Dear readers ,

This document will help you for your preparation of IMO International Mathematical Olympiad , NO National Olympiad. It contains 169 functional equations with the solutions of Patrick "pco" . Many thanks to Patrick for its solutions on Mathlinks , it will help students for IMO.

Moubinool.

1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the set $\{\frac{f(x)}{x} : x \neq 0$ and $x \in \mathbb{R}\}$ is finite, and for all $x \in \mathbb{R}$, $f(x-1-f(x)) = f(x) - x - 1$

   solution

   Let $P(x)$ be the assertion $f(x-1-f(x)) = f(x) - x - 1$ Let $a \in \mathbb{R}$ and $b = f(a)$

   $P(a) \implies f(a-b-1) = b-a-1$ $P(a-b-1) \implies f(2(a-b)-1) = 2(b-a)-1$ And we get easily $f(2^n(a-b)-1) = 2^n(b-a)-1$ $\forall n \in \mathbb{N}$

   It's then immediate to see that the set $\{\frac{f(x)}{x} : x \neq 0$ and $x = 2^n(a-b)-1$ $\forall n \in \mathbb{N}\}$ is finite iff $b = a \iff f(a) = a$

   Hence the unique solution $\boxed{f(x) = x\ \forall x}$ which indeed is a solution

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

   $f(f(y+f(x))) = f(x+y) + f(x) + y$

   solution

   Let $P(x,y)$ be the assertion $f(f(y+f(x))) = f(x+y) + f(x) + y$

   $P(x, f(y)) \implies f(f(f(x)+f(y))) = f(x+f(y)) + f(x) + f(y)$ $P(y, f(x)) \implies f(f(f(x)+f(y))) = f(y+f(x)) + f(x) + f(y)$ Subtracting, we get $f(x+f(y)) = f(y+f(x))$

   So $f(f(x+f(y))) = f(f(y+f(x)))$ So (using $P(x,y)$ and $P(y,x)$) : $f(x+y) + f(y) + x = f(x+y) + f(x) + y$

   So $f(x) - x = f(y) - y$ and so $f(x) = x + a$, which is never a solution.

   $f(f(y+f(x))) = f(x+y) + f(x) + y$

3. Find all functions $f : \mathbb{R}+ \to \mathbb{R}+$ such that $f(1+xf(y)) = yf(x+y)$ for all $x, y \in \mathbb{R}+$.

   solution

   Let $P(x,y)$ be the assertion $f(1+xf(y)) = yf(x+y)$

   1) $f(x)$ is a surjective function $====$ $P(\frac{1}{f(\frac{f(2)}{x})}, \frac{f(2)}{x}) \implies$

   $$f(2) = \frac{f(2)}{x} f(\frac{1}{f(\frac{f(2)}{x})} + \frac{f(2)}{x})$$

And so $x = f(something)$ Q.E.D.

2) $f(x)$ is an injective function ===

Let $a > b > 0$ such that $f(a) = f(b)$ Let $T = b - a > 0$

Comparing $P(x, a)$ and $P(x, b)$, we get $af(x + a) = bf(x + b)$

and so $f(x) = \frac{b}{a}f(x + T)\ \forall x > a$

And so $f(x) = \left(\frac{b}{a}\right)^n f(x + nT)\ \forall x > a, n \in \mathbb{N}$

Let then y such that $f(y) > 1$ (such y exists since $f(x)$ is a surjection, according to 1) above) Let n great enough to have $y + nT - 1 > 0$

$P(\frac{y+nT-1}{f(y)-1}, y) \implies f(1 + \frac{yf(y)+(nT-1)f(y)}{f(y)-1}) = yf(\frac{y+nT-1}{f(y)-1} + y)$ which may be written :

$f(\frac{yf(y)+nT-1}{f(y)-1} + nT) = yf(\frac{yf(y)+nT-1}{f(y)-1})$

and since $f(\frac{yf(y)+nT-1}{f(y)-1} + nT) = \left(\frac{a}{b}\right)^n f(\frac{yf(y)+nT-1}{f(y)-1})$, we get $y = \left(\frac{a}{b}\right)^n$ $\forall n$, which is impossible Q.E.D.

3) $f(1) = 1$ === $P(1, 1) \implies f(1 + f(1)) = f(2)$ and so, since $f(x)$ is injective, $f(1) = 1$ Q.E.D.

4) The only solution is $f(x) = \frac{1}{x}$ === $P(1, x) \implies f(1+f(x)) = xf(1+x)$ and so $f(1 + x) = \frac{1}{x}f(1 + f(x))$

$P(\frac{x}{f(\frac{1}{x})}, \frac{1}{x}) \implies f(1 + x) = \frac{1}{x}f(\frac{x}{f(\frac{1}{x})} + \frac{1}{x})$

And so (comparing these two lines) : $f(1 + f(x)) = f(\frac{x}{f(\frac{1}{x})}\frac{1}{x})$

And so (using injectivity) : $1 + f(x) = \frac{x}{f(\frac{1}{x})} + \frac{1}{x}$ and so $f(\frac{1}{x}) = \frac{x}{f(x)+1-\frac{1}{x}}$

This implies (changing $x \to \frac{1}{x}$) : $f(x) = \frac{\frac{1}{x}}{f(\frac{1}{x})+1-x}$

And so $f(x) = \frac{\frac{1}{x}}{\frac{x}{f(x)+1-\frac{1}{x}}+1-x}$

Which gives $x^2f(x)^2 - 2xf(x) + 1 = 0$

And so $\boxed{f(x) = \dfrac{1}{x}}$, which indeed is a solution

4. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equality $f(y) + f(x + f(y)) = y + f(f(x) + f(f(y)))$

solution

Let $P(x, y)$ be the assertion $f(y) + f(x + f(y)) = y + f(f(x) + f(f(y)))$

$P(f(x), 0) \implies f(0) + f(f(x) + f(0)) = f(f(f(x)) + f(f(0)))$ $P(f(0), x)$ $\implies f(x) + f(f(x) + f(0)) = x + f(f(f(x)) + f(f(0)))$

Subtracting, we get $f(x) = x + f(0)$

Plugging back $f(x) = x + a$ in original equation, we get $a = 0$ and the unique solution $\boxed{f(x) = x \forall x}$

5. Find all non-constant real polynomials $f(x)$ such that for any real x the following equality holds: $f(sinx + cosx) = f(sinx) + f(cosx)$

solution

If $f(x)$ is non constant, let $n > 0$ its degree and Wlog consider $f(x)$ is monic.

Using half-tangent, the equation may be written $f\left(\frac{1+2x-x^2}{1+x^2}\right) = f\left(\frac{2x}{1+x^2}\right)$ $+f\left(\frac{1-x^2}{1+x^2}\right)$ $\forall x$

Multiplying by $(1+x^2)^n$, and setting then $x = i$, we get $(2+2i)^n = (2i)^n + 2^n$ and so $n = 1$ (look at modulus).

Hence the solutions: $f(x) = ax$ $\forall a \in \mathbb{R}^*$

6. Find all functions $f\colon \mathbb{N} \to \mathbb{Z}$ such that for all $x, y \in N$ holds $f(x+|f(y)|) = x + f(y)$

solution

Let $P(x, y)$ be the assertion $f(x + |f(y)|) = x + f(y)$

If $|f(a)| < a$ for some $a \in \mathbb{N}$, then $P(a - |f(a)|, a) \implies |f(a)| = a$ and so contradiction. So $|f(x)| \geq x$ $\forall x \in \mathbb{N}$

If $f(a) < 0$ for some $a \in \mathbb{N}$, then $P(-f(a), a) \implies f(-2f(a)) = 0$ and so contradiction with $f(x) \geq x$ $\forall x \in \mathbb{N}$ So $f(x) \geq 0$ $\forall x \in \mathbb{N}$

As a consequence $|f(x)| = f(x)$ and the problem becomes :

Find all functions $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ such that $f(x + f(y)) = x + f(y)$ $\forall x, y \in \mathbb{N}$ Let then $m = \min(f(\mathbb{N}))$ and we get $f(x) = x$ $\forall x > m$

[Hence the solutions

Let $a \in \mathbb{N}$ $f(x) = x$ $\forall x \geq a$ $f(x)$ can take any value in $[a - 1, +\infty)$ for $x \in [1, a - 1]$

7. Determine all pairs of functions $f, g : \mathbb{Q} \to \mathbb{Q}$ satisfying the following equality

$$f(x + g(y)) = g(x) + 2y + f(y),$$

for all $x, y \in \mathbb{Q}$.

solution

If $f(x)$ is a solution, then so is $f(x) + c$. So Wlog consider that $f(0) = 0$ Let $P(x, y)$ be the assertion $f(x + g(y)) = g(x) + 2y + f(y)$

$P(-g(0), 0) \implies g(-g(0)) = 0$ $P(-g(0), -g(0)) \implies g(0) = 0$ $P(x, 0)$ $\implies f(x) = g(x)$

So we are looking for $f(x)$ such that $f(0)=0$ and $f(x+f(y))=f(x)+2y+f(y)$ Let $Q(x,y)$ be the assertion $f(x+f(y))=f(x)+2y+f(y)$

$Q(x-f(x),x) \implies f(x-f(x))=-2x$ and so $f(x)$ is surjective

$Q(x,y) \implies f(x+f(y))=f(x)+2y+f(y)$ $Q(0,y) \implies f(f(y))=2y+f(y)$ Subtracting, we get $f(x+f(y))=f(x)+f(f(y))$ and, since surjective : $f(x+y)=f(x)+f(y)$

Since $f(x)$ is from $\mathbb{Q}\to\mathbb{Q}$, this immediately gives $f(x)=ax$ and, plugging this in $Q(x,y)$: $a^2-a-2=0$

Hence the two solutions : $f(x)=2x+c$ and $g(x)=2x$ $\forall x$ and for any real c, which indeed is a solution

8. Given two positive real numbers a and b, suppose that a mapping $f:\mathbb{R}^+\to\mathbb{R}^+$ satisfies the functional equation

$$f(f(x))+af(x)=b(a+b)x.$$

Prove that there exists a unique solution of this equation.

solution

$a+2b>0$ and we get thru simple induction : $f^{[n]}(x)=\frac{((a+b)x+f(x))b^n+(bx-f(x))(-a-b)^n}{a+2b}$

If, for some x, $f(x)-bx\neq 0$, we get that, for some n great enough, $f^{[n]}(x)<0$, which is impossible.

Hence the unique solution : $f(x)=bx$ which indeed is a solution

9. Find all non-constant functions $f\colon\mathbb{Z}\to\mathbb{N}$ satisfying all of the following conditions: a)$f(x-y)+f(y-z)+f(z-x)=3(f(x)+f(y)+f(z))-f(x+y+z)$ b)$\sum_{k=1}^{15}f(k)\leq 1995$

solution

Setting $x=y=z=0$ in the equation, we get $f(0)=0\notin\mathbb{N}$ and so no solution Since OP is a brand new user on this forum, I'll consider that he ignored that we use here the notation $\mathbb{N}$ for positive integers and that he meant $\mathbb{N}_0$, set of all non negative integers. If so :

Let $P(x,y,z)$ be the assertion $f(x-y)+f(y-z)+f(z-x)=3(f(x)+f(y)+f(z))-f(x+y+z)$

$P(0,0,0) \implies f(0)=0$ $P(x,0,0) \implies f(-x)=f(x)$ $P(x,-x,0) \implies f(2x)=4f(x)$ $P(x+1,-1,-x-1) \implies f(x+2)=2f(x+1)-f(x)+2f(1)$

This recurrence definition (plus $f(0)=0$) is quite classical and has simple general solution $f(x)=ax^2$

$f(x)\in\mathbb{N}_0$ $\forall x\in\mathbb{Z} \implies a\geq 0$ $f(x)$ non constant $\implies a>0$ $\sum_{k=1}^{15}f(k)=a\sum_{k=1}^{15}k^2=1240a\leq 1995 \implies a\leq 1$

[u][b]Hence the unique solution of the modified problem[/b][/u] : $f(x)=x^2$ $\forall x$,

10. .Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$f(x + yf(x)) + f(xf(y) - y) = f(x) - f(y) + 2xy$

Here is a rather heavy

$$solution$$

:

Let $P(x, y)$ be the assertion $f(x+yf(x))+f(xf(y)-y) = f(x)-f(y)+2xy$

1) $f(x)$ is an odd function and $f(x) = 0 \iff x = 0$ ==

$P(0, 0) \implies f(0) = 0$ $P(0, x) \implies f(-x) = -f(x)$

Suppose $f(a) = 0$. Then $P(a, a) \implies 0 = 2a^2 \implies a = 0$ and so $f(x) = 0 \iff x = 0$ Q.E.D

2) $f(x)$ is additive

===

Let then $x \neq 0$ such that $f(x) \neq 0$: $P(x, \frac{x+y}{f(x)}) \implies f(2x + y) + f(xf(\frac{x+y}{f(x)}) - \frac{x+y}{f(x)}) = f(x) - f(\frac{x+y}{f(x)}) + 2x\frac{x+y}{f(x)}$

$P(\frac{x+y}{f(x)}, -x) \implies -f(xf(\frac{x+y}{f(x)}) - \frac{x+y}{f(x)}) - f(y) = f(\frac{x+y}{f(x)}) + f(x) - 2x\frac{x+y}{f(x)}$

Adding these two lines, we get : $f(2x+y) = 2f(x)+f(y)$ which is obviously still true for $x = 0$ and so :

New assertion $Q(x, y)$: $f(2x + y) = 2f(x) + f(y)$ $\forall x, y$

$Q(x, 0) \implies f(2x) = 2f(x)$ and so $Q(x, y)$ becomes $f(2x + y) = f(2x) + f(y)$ and so $f(x + y) = f(x) + f(y)$ and $f(x)$ is additive. Q.E.D.

3) $f(x)$ solution implies $-f(x)$ solution and so wlog consider from now $f(1) \geq 0$ ====

$P(y, x) \implies f(y + xf(y)) + f(yf(x) - x) = f(y) - f(x) + 2xy \implies -f(-y + x(-f(y))) - f(y(-f(x)) + x) = -f(x) - (-f(y)) + 2xy$ Q.E.D

4) $f(x)$ is bijective and $f(1) = 1$ ====

Using additive property, the original assertion becomes $R(x, y)$: $f(xf(y)) + f(yf(x)) = 2xy$

$R(x, \frac{1}{2}) \implies f(xf(\frac{1}{2}) + \frac{f(x)}{2}) = x$ and $f(x)$ is surjective.

So $\exists a$ such that $f(a) = 1$ Then $R(a, a) \implies a^2 = 1$ and so $a = 1$ (remember that in 3) we choosed $f(1) \geq 0$)

5) $f(x) = x$ ====

$R(x, 1) \implies f(x)+f(f(x)) = 2x$ and so $f(x)$ is injective, and so bijective.

$R(xf(x), 1) \implies f(xf(x))+f(f(xf(x))) = 2xf(x)$ $R(x, x) \implies f(xf(x)) = x^2$ and so $f(x^2) = f(f(xf(x)))$ Combining these two lines, we get $f(x^2) + x^2 = 2xf(x)$

So $f((x+y)^2)+(x+y)^2 = 2(x+y)f(x+y)$ and so $f(xy)+xy = xf(y)+yf(x)$

So we have the properties : $R(x,y)$: $f(xf(y))+f(yf(x))=2xy$ $A(x,y)$: $f(xy)=xf(y)+yf(x)-xy$ $B(x)$: $f(f(x))=2x-f(x)$

So :

(a) : $R(x,x) \implies f(xf(x))=x^2$ (b) : $A(x,f(x)) \implies f(xf(x))=xf(f(x))+f(x)^2-xf(x)$ (c) : $B(x) \implies f(f(x))=2x-f(x)$

And so -(a)+(b)+x(c) : $0=x^2+f(x)^2-2xf(x)=(f(x)-x)^2$ Q.E.D.

6) synthesis of solutions ==== Using 3) and 5), we get two solutions (it's easy to check back that these two functions indeed are solutions) : $f(x)=x\ \forall x$ $f(x)=-x\ \forall x$[/quote]

11. Find all functions f defined on real numbers and taking real values such that $f(x)^2+2yf(x)+f(y)=f(y+f(x))$ for all real numbers x,y. [

solution

Let $P(x,y)$ be the assertion $f(x)^2+2yf(x)+f(y)=f(y+f(x))$

$f(x)=0\ \forall x$ is a solution. So we'll look from now for non all-zero solutions.

Let $f(a)\neq 0$: $P(a,\frac{u-f(a)^2}{2f(a)}) \implies u=f(\text{something})-f(\text{something else})$ and so any real may be written as a difference $f(v)-f(w)$

$P(w,-f(w)) \implies -f(w)^2+f(-f(w))=f(0)$ $P(v,-f(w)) \implies f(v)^2-2f(v)f(w)+f(-f(w))=f(f(v)-f(w))$

Subtracting the first from the second implies $f(v)^2-2f(v)f(w)+f(w)^2=f(f(v)-f(w))-f(0)$ and so $f(f(v)-f(w))=(f(v)-f(w))^2+f(0)$

And so $f(x)=x^2+f(0)\ \forall x\in\mathbb{R}$ which indeed is a solution.

Hence the two solutions : $f(x)=0\ \forall x$ $f(x)=x^2+a\ \forall x$

12. Prove that $f(x+y+xy)=f(x)+f(y)+f(xy)$ is equivalent to $f(x+y)=f(x)+f(y)$.

solution

Let $P(x,y)$ be the assertion $f(x+y+xy)=f(x)+f(y)+f(xy)$

1) $f(x+y)=f(x)+f(y) \implies P(x,y)$ ======= Trivial.

2) $P(x,y) \implies f(x+y)=f(x)+f(y)\ \forall x,y$ ====== $P(x,0) \implies f(0)=0$ $P(x,-1) \implies f(-x)=-f(x)$

2.1) new assertion $R(x,y)$: $f(x+y)=f(x)+f(y)\ \forall x,y$ such that $x+y\neq -2$ —————————

Let x,y such that $x+y\neq -2$: $P(\frac{x+y}{2},\frac{x-y}{x+y-2}) \implies f(x)=f(\frac{x+y}{2})+f(\frac{x-y}{x+y-2})+f(\frac{x^2-y^2}{x+y-2})$

$P(\frac{x+y}{2},\frac{y-x}{x+y-2}) \implies f(y)=f(\frac{x+y}{2})-f(\frac{x-y}{x+y-2})-f(\frac{x^2-y^2}{x+y-2})$

Adding these two lines gives new assertion $Q(x,y): f(x)+f(y) = 2f(\frac{x+y}{2})$ $\forall x,y$ such that $x+y \neq -2$ $Q(x+y,0) \implies f(x+y) = 2f(\frac{x+y}{2})$ and so $f(x+y) = f(x)+f(y)$ Q.E.D.

2.2) $f(x+y) = f(x)+f(y)$ $\forall x,y$ such that $x+y=-2$

If $x=-2$, then $y=0$ and $f(x+y)=f(x)+f(y)$ If $x \neq -2$, then $(x+2)+(-2) \neq -2$ and then $R(x+2,-2) \implies f(x) = f(x+2)+f(-2)$ and so $f(x)+f(-2-x) = f(-2)$ and so $f(x)+f(y) = f(x+y)$

Q.E.D.

13. find all functions $f: R \longrightarrow R$ such that $f(f(x)+y) = 2x + f(f(y)-x)$ for all x,y reals

solution

Let $P(x,y)$ be the assertion $f(f(x)+y) = 2x + f(f(y)-x)$

$P(\frac{f(0)-x}{2}, -f(\frac{f(0)-x}{2})) \implies x = f(f(-f(\frac{f(0)-x}{2})) - \frac{f(0)-x}{2})$ and so $f(x)$ is surjective.

So : $\exists u$ such that $f(u)=0$ $\exists v$ such that $f(v) = x+u$

And then $P(u,v) \implies f(x) = x-u$ which indeed ,is a solution

Hence the answer : $\boxed{f(x) = x+c}$

14. find all functions $f: R \longrightarrow R$ such that $f(x^2+f(y)) = y + f(x)^2$ for all x,y reals

solution

Let $P(x,y)$ be the assertion $f(x^2+f(y)) = y+f(x)^2$

$P(0,y) \implies f(f(y)) = y+f(0)^2$ and then : $P(x, f(y-f(0)^2)) \implies f(x^2+y) = f(y-f(0)^2)+f(x)^2$ Setting $x=0$ in this last equality, we get $f(y) = f(y-f(0)^2)+f(0)^2$ and so $f(x^2+y) = f(y)+f(x)^2-f(0)^2$ Setting $y=0$ in this last equality, we get $f(x^2) = f(0)+f(x)^2-f(0)^2$ and so $f(x^2+y) = f(y)+f(x^2)-f(0)$

Let then $g(x) = f(x)-f(0)$. We got $g(x+y) = g(x)+g(y)$ $\forall x \geq 0, \forall y$ It's immediate to establish $g(0)=0$ and $g(-x) = -g(x)$ and so $g(x+y) = g(x)+g(y)$ $\forall x,y$

$P(x,0) \implies f(x^2+f(0)) = f(x)^2 \implies f(x^2+f(0))-f(0) = f(x)^2-f(0)$ and so $g(x) \geq -f(0)$ $\forall x \geq f(0)$

So $g(x)$ is a solution of Cauchy equation with a lower bound on some non empty open interval. So $g(x) = ax$ and $f(x) = ax+b$

Plugging this back in original equation, we get $a=1$ and $b=0$ and the unique solution $\boxed{f(x) = x}$

15. Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f : (0,1] \to \mathbb{R}$ such that

$$a + f(x+y-xy) + f(x)f(y) \le f(x) + f(y)$$

for all $x, y \in (0,1]$.

solution

Let $g(x)$ from $[0,1) \to \mathbb{R}$ such that $g(x) = f(1-x) - 1$ $a + f(x+y-xy) + f(x)f(y) \le f(x) + f(y) \iff g((1-x)(1-y)) + g(1-x)g(1-y) \le -a$ $\iff g(xy) + g(x)g(y) \le -a$ $\forall x, y \in [0,1)$

Let $P(x,y)$ be the assertion $g(xy) + g(x)g(y) \le -a$

$P(0,0) \implies g(0) + g(0)^2 \le -a \iff a \le \frac{1}{4} - (g(0) + \frac{1}{2})^2$ and so $a \le \frac{1}{4}$

If $a < \frac{1}{4}$: Let us consider $g(x) = -\frac{1}{2}$ $\forall x \in (0,1)$ and $g(0) = -\frac{1}{2} - \sqrt{\frac{1}{4} - a} \ne -\frac{1}{2}$ (so that $g(x)$ is not constant) : If $x = y = 0$: $g(xy) + g(x)g(y) = -a \le -a$ If $x = 0$ and $y \ne 0$: $g(xy) + g(x)g(y) = -\frac{1}{4} - \frac{1}{2}\sqrt{\frac{1}{4} - a} < -\frac{1}{4} < -a$ If $x, y \ne 0$: $g(xy) + g(x)g(y) = -\frac{1}{4} < -a$

If $a = \frac{1}{4}$: $P(0,0) \implies g(0) + g(0)^2 \le -\frac{1}{4}$ and so $g(0) = -\frac{1}{2}$ $P(x,0) \implies g(x) \ge -\frac{1}{2}$ $P(\sqrt{x}, \sqrt{x}) \implies g(x) + g(\sqrt{x})^2 \le -\frac{1}{4} \implies g(x) \le -\frac{1}{4}$ Let then the sequence u_n defined as : $u_0 = -\frac{1}{4}$ $u_{n+1} = -\frac{1}{4} - a_n^2$ It's easy to show with induction that $-\frac{1}{2} \le g(x) \le a_n < 0$ $\forall x \in [0,1)$ It's then easy to show that a_n is a decreasing sequence whose limit is $-\frac{1}{2}$ And so the unique solution for $a = \frac{1}{4}$ is $g(x) = -\frac{1}{2}$ which is not a solution (since constant).

Hence the answer : $\boxed{a \in (-\infty, \frac{1}{4})}$

16. Find all functions $f : \mathbb{Q} \mapsto \mathbb{C}$ satisfying

(i) For any $x_1, x_2, \ldots, x_{2010} \in \mathbb{Q}$, $f(x_1+x_2+\ldots+x_{2010}) = f(x_1)f(x_2)\ldots f(x_{2010})$.

(ii) $\overline{f(2010)}f(x) = f(2010)\overline{f(x)}$ for all $x \in \mathbb{Q}$. [

solution

Let $a = f(0)$

Using $x_1 = x_2 = \ldots = x_p = x$ and $x_{p+1} = \ldots = x_{2010} = 0$, (i) $\implies$ $f(px) = a^{2010-p} f(x)^p$ $\forall x \in \mathbb{Q}, \forall 0 \le p \le 2010 \in \mathbb{Z}$

Setting $x = 0$ in the above equation, we get $a = a^{2010}$ and so : Either $a = 0$ and so $f(x) = 0$ $\forall x$, which indeed is a solution. Either $a^{2009} = 1$ and we get $f(px) = a^{1-p} f(x)^p$

Let then $g(x) = \frac{f(x)}{a}$ and we got $g(px) = g(x)^p$ $\forall 0 \le p \le 2010 \in \mathbb{Z}$ A simple induction using (i) shows that $g(px) = g(x)^p$ $\forall p \in \mathbb{N} \cup \{0\}$

And it's then immediate to get $g(\frac{x}{p}) = g(x)^{\frac{1}{p}}$ and so $g(x) = c^x$ $\forall x \in \mathbb{Q}$

So $f(x) = a \cdot c^x$ (ii) implies then $c = \bar{c}$ and so $c \in \mathbb{R}$

Hence the solutions : $f(x) = 0\ \forall x$

$f(x) = e^{i\frac{2k\pi}{2009}}c^x$ with $k \in \mathbb{Z}$ and $c \in \mathbb{R}$ (according to me, better to say $c \in \mathbb{R}^+$)

17. Find all functions $f : \mathbb{R} \to \mathbb{R}$, satisfying: $f(x) = max_{y\in\mathbb{R}}(2xy - f(y))$ for all $x \in \mathbb{R}$.

solution

1) $f(x) \geq x^2\ \forall x$ ==== $f(x) \geq 2xy - f(y)\ \forall x, y$. Choosing $y = x$, we get $f(x) \geq x^2$ Q.E.D

2) $f(x) \leq x^2\ \forall x$ ==== Let $x \in \mathbb{R}$ Since $f(x) = \max_{y\in\mathbb{R}}(2xy - f(y))$, $\exists$ a sequence y_n such that $\lim_{n\to+\infty}(2xy_n - f(y_n)) = f(x)$

So $\lim_{n\to+\infty}(f(y_n) - y_n^2 + (x - y_n)^2) = x^2 - f(x)$ And since we know that $f(y_n) - y_n^2 \geq 0$, then $LHS \geq 0$ and so $RHS \geq 0$ Q.E.D

So $\boxed{f(x) = x^2}$ which indeed is a solution

18. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$. [

solution

Let $P(x, y)$ be the assertion $f(f(x) + y) = f(x^2 - y) + 4f(x)y$

$P(x, \frac{x^2-f(x)}{2}) \implies f(x)(f(x) - x^2) = 0$ and so : $\forall x$, either $f(x) = 0$, either $f(x) = x^2$

$f(x) = 0\ \forall x$ is a solution $f(x) = x^2\ \forall x$ is also a solution.

