 - 14[u] + 21 = 0 \iff u^2 - 7[u] + \frac{21}{2} = 0$

Let $u = n + \alpha$ where $n = [u], \alpha = \{u\}$. By the definition of u we have $n \geqslant 0$

$\alpha^2 + 2n\alpha + n^2 - 7n + \frac{21}{2} = 0$

$\alpha = -n + \sqrt{7n - \frac{21}{2}}$ (we discard the minus sign since $\alpha \geqslant 0$)

$-n + \sqrt{7n - \frac{21}{2}} \geqslant 0 \implies n \in \{3, 4\}$

$-n + \sqrt{7n - \frac{21}{2}} < 1 \implies n \geqslant 0$

Therefore $n \in \{3, 4\}$. Now $u = n + \alpha = \sqrt{7n - \frac{21}{2}} \implies 2x + 19 = 7n - \frac{21}{2}$, hence $x = \frac{14n-59}{4} \in \left\{-\frac{17}{4}, -\frac{3}{4}\right\}$

□ Find the value of a in order that the equation $1 + \sin^2 ax = \cos x$ has only one root.

Solution

$\sin^2 ax + 2\sin^2\frac{x}{2} = 0$

Therefore $x \in \left\{\frac{k\pi}{a} : k \in \mathbb{Z}\right\} \cap \{2l\pi : l \in \mathbb{Z}\}$

Obviously, $x = 0$ is always a solution, no matter the value of a.

If $(\exists k, l \in \mathbb{Z} \setminus \{0\})\frac{k}{a} = 2l$, then $a = \frac{k}{2l}$. If a is rational, we can always find such k, l. If a is irrational, we can never find such k, l.

Therefore a must be irrational in order to satisfy the problem condition.

□ Let n be a positive integer such that $a + b^2 \mid a^2 + b + n$ has exactly one solution (a, b) with $a, b \in \mathbb{Z}^+$. Prove that either $b + n \leq ab^2$ or $b + n = 2ab^2 + a^2$.

Solution

Let k be the positive integer such that $ka + kb^2 = a^2 + b + n$. Rearranging gives $a^2 - ka + b + n - kb^2 = 0$. Let $P(x) = x^2 - kx + b + n - kb^2$. Clearly, a is one of the two roots to this equation. Let the other root be x_2. By Vieta's formulas, $a + x_2 = k$, so x_2 is also an integer. If $0 < x_2 \neq a$, then (x_2, b) is another solution to the divisibility relation, which contradicts our assumption that (a, b) is unique. Therefore $x_2 \leq 0$ or $x_2 = a$.

In the first case, this implies that $k - a \leq 0$, or that $k \leq a$. Thus $a(a+b^2) \geq k(a+b^2) = a^2+b+n$. Therefore, $a^2 + ab^2 \geq a^2 + b + n$, or $b + n \leq ab^2$. In the second case, this implies that $k - a = a$, or that $k = 2a$. Thus $2a(a + b^2) = k(a + b^2) = a^2 + b + n$. Therefore, $2a^2 + 2ab^2 = a^2 + b + n$, or

$b+n=2ab^2+a^2$. As one of these cases must be true, one of these two results must be true, so either $b+n \leq ab^2$ or $b+n=2ab^2+a^2$.

$\square$ In a regular 3982-gon the vertices are divided into pairs and both vertices in every pair are then joined by a straight line. Prove that the 1991 lines not all can have diffrent lenghts.

Solution

Assume for contradiction that all 1991 lines have different lengths. First, if we label the vertices $A_1, A_2, ..., A_{3982}$, we can see that there are exactly 1991 possible lengths, namely $A_1A_2, A_1A_3, ..., A_1A_{1992}$ (because A_1 and A_{1992} are diametrically opposite points). Therefore, the length of each vertex pair must equal a distinct length in the list $A_1A_2, ..., A_1A_{1992}$. Color the vertices of the 3982-gon alternately white and black. Note that the lengths $A_1A_2, A_1A_4, ..., A_1A_{1992}$ will always connect a white vertex to a black vertex, and the lengths $A_1A_3, A_1A_5, ..., A_1A_{1991}$ will always connect two vertices of the same color. Let W equal the number of white vertices used up in these lengths. Because there are 996 lines that connect two vertices of different colors and 995 that connect two vertices of the same color, we have that $W = 996 + 2 \times x$, where x is the number of lines that connect two white vertices. However, this number is even, while the total number of white vertices is 1991, an odd number, so we have a contradiction.

$\square$ Let $f : \mathbb{R} \to [a, b]$ such that $f(x) = \dfrac{x+m}{x^2+x+1}$, $a, b \in \mathbb{Q}, m \in \mathbb{Z}$. Determine a, b, m for which f is surjective.

Solution

We must have $(\forall y \in [a, b])(\exists x \in \mathbb{R}) y = f(x)$, hence $yx^2+yx+y = x+m \iff yx^2+(y-1)x+y-m = 0$ must have a solution for x. Therefore $(y-1)^2 - 4y(y-m) \geqslant 0$ for all the y's in the codomain. The inequality is equivalent to $-3y^2 + (4m-2)y + 1 \geqslant 0$. Obviously, the discriminant must be non-negative, and the boundaries of the codomain are the solutions of $-3y^2 + (4m-2)y + 1 = 0$.

The discriminant condition yields $4(2m-1)^2 + 12 \geqslant 0$, which is satisfied for all real m, and the roots of the equation are

$y_{1,2} = \frac{2m-1 \pm \sqrt{(2m-1)^2+3}}{3}$

The radicand, being an integer, must be a perfect square, hence for some $t \in \mathbb{Z}$ we must have $(t+2m-1)(t-2m+1) = 3$. Since the possible factorizations of 3 are $1 \cdot 3, 3 \cdot 1, (-1) \cdot (-3)$ and $(-3) \cdot (-1)$, we get $4m-2 \in \{2, -2\} \iff m \in \{0, 1\}$. Finding $y_{1,2}$ in those cases yields the solutions:

$(a, b, m) \in \left\{ \left(-1, \frac{1}{3}, 0\right), \left(-\frac{1}{3}, 1, 1\right) \right\}$

$\square$ For any $n \geq 1993$, prove that:

$$\left(1 - \frac{1}{1993^3}\right)\left(1 - \frac{1}{1994^3}\right) \cdot \ldots \cdot \left(1 - \frac{1}{n^3}\right) > \frac{1992}{1993}$$

Solution

We will prove a generalisation.

$\left(1 - \frac{1}{1993^3}\right)\left(1 - \frac{1}{1994^3}\right) \cdots \left(1 - \frac{1}{n^3}\right) > \frac{1992}{1993} \cdot \frac{n+1}{n}$

We use induction on n

The result is true for $n = 1$

This forms the base case for induction.

Assume the result for some natural n

Consider

$\left(1 - \frac{1}{1993^3}\right) \cdots \left(1 - \frac{1}{(n+1)^3}\right)$

$> \frac{1992}{1993} \cdot \frac{n+1}{n} \left(1 - \frac{1}{(n+1)^3}\right)$

So, it suffices to prove that

$\frac{1992}{1993} \cdot \frac{n+1}{n} \left(1 - \frac{1}{(n+1)^3}\right) \geq \frac{1992}{1993} \cdot \frac{n+2}{n+1}$

$\Longleftrightarrow \frac{(n+1)(n^3+3n^2+3n)}{n(n+1)^3} \geq \frac{n+2}{n+1}$

$\Longleftrightarrow n^3 + 3n^2 + 3n \geq n(n+1)(n+2) = n^3 + 3n^2 + 2n$

$\Longleftrightarrow n \geq 0$ which is true.

□ Find all integers x,y such that $x^6 + x^3y = y^3 + 2y^2$. Thank you in advance

Solution

Taking this as a quadratic in x^3, we get

$x^6 + x^3y - (y^3 + 2y^2) = 0 \iff (x^3)_{1,2} = \frac{-y \pm y\sqrt{4y+9}}{2}$

Therefore for some integer n, $y = \frac{n^2-9}{4} \iff (x^3)_{1,2} = \frac{(n-3)(n+3)(-1\pm n)}{8}$

The factors in the numerator are either all odd or all even, but as the product is divisible by 8, they must be all even, hence $n = 2m+1$, which yields

$(x^3)_{1,2} = (m-1)(m+2)\frac{-1\pm(2m+1)}{2}, y = m^2 + m - 2$

Case 1. $x^3 = (m-1)(m+2)m = m^3 + m^2 - 2m$

By technique of "sandwiching" the expression $m^3 + m^2 - 2m$ between two consecutive cubes, we find that it's sufficient to check $m \in \{-3,-2,-1,0,1,2,3\}$. Of those, only $m \in \{-2,0,1,2\}$ yield an integer $\sqrt[3]{m^3+m^2-2m}$, hence the solutions are $(x,y) \in \{(0,-2),(0,0),(2,4)\}$

Case 2. $x^3 = (m-1)(m+2)(-m-1) \iff -x^3 = m^3 + 2m^2 - m - 2$

The sandwiching technique yields no new solutions.

Conclusion. $(x,y) \in \{(0,-2),(0,0),(2,4)\}$

□ Let ABC is a triangle and M is the midpoint of BC, <BAM=30grad, and <MAC=15, find the angles of ABC.

Solution

Let $CK \perp AB$ and $(AB) \cap (CK) = \{K\}$, $MN \perp AB$ and $(AB) \cap (MN) = \{N\}$. Hence, $MN = \frac{1}{2}CK$, $MN = \frac{1}{2}AM$ and $AK = CK$. Thus, $AK = AM$, which gives $\measuredangle KBM = \measuredangle AKM = \measuredangle AMK = 75^\circ$. Id est, $\measuredangle ABC = 105^\circ$ and $\measuredangle ACB = 30^\circ$.

□ Caculate $\frac{2^3-1}{2^3+1} \cdot \frac{3^3-1}{3^3+1} \cdot \frac{4^3-1}{4^3+1} \cdot \ldots \cdot \frac{n^3-1}{n^3+1}$

Solution

$S = \prod_{k=2}^{n} \frac{k^3-1}{k^3+1} = \prod_{k=2}^{n} \frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)}$

$\prod_{k=2}^{n} \frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)} = \frac{2}{n(n+1)} \prod_{k=2}^{n} \frac{((k+1)^2-(k+1)+1))}{(k^2-k+1)}$

$\frac{2}{n(n+1)} \prod_{k=2}^{n} \frac{((k+1)^2-(k+1)+1))}{(k^2-k+1)} = \frac{2(n^2+n+1)}{3n(n+1)}$

□ Find the smallest natural number n which satisfies the inequality

$2006^{1003} < n^{2006}$

Solution

$2006^{1003} < n^{2006}$

$\implies (2006^{1003})^{\frac{1}{1003}} < (n^{2006})^{\frac{1}{1003}}$

$\implies 2006 < n^2$.

Now, $44^2 = 1936 < 2006 < 2025 = 45^2$, so the smallest possible value of n is $\boxed{45}$.

□ $abc = 1, a,c,b > 0$ prove that $1 + \frac{3}{a+b+c} \geq \frac{6}{ab+ac+bc}$

This rewrites into

$$[(a+b+c)+3](ab+bc+ca) \geq 6(a+b+c);$$

Note that $ab+bc+ca \geq \sqrt{3abc(a+b+c)} = \sqrt{3}\sqrt{a+b+c}$; and letting $a+b+c = t \geq 3$ we have to show that

$$3t(t+3)^2 - 36t^2 \geq 0;$$

Which equivalents $3t(t-3)^2 \geq 0$; which is perfectly true.□

□ Evaluate $\sum_{n=1}^{2010}[n^{\frac{1}{5}}]$ where [.] denotes the greatest integer function.

Solution

$\left[1^{\frac{1}{5}}\right] + \left[2^{\frac{1}{5}}\right] + ... + \left[31^{\frac{1}{5}}\right] = (31)(1) = 31$. $\left[32^{\frac{1}{5}}\right] + \left[33^{\frac{1}{5}}\right] + ... + \left[242^{\frac{1}{5}}\right] = (211)(2) = 422$. $\left[243^{\frac{1}{5}}\right] + \left[244^{\frac{1}{5}}\right] + ... + \left[1023^{\frac{1}{5}}\right] = (781)(3) = 2343$. $\left[1024^{\frac{1}{5}}\right] + \left[1025^{\frac{1}{5}}\right] + ... + \left[2010^{\frac{1}{5}}\right] = (987)(4) = 3948$. So

$$\sum_{n=1}^{2010}\left[n^{\frac{1}{5}}\right] = 31 + 422 + 2343 + 3948 = 6744$$

.

□ prove that:

$cos(sin(x)) > sin(cos(x))$

Solution

$\cos\sin x > \sin\cos x \Leftrightarrow \sin\left(\frac{\pi}{2} - \sin x\right) - \sin\cos x > 0 \Leftrightarrow \Leftrightarrow 2\sin\frac{\frac{\pi}{2}-\sin x-\cos x}{2}\cos\frac{\frac{\pi}{2}-\sin x+\cos x}{2} > 0$, which is true because $|\sin x + \cos x| \leq \sqrt{2}$ and $|\sin x - \cos x| \leq \sqrt{2}$, which gives $0 < \frac{\frac{\pi}{2}-\sqrt{2}}{2} \leq \frac{\frac{\pi}{2}-\sin x-\cos x}{2} \leq \frac{\frac{\pi}{2}+\sqrt{2}}{2} < \frac{\pi}{2}$ and $0 < \frac{\frac{\pi}{2}-\sqrt{2}}{2} \leq \frac{\frac{\pi}{2}-\sin x+\cos x}{2} \leq \frac{\frac{\pi}{2}+\sqrt{2}}{2} < \frac{\pi}{2}$.

□ Let f, g be two functions defined on $[0, 2c]$ where c>0. Show that there exists x, y, is an element of $[0, 2c]$ such that $|xy - f(x) + g(y)| \geq c^2$

Solution

Assume that $|xy - f(x) + g(y)| < c^2$ $\forall x, y \in [0, 2c]$ This is equivelant to $-c^2 < xy - f(x) + g(y) < c^2$ $\forall x, y \in [0, 2c]$

From the last we have

for $x = y = 0$: $-c^2 < f(0) - g(0) < c^2$ (1)

for $x = y = 2c$: $-c^2 < f(2c) - g(2c) - 4c^2 < c^2$ (2)

for $x = 0, y = 2c$: $-c^2 < g(2c) - f(0) < c^2$ (3)

for $x = 2c, y = 0$: $-c^2 < g(0) - f(2c) < c^2$ (4)

Adding (1),(2),(3),(4) we get $-4c^2 < -4c^2 < 4c^2$ contraction. So exist $x, y \in [0, 2c]$ such that $|xy - f(x) + g(y)| \geq c^2$ □ Let a_1, a_2, a_3, a_4, a_5 be positive real numbers which satisfy

(i) $2a_1, 2a_2, 2a_3, 2a_4, 2a_5$ are positive integers (ii) $a_1 + a_2 + a_3 + a_4 + a_5 = 99$

Find the minimum and maximum of $P = a_1a_2a_3a_4a_5$

Solution

This is equivalent to:

If b_1, b_2, b_3, b_4, b_5 are positive integers such that $b_1 + b_2 + b_3 + b_4 + b_5 = 198$, then find the extrema of $P = \frac{b_1b_2b_3b_4b_5}{32}$

By AM-GM principles, the product will be maximized for numbers which are as close as possible to their average, hence we must take $\{b_1, b_2, b_3, b_4, b_5\} = \{40, 40, 40, 39, 39\}$ for $P_{\max} = 3042000$

To minimize the product, we must take as many 1's as possible, hence $\{b_1, b_2, b_3, b_4, b_5\} = \{1, 1, 1, 1, 194\}$, for $P_{\min} = \frac{97}{16}$

□ A,B,C - angles of non-isosceles triangle. Solve an equation system: $\begin{cases} \sin A = 2\sin C\sin(B - 30°) \\ \sin C = 2\sin B\sin(A - 30°) \end{cases}$

Solution

$\sin A = 2\sin C\sin(B - 30°)$ (1) $\sin C = 2\sin B\sin(A - 30°)$ (2)

Using Sine Law and Cosine Law on (1), we get:
$\frac{a}{2R} = 2\frac{c}{2R}\left(\frac{b}{2R}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cos B\right)$
$\frac{a}{c} = \frac{b\sqrt{3}}{2R} - \cos B$
$\frac{a^2+c^2-b^2}{2ac} = \frac{b\sqrt{3}}{2R} - \frac{a}{c}$
$Ra^2 + Rc^2 - Rb^2 = abc\sqrt{3} - 2Ra^2$
$3a^2 - b^2 + c^2 = 4P\sqrt{3} \quad (3)$
where $P = [ABC]$
The similar procedure applied to (2) yields
$-a^2 + b^2 + 3c^2 = 4P\sqrt{3} \quad (4)$
Now $(3) - (4) \implies b^2 + c^2 = 2a^2 \iff c^2 = 2a^2 - b^2 \quad (5)$
Plugging (5) into (3) we get
$5a^2 - 2b^2 = 4P\sqrt{3}$
Squaring that and using Heron's, we get
$25a^4 - 20a^2b^2 + 4b^4 = 3[(a+b)^2 - c^2][c^2 - (a-b)^2]$
Using (5), after a lengthy but trivial simplification, we get
$7a^4 - 11a^2b^2 + 4b^4 = 0$
From there, $\left(\frac{a^2}{b^2}\right)_{1,2} = \frac{11\pm 3}{14}$, but as $a \neq b$ by the problem condition, we take $\frac{a^2}{b^2} = \frac{4}{7} \iff a = \frac{2b}{\sqrt{7}}$
Now (5) yields $c = \frac{b}{\sqrt{7}}$
Therefore $\boxed{a : b : c = 2 : \sqrt{7} : 1}$
Solving for the angles, we find $\boxed{\angle A = \arctan\frac{\sqrt{3}}{2}, \angle B = 120^\circ, \angle C = \arctan\frac{\sqrt{3}}{5}}$
NOTE: Without the non-isosceles condition, we have another solution in the form of an equilateral triangle.