Suppose now that $\exists a \neq 0$ such that $f(a) = 0$ Then if $\exists b \neq 0$ such that $f(b) \neq 0$: $f(b) = b^2$ and $P(a, b) \implies b^2 = f(a^2 - b)$ and so $b^2 = (a^2 - b)^2$ and so $b = \frac{a^2}{2}$ So there is a unique such b (equal to $\frac{a^2}{2}$) But then there at at most two such a (a and $-a$) And it is is impossible to have at most one $x \neq 0$ such that $f(x) = x^2$ and at most two $x \neq 0$ such that $f(x) = 0$

So we have only two solutions : $f(x) = 0\ \forall x$ $f(x) = x^2\ \forall x$

19. Find all continous functions $\mathbb{R} \to \mathbb{R}$ such that :
$f(x + f(y + f(z))) = f(x) + f(f(y)) + f(f(f(z)))$

solution

Let $P(x, y, z)$ be the assertion $f(x + f(y + f(z))) = f(x) + f(f(y)) + f(f(f(z)))$

Subtracting $P(0, y - f(z), z)$ from $P(x, y - f(z), z)$, we get $f(x + f(y)) = f(x) + f(f(y)) - f(0)$ Let $g(x) = f(x) - f(0)$ and $A = f(\mathbb{R})$

We got $g(x+y)=g(x)+g(y)\ \forall x\in\mathbb{R},\ \forall y\in A$ And also $g(x-y)=g(x)-g(y)\ \forall x\in\mathbb{R},\ \forall y\in A$

$g(x+y_1+y_2)=g(x+y_1)+g(y_2)=g(x)+g(y_1)+g(y_2)=g(x)+g(y_1+y_2)$ $\forall x\in\mathbb{R},\ \forall y_1,y_2\in A$ $g(x+y_1-y_2)=g(x+y_1)-g(y_2)=g(x)+g(y_1)-g(y_2)=g(x)+g(y_1-y_2)\ \forall x\in\mathbb{R},\ \forall y_1,y_2\in A$

And, with simple induction, $g(x+y)=g(x)+g(y)\ \forall x,\ \forall y$ finite sums and differences of elements of A

If cardinal of A is 1, we get $f(x)=c$ and so $f(x)=0$ If cardinal of A is not 1 and since $f(x)$ is continuous, $\exists u<v$ such that $[u,v]\subseteq A$ and any real may be represented as finite sums and differences of elements of $[u,v]$

So $g(x+y)=g(x)+g(y)\ \forall x,y$ and so, since continuous, $g(x)=ax$ and $f(x)=ax+b$

Plugging this in original equation, we get $b(a+2)=0$

Hence the solutions : $f(x)=ax$ $f(x)=b-2x$

20. Let a be a real number and let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying: $f(0)=\frac{1}{2}$ and $f(x+y)=f(x)f(a-y)+f(y)f(a-x)$, $\forall x,y\in\mathbb{R}$. Prove that f is constant.

solution

Let $P(x,y)$ be the assertion $f(x+y)=f(x)f(a-y)+f(y)f(a-x)$

$P(0,0)\implies f(a)=\frac{1}{2}$ $P(x,0)\implies f(x)=f(a-x)$ and so $P(x,y)$ may also be written $Q(x,y):\ f(x+y)=2f(x)f(y)$

$Q(a,-x)\implies f(a-x)=f(-x)$ and so $f(x)=f(-x)$

Then, comparing $Q(x,y)$ and $Q(x,-y)$, we get $f(x+y)=f(x-y)$ and choosing $x=\frac{u+v}{2}$ and $y=\frac{u-v}{2}$, we get $f(u)=f(v)$

21. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)^3=-\frac{x}{12}\cdot\left(x^2+7x\cdot f(x)+16\cdot f(x)^2\right),\ \forall x\in\mathbb{R}.$$

solution

This equation may be written $(f(x)+\frac{x}{2})^2(f(x)+\frac{x}{3})=0$ and so 4 solutions :

S1 : $f(x)=-\frac{x}{2}\ \forall x$

S2 : $f(x)=-\frac{x}{3}\ \forall x$

S3 : $f(x)=-\frac{x}{2}\ \forall x<0$ and $f(x)=-\frac{x}{3}\ \forall x\geq 0$

S4 : $f(x)=-\frac{x}{2}\ \forall x>0$ and $f(x)=-\frac{x}{3}\ \forall x\leq 0$

22. Let $f(x)$ be a real-valued function defined on the positive reals such that

(1) if $x < y$, then $f(x) < f(y)$,

(2) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$ for all x.

Show that $f(x) < 0$ for some value of x. [

solution

1) $f(x)$ is concave. ====

If $x < y$: $\frac{x+y}{2} > \frac{2xy}{x+y}$ and so $f(\frac{x+y}{2}) > \frac{f(x)+f(y)}{2}$ Using this plus the fact that $f(x)$ is stricly increasing, we get immediately the result.

2) $\frac{f(x)-f(\frac{x}{2})}{\frac{x}{2}} \geq 2\frac{f(2x)-f(x)}{x}$ ==

Let $a > 1$. From the original inequality, using $y = ax$, we get $f(\frac{2a}{a+1}x) \geq \frac{f(x)+f(ax)}{2}$

$\implies f(\frac{2a}{a+1}x) - f(x) \geq \frac{f(ax)-f(x)}{2}$

$\implies \frac{f(\frac{2a}{a+1}x)-f(x)}{\frac{2a}{a+1}x-x} \geq \frac{a+1}{2}\frac{f(ax)-f(x)}{ax-x}$

Let then the sequence a_n defined as $a_1 = 2$ and $a_{n+1} = \frac{2a_n}{a_n+1}$. We got : $\frac{f(a_{n+1}x)-f(x)}{a_{n+1}x-x} \geq \frac{a_n+1}{2}\frac{f(a_nx)-f(x)}{a_nx-x}$

And, since $f(x)$ is concave, we get also $\frac{f(x)-f(\frac{x}{2})}{\frac{x}{2}} \geq \frac{f(a_nx)-f(x)}{a_nx-x}$

And so $\frac{f(x)-f(\frac{x}{2})}{\frac{x}{2}} \geq \left(\prod_{k=1}^{n}\frac{a_k+1}{2}\right)\frac{f(2x)-f(x)}{x}$

And since $\prod_{k=1}^{+\infty}\frac{a_k+1}{2} = 2$, we got the required result in title of paragraph 2. (just write $\frac{a_k+1}{2} = \frac{a_k}{a_{k+1}}$).

3) Final result ==

From 2), we got $f(x) - f(\frac{x}{2}) \geq f(2x) - f(x)$

And so $f(\frac{x}{2}) - f(\frac{x}{4}) \geq f(x) - f(\frac{x}{2}) \geq f(2x) - f(x)$ ... $f(\frac{x}{2^{n-1}}) - f(\frac{x}{2^n}) \geq f(2x) - f(x)$

... and so (summing these lines) : $f(x) - f(\frac{x}{2^n}) \geq n(f(2x) - f(x))$

Which may be written $f(\frac{x}{2^n}) \leq f(x) - n(f(2x) - f(x))$

And, since $f(2x) > f(x)$, and choosing n great enough, we get $f(\frac{x}{2^n}) < 0$

23. Find all functions f: $\mathbb{R} \to \mathbb{R}$ satisfying:

$$f(xf(y) + f(x)) = 2f(x) + xy$$

solution

Let $P(x,y)$ be the assertion $f(xf(y)+f(x)) = 2f(x)+xy$

$P(1,x-2f(1)) \implies f(\text{something}) = x$ and $f(x)$ is surjective. If $f(a) = f(b)$, subtracting $P(1,a)$ from $P(1,b)$ implies $a=b$ and $f(x)$ is injective, and so bijective.

Let $f(0) = a$ and u such that $f(u) = 0$

$P(u,0) \implies f(au) = 0 = f(u)$ and so, since injective, $au = u$

If $u = 0$, then $a = 0$ and $P(x,0) \implies f(f(x)) = 2f(x)$ and so, since surjective, $f(x) = 2x$ which is not a solution.

So $u \neq 0$ and $a = 1$. Then $P(u,u) \implies 1 = u^2$ and so $u = \pm 1$ If $u = 1$, $P(0,-1) \implies 0 = 2$, impossible.

So $a = 0$ and $u = -1$: $f(-1) = 0$ and $f(0) = 1$ and $P(0,-1) \implies f(1) = 2$

$P(-1,x) \implies f(-f(x)) = -x$ $P(x,-f(1)) \implies f(f(x)-x) = 2(f(x)-x)$

Let then $x \in \mathbb{R}$ and z such $f(z) = f(x)-x$ which exists since $f(x)$ is surjective. Using last equation, we get $f(f(z)) = 2f(z)$ $P(z,-1) \implies f(f(z)) = 2f(z)-z$

And so $z = 0$ and $f(z) = 1$ and $f(x) = x+1$, which indeed is a solution.

Hence the answer : $\boxed{f(x) = x+1}$

24. Find all one-one (injective)functions $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, which satisfies

$$f(f(n)) \leq \frac{f(n)+n}{2}$$

solution

It's easy to show with induction that $f^{[k]}(n) \leq \frac{2f(n)+n}{3} + \frac{2}{3(-2)^k}(n-f(n))$

So, for k great enough : $f^{[k]}(n) \leq \frac{2f(n)+n}{3} + 1$ and so $\exists\ k_1 > k_2$ such that $f^{[k_1]}(n) = f^{[k_2]}(n)$ and, since injective :

$\forall n\ \exists p_n \geq 1$ such that $f^{[p_n]}(n) = n$

Then, setting $k = p_n$ in the above inequality, we get $n \leq \frac{2f(n)+n}{3} + \frac{2}{3(-2)^{p_n}}(n-f(n))$

$\iff 0 \leq (f(n)-n)(1-\frac{1}{(-2)^{p_n}})$ and so $f(n) \geq n\ \forall n$

But $f(n) > n$ for some n and injectivity would imply $f^{[p_n]}(n) > n$ and so $f(n) = n\ \forall n$ which indeed is a solution

25. For a given natural number $k > 1$, find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $f[x^k + f(y)] = y + [f(x)]^k$.

solution

Let $P(x,y)$ be the assertion $f(x^k+f(y))=y+f(x)^k$ Let $f(0)=a$

$P(0,y) \implies f(f(y))=y+a^k$ $P(x,0) \implies f(x^k+a)=f(x)^k$ $P(x,f(y)))$ $\implies f(x^k+y+a^k)=f(y)+f(x^k+a)$

Let then $g(x)=f(x-a^k+a)$. This last equality becomes $g(x^k+y+2a^k-a)=g(y+a^l-a)+g(x^k+a^k) \iff g(x^k+a^k+y)=g(y)+g(x^k+a^k)$

And so $g(x+y)=g(x)+g(y)$ $\forall x\ge a^k,\forall y$ Let then $x\ge 0:$ $g(a^k+x+y)=g(a^k+(x+y))=g(a^k)+g(x+y)$ $g(a^k+x+y)=g((a^k+x)+y)=g(a^k+x)+g(y)=g(a^k)+g(x)+g(y)$ And so $g(x+y)=g(x)+g(y)$ $\forall x\ge 0,\forall y$

So $g(0)=0$ and $g(-x)=-g(x)$. Then : $\forall x\ge 0,\forall y:$ $-g(x-y)=-g(x)-g(-y) \implies g(-x+y)=g(-x)+g(y)$ and so $g(x+y)=g(x)+g(y)$ $\forall x,y$

And so $g(px)=pg(x)$ $\forall p\in\mathbb{Q},\forall x$

Then $f(x^k+a)=f(x)^k$ implies $g(x^k+a^k)=g(x+a^k-a)^k \implies g(x^k)+g(a^k)=(g(x)+g(a^k-a))^k$ Notice that $g(a^k-a)=f(0)=a$ and replace x with $x+y$ and we get :

$g((x+y)^k)+g(a^k)=(g(x)+g(y)+a)^k$

$g(\sum_{i=0}^{k}\binom{k}{i}x^iy^{k-i})+g(a^k)=\sum_{i=0}^{k}\binom{k}{i}g(x)^i(g(y)+a)^{k-i}$

Let then $x\in\mathbb{Q}$ and this equation becomes :

$\sum_{i=0}^{k}\binom{k}{i}x^ig(y^{k-i})+g(a^k)=\sum_{i=0}^{k}\binom{k}{i}g(1)^ix^i(g(y)+a)^{k-i}$

And so we have two polynomials in x (LHS and RHS) which are equal for any $x\in\mathbb{Q}$. So they are identical and all their coefficients are equal.

Since $k\ge 2$, consider the equality of coefficients of x^{k-2} : If $k>2$, this equality is $g(y^2)=g(1)^{k-2}(g(y)+a)^2$ and $g(x)$ has a constant sign over $\mathbb{R}^+$ If $k=2$, this equality becomes $g(y^2)+g(a^2)=(g(y)+a)^2$ and $g(x)\ge -g(a^2)$ $\forall x\ge 0$

In both cases, we have $g(x)$ either upper bounded, either lower-bounded on a non empty open interval, and this a classical condition to conclude to continuity and $g(x)=cx$ $\forall x$

And so $f(x)=cx+d$ for some real c,d

Plugging this back in original equation, we get :

$f(x)=x$ $\forall x$ which is a solution for any k $f(x)=-x$ $\forall x$ which is another solution if k is odd

26. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x,y\in\mathbb{R}$,

    $(x+y)(f(x)-f(y))=(x-y)(f(x)+f(y))$

solution

Expanding, we get

$xf(x) - xf(y) + yf(x) - yf(y) = xf(x) - yf(x) + xf(y) - yf(y)$

Simplifying,

$2yf(x) = 2xf(y)$

$yf(x) = xf(y)$

$\dfrac{f(x)}{x} = \dfrac{f(y)}{y}$

Let $g(x) = \dfrac{f(x)}{x}$. Since $g(x) = g(y)$ for all x and y, $g(x) = k$ where k is a constant. Thus,

$g(x) = k = \dfrac{f(x)}{x}$

$f(x) = kx$

27. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$:

$$f(x - y + f(y)) = f(x) + f(y).$$

solution

Let $P(x, y)$ be the assertion $f(x - y + f(y)) = f(x) + f(y)$ Let $f(0) = a$

$P(0, 0) \implies f(a) = 2a$ and so $f(a) - a = a$ $P(0, a) \implies f(f(a) - a) = f(0) + f(a)$ and so $f(0) = 0$

$P(0, x) \implies f(f(x) - x) = f(x)$ $P(x, f(y) - y) \implies f(x - f(y) + y + f(f(y) - y)) = f(x) + f(f(y) - y)$ and so $f(x + y) = f(x) + f(y)$ and so $f(x) = xf(1)$ (remember we are in $\mathbb{Z}$)

Plugging this in original equation, we get two solutions :

$f(x) = 0\ \forall x$ $f(x) = 2x\ \forall x$

28. We denote by $\mathbb{R}+$ the set of all positive real numbers.

Find all functions $f : \mathbb{R}+ \to \mathbb{R}+$ which have the property: $f(x)f(y) = 2f(x + yf(x))$

for all positive real numbers x and y.

solution

Let $P(x, y)$ be the assertion $f(x)f(y) = 2f(x + yf(x))$

Let $u, v > 0$. Let $a \in (0, u)$

Let $x = a > 0$ and $y = \frac{u-a}{f(a)} > 0$ and $z = \frac{2v}{f(x)f(y)} > 0$

$f(x)f(y) = 2f(x + yf(x)) = 2f(u)$ and so $f(x)f(y)f(z) = 2f(u)f(z) = 4f(u + zf(u)) = 4f(u + v)$

$f(y)f(z) = 2f(y + zf(y))$ and so $f(x)f(y)f(z) = 2f(x)f(y + zf(y)) = 4f(x + (y + zf(y))f(x)) = 4f(x + yf(x) + zf(x)f(y)) = 4f(u + 2v)$

And so $f(u+v)=f(u+2v)\ \forall u,v>0$ and so $f(x)=f(y)\ \forall x,y$ such that $2x>y>x>0$

And it's immediate from there to conclude $f(x)=f(y)\ \forall x,y>0$

Hence the unique solution $\boxed{f(x)=2\forall x>0}$

29. Find all continuous functions $\mathbb{R}\to\mathbb{R}$ satisfying the equation: $f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$

solution

Let $P(x,y,z)$ be the assertion

$f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$

$P(x,y,y)\implies f(x+2y)-f(x+y)=f(x+y)-f(x)+(f(2y)+f(0)-2f(y))$
$P(x+y,y,y)\implies f(x+3y)-f(x+2y)=f(x+2y)-f(x+y)+(f(2y)+f(0)-2f(y))$ ... $P(x+(n-1)y,y,y)\implies f(x+(n+1)y)-f(x+ny)=f(x+ny)-f(x+(n-1)y)+(f(2y)+f(0)-2f(y))$

Adding these lines gives $f(x+(n+1)y)-f(x+ny)=f(x+y)-f(x)+n(f(2y)+f(0)-2f(y))$

And so (adding this last lines for $n=0,...,k-1)$: $f(x+ky)-f(x)=k(f(x+y)-f(x))+\frac{k(k-1)}{2}(f(2y)+f(0)-2f(y))$

Setting $x=0$ in this last equality and renaming $y\to x$ and $k\to n$, we get :

$f(nx)=\frac{f(2x)+f(0)-2f(x)}{2}n^2+\frac{4f(x)-f(2x)-3f(0)}{2}n+f(0)$

So : $f(q\frac{p}{q})=\frac{f(2\frac{p}{q})+f(0)-2f(\frac{p}{q})}{2}q^2+\frac{4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0)}{2}q+f(0)$

And since $f(q\frac{p}{q})=f(p)=\frac{f(2)+f(0)-2f(1)}{2}p^2+\frac{4f(1)-f(2)-3f(0)}{2}p+f(0)$, we get :

$(f(2)+f(0)-2f(1))p^2+(4f(1)-f(2)-3f(0))p=(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2+(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))q$

Replacing $p\to np$ and $q\to nq$ in this equation, we get :

$(f(2)+f(0)-2f(1))p^2n^2+(4f(1)-f(2)-3f(0))pn=(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2n^2+(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))qn$ and so :

$n^2\left((f(2)+f(0)-2f(1))p^2-(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2\right)+n\left((4f(1)-f(2)-3f(0))p-(4f(\frac{p}{q})-f\right.$
0

And since this is true for any n, we get : $(f(2)+f(0)-2f(1))p^2-(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2=0$ $(4f(1)-f(2)-3f(0))p-(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))q=0$

From these two lines, we get $f(\frac{p}{q})=\frac{f(2)+f(0)-2f(1)}{2}\frac{p^2}{q^2}+\frac{4f(1)-f(2)-3f(0)}{2}\frac{p}{q}+f(0)$

And so $f(x) = ax^2 + bx + c \ \forall x \in \mathbb{Q}^+$ which indeed fits whatever are a, b, c.

So $f(x) = ax^2 + bx + c \ \forall x \in \mathbb{R}^+$ (using continuity)

Let then $x > 0$: $P(-x, x, x) \implies f(-x) + 3f(x) = f(2x) + 3f(0)$ and, since $x \geq 0$ and $2x \geq 0$:

$f(-x) = (4ax^2 + 2bx + c) + 3c - 3(ax^2 + bx + c) = ax^2 - bx + c$

And so $f(x) = ax^2 + bx + c \ \forall x \in \mathbb{R}$

30. Find all continuous functions $f : R \to R$ that satisfy $f(x+y)+f(xy)+1 = f(x) + f(y) + f(xy + 1) \ \forall x, y \in R$.

solution

Let $P(x, y)$ be the assertion $f(x+y)+f(xy)+1 = f(x)+f(y)+f(xy+1)$

1) Let us solve the easier equation $(E1)$: ============================ "Find all functions $g(x)$ from $\mathbb{N} \to \mathbb{R}$ such that : $g(2x+y)-g(2x)-g(y) = g(2y + x) - g(2y) - g(x) \ \forall x, y \in \mathbb{N}$"

The set $\mathbb{S}$ of solutions is a $\mathbb{R}$-vector space. Setting $y = 1$, we get $g(2x+1) = g(2x) + g(1) + g(x + 2) - g(2) - g(x)$ Setting $y = 2$, we get $g(2x + 2) = g(2x) + g(2) + g(x + 4) - g(4) - g(x)$ From these two equations, we see that knowledge of $g(1), g(2), g(3), g(4)$ and $g(6)$ gives knowledge of $g(x)$ $\forall x \in \mathbb{N}$ and so dimension of $\mathbb{S}$ is at most 5. But the 5 functions below are independant solutions : $g_1(x) = 1$ $g_2(x) = x$ $g_3(x) = x^2$ $g_4(x) = 1$ if $x = 0 \pmod 2$ and $g_4(x) = 0$ if $x \neq 0 \pmod 2$ $g_5(x) = 1$ if $x = 0 \pmod 3$ and $g_5(x) = 0$ if $x \neq 0 \pmod 3$ And the general solution of $(E1)$ is $g(x) = a \cdot x^2 + b \cdot x + c + d \cdot g_4(x) + e \cdot g_5(x)$

2) Solutions of the original equation : ========================= $P(x, 0) \implies f(1) = 1$ Comparing $P(xy, z)$ and $P(xz, y)$, we get $Q(x, y, z)$: $f(xy + z) - f(xy) - f(z) = f(xz + y) - f(xz) - f(y)$

2.1) $f(x) = ax^2 + bx + c \ \forall x > 0$ ————————————— Let p a positive integer. $Q(2, \frac{m}{p}, \frac{n}{p}) \implies f(\frac{2m+n}{p}) - f(\frac{2m}{p}) - f(\frac{n}{p}) = f(\frac{2n+m}{p}) - f(\frac{2n}{p}) - f(\frac{m}{p})$

So $f(\frac{x}{p})$ is a solution of $(E1)$ and so $f(\frac{x}{p}) = a_p \cdot x^2 + b_p \cdot x + c_p + d_p \cdot g_4(x)e_p \cdot g_5(x) \ \forall x \in \mathbb{N}$ Choosing $x = kp$, it's easy to see that $a_p = \frac{a}{p^2}$, then that $b_p = \frac{b}{p}$ Choosing $x = 2kp$, $x = 3kp$ and $x = 6kp$, it's easy to see that $c_p = c$ and $d_p = e_p = 0$

And so $f(\frac{x}{p}) = a(\frac{x}{p})^2 + b(\frac{x}{p}) + c \ \forall x, p \in \mathbb{N}$ And so $f(x) = ax^2 + bx + c$ $\forall x \in \mathbb{Q}^{+*}$

Now, $f(x)$ continuous implies $f(x) = ax^2 + bx + c \ \forall x \in \mathbb{R}^+$ Q.E.D.

2.2) $f(x) = a'x^2 + b'x + c' \ \forall x < 0$ ————————————— - $Q(2, -\frac{m}{p}, -\frac{n}{p}) \implies f(-\frac{2m+n}{p}) - f(-\frac{2m}{p}) - f(-\frac{n}{p}) = f - \frac{2n+m}{p}) +$

$f(-\frac{2n}{p}) - f(-\frac{m}{p})$ So $f(-\frac{x}{p})$ is a solution of $(E1)$ and the same method as in 2.1 above gives the result.

2.3) $f(x) = ax^2 + bx + 1 - a - b\ \forall x$ ———————————————————— We got $f(x) = ax^2 + bx + c\ \forall x > 0$ and $f(x) = a'x^2 + b'x + c'\ \forall x < 0$

Continuity at 0 implies $c = c'$ and $f(1) = 1$ implies $c = 1 - a - b$ $P(-1, -1) \implies a' = a$ $P(-2, 3) \implies b' = b$ Q.E.D

It is then easy to check back that this necessary form is indeed a solution and we got the result :

$\boxed{f(x) = ax^2 + bx + 1 - a - b}\ \forall x$

31. Find all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ such that:

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)}.$$

solution

Let $P(x, y)$ be the assertion $f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)}$ Let $f(1) = a$

$P(1, 1) \implies f(2) = \frac{1}{4}$ $P(2, 2) \implies f(4) = \frac{1}{16}$ $P(2, 1) \implies f(3) = \frac{1}{4a+5}$ $P(3, 1) \implies f(4) = \frac{1}{4a^2+5a+7}$ and so $4a^2 + 5a + 7 = 16$ and so $a = 1$ (remember $f(x) > 0$)

$P(x, 1) \implies \frac{1}{f(x+1)} = \frac{1}{f(x)} + 2x + 1$ and so $\frac{1}{f(x+n)} = \frac{1}{f(x)} + 2nx + x^2$ and $f(n) = \frac{1}{n^2}$

$P(x, n) \implies f(nx) = \frac{f(x)+\frac{1}{n^2}}{\frac{1}{f(x)}+n^2}$

Setting $x = \frac{p}{n}$ in this last equality, we get $f(\frac{p}{n}) = \frac{n^2}{p^2}$ (remember $f(x) > 0$)

Hence the answer : $\boxed{f(x) = \frac{1}{x^2}}$ $\forall x \in \mathbb{Q}^+$ which indeed is a solution.

32. Find all continuous $f : R \to R$ such that for reals x, y - $f(x + f(y)) = y + f(x + 1)$

solution

Let $P(x, y)$ be the assertion $f(x + f(y)) = y + f(x + 1)$

$P(0, y+1-f(1)) \implies f(f(y+1-f(1))) = y+1$ $P(x-f(1), f(y+1-f(1))) \implies f(x - f(1) + f(f(y + 1 - f(1)))) = f(y + 1 - f(1)) + f(x + 1 - f(1))$ and so $f(x + y + 1 - f(1)) = f(y + 1 - f(1)) + f(x + 1 - f(1))$

Let then $g(x) = f(x + 1 - f(1))$ and we get $g(x + y) = g(x) + g(y)$ and so, since continuous, $g(x) = ax$ and $f(x) = a(x + f(1) - 1)$

Plugging $f(x) = ax + b$ in original equation, we get two solutions : $f(x) = 1 + x\ \forall x$ $f(x) = 1 - x\ \forall x$

33. $f : Z \to Z$ $f(m+n) + f(mn-1) = f(m)f(n) + 2$

solution

Let $P(x, y)$ be the assertion $f(x+y) + f(xy-1) = f(x)f(y) + 2$

$P(x, 0) \implies f(x)(f(0)-1) = f(-1) - 2$

If $f(0) \neq 1$, this implies $f(x) = c$ and $2c = c^2 + 2$ and no solution. So $f(0) = 1$ and $f(-1) = 2$

Let then $f(1) = a$ $P(1, 1) \implies f(2) = a^2 + 1$ $P(2, 1) \implies f(3) = a^3 + 2$ $P(3, 1) \implies f(4) = a^4 - a^2 + 2a + 1$ $P(2, 2) \implies f(4) = a^4 - a^3 + 2a^2 + 1$

And so $a^4 - a^2 + 2a + 1 = a^4 - a^3 + 2a^2 + 1 \iff a(a-1)(a-2) = 0$

If $a = 0$: Previous lines imply $f(2) = 1$ and $f(3) = 2$ and $f(4) = 1$ $P(4, 1) \implies f(5) = 0$ But $P(3, 2) \implies f(5) = 2$ and so contradiction

If $a = 1$: Previous lines imply $f(2) = 2$ and $f(3) = 3$ and $f(4) = 3$ $P(4, 1) \implies f(5) = 2$ But $P(3, 2) \implies f(5) = 4$ and so contradiction

If $a = 2$, then $P(m+1, 1) \implies f(m+2) = 2f(m+1) - f(m) + 2$ which is easily solved in $f(m) = m^2 + 1$ which indeed is a soluion.

Hence the unique solution : $\boxed{f(x) = x^2 + 1}$ $\forall x \in \mathbb{Z}$

34. Find All Functions $f : \mathbb{R} \to \mathbb{R}$ Such That $f(x-y) = f(x+y)f(y)$

solution

Let $P(x, y)$ be the assertion $f(x-y) = f(x+y)f(y)$

$P(0, 0) \implies f(0)^2 = f(0)$ and so $f(0) = 0$ or $f(0) = 1$

If $f(0) = 0$: $P(x, 0) \implies f(x) = 0$ $\forall x$ which indeed is a solution

If $f(0) = 1$: $P(x, x) \implies f(x)f(2x) = 1$ and so $f(x) \neq 0$ $\forall x$ $P(\frac{2x}{3}, \frac{x}{3}) \implies f(\frac{x}{3}) = f(x)f(\frac{x}{3})$ and, since $f(\frac{x}{3}) \neq 0$: $f(x) = 1$ which indeed is a solution.

Hence the two solutions : $f(x) = 0$ $\forall x$ $f(x) = 1$ $\forall x$

35. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) \cdot f(y) = f(x) + f(y) + f(xy) - 2 \quad \forall x, y \in \mathbb{R}.$$

solution

Setting $f(x) = g(x) + 1$, the equation becomes $g(xy) = g(x)g(y)$, very classical equation whose general solutions are : $g(x) = 1$ $\forall x$ $g(0) = 0$ and $g(x) = |x|^a$ $\forall x \neq 0$ where a is any non zero real. $g(0) = 0$ and $g(x) = sign(x)|x|^a$ $\forall x \neq 0$ where a is any non zero real.

Hence the three solutions of the required equation : $f(x) = 2\ \forall x$ $f(0) = 1$ and $f(x) = 1 + |x|^a\ \forall x \neq 0$ where a is any non zero real. $f(0) = 1$ and $f(x) = 1 + sign(x)|x|^a\ \forall x \neq 0$ where a is any non zero real

And so : ...$g(xy) = g(x)g(y)$, very classical :) equation whose general solutions are : $g(x) = 0\ \forall x$ $g(x) = 1\ \forall x$ $g(0) = 0$ and $g(x) = e^{h(\ln|x|)}$ $\forall x \neq 0$ where $h(x)$ is any solution of Cauchy's equation. $g(0) = 0$ and $g(x) = sign(x)e^{h(\ln|x|)}\ \forall x \neq 0$ where $h(x)$ is any solution of Cauchy's equation.

Hence the four solutions of the required equation : $f(x) = 1\ \forall x$ $f(x) = 2$ $\forall x$ $f(0) = 1$ and $f(x) = 1 + e^{h(\ln|x|)}\ \forall x \neq 0$ where $h(x)$ is any solution of Cauchy's equation. $f(0) = 1$ and $f(x) = 1 + sign(x)e^{h(\ln|x|)}\ \forall x \neq 0$ where $h(x)$ is any solution of Cauchy's equation

36. Find all functional $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy: $f(x^3+2y)+f(x+y) = g(x+2y)\ \forall x, y \in \mathbb{R}$

solution

If (f, g) is a solution, so is $(f + c, g + 2c)$ and so Wlog say $f(0) = 0$

Setting $y = 0$ in the equation gives $g(x) = f(x^3) + f(x)$ Pluging this in original equation, we get assertion $P(x, y)$: $f(x^3 + 2y) + f(x + y) = f((x + 2y)^3) + f(x + 2y)$

Setting $x = -y$ in the equation gives $g(y) = f(2y - y^3)$ and so $g(x) = f(2x - x^3)$ Pluging this in original equation, we get assertion $Q(x, y)$: $f(x^3 + 2y) + f(x + y) = f(2(x + 2y) - (x + 2y)^3)$

1) $f(x+\frac{1}{2}) = f(x)\ \forall x$ ==================== $P(1, x-\frac{1}{2}) \implies f(x+\frac{1}{2}) = f((2x)^3)$ $P(0, x) \implies f(x) = f((2x)^3)$ And so $f(x+\frac{1}{2}) = f(x)$ Q.E.D.

2) $f(x) = 0\ \forall x \in [0, 1]$ ============== Let $y \in (0, 1]$ $Q(x, y - x) \implies f(x^3 - 2x + 2y) + f(y) = f(2(2y - x) - (2y - x)^3)$ Consider now the equation $x^3 - 2x + 2y = 2(2y - x) - (2y - x)^3$ It may be written $(x - y)^2 = \frac{1-y^2}{3}$ and it has always at least one solution x since $y \in (0, 1]$

Choosing this value x, $f(x^3 - 2x + 2y) + f(y) = f(2(2y - x) - (2y - x)^3)$ becomes $f(y) = 0$ Q.E.D.

3) Solutions ======== 2) gave $f(x) = 0\ \forall x \in [0, 1]$ 1) gave $f(x+\frac{1}{2}) = f(x)$ So $f(x) = 0\ \forall x$ So $g(x) = 0\ \forall x$

Hence the answer : $\boxed{(f(x), g(x)) = (c, 2c)}$ for any real c

37. Find all functional $f : \mathbb{R} \to \mathbb{R}$ satisfy:$xf(x) - yf(y) = (x - y)f(x + y)$ for all $x, y \in \mathbb{R}$

solution

Let $P(x, y)$ be the assertion $xf(x) - yf(y) = (x - y)f(x + y)$

$P(\frac{x-1}{2}, \frac{1-x}{2}) \implies \frac{x-1}{2}f(\frac{x-1}{2}) - \frac{1-x}{2}f(\frac{1-x}{2}) = (x - 1)f(0)$

$P(\frac{1-x}{2},\frac{x+1}{2}) \implies \frac{1-x}{2}f(\frac{1-x}{2})-\frac{x+1}{2}f(\frac{x+1}{2})=-xf(1)$
$P(\frac{x+1}{2},\frac{x-1}{2}) \implies \frac{x+1}{2}f(\frac{x+1}{2})-\frac{x-1}{2}f(\frac{x-1}{2})=f(x)$
Adding these three lines, we get $f(x)-xf(1)+(x-1)f(0)=0$ and so $f(x)=(f(1)-f(0))x-f(0)$

And so $\boxed{f(x)=ax+b}$ which indeed is a solution

38. Find all continuous, strictly increasing functions $f:\mathbb{R}\to\mathbb{R}$ *such that* 1) $f(0)=0$ 2) $f(1)=1$ 3) $[f(x+y)]=[f(x)]+[f(y)]\ \ \forall\ \ x,y\in\mathbb{R}$ *such that* $[x+y]=[x]+[y]$

solution

a) $f(x)\in(0,1)\ \forall x\in(0,1)$ Trivial using 1) 2) and increasing property

b) $[f(n)]=n\ \forall n\in\mathbb{Z}$ $[m+n]=[m]+[n]\ \forall m,n\in\mathbb{Z}$ and so $[f(m+n)]=[f(m)]+[f(n)]$ and so $[f(n)]=n[f(1)]=n$

c) $[f(x)]\geq[x]\ \forall x$ $x\geq[x]$ and $f(x)$ increasing implies $f(x)\geq f([x])$ and so $[f(x)]\geq[f([x])]=[x]$

d) $[f(x)]<[x]+1\ \forall x$ If $[f(a)]\geq[a]+1$ for some a, then : $[f([a])]=[a]$ and so $f([a])<[a]+1$ Then continuity implies $\exists u\in([a],a)$ such that $f(u)=[a]+1$ Choosing then some $x\in([a],u)$ and $y=a-x\in(0,1)$ we get $[x+y]=[a]=[x]+[y]$ and so : $[f(x+y)]=[f(x)]+[f(y)]$ which is $[f(a)]=[f(x)]+[f(y)]$ which is wrong since $[f(a)]\geq[a]+1$ while $[f(x)]=[a]$ and $[f(y)]=0$ So no such a

From c),d) we get $[f(x)]=[x]$ and, plugging this in original equation, we get that any strictly increasing continuous function matching 1) and 2) and $[f(x)]=[x]$ matches 3) too.

$[f(x)]=[x]$ and continuity imply $f(n)=n$

[u][b]Hence the answer[/b][/u]: $f(x)$ solution if and only if : $f(x)=x$ $\forall x\in\mathbb{Z}$ $f(x)$ may take any values in $(n,n+1)$ when $x\in(n,n+1)$ with respect to the properties "strictly increasing and continuous"

39. Find All Functions $f:\mathbb{N}\to\mathbb{N}$ $f(m+f(n))=n+f(m+k)\ \ \forall\ m,n,k\in\mathbb{N}$ With k Being Fixed Natural Number

solution

If $f(n)<k$ for some n, then the equation may be written $f(m+(k-f(n)))=f(m)-n\ \forall m>f(n)$ So $f(m+p(k-f(n)))=f(m)-pn$, which is impossible, since this would imply $f(x)<0$ for some x great enough.

If $f(n)=k$ for some n, then the equation implies $n=0$, impossible

So $f(n)>k\ \forall n$ and the equation may be written $f(m+(f(n)-k))=n+f(m)\ \forall m>k$ And so $f(m+p(f(n)-k))=pn+f(m)$ Choosing then

$p = f(q) - k$, we get $f(m + (f(q) - k)(f(n) - k)) = (f(q) - k)n + f(m)$ and so, by symetry : $(f(q) - k)n = (f(n) - k)q$ $\forall q, n$ And so $\frac{f(q)-k}{q} = \frac{f(n)-k}{n}$ and so $f(n) = k + cn$ for some constant c

Plugging this in original equation, we get $c = 1$ and so solution $\boxed{f(n) = n + k}$

40. find all $f : R \to R$ such that $f(x)f(yf(x) - 1) = x^2 f(y) - f(x)$ for real x,y.

solution

$f(x) = 0$ $\forall x$ is a solution and let us from now look for non all-zero solutions.

Let $P(x, y)$ be the assertion $f(x)f(yf(x) - 1) = x^2 f(y) - f(x)$ Let u such that $f(u) \neq 0$

$P(1, 1) \implies f(1)f(f(1) - 1) = 0$ and so $\exists v$ such that $f(v) = 0$ $P(v, u)$ $\implies v^2 f(u) = 0$ and so $v = 0$

So $f(x) = 0 \iff x = 0$ and we got $f(1) = 1$

$P(1, x) \implies f(x - 1) = f(x) - 1$ and so $P(x, y)$ may be written : New assertion $Q(x, y)$: $f(x)f(yf(x)) = x^2 f(y)$

Let $x \neq 0$: $Q(x, x) \implies f(xf(x)) = x^2$ and so any $x \geq 0$ is in $f(\mathbb{R})$

$Q(x, y) \implies f(x)f(yf(x)) = x^2 f(y)$ $Q(x, 1) \implies f(x)f(f(x)) = x^2$ $Q(x, y + 1) \implies f(x)f(yf(x) + f(x)) = x^2 f(y) + x^2$

And so $f(x)f(yf(x) + f(x)) = f(x)f(yf(x)) + f(x)f(f(x))$

Choosing then $z > 0$ and x such that $f(x) = z$, we get : $f(yz + z) = f(yz) + f(z)$ and so $f(x + y) = f(x) + f(y)$ $\forall x > 0$, $\forall y$

And this immediately implies $f(x + y) = f(x) + f(y)$ $\forall x, y$ ($x = 0$ is obvious and using $y = -x$, we get $f(-x) = -f(x)$)

$Q(x, 1) \implies f(x)f(f(x)) = x^2$ $Q(x+1, 1) \implies (f(x)+1)(f(f(x))+1) = x^2 + 2x + 1$ And so $f(f(x)) + f(x) = 2x$

And combinaison of $f(x)f(f(x)) = x^2$ and $f(f(x)) + f(x) = 2x$ implies $(f(x) - x)^2 = 0$ and so $f(x) = x$ $\forall x$, which indeed is a solution

[u][b]Hence the solutions [/b][/u]: $f(x) = 0$ $\forall x$ $f(x) = x$ $\forall x$

41. Prove that there is no function like $f : \mathbb{R}_+ \to \mathbb{R}$ such that for all positive x, y :

$f(x + y) > y(f(x)^2)$

solution

Let $P(x, y)$ be the assertion $f(x + y) > yf(x)^2$

Let $x > 0$: $P(\frac{x}{2}, \frac{x}{2}) \implies f(x) > 0$ $\forall x$

Let then $a > 0$ and $x \in [0, a]$: $P(x, 2a - x) \implies f(2a) > (2a - x)f(x)^2 \geq af(x)^2$ and so $f(x)^2 < \frac{f(2a)}{a}$

And so $f(x)$ is upper bounded over any interval $(0,a]$

Let then $f(1)=u>0$ and the sequence $x_0=1$ and $x_{n+1}=x_n+\frac{2}{f(x_n)}$ $\forall n\ge 0$:

$P(x_n,\frac{2}{f(x_n)})$ $\implies$ $f(x_{n+1})>2f(x_n)$ and so $f(x_n)>2^nu$ $\forall n>0$

So $x_1=1+\frac{2}{u}$ and $x_{n+1}<x_n+\frac{1}{2^{n-1}u}$ $\forall n>0$

So $x_n<1+\frac{1}{u}(2+1+\frac{1}{2}+\frac{1}{4}+...\frac{1}{2^{n-1}})<1+\frac{4}{u}$

But $f(x_n)>2^nu$ and $x_n<1+\frac{4}{u}$ shows that $f(x)$ is not upper bounded over $(0,1+\frac{4}{u}]$, and so contradiction with the first sentence of this proof.

So no such function.

42. Let f be a function defined for positive integers with positive integral values satisfying the conditions:
[b](i)[/b] $f(ab)=f(a)f(b)$,
[b](ii)[/b] $f(a)<f(b)$ if $a<b$,
[b](iii)[/b] $f(3)\ge 7$.
Find the minimum value for $f(3)$.

solution

Let $m>n>1$ two integers :

If $\frac{p}{q}<\frac{\ln m}{\ln n}<\frac{r}{s}$, with $p,q,r,s\in\mathbb{N}$, we get :

$n^p<m^q$ and so $f(n)^p<f(m)^p$ and so $\frac{p}{q}<\frac{\ln f(m)}{\ln f(n)}$

$m^s<n^r$ and so $f(m)^s<f(n)^r$ and so $\frac{\ln f(m)}{\ln f(n)}<\frac{r}{s}$

And so $\frac{\ln f(m)}{\ln f(n)}=\frac{\ln m}{\ln n}$ and $\frac{\ln f(m)}{\ln m}=\frac{\ln f(n)}{\ln n}=c$

And $f(n)=n^c$

And so $f(3)=3^c\ge 7$

So $c=2$ and minimum value for $f(3)$ is nine, which is reached for function $f(n)=n^2$

43. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(a^3)+f(b^3)+f(c^3)=f(3abc)\quad\forall a,b,c\in\mathbb{R}.$$

solution

Setting $b=c=0$, we get $f(a^3)=-f(0)$ and so $f(x)$ is constant and the only constant solution is $\boxed{f(x)=0}$ $\forall x$

44. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a^3) + f(b^3) + f(c^3) = a \cdot f(a^2) + b \cdot f(b^2) + c \cdot f(c^2) \quad \forall a, b, c \in \mathbb{R}.$$

solution

This is equivalent to $f(x^3) = xf(x^2)$ and there are infinitely many solution.

Let $x \sim y$ the relation defined on $(1, +\infty)$ as $\frac{\ln(\ln x) - \ln(\ln y)}{\ln 3 - \ln 2} \in \mathbb{Z}$

This is an equivalence relation. Let $c(x)$ any choice function which associates to any real in $(1, +\infty)$ a representant (unique per class) of its class. Let $n(x) = \frac{\ln(\ln x) - \ln(\ln c(x))}{\ln 3 - \ln 2} \in \mathbb{Z}$ We get $x = c(x)^{\left(\frac{3}{2}\right)^{n(x)}}$ and so $f(x) = \frac{xf(c(x))}{c(x)}$

And so we can define $f(x)$ only over $c((1, +\infty))$ Let $g(x)$ any finction from $\mathbb{R} \to \mathbb{R}$

$f(x) = \frac{xg(c(x))}{c(x)}$

We can define in the same way $f(x)$ over $(0, 1)$ We can define then $f(1)$ as any value, $f(0)$ as 0 and $f(-x) = -f(x)$

And we have got all suitable $f(x)$

45. Determine all monotone functions $f : \mathbb{R} \to \mathbb{Z}$ such that $f(x) = x, \forall x \in \mathbb{Z}$ and $f(x + y) \geq f(x) + f(y), \forall x, y \in \mathbb{R}$.

solution

Induction gives $f(qx) \geq qf(x)$ $\forall q \in \mathbb{N}$ and so, setting $x = \frac{p}{q}$, $f(\frac{p}{q}) \leq \frac{p}{q}$.

Since $f(x)$ is non decreasing and $f(x) \in \mathbb{Z}$, this implies $f(x) = [x]$ $\forall x \in \mathbb{Q}$

Since $f(x)$ is non decreasing, this implies $f(x) = [x]$ $\forall x \in \mathbb{R}$

46. Find all monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(4x) - f(3x) = 2x$, for each $x \in \mathbb{R}$.

solution

Forget the "monotone" constraint and the general solution of functional equation is :

$\forall x > 0 :$ $f(x) = 2x + h(\frac{\ln x}{\ln 4 - \ln 3})$ where $h(x)$ is any function defined over $[0, 1)$ $f(0) = a$ $\forall x < 0 :$ $f(x) = 2x + k(\frac{\ln -x}{\ln 4 - \ln 3})$ where $k(x)$ is any function defined over $[0, 1)$

Adding then monotone constraint and looking at $f(x)$ when $x \to 0$, we see that we must have $\sup h([0, 1)) = \inf h([0, 1)$ and so $h(x) = c$ constant.

And then, continuity at 0 implies that $h(x) = k(x) = a$ and so $\boxed{f(x) = 2x + a}$

47. Let $n \in \mathbb{N}$, such that $\sqrt{n} \notin \mathbb{N}$ and $A = \{a + b\sqrt{n} | a, b \in \mathbb{N}, a^2 - nb^2 = 1\}$. Prove that the function $f : A \to \mathbb{N}$, such that $f(x) = [x]$ is injective but not surjective.

    $(\mathbb{N} = \{1, 2, ...\})$

    solution

    If $[a + b\sqrt{n}] = p \geq 1$, then :

    $p \leq a + b\sqrt{n} < p + 1$ $\frac{1}{p+1} < a - b\sqrt{n} < \frac{1}{p}$

    Adding, we get $p + \frac{1}{p+1} < 2a < p + 1 + \frac{1}{p}$

    And since $(p + 1 + \frac{1}{p}) - (p + \frac{1}{p+1}) = 1 + \frac{1}{p(p+1)} < 2$, this interval may contain at most one even integer.

    So knowledge of $f(x)$ implies knowledge of a and so (using $a^2 - nb^2 = 1$), knowledge of b

    So $f(x)$ is injective.

    Consider then $p = 2$ and the equation becomes $2 + \frac{1}{3} < 2a < 3 + \frac{1}{2}$ and so $1 < \frac{7}{6} < a < \frac{7}{4} < 2$ and so no such a. So $f(x) = 2$ is impossible and $f(x)$ is not surjective.

48. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that :

    $f(x^2 + y^2) = f(xy)$

    solution

    The system $x^2 + y^2 = u$ and $xy = v$ has solution with $x, y > 0$ iff $u > 2v > 0$

    And so $f(u) = f(v)$ $\forall u > 2v > 0$

    Let then $x > y > 0$: $x > 2\frac{y}{4}$ and so $f(x) = f(\frac{y}{4})$

    $y > 2\frac{y}{4}$ and so $f(y) = f(\frac{y}{4})$

    And so $f(x) = f(y)$ and so $f(x)$ is constant

49. find all functions f : Z —–>Z such that f(-1) = f(1) and f(x) + f(y) = f(x+2xy) + f(y-2xy) for all integers x,y.

    solution

    Let $f(1) = f(-1) = a$ Let $P(x, y)$ be the assertion $f(x) + f(y) = f(x + 2xy) + f(y - 2xy)$

    Let $A = \{x$ such that $f(x) = f(-x) = a\}$ $1 \in A$

    1) $x \in A \implies 2x{+}1 \in A$ ==================================
    Let $x \in A$ $P(-x, -1) \implies f(-x) + f(-1) = f(x) + f(-1 - 2x) \implies f(-2x - 1) = a$ $P(1, x) \implies f(1) + f(x) = f(1 + 2x) + f(-x) \implies f(2x + 1) = a$ So $f(2x + 1) = f(-2x - 1) = a$ and so $2x + 1 \in A$ Q.E.D.

    2) $f(x) = f(-x) \implies f(x{-}1) = f(1{-}x)$ ==================================
    Let x such that $f(x) = f(-x)$ $P(1, -x) \implies f(1) + f(-x) = f(1 - 2x) +$

$f(x)$ and so $f(1-2x) = a$ $P(1-x,-1) \implies f(1-x) + f(-1) = f(x-1) + f(1-2x)$ and so $f(1-x) = f(x-1)$ Q.E.D

3) $f(x) = f(-x)\ \forall x$ and $f(2x+1) = a\ \forall x$ === From 1) and since $1 \in A$, we deduce $1 \in A$, $3 \in A$, $7 \in A$, ..., $2^n - 1 \in A$ ... So we can find in A numbers as great as we want. Using then 2) as many times as we want, we get thet $f(x) = f(-x)\ \forall x$ Then $P(1,x) \implies f(1) + f(x) = f(1+2x) + f(-x) \implies f(2x+1) = a$ Q.E.D.

4)$f((2k+1)x) = f(x)\ \forall x, k$ ================================= $P(x, 2k+1) \implies f(x) + f(2k+1) = f(x(4k+3)) + f((2k+1)(1-2x))$ and so, using 3) : $f(x) = f(x(4k+3))$ $P(-x, -2k-1) \implies f(-x) + f(-2k-1) = f(x(4k+1)) + f(-(2k+1)(2x+1))$ and so, using 2) and 3) : $f(x) = f(-x) = f(x(4k+1))$ So $f(x) = f(x(2k+1))$ Q.E.D.

5) General solution =================== From $f(x) = f(x(2k+1))$, we get that $f(x) = h(v_2(x))$ And since $v_2(x) = v_2(x(2y+1))$ and $v_2(y) = v_2(y(1-2x))$, we get that any $h(x)$ is a solution. Hence the answer :

$\boxed{f(x) = h(v_2(x))}$ where $h(x)$ is any function from $\mathbb{N} \cup \{0\} \to \mathbb{Z}$

50. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) \le f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and $f(x) \le e^x - 1$ for each $x \in \mathbb{R}$.

solution

$f(x+0) \le f(x) + f(0)$ and so $f(0) \ge 0$ and since $f(0) \le e^0 - 1 = 0$, we get $f(0) = 0$ $f(x + (-x)) \le f(x) + f(-x)$ and so $f(x) + f(-x) \ge 0$

$f(x) \le e^x - 1 \implies f(x) \le f(\frac{x}{2}) + f(\frac{x}{2}) \le 2(e^{\frac{x}{2}} - 1)$

$f(x) \le 2(e^{\frac{x}{2}} - 1) \implies f(x) \le f(\frac{x}{2}) + f(\frac{x}{2}) \le 4(e^{\frac{x}{4}} - 1)$

And immediate induction gives $f(x) \le 2^n(e^{\frac{x}{2^n}} - 1)$

Setting $n \to +\infty$, we get $f(x) \le x$

So $f(x) + f(-x) \le x + (-x) = 0$ and so, since we already got $f(x) + f(-x) \ge 0$, we get $f(x) + f(-x) = 0$

Then $f(-x) \le -x \implies -f(x) \le -x \implies f(x) \ge x$

And so $\boxed{f(x) = x}$ which indeed is a solution

51. find all continues functions $f : R \longrightarrow R$ for each two real numbers x, y:

$f(x+y) = f(x + f(y))$

solution

If $f(x) = x\ \forall x$, we got a solution. If $\exists a$ such that $f(a) \ne a$, then $f(x+a) = f(x+f(a))$ implies that $f(x)$ is periodic and one of its periods is $|f(a) - a|$.

Let $T = \inf\{\text{positive periods}\}$ If $T = 0$, then $f(x) = c$ is constant and we got another solution. if $T \ne 0$, then T is a period of $f(x)$ (since continuous) and, since any $f(y)-y$ is also a period, we get that $f(y)-y=n(y)T$ where $n(y)\in\mathbb{Z}$ but then $n(y)$ is a continuous function from $\mathbb{R}\to\mathbb{Z}$ and so is constant and $f(y)=y+kT$ which is not a periodic function. Hence the two solutions : $f(x)=x\ \forall x\ f(x)=c\ \forall x$ for any $c\in\mathbb{R}$

52. • $f\left(f(x)y+x\right)=xf(y)+f(x)$, for all real numbers x,y and • the equation $f(t)=-t$ has exactly one root.

solution

Let $P(x,y)$ be the assertion $f(f(x)y+x)=xf(y)+f(x)$ Let t be the unique real such that $f(t)=-t$

$f(x)=0\ \forall x$ is a solution. Let us from now look for non all-zero solutions. Let u such that $f(u)\neq 0$

$P(1,0)\implies f(0)=0$ and so $t=0$ If $f(a)=0$, then $P(a,u)\implies af(u)=0$ and so $a=0$ So $f(x)=0\iff x=0$

If $f(1)\neq 1$, then $P(1,\frac{1}{1-f(1)})\implies f(\frac{f(1)}{1-f(1)}+1)=f(\frac{1}{1-f(1)})+f(1)$ and so $f(1)=0$, which is impossible. So $f(1)=1$

$P(1,-1)\implies f(-1)=-1$ $P(x,-1)\implies f(x-f(x))=f(x)-x$ and so, since the only solution of $f(t)=-t$ is $t=0$: $f(x)=x$ which indeed is a solution.

[u][b]Hence the two solutions [/b][/u]: $f(x)=0\ \forall x\ f(x)=x\ \forall x$

53. Find all function $f:R\to R$ $f(x+f(y))+f(f(y))=f(f(x))+2f(y)$ $f(x+f(x))=2f(x)$ and $f(f(x))=f(x)$ while $f(0)=0$

solution

1) It's not very fair to transform a problem and claim that there exists a solution when your transformation is not an equivalence and so you dont know if there is such an olympiad level solution.

2) Solution of the original problem : Let $P(x,y)$ be the assertion $f(x+f(y))+f(f(y))=f(f(x))+2f(y)$

$P(0,y)\implies f(f(y))=\frac{f(f(0))}{2}+f(y)$ $P(0,x)\implies f(f(x))=\frac{f(f(0))}{2}+f(x)$ Plugging this in $P(x,y)$, we get new assertion $Q(x,y)$: $f(x+f(y))=f(x)+f(y)$ It's immediate to see that the two assertions are equivalent.

The new assertion has been solved many times in mathlinks :

Let $A=f(\mathbb{R})$. Using $f(x)+f(y)=f(x+f(y))$ and $f(x)-f(y)=f(x-f(y))$ (look at $Q(x-f(y),y)$), we see that A is an additive subgroup of $\mathbb{R}$

Then the relation $x\sim y\iff x-y\in A$ is an equivalence relation and let $c(x)$ any choice function which assocoates to a real x a representant (unique per class) of it's equivalence class.

Setting $g(x) = f(x) - x$, $Q(x, y)$ may be written $g(x+f(y)) = g(x)$ and so $g(x)$ is constant in any equivalence class and so $f(x) - x = f(c(x)) - c(x)$ and so $f(x) = h(c(x)) + x - c(x)$ where $h(x)$ is a function from $\mathbb{R} \to A$

[u][b]So, any solution may be written as [/b][/u]$f(x) = x - c(x) + h(c(x))$ where : $A \subseteq \mathbb{R}$ is an additive subgroup of $\mathbb{R}$ $c(x)$ is any choice function associating to a real x a representant (unique per class) of it's equivalence class for the equivalence relation $x - y \in A$ $h(x)$ is any function from $\mathbb{R} \to A$

[u][b]Let us show now that this mandatory form is sufficient and so that we got a general solution [/b][/u]: Let $A \subseteq \mathbb{R}$ any additive subgroup of $\mathbb{R}$ Let $c(x)$ any choice function associating to a real x a representant (unique per class) of it's equivalence class for the equivalence relation $x - y \in A$ Let $h(x)$ any function from $\mathbb{R} \to A$ Let $f(x) = x - c(x) + h(c(x))$

$x - c(x) \in A$ and $h(c(x)) \in A$ and A subgroup imply that $f(x) \in A$ So $x + f(y) \sim x$ and $c(x + f(y)) = c(x)$ So $f(x + f(y)) = x + f(y) - c(x + f(y)) + h(c(x + f(y))) = x + f(y) - c(x) + h(c(x)) = f(x) + f(y)$ Q.E.D.