□ Find the polynomial $P(x)$, which satisfies the identity $P(x^2) + 2x^2 + 10x = 2xP(x+1) + 3$

Solution

If $n = \deg P$, then $\max\{2n, 2\} = n + 1$, which can be satisfied only for $n = 1$. Hence $P(x) = ax + b$. Plugging that into the equation and equating the coefficients, we find $P(x) = 2x + 3$.

□ Let a be a positive integer and define a sequence $\{u_n\}$ is defined as follows.

$$u_1 = 2,\ u_2 = a^2 + 2,\ u_n = au_{n-2} - u_{n-1},\ n = 3,\ 4,\ 5,\ \cdots$$

Find the necessary and sufficient condition for a such that a multiple of 4 doesn't appear in the term of the sequence $\{u_n\}$.

Solution

Obviously, we can simply consider the whole sequence $\bmod 4$ and ask when 0 will never occur in the sequence.

Next, note that the sequence $\bmod 4$ can be calculated from $a \bmod 4$, so we can simply consider the four cases of a.

If $a \equiv 0 \bmod 4$ then the sequence goes 2, 2, 2, 2, 2, ... so 0 never occurs.
If $a \equiv 1 \bmod 4$ then the sequence goes 2, 3, 3, 0, 3, ... so u_4 is divisible by 4.
If $a \equiv 2 \bmod 4$ then the sequence goes 2, 2, 2, 2, 2, ... so 0 never occurs.
If $a \equiv 3 \bmod 4$ then the sequence goes 2, 3, 3, 2, 3, 3, 2, ... so 0 never occurs.
Therefore, no multiple of 4 occurs in $\{u_n\}$ iff $a \not\equiv 1 \bmod 4$.

□ The product of two numbers '231' and 'ABA' is 'BA4AA' in a certain base system (where base is less than 10), where A and B are distinct digits. What is the base of that system?

Solution

Let x be the base. Due to the presence of the digit 4, we have $5 \leqslant x \leqslant 9$, and also $1 \leqslant a, b \leqslant x-1$.

The given equation is equivalent to $(x+1)(2x+1)(ax^2+bx+a) = bx^4+ax^3+4x^2+ax+a$. By Bezout we get $bx^4+ax^3+4x^2+ax+a \equiv b-a+4 \pmod{x+1}$ and $bx^4+ax^3+4x^2+ax+a \equiv \frac{b}{16}+\frac{3a}{8}+1 \pmod{2x+1}$. Since $2x+1$ is odd, the last condition is equivalent to $bx^4+ax^3+4x^2+ax+a \equiv b+6a+16 \pmod{2x+1}$.

Therefore we have two conditions:

$\frac{b-a+4}{x+1} \in \mathbb{N} \quad (1)$

$\frac{b+6a+16}{2x+1} \in \mathbb{N} \quad (2)$

Since $b \leqslant x-1 \wedge a \geqslant 1$, we have $\frac{b-a+4}{x+1} \leqslant \frac{x+2}{x+1} = 1+\frac{1}{x+1}$, hence $\frac{b-a+4}{x+1} = 1 \implies b-a = x-3$. Since $b \leqslant x-1$ and $a \geqslant 1$, we have only two possibilities: $(a,b) = (2, x-1)$ and $(a,b) = (1, x-2)$.

Plugging the first possibility into (2) we get $\frac{x+27}{2x+1} = \frac{1}{2}\left(1+\frac{53}{2x+1}\right)$, hence $2x+1$ must be a divisor of 53, which can't be fulfilled for $5 \leqslant x \leqslant 9$ as 53 is prime.

The second possibility yields $\frac{x+20}{2x+1} = \frac{1}{2}\left(1+\frac{39}{2x+1}\right)$. The only divisor of 39 for $5 \leqslant x \leqslant 9$ is 13 for $x=6$, which in turn yields $(a,b) = (1,4)$.

Hence the solution is $231_6 \cdot 141_6 = 41411_6$ (in the decimal system: $91 \cdot 61 = 5551$)

□ Find all $n \in \mathbb{N}$ satisfy : $x^{2n} - x^n + 1 \equiv 0 (mod x^2 - x + 1)$

Solution

Put $P(x) = x^{2n} - x^n + 1$. The zeroes of x^2-x+1 are $e^{\pm i\pi/3}$, so we must have $P(e^{\pm i\pi/3}) = 0$, which reduces to a system:

$\cos\frac{2n\pi}{3} - \cos\frac{n\pi}{3} + 1 = 0 \iff 2\cos^2\frac{n\pi}{3} - \cos\frac{n\pi}{3} = 0$

$\sin\frac{2n\pi}{3} - \sin\frac{n\pi}{3} = 0 \iff \sin\frac{n\pi}{3}(2\cos\frac{n\pi}{3} - 1) = 0$

The first equation yields $\cos\frac{n\pi}{3} = 0 \vee \cos\frac{n\pi}{3} = \frac{1}{2}$, and the second one yields $\sin\frac{n\pi}{3} = 0 \vee \cos\frac{n\pi}{3} = \frac{1}{2}$.

Assume $\cos\frac{n\pi}{3} = 0$. Then $\sin\frac{n\pi}{3} = \pm 1 \wedge \cos\frac{n\pi}{3} \neq \frac{1}{2}$, hence the second set of the conditions can't be satisfied. Therefore $\cos\frac{n\pi}{3} = \frac{1}{2}$, which also satisfies the second set of the conditions.

$\cos\frac{n\pi}{3} = \frac{1}{2} \implies \frac{n\pi}{3} = \pm\frac{\pi}{3} + 2k\pi \implies n = 6k \pm 1$

Therefore $n \in (6\mathbb{N}_0 + 1) \cup (6\mathbb{N}_0 + 5)$

□ Given a triangle ABC and a point P in the same plane as $\triangle ABC$, let the directed distance from P to AB, BC, CA be c, a, b respectively, where negative means that P is on the opposite side of the edge as the other vertex. Prove that

$\frac{a}{h_a} + \frac{b}{h_b} + \frac{c}{h_c} = 1$

where h_a, h_b, h_c are the lengths of the altitudes to BC, CA, AB respectively. No using barycentric coordinates, because that makes the problem trivial.

Solution

First let's assume P is an internal point of the triangle. If $S := [ABC]$, then

$\frac{aBC}{2} + \frac{bCA}{2} + \frac{cAB}{2} = S$

Putting $BC = \frac{2S}{h_a}, CA = \frac{2S}{h_b}, AB = \frac{2S}{h_c}$, we obtain the result.

Now assume A and P are on the opposite sides of BC, and that P, B and P, C are on the same side of AC, AB respectively. Then $[ABC] = [ABP] + [APC] - [BPC]$, hence

$-\frac{|a|BC}{2} + \frac{bCA}{2} + \frac{cAB}{2} = S \implies -\frac{|a|}{h_a} + \frac{b}{h_b} + \frac{c}{h_c} = 1$

But $-|a| = a$, hence the result follows.

We proceed similarly in the remaining cases.

□ in a warehouse N containersmarked 1 through N are arranged in two piles. A forklift can take several containers from the top of one pile and place them on the top of other pile. Prove that all the containers can be arranged in one pile in increasing order of their numbers with 2N-1 such operations of the forklift.

Solution

We assume that during the moves some pile is allowed to be empty (otherwise there can be no solution, for example for pile A being 23 and pile B being 41). The correct formula is $2(N-1)$ and is proved by induction. For $N=1$ it is trivial 0 moves are needed. Assume $2(N-1)$ moves are enough for N containers. When having $N+1$ containers, assume 1 is in some position in pile A. Put pile A over B, then cut at 1 and recreate pile A, now having 1 at the bottom (it took 2 moves). Now the remaining N containers (from 2 to $N+1$) require at most $2(N-1)$ moves (by induction hypothesis), so $2+2(N-1)=2((N+1)-1)$ moves are enough for $N+1$ containers.

□ Find min of m^2+n^2 with m,n satisfies that following equation have solution. $x^4+mx^3+nx^2+mx+1=0$

$m,n \in \mathbb{R}$

Solution

$x=0$ is not a solution, so we can divide by x^2 to get $x^2+mx+n+\frac{m}{x}+\frac{1}{x^2}=0$.

Let $k=x+\frac{1}{x}$. Then $|k|\geq 2$ by AM-GM. We have $k^2-2+mk+n=0$. We can assume that $k>0$, because we can do the transformation $(k,m)\to(-k,-m)$ and keep the rest of the equation true.

We see that

$(km+n)^2+(m-kn)^2=(k^2+1)(m^2+n^2)$.

Keeping k constant, we see that the min of m^2+n^2 is when $m=kn$, because $km+n$ is constant, equal to $2-k^2$. Then

$m^2+n^2=\frac{(2-k^2)^2}{k^2+1}=\frac{(k^2-5)(k^2+1)+9}{k^2+1}=k^2-5+\frac{9}{k^2+1}$.

This is an increasing function in k because $a+\frac{1}{a}-4$ is, where $a=k^2+1$, so the min is when $k=2$ and $m^2+n^2=2^2-5+\frac{9}{2^2+1}=-1+\frac{9}{5}=\frac{4}{5}$, when $n=-1$ and $m=-\frac{1}{2}$. This has solution $x=1$.

□ There exists a polynomials P of degree 5 with the property that If Z is a complex no. such that $Z^5+2004Z=1$,then $P(Z^2)=0$. Then find the value of $|\frac{P(1)}{P(-1)}|$

Solution

Let $Q(z)=z^5+2004z-1$ and let $z_{1,2,3,4,5}$ be its zeroes. By the definition of $P(x)$, we have

$P(x)=C(x-z_1^2)(x-z_2^2)(x-z_3^2)(x-z_4^2)(x-z_5^2)$ where C is a constant.

Since we need to find a ratio, WLOG we can take $C=1$.

Then $P(1)=(1-z_1^2)(1-z_2^2)(1-z_3^2)(1-z_4^2)(1-z_5^2)$. By factoring this, we find that $P(1)=Q(1)\cdot(-1)^5Q(-1)=-2004\cdot(-2006)=2005^2-1$

For $P(-1)$ we get $P(-1)=-(1+z_1^2)(1+z_2^2)(1+z_3^2)(1+z_4^2)(1+z_5^2)$. By factoring this in $\mathbb{C}$ - i.e. using $1+a^2=(-i+a)(i+a)$ - we find that $P(-1)=-(-1)^5Q(i)(-1)^5Q(-i)=-(-1+2005i)(-1-2005i)=-(1+2005^2)$ Hence $\frac{P(1)}{P(-1)}=\frac{1-2005^2}{1+2005^2}$

□ Twenty five boys and twenty five girls sit around a table. Prove that is always possible to find a person both whose neighbors are girls.

Solution

Denote the positions around the table by $0,1,\ldots,49$; notice 49 also neighbors 0. Either the odd positions $1,3,\ldots,49$ or the even positions $0,2,\ldots,48$ accommodate 12 or less boys (pigeonhole principle). WLOG, assume it's the 25 odd positions. If no boys are seated there, the triplet $(1,2,3)$ (for example; in fact many other) has two girls at its ends. So at least a boy must be seated there.

Fix one of them, say k, and group the remaining 24 positions in 12 pairs $(k+2,k+4),(k+6,k+8),\ldots,(k-4,k-2)$, the indices being taken modulo 50 (for example, $k=15$ and group in 12 pairs $(17,19),(21,23),\ldots,(45,47),(49,1),(3,5),\ldots,(11,13)$). If any pair contains no boy, then it's made of two girls, and, together with the middle person, it's a triplet with two girls at its ends. So we need a boy in each pair, which is impossible, since we have at most $12-1=11$ boys left.

Of course, the problem being symmetric in boys and girls, the same conclusion is valid for genders reverted.

□ Prove that $\sum_{k=1}^n \tan^2\frac{k\pi}{2n+1}=n(2n+1)$ $\sum_{k=1}^n \cot^2\frac{k\pi}{2n+1}=\frac{n(2n-1)}{3}$

Solution

$\sin(2n+1)\theta=\sum_{k=0}^n(-1)^k\binom{2n+1}{2k+1}\sin^{2k+1}\theta\cos^{2n-2k}\theta=\cos^{2n+1}\theta\sum_{k=0}^n(-1)^k\binom{2n+1}{2k+1}\tan^{2k+1}\theta$ So $\sum_{k=0}^n(-1)^k\binom{2n+1}{2k+1}x^{2k+1}=0\iff\sum_{k=0}^n(-1)^k\binom{2n+1}{2k}x^{2n-2k}=0\iff x=\tan\frac{k\pi}{2n+1}\quad(k=1,\ldots 2n)$ Therefore $\sum_{k=0}^n(-1)^k\binom{2n+1}{2k}x^{n-k}=0\iff x=\tan^2\frac{k\pi}{2n+1}\quad(k=1,\ldots n)$

Thus $\sum_{k=1}^n\tan^2\frac{k\pi}{2n+1}=\frac{\binom{2n+1}{2}}{\binom{2n+1}{0}}=n(2n+1)$ and $\sum_{k=1}^n\cot^2\frac{k\pi}{2n+1}=\frac{\binom{2n+1}{2n-2}}{\binom{2n+1}{2n}}=\frac{n(2n-1)}{3}$

The coefficients of x^{13} and x^0 match. The coefficient of x in the LHS is 1 which is equal to the coefficient of x in the RHS which is $a[x]_T-b$. Therefore $[x]_T=\frac{1+b}{a}$. But $[x]_T$ must be an integer, so from the factors of 90, the only possible values of a are $-9,-2,-1,1,2,10$. Checking all of these cases yields that $a=2$ is the solution.

(Note: We could have either gone further, calculating more coefficients of $T(x)$, or we could have just checked all the factors of 90.)

□ Let $z\in\mathbb{C}$ and $a,b\geq 0$. If $\omega=\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}$, then prove that : $|z-1|+|z-a\omega|+|z-b\omega^2|\geq 1+a+b$.

Solution

let $Arg(\omega)=\theta=\frac{2\pi}{3}, Arg(\omega^2)=Arg(\overline{\omega})=-\theta=-\frac{2\pi}{3}$

$A=(1,0),B=(a\cos\theta,a\sin\theta),C=(b\cos(-\theta),b\sin(-\theta))=(b\cos\theta,-b\sin\theta)$ For any ponit P_z on the complex plane, we have

$\overline{P_zA}+\overline{P_zB}\geq\overline{AB}$ which is true by triangular ineq. i.e.

$|z-1|+|z-a\omega|\geq\sqrt{(a\cos\theta-1)^2+(a\sin\theta)^2}=\sqrt{a^2-2a\cos\theta+1}=a+1$, since $\cos\frac{2\pi}{3}=-\frac{1}{2}$

Similarly, $\overline{P_zA}+\overline{P_zC}\geq\overline{AC}$

$|z-1|+|z-b\omega^2|\geq\sqrt{(b\cos\theta-1)^2+(-b\sin\theta)^2}=\sqrt{b^2-2b\cos\theta+1}=b+1$,and

$\overline{P_zB}+\overline{P_zC}\geq\overline{BC}$

$|z-a\omega|+|z-b\omega^2|\geq\sqrt{(a\cos\theta-b\cos\theta)^2+(a\sin\theta+b\sin\theta)^2}$

$=\sqrt{a^2+b^2-2ab\cos^2\theta+2ab\sin^2\theta}=\sqrt{a^2+b^2-2ab\cos 2\theta}=a+b$, since $\cos 2\theta=\cos\frac{4\pi}{3}=-\frac{1}{2}$

Hence ,adding the above three ineq., then $|z-1|+|z-a\omega|+|z-b\omega^2|\geq\frac{1}{2}(a+1+b+1+a+b)=1+a+b$

The equality holds only when $P_z=(0,0)$

□ Find $\max_{n\in\mathbb{N}^*}a_n$, where $a_n=\sqrt[n]{n}$.

Solution

Define over $\mathbb{R}$ the following equivalence relation: $x\sim y\iff\operatorname{sign}(x)=\operatorname{sign}(y)\iff x=y=0\ \vee\ xy>0\iff$

x and y [u]have same sign[/u] . Denote $a_n=\sqrt[n]{n}$, $n\in\mathbb{N}^*$. Thus, for any $n\geq 3$, $a_{n+1}-a_n=\left(\sqrt[n]{n+1}-\sqrt[n]{n}\right)\sim$

$[(n+1)^n-n^{n+1}]\sim\left[\left(1+\frac{1}{n}\right)^n-n\right]<0$ because $\left(1+\frac{1}{n}\right)^n\nearrow e\implies\left(1+\frac{1}{n}\right)^n<e<3\leq n$.

□ Given the equation: $\sin kx=\sin x$ Find the value of k for which this equation and the equation $\cos 3x=\cos 2x$ have, within the range $(0,360]$ (degrees), one and only one common solution

Solution

Angles will be in degrees.

When is $\sin a = \sin b$? When either $a \equiv b \pmod{360}$ or $a \equiv 180 - b \pmod{360}$.

When is $\cos a = \cos b$? When either $a \equiv b \pmod{360}$ or $a \equiv -b \pmod{360}$.

So the equation

$sinkx =$

$sinx$ can be written as $kx \equiv x \pmod{360}$ or $kx \equiv 180 - x \pmod{360}$. That gets us two families of solutions:

$x = \frac{360j}{k-1}$ or $x = \frac{180+360j}{k+1}$ for $j \in \mathbb{Z}$.