And so we got a general solution.

[u][b]Some examples [/b][/u]: 1) Let $A = \mathbb{R}$ and so a unique class and $c(x) = a$ and $f(x) = x - a + h(a)$ and so the solution $\boxed{f(x) = x + b}$ (notice that $f(0) = 0$ is not mandatory.

2) Let $A = \{0\}$ and so equivalence classes are $\{x\}$ and so $c(x) = x$ and $h(x) = 0$ and $f(x) = x - x + 0$ and so the solution $\boxed{f(x) = 0}$

3) Let $A = \mathbb{Z}$ and $c(x) = x - \lfloor x \rfloor$ and $h(x) = \lfloor 2x \rfloor$ $f(x) = x - x + \lfloor x \rfloor + \lfloor 2x - 2\lfloor x \rfloor \rfloor$ and so the solution $\boxed{f(x) = \lfloor 2x \rfloor - \lfloor x \rfloor}$

and infinitely many other

54. Find all functions $f : \mathbb{R}_0 \to \mathbb{R}_0$ satisfying the functional relation $f(f(x) - x) = 2x$ $\forall x \in \mathbb{R}_0$

solution

Ok, so $\mathbb{R}_0$ here is the set of non negative real numbers. Then : In order to LHS be defined, we get $f(x) \geq x$ $\forall x$ So $f(f(x) - x) \geq f(x) - x$ $\forall x \iff f(x) \leq 3x$

So we got $x \leq f(x) \leq 3x$

If we consider $a_n x \leq f(x) \leq b_n x$, we get $a_n(f(x) - x) \leq 2x \leq b_n(f(x) - x)$ and so $\frac{b_n+2}{b_n} x \leq f(x) \leq \frac{a_n+2}{a_n} x$

And so the sequences : $a_1 = 1$ $b_1 = 3$ $a_{n+1} = \frac{b_n+2}{b_n}$ $b_{n+1} = \frac{a_n+2}{a_n}$

And it's easy to show that : a_n is a non decreasing sequence whose limit is 2 b_n is a non increasing sequence whose limit is 2

And so $\boxed{f(x) = 2x}$ which indeed is a solution.

55. (Romania District Olympiad 2011 - Grade XI)

Find all functions $f : [0,1] \to \mathbb{R}$ for which we have:

$$|x-y|^2 \le |f(x) - f(y)| \le |x-y|,$$

for all $x, y \in [0,1]$.

solution

Let $P(x,y)$ be the assertion $|x-y|^2 \le |f(x) - f(y)| \le |x-y|$

Setting $y \to x$ in $P(x,y)$, we conclude that $f(x)$ is continuous. If $f(a) = f(b)$, then $P(a,b) \implies (a-b)^2 \le 0$ and so $a = b$ and $f(x)$ is injective

$f(x)$ continuous and injective implies monotonous. $f(x)$ solution implies $f(x) + c$ and $c - f(x)$ solutions too. So Wlog say $f(0) = 0$ and $f(x)$ increasing.

Then : $P(1,0) \implies f(1) = 1$ and so $f(x) \in [0,1]$ $P(x,0) \implies f(x) \le x$ $P(x,1) \implies 1 - f(x) \le 1 - x$

And so $f(x) = x$ which indeed is a solution.

[u][b]Hence the solutions [/b][/u]: $f(x) = x + a$ for any real a $f(x) = a - x$ for any real a

56. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 - f^2(y)) = xf(x) - y^2$, for all real numbers x, y.

solution

Let $P(x,y)$ be the assertion $f(x^2 - f^2(y)) = xf(x) - y^2$

1) $f(x) = 0$ iff $x = 0$ ================== Let $u = -f^2(0)$: $P(0,0) \implies f(u) = 0$

$P(0,u) \implies f(0) = -u^2$ and so $u = -f^2(0) = -u^4$ and so $u \in \{-1, 0\}$

If $u = -1$: $f(0) = -1$ and $P(-1,0) \implies f(0) = -f(-1)$ and so contradiction since $f(0) = -1$ while $f(-1) = f(u) = 0$. So $u = 0$ and $f(0) = 0$ Then $P(x,0) \implies f(x^2) = xf(x)$ and if $f(y) = 0$, then $P(x,y)$ $\implies y = 0$ Q.E.D.

2) $f(x)$ is odd and surjective =================== $P(0,x) \implies$ $f(-f^2(x)) = -x^2$ and so any non positive real may be reached Comparing $P(x,0)$ and $P(-x,0)$, we get $xf(x) - xf(-x)$ and si $f(-x) = -f(x)$ $\forall x \ne 0$, still true if $x = 0$ and so $f(x)$ is odd. So any non negative real may be reached too. And since $f(0) = 0$, $f(x)$ is surjective. Q.E.D.

3) $f(x) = x$ $\forall x$ ============ $P(x,0) \implies f(x^2) = xf(x)$ $P(0,y)$ $\implies f(-f^2(y)) = -y^2$ And so $f(x^2 - f^2(y)) = f(x^2) + f(-f^2(y))$

And so, since surjective : $f(x+y) = f(x) + f(y)$ $\forall x \ge 0, y \le 0$ And so, since odd, $f(x+y) = f(x) + f(y)$ $\forall x, y$

Then from $f(x^2) = xf(x)$, we get $f((x+1)^2) = (x+1)f(x+1)$ and so $f(x^2) + 2f(x) + f(1) = xf(x) + xf(1) + f(x) + f(1)$

And so $2f(x) = xf(1) + f(x)$ and $f(x) = ax$ Plugging this back in original equation, we get $a = 1$

And so the unique solution $\boxed{f(x) = x}$ $\forall x$

57. Find all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ such that $f(2x + 3y) = 2f(x) + 3f(y) + 4$, for all integers $x, y \geq 1$.

solution

I suppose that $N^* = \mathbb{N}$ is the set of natural numbers (positive integers) Let $P(x, y)$ be the assertion $f(2x + 3y) = 2f(x) + 3f(y) + 4$

Subtracting $P(x + 3, y)$ from $P(x, y + 2)$, we get $2(f(x + 3) - f(x)) = 3(f(y + 2) - f(y))$

And so these two quantities are constant and multiple of 6 and so : $f(x + 3) = f(x) + 3c$ $f(y + 2) = f(y) + 2c$ and (using $y = x + 1$ in this last equation) : $f(x + 3) = f(x + 1) + 2c$

and so $f(x + 1) = f(x) + c$ and $f(x) = cx + d$

Plugging this in $P(x, y)$, we get $\boxed{f(x) = ax - 1}$ for any real $a > 1$ (the case $a = 1$ must be excluded in order to have $f(1) \in \mathbb{N}$)

58. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(m + f(n)) = f(m + n) + 2n + 1$, for all integers m, n.

solution

The equation may be written $f(m + (f(n) - n)) = f(m) + 2n + 1$

And so $f(m + k(f(n) - n)) = f(m) + k(2n + 1)$ Setting $k = f(p) - p$, this becomes $f(m + (f(p) - p)(f(n) - n)) = f(m) + (f(p) - p)(2n + 1)$

And using symetry between n and p, we get $(f(p) - p)(2n + 1) = (f(n) - n)(2p + 1)$

And so $\frac{f(n)-n}{2n+1} = c$ and so $f(n) = n(2c + 1) + c$ with $c = f(0) \in \mathbb{Z}$

Plugging this in original equation, we get $c = -1$ and so the solution $\boxed{f(x) = -x - 1}$

59. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(0) = 2$ and $f(x + f(x + 2y)) = f(2x) + f(2y)$, for all integers x, y.

solution

Let $P(x, y)$ be the assertion $f(x + f(x + 2y)) = f(2x) + f(2y)$

$P(0, 2) \implies f(2) = 4$ $P(0, 1) \implies f(4) = 6$ And so, using induction with $P(0, n)$, we get $f(2n) = 2n + 2$ $\forall n \geq 0$

Let $x \geq 0$: $P(2x, -x) \implies f(-2x) = f(2x + 2) - f(4x) = (2x + 4) - (4x + 2) = -2x + 2$

So $f(2x) = 2x + 2$ $\forall x \in \mathbb{Z}$ and $P(x, y)$ may be written $f(x + f(x + 2y)) = 2x + 2y + 4$

If $\exists$ odd $2a+1$ such that $f(2a+1) = 2b$ is even, then : $P(2a-2b+1, b)$ $\implies 4b = 4a+6$, which is impossible modulus 4

So $f(y)$ is odd for any odd y Let then odd x : $f(x+2y)$ is odd and so $x+f(x+2y)$ is even and so $f(x+f(x+2y)) = x+f(x+2y)+2$ So $x+f(x+2y)+2 = 2x+2y+4$ and $f(x+2y) = x+2y+2$

And so $\boxed{f(x) = x+2}$ $\forall x \in \mathbb{Z}$, which indeed is a solution

60. For wich integer k does there exist a function $f : \mathbb{N} \to \mathbb{Z}$ with $f(1995) = 1996$ and $f(xy) = f(x) + f(y) + kf(\gcd(x, y))$ for all $x, y \in \mathbb{N}$

solution

Let $P(x, y)$ be the assertion $f(xy) = f(x) + f(y) + kf(\gcd(x, y))$

$P(x, x) \implies f(x^2) = (k+2)f(x)$ $P(x^2, x) \implies f(x^3) = (2k+3)f(x)$ $P(x^3, x) \implies f(x^4) = (3k+4)f(x)$ $P(x^2, x^2) \implies f(x^4) = (k+2)^2 f(x)$

So $(3k+4)f(x) = (k+2)^2 f(x)$ and setting $x = 1995$, we get $(k+2)^2 = (3k+4)$ and so $k \in \{-1, 0\}$

For $k = -1$, solutions exist. For example $f(n) = 1996$ $\forall n$.

For $k = 0$, solutions exist. For example $f(1) = 0$ and $f(\prod_{k=1}^{n} p_i^{n_i}) = 499 \sum_{k=1}^{n} n_i$ (where p_i are distinct primes and $n_i \in \mathbb{N}$).

Hence the answer : $\boxed{k \in \{-1, 0\}}$

61. Find all functions $f, g : \mathbb{Z} \to \mathbb{Z}$ such that g is bijective and

$$f(g(x) + y) = g(f(y) + x).$$

solution

We just need $g(x)$ injective and we dont need the restriction $\mathbb{Z} \to \mathbb{Z}$ (it's the same result for $\mathbb{R} \to \mathbb{R}$) :

Let $P(x, y)$ be the assertion $f(g(x) + y) = g(f(y) + x)$

$P(x, g(0)) \implies f(g(x) + g(0)) = g(f(g(0)) + x)$ $P(0, g(x)) \implies f(g(0) + g(x)) = g(f(g(x)))$

So $g(f(g(0)) + x) = g(f(g(x)))$ and, since $g(x)$ is injective : $f(g(x)) = x + f(g(0))$

$P(x, 0) \implies f(g(x)) = g(f(0) + x)$ and so $g(x + f(0)) = x + f(g(0))$ and so $g(x) = x + a$ for some a

(We previously got $f(g(x)) = x + f(g(0))$ Then $P(x, 0) \implies f(g(x)) = g(f(0) + x)$ and so $g(x + f(0)) = x + f(g(0))$

From there we immediately get $g(x) = (x - f(0)) + f(g(0))$ and so $g(x) = x + a$ for some $a = f(g(0)) - f(0)$)

Then $f(g(x)) = x + f(g(0))$ becomes $f(x+a) = x + f(g(0))$ and so $f(x) = x + b$ for some b

Plugging back in original equation we get that these are solutions whatever are $a, b \in \mathbb{Z}$

Hence the answer : $f(x) = x + b$ $\forall x$ and for any $b \in \mathbb{Z}$ (or $\mathbb{R}$ is we move the problem in $\mathbb{R}$) $g(x) = x + a$ $\forall x$ and for any $a \in \mathbb{Z}$ (or $\mathbb{R}$ is we move the problem in $\mathbb{R}$)

62. (Belarus 1995) Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy \qquad \forall x, y \in \mathbb{R}.$$

solution

Let $P(x, y)$ be the assertion $f(f(x+y)) = f(x+y) + f(x)f(y) - xy$ Let $f(0) = a$

$P(x, y) \implies f(f(x+y)) = f(x+y) + f(x)f(y) - xy$ $P(x+y, 0) \implies f(f(x+y)) = f(x+y) + af(x+y)$ Subtracting, we get new assertion $Q(x, y) : af(x+y) = f(x)f(y) - xy$

$Q(x, -x) \implies a^2 = f(x)f(-x) + x^2$ $Q(x, x) \implies af(2x) = f(x)^2 - x^2$ $Q(-x, 2x) \implies af(x) = f(-x)f(2x) + 2x^2 \implies a^2 f(x) = f(-x)(f(x)^2 - x^2) + 2ax^2 \implies a^2 f(x)^2 = f(x)f(-x)(f(x)^2 - x^2) + 2ax^2 f(x) = (a^2 - x^2)(f(x)^2 - x^2) + 2ax^2 f(x)$

And so $x^2(f(x) - a - x)(f(x) - a + x) = 0$

So : $\forall x$, either $f(x) = a + x$, either $f(x) = a - x$ (the case $x = 0$ is true too)

Suppose now that $f(x) = a + x$ for some x $P(x, 0) \implies f(a + x) = (a+1)x + a(a+1)$ and so : either $(a+1)x + a(a+1) = a + (a+x) \iff a(x + a - 1) = 0$ either $(a+1)x + a(a+1) = a - (a+x) \iff (a+2)x + a(a+1) = 0$ And so either $a = 0$, either there are at most two such x : $1 - a$ and $-\frac{a(a+1)}{a+2}$

Suppose now that $f(x) = a - x$ for some x $P(x, 0) \implies f(a - x) = -(a+1)x + a(a+1)$ and so : either $-(a+1)x + a(a+1) = a + (a-x) \iff a(x - a + 1) = 0$ either $-(a+1)x + a(a+1) = a - (a-x) \iff (a+2)x - a(a+1) = 0$ And so either $a = 0$, either there are at most two such x : $a - 1$ and $\frac{a(a+1)}{a+2}$

And so $a = 0$ and either $f(x) = x$, either $f(x) = -x$ If $f(1) = 1$, then $Q(x, 1) \implies f(x) = x$ $\forall x$ which indeed is a solution If $f(1) = -1$, then $Q(x, -1) \implies f(x) = -x$ $\forall x$ which is not a solution

Hence the answer : $\boxed{f(x) = x}$ $\forall x$

63. Find all numbers $d \in [0,1]$ such that if $f(x)$ is an arbitrary continues function with domain $[0,1]$ and $f(0) = f(1)$,there exist number $x_0 \in [0, 1-d]$ such that $f(x_0) = f(x_0 + d)$

solution

1) $d = 0$ fits ========== Just choose $x_0 = 0$:)

2) $d = \frac{1}{n}$ fits =========== Let $g(x) = f(x+d) = f(x+\frac{1}{n})$ Let the sequence $a_k = f(\frac{k}{n})$ $a_0 = a_n = f(0)$ and so : either $\exists k \in [0, n-1]$ such that $a_k = a_{k+1}$ and just choose $x_0 = \frac{k}{n}$ either $a_k \neq a_{k+1}$ $\forall k \in [0, n-1]$ and then :

If $a_1 > a_0$, the sequence cannot be increasing for any k and then $\exists k \in [0, n-1]$ such that $a_k < a_{k+1}$ and $a_{k+2} < a_{k+1}$ and then : $f(\frac{k}{n}) < g(\frac{k}{n})$ and $g(\frac{k}{n}+d) < f(\frac{k}{n}+d)$ and so $\exists x_0 \in (\frac{k}{n}, \frac{k}{n}+d)$ such that $f(x_0) = g(x_0)$ (since continuous).

If $a_1 < a_0$, the sequence cannot be decreasing for any k and then $\exists k \in [0, n-1]$ such that $a_k > a_{k+1}$ and $a_{k+2} > a_{k+1}$ and then : $f(\frac{k}{n}) > g(\frac{k}{n})$ and $g(\frac{k}{n}+d) > f(\frac{k}{n}+d)$ and so $\exists x_0 \in (\frac{k}{n}, \frac{k}{n}+d)$ such that $f(x_0) = g(x_0)$ (since continuous). Q.E.D

3) no other d fit =========== Let $d \in (0,1)$ and n, r such that $1 = nd + r$ with n non negative integer and $r \in (0, d)$ Choose any $u > 0$ and any continuous $h(x)$ defined over $[0, d]$ such that : $h(0) = 0$ $h(r) = nu$ $h(d) = -u$

And define $f(x)$ in a recursive manner : $\forall x \in [0, d]$: $f(x) = h(x)$ $\forall x > d$: $f(x) = f(x-d) - u$

We have : $f(x)$ continuous $f(0) = f(1) = 0$ And the equation $f(x) = f(x+d)$ is equivalent to $f(x) = f(x) - u$ and has no solution. Q.E.D.

[u][b]Hence the result[/b][/u] : $\boxed{d \in \{0\} \cup \left(\bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\} \right)}$

64. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$

$f(x + f(xy)) = f(x + f(x)f(y)) = f(x) + xf(y)$

solution

Let $P(x, y)$ be the assertions $f(x+f(xy)) = f(x+f(x)f(y)) = f(x)+xf(y)$

$f(x) = 0$ $\forall x$ is a solution and let us from now look for non allzero solutions. Let u such that $f(u) \neq 0$

1) $f(x) = 0 \iff x = 0$ =================== $P(-1, -1) \implies f(-1 + f(1)) = f(-1 + f(-1)^2) = 0$ and so $\exists v$ such that $f(v) = 0$ $P(v, u) \implies 0 = vf(u)$ and so $v = 0$ Q.E.D.

2) $f(n)=n\ \forall n\in\mathbb{N}$ ============== $P(-1,-1) \implies f(-1+f(1))=f(-1+f(-1)^2)=0$ and so, using 1) : $-1+f(1)=-1+f(-1)^2=0$ So $f(1)=1$

$P(1,x) \implies f(1+f(x))=1+f(x)$ and so from $f(1)=1$, we get $f(n)=n$ $\forall n\in\mathbb{N}$ Q.E.D.

3) $f(-1)=-1$ =========== $P(-1,-1) \implies f(-1+f(1))=f(-1+f(-1)^2)=0$ and so, using 1) : $-1+f(1)=-1+f(-1)^2=0$ So $f(-1)=\pm1$ If $f(-1)=1$, then :

$P(\frac{1}{n},n) \implies f(\frac{1}{n}+1)=f(\frac{1}{n})+\frac{1}{n}f(n)$ $P(\frac{1}{n},-n) \implies f(\frac{1}{n}+1)=f(\frac{1}{n})+\frac{1}{n}f(-n)$ And so $f(-n)=f(n)=n$ Then $P(-1,2) \implies f(-1+f(-2))=f(-1+f(-1)f(2))=f(-1)-f(2) \implies 1=1=-1$, contradiction So $f(-1)=-1$ Q.E.D.

4) $f(x)$ is injective =========== If $f(y_1)=f(y_2)$ and $y_2=0$ then $f(y_1)=0$ and 1) gives $y_1=y_2=0$ If $f(y_1)=f(y_2)$ and $y_2\neq0$, let $a=\frac{y_1}{y_2}$ $P(y_2,1) \implies f(y_2+f(y_2))=f(y_2)+y_2$ $P(y_2,a) \implies f(y_2+f(y_1))=f(y_2)+y_2f(a)$ And so $f(a)=1$

$P(a,1) \implies f(a+1)=a+1$

Notice that if $f(x)=x$, then : $P(1,x) \implies f(x+1)=x+1$ $P(-1,x) \implies f(-1+f(-1)f(x))=f(-1)-f(x) \implies f(-x-1)=-x-1$

Applying this to $f(a+1)=a+1$, we get $f(-a-2)=-a-2$ (second property) $f(-a-1)=-a-1$ (then first property) $f(a)=a$ (then second property) And so $a=1$ And so $y_1=y_2$ Q.E.D.

5) $f(xy)=f(x)f(y)$ =========== This is an immediate consequence of $f(x+f(xy))=f(x+f(x)f(y))$ and $f(x)$ injective

6) $f(x)=x\ \forall x$ ========== Let $x\neq0$ We trivially have from 5) that $f(\frac{1}{x})=\frac{1}{f(x)}$

Then $P(\frac{1}{x},x) \implies f(\frac{1}{x}+1)=\frac{1}{f(x)}+\frac{f(x)}{x}$

Then $f(x+1)=f(x(\frac{1}{x}+1))=f(x)f(\frac{1}{x}+1)=1+\frac{f(x)^2}{x}$
But $P(x,\frac{1}{x}) \implies f(x+1)=f(x)+xf(\frac{1}{x})=f(x)+\frac{x}{f(x)}$

So $1+\frac{f(x)^2}{x}=f(x)+\frac{x}{f(x)}$

$\implies xf(x)+f(x)^3=xf(x)^2+x^2$

$\implies (f(x)^2+x)(f(x)-x)=0$

And so $f(x)=x\ \forall x>0$ And since $f(-x)=f((-1)x)=f(-1)f(x)=-f(x)$, we get $f(x)=x\ \forall x$ which indeed is a solution

7) Synthesis of solutions ================= And so we got two solutions : $f(x)=0\ \forall x$ $f(x)=x\ \forall x$

65. Let $f:[0,1]\to\mathbb{R}_+^*$ be a continous function such that $f(x_1)f(x_2)...f(x_n)=e$, for all $n\in\mathbb{N}^*$ and for all $x_1,x_2,...,x_n\in[0,1]$ with $x_1+x_2+...+x_n=1$.

Prove that $f(x) = e^x,\ x \in [0, 1]$.

solution

Choosing $x_i = \frac{1}{n}$, we get $f(\frac{1}{n})^n = e$ and so $f(\frac{1}{n}) = e^{\frac{1}{n}}$
Let $q > p \geq 1$: choosing $n = q - p + 1$ and $x_1 = x_2 = ... = x_{n-1} = \frac{1}{q}$ and $x_n = \frac{p}{q}$, we get : $f(\frac{1}{q})^{q-p} f(\frac{p}{q}) = e$ and so $e^{\frac{q-p}{q}} f(\frac{p}{q}) = e$ and so $f(\frac{p}{q}) = e^{\frac{p}{q}}$
And so $f(x) = e^x$ $\forall x \in \mathbb{Q} \cap (0, 1)$ and continuity implies $f(x) = e^x$ $\forall x \in [0, 1]$ which indeed is a solution

66. Find all functions $f : R \to R$:
$f(xy)f(f(x) - f(y)) = (x - y)f(x)f(y)$

solution

There are infinitely many solutions but I did not succeed up to now finding all of them.

[u][b]Some solutions [/b][/u]:

1) trivial solution $f(x) = x\ \forall x$

2) trivial solution $f(x) = 0\ \forall x$

3) $f(a) = b$ and $f(x) = 0\ \forall x \neq a$ where a is any nonzero real and $b \neq \pm a$

4) $f(x) = x\ \forall x \in \mathbb{Q}$ and $f(x) = 0$ anywhere else

5) $f(x) = x\ \forall x \in \mathbb{Q}[\sqrt{2}]$ and $f(x) = 0$ anywhere else

In fact 4) and 5) may be merged in :

$f(x) = x\ \forall x \in \mathbb{K}$ and $f(x) = 0$ anywhere else where $\mathbb{K}$ is any subfield of $\mathbb{R}$

... and a lot of other.

67. find all functions f from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ which satisfy for all $x, y, z \in \mathbb{R}$ the identity
$f(f(x) + f(y) + f(z)) = f(f(x) - f(y)) + f(2xy + f(z)) + 2f(xz - yz)$

solution

$f(x)$ constant implies $f(x) = 0\ \forall x$ which indeed is a solution. Let us from now look for non constant solutions.

Let $P(x, y, z)$ be the assertion $f(f(x) + f(y) + f(z)) = f(f(x) - f(y)) + f(2xy + f(z)) + 2f(xz - yz)$ Let $f(0) = a$

1) $f(x)$ is even ============ Subtracting $P(2, 1, 0)$ from $P(1, 2, 0)$, we get $f(f(2) - f(1)) = f(f(1) - f(2))$ Subtracting then $P(2, 1, x)$ from $P(1, 2, x)$ and using the above result, we get $f(x) = f(-x)$ and so $f(x)$ is an even function. Q.E.D.

2) $f(x)=0 \iff f(0)=0$ ==================== 2.1) $f(0)=0$ ———- Subtracting $P(-x-a,\frac{1}{2},0)$ from $P(x+a,\frac{1}{2}y,0)$, we get $f(x+2a)=f(-x)=f(x)$ and so, if $a\neq 0$, $f(x)$ is periodic and one period is $2a$

But then comparing $P(x,y,z+2a)$ and $P(x,y,z)$, we get $f((x-y)z)=f((x-y)(z+2a))$ and so $f(x)$ is constant, impossible

So $a=0$ Q.E.D.

2.2) $f(x)=0 \implies x=0$ ——————— If $f(u)=0$ for some u, then comparing $P(x,y,u)$ and $P(x,-y,u)$, we get $f((x-y)u)=f(x+y)u)$ And so, if $u\neq 0$, we get that $f(x)$ is constant, impossible So $u=0$ Q.E.D.

3) $f(x_1)=f(x_2) \implies x_1=\pm x_2$ ================================== If $f(x_1)=f(x_2)=0$, then $x_1=x_2=0$, according to 2) above.

If $f(x_1)=f(x_2)\neq 0$, then $x_1\neq 0$ and $x_2\neq 0$ Comparing $P(x_1,x,0)$ and $P(x_2,x,0)$, we get $f(2x_1x)=f(2x_2x)$ and so $f(tx)=f(x)$ $\forall x$, with $t=\frac{x_1}{x_2}$
Comparing then $P(tx,y,1)$ with $P(x,ty,1)$, we get $f(tx-y)=f(x-ty)$ $\forall x,y$ If $t\neq\pm 1$, this implies that $f(x)$ is constant, impossible.

Q.E.D

4) $f(x)=x^2$ $\forall x$ ============= Suppose $f(u)\neq u^2$ for some u. Then : $P(u,u,x) \implies f(2f(u)+f(x))=f(2u^2+f(x))$ and so :

either $2f(u)+f(x)=2u^2+f(x)$ and so $f(u)=u^2$, impossible either $2f(u)+f(x)=-2u^2-f(x)$ and so $f(x)=-f(u)-u^2$ and $f(x)$ is constant, impossible.

And so $f(x)=x^2$ $\forall x$, which indeed is a solution.

5) Synthesis of solutions : ================== So we found two solutions : $f(x)=0$ $\forall x$ $f(x)=x^2$ $\forall x$

68. Determine all functions $f:\mathbb{R}^*\to\mathbb{R}^*$ such that $f\left(\frac{f(x)}{f(y)}\right)=\frac{1}{y}\cdot f(f(x))$, for each $x,y\in\mathbb{R}^*$ and are strictly monotonic on $(0,+\infty)$.

solution

Let $P(x,y)$ be the assertion $f(\frac{f(x)}{f(y)})=\frac{f(f(x))}{y}$

$f(x)$ is injective and then $P(x,1)$ implies $f(1)=1$

$P(1,x) \implies f(\frac{1}{f(x)})=\frac{1}{x}$

$P(\frac{1}{f(\frac{1}{x})},\frac{1}{f(y)}) \implies f(xy)=f(x)f(y)$

This implies $f(x)>0$ $\forall x>0$ and so $g(x)=\ln(f(e^x))$ is a monotonous function such that $g(x+y)=g(x)+g(y)$ and so $g(x)=ax$ and so $f(x)=x^a$ $\forall x>0$

Plugging this in $f(\frac{1}{f(x)})=\frac{1}{x}$, we get $f(x)=x\ \forall x>0$ or $f(x)=\frac{1}{x}\ \forall x>0$

$f(xy)=f(x)f(y)$ implies $f(-1)=\pm 1$ and so $f(-1)=-1$ (since $f(x)$ is injective an $f(1)=1$) and so $f(-x)=-f(x)$.