The equation $\cos 3x = \cos 2x$ can be solved as follows:

$3x \equiv 2x \pmod{360}$ which implies $x \equiv 0 \pmod{360}$ or $x = 360n$.

or

$3x \equiv -2x \pmod{360}$, which implies $5x \equiv 0 \pmod{360}$ or $x = \frac{360n}{5}$.

That second equation includes the first.

So, when do solutions coincide?

Either $\frac{360n}{5} = \frac{360j}{k-1}$ or $\frac{360n}{5} = \frac{180+360j}{k+1}$.

Take the first equation, divide by 360 and multiply by $5(k-1)$ to get $(k-1)n = 5j$.

This always has $n = 0, j = 0$ as a solution. We also have solutions whenever $5 \mid n$ (but that's the same place on the circle). If 5 doesn't divide n, then we would need $5 \mid (k-1)$ or $k \equiv 1 \pmod{5}$. Then $j = \frac{(k-1)n}{5}$, and as n ranges over all integers not equivalent to 5, then j will always be an integer.

Now let's look at the other equation. This time, divide by 180 and multiply by $5(k+1)$. That leaves $2(k+1)n = 5 + 10j$.

If 5 divides $k+1$, we get no solution, as one side is divisible by 10 and the other side is $\equiv 5 \pmod{10}$. But if 5 doesn't divide $k+1$, then we would have $5 \mid n$, which gets us back to $x \equiv 0 \pmod{360}$.

So:

If $k \not\equiv 1 \pmod{5}$, then the only solution in the circle is $x \equiv 0 \pmod{360}$. However, if $k \equiv 1 \pmod{5}$, then $\{0, 72, 144, 216, 288\}$ and their equivalents mod 360 are all solutions.

The question asked for the k that produce a unique solution in the circle; that would be $\{k : k \not\equiv 1 \pmod{5}\}$.

□ (x) is a polynomial of degree 998.p(k)=1/k for K is integral varying from 1 to 999. Find the value of P (1001).

Solution

a. 1 b. 1001 c. 1/1001 d.1/(1001!)

Your definition is equivalent to $kP(k) = 1$ for all the integers between 1 and 999. So, $kP(k) - 1 = A(k-1)(k-2)\ldots(k-999)$, where A is some unknown constant. For $k = 0$, we have that $-1 = -A(999!)$, so $A = \frac{1}{999!}$. Now, $1001P(1001) - 1 = \frac{1000!}{999!}$. $1001P(1001) = 1001$, so $\boxed{P(1001) = 1}$. The answer: $\boxed{A}$.

□ Given a, b, c and $\frac{ab+bc+ac}{\sqrt{abc}}$ are all positive integers, does that imply that $\sqrt{\frac{ac}{b}}, \sqrt{\frac{ab}{c}}, \sqrt{\frac{bc}{a}}$ must all be integers?

Solution

Clearly $\sqrt{abc} \in \mathbb{N}$ so $abc = k^2, k \in \mathbb{N}$

Write $M = (a, b, c) = (\alpha^2 xy, \beta^2 yz, \gamma^2 zx)$
With $\gcd(\alpha, \beta) = \gcd(\beta, \gamma) = \gcd(\gamma, \alpha) = 1$
constructive proof Take $M = (a, b, c)$ and let $\gcd(a, b) = y \implies M = (a'y, b'y, c)$
Let $\gcd(a', c) = x \implies M = (a''yx, b'y, c'x)$
Let $\gcd(b', c') = z \implies M = (a''xy, b''yz, c''zx)$
since $\gcd(a'', b'') = \gcd(b'', c'') = \gcd(c'', a'') = 1$ it follows that a'', b'', c'' are perfect squares.
$\therefore M = (\alpha^2 xy, \beta^2 yz, \gamma^2 zx)$
This gives
$\frac{ab+bc+ca}{\sqrt{abc}} = \frac{\sum \alpha^2\beta^2 y}{\alpha\beta\gamma}$
Hence $\alpha|z, \beta|x$ and $\gamma|y$
Therefore
$\sqrt{\frac{ab}{c}} = \sqrt{\frac{\alpha^2 xy\beta^2 yz}{\gamma^2 zx}} = \frac{\alpha\beta y}{\gamma} \in \mathbb{N}$ because $\gamma|y$
$\square$ Prove that every $f : \mathbb{N} \to \mathbb{N}$ which is a bijection can be written as the sum of two involutions.

Solution

I assume that should read "composition of two involutions".

Let $X_1 = \mathbb{N}$. We define X_n iteratively as follows: let $S_n = \{x : \exists \min(X_n)\}$, and set $X_{n+1} = X_n \backslash S_n$; thus, $\bigcup S_n = \mathbb{N}$. (here f^n refers to the composition of f, n times)

Suppose $|S_n| = k \in \mathbb{N}$. If $k = 1$, then define $g_n(x) = h_n(x) = x$ where $x \in S_n$. Otherwise, $S_n = \{x_1, \ldots, x_k\}$ where $f(x_i) = x_{i+1}, x_{k+1} := x_1$, define the involutions $g_n, h_n : S_n \to S_n$ as follows: $g_n(x_i) = x_{k+2-i}, h_n(x_i) = x_{k+3-i}$ (they are involutions due to the definition of x_{k+1}, though this is shown in more detail in the hidden tag); obviously $f_n(x_i) = h_n(g_n(x_i))$. More specifically $g_n(x_1) = x_1, g_n(x_i) = x_{k+2-i}$ for $2 \le i \le k$; and $h_n(x_1) = x_2, h_n(x_2) = x_1$, and $h_n(x_i) = x_{k+3-i}$ for $3 \le i \le k$. Observe that

$$h_n(g_n(x_i)) = \begin{cases} x_{k+3-(k+2-i)} = x_{i+1}, & 2 \le i \le k-1 \\ x_2, & i = 1 \\ x_1, & i = k \end{cases} = f_n(x_i).$$

Here's an example, for $k = 5$ and $S_n = \{1, 2, 3, 4, 5\}$:

x	g(x)	h(g(x))
1	1	2
2	5	3
3	4	4
4	3	5
5	2	1

where $g(1) = 1$, and the remaining elements are 'reflected' by g; and all the elements are 'reflected' by h. If S_n is countably infinite, select an arbitrary element $x_1 \in S_n$, and let $S_n = \{x_1, \ldots\}$ where $x_{2k+1} = f^k(x_1)$ and $f^k(x_{2k}) = x_1$, $k \in \mathbb{N}$. Then define the involutions $g_n, h_n : S_n \to S_n$ as follows: $g_n(x_1) = x_1, g_n(x_{2k}) = x_{2k+1}, g_n(x_{2k+1}) = x_{2k}$; and $h_n(x_1) = x_3, h_n(x_3) = x_1, h_n(x_{2k}) = x_{2k+3}, h_n(x_{2k+3}) = x_{2k}$. Verify, much like above, that $f_n(x_i) = h_n(g_n(x_i))$.

Then, naturally, we have $f = h(g(x))$, where $g(x) = g_n(x)$ if $x \in S_n$ and $h(x) = h_n(x)$ if $x \in S_n$.

Note in essence that the involutions defined are similar to slightly shifted reflections; will post a more informal explanation.

$2 \equiv 1 \pmod{30}$ or $p^2 \equiv 19 \pmod{30}$

Solution

It's only true for $p > 5$. We have to show that either $p^2 - 1$ is divisible by 30 or $p^2 - 19$ is. Both are even for $p > 5$.Since p is either 1 or 2 mod 3 for p >5, both are divisible by 3. So we have to show 5 divides one of the two. If $p > 5$ then it is either 1,2,3, or 4 mod 5. If it is 1 or 4 mod 5, then 5 divides $p^2 - 1$. If it is 2 or 3 mod 5, then 5 divides $p^2 - 19$. SoSince 2,3, and 5 all divide one of $p^2 - 1$ or $p^2 - 19$, one of them must be divisible by 30.

□ Find the smallest natural number n,such that there exist positive integrs $x_1, x_2, \dots, x_n$, such that $x_1^3 + x_2^3 + \ldots + x_n^3 = 2008$

Solution

Assume there are two positive integers a, b such that $a^3 + b^3 = 2008$

Then $2008 = a^3 + b^3 \geq \frac{(a+b)^3}{4} \implies a + b < 2\sqrt[3]{1004} < 2 \cdot 11 = 22$

Since $2008 = 2^3 \cdot 251$ we have $a + b = 1, 2, 4$ or 8

But $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ so $a^2 - ab + b^2 \geq 251$ but $a^2 - ab + b^2 = (a+b)^2 - 3ab < 8^2 = 64$

Contradiction

□ prove: $lcm(1, 2, \dots, 2n) = lcm(n+1, n+2, \dots, n+n)$

Solution

This is obvious,Since for every number $a \in \{1, 2, 3, \dots, n\}$ there exist a number $b \in \{n+1, n+2, \dots, 2n\}$ such that $a \mid b$. The claim easily follows.

□ Prove that: in eight integers have three digits,∃ $\overline{a_1a_2a_3}$ and $\overline{b_1b_2b_3}$ satisfy $a_1a_2a_3b_1b_2b_3 \equiv 0 \pmod 7$

Solution

Just note that $10^3 \equiv -1 \mod 7$, By the box principle there are two integers a_i, a_j with the same residue $\mod 7$ so $10^3 a_j + a_i \equiv a_i - a_j \equiv 0 \mod 7$

□ $a_1, a_2, \dots, a_n$ are positive numbers such that their sum is one. Find the minimum of: $a_1/(1 + a_2 + \ldots + n) + a_2/(1 + a_1 + a_3 + \ldots + a_n) + \ldots + a_n/(1 + a_1 + \ldots + a_{n-1})$(and please prove it!).

Solution

Assuming you meant to have $1 + a_2 + \cdots + a_n$ in the denominator of the first term, Let $S = \frac{a_1}{1 + a_2 + \cdots + a_n} + \frac{a_2}{1 + a_1 + \cdots + a_n} + \cdots + \frac{a_n}{1 + a_1 + \cdots + a_{n-1}}$. We have that $a_1 + a_2 + \cdots + a_n = 1$, Thus we can rewrite the original expression as,

$$S = \sum_{i=1}^{n} \frac{a_i}{2 - a_i}$$

We can then add one to each term then subtract n to get,

$$S = -n + \sum_{i=1}^{n} \frac{2}{2 - a_1}$$

Take out a factor of 2 from the sum,

$$S = -n + 2\left(\sum_{i=1}^{n} \frac{1}{2 - a_1}\right)$$

Use Cauchy-Schwarz to show that,

$$(2n - 1)\left(\sum_{i=1}^{n} \frac{1}{2 - a_1}\right) \geq n^2 \implies \sum_{i=1}^{n} \frac{1}{2 - a_1} \geq \frac{n^2}{2n - 1}$$

Hence,

$$S = -n + 2\left(\sum_{i=1}^{n} \frac{1}{2-a_1}\right) \geq -n + 2\left(\frac{n^2}{2n-1}\right) = \frac{2n^2}{2n-1} - n = \boxed{\frac{n}{2n-1}}$$

And that's our answer. Equality occurs when $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$

□ Prove that there are infinitely many solutions: $a^2b^2 - 4b(b+1) = c^2$

Solution

Let $a = 3$ and $b = F_{2n-1}^2$ where $n \in \mathbb{N}$. (Note: F_n is the Fibonacci sequence). Using the well-known fact that $5F_{2n-1}^2 - 4$ is a perfect square for $n \in \mathbb{N}$, we have:

$$\begin{aligned} a^2b^2 - 4b(b+1) &= 9F_{2n-1}^4 - 4F_{2n-1}^4 - 4F_{2n-1}^2 \\ &= 5F_{2n-1}^4 - 4F_{2n-1}^2 \\ &= F_{2n-1}^2(5F_{2n-1}^2 - 4), \end{aligned}$$

which is a perfect square.

Therefore,Since there are infinitely many numbers of the form F_{2n-1}^2, there are infinitely many integer solutions.

□ Find all x such that:
$\sqrt{\cos 2x - \sin 4x} = \sin x - \cos x$

Solution

$\sqrt{\cos 2x - \sin 4x} = \sin x - \cos x$

$\Leftrightarrow \begin{cases} \sin x - \cos x \geq 0 \\ \cos 2x - \sin 4x = 1 - \sin 2x \end{cases}$

$\Leftrightarrow \begin{cases} \sin(x - \frac{\pi}{4}) \geq 0 \\ (\cos 2x + \sin 2x)(\cos 2x + \sin 2x - 1) = 0 \end{cases}$

$\Leftrightarrow \begin{cases} k2\pi \leq x - \frac{\pi}{4} \leq \pi + k2\pi \, (k \in Z) \quad (*) \\ \left[\begin{array}{l} \sin(2x + \frac{\pi}{4}) = 0 \qquad (1) \\ \sin(2x + \frac{\pi}{4}) = \frac{\sqrt{2}}{2} = \sin\frac{\pi}{4} \qquad (2) \end{array} \right. \end{cases}$

We have

$(1) : \sin(2x + \frac{\pi}{4}) = 0$

$\Leftrightarrow x = -\frac{\pi}{8} + \frac{l\pi}{2} \;\; (l \in Z)$

Because the condition (*) must be satisfied by x, therefore :

$k2\pi \leq -\frac{\pi}{8} + \frac{l\pi}{2} - \frac{\pi}{4} \leq \pi + k2\pi \;\; (l, k \in Z)$

$\Rightarrow \frac{3\pi}{8} \leq \frac{l\pi}{2} - k2\pi \leq \frac{11\pi}{8}$

$\Rightarrow \frac{3}{8} \leq \frac{l}{2} - 2k \leq \frac{11}{8}$

$\Rightarrow l = 2(2k+1) = 2a(a \in Z)$

$\Rightarrow x = -\frac{\pi}{8} + a\pi(a \in Z)$

We have

$(2) : \sin(2x + \frac{\pi}{4}) = \sin\frac{\pi}{4}$

$\Leftrightarrow \left[\begin{array}{l} x = m\pi \\ x = \frac{\pi}{4} + m\pi \end{array} \right. \quad (m \in Z)$

Similarly, we obtain $x = \frac{\pi}{4} + m\pi$ and $x = (2k+1)\pi$ where $k, m \in Z$

Conclusion, the solutions for the given equation are: $x = -\frac{\pi}{8} + a\pi, x = \frac{\pi}{4} + m\pi, x = (2k+1)\pi$ where $a, m, k \in Z$.

□ I mean that the number of digits of a, plus the number of digits of a^n equals 361

Solution

$\lfloor \log_{10} a \rfloor + \lfloor n \log_{10} a \rfloor = 359 \qquad (1)$

so $(n+1)\lfloor \log_{10} a \rfloor \le \lfloor \log_{10} a \rfloor + \lfloor n \log_{10} a \rfloor \le \lfloor (n+1) \log_{10} a \rfloor$

let $\log_{10} a = p + r$ with $p \in \mathbb{N}$ and $0 < r < 1$ then

$(n+1)p \le 359 \le (n+1)p + (n+1)r < (n+1)(p+1) \Longrightarrow p \le \frac{359}{n+1} < p+1$

so $p = \left\lfloor \frac{356}{n+1} \right\rfloor$

from (1): $p + np + \lfloor nr \rfloor = 359$ butSince $0 \le \lfloor nr \rfloor < n$ we have

$359 < (n+1)\left\lfloor \frac{356}{n+1} \right\rfloor + n \qquad (2)$

but the only value of $n \in \{1, 2, \ldots, 9\}$ for which (2) is true is $\boxed{n = 6}$

□ Solve the equation $x^x + y^y = \overline{xy} + 3$ where $\overline{xy} = 10x + y$

Solution

$\overline{xy} + 3 \le 99 + 3 \le 102 \implies x^x + y^y \le 102 \implies x, y \le 3.$

Furthermore, 0^0 is undefined so neither digit can be 0.

Case $x = 1$: $1 + y^y = 13 + y \implies y^y - y = 12 \implies y \notin \mathbb{N}.$

Case $x = 2$: $4 + y^y = 23 + y \implies y^y - y = 19 \implies y \notin \mathbb{N}.$

Case $x = 3$: $27 + y^y = 33 + y \implies y^y - y = 6 \implies y \notin \mathbb{N}.$

So, there are no solutions in $\mathbb{N}$.

□ Find all integer solutions (n,m) to - $n^4 + 2n^3 + 2n^2 + 2n + 1 = m^2$

Solution

we factor the left side of the equation, we obtain

$(n+1)^2(n^2+1) = m^2$

Now $n^2 + 1$ needs to be perfect square, because $(n+1)^2$ and m^2 are perfect squares.

From $n^2 + 1 = x^2$ we get $n = 0$ and $x = + - 1$, from there $m = + - 1$

And second solution would be for $m = 0$, then we have $n = -1$.

□

For a math contest there is a shortlist with 46 problems, of which 10 are geometry problems. The difficulty of every two problems is different (so there are no two problems with the same difficulty). Let N be the number of ways the selection committee can select 3 problems, such that - Problem 1 is easier than problem 2, - Problem 2 is easier than problem 3, - There is at least one geometry problem in the test. Calculate $\frac{N}{4}$.

Solution

Given an arbitrary selection of three problems, there is only one way to order them such that they are in ascending order of difficulty. Therefore, there are $\binom{46}{3} = 15180$ possible tests. However, we must compute the number of tests with no geometry problems. This is $\binom{36}{3} = 7140$. $N = \frac{15180-7140}{4} = \boxed{2010}$.

□

Show, using the binomial expansion, that $(1+\sqrt{2})^5 < 99$. Show also that $\sqrt{2} > 1.4$. Deduce that $2^{\sqrt{2}} > 1 + \sqrt{2}$.

Solution

first we will prove that $\sqrt{2} > 1.4$. Squaring that we get that $2 > 1.96$ which is true and we 'll prove that $1.5 > \sqrt{2}$, which is also trivial when we square it.

Now $(1+\sqrt{2})^5 < 99.$ –> $(1+\sqrt{2})^5 < (1+1.5)^5 = 97.65625 < 99$

$2^{\sqrt{2}} > 1 + \sqrt{2}$ is trivial by Bernoulli's inequality ... Rewrite number 2 from left side of inequality in form $(1+1)$

□

Prove that: p is prime, $p \geq 3$, the equation $x^2 + 1 \equiv \pmod p$ have solution if $p = 4k + 1$

Solution

Assume $p = 4k + 3$, then obviously p does not divide x so

$$x^2 \equiv -1 \implies 1 \equiv x^{p-1} \equiv x^{2 \cdot \frac{p-1}{2}} = (-1)^{\frac{p-1}{2}} = -1 \pmod p.$$

□ Let $a > 2$ be an odd number and let n be a positive integer. Prove that a divides $1^{a^n} + 2^{a^n} + \cdots + (a-1)^{a^n}$

Solution

"Solution 1 Let $S = 1^{a^n} + 2^{a^n} + \cdots + (a-1)^{a^n}$. We can express S as,

$$(1^{a^n} + (a-1)^{a^n}) + (2^{a^n} + (a-2)^{a^n}) + \cdots + \left(\left(\frac{a-1}{2}\right)^{a^n} + \left(\frac{a+1}{2}\right)^{a^n}\right)$$

Then use the fact that

$$a + b | a^k + b^k$$

for positive integers a and b whenever k is an odd positive integer. Click here for proof of this fact We know that,

$$a + b \equiv 0 \pmod{a+b}$$

Thus,

$$a \equiv -b \pmod{a+b} \implies a^k \equiv (-b)^k \pmod{a+b}$$

Since k is odd we know that, $(-b)^k \equiv -b^k \pmod{a+b}$ and therefore,

$$a^k \equiv -b^k \pmod{a+b} \implies a^k + b^k \equiv 0 \pmod{a+b}$$

From where we get that $a + b | a^k + b^k$.

Thus,

$$\begin{array}{rcl} 1 + (a-1) & | & 1^{a^n} + (a-1)^{a^n} \\ 2 + (a-2) & | & 2^{a^n} + (a-2)^{a^n} \\ & \vdots & \\ \left(\frac{a-1}{2}\right) + \left(\frac{a+1}{2}\right) & | & \left(\frac{a-1}{2}\right)^{a^n} + \left(\frac{a+1}{2}\right)^{a^n} \end{array}$$

We can use this fact because a^n is always an odd integer when n is a positive integer and a is odd. Hence S is divisible by a.

Solution 2 Note that,

$$\begin{aligned} 1^{a^n} + (a-1)^{a^n} &\equiv 1^{a^n} + (-1)^{a^n} \equiv 1^{a^n} - 1^{a^n} \equiv 0 \pmod a \\ 2^{a^n} + (a-2)^{a^n} &\equiv 1^{a^n} + (-2)^{a^n} \equiv 2^{a^n} - 2^{a^n} \equiv 0 \pmod a \\ &\vdots \\ \left(\frac{a-1}{2}\right)^{a^n} + \left(\frac{a+1}{2}\right)^{a^n} &\equiv \left(\frac{a-1}{2}\right)^{a^n} + \left(a - \frac{a-1}{2}\right)^{a^n} \equiv \left(\frac{a-1}{2}\right)^{a^n} - \left(\frac{a-1}{2}\right)^{a^n} \equiv 0 \pmod a \end{aligned}$$

Adding all the equations up we get that,

$$S \equiv 0 + 0 + \cdots + 0 \equiv 0 \pmod{a}$$

$\square$

$\Sigma x_i \leq \Sigma x_i^2$ for $x_i > 0$

Prove that

Solution

$\Sigma x_i^p \leq \Sigma x_i^{p+1}$ for $p > 1$, $p \in R$

$\Sigma x_i \leq \Sigma x_i^2 \implies \Sigma x_i^2 - x_i \geq 0 \implies \Sigma x_i(x_i - 1) \geq 0$

So it is only natural to divide the terms depending on whether or not they are positive or negative, i.e.:

$\sum_{i:x_i>1} x_i(x_i-1) + \sum_{i:x_i<1} x_i(x_i-1) \geq 0$

Clearly all the terms in the first summand on LHS are positive, whereas all the terms in the second one are negative.

Since $x_i > 1 \implies x_i^{p-1} > 1$ we have, $\sum_{i:x_i>1} x_i^p(x_i-1) \geq \sum_{i:x_i>1} x_i(x_i-1)$

Similarly, $x_i < 1 \implies x_i^{p-1} < 1 \implies \sum_{i:x_i<1} x_i^p(x_i-1) \geq \sum_{i:x_i<1} x_i(x_i-1)$ (recall that both sides are negative)

Adding the two inequalities, we get: $\sum_{i:x_i>1} x_i^p(x_i-1) + \sum_{i:x_i<1} x_i^p(x_i-1) \geq \sum_{i:x_i>1} x_i(x_i-1) + \sum_{i:x_i<1} x_i(x_i-1) \geq 0 \implies \sum_{i:x_i>1} x_i^p(x_i-1) + \sum_{i:x_i<1} x_i^p(x_i-1) = \sum x_i^p(x_i-1) \geq 0 \implies \sum x_i^p \leq \sum x_i^{p+1}$ as desired

$\square$ Let $a_1, a_2, \ldots, a_n$ be postive real numbers. Prove: $(a_1 + \ldots + a_n)^2 \leq \frac{\pi^2}{6}(1^2a_1^2 + 2^2a_2^2 + \ldots + n^2a_n^2)$

Solution

From Cauchy-Schwarz inequality,

$$\frac{\pi^2}{6}\left(\sum_{i=1}^{n} i^2a_i^2\right) = \left(\sum_{i=1}^{\infty}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n} i^2a_i^2\right) \geq \left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n} i^2a_i^2\right) \geq \left(\sum_{i=1}^{n} a_i\right)^2.$$

$\square$ Find all pairs of integers (m, n) such that the numbers $A = n^2 + 2mn + 3m^2 + 2$, $B = 2n^2 + 3mn + m^2 + 2$, $C = 3n^2 + mn + 2m^2 + 1$ have a common divisor greater than 1.

Solution

Suppose p is prime and $p|A, B, C$.

$A - B = 2m^2 - mn - n^2 = (m-n)(2m+n) \qquad (1)$

$C - B = m^2 - 2mn + n^2 - 1 = (m-n)^2 - 1 \qquad (2)$

From (1), $p|(m-n)$ or $p|(2m+n)$ but clearly $p \not|(m-n)$ because of (2)

replacing $n \equiv -2m \mod p$ in A and C gives $3m^2 + 2 \equiv 12m^2 + 1 \mod p$

But $\gcd(3m^2+2, 12m^2+1) = \gcd(3m^2+2, 7)$ so the greatest common denominator is at most 7

So $3m^2 + 1 \equiv 0 \mod 7 \Longrightarrow m \equiv 2, 5 \mod 7 \Longrightarrow n \equiv 3, 4 \mod 7$

Hence $(m, n) = (7k_1 + 2, 7k_2 + 3) or (7k_1 + 5, 7k_2 + 4)$

$\square$ 100 lines lie in the plane. Is it possible for them to have exactly 2010 points of intersection?

Solution

Let $(a, b, c, d, e, ...)$ be the parallel line sets and numbers of lines parallel. (suppose there are 7 line, Parallel sets are (1,2,3) (4,5) (6) (7), then the code will be (3,2,1,1)) It is easy to see that the intersections are in form $\frac{a(100-a)+b(100-b)+c(100-c)...}{2} = 2010$ Where $a + b + c + \ldots = 100$

$100(a + b + c + \ldots) - a^2 + b^2 + c^2 \ldots = 4020$ $5980 = a^2 + b^2 + c^2 \ldots$

Then using trial and error, I obtained a set (77,4,2,2,2,2,2,2,2,2,1,1,1) so it is possible

□

Find the values of k such that the equations are equivalent. $kx^2-(2k-3)x+k+3=0$ $x^2-2(k-1)x+k+1=0$

Solution

Expanding each equation,

$$kx^2-2kx+3x+k+3=0$$
$$x^2-2kx+2x+k+1=0.$$

since they are both equal to 0, we can set them equal to each other to get (including some simplifying)

$$\begin{aligned} kx^2-2kx+3x+k+3&=x^2-2kx+2x+k+1\\ kx^2+x+2&=x^2\\ (k-1)x^2+x+2&=0. \end{aligned}$$

The solutions to this quadratic must be real, so using the quadratic formula, the roots are

$$\frac{-1\pm\sqrt{1^2-4\cdot 2(k-1)}}{2(k-1)}=\frac{-1\pm\sqrt{-8k+9}}{2k-2}.$$

We need the radicand to be positive, but at the same time, we can't have $k=1$ otherwise the denominator is undefined. In order for the radicand to be positive,

$$-8k+9\le 0\Rightarrow k\le\frac{9}{8}$$

so

$$\boxed{k\in(-\infty,1)\cup\left(1,\frac{9}{8}\right]}.$$

□

Solve for r, w, b, and g where $n=r+w+b+g$. $\frac{\binom{r}{4}}{\binom{n}{4}}=\frac{\binom{r}{3}\binom{w}{1}}{\binom{n}{4}}=\frac{\binom{r}{2}\binom{w}{1}\binom{b}{1}}{\binom{n}{4}}=\frac{\binom{r}{1}\binom{w}{1}\binom{b}{1}\binom{g}{1}}{\binom{n}{4}}$

Solution

First, you can get rid of the denominator.

Next, expand the expressions. You get:

$$\frac{1}{24}r(r-1)(r-2)(r-3)=\frac{1}{6}r(r-1)(r-2)w=\frac{1}{2}r(r-1)wb=rwbg$$

Getting rid of the fractions:

$$r(r-1)(r-2)(r-3)=4r(r-1)(r-2)w=12r(r-1)wb=24rwbg$$

From this, you can conclude the following:

$$\begin{aligned} r-3&=4w\\ r-2&=3b\\ r-1&=2g \end{aligned}$$

And therefore, $r=2g+1=3b+2=4w+3$. In other words, $r\equiv 1 \pmod 2 \equiv 2 \pmod 3 \equiv 3 \pmod 4$, and the smallest possible value for r that satisfies the above is 11.

Working backwards, $g = 5$, $b = 3$, and $w = 2$. So the sum is $11 + 5 + 3 + 2 = \boxed{21}$.

□ Let f,g: R >R be functions like that so **f(g(x))=g(f(x))=-x** for any x is element of R a) prove that f and g are odd functions b) Make an example of these two functions **f** isn't equal to **g**

Solution

a) : $g(f(g(x))) = g(u)$ where $u = f(g(x)) = -x$ and so $g(f(g(x))) = g(-x)$ $g(f(g(x))) = g(f(v)) = -v$ where $v = g(x)$ and so $g(f(g(x))) = -g(x)$ So $g(-x) = -g(x)$ and $g(x)$ is an odd function.

Same computation with $f(g(f(x)))$ shows that $f(x)$ is an odd function.

b) Choose $f(x) = 2x$ and $g(x) = -\frac{x}{2}$

□ If a+b+c=1, a,b,c>0, prove that
$\frac{ab+\sqrt{a^3c}+\sqrt{b^3c}}{a+b} + \frac{bc+\sqrt{b^3a}+\sqrt{c^3a}}{b+c} + \frac{ca+\sqrt{a^3b}+\sqrt{c^3b}}{c+a} \le \frac{3}{2}$

Solution

By AM-GM , $\sqrt{a^3c} \le \frac{a^2+ac}{2}$ and, $\sqrt{b^3c} \le \frac{b^2+bc}{2}$, therefore - $\sum_{cyclic} \frac{ab+\sqrt{a^3c}+\sqrt{b^3c}}{a+b} \le \sum_{cyclic} \frac{2ab+a^2+b^2+c(a+b)}{2(a+b)} = \sum_{cyclic} \frac{(a+b)(a+b+c)}{2(a+b)} = \sum_{cyclic} \frac{a+b+c}{2} = \frac{3}{2}$ Equality for $a = b = c = \frac{1}{3}$ $Q.E.D$

□ Solve for x, y such that $2x > y > x$,if $2(2x - y)^2 = (y - x)$

Solution

Let $z = y - x$, so $0 < z < x$ and $2(x - z)^2 = z$. Solving for z using the quadratic formula gives:

$$z = \frac{4x + 1 \pm \sqrt{8x + 1}}{4}$$

The positive sign gives $z > x$, so take the negative sign. For z to be an integer, $8x + 1 = (4k + 1)^2$ for some k. Solving for x gives $x = 2k^2 + k$ for some k, so $z = 2k^2$, so $\boxed{(x, y) = (2k^2 + k, 4k^2 + k)}$ for $k \in \mathbb{N}$

□ Find the sum
$\sum_{k=1}^{89} \tan^2 k$

Solution

Let's find a polynomial such that this 89 numbers are the roots of it, then the coefficients will give the sum. We have $(\cos(x)+i\cdot\sin(x))^n = \cos(nx)+i\cdot\sin(nx) \implies (1+i\cdot tan(x))^n = \frac{1}{\cos(x)^n}(\cos(nx) + i\cdot\sin(nx))$. Write $z := tan(x)$. Thus, $\sum_{k=0}^{n} \binom{n}{k} i^k z^k = \frac{1}{\cos(x)^n}(\cos(nx) + i\sin(nx))$. Now let $n = 180$ and let x having 'integer-valued degree', so $\sum_{k=0}^{180} \binom{180}{k} i^k z^k = \frac{1}{\cos(x)^n}(\cos(nx) + i\cdot\sin(nx)) = \frac{(-1)^x}{\cos(x)^n}$. Now look at the imaginary part, giving: $z\sum_{k=0}^{89} \binom{180}{2k+1}(-1)^k(z^2)^k = 0$. But this is the polynomial we wanted, since its roots are $tan(k^\circ)^2$ (we also counted $tan(0) = 0$, which can be neglected). So $\sum_{k=1}^{89} tan(k^\circ)^2 = \frac{\binom{180}{177}}{\binom{180}{179}} = \frac{15931}{3}$.

□

Find positive integers a, b, c, d such that $a + b + c + d - 3 = ab = cd$.

Solution

Without loss of generality, $1 \le a \le b \le c \le d$ so we have $a + b + c + d - 3 \le 4d - 3$. We also have $a + b + c + d - 3 = cd \le 4d - 3 \implies 3 \le (4 - c)d$. The product on the RHS must be positive and it follows that each factor must be positive because d must be a positive integer. Therefore, we have $1 \le c \le 3$. From here, we have 3 cases.

Case 1: $c = 1$ If $c = 1$, we must have $a = b = 1$ from our inequality chain. The equality chain becomes $d = 1 = d$ so the solution for this case is $a = b = c = d = 1$. Substituting values, we find that this solution works.

Case 2: $c = 2$ If $c = 2$, we have $a + b + d - 1 = 2d \implies a + b - 1 = d$. Note that $a + b \leq 4 \implies a + b - 1 = d \leq 3$. Now suppose that $d = 3$. Then we have $a + b = 4$ which is only satisfied by $a = b = 2$. Quickly checking, we find that this does not work. If $d = 2$, then we have $ab = 4$, which again is satisfied by $a = b = 2$, so there are no solutions for this case.

Case 3: $c = 3$ If $c = 3$, we have $a + b + d = 3d \implies a + b = 2d$. Note that $a + b \leq 6$ so that $d \leq 3$. using the equation $ab = cd$ and checking $d = 3$, we find that no a, b exist. Thus, there are no solutions for this case.

The only solution is $\boxed{(a, b, c, d) = (1, 1, 1, 1)}$.

□

The age of the father is 5.5 times as that of the second daughter. Mom got married at 20; at that time grandfather was 57. The first son was born when mom was 22. At present, the first daughter is 19; her age differs from the second son by 5 and from the second daughter by 9. The last year, age of the third son was half of the first son. The sum of the age of the second daughter and the third son equal the age of the second son. What is the the age of the first son?

Solution

Let the first son be x years old.

We know that the second daughter must be 10 years old and the third son's is $\frac{x+1}{2}$ years old.

Also, $10 + \frac{x+1}{2} = 14$ or 24 since the second son is 5 years older or younger than the first daughter.

If $\frac{x+1}{2} = 14, x = 27$ and if $\frac{x+1}{2} = 4, x = 7$. since the first son must be older than the second, then $\boxed{27}$.

□ Let ABC be an A-right triangle and let M be a point of $[BC]$. Denote $\left\{\begin{array}{c} E \in (AB)\ ;\quad \widehat{EMA} \equiv \widehat{EMB} \\ \\ F \in (AC)\ ;\quad \widehat{FMA} \equiv \widehat{FMC} \end{array}\right\|$.

Prove that $\left\{\begin{array}{c} \frac{c^2}{AM+MB} + \frac{b^2}{AM+MC} = a \\ \\ c \cdot AE + b \cdot AF = a \cdot AM \\ \\ c^2 \cdot MC^2 + b^2 \cdot MB^2 = a^2 \cdot AM^2 \end{array}\right\|$, where $BC = a\ ,\ AC = b\ ,\ AB = c$.