[u][b]Hence the two solutions[/b][/u] : $f(x)=x\ \forall x\neq 0$ $f(x)=\frac{1}{x}\ \forall x\neq 0$ which indeed are solutions

69. Find all functions, $f:\mathbb{R}^+\to\mathbb{R}^+$ such that: $x^2f(f(x)+f(y))=(x+y)f(yf(x))$ for all x,y in $\mathbb{R}^+$

solution

I consider that $\mathbb{R}^+$ is the set of all positive real numbers. Let $P(x,y)$ be the assertion $x^2f(f(x)+f(y))=(x+y)f(yf(x))$

If $f(u)=f(v)$ then, comparing $P(u,1)$ and $P(v,1)$ we get $\frac{u+1}{u^2}=\frac{v+1}{v^2}$ $\iff$ $(v-u)(uv+v+u)=0$ and so $u=v$ and $f(x)$ is injective.

Then $P(\frac{3}{2},\frac{3}{4})\implies f(f(\frac{3}{2})+f(\frac{3}{4}))=f(\frac{3}{4}f(\frac{3}{2}))$

And so, since injective : $f(\frac{3}{2})+f(\frac{3}{4})=\frac{3}{4}f(\frac{3}{2})$

And so $\frac{1}{4}f(\frac{3}{2})+f(\frac{3}{4})=0$, impossible since $f(x)>0\ \forall x$

So no solution.

70. Let $f:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ be a function which is bounded on the interval $[0,1]$ and obeys the inequality

$$f(x)f(y)\leq x^2f\left(\frac{y}{2}\right)+y^2f\left(\frac{x}{2}\right)$$

for each pair of nonnegative reals x and y. Prove that $f(x)\leq\frac{x^2}{2}$ for all nonnegative reals x.

solution

Setting $x=y$ in the inequality, we get $2x^2f(\frac{x}{2})\geq f(x)^2$

Setting $g(x)=\frac{2f(x)}{x^2}$ this becomes $g(\frac{x}{2})\geq g(x)^2$ and so $g(\frac{x}{2^n})\geq g(x)^{2^n}$

Suppose then that $g(u)=a>1$ for some u, then $g(\frac{u}{2^n})\geq a^{2^n}$

And so $f(\frac{u}{2^n})\geq u^2\frac{a^{2^n}}{2^{2n+1}}$

Setting $n\to+\infty$ in the above inequation, we get thet LHS is clearly unbounded, and so contradiction with the fact that $f(x)$ is bounded on $[0,1]$

So $g(x)\leq 1\ \forall x$

So $f(x)\leq\frac{x^2}{2}\ \forall x$

71. Find all strictly increasing bijective function $f:R-->R$ such that $f(x)+f^{-1}(x)=2x$ for all real x.

solution

$f(x)$ increasing bijection implies $f(x)$ continuous. The equation may be written $f(f(x)) - f(x) = f(x) - x$ and so $g(x + g(x)) = g(x)$ where $g(x) = f(x) - x$ is continuous.

Let us look for continuous solutions of $g(x + g(x)) = g(x)$

$g(x) = 0$ $\forall x$ is a solution and let us from now look for non all zero solutions. If $g(x)$ is solution, then $-g(-x)$ is solution too and so Wlog say $g(u) = v > 0$ for some u

Let $A = \{x \geq u$ such that $g(x) = g(u) = v\}$

From $g(x + g(x)) = g(x)$, we get $g(x + ng(x)) = g(x)$ and so $u + nv \in A$ $\forall n \in \mathbb{N} \cup \{0\}$

If A is not dense in $[u, +\infty)$, let then $a, b \in A$ such that $u \leq a < b$ and $(a, b) \cap A = \emptyset$. (existence of a, b needs continuity of $g(x)$)

Let then $y \in (a, b)$. So $g(y) \neq v$ Consider then $y - a + n(g(y) - v)$ for $n \in \mathbb{N}$ Since $g(y) \neq v$, this quantity, for n great enough is out of $[-v, +v]$ and so let $m > 0$ such that $y - a + m(g(y) - v) \notin [-v, +v]$ and so such that $y + mg(y) \notin [a + (m-1)v, a + (m+1)v]$

Looking at the continuous function $h(x) = x + mg(x)$, we get : $h(a) = a + mv \in (a + (m-1)v, a + (m+1)v)$ $h(y) = y + mg(y) \notin [a + (m-1)v, a + (m+1)v]$

So (using continuity of $h(x)$), $\exists z \in (a, y)$ such that $h(z) = a + (m-1)v$ or $h(z) = a + (m+1)v$ But then $g(h(z)) = v$ and so $g(z + mg(z)) = g(z) = v$, impossible since $z \in (a, b)$ and $(a, b) \cap A = \emptyset$.

So A is dense in $[u, +\infty)$

Then continuity of $g(x)$ implies $g(x) = v$ $\forall x \geq u$. Let then any $w < u$: If $g(w) > 0$, then $\exists n \in \mathbb{N}$ such that $w + ng(w) > u$ and so $g(w) = v$. So $\forall x < u$: either $g(x) = v$, either $g(x) \leq 0$ and continuity gives the conclusion $g(x) = v$ $\forall x$

So $g(x) = c$ and $\boxed{f(x) = x + c}$ which indeed is a solution.

72. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying (a) $f(0) = 0$ (b) $f\left(\frac{x^2+y^2}{2xy}\right) = \frac{f(x)^2+f(y)^2}{2xy}$ $\forall$ $x, y \in \mathbb{R}, x \neq 0, y \neq 0$

solution

Let $P(x, y)$ be the assertion $f\left(\frac{x^2+y^2}{2xy}\right) = \frac{f(x)^2+f(y)^2}{2xy}$

$P(1, 1) \implies f(1) = f(1)^2$ and so $f(1) \in \{0, 1\}$

If $f(1) = 0$, then $P(x, x) \implies f(x) = 0$ $\forall x \neq 0$ and so $f(x) = 0$ $\forall x$

If $f(1) = 1$, then $P(x, x) \implies f(x)^2 = x^2$ $\forall x \neq 0$ and so $f(x)^2 = x^2$ $\forall x$

Then $P(x, y)$ becomes $f\left(\frac{x^2+y^2}{2xy}\right) = \frac{x^2+y^2}{2xy}$

And so $f(x) = x\ \forall x$ such that $|x| \geq 1$ and obviously $f(x)$ may be either x, either $-x$ for any other x

And so the solutions : 1) $f(x) = 0\ \forall x$

2) $f(x) = e(x)x\ \forall x \in (-1, 1)$ and $f(x) = x\ \forall x \in (-\infty, -1] \cup [1, +\infty)$ where $e(x)$ is any function from $(-1, 1) \to \{-1, 1\}$

73. Find all $f : \mathbb{R} \to \mathbb{R}$ such that $xf(y) - yf(x) = f(\frac{y}{x})$ for $x, y \in \mathbb{R}, x \neq 0$

solution

Let $P(x, y)$ be the assertion $xf(y) - yf(x) = f(\frac{y}{x})$

$P(2, 0) \implies f(0) = 0$ $P(1, 1) \implies f(1) = 0$ $P(x, 1) \implies f(x) = -f(\frac{1}{x})$ $\forall x \neq 0$

$P(\frac{1}{x}, 2) \implies \frac{f(2)}{x} + 2f(x) = f(2x)\ \forall x \neq 0$

$P(\frac{1}{2}, x) \implies \frac{f(x)}{2} + xf(2) = f(2x)\ \forall x \neq 0$

Subtracting, we get $f(x) = \frac{2f(2)}{3}\frac{x^2-1}{x}\ \forall x \neq 0$

Hence the solution : $f(0) = 0$ and $f(x) = a\frac{x^2-1}{x}\ \forall x \neq 0$ which indeed is a solution (where a is any real)

74. Find all $k \in \mathbb{N}$ such that there exist exactly k functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying: $f(x + y) = kf(x)f(y) + f(x) + f(y)$ for all x, y in $\mathbb{Q}$

solution

Let $h(x) = kf(x) + 1$. The equation becomes $h(x + y) = h(x)h(y)$ and so two solutions : $h(x) = 0\ \forall x$ $h(x) = 1\ \forall x$ The other solutions $h(x) = a^x$ do not fit since they are not from $\mathbb{Q} \to \mathbb{Q}$

Hence the answer $\boxed{k = 2}$

75. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that: $f(x+y^2+z) = f(f(x))+yf(x)+f(z)\ \forall x, y, x \in \mathbb{R}$

solution

I suppose we must read $\forall x, y, z \in \mathbb{R}$ and not $\forall x, y, x \in \mathbb{R}$

$f(x) = 0\ \forall x$ is a solution. Let us from now look for non allzero solutions.

Let $P(x, y)$ be the assertion $f(x + y^2 + z) = f(f(x)) + yf(x) + f(z)$ Let u such that $f(u) \neq 0$

$P(u, \frac{x-f(f(u))-f(0)}{f(u)}, 0) \implies f(something) = x$ and so $f(x)$ is surjective.

$P(x, 0, 0) \implies f(x) = f(f(x)) + f(0)$ and so $f(x) = x - f(0)\ \forall x \in f(\mathbb{R})$

And since $f(x)$ is surjective, we get $f(x) = x - f(0)\ \forall x \in \mathbb{R}$.

Setting then $x = 0$, we get $f(0) = 0$ and hence the result :

$f(x) = 0\ \forall x$ $f(x) = x\ \forall x$ which indeed is a solution

76. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2-y^2)=x^2-f(y^2)$ for all reals x,y

solution

Let $P(x,y)$ be the assertion $f(x^2-y^2)=x^2-f(y^2)$

$P(0,0) \implies f(0)=0$

(a) : $P(\frac{x+1}{2},\frac{x-1}{2}) \implies f(x)=\frac{(x+1)^2}{4}-f(\frac{(x-1)^2}{4})$

(b) : $P(\frac{x-1}{2},\frac{x-1}{2}) \implies 0=\frac{(x-1)^2}{4}-f(\frac{(x-1)^2}{4})$

(a)-(b) : $\boxed{f(x)=x}$ which indeed is a solution

77. Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that: $xf(yz)+yf(z)+z=f(f(x)yz+f(y)z+f(z))$ $\forall x,y\in\mathbb{Q}$

solution

Let $P(x,y,z)$ be the assertion $xf(yz)+yf(z)+z=f(f(x)yz+f(y)z+f(z))$

$P(x,0,0) \implies xf(0)=f(f(0))$ $\forall x$ and so $f(0)=0$ $P(0,0,x) \implies f(f(x))=x$ and so $f(x)$ is an involutive bijection.

$P(-1,1,1) \implies 1=f(f(-1)+2f(1))=f(f(1))$ and so, since injective, $f(-1)+2f(1)=f(1)$ and so $f(1)+f(-1)=0$ $P(0,-1,1) \implies -f(1)+1=f(f(-1)+f(1))=0$ and so $f(1)=1$

$P(0,x,1) \implies x+1=f(f(x)+1)=f(f(x+1))$ and so, since injective, $f(x+1)=f(x)+1$ And so $f(x+n)=f(x)+n$ and $f(n)=n$ $\forall x$, $\forall n\in\mathbb{Z}$

Let then $p,q\in\mathbb{Z}$ with $q\neq 0$: $P(0,f(\frac{p}{q}),q) \implies qf(\frac{p}{q})+q=f(p+q)=p+q$ and so $f(\frac{p}{q})=\frac{p}{q}$

So $\boxed{f(x)=x}$ $\forall x\in\mathbb{Q}$ which indeed is a solution.

78. Find all such functions $f:\mathbb{R}\to\mathbb{R}$ such that: $f(x+y+f(y))=f(f(x))+2y$ for all real x,y

solution

Let $P(x,y)$ be the assertion $f(x+y+f(y))=f(f(x))+2y$

If $f(a)=f(b)=c$ for some a,b, then : $P(a,b) \implies f(a+b+c)=f(c)+2b$ $P(b,a) \implies f(b+a+c)=f(c)+2a$ And so $a=b$ and $f(x)$ is injective.

Then $P(x,0) \implies f(x+f(0))=f(f(x))$ and so, since injective : $f(x)=x+f(0)$ which indeed is a solution whatever is $f(0)$

Hence the answer : $\boxed{f(x)=x+a}$ $\forall x$ and for any real a

79. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$x^2f(x)+y^2f(y)-(x+y)f(xy)=(x-y)^2f(x+y)$$

holds for every pair $(x,y)\in\mathbb{R}^2$.

solution

Let $P(x,y)$ be the assertion $x^2f(x)+y^2f(y)-(x+y)f(xy)=(x-y)^2f(x+y)$ Let $a=f(1)$

$P(1,0) \implies f(0)=0$ $P(x,-x) \implies x^2(f(x)+f(-x))=0$ and so $f(-x)=-f(x)$ $\forall x\neq 0 \implies f(-x)=-f(x)$ $\forall x$

$P(x,1) \implies x^2f(x)+a-(x+1)f(x)=(x-1)^2f(x+1)$ $P(x+1,-1) \implies (x+1)^2f(x+1)-a+xf(x+1)=(x+2)^2f(x)$ Adding : $xf(x+1)=(x+1)f(x)$ and so $f(x+1)=\frac{x+1}{x}f(x)$ $\forall x\neq 0$

Plugging this in $P(x,1)$, we get $a=\frac{1}{x}f(x)$ $\forall x\neq 0$ and so $f(x)=ax$ $\forall x\neq 0$ and so $f(x)=ax$ $\forall x$

And it is easy to check back that this indeed is a solution, whatever is a

Hence the answer : $\boxed{f(x)=ax}$ $\forall x$ and for any $a\in\mathbb{R}$

80. Find all $f:\mathbb{Z}^+ -->\mathbb{Z}^+$ such that

$$xf(y)+yf(x)=(xf(f(x))+yf(f(y)))f(xy)$$

and f is increasing(not necessarily strictly increasing).

solution

Let $P(x,y)$ be the assertion $xf(y)+yf(x)=(xf(f(x))+yf(f(y)))f(xy)$

$P(1,1) \implies f(f(1))=1$ and so $f(1)\leq f(f(1))=1$ (since non decreasing) and so $f(1)=1$ $P(x,1) \implies f(f(x))f(x)=1$ and so $f(x)=f(f(x))=1$

Hence the unique solution : $\boxed{f(x)=1\ \forall x}$

81. Find all pairs of functions $f,g:R\to R$ such that f is strictly increasing and for all $x,y\in R$ we have $f(xy)=g(y)f(x)+f(y)$

solution

Let $P(x,y)$ be the assertion $f(xy)=g(y)f(x)+f(y)$

$f(x)$ strictly increasing implies $\exists u$ such that $f(u)\neq 0$

$P(x,u) \implies f(xu)=g(u)f(x)+f(u)$ $P(u,x) \implies f(xu)=g(x)f(u)+f(x)$ Subtracting, we get $g(x)=\frac{g(u)-1}{f(u)}f(x)+1$ and so $g(x)=af(x)+1$ for some real a

Plugging this in original equation, we get new assertion $Q(x,y)$: $f(xy)=af(x)f(y)+f(x)+f(y)$

If $a=0$, we get $f(xy)=f(x)+f(y)$ but then : $Q(1,1) \implies f(1)=0$ $Q(-1,-1) \implies f(-1)=0$ And so $f(-1)=f(1)$ which is impossible since $f(x)$ is strictly increasing

So $a\neq 0$. Let then $h(x)=af(x)+1$ $h(x)$ is strictly monotonous (increasing if $a>0$ and decreasing if $a<0$) and $Q(x,y)$ becomes $h(xy)=h(x)h(y)$ This is a well known functional equation whose only monotonous solutions are $h(x) = sign(x)|x|^t$ where $t \in \mathbb{R}^+$ (where $sign(x) = -1\ \forall x < 0$, $sign(0) = 0$, $sign(x) = 1\ \forall x > 0$)

Then $a > 0$ and[b][u] the solutions of original equation are[/u][/b] : Let any $c, t \in \mathbb{R}^+$ $f(x) = c(sign(x)|x|^t - 1)\ \forall x$ $g(x) = sign(x)|x|^t\ \forall x$ which inded are solutions

Notice that hungnguyenvn'solution is not well defined for $x < 0$ and, if he/she adds the condition $t \in \mathbb{N}$ in order to have the function fully defined, then a lot of solutions are missing

82. find all functions $f, g, h : \mathbb{R} \to \mathbb{R}$ such that for all $x, y, z \in \mathbb{R}$:
$f(h(g(x) + y)) + g(z + f(y)) = h(y) + g(y + f(z)) + x$

solution

It's easy to show that $f(x) = x + a$ But then, infinitely many solutions exist. For example, Choose as $h(x)$ any bijective solution of Cauchy equation and choose $g(x) = h^{-1}(x - a)$

And I think that a lot of other exist.

83. $f : R^+ - > R^+$ $f(x)f(yf(x)) = f(x + y)$ determine f.

solution

[i][b]Modified problem where the function if from $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$[/b][/i]

Let $P(x, y)$ be the assertion $f(x)f(yf(x)) = f(x + y)$

$P(0, 0) \implies f(0) \in \{0, 1\}$ If $f(0) = 0$ then $P(0, x) \implies f(x) = 0\ \forall x$ which indeed is a solution.

Let us from now consider $f(0) = 1$

If $f(x) > 0\ \forall x > 0$, then : The previous posts imply $f(x) = \frac{1}{1+ax}$ for some $a \geq 0$ and for any $x > 0$ And since $f(0) = 1$, this formula is true again for $x = 0$ and it's easy to see that this indeed is a solution.

If $\exists u > 0$ such that $f(u) = 0$, then $P(u, x) \implies f(u + x) = 0\ \forall x \geq 0$ Let then $a = \inf\{x > 0$ such that $f(x) = 0\}$

If $a = 0$, we get $f(x) = 0\ \forall x > 0$ and it's immediate to see that this indeed is a solution (including the fact that $f(0) = 1$).

If $a > 0$, we get $f(x) = 0\ \forall x > a$ and $f(x) > 0\ \forall x < a$

Consider now $x < a$ and $x + y > a$: $P(x, y) \implies f(yf(x)) = 0$ and so $yf(x) \geq a$ So $f(x) \geq \frac{a}{y}\ \forall y \in (a - x, +\infty)$ So $f(x) \geq \frac{a}{a-x}\ \forall x \in (0, a)$

Consider now $x < a$ and $x+y < a$ with $y \neq 0$: $P(x, y) \implies f(yf(x)) \neq 0$ and so $yf(x) \leq a$ So $f(x) \leq \frac{a}{y}\ \forall y \in (0, a - x)$ So $f(x) \leq \frac{a}{a-x}\ \forall x \in (0, a)$

So we got a mandatory condition : $f(x) = \frac{a}{a-x}\ \forall x \in (0, a)$, still true for $x = 0$ Then $P(\frac{a}{2}, \frac{a}{2}) \implies f(a) = 0$ and we got the function : $f(x) = \frac{a}{a-x}$ $\forall x \in [0, a)$ and $f(x) = 0\ \forall x \geq a$ which indeed is a solution.

[u][b]Hence the solutions [/b][/u]: S1 : $f(x) = 0\ \forall x$

S2 : $f(x) = 0\ \forall x > 0$ and $f(0) = 1$

S3 : $f(x) = \frac{1}{1+ax}\ \forall x$ and for any $a \geq 0$

S4 : $f(x) = \frac{a}{a-x}\ \forall x \in [0, a)$ and $f(x) = 0\ \forall x > a$ for any $a > 0$

84. Determine all injective functions $f : \mathbb{N}^* \to \mathbb{N}$ such that $f\left(C_n^m\right) = C_{f(n)}^{f(m)}$, for all $m, n \in \mathbb{N}^*, n \geq m$,

    where $C_n^m = \begin{pmatrix} n \\ m \end{pmatrix}$.

    solution

    If $f(1) \neq 1$, then $f(n) = f(\binom{n}{1}) = \binom{f(n)}{f(1)}$ implies $f(1) = f(n) - 1$ which is impossible for any n since $f(x)$ is injective.

    So $f(1) = 1$ Let then $n > 2$: $f(n) = f(\binom{n}{n-1}) = \binom{f(n)}{f(n-1)}$ and so either $f(n-1) = 1$, impossible since injective, either $f(n-1) = f(n) - 1$ So $f(n) = f(n-1) + 1$ and we get $f(n) = n + c\ \forall n > 1$ where $c = f(2) - 2$

    Using then $f(\binom{4}{2}) = \binom{f(4)}{f(2)}$, we get $f(6) = \binom{c+4}{c+2}$ and so $c + 6 = \frac{(c+4)(c+3)}{2}$ which gives $c \in \{-5, 0\}$ and so $c = 0$

    Hence the unique solution $\boxed{f(n) = n}\ \forall n$, which indeed is a solution.

85. Find all $f : \mathbb{R} \to \mathbb{R}$ such that: $f(x^5 - y^5) = x^2 f(x^3) - y^2 f(y^3)$

    solution

    Let $P(x, y)$ be the assertion $f(x^5 - y^5) = x^2 f(x^3) - y^2 f(y^3)$

    $P(0,0) \implies f(0) = 0$ $P(x,0) \implies f(x^5) = x^2 f(x^3)$ $P(0,x) \implies f(-x^5) = -x^2 f(x^3) = -f(x^5)$ and so $f(x)$ is an odd function.

    So $P(x, -y) \implies f(x^5+y^5) = f(x^5)+f(y^5)$ and so $f(x+y) = f(x)+f(y)$ $\forall x, y$ and so $f(qx) = qf(x)\ \forall q \in \mathbb{Q}$

    Writing then $P(x+q, 0)$, we get $f(x^5+5qx^4+10q^2x^3+10q^3x^2+5q^4x+q^5) = (x^2 + 2qx + q^2)f(x^3 + 3qx^2 + 3q^2x + q^3)$

    So $f(x^5) + 5qf(x^4) + 10q^2 f(x^3) + 10q^3 f(x^2) + 5q^4 f(x) + q^5 f(1) - (x^2 + 2qx + q^2)(f(x^3) + 3qf(x^2) + 3q^2 f(x) + q^3 f(1)) = 0$

    This is a polynomial in q which is zero for any $q \in \mathbb{Q}$. So this is the allzero polynomial and all its coefficients are zero.

    Looking at coefficient of q^4, we get then $5f(x) - 3f(x) - 2xf(1) = 0$ and so $f(x) = xf(1)\ \forall x$

    Hence the solution : $\boxed{f(x) = ax}\ \forall x$ and for any $a \in \mathbb{R}$, which indeed is a solution

86. Find all $f:\mathbb{R}\to\mathbb{R}$, such that: $f(xf(y))=yf(x)$, $lim_{x\to+\infty}f(x)=0$.

solution

$f(x)=0\ \forall x$ is a solution. So let us from now look for non allzero solutions. Let $P(x,y)$ be the assertion $f(xf(y))=yf(x)$ Let u such that $f(u)\neq 0$

$P(0,0)\implies f(0)=0$ and so $u\neq 0$ $P(u,x)\implies f(uf(x))=xf(u)$ and so $f(x)$ is a bijection $P(1,1)\implies f(f(1))=f(1)$ and, since injective, $f(1)=1$ $P(1,x)\implies f(f(x))=x$ $P(-1,f(-1))\implies 1=f(-1)^2$ and so $f(-1)=-1$ (since injective)

$P(x,f(y))\implies f(xy)=f(x)f(y)$ So $f(x)>0\ \forall x>0$ Setting then $f(x)=e^{h(\ln x)}$ for $x>0$, we get $h(x+y)=h(x)+h(y)$ and $\lim_{x\to+\infty}h(x)=-\infty$ So $h(x)$ is a solution of Cauchy equation which is upper bounded from a given point, and so $h(x)=cx$ with $c<0$

So $f(x)=x^c\ \forall x>0$ and then $f(f(x))=x$ implies $c=-1$

[u][b]Hence the solutions[/b][/u] (which indeed are solutions) : $f(x)=0$ $\forall x$ $f(0)=0$ and $f(x)=\frac{1}{x}\ \forall x\neq 0$

87. Find all $f:\mathbb{R}\to\mathbb{R}$, such that: $f(x+y)=\frac{f(x)+f(y)}{1+f(x)f(y)}$ and f is continuous.

solution

Let $P(x,y)$ be the assertion $f(x+y)=\frac{f(x)+f(y)}{1+f(x)f(y)}$

$P(x,x)\implies f(2x)(1+f(x)^2)=2f(x)$ and so : either $f(2x)=0$, either $f(x)^2-\frac{2}{f(2x)}f(x)+1=0$ and so the discriminant of the quadratic must be ≥ 0 : $|f(2x)|\leq 1$

So $|f(x)|\leq 1$.

If $f(u)=+1$ for some u : $P(x-u,u)\implies f(x)=1\ \forall x$ and we got a solution If $f(u)=-1$ for some u : $P(x-u,u)\implies f(x)=-1\ \forall x$ and we got another solution If $|f(x)|<1\ \forall x$, let then $g(x)=\ln(1+f(x))-\ln(1-f(x))$

$g(x)$ is continuous and $f(x)=\frac{e^{g(x)}-1}{e^{g(x)}+1}$

$P(x,y)$ becomes then $\frac{e^{g(x+y)}-1}{e^{g(x+y)}+1}=\frac{e^{g(x)+g(y)}-1}{e^{g(x)+g(y)}+1}$ and so $g(x+y)=g(x)+g(y)$

And since $g(x)$ is continuous, we get $g(x)=ax$

[u][b]Hence the solutions[/b][/u] (which indeed are solutions) : $f(x)=-1$ $\forall x$

$f(x)=+1\ \forall x$

$f(x)=\frac{e^{ax}-1}{e^{ax}+1}\ \forall a$ (notice that $a=0$ gives the solution $f(x)=0\ \forall x$

88. Find all functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy the equality, $f(x+f(y))=f(x-f(y))+4xf(y)$ for any $x,y\in R$ (Here R denote the set of real numbers)

solution

A classical solution : $f(x) = 0\ \forall x$ is a solution. Let us from now look for non allzero solutions :

Let $P(x, y)$ be the assertion $f(x+f(y)) = f(x-f(y))+4xf(y)$ Let t such that $f(t) \neq 0$

Let $u \in \mathbb{R}$: $P(\frac{u}{8f(t)}, t) \implies u = 2f(\frac{u}{8f(t)} + f(t)) - 2f(\frac{u}{8f(t)} - f(t))$

Let us call $a = \frac{u}{8f(t)} + f(t)$ and $b = \frac{u}{8f(t)} - f(t)$ so that $2f(a) - 2f(b) = u$

$P(2f(a) - f(b), b) \implies f(2f(a)) = f(2f(a) - 2f(b)) + 8f(a)f(b) - 4f(b)^2$
$P(f(a), a) \implies f(2f(a)) = f(0) + 4f(a)^2$

Subtracting these two lines, we get $f(2f(a) - 2f(b)) = f(0) + (2f(a) - 2f(b))^2$ and so $f(u) = f(0) + u^2\ \forall u$ which indeed is a solution.

Hence the only solutions $f(x) = 0\ \forall x$ $f(x) = x^2 + c\ \forall x$ and for any real c

89. Show that for all integers $a, b > 1$ there is a function $f : \mathbb{Z}_+^* \to \mathbb{Z}_+^*$ such that $f(a \cdot f(n)) = b \cdot n$ for all positive integer n.

solution

Consider the three sets : $U_a = \mathbb{N} \setminus a\mathbb{N}$: the set of all positive integers not divisible by a $U_b = \mathbb{N} \setminus b\mathbb{N}$: the set of all positive integers not divisible by b $V = a\mathbb{N} \setminus ab\mathbb{N}$: the set of all positive integers divisible by a and not divisible by ab

U_a and U_b both are infinite countable (since $a, b > 1$) and so $\exists$ a bijection $u(n)$ from $U_a \to U_b$

Define then $f(n)$ as : $\forall n \in U_a$: $f(n) = u(n)$ $\forall n \in V$: $f(n) = b \times u^{-1}(\frac{n}{a})$ (notice that $n \in V \implies a|n$ and $b \not| \frac{n}{a}$) $\forall n \notin U_a \cup V$: $f(n) = ab \times f(\frac{n}{ab})$ (notice that $n \notin U_a \cup V \implies ab|n$)

Easy to check that this function matches all requirements.

90. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)}.$$

Where $\mathbb{Q}^+$ is the set of positive rational numbers.

solution

Let $P(x, y)$ be the assertion $f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)}$

$P(1, 1) \implies f(2) = \frac{1}{4}$ $P(2, 1) \implies f(3) = \frac{1}{5+4f(1)}$ $P(3, 1) \implies f(4) = \frac{f(3)}{7f(3)+1} = \frac{1}{12+4f(1)}$ $P(2, 2) \implies f(4) = \frac{1}{16}$

And so $f(1)=1$ and an easy induction using $P(x,1)$: $\frac{1}{f(x+1)}=\frac{1}{f(x)}+2x+1$ gives $\frac{1}{f(x+n)}=2nx+n^2+\frac{1}{f(x)}$

And $f(n)=\frac{1}{n^2}$

Then $P(\frac{p}{q},q) \implies f(\frac{p}{q})+f(q)+2pf(p)=\frac{f(p)}{f(\frac{p}{q}+q)}$

Which becomes, using $f(p)=\frac{1}{p^2}$ and $f(q)=\frac{1}{q^2}$ and $\frac{1}{f(x+q)}=2qx+q^2+\frac{1}{f(x)}$:

$p^2f(\frac{p}{q})^2+(\frac{p^2}{q^2}-q^2)f(\frac{p}{q})-1=0$ whose unique positive root is $f(\frac{p}{q})=\frac{q^2}{p^2}$

Hence the answer : $\boxed{f(x)=\frac{1}{x^2}}$ which indeed is a solution.

91. Find all functions $f:\mathbb{R}\to\mathbb{R}$ for which

$$x(f(x+1)-f(x))=f(x),$$

for all $x\in\mathbb{R}$ and

$$|f(x)-f(y)|\le|x-y|,$$

for all $x,y\in\mathbb{R}$.

solution

We get easily from first equation that $\frac{f(x+1)}{x+1}=\frac{f(x)}{x}$ $\forall x\notin\{-1,0\}$ and so $f(x)=xp(x)$ $\forall x\notin\{-1,0\}$ where $p(x)$ is a periodic function whose 1 is a period.

The second inequation implies that $f(x)$ is continuous and so $p(x)$ is too and so $f(x)=xp(x)$ $\forall x$

Let then $u,v\in\mathbb{R}$ and $n\in\mathbb{Z}$ Using $x=u+n+1$ and $y=v+n$ in the second inequation, we get (remember that $p(x)$ has period 1) : $|(u+n+1)p(u)-(v+n)p(v)|\le|u+n+1-v-n|$

$\implies |(u+1)p(u)-vp(v)+n(p(u)-p(v))|\le|u+1-v|$

$\implies \left|\frac{(u+1)p(u)-vp(v)}{n}+p(u)-p(v)\right|\le\left|\frac{u+1-v}{n}\right|$ $\forall n\ne 0$

Setting $n\to+\infty$ in this last line, we get $p(u)=p(v)$ and so $p(x)$ is the constant function.

[b][u]Hence the result[/u][/b] : $f(x)=cx$ $\forall x$ where c is any real $\in[-1,1]$

92. Determine all functions $f:R\to R$ continuous on R such that: $f(\sqrt{2}x)=2f(x)$, $f(x+1)=f(x)+2x+1$ for all $x\in R$

solution

The general solution of second part is quite classical and is $f(x)=x^2+p(x)$ where $p(x)$ is any periodical function for which 1 is a period.

Plugging this general form in first part, we get $p(\sqrt{2}x) = 2p(x)$ This shows that either $p(x) = 0\ \forall x$, either $p(x)$ is unbounded. But $f(x)$ continuous implies $p(x)$ continuous and any periodical continuous function is bounded. So $p(x) = 0\ \forall x$

Hence the unique solution : $f(x) = x^2\ \forall x$

93. Find all functions on real numbers such that :
$f(2x + f(y)) = f(2x) + xf(2y) + f(f(y))$

solution

$f(x) = 0\ \forall x$ is a solution. Let us from now look for non all-zero solutions. Let $P(x, y)$ be the assertion $f(2x + f(y)) = f(2x) + xf(2y) + f(f(y))$ Let u such that $f(u) \neq 0$ and let $a = \frac{f(2u)}{4f(u)}$

$P(0, 0) \implies f(0) = 0$

1) $f(2x) = 4af(x)\ \forall x$ and $a \neq 0$ ===========================

$P(\frac{f(x)}{2}, u) \implies f(f(x)+f(u)) = f(f(x))+\frac{1}{2}f(x)f(2u)+f(f(u))$ $P(\frac{f(u)}{2}, x) \implies f(f(x)+f(u)) = f(f(x))+\frac{1}{2}f(u)f(2x)+f(f(u))$ And so $f(x)f(2u) = f(u)f(2x)$ and so $f(2x) = 4af(x)\ \forall x$

Setting $x = \frac{u}{2}$ in this equation shows that $a \neq 0$ and ends this part

2) $a = 1$ and $f(f(x)) = f(x)^2\ \forall x$ ======================== $P(\frac{x}{2}, y)$ becomes $f(x + f(y)) = f(x) + 2axf(y) + f(f(y))$ Using this equation, it's easy to show thru induction that $f(nf(y)) = an^2f(y)^2 + n(f(f(y)) - af(y)^2)$

Replacing $n \to 2n$ in this equation, we get $f(2nf(y)) = 4an^2f(y)^2 + 2n(f(f(y)) - af(y)^2)$

But $f(2nf(y)) = 4af(nf(y)) = 4a^2n^2f(y)^2 + 4an(f(f(y)) - af(y)^2)$

And so $4an^2f(y)^2 + 2n(f(f(y)) - af(y)^2) = 4a^2n^2f(y)^2 + 4an(f(f(y)) - af(y)^2)$

These are two polynomials in n which take the same values for any positive integer n and so we can equate their coefficients : 1) coefficient of n^2 : $4af(y)^2 = 4a^2f(y)^2$ and so $a = 1$ (since $a \neq 0$ and we can choose $y = u$ so that $f(y) \neq 0$) 2) coefficient of n : $f(f(y)) - f(y)^2 = 2(f(f(y)) - f(y)^2)$ and so $f(f(y)) = f(y)^2\ \forall y$ Q.E.D.

3) $f(x) = x^2\ \forall x$ ============== $P(-\frac{f(x)}{2}, x) \implies f(-f(x)) = f(x)^2$

$P(-\frac{f(x)}{2}, y) \implies f(f(y) - f(x)) = f(-f(x)) - 2f(x)f(y) + f(f(y)) = f(x)^2 - 2f(x)f(y) + f(y)^2 = (f(y) - f(x))^2$

$P(\frac{x}{2}, y) \implies f(x + f(y)) = f(x) + 2xf(y) + f(y)^2$ Setting $x = \frac{z-f(u)^2}{2f(u)}$ and $y = u$ in the previous line, we get that any real z may be written as

$f(r)-f(s)$ for some real r,s And since we previously got $f(f(y)-f(x)) = (f(y)-f(x))^2\ \forall x,y$, we get $f(z)=z^2\ \forall z$ Q.E.D.

4) Synthesis of solutions ================= We got two solutions : $f(x)=0\ \forall x$ which indeed is a solutio,n $f(x)=x^2\ \forall x$ which indeed is too a solution

94. Let $f:\mathbb{R}\to\mathbb{R}$ be such that for all $x,y\in\mathbb{R}$, $|f(x-y)|=|f(x)-f(y)|$. Can we conclude that $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{R}$? Justify your answer.

solution

$f(x)=0\ \forall x$ is a solution of the functional equation and is such that $f(x+y)=f(x)+f(y)\ \forall x,y$ So, let us look from now only for non allzero solutions. Let $P(x,y)$ be the assertion $|f(x-y)|=|f(x)-f(y)|$ Let w such that $f(w)\neq 0$

$P(0,0) \implies f(0)=0$ $P(0,x) \implies |f(-x)|=|f(x)|$

Suppose now that $\exists u,v$ such that $f(-u)=-f(u)$ and $f(-v)=f(v)$ $P(-u,-v) \implies |f(-u+v)|=|f(u)+f(v)|$ and so $|f(u-v)|=|f(u)+f(v)|$ and since $|f(u-v)|=|f(u)-f(v)|$: either $f(u)=0$ and so $f(-u)=f(u)$ and so both u,v are such that $f(-x)=f(x)$ either $f(v)=0$ and so $f(-v)=-f(v)$ and so both u,v are such that $f(-x)=-f(x)$

So $f(-x)=f(x)\ \forall x$ or $f(-x)=-f(x)\ \forall x$

But if $f(-x)=f(x)\ \forall x$, then : $P(\frac{w}{2},-\frac{w}{2}) \implies |f(w)|=|f(\frac{w}{2})-f(-\frac{w}{2})|$ $=|f(\frac{w}{2})-f(\frac{w}{2})|=0$, impossible (definition of w)

So $f(-x)=-f(x)\ \forall x$

Let us call $(x,y)\in\mathbb{R}^2$: "white" if $f(x)=f(y)$ and so $f(x-y)=0$ "green" if $f(x-y)=f(x)-f(y)\neq 0$ "red" if $f(x-y)=f(y)-f(x)\neq 0$ Notice that $f(-x)=-f(x)$ implies that (x,y) and (y,x) have same colours

Let then (a,b) and (b,c) two non white pairs. If (a,b) and (c,b) dont have the same color, then : $|f(a)-f(c)|=|f(a-c)|=|f((a-b)-(c-b))|$ $=|f(a-b)-f(c-b)|=|f(a)+f(c)-2f(b)|$ and so : either $f(a)-f(c)=f(a)+f(c)-2f(b)$ and so $f(c)=f(b)$, impossible since (c,b) is not white either $f(a)-f(c)=-f(a)-f(c)+2f(b)$ and so $f(a)=f(b)$, impossible since (a,b) is not white So (a,b) and (c,b) have same color

Let then (x,y) and (z,t) two non white pairs. : $P(w,-w) \implies |f(2w)|=2|f(w)|\neq 0$ So $f(w),f(2w),f(4w)$ are pairwise different So one of these three numbers (let us call it $f(u)$) is different from $f(y)$ and from $f(z)$ and so (y,u) and (z,u) both are non white.

(x,y) and (y,u) are both non white, so have same colours (y,u) and (u,z) are both non white, so have same colours (z,u) and (z,t) are both non white, so have same colours

So (x,y) and (z,t) both have same colours and so : either all pairs are either white, either green either all pairs are either white, either red

In the first case, we get $f(x-y)=f(x)-f(y)\ \forall x,y$ and so $f(x+y)=f(x)+f(y)\ \forall x,y$ In the second case, we get $f(x-y)=f(y)-f(x)\ \forall x,y$ and so (choose $x=w$ and $y=0$) contradiction

[u][b]Hence the result [/b][/u]: $f(x+y)=f(x)+f(y)\ \forall x,y$

95. Find all function $f:\mathbb{R}\to\mathbb{R}$ such that: $+$, $f(x)\in\mathbb{Z}\Leftrightarrow x\in\mathbb{Z}+$, $f(f(xf(y))+x)=yf(x)+x\ \forall x\in\mathbb{Q}^+$

solution

It's rather easy to establish that $f(x)=x\ \forall x\in\mathbb{Z}\cup\mathbb{Q}^+$

But there are a lot of solutions out of the trivial $f(x)=x$: for example any solution of Cauchy equation such that $f(1)=1$ and $f(f(f(x)))=x$ (easy to build infinitely many such functions using Hamel basis)

And I'm not sure at all that these are the only solutions :?:

96. Find all function $f:R\to R$ such that: $f(x+f(y))=f(x)+\frac{1}{8}xf(4y)+f(f(y))$

solution

$f(x)=0\ \forall x$ is a solution. Let us from now look for non allzero solutions. Let $P(x,y)$ be the assertion $f(x+f(y))=f(x)+\frac{1}{8}xf(4y)+f(f(y))$ Let t such that $f(t)\neq 0$

$P(0,0)\implies f(0)=0$

$P(f(x),f(t))\implies f(f(x)+f(t))=f(f(x))+\frac{1}{8}f(x)f(4t)+f(f(t))$ $P(f(t),f(x))\implies f(f(x)+f(t))=f(f(x))+\frac{1}{8}f(t)f(4x)+f(f(t))$

So $f(x)f(4t)=f(t)f(4x)$ and so $f(4x)=8af(x)$ for some $a\in\mathbb{R}$ (remember $f(t)\neq 0$)

$P(x,y)$ implies then new assertion $Q(x,y)$: $f(x+f(y))=f(x)+axf(y)+f(f(y))$

Choosing $y=t$ and the appropriate x in $Q(x,y)$, we immediately get that any real may be written as $f(u)-f(v)$ for some real u,v

$Q(f(u)-f(v),v)\implies f(f(u))=f(f(u)-f(v))+af(u)f(v)-af(v)^2+f(f(v))$ $Q(f(v)-f(u),u)\implies f(f(v))=f(f(v)-f(u))+af(v)f(u)-af(u)^2+f(f(u))$ Adding these two lines, we get $f(f(u)-f(v))+f(f(v)-f(u))=a(f(u)-f(v))^2$

And so $f(x)+f(-x)=ax^2\ \forall x$ Using then $4x$ instead of x in this equality and remembering that $f(4x)=8af(x)$, we get $a=2$ and so we now have :

$Q(x,y)$: $f(x+f(y))=f(x)+2xf(y)+f(f(y))$ $f(4x)=16f(x)$ $f(x)+f(-x)=2x^2$

$Q(f(x),x)\implies f(2f(x))=2f(f(x))+2f(x)^2$ $Q(2f(x),x)\implies f(3f(x))=3f(f(x))+6f(x)^2$ $Q(3f(x),x)\implies f(4f(x))=4f(f(x))+12f(x)^2$ And since $f(4f(x))=16f(f(x))$, we get $f(f(x))=f(x)^2$

And so $Q(x,y)$ becomes new assertion $R(x,y)$: $f(x+f(y)) = f(x) + 2xf(y) + f(y)^2$

$R(-f(v),v) \implies 0 = f(-f(v)) - 2f(v)^2 + f(v)^2$ and so $f(-f(v)) = f(v)^2$ $R(-f(v),u) \implies f(f(u)-f(v)) = f(-f(v)) - 2f(u)f(v) + f(u)^2 = f(u)^2 - 2f(u)f(v) + f(v)^2 = (f(u)-f(v))^2$

And so $f(x) = x^2$ which indeed is a solution.

[u][b]Hence the solutions [/b][/u]: $f(x) = 0\ \forall x$ $f(x) = x^2\ \forall x$

97. Find all function $f : \mathbb{R} \to \mathbb{R}$ such that: $f(f(x+y)) = f(x+y) + f(x)f(y) - xy$

solution

Let $P(x,y)$ be the assertion $f(f(x+y)) = f(x+y) + f(x)f(y) - xy$

$P(x+y,0) \implies f(f(x+y)) = f(x+y) + f(0)f(x+y)$ Subtracting this from $P(x,y)$, we get new assertion $Q(x,y)$: $f(0)f(x+y) = f(x)f(y) - xy$

$Q(1,1) \implies f(0)f(2) = f(1)^2 - 1$ $Q(x,1) \implies f(0)f(x+1) = f(x)f(1) - x$ $Q(x+1,1) \implies f(0)f(x+2) = f(x+1)f(1) - (x+1) \implies f(0)^2 f(x+2) = f(x)f(1)^2 - xf(1) - f(0)x - f(0)$ $Q(2,x) \implies f(0)f(x+2) = f(2)f(x) - 2x \implies f(0)^2 f(x+2) = (f(1)^2-1)f(x) - 2f(0)x$

And so $f(x)f(1)^2 - xf(1) - f(0)x - f(0) = (f(1)^2-1)f(x) - 2f(0)x$ which implies $f(x) = x(f(1)-f(0)) + f(0)$

So $f(x) = ax+b$ and plugging this in original equation, we get $a = 1$ and $b = 0$

Hence the solution $\boxed{f(x) = x}$ $\forall x$

98. Let $f(x)$ a continuous strictly decreasing function from $\mathbb{R}^+ \to \mathbb{R}^+$ such that : $f(x+y) + f(f(x)+f(y)) = f(f(x+f(y)) + f(y+f(x)))\ \forall x,y \in \mathbb{R}^+$ Prove that $f(f(x)) = x\ \forall x \in \mathbb{R}^+$

solution

$f(x)$ from $\mathbb{R}^+ \to \mathbb{R}^+$, continuous, strictly decreasing $\implies$ equation $f(x) = x$ has a unique root $a > 0$ Setting $y = a$ in the functional equation implies $f(x+a) + f(f(x)+a) = f(f(x+a) + f(f(x)+a))$ And so $f(x+a) + f(f(x)+a)$ is also root of $f(X) = X$ and so is a : $f(x+a) + f(f(x)+a) = a$ Setting $x \to f(x)$ in this expression, we get $f(f(x)+a) + f(f(f(x))+a) = a$ And so $f(f(f(x))+a) = f(x+a)$ and, since injective (since strictly decreasing) : $\boxed{f(f(x)) = x}$ Q.E.D and, btw, such a function exists : choose for example $f(x) = \frac{1}{x}$

99. Is there any systematic set of solutions to $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = g(f(x)) = 0$$

for all $x \in \mathbb{R}$?

solution

Choose any sets A, B such that $0 \in A$ and $0 \in B$ Let $u(x)$ any function from $\mathbb{R} \to A$ and $v(x)$ any function from $\mathbb{R} \to B$ Define f, g as :

$\forall x \in A : f(x) = 0$ $\forall x \notin A : f(x) = v(x)$

$\forall x \in B : g(x) = 0$ $\forall x \notin B : f(x) = u(x)$

It's easy to show that this is a general solution (it's a solution and any solution may be put in this form)

100. Find those values of the real parameter α such that there exists only one function f from reals to reals satisfying the following functional equation :

$$f(x^2 + y + f(y)) = (f(x))^2 + \alpha y.$$

solution

Let $P(x, y)$ be the assertion $f(x^2 + y + f(y)) = f(x)^2 + \alpha y$ Let $f(0) = a$

If $\alpha = 0$, then we get at least the two solutions $f(x) = 0$ $\forall x$ and $f(x) = 1$ $\forall x$. So $\alpha \neq 0$

Since $\alpha \neq 0$, $P(0, \frac{x-a^2}{\alpha}) \implies f(\frac{x-a^2}{\alpha} + f(\frac{x-a^2}{\alpha})) = x$ and so $\boxed{f(x) \text{ is surjective}}$. Comparing $P(x, y)$ and $P(-x, y)$, we get $f(-x)^2 = f(x)^2$ and so $\forall x$: either $f(-x) = -f(x)$, either $f(-x) = f(x)$

Let $x > 0$ and b such that $f(b) = -x$: $P(\sqrt{x}1, b) \implies -x = f(\sqrt{x})^2 + \alpha b$ and so $b = -\frac{x+f(\sqrt{x})^2}{\alpha} \neq 0$ So there is a unique $b \neq 0$ such that $f(b) = -x$ and so $f(-b)$ cant be equal to $f(b)$ and so $f(-b) = x = -f(b)$ $P(0, b)$ $\implies f(b + f(b)) = a^2 + \alpha b$ $P(0, -b) \implies f(-b - f(b)) = a^2 - \alpha b$ And since $f(-b - f(b)) = \pm f(b + f(b))$, we get $a^2 + \alpha b = \pm(a^2 - \alpha b)$ and so $a = \boxed{f(0) = 0}$ (since $b \neq 0$)

If $f(u) = f(v) = w < 0$, then the previous lines proved that $a = b$ $(= -\frac{-w+f(\sqrt{-w})^2}{\alpha} \neq 0)$ If $f(u) = f(v) = w > 0$, then $\exists$ unique t such that $f(t) = -w$ and $f(-t) = w$ and so $u = \pm t$ but $f(t) = -w$ and so $u = v = -t$ If $f(u) = 0$, then the previous lines proved that there is a unique b such that $f(b) = 0$ and since $f(0) = 0$, we get $b = 0$

So $\boxed{f(x) \text{ is an odd bijection}}$.

$P(0, y) \implies f(y + f(y)) = \alpha y$ $P(x, 0) \implies f(x^2) = f(x)^2$

And so $P(x, y)$ becomes $f(x^2 + y + f(y)) = f(x^2) + f(y + f(y))$ And since $f(x + f(x)) = \alpha x$ and $f(x)$ is bijective, we get that $x + f(x)$ is bijective too

And so $f(x^2+y+f(y)) = f(x^2)+f(y+f(y))$ becomes $f(u+v) = f(u)+f(v)$ $\forall u \geq 0$ and $\forall v$ So (since odd) : $\boxed{f(u+v) = f(u)+f(v)}$ $\forall u, v$ But $f(x^2) = f(x)^2$ implies $f(v) \geq 0$ $\forall v \geq 0$ and then $f(u+v) = f(u)+f(v)$ implies $f(x)$ non decreasing.

So $f(x) = cx$ (monotonous solution of Cauchy equation) and, plugging in original equation, we get : $c^2 = c$ and $\alpha = 2c$ and so $c = 1$ and $\alpha = 2$

[u][b]Hence the answer [/b][/u]: If $\alpha \notin \{0, 2\}$: no solution If $\alpha = 0$: at least two solutions If $\boxed{\alpha = 2}$: exactly one solution $f(x) = x$

101. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) + xy = f(x)f(y)$.

solution

Let $P(x, y)$ be the assertion $f(x+y) + xy = f(x)f(y)$

$P(x, 1) \implies f(x+1) + x = f(x)f(1)$ and so $f(x+1) = f(1)f(x) - x$ $P(x+1, 1) \implies f(x+2) + x + 1 = f(x+1)f(1)$ and so $f(x+2) = f(1)f(x+1) - x - 1$ and so $f(x+2) = f(1)^2 f(x) - x(f(1)+1) - 1$ $P(x, 2)$ $\implies f(x+2) + 2x = f(x)f(2)$ and so $f(x+2) = f(2)f(x) - 2x$

So $f(1)^2 f(x) - x(f(1)+1) - 1 = f(2)f(x) - 2x$ and $(f(1)^2 - f(2))f(x) = x(f(1)-1)+1$

$f(1)^2 - f(2) = 0$ would imply $x(f(1)-1)+1 = 0$ $\forall x$, which is impossible

So $f(x) = ax + b$ for some a, b and plugging this in original equation, we get $a = \pm 1$ and $b = 1$

[u][b]Hence the solutions [/b][/u]: $f(x) = x + 1$ $\forall x$ $f(x) = 1 - x$ $\forall x$

102. Let a and b be reals numbers, $b < 0$. Let f be a function from the real line R into R and satisfying: $(x \in R)$, $f(f(x)) = a + bx$ Prove that f has infinitly discontinuities.

solution

Writing $f(x) = g(x - \frac{a}{1-b}) + \frac{a}{1-b}$, the equation becomes $g(g(x)) = bx$

If $b = -1$ First, we note that $g(0) = 0$. Suppose g has only n discontinuities $x_1, \ldots, x_n$ (including zero), and let $S = \{x : x = g^i(x_j) \text{ for some } i, j\} \cup \{0\}$. S is still finite, and contains $4k+1$ elements for some integer $k \leq n$. Also, $g(S) = S$ and $g^{-1}(S) = S$. $\mathbb{R} \backslash S$ is the union of $4k+2$ open intervals, and g is continuous on each of these intervals. Since g maps $\mathbb{R} \setminus S$ to itself bijectively, these intervals must be mapped to each other by g. Let A be the set of these intervals; we define g on A in the natural way. Since each element of A is either entirely positive or entirely negative, $g^2(U) \neq U$ for each $U \in A$. On the other hand, g^4 is the identity on U, so each orbit in U has exactly four elements. The number of elements in U is not divisible by 4, and we have a contradiction.

If $b \neq -1$:

From $g(g(x)) = bx$, we get $g(bx) = bg(x)$ and $g(0) = 0$ Notice that $g(x)$ is a bijection and so $g(x) = 0 \iff x = 0$

Let $u > 0$ and $v = g(u) \neq 0$ If $v > 0$, then $g(v) = bu < 0$ and so there is a discontinuity in $[u, v]$ (or $[v, u]$) else we would have some $t \in (u, v)$ or (v, u) such that $g(t) = 0$, impossible If $v < 0$, then $g(v) = bu < 0$ and $g(bu) = bv > 0$ so there is a discontinuity in $[v, bu]$ (or $[bu, v]$) else we would have some $t \in (v, bu)$ or (bu, v) such that $g(t) = 0$, impossible

So there is at least a discontinuity $x_0 \neq 0$ Since $f(bx) = bf(x)$, a discontinuity point at x_0 implies a discontinuity point at bx_0 and so, since $b \neq -1$ and $x_0 \neq 0$, infinitely many discontinuity points. Q.E.D.

103. Find all functions f and g that satisfies:
$f(g(x)) = 2x^2 + 1$ and $g(f(x)) = (2x + 1)^2$

solution

Still a strange problem which strongly seems to be a crazy invented one :(:(In what contest did you get it ?

Obviously there is the trivial solution $f(x) = 2x + 1$ and $g(x) = x^2$ but there are infinitely many other solutions and I dont think we can give a form for all of them ..

Let the sequence a_n defined as $a_0 = 0$ and $a_{n+1} = 2a_n^2 + 1$ Choose then $u(x)$ as any continuous strictly increasing bijection from $[0, 1] \to [0, 1]$ Define $g(x)$ as : $\forall x \in [a_0, a_1)$: $g(x) = u(x)$ $\forall x \in [a_{n+1}, a_{n+2})$: $g(x) = (2g(\sqrt{\frac{x-1}{2}}) + 1)^2$ (notice that $\sqrt{\frac{x-1}{2}} \in [a_n, a_{n+1})$) $\forall x < 0$: $g(x) = g(-x)$ So $g(x)$ is even and is also a continuous increasing bijection from $[0, +\infty) \to [0, +\infty)$

For any $x \geq -\frac{1}{2}$, the equation $g(z) = (2x + 1)^2$ has two roots $\pm z$ and let $f(x) = |z|$ For any $x < -\frac{1}{2}$, the equation $g(z) = (2x + 1)^2$ has two roots $\pm z$ and let $f(x) = -|z|$

$f(x)$ and $g(x)$ are fully defined By construction of $f(x)$, we clearly have $g(f(x)) = (2x + 1)^2$ $\forall x$ It remains to check $f(g(x))) = 2x^2 + 1$:

Since $g(f(x)) = (2x+1)^2$, we get $g(f(g(x))) = (2g(x)+1)^2$ By construction of $g(x)$, we had $g(2x^2 + 1) = (2g(x) + 1)^2$ So $g(f(g(x))) = g(2x^2 + 1)$ But $g(x) \geq 0$ and so $f(g(x)) \geq 1$ And so $f(g(x)) = 2x^2 + 1$ (remember that $g(x)$ is even and is also a continuous increasing bijection from $[0, +\infty) \to [0, +\infty)$) Q.E.D.

So we built infinitely many solutions (f, g) to the problem.

Caution : these are not all the solutions. There are certainly a lot of other solutions

104. If $f(x)$ is a continuous function and $f(f(x)) = 1 + x$ then find $f(x)$.

solution

$f(x)$ is a continuous bijection and so is monotonic. If $f(x)$ is decreasing, then $\exists u$ such that $f(u)=u$ but then $f(f(u))=u\neq u+1$ and so $f(x)$ is increasing.