Solution

By applying Stewart's theoremand the Pytagorean theorem in triangle ABC, we obtain that $\frac{c^2}{AM+MB} + \frac{b^2}{AM+MC} = \frac{c^2(AM+MC)+b^2(AM+MB)}{(AM+MB)(AM+MC)} = \frac{AM(b^2+c^2)+c^2MC+b^2MB}{AM^2+MB\cdot MC+AM(MB+MC)} == \frac{AM\cdot a^2+a\cdot AM^2+a\cdot MB\cdot MC}{AM^2+MB\cdot MC+a\cdot AM} = \frac{a(AM^2+MB\cdot MC+a\cdot A}{AM^2+MB\cdot MC+a\cdot A}$ a. since ME and MF are the bisectors of the angles $\angle AMB\ \angle AMC$, we have that $c \cdot AE + b \cdot AF = c \cdot \frac{AM\cdot AB}{AM+MB} + b \cdot \frac{AM\cdot AC}{AM+MC} = AM\left(\frac{c^2}{AM+MB} + \frac{b^2}{AM+MC}\right) = a \cdot AM$. Again, with Stewart's theorem, $a^2 \cdot AM^2 = a(c^2MC + b^2MB - a \cdot MB \cdot MC) = (MB + MC)(c^2MC + b^2MB) - a^2 \cdot MB \cdot MC = = c^2 \cdot MC^2 + b^2 \cdot MB^2 + MB \cdot MC(b^2 + c^2 - a^2) = c^2 \cdot MC + b^2 \cdot MB$.

□ Let ABC be an A-isosceles triangle with the circumcentre O and the incentre I . Denote $D \in AC$ for which $DO \perp CI$. Prove that $ID \parallel AB$.

Solution

Denote the midpoint M of $[BC]$ and $K \in CI \cap DO$. Thus, the quadrilateral $OKMC$ is inscribed in the circle with the diameter $[OC] \implies \widehat{DOA} \equiv \widehat{MOK} \equiv \widehat{MCK} \equiv \widehat{DCI} \implies \widehat{DOA} \equiv \widehat{DCI} \implies DOIC$ is cyclically $\implies \widehat{DIA} \equiv \widehat{DIO} \equiv \widehat{DCO} \equiv \widehat{CAM} \equiv \widehat{MAB} \implies ID \parallel AB$.

□ Prove that $\sum_{r=0}^{m}(-1)^r\binom{n}{r} = (-1)^m\binom{n-1}{m}$ where $m < n$.

Solution

$\sum_{r=0}^{m}(-1)^r\binom{n}{r} = \binom{n}{0} + \sum_{r=1}^{m}(-1)^r\binom{n}{r} = \binom{n-1}{0} + \sum_{r=1}^{m}(-1)^r\left[\binom{n-1}{r} + \binom{n-1}{r-1}\right] =$

$\sum_{r=0}^{m}(-1)^r\binom{n-1}{r} - \sum_{r=1}^{m}(-1)^{r-1}\binom{n-1}{r-1} = \sum_{r=0}^{m}(-1)^r\binom{n-1}{r} - \sum_{r=0}^{m-1}(-1)^r\binom{n-1}{r} = (-1)^m\binom{n-1}{m}$.
Remark. I used the **Pascal**'s relation : $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ and $\sum_{r=s}^{m} f(r) = \sum_{r=s-p}^{m-p} f(r+p)$.

□ A polynomial p has remainder 7 when it is divided by $X+2$ and remainder $X+3$ when it is divided by X^2+2 . Determine the remainder when p is divided by $(X+2)(X^2+2)$.

Solution

Proof 1. The polynomial p has remainder $X+3$ when it is divided by $X^2+2 \iff$ exist a polynomial $q \in \mathbb{C}[X]$ so that

$\boxed{p = \left(X^2+2\right)q + X + 3}$ $(*)$. The polynomial p has remainder 7 when it is divided by $X+2 \iff$ $p(-2) = 7 \overset{(*)}{\iff}$

$6q(-2) + 1 = 7 \iff q(-2) = 1 \iff$ exists $s \in \mathbb{C}[X]$ so that $\boxed{q = (X+2)s + 1}$ (1) . Thus, from the relatio $(*)$

obtain that $p = (x^2+2)\left[(X+2)s+1\right] + X + 3 \iff \boxed{p = (X+2)\left(X^2+2\right)s + X^2 + X + 5}$.

In conclusion, from the oneness of the remainder obtain that the required remainder is $\boxed{r = X^2 + X + 5}$.

Proof 2. Exist uniquelly $\{a,b,c\} \subset \mathbb{C}$ and $q \in \mathbb{C}[X]$ so that $p = (X+2)\left(X^2+2\right)q + aX^2 + bX + c$.

From the hypothesis obtain that $\left\{\begin{array}{ccc} p(-2) & = & 7 \\ p\left(X^2 := -2\right) & \equiv & X+3 \end{array}\right| \iff \left\{\begin{array}{ccc} 4a-2b+c & = & 7 \\ bX + (c-2a) & \equiv & X+3 \end{array}\right| \iff$

$\left\{\begin{array}{ccc} 4a-2b+c & = & 7 \\ b & = & 1 \\ -2a+c & = & 3 \end{array}\right| \iff \left\{\begin{array}{ccc} a & = & 1 \\ b & = & 1 \\ c & = & 5 \end{array}\right| \implies$ the required remainder is $\boxed{X^2 + X + 5}$.

An easy extension. A polynomial p has remainder k when it is divided by $X+\alpha$ and remainder $mX+n$

when it is divided by $\beta X^2 + \gamma$. Determine the remainder when p is divided by $(X+\alpha)\left(\beta X^2+\gamma\right)$.

Proof 2. Exist uniquelly $\{a,b,c\} \subset \mathbb{C}$ and $q \in \mathbb{C}[X]$ so that $p = (X+\alpha)\left(\beta X^2+\gamma\right)q + aX^2 + bX + c$.

From the hypothesis obtain that $\left\{\begin{array}{ccc} p(-\alpha) & = & k \\ p\left(X^2 := -\frac{\gamma}{\beta}\right) & \equiv & mX+n \end{array}\right| \iff \left\{\begin{array}{ccc} \alpha^2 a - \alpha b + c & = & k \\ bX + \left(c - \frac{\gamma a}{\beta}\right) & \equiv & mX+n \end{array}\right.$

$\left\{\begin{array}{ccc} \alpha^2 a - \alpha b + c & = & k \\ b & = & m \\ c - \frac{\gamma a}{\beta} & = & n \end{array}\right| \iff \left\{\begin{array}{ccc} a & = & \frac{\beta(\alpha m + k - n)}{\alpha^2\beta + \gamma} \\ b & = & m \\ c & = & \frac{\gamma(\alpha m + k) + \alpha^2\beta n}{\alpha^2\beta + \gamma} \end{array}\right| \implies$ the required remainder is

$\boxed{\frac{\beta(\alpha m + k - n)}{\alpha^2\beta + \gamma}X^2 + mX + \frac{\gamma(\alpha m + k) + \alpha^2\beta n}{\alpha^2\beta + \gamma}}$.

□ Solve the inequation $|x+1| - |2x-1| > -1$.

Solution

$|x+1| - |2x-1| > -1 \iff |2x-1| < |x+1| + 1$. Thus, appear two cases: **Case 1.** $x \ge \frac{1}{2} \implies$

$2x-1<x+2 \implies x<3 \implies x\in S_1=\left[\frac{1}{2},3\right)$.

Case 2. $x<\frac{1}{2} \implies 1-2x<|x+1|+1 \implies -2x<|x+1|$. Appear two subcases:

......**Case 2.1** $x\geq 0 \implies x\in S_{21}=\left[0,\frac{1}{2}\right)$.

......**Case 2.2** $x<0 \implies 4x^2<(x+1)^2 \implies 3x^2-2x-1<0 \implies x\in S_{22}=\left(-\frac{1}{3},0\right)$.

Therefore, $S_2=S_{21}\cup S_{22}=\left[0,\frac{1}{2}\right)\cup\left(-\frac{1}{3},0\right) \implies S_2=\left(-\frac{1}{3},\frac{1}{2}\right)$. In conclusion, the solution of the proposed inequation is $S=S_1\cup S_2=\left[\frac{1}{2},3\right)\cup\left(-\frac{1}{3},\frac{1}{2}\right) \implies S=\left(-\frac{1}{3},3\right)$, i.e. $-\frac{1}{3}<x<3$.

□ 3. Let P be an interior point of an equilateral triangle ABC such that $AP^2=BP^2+CP^2$. Prove that $\angle BPC=150^o$.

Solution

Let R be the reflection of the point P w.r.t. the midpoint M of the side $[BC]$.

$PA^2=PB^2+PC^2$, $4AM^2=3a^2$, $4AM^2=2(AR^2+PA^2)-4MP^2$, $4MP^2=2(PB^2+PC^2)-a^2$;

$2AR^2=4AM^2-2PA^2+4MP^2=3a^2-2(PB^2+PC^2)+[2(PB^2+PC^2)-a^2]=2a^2 \Longrightarrow AR=a$.

Therefore $R\in C(A,a)$, the quadrilateral $BPCR$ is a parallelogram and $A=60°$.

Thus, $m(\widehat{BPC})=m(\widehat{BRC})=\frac{1}{2}(360°-A)=150°$.

□□□□□ bdt □

□ Here is a inequality stronger than the well-known inequality $\sum$
$cosA\leq\frac{3}{2}$ in any ABC :

$\boxed{12\cdot(cosA+cosB+cosC)\leq 15+cos(A-B)+cos(B-C)+cos(C-A)\leq 18}$.

Solution

$0\leq 2\cdot\sum\left(2sin\frac{A}{2}-cos\frac{B-C}{2}\right)^2=$

$4\sum 2$
$sin^2\frac{A}{2}+\sum 2$
$cos^2\frac{B-C}{2}-4\sum 2$
$cos\frac{B+C}{2}$
$cos\frac{B-C}{2}=$

$4\left(3-\sum cosA\right)+3+\sum$
$cos(B-C)-4\sum($
$cosB+$
$cosC)=$

$15+\sum$
$cos(B-C)-12\sum$
$cosA$. In conclusion,

$12\sum$
$cosA\leq 15+\sum$
$cos(B-C)$.

Prove easily that we'll have equality iff $A=B=C$. Another way: We have: $\cos(A-B)+\cos(B-C)+\cos(C-A)=\sum\cos A\cos B+\sum\sin A\sin B=\frac{p^2+r^2-4R^2}{4R^2}+\frac{p^2+r^2+4Rr}{4R^2}=\frac{p^2+r^2+2Rr-2R^2}{2R^2}$ And: $\sum\cos A=\frac{R+r}{R}$ The inequality is equivalent to: $\frac{p^2+r^2+2Rr-2R^2}{2R^2}+15\geq 12.\frac{R+r}{R}\Leftrightarrow 28R^2+p^2+r^2+2Rr\geq 24R(R+r)\Leftrightarrow 4R^2+p^2+r^2\geq 22Rr$ By Gerretsen inequality, we have: $p^2\geq r(16R-5r)$ So we need to prove that: $4R^2+r(16R-5r)+r^2\geq 22Rr\Leftrightarrow 4R^2\geq 4r^2+6Rr\Leftrightarrow(R-2r)(2R+r)\geq 0$ Which is clearly true because $R\geq 2r$ so we are done!

□ Let ABC be a triangle and let $D\in(BC)$ be a point for which $\widehat{BAD}\equiv\widehat{CAD}$. Then $\boxed{AD^2=AB\cdot AC-DB\cdot DC}$.

Solution

Proof 1. Denote the second intersection E between the line AD and the circumcircle w of the triangle ABC . From the relation $p_w(D) = DA \cdot DE = DB \cdot DC$- the power of the point D w.r.t. the circle w and $\triangle ABE \sim \triangle ADC$ obtain : $\frac{AB}{AD} = \frac{AE}{AC} \implies AE \cdot (AD + DE) = AB \cdot AC \implies AD^2 = AB \cdot AC - DB \cdot DC$.

Proof 2. From the bisector theorem $\frac{DB}{c} = \frac{DC}{b} = \frac{a}{b+c}$ and the Stewart's theorem $a \cdot AD^2 + a \cdot DB \cdot DC = c^2 \cdot DC + b^2 \cdot DB$ obtain $a \cdot (AD^2 + DB \cdot DC) = c^2 \cdot \frac{ab}{b+c} + b^2 \cdot \frac{ac}{b+c}$, i.e. $a \cdot (AD^2 + DB \cdot DC) = \frac{abc(b+c)}{b+c} \implies AD^2 = bc - DB \cdot DC$.

Remark. If the point $D_1 \in BC$ so that the ray $[AD_1$ is the exterior bisector of the angle $\angle BAC$, then $\boxed{AD_1^2 = D_1B \cdot D_1C - AB \cdot AC}$.

□ Circles with centers O and O' are disjoint. A tangent from O to the second circle intersects the first in A and B . A tangent from O' to the first circle intersects the second circle in A' and B' such that A and A' lie on the same side of OO' .Prove that $AA'B'B$ is a trapezoid .

Solution

Denote : the circles $w = C(O)$, $w' = C(O')$; the tangent points $T \in w$, $T' \in w'$ so that the line OO' separates these points T, T'; the intersection $S \in \overline{BOAT'} \cap \overline{A'O'B'T}$. Therefore, the quadrilateral $OTO'T'$ is cyclically , i.e. $\widehat{AOT} \equiv \widehat{A'O'T'} \implies$ the isosceles triangles AOT, $A'O'T'$ are similarly $\implies \widehat{OAT} \equiv \widehat{T'A'T}$, $\widehat{STB} \equiv \widehat{BT'B'} \implies$ the quadrilaterals $ATA'T'$, $BTB'T'$ are cyclically $\implies SA \cdot ST' = ST \cdot SA'$, $SB \cdot ST' = ST \cdot SB' \implies \frac{SA}{SB} = \frac{SA'}{SB'} \implies \boxed{AA' \parallel BB'}$.

Remarks. Prove easily that : $BT \parallel A'T'$, $AT \parallel B'T'$, i.e. and the quadrilaterals $BTA'T'$, $ATB'T'$ are trapezoids ; if denote the intersections $U \in AT \cap A'T'$ and $V \in BT \cap B'T'$ then the quadrilateral $TUT'V$ is rectangle and the line UV is radical axis between the circles w, w'.

□ $\{x, y, z\} \subset R$, $x \ge y$, $x \ge z$; $x + y + z = 1$, $xy + yz + zx = \frac{1}{4} \implies \sqrt{x} = \sqrt{y} + \sqrt{z}$.

Solution

$p^4 + q^4 + r^4 - 2p^2q^2 - 2q^2r^2 - 2p^2r^2 = (p + q + r)(p + q - r)(p - q + r)(p - q - r)$

Hence

$$x^2 + y^2 + z^2 - 2xy - 2yz - 2xz = (\sqrt{x} + \sqrt{y} + \sqrt{z})(\sqrt{x} + \sqrt{y} - \sqrt{z})(\sqrt{x} - \sqrt{y} + \sqrt{z})(\sqrt{x} - \sqrt{y} - \sqrt{z})$$

But the LHS is equal to $(x + y + z)^2 - 4(xy + yz + xz) = 0$, hence either $\sqrt{z} = \sqrt{x} + \sqrt{y}$ or $\sqrt{y} = \sqrt{x} + \sqrt{z}$ or $\sqrt{x} = \sqrt{y} + \sqrt{z}$. However, because of $x \geqslant y \wedge x \geqslant z$, we have the third option. QED

PROVE that for any $n \in N^*$ and $p \in N$ there is the inequality $\boxed{A_{2(n+p)}^{2n} \le 2^{2n-1} \cdot \dfrac{n + 2p}{n + p} \cdot \left[A_{n+p}^n\right]^2}$.

Particular case. $p := n \implies \boxed{A_{4n}^{2n} \le 3 \cdot 2^{2(n-1)} \cdot (A_{2n}^n)^2}$.

Notation. $A_m^n = m(m-1)(m-2)\ldots(m-n+2)(m-n+1) = \frac{m!}{(m-n)!}$, for $m, n \in N$, $0 \le n \le m$.

□ Given are $\{\ m\ ,\ n\ ,\ p\ \} \subset N^*$ so that $m > np$ and $k > 0$. Ascertain **without derivatives** the ratio $\frac{x}{y}$ so that the expression $x^p \cdot (x + y)$ is minimum for the all positive real numbers x , y for which $x^m \cdot y^n = k$.

Solution

We want to use an inequality to reduce $x^p(x + y)$ to a constant with an equality condition. Basically, some multiple/power of $x^m y^n$. We can use AM-GM to get

$(m + n)(x + y) = \sum_{i=1}^{m-np} \frac{m+n}{m-np} x + \sum_{i=1}^{n(p+1)} \frac{m+n}{n(p+1)} y \ge C x^{\frac{m-np}{m+n}} y^{\frac{n(p+1)}{m+n}}$

$x + y \ge \frac{C}{m+n} x^{\frac{m-np}{m+n}} y^{\frac{n(p+1)}{m+n}}$

for some constant C. Then our expression is

$x^p(x + y) \ge \frac{C}{m+n} x^{\frac{m(p+1)}{m+n}} y^{\frac{n(p+1)}{m+n}} = \frac{C}{m+n} k^{\frac{p+1}{m+n}}$,

which is constant. Equality holds when
$\frac{m+n}{m-np}x = \frac{m+n}{n(p+1)}y \Rightarrow \frac{x}{m-np} = \frac{y}{n(p+1)}$
as desired.

□ Prove that the polynomial

$$x^{9999} + x^{8888} + x^{7777} + \ldots + x^{1111} + 1$$

is divisible by

$$x^9 + x^8 + x^7 + \ldots + x + 1$$

Solution

Denote the first polynomial by $P(x)$ and the second one by $Q(x)$. It's obvious that $P(x) = Q(x^{1111})$. The roots of $Q(x)$ are $x_k =$
$cos\frac{2k\pi}{10} + i$
$sin\frac{2k\pi}{10}, k = 1, 2, \ldots, 9$. Since

$$x_k^{1111} = cos\frac{2222k\pi}{10} + i\, sin\frac{2222k\pi}{10} = cos(222k\pi + \frac{2k\pi}{10}) + i\, sin(222k\pi + \frac{2k\pi}{10}) = x_k$$

, it follows that $P(x^k) = Q(x_k^{1111}) = Q(x_k) = 0$, i.e. all the roots of $Q(x)$ are also the roots of $P(x)$. Therefore, $P(x)$ is divisible by $Q(x)$.

□ Let a and b be integers. Prove that a and b are relatively prime if and only if there exists x and y such that $ax + by = 1$.

Solution

Let $S = \{s \in \mathbb{N} | \exists x, y \in \mathbb{Z} : s = ax + by\}$. It is non-empty since $a^2 + b^2 \in S$, so consider its minimal element d.

Consider the remainders r, s when a, b are divided by d. We have
$a = dp + r, 0 \leq r < d$ $b = dt + s, 0 \leq s < d$

So $r = a - dp, s = b - dt$ are also linear combinations of a, b. But we assumed d was minimal, so $r, s \notin S$. It follows that $r, s \notin \mathbb{N}$, so they equal 0.

Then $d|a, d|b, gcd(a, b) = 1 \Rightarrow d = 1$. QED. Another way First, let's assume that a and b aren't coprime. Then there exists $d > 1$ such that $a = da_1$ and $b = db_1$. But then $1 = ax+by = d(a_1x+b_1y)$, which means that $d|1$, and that's impossible.

Now, let's assume that a and b are coprime. We'll prove that the numbers $0, b, 2b, \ldots, (a-1)b$ give pairwise distinct remainders modulo a. Assume the opposite, that there are distinct $k, l \in \{0, 1, \ldots, a-1\}$ such that $kb \equiv lb \pmod{a}$. Then $a|(k-l)b$, but as no prime factor of a is a prime factor of b, it means that $a|k-l$. Since $|k-l| < a$, that implies $k = l$, contrary to the presumption.

Therefore, a different numbers give a different remainders modulo a, so one of those remainders must be 1. Hence there exists $k_0 < a$ such that $k_0 b \equiv 1 \pmod{a}$, which means that there exists $l_0 \in \mathbb{Z}$ such that $k_0 b - 1 = l_0 a$, so we can take $x = -l_0$ and $y = k_0$. QED.

$\square$ Find $\sum_{k=0}^{49}(-1)^k\binom{99}{2k}$ where $\binom{n}{j}=\frac{n!}{j!(n-j)!}$.

Solution

The easiest way is to rewrite the sum as

$S=\sum_{k=0}^{49}(-1)^k\binom{99}{2k}=\sum_{k=0}^{49}i^{2k}\binom{99}{2k}=\sum_{k=0}^{49}\binom{99}{2k}i^{2k}1^{99-2k}$, and from there it's obvious that the sum is actually the real part of $(1+i)^{99}$, and that's easy to calculate:

$$\begin{aligned} S &= \Re\{(1+i)^{99}\} \\ &= \Re\{(\sqrt{2}e^{i\pi/4})^{99}\} \\ &= 2^{49}\sqrt{2}\cdot\Re\{e^{i99\pi/4}\} \\ &= 2^{49}\sqrt{2}\cdot\Re\{e^{i3\pi/4}\} \\ &= 2^{49}\sqrt{2}\cdot\Re\{-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\} \\ &= -2^{49} \end{aligned}$$

$\square$ Let $A=\{x\in\mathbb{R}|\ x^2-(1-m)x-2m-2=0\}$ and $B=\{x\in\mathbb{R}|(m-1)x^2+mx+1=0\}$. Find all m such that $M=A\cup B$ has 3 elements.

Solution

We have $\Delta_A=(1-m)^2+4(2m+2)=1-2m+m^2+8m+8=m^2+6m+9=(m+3)^2$, and $\Delta_B=m^2-4(m-1)=m^2-4m+4=(m-2)^2$. Therefore, for $m\neq 1$ we have

$A=\{\frac{1-m+m+3}{2},\frac{1-m-m-3}{2}\}=\{2,-m-1\}$ and $B=\{\frac{-m+m-2}{2(m-1)},\frac{-m-m+2}{2(m-1)}\}=\{\frac{1}{1-m},-1\}$

Now we have the following possibilities:

1. $2=\frac{1}{1-m}\iff m=\frac{1}{2}$ 2. $-m-1=\frac{1}{1-m}\iff m^2-1=1\iff m=\pm\sqrt{2}$ 3. $-m-1=-1\iff m=0$ 4. $2=-m-1\iff m=-3$ 5. $\frac{1}{1-m}=-1\iff m=2$

(Options 1, 2 and 3 cover for equal elements between the sets, and 4 and 5 for double roots.)

For all those values we manually check that $A\cup B$ indeed has 3 elements.

It remains to be seen what happens with $m=1$. Then $A=\{x\in\mathbb{R}|x^2-4=0\}=\{2,-2\}$ and $B=\{x\in\mathbb{R}|x+1=0\}=\{-1\}$. We see that in this case $A\cup B$ clearly has 3 elements too.

Therefore, $\mathcal{M}=\{-3,-\sqrt{2},0,\frac{1}{2},1,\sqrt{2},2\}$

$\square$ How many 3-element subsets of the set $\{3,3^2,3^3,\dots,3^{1000}\}$ consist of three numbers which form a geometric sequence?

Solution

This can be rephrased as: How many three-element subsets of $\{1,2,\dots,1000\}$ form an arithmetic sequence.

Let the elements be a,b,c where $a<b<c$. We must have $a+c=2b$, hence a and c must be of the same parity.

1) Among 500 even numbers in the given set, for every even c we have $\frac{c}{2}-1$ even numbers smaller than it. Choosing one of them as a will automatically determine b. The number of such choices for all possible c is $\sum_{k=1}^{500}(k-1)=\sum_{k=0}^{499}k=\frac{499\cdot 500}{2}$

2) Among 500 odd numbers in the given set, for every odd c we have $\frac{c-1}{2}$ odd numbers smaller than it. Similarly as in the previous case, we find that the number of choices is $\sum_{k=1}^{500}(k-1)=\sum_{k=0}^{499}k=\frac{499\cdot 500}{2}$

Therefore, the total number of choices is $499\cdot 500=249500$

$\square$ Solve in Z the equation $3^x=x^2+3x+1$.

Solution

For $x < 0$ LHS isn't integer.

For $x \geqslant 3$ we can use induction to prove $3^x > x^2 + 3x + 1$:

1° For $x = 3$ we have $3^3 > 3^2 + 3 \cdot 3 + 1$ which is true.

2° Inductive step:

$$\begin{aligned} x \geqslant 3 &\implies (x+1)^2 - 2 > 0 \\ &\implies 2x^2 + 4x - 2 > 0 \\ &\implies 3x^2 + 9x + 3 > x^2 + 5x + 5 \\ &\implies 3(x^2 + 3x + 1) > (x+1)^2 + 3(x+1) + 1 \end{aligned}$$

Hence $3^x > x^2 + 3x + 1 \implies 3^{x+1} > 3(x^2 + 3x + 1) > (x+1)^2 + 3(x+1) + 1$

Therefore, we have to check $x = 0, 1, 2$ and we find that the unique solution is $x = 0$. $\square$ Find $n \in \mathbb{N}$ such that $(x+k)^n - x^n - k^n = 0$,$k \in \Re, k \neq 0, x = ke^{\frac{2\pi i}{3}}$

Solution

The equation is equivalent to

$(1 + e^{i2\pi/3})^n - e^{i2n\pi/3} - 1 = 0$, or

$e^{in\pi/3} - e^{i2n\pi/3} - 1 = 0$

This produces the system

$$\begin{cases} cos\frac{n\pi}{3} - \\ cos\frac{2n\pi}{3} - 1 = 0 \\ \\ sin\frac{n\pi}{3} - \\ sin\frac{2n\pi}{3} = 0 \end{cases}$$

The second equation transforms into

$sin\frac{n\pi}{3}(1 - 2$
$cos\frac{n\pi}{3}) = 0$, which gives $\frac{n\pi}{3} = k\pi \vee \frac{n\pi}{3} = \pm\frac{\pi}{3} + 2k\pi, k \in \mathbb{Z}$. This in turn gives $n = 3k \vee n = 6k \pm 1$

Substituting these values into the first equation, we see that it's satisfied only by $n = 6k \pm 1$.

Hence, the desired set is $n \in \{6k + 1, 6k + 5 | k \in \mathbb{Z}_0^+\}$

$\square$ Each side of a rhombus has length 12 and one of its angles is 150°. External squares are drawn on each of the four sides of the rhombus. A point is marked at the center of each square and they are connected to form a quadrilateral. Find the area of this quadrilateral.

Solution

Thiếu hình vẽ $\angle O_1AO_4 = 30° + 2 \cdot 45° = 120°$ and $AO_1 = AO_4$ $\angle O_1BO_2 = 360° - 2 \cdot 45° - 150° = 120°$ and $BO_1 = BO_2$

Hence triangles O_1AO_4 and O_1BO_2 are isosceles and congruent because $AO_1 = BO_1$. Hence

1° $O_1O_4 = O_1O_2$ 2° $\angle AO_1O_4 = \angle BO_1O_2 \implies \implies \angle O_4O_1O_2 = \angle AO_1B - \angle AO_1O_4 + \angle BO_1O_2 = \angle AO_1B = 90°$

Hence $O_1O_2O_3O_4$ is a square and $AO_1 = 6\sqrt{2}$

Applying Cosine Law to $\triangle AO_1O_4$ we get $O_1O_4^2 = (6\sqrt{2})^2 + (6\sqrt{2})^2 - 2 \cdot 6\sqrt{2} \cdot 6\sqrt{2} \cdot cos120° = 216$, and in the same time that's the area of the square.

□ Given three real numbers p,q, and r where $0 < p, q, r < 1$. Show that $pq + qr + rp - 2pqr < 1$

Solution

For positive reals u, v, t it holds

$uvt + uv + ut + vt > 0$

Adding $u + v + t + 1 + 2$ to the both sides and simplifying we get

$(u + 1)(v + 1)(t + 1) + 2 > u + v + t + 3$

Substitute $a = u + 1, b = v + 1, c = t + 1$ where $a, b, c > 1$:

$abc + 2 > a - 1 + b - 1 + c - 1 + 3 \iff abc + 2 > a + b + c \iff a + b + c - 2 < abc$

Substitute $p = \frac{1}{a}, q = \frac{1}{b}, r = \frac{1}{c}$ where $p, q, r < 1$ and the result follows.

□ Prove that $R = \frac{1}{4} \cdot \frac{\sqrt{(AB+CD)(AD+BC)(AC+BD)}}{A}$

for cyclic quad $ABCD$, where A is the area of $ABCD$.

Solution

We have

$$\begin{aligned} [ABC] &= \frac{abe}{4R} \\ ADC &= \frac{cde}{4R} \\ \Rightarrow [ABCD] &= \frac{e(ab + cd)}{4R} \\ ABD &= \frac{adf}{4R} \\ CBD &= \frac{bcf}{4R} \\ \Rightarrow [ABCD] &= \frac{f(ad + bc)}{4R} \end{aligned}$$

(had to leave out brackets on some of those...the formatting get's screwed up...)

So then

$$[ABCD]^2 = \frac{ef(ab + cd)(ad + bc)}{(4R)^2}.$$

But Ptolemy's Thorem gives us $ef = ac + bd$ which leads us to conclude that

$$[ABCD] = \frac{\sqrt{(ab + cd)(ad + bc)(ac + bd)}}{4R}.$$

□[/img]

□ The figure shows a rectangle divided into 9 squares. The squares have integral sides and adjacent sides of the rectangle are coprime. Find the perimeter of the rectangle.

Solution

Thiếu hình vẽ On the attached picture, the letters denote the sides of the corresponding squares.

We have

$c = a + b$ $d = a + c = 2a + b$ $e = c + d = 3a + 2b$ $f = d + e = 5a + 3b$ $g = a + d + f = 8a + 4b$ $h = g + a - b = 9a + 3b$ $i = b + c + e = 4a + 4b$

On the other hand, $i = h - b = 9a + 2b$, therefore $9a + 2b = 4a + 4b \implies 5a = 2b \implies b = \frac{5a}{2}$

One side of the rectangle is $s_1 = g+h = 17a+7b = \frac{69a}{2}$, and the other is $s_2 = f+g = 13a+7b = \frac{61a}{2}$. Since s_1 and s_2 are integer and coprime, a must be 2, and the perimeter is $P = 2(s_1 + s_2) = 260$ □

Find the smallest positive integer whose cube ends in 888. (do this without a calculator or computer.)

Solution

We have to find x such that $x^3 = 1000n + 888$ for some $n \in \mathbb{N}$

Obviously x is even, hence we can put $x = 2y$, which gives

$125n + 111 = y^3 \implies 125n + 110 = y^3 - 1$

From this $y^3 - 1 \equiv 0 \pmod 5$. Checking the cubes of the residues modulo 5, we see that only 1 satisfies the condition. Hence $y = 5a + 1$. Substituting we get

$125n + 110 = 125a^3 + 75a^2 + 15a \implies 25n + 22 = 25a^3 + 15a^2 + 3a \implies 25n + 20 = 25a^3 + 15a^2 + 3a - 2$

Now $3a - 2 \equiv 0 \pmod 5$, hence we can write $3a - 2 = 5b$. From there $a = \frac{5b+2}{3} = b + \frac{2(b+1)}{3}$. This means that $b + 1 = 3c$, which gives $a = 3c - 1 + 2c = 5c - 1$. Substituting a we get

$25n + 20 = 25(125c^3 - 75c^2 + 15c - 1) + 15(25c^2 - 10c + 1) + 15c - 3 - 2 \implies 25n + 25 = 25(125c^3 - 75c^2 + 15c - 1 + 15c^2 - 6c) + 15 + 15c \implies n + 1 = 125c^3 - 60c^2 + 9c - 1 + \frac{3(c+1)}{5} \qquad (*)$

From there it's clear that $c + 1 \equiv 0 \pmod 5$, hence $c = 5k - 1$. Now $a = 5c - 1 = 25k - 6$, and $x = 2y = 10a + 2 = 250k - 58$. Obviously, the smallest x is obtained for $k = 1$, which gives $x = 192$.

From $k = 1$ we get $c = 4$. Substituting that into $(*)$ we get $n = 7077$

Therefore, the desired number is 192 and $192^3 = 7077888$

Note: All numbers such that their cube ends in 888 are given by $x = 250k - 58, k \in \mathbb{N}$

□ Find real numbers a, b such that for every $x, y \in \mathbb{R}$ we have $|ax + by| + |ay + bx| = |x| + |y|$.

Solution

Putting $x = 1, y = 0$ and $x = b, y = -a$ we get

$|a| + |b| = 1 \ |a^2 - b^2| = |a| + |b| = 1$

The second equation can be rewritten

$||a| - |b|| \cdot (|a| + |b|) = 1 \implies |a| - |b| = \pm 1$

From $|a| + |b| = 1 \wedge |a| - |b| = \pm 1$ we find all the solutions: $(\pm 1, 0), (0, \pm 1)$

□

If w and z are complex numbers, prove that:

$2|w||z||w - z| \geq (|w| + |z|) \left| w|z| - z|w| \right|$

Solution

If we write $w = We^{i\alpha}$ and $z = Ze^{i\beta}$ where $W, Z \geqslant 0$, then the given inequality simplifies to

$2|We^{i\alpha} - Ze^{i\beta}| \geqslant (W + Z)|e^{i\alpha} - e^{i\beta}|$

From there we have a chain of equivalent inequalities:

$2(W^2 + Z^2 - 2WZ$

$cos(\alpha - \beta)) \geqslant W^2 + Z^2 + 2WZ - (W^2 + Z^2 + 2WZ)$

$cos(\alpha - \beta)$

$W^2 + Z^2 - 2WZ$

$cos(\alpha - \beta) \geqslant 2WZ - W^2$

$cos(\alpha - \beta) - Z^2$

$cos(\alpha - \beta)$

$W^2 + Z^2 - 2WZ \geqslant -$

$cos(\alpha - \beta)(W^2 + Z^2 - 2WZ)$

$cos(\alpha - \beta) \geqslant -1$

and the last line is obviously true.

□ In quadrilateral $ABCD$, $\angle A = \angle C = 90°$, $AB + AD = 7$ and $BC - CD = 3$. Find the area of quadrilateral $ABCD$.

Solution

Denote $x = AB$ and $y = BC$. Then $AD = 7 - x$ and $CD = y - 3$

From Pythagoras we have

$$\begin{aligned} x^2 + (7-x)^2 = y^2 + (y-3)^2 \quad &\Longrightarrow \quad 2x^2 - 2y^2 = 14x - 6y - 40 \\ &\Longrightarrow \quad x^2 - y^2 = 7x - 3y - 20 \qquad (*) \end{aligned}$$

From the formula for the area of the right triangle, we have that the quadrilateral area is

$$\begin{aligned} A &= \frac{x(7-x)}{2} + \frac{y(y-3)}{2} \\ &= \frac{7x - 3y - x^2 + y^2}{2} \\ &= \frac{7x - 3y - (x^2 - y^2)}{2} \end{aligned}$$

using $(*)$ we get $A = \frac{7x-3y-(7x-3y-20)}{2} = 10$

$\square$ Let $f(1) = 1$ and for all natural numbers n, $f(1)+f(2)+\ldots+f(n) = n^2 f(n)$. What is $f(2006)$?