If $f(x)\leq x$ for some x, then $f(f(x))\leq f(x)\leq x$ and so $f(f(x))\neq x+1$. So $f(x)>x\ \forall x$ If $f(x)\geq x+1$ for some x, then $f(f(x))\geq f(x+1)$ and so $f(x+1)\leq x+1$, impossible (see previous line)

So $f(x)$ is a continuous increasing function such that $x<f(x)<x+1\ \forall x$

Let then $f(0)=a\in(0,1)$ $f(a)=f(f(0))=1$ and so $f([0,a))=[a,1)$ Using then $f(x)=1+f^{-1}(x)$, we get that knowledge of $f(x)$ in $[0,a)$ implies knowledge of $f(x)$ in $[a,1)$ Using then $f(x+1)=f(x)+1$, we get that knowledge of $f(x)$ in $[0,1)$ implies knowledge of $f(x)$ in $\mathbb{R}$

So $f(x)$ is full defined by its values over $[0,a)$

And obviously, the only constraints for these values are : increasing, continuous, and $f(a)=1$

[u][b]Hence the solutions [/b][/u]: Let any $a\in(0,1)$ Let any continuous increasing bijection $h(x)$ from $[0,a]\to[a,1]$ $h^{-1}(x)$ is a continuous increasing bijection from $[a,1]\to[0,a]$

Define $f(x)$ as : $\forall x\in[0,a):\ f(x)=h(x)$ $\forall x\in[a,1):\ f(x)=1+h^{-1}(x)$ $\forall x\notin[0,1):\ f(x)=f(\{x\})+\lfloor x\rfloor$

And so obviously infinitely many solutions (the simplest is trivially $x+\frac{1}{2}$)

Just for complementary info : here is a rather nice general family of solutions :

Let $u(x)$ any increasing continuous bijection from $[0,1]\to[0,1]$

Let $h(x)=\lfloor x\rfloor+u(\{x\})$ $h(x)$ is an increasing continuous bijection from $\mathbb{R}\to\mathbb{R}$

Then $f(x)=h^{-1}(h(x)+\frac{1}{2})$ is a continuous solution of the functional equation $f(f(x))=x+1$

The problem is that I'm not sure that this is a general solution (I mean that I'm not sure that all solutions may be obtained in this form). My previous post gives all the solutions

105. Given a real number A and an integer n with $2\leq n\leq 19$, find all polynomials $P(x)$ with real coefficients such that $P(P(P(x)))=Ax^n+19x+99$.

solution

Let $m=$ degree of $P(x)$. We know that degree of $P(P(P(x)))$ is m^3

If $A=0$ we get then $m^3=1$ and so $m=1$ and $P(x)=ax+b$ and we get $P(P(P(x)))=a^3x+b(a^2+a+1)=19x+99$ and so $P(x)=\sqrt[3]{19}x+\frac{99(\sqrt[3]{19}-1)}{18}$

If $A\neq 0$, we get then $m^3=n$ and, since $n\in[2,19]$, we get $m=2$ and $n=8$

So $P(x)=ax^2+bx+c$ The two highest degree summands of $P(P(x))$ are then $a^3x^4+2a^2bx^3$ The two highest degree summands of $P(P(P(x)))$ are then $a^7x^8+4a^6bx^7$ and so $b=0$ But then $P(x)$ is even, and so must be $P(P(P(x)))$, which is wrong. So no solution if $A\neq 0$

Hence the unique answer : $A=0$ and $P(x)=\sqrt[3]{19}x+\frac{99(\sqrt[3]{19}-1)}{18}$

106. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$f(xf(y)+f(x))=2f(x)+xy\ \forall x,y\in\mathbb{R}.$

solution

Let $P(x,y)$ be the assertion $f(xf(y)+f(x))=2f(x)+xy$

If $f(a)=f(b)$, comparing $P(1,a)$ and $P(1,b)$ implies $a=b$ and $f(x)$ is an injection. $P(1,x-2f(1)) \implies f(f(x-2f(1))+f(1))=x$ and $f(x)$ is a surjection Let then u,v such that $f(u)=0$ and $f(v)=1$: $P(u,v) \implies 0=uv$ and so either $f(0)=0$, either $f(0)=1$

If $f(0)=0$, then $P(x,0) \implies f(f(x))=2f(x)$ and so, since surjective, $f(x)=2x$ which is not a solution So $f(0)=1$

Let then $x\neq 0$ and y such that $f(y)=-\frac{f(x)}{x}$ (which exists since $f(x)$ is surjective) $P(x,y) \implies y=\frac{1-2f(x)}{x}$ and so : (i) : $f(\frac{1-2f(x)}{x})=-\frac{f(x)}{x}$ $\forall x\neq 0$

$P(x,-\frac{f(x)}{x}) \implies f(xf(-\frac{f(x)}{x})+f(x))=f(x)$ and so, since injective, $xf(-\frac{f(x)}{x})+f(x)=x$ and so : (ii) : $f(-\frac{f(x)}{x})=1-\frac{f(x)}{x}$ $\forall x\neq 0$

$P(-1,-1) \implies f(-1)=0$ $P(x,-1) \implies f(f(x))=2f(x)-x$ Setting $x\to\frac{1-2f(x)}{x}$ in this expression and, using (i) and (ii), we get : $f(f(\frac{1-2f(x)}{x}))=2f(\frac{1-2f(x)}{x})-\frac{1-2f(x)}{x}$ $f(-\frac{f(x)}{x})=-2\frac{f(x)}{x}-\frac{1-2f(x)}{x}$ $1-\frac{f(x)}{x}=-2\frac{f(x)}{x}-\frac{1-2f(x)}{x}$ $x-f(x)=-2f(x)-(1-2f(x))$ $f(x)=x+1$ $\forall x\neq 0$ And since $f(0)=1=0+1$, we get $\boxed{f(x)=x+1}$ $\forall x$, which indeed is a solution

107. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all reals x,y,z it holds that

$$f(x+f(y+z))+f(f(x+y)+z)=2y.$$

solution

Let $P(x,y,z)$ be the assertion $f(x+f(y+z))+f(f(x+y)+z)=2y$ Let $f(0)=a$

$P(x-a,\frac{a-x}{2},\frac{x-a}{2}) \implies f(x)+f(f(\frac{x-a}{2})+\frac{x-a}{2})=a-x$

$P(\frac{x-a}{2},0,\frac{x-a}{2}) \implies f(\frac{x-a}{2}+f(\frac{x-a}{2}))=0$

And so $\boxed{f(x)=a-x}$ which indeed is a solution, whatever is the real a

108. The set of all solutions of the equation $f(xy) = f(x)f(y)$ is :

a) $f(x) = 0\ \forall x$ b) $f(x) = 1\ \forall x$ c) $f(0) = 0$ and $f(x) = e^{h(\ln|x|)}$ where $h(x)$ is any solution of Cauchy equation $h(x+y) = h(x) + h(y)$ d) $f(0) = 0$ and $f(x) = \text{sign}(x)e^{h(\ln|x|)}$ where $h(x)$ is any solution of Cauchy equation $h(x+y) = h(x) + h(y)$

If you restrict to continuous solutions, then you get : a) $f(x) = 0\ \forall x$ b) $f(x) = 1\ \forall x$ c) $f(x) = |x|^a$ where a is any positive real d) $f(x) = \text{sign}(x)|x|^a$ where a is any positive real

109. Does the equation $x + f(y + f(x)) = y + f(x + f(y))$ have a continuous solution $f : \mathbb{R} \to \mathbb{R}$?

solution

Let $P(x, y)$ be the assertion $x+f(y+f(x)) = y+f(x+f(y))$ Let $g(x) = f(x)-x$ $P(x, y)$ becomes new assertion $Q(x, y) :\ x+g(x)+g(x+y+g(x)) = y+g(y)+g(x+y+g(y))$ From this equation, we get that $g(x)$ is injective and so, since continuous, monotonous.

$Q(x, -x) \implies x+g(x)+g(g(x)) = -x+g(-x)+g(g(-x))$ and so $x+g(x)+g(g(x))$ is an even function. But if $g(x)$ is increasing, $x + g(x) + g(g(x))$ is increasing, so injective, and so cant be even. So $g(x)$ is decreasing. Looking at $Q(x, y)$, we immediately get then that $\lim_{x\to-\infty} g(x) = +\infty$ and $\lim_{x\to+\infty} g(x) = -\infty$ (il any of these limits was a finite value, $Q(x, y)$ would lead to contradiction : one side infinite, the other finite).

Writing $Q(x, y)$ as $f(x)+g(y+f(x)) = f(y)+g(x+f(y))$, we get that $f(x)$ is injective too, and so monotonous. Writing $Q(x, y)$ as $-y+f(y+f(x)) = f(y)+g(x+f(y))$, we get that $\lim_{x\to+\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, in contradiction with the fact that $f(x)$ is monotonous.

[u][b]So no such continuous solution.[/b][/u]

110. Find all polynomials $P(x)$ of the smallest possible degree with the following properties:

[b](i)[/b] The leading coefficient is 200; [b](ii)[/b] The coefficient at the smallest non-vanishing power is 2; [b](iii)[/b] The sum of all the coefficients is 4; [b](iv)[/b] $P(-1) = 0, P(2) = 6, P(3) = 8$.

solution

(iii) implies $f(1) = 4$ (iii)+(iv) imply $f(x) = 2(x+1) + (x+1)(x-1)(x-2)(x-3)Q(x)$ (i) implies $f(x) = 2(x+1)+200(x+1)(x-1)(x-2)(x-3)Q(x)$ with $Q(x)$ monic

$Q(x) = 1$ is not a solution (smallest non vanishing power summand is -1998) $Q(x) = x + c$ implies that the powers 1 and 0 summands are $(1000c - 1198)x + 2 - 1200c$

$c = 0$ gives smallest non vanishing power summand is 2 and so is a solution $c = \frac{1}{600}$ gives smallest non vanishing power summand is $(\frac{5}{3} - 1198)x$ and so is not a solution

Hence the unique answer : $\boxed{f(x) = 2(x+1) + 200x(x+1)(x-1)(x-2)(x-3)}$

111. Find all functions $f : R \to R$ satisfy the following equation $f(f(x-y)) = f(x)f(y) + f(x) - f(y) - xy$

solution

Let $P(x,y)$ be the assertion $f(f(x-y)) = f(x)f(y) + f(x) - f(y) - xy$
Let $f(0) = a$

Notice that the summand xy in RHS implies that $f(x)$ can not be bounded.

$P(x,0) \implies f(f(x)) = (a+1)f(x) - a$ And so (squaring) : $f(f(x))^2 = (a+1)^2 f(x)^2 - 2a(a+1)f(x) + a^2$ $P(f(x), f(x)) \implies f(f(x))^2 = f(x)^2 + f(a)$

And so $(a+1)^2 f(x)^2 - 2a(a+1)f(x) + a^2 = f(x)^2 + f(a)$ And since $P(0,0)$ implies $a^2 = f(a)$, we get : $af(x)((a+2)f(x) - 2(a+1)) = 0$

Setting $x = 0$ in this last equality, we get $a^2(a^2 - 2) = 0$ and so $a = 0$ or $a^2 = 2$

If $a^2 = 2$, then $af(x)((a+2)f(x) - 2(a+1)) = 0$ implies $f(x) \in \{0, 2\frac{a+1}{a+2}\}$ bounded, in contradiction with original equation. So $a = 0$ and $P(x,x)$ $\implies f(x)^2 = x^2 \ \forall x$

Let then $x, y \notin \{0,1\}$ such that $f(x) = x$ and $f(y) = -y$: If $f(f(x-y)) = x - y$, $P(x,y)$ becomes $xy = y$, impossible If $f(f(x-y)) = y - x$, $P(x,y)$ becomes $xy = x$, impossible So : either $f(x) = x \ \forall x \neq 1$ either $f(x) = -x$ $\forall x \neq 1$

If $f(x) = x \ \forall x \neq 1$, then $P(3,1) \implies 2 = 3f(1) + 3 - f(1) - 3$ and so $f(1) = 1$ and so $f(x) = x \ \forall x$ If $f(x) = -x \ \forall x \neq 1$ then $P(2,0) \implies$ $2 = -2$, impossible

Hence the unique solution : $\boxed{f(x) = x}$ $\forall x$ which indeed is a solution

112. Find polynomials $f(x), g(x)$ and $h(x)$, if they exist, such that for all x,

$|f(x)| - |g(x)| + h(x) = -1$ if $x < -1$; $|f(x)| - |g(x)| + h(x) = 3x + 2$ if $-1 \leq x \leq 0$;

$|f(x)| - |g(x)| + h(x) = -2x + 2$ if $x > 0$

solution

If (f, g, h) is solution, so are $(\pm f, \pm g, h)$. So wlog say highest degrees coefficicients of f, g are positive.

1) If both f,g have even degrees : Then $|f(x)| = f(x)$ and $|g(x)| = g(x)$ when $x \to \pm\infty$, which is impossible (values of $|f| - |g| + h$ are different when $x \to \pm\infty$)

2) If both f,g have odd degrees : When $x \to -\infty$, we get $|f| = -f$ and $|g| = -g$ and so $-f + g + h = -1$ When $x \to +\infty$, we get $|f| = f$ and $|g| = g$ and so $f - g + h = 2 - 2x$ So $h(x) = \frac{1}{2} - x$ and $f - g = \frac{3}{2} - x$

Then $3x+2$ can only be $f+g+h$ or $-f-g+h$: 2.1) $f+g+h=3x+2$ Then $f(x)=\frac{3}{2}(x+1)$ and $g(x)=\frac{5}{2}x$ which is a solution

2.2) $-f-g+h=3x+2$ Then $f(x)=-\frac{5}{2}x$ and $g(x)=-\frac{3}{2}(x+1)$, impossible (we choosed highest coefficients positive)

3) If degree of f is even and degree of g is odd : When $x\to-\infty$, we get $|f|=f$ and $|g|=-g$ and so $f+g+h=-1$ When $x\to+\infty$, we get $|f|=f$ and $|g|=g$ and so $f-g+h=2-2x$ So $g(x)=x-\frac{3}{2}$ and $f+h=\frac{1}{2}-x$

Then $3x+2$ can only be $-f+g+h$ or $-f-g+h$: 3.1) $-f+g+h=3x+2$ Then $f(x)=-\frac{3}{2}(x+1)$, impossible (we choosed highest coefficients positive)

3.2) $-f-g+h=3x+2$ Then $f(x)=-\frac{5}{2}x$, impossible (we choosed highest coefficients positive)

4) If degree of f is odd and degree of g is even : When $x\to-\infty$, we get $|f|=-f$ and $|g|=g$ and so $-f-g+h=-1$ When $x\to+\infty$, we get $|f|=f$ and $|g|=g$ and so $f-g+h=2-2x$ So $f(x)=\frac{3}{2}-x$, impossible (we choosed highest coefficients positive)

Hence the four solutions : $\boxed{(f,g,h)=(\pm\frac{3}{2}(x+1),\pm\frac{5}{2}x,\frac{1}{2}-x)}$

113. Find all functions $f:\mathbb{Z}\backslash\{0\}\mapsto\mathbb{Q}$, satisfying $f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{2}$ whenever $x,y,\frac{x+y}{3}\in\mathbb{Z}\setminus\{0\}$.

solution

Let $P(x,y)$ be the assertion $f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{2}$

$P(1,2)\implies f(2)=f(1)$ $P(3,3)\implies f(3)=f(2)=f(1)$ $P(2,4)\implies f(4)=f(2)=f(1)$

Let then integer $n\geq 2$: $P(n,2n)\implies f(2n)=f(n)$ $P(n-1,2n+1)\implies f(2n+1)=2f(n)-f(n-1)$

And so (induction) $f(n)=f(1)$ $\forall n\in\mathbb{N}$

Let then $n\in\mathbb{N}$: $P(n+3,-n)\implies f(-n)=2f(1)-f(n+3)=f(1)$

Hence the solution : $\boxed{f(x)=a}$ $\forall x\in\mathbb{Z}\setminus\{0\}$ and for any $a\in\mathbb{Q}$

114. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real x,y

$$f(f(x)^2+f(y))=xf(x)+y.$$

solution

Let $P(x,y)$ be the assertion $f(f(x)^2+f(y))=xf(x)+y$ Let $f(0)=a$

$P(0,0)\implies f(a^2+a)=0$ and then $P(a^2+a,x)\implies f(f(x))=x$ and $f(x)$ is bijective and involutive.

Then $P(f(1),a)\implies f(1)=f(1)+a$ and so $a=0$

$P(f(x), f(y)) \implies f(x^2+y) = xf(x) + f(y)$ $P(f(x), 0) \implies f(x^2) = xf(x)$ Subtracting, we get $f(x^2+y) = f(x^2) + f(y)$

So $f(x+y) = f(x) + f(y)$ $\forall x \geq 0, \forall y$ and it's immediate to conclude $f(x+y) = f(x) + f(y)$ $\forall x, y$.

$P(f(x), 0) \implies f(x^2) = xf(x)$ $P(f(x+1), 0) \implies f(x^2+2x+1) = (x+1)f(x+1)$

Subtracting, we get $2f(x)+f(1) = xf(1)+f(x)+f(1)$ and so $f(x) = xf(1)$ $\forall x$

Plugging back in original equation, we get two solutions : $f(x) = x$ $\forall x$ $f(x) = -x$ $\forall x$

115. Find all continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x)+f(y)+xy(x+y)(x^2+xy+y^2)$.

solution

Let $g(x) = f(x) - \frac{x^5}{5}$ and the equation becomes $g(x+y) = g(x) + g(y)$ and so $g(x) = ax$ since continuous

Hence the solutions : $\boxed{f(x) = \dfrac{x^5}{5} + ax}$ $\forall x$ and for any real a

116. Find polynomial $P(x)$ such that $P(x)$ is divisible by (x^2+1) and $P(x)+1$ is divisible by x^3+x^2+1

solution

So $P(x) = (x^2+1)Q(x)$ and $P(x)+1 = (x^3+x^2+1)R(x)$

$\implies (x^2+1)Q(x)+1 = (x^3+x^2+1)R(x)$

$\implies R(i) = i$ and $R(-i) = -i$ and so $R(x) - x = (x^2+1)S(x)$

$\implies Q(x) = x^2+x-1+(x^3+x^2+1)S(x)$

$\implies \boxed{P(x) = (x^2+1)(x^2+x-1)+(x^2+1)(x^3+x^2+1)S(x)}$ which indeed is a solution whatever is polynomial $S(x)$

117. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+f(y)) = 2f(xf(y))$

solution

Let $P(x, y)$ be the assertion $f(x+f(y)) = 2f(xf(y))$

$f(x) = 1$ $\forall x$ is not a solution and so $\exists u$ such that $f(u) \neq 1$

$P(\frac{f(u)}{f(u)-1}, u) \implies f(v) = 0$ with $v = \frac{f(u)^2}{f(u)-1}$

$P(0, v) \implies f(0) = 0$ and then $P(x, v) \implies \boxed{f(x) = 0}$ $\forall x$ which indeed is a solution.

118. Find all functions $f:\mathbb{Q}^+\to\mathbb{Q}^+$ such that for all x,y in $\mathbb{Q}$ $f(f^2(x)y)=x^3f(xy)$ Here $f^2(x)$ means $f(x)*f(x)$

solution

Let $P(x,y)$ be the assertion $f(f^2(x)y)=x^3f(xy)$

$P(x,1) \implies f(f^2(x))=x^3f(x)$ and so $f(x)$ is injective.

$P(x,f^2(y)) \implies f(f^2(x)f^2(y))=x^3f(xf^2(y))$ $P(y,x) \implies f(f^2(y)x)=y^3f(xy)$ $P(xy,1) \implies x^3y^3f(xy)=f(f^2(xy))$

Multiplying these lines (and since no factor may be zero), we get $f(f^2(x)f^2(y))=f(f^2(xy))$ and so, since injective and positive : $f(xy)=f(x)f(y)$

((If you agree with $f(x)$ injective and $f(f^2(x)f^2(y))=f(f^2(xy))$ then, since $f(u)=f(v)$ implies $u=v$, we get $f^2(x)f^2(y)=f^2(xy)$

And since $f(x)>0\ \forall x$, we can just take square root and we get $f(x)f(y)=f(xy)))$ $P(x,y)$ becomes then $(f(f(x)))^2=x^3f(x)$ and $f(xy)=f(x)f(y)$

Setting $g_1(x)=xf(x)$, this is equivalent to $(g_1(g_1(x))^2=g_1^5(x)$ and $g_1(xy)=g_1(x)g_1(y)$

From there we get that $g_1(x)$ must always be the square of a rational and so it exists a function $g_2(x)$ from $\mathbb{Q}^+\to\mathbb{Q}^+$ such that : $g_1(x)=g_2(x)^2$ and so : $(g_2(g_2(x))^4=g_2^5(x)$ and $g_2(xy)=g_2(x)g_2(y)$

And this may be repeated infinitely, building a sequence of multiplicative functions $g_n(x)$ such that : $g_{n-1}(x)=g_n^2(x)$ and $(g_n(g_n(x)))^{2n}=g_n^5(x)$

And so the only possibility is $g_n(x)=1\ \forall x$ and $g(x)=1$ and so $\boxed{f(x)=\frac{1}{x}}$ which indeed is a solution.

119. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f\left(x+f(y)\right)=f(x+xy)+yf(1-x)$ for all real numbers x and y.

solution

Let $P(x,y)$ be the assertion $f(x+f(y))=f(x+xy)+yf(1-x)$

1) If $f(1)\neq 0 === P(0,x) \implies f(f(x))=f(0)+xf(1)$ and so $f(x)$ is injective. $P(0,0) \implies f(f(0))=f(0)$ and so $f(0)=0$ (since injective)

Let then $x\neq 0$: $P(\frac{f(x)}{x},x) \implies f(1-\frac{f(x)}{x})=0$ and so $1-\frac{f(x)}{x}=0$ (since injective) So $f(x)=x\ \forall x$ which indeed is a solution.

2) If $f(1)=0 === P(0,0) \implies f(f(0))=f(0)$ $P(1,f(0)) \implies f(0)^2=0$ and so $f(0)=0$ $P(0,x) \implies f(f(x))=0$

$P(1,f(x-1)) \implies f(f(x-1)+1)=0$ $P(1,x-1) \implies f(f(x-1)+1)=f(x)$

And so $f(x)=0\ \forall x$ which indeed is a solution.

[u][b]Hence the solutions [/b][/u]: $f(x)=x\ \forall x$ $f(x)=0\ \forall x$

120. Find all functions f such that $[f(x).f(y)]^2 = f(x+y).f(x-y)$ (x, y Reals)

solution

As is, we have at least infinitely many solutions : $f(x) = 0$ $\forall x$ $f(x) = e^{ah(x)^2}$ where $h(x)$ is any solution of Cauchy equation $f(x) = -e^{ah(x)^2}$ where $h(x)$ is any solution of Cauchy equation And also any product of such solutions

If we add the statment of continuity Let $P(x, y)$ be the assertion $f(x)^2 f(y)^2 = f(x+y)f(x-y)$

$f(x) = 0$ $\forall x$ is a solution and let us from now look for non all-zero solutions. Let u such that $f(u) \neq 0$

$P(u, 0) \implies f(u)^2 f(0)^2 = f(u)^2$ and so $f(0) = \pm 1$ $f(x)$ solution implies $-f(x)$ solution and so wlog say $f(0) = +1$

If $f(t) = 0$ for some $t \neq 0$, then $P(\frac{t}{2}, \frac{t}{2}) \implies f(\frac{t}{2})^4 = f(t)$ and so $f(\frac{t}{2}) = 0$ and so $f(\frac{t}{2^n}) = 0$ $\forall n \in \mathbb{N}$ So continuity would imply $f(0) = 0$, impossible.

So $f(x) > 0$ $\forall x$ and we can write $f(x) = e^{g(x)}$ for some continuous function $g(x)$ such that : $g(0) = 0$ New assertion $Q(x, y)$: $2g(x) + 2g(y) = g(x+y) + g(x-y)$ $\forall x, y$

Let $x \in \mathbb{R}$ and the sequence $a_n = g(nx)$ with $a_0 = 0$ $Q((n+1)x, x) \implies a_{n+2} = 2a_{n+1} - a_n + 2a_1$ whose solution is $a_n = a_1 n^2$

So $g(nx) = n^2 g(x)$ $\forall x, \forall n \in \mathbb{N}$ It's immediate to show that this is still true for $n \in \mathbb{Z}$

$g(p) = p^2 g(1)$ $\forall p \in \mathbb{Z}$ and so $p^2 g(1) = g(q\frac{p}{q}) = q^2 g(\frac{p}{q})$

So $g(x) = x^2 g(1)$ $\forall x \in \mathbb{Q}$ and continuity again gives $g(x) = ax^2$ $\forall x \in \mathbb{R}$

[u][b]Hence the continuous solutions of the equation [/b][/u] (it's easy to check back that they indeed are solutions) : $f(x) = 0$ $\forall x$ $f(x) = e^{ax^2}$ $\forall x \in \mathbb{R}$ and for any real a $f(x) = -e^{ax^2}$ $\forall x \in \mathbb{R}$ and for any real a

121. Find all functions $f : \mathbb{R} \to [0; +\infty)$ such that:

$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$$

for all real numbers x and y.

solution

Let $P(x, y)$ be the assertion $f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$

$P(0, 0) \implies f(0) = 0$ $P(0, x) \implies f(x^2) = f(-x^2)$ and so $f(x)$ is even.

Let $x \geq y \geq z \geq 0$

(a) : $P(\sqrt{\frac{x+y}{2}}, \sqrt{\frac{x-y}{2}}) \implies f(x) = f(y) + f(\sqrt{x^2 - y^2})$

(b) : $P(\sqrt{\frac{y+z}{2}},\sqrt{\frac{y-z}{2}}) \implies f(y)=f(z)+f(\sqrt{y^2-z^2})$

(c) : $P(\sqrt{\frac{x+z}{2}},\sqrt{\frac{x-z}{2}}) \implies f(x)=f(z)+f(\sqrt{x^2-z^2})$

(a)+(b)-(c) : $f(\sqrt{x^2-z^2})=f(\sqrt{x^2-y^2})+f(\sqrt{y^2-z^2})$

Writing $f(x)=g(x^2)$, this becomes $g(x+y)=g(x)+g(y)$ $\forall x,y\geq 0$ And since $g(x)\geq 0$, we get $g(x)=ax$ and so $f(x)=ax^2$ $\forall x\geq 0$ and for some $a\geq 0$

And since $f(x)$ is even, we get $\boxed{f(x)=ax^2}$ $\forall x$ and for any real $a\geq 0$ which indeed is a solution.

122. Given two function $f,g:\mathbb{R}\to\mathbb{R}$, such that $f(x+g(y))=3x+y+12$ for all $x,y\in R$. Find the value of $g(2004+f(2004))$

solution

Let $P(x,y)$ be the assertion $f(x+g(y))=3x+y+12$

$P(x-g(0),0) \implies f(x)=3x-3g(0)+12$

$P(-g(x),x) \implies f(0)=-3g(x)+x+12$

So $f(x)=3x+a$ and $g(x)=\frac{x}{3}+b$ with $a+3b=12$ which indeed are solutions

Then $g(x+f(x))=g(4x+a)=\frac{4x}{3}+\frac{a+3b}{3}=\frac{4x}{3}+4$

And so $\boxed{g(2004+f(2004))=2676}$

123. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+f(y))=f(y^2+3)+2x\cdot f(y)+f(x)-3, \qquad \forall x,y\in\mathbb{R}.$$

solution

Let $P(x,y)$ be the assertion $f(x+f(y))=f(y^2+3)+2xf(y)+f(x)-3$
Let $f(0)=a$

$P(x,y)$ may be written $f(x+f(y))-f(x)=(f(y^2+3)-3)+2xf(y)$ So, since $f(x)=0$ $\forall x$ is not a solution, we get that any real x may be written $x=f(u)-f(v)$ for some u,v

Let $g(x)=f(x)-x^2-a$. $P(x,y)$ becomes $g(x+f(y))=g(x)+f(y^2+3)-f(y)^2-3$ $P(0,y)$ becomes $g(f(y))=f(y^2+3)-f(y)^2-3$ Subtracting, we get new assertion $Q(x,y)$: $g(x+f(y))=g(x)+g(f(y))$

(a) : $Q(x-f(z),y) \implies g(x+f(y)-f(z))=g(x-f(z))+g(f(y))$ (b) : $Q(x-f(z),z) \implies g(x)=g(x-f(z))+g(f(z))$ (c) : $Q(f(y)-f(z),z) \implies g(f(y))=g(f(y)-f(z))+g(f(z))$ (a)-(b)+(c) : $g(x+f(y)-f(z))-g(x)=g(f(y)-f(z))$

And since any real may be written as $f(y)-f(z)$, we get $g(x+y)=g(x)+g(y)$

And so we get $f(x) = x^2 + a + g(x)$ where $g(x)$ is some solution of additive Cauchy equation.