Solution

The equation is equivalent to

$f(1) + \cdots + f(n-1) = (n^2-1)f(n) \iff f(n) = \frac{1}{n^2-1}(f(1) + \cdots + f(n-1))$

Calculating the first few terms, we get $f(2) = \frac{1}{3}, f(3) = \frac{1}{6}, f(4) = \frac{1}{10}, f(5) = \frac{1}{15}$. We note that the denominators are the triangular numbers, hence we assume $f(n) = \frac{2}{n(n+1)}$

For $n = 1$ we have $f(1) = \frac{2}{1 \cdot 2} = 1$

Inductive step:

$$\begin{aligned} f(n+1) &= \frac{1}{(n+1)^2 - 1}\left(\frac{2}{1 \cdot 2} + \frac{2}{2 \cdot 3} + \cdots + \frac{2}{n(n+1)}\right) \\ &= \frac{2}{(n+1)^2 - 1}\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{2}{(n+1)^2 - 1}\left(1 - \frac{1}{n+1}\right) \\ &= \frac{2}{n(n+2)} \cdot \frac{n}{n+1} \\ &= \frac{2}{(n+1)(n+2)} \end{aligned}$$

Hence $f(2006) = \frac{1}{1003 \cdot 2007}$ $\square$

Find $x \in \mathbb{R}$ such that $\displaystyle\sum_{i=0}^{2n}(-|i-n| + n + 1)x^i = 0$

Solution

The polynomial is

$$P(x) = x^{2n} + 2x^{2n-1} + 3x^{2n-2} + \cdots + nx^{n+1} + (n+1)x^n + nx^{n-1} + \cdots + 3x^2 + 2x + 1$$

.

We can write it as the sum of the following polynomials:

$$\begin{aligned}
Q_1(x) &= x^{2n} + x^{2n-1} + \cdots + x + 1 \\
Q_2(x) &= x^{2n-1} + x^{2n-2} + \cdots + x^2 + x \\
Q_3(x) &= x^{2n-2} + x^{2n-3} + \cdots + x^3 + x^2 \\
&\vdots \\
Q_n(x) &= x^{n+1} + x^n + x^{n-1} \\
Q_{n+1}(x) &= x^n
\end{aligned}$$

It's obvious that $(\forall i \in \{1,2,\ldots,n\})Q_i(1) \neq 0$ and $P(1) \neq 0$. Hence we can expand $Q_i(x)$ with $x-1$:

$$\begin{aligned}
Q_1(x) &= \frac{x^{2n+1}-1}{x-1} \\
Q_2(x) &= x \cdot \frac{x^{2n-1}-1}{x-1} = \frac{x^{2n}-x}{x-1} \\
Q_3(x) &= x^2 \cdot \frac{x^{2n-3}-1}{x-1} = \frac{x^{2n-1}-x^2}{x-1} \\
&\vdots \\
Q_n(x) &= x^{n-1} \cdot \frac{x^3-1}{x-1} = \frac{x^{n+2}-x^{n-1}}{x-1} \\
Q_{n+1}(x) &= x^n \cdot \frac{x-1}{x-1} = \frac{x^{n+1}-x^n}{x-1}
\end{aligned}$$

Adding these fractions we get

$P(x) = \frac{x^{2n+1}+x^{2n}+x^{2n-1}+\cdots+x^{n+1}-x^n-x^{n-1}-\cdots-x-1}{x-1}$

Extracting the factor x^{n+1} from the first $n+1$ terms and the factor -1 from the last $n+1$ terms of the numerator, we get

$P(x) = \frac{(x^{n+1}-1)(x^n+x^{n-1}+\cdots+x+1)}{x-1}$

Dividing the first term of the numerator with the denominator, we get

$P(x) = (x^n + x^{n-1} + \cdots + x + 1)^2$

Hence, the roots of this polynomial are (1) all double and (2) equal to strictly complex roots of unity of the order $n+1$:

$x_{2k-1} = x_{2k} =$
$cos\frac{2k\pi}{n+1} + i$
$sin\frac{2k\pi}{n+1}, k = \overline{1,n}$

□ Solve the equation: $\sqrt{x-\sqrt{x-\sqrt{x-\sqrt{x-5}}}} = 5$.

Solution

Just take $t = \sqrt{x-\sqrt{x-5}}$ and consider two cases: $t > 5$, $t < 5$ which will give you impossible result. So the only case is $t = 5$ which gives $x = 30$.

□ If $f(N+1) = N(-1)^{N+1} - 2f(N)$ for all N$\geqslant$1 ,and $f(1) = f(2005)$ so what is $f(1) + f(2) + f(3) + \ldots + f(2004)$?

Solution

Denote $S = f(1) + f(2) + \cdots + f(2004)$. We have

$$
\begin{aligned}
f(2) &= 1-2f(1)\\
f(3) &= -2-2f(2)\\
f(4) &= 3-2f(3)\\
&\cdots\\
f(2005) &= -2004-2f(2004)
\end{aligned}
$$

First, observe that

$$1-2+3-4+5-6+\cdots+2003-2004=(1-2)+(3-4)+(5-6)+\cdots+(2003-2004)=1002\cdot(-1)$$

Second, $f(2)+f(3)+\cdots+f(2005)=S-f(1)+f(2005)=S$, since $f(1)=f(2005)$.

Now, if we add up all the equations above, we get

$S=-1002-2S \implies 3S=-1002 \implies S=-334.$

□ x,y,z are all positive real such that

$x+[y]+\{z\}=13.2$

$[x]+\{y\}+z=14.3$

$\{x\}+y+[z]=15.1$

Find x,y,z .

$[x]$ =integer part of x . $\{x\}$ = fraction part of x

Solution

As $\{a\}+[a]=a$, adding up the equations we get

$2(x+y+z)=42.6 \iff x+y+z=21.3$

Subtracting the first equation we get

$\{y\}+[z]=8.1$

Since $0\leqslant\{y\}<1$ and $[z]$ is integer, the only possibility is $\{y\}=0.1,[z]=8$

In the same manner we find $\{x\}=0,[y]=7$ and $[x]=6,\{z\}=0.2$

Now $x=[x]+\{x\}=6,y=7.1,z=8.2$

□ What is the remainder of $\frac{x^{203}-1}{x^4-1}$?

Solution

Since the denominator is of the fourth degree, the remainder will be of the third degree, hence

$x^{203}-1=(x^4-1)Q(x)+ax^3+bx^2+cx+d$

for some polynomial $Q(x)$

Now substitute the roots of x^4-1, which are $\pm1,\pm i$:

$a+b+c+d=0$ $-a+b-c+d=-2$ $-ia-b+ic+d=-i-1$ $ia-b-ic+d=i-1$

From the last two equations $-a+c=-1 \implies c=-1+a$ and $-b+d=-1 \implies d=-1+b$.

Substituting that into the first two equations we get

$a+b-1+a-1+b=0 \implies a+b=1$ $-a+b+1-a-1+b=-2 \implies -a+b=-1$

From these two we get $a=1,b=0$, which gives $c=0,d=-1$

Hence the remainder is x^3-1

□ On what intervals is it true that $\frac{||x-2|-|x+2||+||x-2|+|x+2||}{x}>1$?

Solution

The numerator is never negative, hence the denominator must be positive in order for the fraction to be greater than one. Therefore $x>0$.

Now let's analyze the numerator. It's of the form $|a-b|+|a+b|$. For $a \geqslant b$ this simplifies to $2a$, and for $a<b$ to $2b$. Hence we can write $|a-b|+|a+b|=2\max\{a,b\}$.

This turns the numerator into $2\max\{|x-2|,|x+2|\}$. For $x>0$ it's easy to check that $|x+2|$ is always greater than $|x-2|$: For $0<x<2$ we have to compare $x+2$ to $2-x$, or equivalently x to $-x$, and for $x \geqslant 2$ we have to compare $x+2$ to $x-2$, or equivalently 2 to -2. In both cases, we get $2\max\{|x-2|,|x+2|\}=2|x+2|=2(x+2)$ (x being positive). Therefore the inequality becomes $\frac{2(x+2)}{x}>1 \overset{x>0}{\iff} 2x+4>x \iff x>-4$, which is satisfied for all $x>0$.

□ solve it WITHOUT differential calculus and/or vectors.

Find the minimum and maximum possible values of

$$2\sin x\cos y+3\sin x\sin y+6\cos x$$

where $x,y\in\mathbb{R}$.

Solution

$$\begin{aligned}
2\sin x\cos y+3\sin x\sin y+6\cos x &= \sin x(2\cos y+3\sin y)+6\cos x\\
&= \sqrt{13}\sin x(\frac{2}{\sqrt{13}}\cos y+\frac{3}{\sqrt{13}}\sin y)+6\cos x\\
&= \sqrt{13}\sin x\sin(y+\arctan\frac{2}{3})+6\cos x\\
&= \sqrt{13\sin^2(y+\arctan\frac{2}{3})+36}\cos(x-\arctan\frac{\sqrt{13}\sin(y+\arctan\frac{2}{3})}{6})
\end{aligned}$$

From here it's obvious that the maximum and minimum values of the root are 6 (for $\sin(y+\arctan\frac{2}{3})=0 \iff y+\arctan\frac{2}{3}=k\pi$) and 7 (for $\sin(y+\arctan\frac{2}{3})=\pm1 \iff y+\arctan\frac{2}{3}=\frac{\pi}{2}+k\pi$). In each of these cases, we can choose x independently of y such that the cosine is equal to ±1. Hence, the minimum of the expression is -7 and the maximum is 7.

□ Let x be a real number such that $x+\frac{1}{x}$ is an integer. Prove that $x^n+\frac{1}{x^n}$ is an integer for all positive integers n

Solution

Denote $a_n=x^n+\frac{1}{x^n}$. By the given presumption, we have that $a_1\in\mathbb{Z}$, and hence $a_2=x^2+\frac{1}{x^2}=(x+\frac{1}{x})^2-2=a_1^2-2\in\mathbb{Z}$

Now induction. Assume that for $n\geqslant 2$ both a_n and a_{n-1} are integer. Then $a_na_1=(x^n+\frac{1}{x^n})(x+\frac{1}{x})=x^{n+1}+\frac{1}{x^{n+1}}+x^{n-1}+\frac{1}{x^{n-1}}=a_{n+1}+a_{n-1}$, which gives $a_{n+1}=a_na_1-a_{n-1}$, so by the inductive assumption, a_{n+1} is also an integer. QED

□ x and y are two real numbers, with $x>y\ldots$ Prove or disprove: $x-[x]\geq y-[y]$

Solution

Generally, it can be said that the inequality [u]doesn't[/u] hold for any x,y such that $x>y$ and $[x-y]=[x]-[y]-1$

(Proof: put $x=n+\alpha, y=n-k+\beta$ where n,k are integers and α,β are fractional parts. Then $x-y=k+\alpha-\beta$, from which the condition $\alpha-\beta<0$ gives $[x-y]=k-1$. On the other hand, k can be expressed as $[x]-[y]$, hence the statement.)

□ If $ABCD$ is a trapezoid with DC parallel to AB, $\angle DCB$ is a right angle, $DC=6, BC=4, AB=y$, and $\angle ADB=x$, find y in terms of x.

Solution

Assume $CD < AB$. Find $E \in AB$ such that $DE \perp AB$. Then $AE = y-6, ED = 4, \angle EDA = x-90°$, hence $\frac{y-6}{4} = \tan(x - 90°) = -\cot x \implies y = 6 - 4\cot x$.

If $CD \geqslant AB$, then $AE = 6-y, ED = 4, \angle EDA = 90° - x$, hence $\frac{6-y}{4} = \tan(90° - x) = \cot x \implies y = 6 - 4\cot x$.

Therefore, in any case $y = 6 - 4\cot x$. – Solve in reals:

$9^x - 6^x = 4^{x+\frac{1}{2}}$ – $\square$ Let obtuse triangle ABC satisfy $AB \cdot BC \cdot CA = 3\sqrt{3} \sin A \sin B \sin C$. Find the upper bound of the area of ABC.

Solution

By the sine Law we have

$3\sqrt{3} = \frac{AB}{\sin C} \cdot \frac{BC}{\sin A} \cdot \frac{CA}{\sin B} = (2R)^3$

$(2R)^3 = 3^{3/2} \implies R = \frac{\sqrt{3}}{2}$

Since the triangle is obtuse, the upper bound of its area is the area of the equilateral right triangle with the hypotenuse $2R$, and that's $R^2 = \frac{3}{4}$. Hence $A < \frac{3}{4}$

$\square$ How many real numbers x satisfy the equation $\frac{1}{5} \log_2 x = \sin(5\pi x)$?

Solution

Solution 1

1. On interval $(0, 1)$ function $y = 5\sin(5\pi x)$ has two negative half-periods, on $(1/5, 2/5)$ and $(3/5, 4/5)$, reaching value -5 in $x_1 = 3/10$ and $x_2 = 7/10$. Since $\log_2(3/10) > -5$ and $\log_2(7/10) > -5$, the sinusoid will intersect the logarithm curve in four points (two for each half-period).

2. In the point $x = 1$ the equation is satisfied, since $\log_2 1 = 5\sin(5\pi \cdot 1)$

3. On interval $(1, 32)$ function $y = 5\sin(5\pi x)$ has 77 positive half-periods — 5 on each interval $(2n - 1, 2n + 1)$, plus 2 on $(31, 32)$ — reaching value 5 in the midpoints of these half-periods. Since $\log_2 x \leq 5$ for $1 \leq x \leq 32$, it follows that the logarithmic curve will intersect the sinusiod in 2 points for each positive half-period, giving 154 points.

In total, we have $4 + 1 + 154 = 159$ points.

Solution 2 The range of $\sin x$ is $[-1, 1]$. Hence, we only need to consider $\left|\frac{1}{5}\log_2 x\right| \leq 1$. This is satisfied for $\frac{1}{32} \leq x \leq 32$. First let's consider $\frac{1}{32} \leq x < 1$. In this interval, the left hand side is negative while the right hand side is negative only in $[1/5, 2/5]$ and $[3/5, 4/5]$, so there are 4 solutions.

When $1 < x \leq 32$, the left hand side is positive, and the right hand side is positive only in $[6/5, 7/5]$, $[8/5, 9/5]$, $\ldots$, and $[158/5, 169/5]$. There are then 2 points of intersection in each of these intervals, and there are 77 intervals.

When $x = 1$ both sides of the equation are 0, so in all we have $4 + 77 \cdot 2 + 1 = \boxed{159}$ solutions.

Solution 3 $y = \frac{1}{5}\log_2 x(a)$ - $(A) = \sin(5\pi x)$ - (B) can only be true when $0 < y \leq 1$. $\frac{1}{5}\log_2 x = 1 \Rightarrow x = 32$. B has a period of $\frac{2\pi}{5\pi} = \frac{2}{5}$. Therefore the graph A passes through B $\frac{32}{2/5} = 80$ times. For each period, A passes through B 2 times except first one. So therefore there are $80 * 2 - 1 = \boxed{159}$ such points where A and B intersect..

$\square$ xem them

$\square$ Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

Solution

since $x^3 - x = x(x - 1)(x + 1)$, we'll use the remainders when $P(x)$ is divided by $x, x - 1, x + 1$, and those are $P(0) = 0, P(1) = 5, P(-1) = -5$ respectively.

Now

$P(x) = x(x - 1)(x + 1)Q(x) + ax^2 + bx + c$

Substitute $x = 0, x = 1, x = -1$ to obtain the system of equations:

$$\begin{aligned} c &= 0 \\ a + b + c &= 5 \\ a - b + c &= -5 \end{aligned}$$

which has the solution $a = 0, b = 5, c = 0$

Therefore the desired remainder is $5x$.

$\square$ $a_1, a_2, \ldots a_{100}$ are real numbers that satisfy

$a_1 + \ldots + a_n = n(1 + a_{n+1} + \ldots + a_{100})$

for all integers 1 thru 100. Find a_{13}.

Solution

Put $S_n = a_1 + \cdots + a_n$. By the initial equation, $S_{100} = 100$. Now

$S_n = n(1 + S_{100} - S_n) \iff (n+1)S_n = (S_{100} + 1)n \iff S_n = \frac{101n}{n+1}$

Then

$a_n = S_n - S_{n-1} = \frac{101n}{n+1} - \frac{101(n-1)}{n} = \frac{101}{n(n+1)}$

Therefore $a_{13} = \frac{101}{182}$

$\square$ Prove the following without calculus. $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots < 3$

Solution

Put $S = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$. Then $\frac{1}{2!} + \frac{1}{3!} + \cdots = S - 2$. Now

$$\begin{aligned} 2(S-2) &= \frac{2}{2!} + \frac{2}{3!} + \frac{2}{4!} + \ldots \\ &< \frac{2}{2!} + \frac{3}{3!} + \frac{4}{4!} + \ldots \\ &= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \\ &= S - 1 \end{aligned}$$

Hence $2(S-2) < S - 1 \iff S < 3$

$\square$

If x is a positive real number, simplify:

$$\cos(arctan(\sin(arccot(x))))]^2$$

Solution

$\cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}}$, hence $\cos\arctan t = \frac{1}{\sqrt{1+t^2}}$

Similarly

$\sin\phi = \frac{1}{\sqrt{1+\cot^2\phi}}$, hence $\sin\operatorname{arccot} x = \frac{1}{\sqrt{1+x^2}}$

Therefore

$A = \left(\frac{1}{\sqrt{1+\left(\frac{1}{\sqrt{1+x^2}}\right)^2}}\right)^2 = \frac{1+x^2}{2+x^2}$

$\square$ Find the residue when x^{1000} is divided by $x^3 + x^2 + x + 1$. and the coefficient of x^{100} for the quotient.

Solution

Denote $P(x) = x^{1000}$. We have $x^3 + x^2 + x + 1 = (x+1)(x+i)(x-i)$, hence

$P(x) = (x+1)(x+i)(x-i)Q(x) + R(x)$ for some polynomials $Q(x), R(x)$, where $\deg R(x) = 2$, hence we can write $R(x) = ax^2 + bx + c$

since $P(1) = 1, P(i) = P(-i) = (-1)^{500} = 1$, we have

$1 = a \cdot 1^2 + b \cdot 1 + c \implies a + b + c = 1$ $1 = a \cdot i^2 + b \cdot i + c \implies -a + c + ib = 1$ $1 = a \cdot (-i)^2 + b \cdot (-i) + c \implies -a + c - ib = 1$

Obviously $b = 0$ and from $a + c = 1 \wedge -a + c = 1$ we get $a = 0, c = 1$

Hence $R(x) \equiv 1$

Now

$$\begin{aligned} Q(x) &= \frac{P(x) - R(x)}{x^3 + x^2 + x + 1} \\ &= \frac{(x^{1000} - 1)(x - 1)}{x^4 - 1} \\ &= (x^{996} + x^{992} + \cdots + x^4 + 1)(x - 1) \end{aligned}$$

Obviously, the coefficient of x^{100} is -1.

□ Find m and solve the following equation, knowing that its roots form a geometric sequence: $x^4 - 15x^3 + 70x^2 - 120x + m = 0$.

Solution

If the roots are a, aq, aq^2, aq^3, then by Vieta we have

$$\begin{aligned} a + aq + aq^2 + aq^3 &= 15 \\ a^2q + a^2q^2 + a^2q^3 + a^2q^3 + a^2q^4 + a^2q^5 &= 70 \\ a^3q^3 + a^3q^4 + a^3q^5 + a^3q^6 &= 120 \\ a^4q^6 &= m \end{aligned}$$

(1) simplifies to $a(1 + q + q^2 + q^3) = 15$, and (3) to $a^3q^3(1 + q + q^2 + q^3) = 120$. Dividing those two we get $a^2q^3 = 8$, from which we obtain $m = a^4q^6 = (a^2q^3)^2 = 64$.

The equation becomes

$$\begin{aligned} x^4 - 15x^3 + 70x^2 - 120x + 64 = 0 &\iff x^4 - 3x^3 + 2x^2 - 12x^3 + 36x^2 - 24x + 32x^2 - 96x + 64 = 0 \\ &\iff x^2(x^2 - 3x + 2) - 12x(x^2 - 3x + 2) + 32(x^2 - 3x + 2) = 0 \\ &\iff (x^2 - 12x + 32)(x^2 - 3x + 2) = 0 \end{aligned}$$

The solutions are $1, 2, 4, 8$

□ If $x + y = 3 - \cos 4\alpha, x - y = 4\sin 2\alpha$. Prove $\sqrt{x} + \sqrt{y} = 2$

Solution

As $\cos 2\phi = 1 - 2\sin^2\phi$, we have $\cos 4\alpha = 1 - 2\sin^2 2\alpha$. For shortness put $a = \sin 2\alpha$. Then

$x + y = 3 - (1 - 2a^2) = 2 + 2a^2$ $x - y = 4a$

By adding up the equations we get

$2x = 2(1 + 2a + a^2) \iff x = (1 + a)^2 \iff \sqrt{x} = |1 + a|$

since $a \geqslant -1$ (by the definition of a), we get $|1+a| = 1+a \iff \sqrt{x} = 1+a$

By subtracting the second equation from the first we get

$2y = 2(1-2a+a^2) \iff y = (1-a)^2 \iff \sqrt{y} = |1-a|$

since $a \leqslant 1$ (by the definition of a), we get $|1-a| = 1-a \iff \sqrt{y} = 1-a$

Now $\sqrt{x}+\sqrt{y} = 1+a+1-a = 2$

$$\text{Another way:} \left\| \begin{array}{c} \sin 2\alpha = t \in [-1,1] \\ \\ \cos 4\alpha = 1-2t^2 \end{array} \right. \Longrightarrow \left\| \begin{array}{c} x+y = 2(t^2+1) \\ \\ x-y = 4t \end{array} \right. \Longrightarrow \left\| \begin{array}{c} x = (t+1)^2 \\ \\ y = (t-1)^2 \end{array} \right. \Longrightarrow \left\| \begin{array}{c} \sqrt{x} = t+1 \\ \\ \sqrt{y} = 1-t \end{array} \right.$$

$\Longrightarrow \boxed{\sqrt{x}+\sqrt{y} = 2}$ *right.*

$$\textbf{Remark.} \left\| \begin{array}{c} \sqrt{x} = (\sin\alpha+\cos\alpha)^2 \\ \\ \sqrt{y} = (\sin\alpha-\cos\alpha)^2 \end{array} \right. \Longrightarrow \left\| \begin{array}{c} \sqrt{x}+\sqrt{y} = (\sin\alpha+\cos\alpha)^2 + (\sin\alpha-\cos\alpha)^2 = 2\ . \\ \\ \sqrt[4]{x}+\sqrt[4]{y} = |\sin\alpha+\cos\alpha| + |\sin\alpha-\cos\alpha| \le 2 \Longrightarrow \boxed{\sqrt[4]{x}+\sqrt[4]{}} \end{array} \right.$$

□ Let $ABCD$ be a parallelogram. Denote the point $M \in [CD]$ for which $\widehat{MAC} \equiv \widehat{MAD}$ and the point $N \in [BC]$ for which $\widehat{NAB} \equiv \widehat{NAC}$. Define the points $X \in AB \cap MN$ and $Y \in AD \cap MN$. Prove that the area [XAY] is equally to the area [ABCD] if and only if $ABCD$ becomes a rectangle.

Solution

Thiếu hình vẽ See the attached diagram for additional notation.

AM is the bisector of $\angle DAC$, which gives $\frac{DM}{MC} = \frac{b}{d}$. Similarly, $\frac{BN}{NC} = \frac{a}{d}$

$\triangle YDM \sim \triangle NCM \implies \frac{[YDM]}{[NCM]} = \left(\frac{DM}{MC}\right)^2 = \frac{b^2}{d^2}$. Similarly, $\frac{[XBN]}{[NCM]} = \left(\frac{BN}{NC}\right)^2 = \frac{a^2}{d^2}$

Now

$$[XBN] + [YDM] = [NCM] \iff \frac{a^2+b^2}{d^2}[NCM] = [NCM] \iff a^2+b^2 = d^2$$

which is fulfilled if and only if $ABCD$ is a rectangle.

□ Let ABC be a right triangle ($AB \perp AC$). The its incircle $w = C(I,r)$ touches the sides $[AB]$, $[AC]$ in the points E , F .

Prove that the intersections of the line EF with the lines BI , CI belong to the circumcircle of the triangle ABC .

Solution

Let BI meets EF at point D. since $\angle A = 90^o$, so $AFIE$ is a square. Thus EF is the perpendicular bisector of AI, hence $\triangle DIA$ is an isoseles triangle. Because

$\angle BDE = 180^o - 135^o - \frac{\angle B}{2} = 45^o - \frac{\angle B}{2}$

so

$\angle ADB = 2\angle BDE = 90^o - \angle B = \angle C$

Therefore D lies on the circumcircle of triangle ABC.

□ Let be given a triangle ABC. Prove that : $\frac{5}{3\sqrt{3}} \cdot S \le R^2 + r^2 \quad (*)$

Solution

Denote $2p = a+b+c$. Therefore,

$$\left\{ \begin{array}{l} p \le \frac{3\sqrt{3}}{2} \cdot R \Longrightarrow S = pr \le \frac{3\sqrt{3}}{2} \cdot Rr \Longrightarrow \frac{5}{3\sqrt{3}} \cdot S \le \frac{5}{2} \cdot Rr \\ \\ 2r \le R \Longrightarrow 0 \le (R-2r)(2R-r) \Longrightarrow \frac{5}{2} \cdot Rr \le R^2 + r^2 \end{array} \right\| \Longrightarrow (*)\ .$$

□ Let ABC be a triangle with the centroid G . Prove that $BC+GA = CA+GB = AB+GC \iff AB = BC = CA$.

Solution

Let AA', BB' be the medians. Then

$AA'^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4} \qquad (*)\ BB'^2 = \frac{a^2+c^2}{2} - \frac{b^2}{4}$

giving

$(AA' - BB')(AA' + BB') = \frac{3}{4}(b-a)(b+a) \qquad (1)$

From the given condition

$$GA - GB = CA - CB = b - a \iff \frac{2}{3}(AA' - BB') = b - a \iff AA' - BB' = \frac{3}{2}(b-a) \qquad (2)$$

Plugging (2) into (1):

$\frac{3}{2}(b-a)(AA' + BB') = \frac{3}{4}(b-a)(b+a)$

Assume $a \neq b$. Then the last equation yields

$AA' + BB' = \frac{1}{2}(b+a)$

Together with

$AA' - BB' = \frac{3}{2}(b-a)$

we get

$AA' = b - \frac{a}{2}$

Plugging that into $(*)$ we get

$b^2 - ab + \frac{a^2}{4} = \frac{b^2+c^2}{2} - \frac{a^2}{4} \iff c^2 = (b-a)^2 \iff c = |b-a|$

but that's impossible by the triangle inequality. Therefore $a = b$. Similar argument for $b = c, c = a$.

QED

□ Prove that for all n,

$$\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n}{2}\right)^n$$

Solution

Lemma (well-known). $n \in \mathbb{N}\ ,\ n \geq 2 \Longrightarrow 2 < \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} < e.$

$$\left\{\begin{array}{c} \boxed{a_n = \left(\frac{n}{e}\right)^n} \\ \\ \boxed{b_n = n!} \\ \\ \boxed{c_n = e \cdot \left(\frac{n}{2}\right)^n} \end{array}\right\| \Longrightarrow \left\{\begin{array}{l} x_n = \frac{a_n}{b_n} = \left(\frac{n}{e}\right)^n \cdot \frac{1}{n!} \\ \\ y_n = \frac{b_n}{c_n} = \frac{n!}{e} \cdot \left(\frac{2}{n}\right)^n \end{array}\right\| \Longrightarrow$$

$$\left\{\begin{array}{l} \frac{x_{n+1}}{x_n} = \frac{1}{e} \cdot \left(1 + \frac{1}{n}\right)^n < 1 \\ \\ \frac{y_{n+1}}{y_n} = \frac{2}{\left(1+\frac{1}{n}\right)^n} < 1 \end{array}\right\| \Longrightarrow \left\{\begin{array}{l} x_{n+1} < x_n \\ \\ y_{n+1} < y_n \end{array}\right\| \Longrightarrow$$

$$\left\{\begin{array}{l} \ldots < x_{n+1} < x_n < \ldots < x_1 = \frac{1}{e} < 1 \\ \\ \ldots < y_{n+1} < y_n < \ldots < y_1 = \frac{2}{e} < 1 \end{array}\right\| \Longrightarrow \left\{\begin{array}{l} a_n < b_b \\ \\ b_n < c_n \end{array}\right\| \Longrightarrow$$

$\boxed{a_n < b_n < c_n}$.

□ In ΔABC, draw the angle trisectors of $\angle A$ and $\angle B$. Two of those trisectors intersect in the midpoint of the circumscribed circle. Prove that the other two trisectors intersect in the orthocenter of the triangle.

Solution

In this case : $\begin{cases} \frac{A}{6}=\frac{B}{3}=\frac{C}{5}=\frac{\pi}{14} \quad \vee \quad \frac{A}{6}=\frac{B}{5}=\frac{C}{3}=\frac{\pi}{14}\ ; \\ \\ m(\widehat{ACH})=m(\widehat{BCO})=\frac{\pi}{14} \quad \vee \quad \frac{2\pi}{7} \quad , \quad m(\widehat{OCH})=\frac{3\pi}{14} \end{cases}$

i.e. the rays $[CH$ and $[CO$ arn't the trisectors of the angle $\widehat{ACB}$.

□ Thiếu hình vẽ Let M be the middlepoint of the hypotenuse (BC) of the right triangle ABC. Prove that the line joining the incenters of the triangles ABM and ACM divides the area of the triangle ABC evenly.

Remark. Prove easily that in a right triangle ABC , $AB \perp AC$ there are the identities : $\left\| \begin{array}{l} (a+b+c)^2 = 2(a+b)(a+c)\ . \\ \\ (b+c-a)^2 = 2(a-b)(a-c) \end{array} \right.$

Lemma ([u]one's own[/u]). In the rectangle $ABCD$ denote the points : $\left\| \begin{array}{c} M \in CD\ ,\ \widehat{MAC} \equiv \widehat{MAD} \\ \\ N \in BC\ ,\ \widehat{NAB} \equiv \widehat{NAC} \\ \\ X \in AB \cap MN\ ,\ Y \in AD \cap MN \end{array} \right.$

$\Longrightarrow [XAY]=[ABCD]$ Proof of the lemma. **Proof 1.** Denote : $\left\| \begin{array}{l} AB=b\ , AD=c\ , AC=a \\ \\ b^2+c^2=a^2 \end{array} \right. \Longrightarrow$

$\left\| \begin{array}{l} \frac{MD}{MC}=\frac{AD}{AC} \Longrightarrow \frac{MD}{c}=\frac{MC}{a}=\frac{b}{a+c} \Longrightarrow \boxed{MC=\dfrac{ab}{a+c}} \\ \\ \frac{NB}{NC}=\frac{AB}{AC} \Longrightarrow \frac{NB}{b}=\frac{NC}{a}=\frac{c}{a+b} \Longrightarrow \boxed{NC=\dfrac{ac}{a+b}} \end{array} \right.$ Therefore,

$\left\| \begin{array}{l} \frac{DY}{NC}=\frac{MD}{MC} \Longrightarrow DY=\frac{ac}{a+b}\cdot\frac{c}{a}=\frac{c^2}{a+b} \Longrightarrow AY=AD+DY=c+\frac{c^2}{a+b} \Longrightarrow \boxed{AY=\dfrac{c(a+b+c)}{a+b}} \\ \\ \frac{BX}{MC}=\frac{NB}{NC} \Longrightarrow BX=\frac{ab}{a+c}\cdot\frac{b}{a}=\frac{b^2}{a+c} \Longrightarrow AX=AB+BX=b+\frac{b^2}{a+c} \Longrightarrow \boxed{AX=\dfrac{b(a+b+c)}{a+c}} \end{array} \right.$

$\Longrightarrow AX \cdot AY = \frac{bc(a+b+c)^2}{(a+b)(a+c)}$.

Using the first identity from the above remark obtain $AX \cdot AY = 2bc$, i.e. $[XAY]=[ABCD]$.

Remark. $[XAY]=[ABCD] \Longleftrightarrow [DMY]+[BNX]=[MCN] \Longleftrightarrow DM \cdot DY + BN \cdot BX = CM \cdot CN \Longleftrightarrow$

$\frac{bc}{a+c}\cdot\frac{c^2}{a+b}+\frac{bc}{a+b}\cdot\frac{b^2}{a+c}=\frac{ab}{a+c}\cdot\frac{ac}{a+b} \Longleftrightarrow bc^3+b^3c=a^2bc \Longleftrightarrow b^2+c^2=a^2$, what is truly.

Proof 2 (with areas).

◄ ✠ ►

Proof of the proposed problem. Denote the incircles $w_1=C(I_1,r_1)$, $w_2=C(I_2,r_2)$ of the triangles ABM , ACM respectively , the touch-points $P \in AB \cap w_1$, $N \in AC \cap w_2$ and the intersections $X \in AB \cap I_1I_2$, $Y \in AC \cap I_1I_2$. Apply the above lemma to the rectangle $APMN$ (the ray $[AI_1$ is the bisector of the angle $\widehat{PAM}$ and the ray $[AI_2$ is the bisector of the angle $\widehat{NAM}$) : $AX \cdot AY = 2 \cdot AP \cdot AN \Longrightarrow AX \cdot AY = \frac{AB \cdot AC}{2}$, i.e. $[XAY]=\frac{1}{2}\cdot[ABC]$.

Another way

See the attached diagram for notation. The incenters are P, Q.

$r_a=\frac{\frac{1}{2}\cdot(ab/2)}{(a+\frac{c}{2}+\frac{c}{2})/2}=\frac{ab}{2(a+c)}, r_b=\frac{ab}{2(b+c)}$

$\triangle PP_1M \sim \triangle QQ_2M \implies \frac{r_b}{x} = \frac{\frac{a}{2}}{\frac{b}{2}-r_a+x}$
$\frac{a}{2}x = \frac{ab}{2(b+c)}\left(x + \frac{b}{2} - \frac{ab}{2(a+c)}\right)$
$\frac{a}{2}x = \frac{ab}{2(b+c)}\left(x + \frac{bc}{2(a+c)}\right)$
$x\left(\frac{a}{2} - \frac{ab}{2(b+c)}\right) = \frac{ab^2c}{4(a+c)(b+c)}$
$x\frac{ac}{2(b+c)} = \frac{ab^2c}{4(a+c)(b+c)}$
$x = \frac{b^2}{2(a+c)}$
Then $CM = \frac{b}{2} + x = \frac{b(a+b+c)}{2(a+c)}$
By symmetry, $CN = \frac{a(a+b+c)}{2(b+c)}$
Then

$$\begin{aligned}
[CMN] &= \frac{1}{2}CM \cdot CN \\
&= \frac{ab}{2} \cdot \frac{(a+b+c)^2}{4(a+c)(b+c)} \\
&= [ABC] \cdot \frac{a^2+b^2+c^2+2ab+2ac+2bc}{4(ab+ac+bc+c^2)} \\
&= [ABC] \cdot \frac{2a^2+2b^2+2ab+2ac+2bc}{4(a^2+b^2+ab+ac+bc)} \\
&= \frac{1}{2}[ABC]
\end{aligned}$$

Hết năm 2007-12/6/2013 cho Virgil Nicula