Plugging this in $P(0, x)$: $f(f(x)) = f(x^2 + 3) + a - 3$, we get :

$a^2 + g(x)^2 + 2ax^2 + 2x^2g(x) + 2ag(x) + g(a) + g(g(x)) - 6x^2 - 6 - g(3) - a = 0$

Replacing in the above line $x \to px$ with $p \in \mathbb{Q}$ and remembering that $g(px) = pg(x)$, we get : $a^2 + p^2g(x)^2 + 2ax^2p^2 + 2x^2g(x)p^3 + 2ag(x)p + g(a) + g(g(x))p - 6x^2p^2 - 6 - g(3) - a = 0$

And this is a polynomial in p which is zero for any $p \in \mathbb{Q}$ and so this is the null polynomial. So coefficient of p^3 is zero and so $g(x) = 0\ \forall x$

So $f(x) = x^2 + a$ and plugging this in original equation, we easily get $a = 3$

Hence the unique solution $\boxed{f(x) = x^2 + 3}$

124. Find all functions $f : R \to R$ such that for $x \in R\backslash 0{,}1$:
$f(\frac{1}{x}) + f(1 - x) = x$

solution

Let $P(x)$ be the assertion $f(\frac{1}{x}) + f(1 - x) = x$

(a) : $P(\frac{1}{x}) \implies f(x) + f(\frac{x-1}{x}) = \frac{1}{x}$

(b) : $P(1 - x) \implies f(\frac{1}{1-x}) + f(x) = 1 - x$

(c) : $P(\frac{x}{x-1}) \implies f(\frac{x-1}{x}) + f(\frac{1}{1-x}) = \frac{x}{x-1}$

(a)+(b)-(c) : $\boxed{f(x) = \dfrac{1}{2x} - \dfrac{x}{2} - \dfrac{1}{2(x-1)}}$ $\forall x \notin \{0, 1\}$ and $f(0), f(1)$ taking any value we want. And it's easy to check back that this indeed is a solution.

125. Find all pairs of functions $f, g : \mathbb{Z} \to \mathbb{Z}$ such that:

$$f(g(x) + y) = g(f(y) + x)$$

holds for arbitrary integers x, y and g is injective.

solution

I didn't notice why we need $\mathbb{Z}$ instead of $\mathbb{R}$ in this problem, but anyway.

$f(g(x) + y) = g(f(y) + x) \Leftrightarrow g(f(g(x) + y) + z) = g(g(f(y) + x) + z) \Leftrightarrow$

$\Leftrightarrow f(g(z) + g(x) + y) = g(g(f(y) + x) + z) \Leftrightarrow$

$\Leftrightarrow g(f(g(z) + y) + x) = g(g(f(y) + x) + z) \implies$

$\implies f(g(z)+y)+x = g(f(y)+x)+z \Leftrightarrow g(f(y)+z)+x = g(f(y)+x)+z.$

Put $z = -f(y)$: $g(0) + x + f(y) = g(f(y) + x)$

Put $x=-f(y)+t: g(t)=t+g(0)=t+c$.

Our statement now looks as follows $f(x+y+c)=x+f(y)+c$.

Put $x=-c-y: f(y)=y+f(0)$.

[b]Answer: $f(x)=x+c_1, g(x)=x+c_2$

126. Find all function $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(f(x)+y)=f(x^2-y^2)+4f(x).y, \forall x,y\in\mathbb{R}$$

solution

$f(x)=0\ \forall x$ is a solution. Let us from now look for non allzero solutions. Let $P(x,y)$ be the assertion $f(f(x)+y)=f(x^2-y^2)+4f(x)y$ Let $f(u)=v\neq 0$

1) Any real may be written as $x=f(a)-f(b)$ for some $a,b\in\mathbb{R}$ ======================= (a) : $P(u,\frac{x}{8v}) \implies f(u+\frac{x}{8v})=f(u^2-(\frac{x}{8v})^2)+\frac{x}{2}$ (b) : $P(u,-\frac{x}{8v}) \implies f(u-\frac{x}{8v})=f(u^2-(\frac{x}{8v})^2)-\frac{x}{2}$ (a)-(b) : $x=f(u+\frac{x}{8v})-f(u-\frac{x}{8v})$ Q.E.D.

2) $f(x)$ is even ============== (a) : $P(x,f(y)) \implies f(f(x)+f(y))=f(x^2-f(y)^2)+4f(x)f(y)$ (b) : $P(x,-f(y)) \implies f(f(x)-f(y))=f(x^2-f(y)^2)-4f(x)f(y)$ (c) : $P(y,f(x)) \implies f(f(x)+f(y))=f(y^2-f(x)^2)+4f(x)f(y)$ (d) : $P(y,-f(x)) \implies f(f(y)-f(x))=f(y^2-f(x)^2)-4f(x)f(y)$ (a)-(b)-(c)+(d) : $f(f(x)-f(y))=f(f(y)-f(x))$ Q.E.D. (using 1))

3) If $f(x)=x$ for some x implies $x=0$ ================================== $P(x,-x) \implies f(f(x)-x)=f(0)-4xf(x)$ If $f(x)=x$, this becomes $f(0)=f(0)-4x^2$ Q.E.D.

4) $f(0)=0$ ========== $P(0,0) \implies f(f(0))=f(0)$ and so, using 3) : $f(0)=0$ Q.E.D.

5) No non allzero solution ======================= $P(0,u) \implies f(u)=f(-u^2)=f(u^2)$ $P(u,0) \implies f(f(u))=f(u^2)$ And so $f(f(u))=f(u)$ and so, using 3) : $f(u)=0$ and so contradiction

Hence the unique solution : $\boxed{f(x)=0}\ \forall x$

127. Find all surjective functions $f:\mathbb{N}\to\mathbb{N}$ if $f(n)\geq n+(-1)^n, \forall n\in\mathbb{N}$.

solution

Let S_n be the set of natural numbers solutions of the equation $x+(-1)^x\leq n$: Obviously, this set is the set of all even numbers $\leq n-1$ and all odd numbers $\leq n+1$ and so :

$S_{2p}=\{1,2,3,...,2p-1,2p+1\}$ $S_{2p+1}=\{1,2,3,...,2p+1\}$

So $S_1=\{1\}$ and so $f(1)=1$

We clearly have $f^{-1}([1,n])\subseteq\bigcup_{k\in[1,n]}S_k$ So $f^{-1}([1,2p])\subseteq\{1,2,3,...,2p-1,2p+1\}$ And $f^{-1}([1,2p+1])\subseteq\{1,2,3,...,2p+1\}$

So $|f^{-1}([1,n])| = n$ and this implies that $f^{-1}(\{n\}) = f^{-1}([1,n]) \backslash f^{-1}([1,n-1])$

[u][b]Hence the unique solution [/b][/u]: $f(1) = 1$ $f(2p) = 2p+1$ $\forall p \geq 1$ $f(2p+1) = 2p$ $\forall p \geq 1$

128. It is true,for any quadratic functions $f(x)$ and for any distinct number a, b, c, $f(a) = bc, f(b) = ac, f(c) = ab$. Find $f(a+b+c)$

solution

I had a doubt about the fact that $P(x)$ was monic but I understood that you used the constant term to conclude this (and this was the reason for which you distinguished the case $abc = 0$).

Direct method is less elegant but works fine too : let $f(x) = ux^2 + vx + w$: (1) : $ua^2 + va + w = bc$ (2) : $ub^2 + vb + w = ac$ (3) : $uc^2 + vc + w = ab$

(2)-(1) : $u(b^2 - a^2) + v(b-a) = c(a-b)$ and so, since distincts : $u(a+b) + v = -c$ (3)-(1) : $u(c^2 - a^2) + v(c-a) = b(a-c)$ and so, since distincts : $u(a+c) + v = -b$

Subtracting ; $u(b-c) = b-c$ and so $u = 1$ and so $v = -a-b-c$ and so, using (1) : $w = ab + bc + ca$

And $f(x) = x^2 - (a+b+c)x + ab + bc + ca$ and $f(a+b+c) = ab + bc + ca$

129. Determine all monotone functions $f : [0; +\infty[\to \mathbb{R}$ such that

$f(x+y) - f(x) - f(y) = f(xy+1) - f(xy) - f(1)$, for all $x, y \geq 0$ and

$f(3) + 3f(1) = 3f(2) + f(0)$.

solution

If $f(x)$ is solution, then so is $f(x) + a$ and so Wlog say $f(1) = 1$

Let $P(x,y)$ be the assertion $f(x+y) - f(x) - f(y) = f(xy+1) - f(xy) - 1$

Let $m, n, p \in \mathbb{N}$ and let $g(x) = f(\frac{x}{p})$ Comparing $P(\frac{2m}{p}, \frac{n}{p})$ and $P(\frac{2n}{p}, \frac{m}{p})$, we get : $g(2m+n) - g(2m) - g(n) = g(2n+m) - g(2n) - g(m)$

1) Let us look for all solutions of the following problem : "Find all functions $g(x)$ from $\mathbb{N} \to \mathbb{R}$ such that : $g(2x+y) - g(2x) - g(y) = g(2y+x) - g(2y) - g(x)$ $\forall x, y \in \mathbb{N}$"

The set $\mathbb{S}$ of solutions is a $\mathbb{R}$-vector space. Setting $y = 1$, we get $g(2x+1) = g(2x) + g(1) + g(x+2) - g(2) - g(x)$ Setting $y = 2$, we get $g(2x+2) = g(2x) + g(2) + g(x+4) - g(4) - g(x)$ From these two equations, we see that knowledge of $g(1), g(2), g(3), g(4)$ and $g(6)$ gives knowledge of $g(x)$ $\forall x \in \mathbb{N}$ and so dimension of $\mathbb{S}$ is at most 5. But the 5 functions below are independant solutions : $g_1(x) = 1$ $g_2(x) = x$ $g_3(x) = x^2$ $g_4(x) = 1$ if $x = 0 \pmod 2$ and $g_4(x) = 0$ if $x \neq 0 \pmod 2$ $g_5(x) = 1$ if $x = 0$

(mod 3) and $g_5(x) = 0$ if $x \neq 0 \pmod 3$ And the general solution is $g(x) = a \cdot x^2 + b \cdot x + c + d \cdot g_4(x) + e \cdot g_5(x)$

2) back to our problem So $f(\frac{x}{p}) = a_p x^2 + b_p x + c_p + d_p g_4(x) + e_p g_5(x)$ $\forall x \in \mathbb{N}$

Choosing $x = kp$, we get $f(k) = a_p k^2 p^2 + b_p kp + c_p + d_p g_4(kp) + e_p g_5(kp)$ and so $a_p p^2 = a$ and $b_p p = b$ for some real a, b Choosing $x = 2kp$, $x = 3kp$ and $x = 6kp$, we get $c_p = c$ and $d_p = e_p = 0$

So $f(\frac{x}{p}) = a\frac{x^2}{p^2} + b\frac{x}{p} + c\ \forall x \in \mathbb{N}$

And so $f(x) = ax^2 + bx + c\ \forall x \in \mathbb{Q}^+$

$f(x)$ monotonous implies then $a = 0$ or $\frac{b}{a} \geq 0$

$f(x)$ monotonous implies then $f(x) = ax^2 + bx + c\ \forall x \in \mathbb{R}^+$

$f(3) + 3f(1) = 3f(2) + f(0)$ implies then $f(x) = ax^2 + bx + c\ \forall x \in \mathbb{R}_0^+$ and it's easy to check back that this mandatory form indeed is a solution.

[u][b]Hence the answer [/b][/u]: $\boxed{f(x) = ax^2 + bx + c}$ $\forall x \geq 0$ and for any real a, b, c such that $ab \geq 0$

130. "Find all polynomials $p(x), q(x) \in \mathbb{R}[X]$ such that $p(x)q(x+1) - p(x+1)q(x) = 1\ \forall x \in \mathbb{R}$"

solution

Notice that if the equality is true for any $x \in \mathbb{R}$, it's also true for any $x \in \mathbb{C}$

We get : $p(x)q(x+1) - p(x+1)q(x) = 1$ $p(x-1)q(x) - p(x)q(x-1) = 1$

And so, subtracting $p(x)(q(x-1) + q(x+1)) = q(x)(p(x-1) + p(x+1))$

But no real or complex zero of $p(x)$ may be a zero of $q(x)$ else $p(x)q(x+1) - p(x+1)q(x) = 1$ would be false. So $p(x)|p(x-1) + p(x+1)$ and since they are two polynomials with same degree, we get :

$p(x+1) + p(x-1) = ap(x)$ (and same for $q(x)$ with same constant a).

Writing this as $\frac{p(x+1)}{p(x)} + \frac{p(x-1)}{p(x)} = a$ and setting $x \to +\infty$, we get $a = 2$

So $p(x+1) - p(x) = p(x) - p(x-1)$ and so $p(x+1) - p(x) = b$ constant (since polynomials).

So $p(x) = bx + c$ and $q(x) = b'x + c'$

Plugging this in original equation, we get $cb' - bc' = 1$

Hence the answer $p(x) = ax + b$ $q(x) = cx + d$ for any real a, b, c, d such that $bc - ad = 1$

131. The function $f(x)$ defined by

$f(x) = \frac{ax+b}{cx+d}$. Where a, b, c, d are non zero real number has the properties $f(19) = 19$ and $f(97) = 97$.

And, $f(x(x)) = x$. for all value of x except $-\frac{d}{c}$. Find the range of $f(x)$

solution

$f(x) = x - \frac{cx^2+(d-a)x-b}{cx+d}$ and so $cx^2 + (d-a)x - b = c(x-19)(x-97)$ and so $\frac{d-a}{c} = -116$ and $\frac{b}{c} = -1843$

Setting $\frac{d}{c} = u$, we get $f(x) = \frac{(116+u)x-1843}{x+u}$

Since $f(f(x)) = x\ \forall x \neq -u$, we get $f(x) \neq -u\ \forall x \neq -u$ The equation $f(x) = -u$ is $x \neq -u$ and $(116+2u)x = 1843-u^2$ and so : either $u = -58$ and so we have no solution to this equation either $-(116+2u)u = 1843-u^2$ (and so the only solution is $x = -u$) but then we get $-u \in \{19, 97\}$, impossible

So $u = -58$ and $f(x) = \frac{58x-1843}{x-58}$ and it's easy to check that this function indeed is a solution.

And so $\boxed{f(\mathbb{R} \setminus \{-\frac{d}{c}\}) = \mathbb{R} \setminus \{58\}}$

132. Let E be the set of all bijective mappings from $\mathbb{R}$ to $\mathbb{R}$ satisfying

$$f(t) + f^{-1}(t) = 2t, \qquad \forall t \in \mathbb{R},$$

where f^{-1} is the mapping inverse to f. Find all elements of E that are monotonic mappings.

solution

$f(x)$ strictly (since bijective) monotonic implies $f^{-1}(x)$ strictly monotonic in the same direction (both increasing or both decreasing) and since their sum is increasing, we get that $f(x)$ is increasing.

Suppose now that $f(x) - x$ is not constant. Let then $u \neq v$ such that $f(u) - u = a > b = f(v) - v$

Using $f(x)+f^{-1}(x) = 2x$, it's easy to show that $f(u+na) = u+(n+1)a$ and $f(v + nb) = v + (n + 1)b\ \forall n \in \mathbb{Z}$

Let then $n = \left\lfloor \frac{v-u}{a-b} \right\rfloor$ so that $n + 1 > \frac{v-u}{a-b} \geq n$: $\frac{v-u}{a-b} \geq n \implies v - u \geq na-nb \implies v+nb \geq u+na \implies f(v+nb) \geq f(u+na)$ (since increasing) $\implies v + (n+1)b \geq u + (n+1)a \implies \frac{v-u}{a-b} \geq n+1$ And so contradiction.

So $f(x) - x$ is constant and $\boxed{f(x) = x + c}$ $\forall x$, and for any real c And it's easy to check back that these functions indeed are solutions.

133. If

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right) \qquad \forall x, y \in \mathbb{R} \text{ and } xy \neq 1$$

and

$$\lim_{x \to 0} \frac{f(x)}{x} = 2$$

Then find $f(x)$.

solution

Let $g(x)$ from $=]-\frac{\pi}{2};+\frac{\pi}{2}[\to \mathbb{R}$ defined as $g(x)=f(\tan x)$

The functional equation implies $g(x)+g(y)=g(x+y)$ $\forall x,y,x+y\in A$ The second property implies that $g(x)$ is bounded on some non empty open interval containing 0

So we get $g(x)=ax$ $\forall x\in A$ and second property implies $a=2$ So $f(x)=2\arctan x$ $\forall x$

But this mandatory function obviously does not match the functional equation (set $x=y=\sqrt{3}$ as counterexample)

So no solution for this functional equation.

134. If

$$f(xy)=xf(y)+yf(x)\quad \forall x,y\in \mathbb{R}^+$$

and $f(x)$ is differentiable in $(0,\infty)$. Then find $f(x)$.

solution

Let $g(x)$ from $\mathbb{R}\to\mathbb{R}$ defined as $g(x)=e^{-x}f(e^x)$ and we get $g(x+y)=g(x)+g(y)$

Since $f(x)$ is differentiable in $(0,+\infty)$, $g(x)$ is continuous and so $g(x)=ax$

And so $\boxed{f(x)=ax\ln x}$ $\forall x>0$ and $f(x)=$ any value for $x\leq 0[$

135. Find all functions from non-zero rationals to reals such that f(xy)=f(x)+f(y)

solution

Let $P(x,y)$ be the assertion $f(xy)=f(x)+f(y)$

$P(1,1)\implies f(1)=0$ $P(-1,-1)\implies f(-1)=0$ $P(x,-1)\implies f(-x)=f(x)$

$P(x,\frac{1}{x})\implies f(\frac{1}{x})=-f(x)$

So $f(x^n)=nf(x)$ $\forall x\in\mathbb{Q}^*,\forall n\in\mathbb{Z}$

And since any positive rational may be written in a unique manner as $x=\prod p_i^{n_i}$ with p_i prime and $n_i\in\mathbb{Z}^*$, we get $f(x)=\sum n_i f(p_i)$

And it's easy to see that this indeed is a solution.

[u][b]Hence the answer [/b][/u]: We can choose in any manner the values $f(p_i)$ for all primes and from there : For any rational $x>0:$ $f(1)=0$ For $x\neq 1:$ $x=\prod p_i^{n_i}$ with p_i prime and $n_i\in\mathbb{Z}^*$ and then $f(x)=\sum n_i f(p_i)$

For any rational $x<0:$ $f(x)=f(-x)$

136. Let a function $g : \mathbb{N}_0 \to \mathbb{N}_0$ satisfy $g(0) = 0$ and $g(n) = n - g(g(n-1))$ for all $n \geq 1$. Prove that:

a) $g(k) \geq g(k-1)$ for any positive integer k. b) There is no k such that $g(k-1) = g(k) = g(k+1)$.

solution

First notice that $g(n) \leq n\ \forall n \in \mathbb{N}_0$ Let us then prove with induction that $g(n+1) - g(n) \in \{0, 1\}\ \forall n \in \mathbb{N}_0$

$g(0) = 0$ $g(1) = 1 - g(g(0)) = 1$ $g(2) = 2 - g(g(1)) = 1$

and so $g(k+1) - g(k) \in \{0, 1\}\ \forall k \in [0, 1]$

Suppose now $g(k+1) - g(k) \in \{0, 1\}\ \forall k \in [0, n-1]$ for some $n \geq 2 \in \mathbb{N}$ $g(n+1) - g(n) = 1 - (g(g(n)) - g(g(n-1)))$ We know that $g(n) - g(n-1) \in \{0, 1\}$ and so : If $g(n) - g(n-1) = 0$, we get $g(g(n)) - g(g(n-1)) = 0$ and so $g(n+1) - g(n) = 1$ If $g(n) - g(n-1) = 1$, we get $g(g(n)) - g(g(n-1)) = g(g(n-1)+1) - g(g(n-1)) \in \{0, 1\}$ (since $g(n-1) \leq n-1$ and using then the induction property) And so $g(n+1) - g(n) = 1 - (g(g(n)) - g(g(n-1))) \in \{0, 1\}$ End of induction step

And so $g(n+1) \geq g(n)\ \forall n \in \mathbb{N}_0$ and part $a)$ is proved.

Part b) is quite simple : If $g(n) = g(n-1)$, then $g(g(n)) = g(g(n-1))$ and so $g(n+1) - g(n) = n + 1 - g(g(n)) - n + g(g(n-1)) = 1$ and so $g(n+1) \neq g(n)$ Q.E.D.

137. Does there exist $f : N -> N$ such that $3n \leq f(n) + f(f(n)) \leq 3n + 1$?

solution

$f(1) + f(f(1)) \in [3, 4]$ and so $f(1) \in \{1, 2, 3\}$

If $f(1) = 1$ then $f(1) + f(f(1)) = 2 \notin [3, 4]$ and so impossible If $f(1) = 2$ then $f(f(1)) \in [1, 2]$ – If $f(1) = 2$ and $f(f(1)) = f(2) = 1$ then $f(2) + f(f(2)) = 3 \notin [6, 7]$ and so impossible – If $f(1) = 2$ and $f(f(1)) = f(2) = 2$ then $f(2) + f(f(2)) = 4 \notin [6, 7]$ and so impossible If $f(1) = 3$ then $f(f(1)) = f(3) = 1$ and then $f(3) + f(f(3)) = 4 \notin [9, 10]$ and so impossible

So no such function

138. find the polyminal with coefficient in R such that:

$\forall x, y \in R$

$$P(x^{2010} + y^{2010}) = (P(x))^{2010} + (P(y))^{2010}$$

solution

Let $A(x, y)$ be the assertion $P(x^n + y^n) = P(x)^n + P(y)^n$ where $n = 2010$

$A(x, 0) \implies P(x^n) = P(x)^n + P(0)^n$ $A(y, 0) \implies P(y^n) = P(y)^n + P(0)^n$

Subtracting these two lines from $A(x, y)$, we get $P(x^n + y^n) = P(x^n) + P(y^n) - 2P(0)^n$

And so $P(x+y) = P(x) + P(y) + a\ \forall x, y \geq 0$ and for some $a \in \mathbb{R}$ And so $P(x+y) = P(x) + P(y) + a\ \forall x, y$ and for some $a \in \mathbb{R}$ And so $P(x) - a$ is a continuous solution of Cauchy's equation.

So $P(x) = cx + a$ for some a, b and, plugging in original equation, we get the solutions :

$P(x) = 0\ \forall x$

$P(x) = 2^{-\frac{1}{2009}}\ \forall x$

$P(x) = x\ \forall x$

139. Given $f(x) = ax^3 + bx^2 + cx + d$, such that $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, $f(3) = 8$. Find the value of $f(4)$

solution

$f(0) = 1 \iff d = 1$

(e1) : $f(1) = 2 \iff a + b + c = 1$ (e2) : $f(2) = 4 \iff 8a + 4b + 2c = 3$ (e3) : $f(3) = 8 \iff 27a + 9b + 3c = 7$

(e2)-2(e1) : $6a + 2b = 1$ (e3)-3(e1) : $12a + 3b = 2$ This gives $a = \frac{1}{6}$ and $b = 0$

And so $c = \frac{5}{6}$ and $f(x) = \frac{x^3+5x+6}{6}$ and $\boxed{f(4) = 15}$

140. Does There Exist A Function

$f : N \to N$

$\forall n \geq 2$

$f(f(n-1)) = f(n+1) - f(n)$

solution

$f(n+1) - f(n) \geq 1\ \forall n \geq 2$ and so $f(n) \geq f(2) + n - 2 \geq n - 1\ \forall n \geq 3$

So : $\forall n \geq 5$: $f(n-1) \geq n-2 \geq 3$ and so $f(f(n-1)) \geq f(n-1) - 1 \geq n-3$ and so $f(n+1) - f(n) \geq n - 3$

Adding these lines for $n = 5, 6, 7$, we get $f(8) - f(5) \geq 9$ and so $f(8) \geq 10$. Let then $a = f(8) \geq 10$

Adding then the lines $f(f(n-1)) = f(n+1) - f(n)$ for $n = 2 \to a - 1$, we get $f(a) - f(2) = \sum_{k=1}^{a-2} f(f(k))$

And, since $a \geq 10$, we can write $f(a) - f(2) = f(f(8)) + \sum_{k=1,k\neq 8}^{a-2} f(f(k))$

and so, since $f(8) = a$, this becomes : $-f(2) = \sum_{k=1,k\neq 8}^{a-2} f(f(k))$, clearly impossible since $LHS < 0$ while $RHS > 0$

And so no solution

141. If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x) \cdot f(y)$; then is it necessary for $f(x)$ to be of the form a^x for some $a \in \mathbb{R}$?

solution

No.

First notice that $f(u) = 0 \implies f(x) = f(x-u)f(u) = 0\ \forall x$ and so a solution $f(x) = 0\ \forall w$ Then, if $f(x) \neq 0\ \forall x$, we get $f(x) = f(\frac{x}{2})^2 > 0$ and so we can write $g(x) = \ln f(x)$ and we have the equation $g(x+y) = g(x)+g(y)$

So $g(x)$ is any solution of Cauchy's equation and we have the general solutions :

$f(x) = 0\ \forall x$ $f(x) = e^{g(x)}$ where $g(x)$ is any solution of Cauchy's equation.

[u]If you add some constraints [/u](continuity, or $\ln f(x)$ upper bounded or lower bounded on some interval, then we get $g(x) = cx$ and so $f(x) = a^x$ for some $a > 0$[

142. Give all functions $f : R+ \to R+$ such that $(x+y)f(f(x)y) = x^2(f(f(x)+f(y))$ for all x, y positive real.

solution

Let $P(x, y)$ be the assertion $(x+y)f(f(x)y) = x^2 f(f(x)+f(y))$

If $f(a) = f(b)$, then, comparing $P(a, y)$ and $P(b, y)$, we get $\frac{a+y}{a^2} = \frac{b+y}{b^2}$ and so $a = b$ and $f(x)$ is injective.

$P(\frac{1+\sqrt{5}}{2}, 1) \implies f(f(\frac{1+\sqrt{5}}{2})) = f(f(\frac{1+\sqrt{5}}{2}) + f(1))$

And so, since injective : $f(\frac{1+\sqrt{5}}{2}) = f(\frac{1+\sqrt{5}}{2}) + f(1)$ and $f(1) = 0$, impossible.

So no solution to this equation.

143. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = max(f(x), y) + min(x, f(y))$

solution

Let $P(x, y)$ be the assertion $f(x+y) = \max(f(x), y) + \min(x, f(y))$

(a) : $P(x, 0) \implies f(x) = \max(f(x), 0) + \min(x, f(0))$ (b) : $P(0, x) \implies f(x) = \min(0, f(x)) + \max(f(0), x)$

Using the fact that $\max(u, v) + \min(u, v) = u+v$, the sum (a)+(b) implies $f(x) = x + f(0)$

Then $P(0, f(0)) \implies f(0) = \min(0, 2f(0))$ and so $f(0) = 0$

Hence the unique solution : $\boxed{f(x) = x}$ $\forall x$, which indeed is a solution

144. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in R$ $f(x+f(y)) = f(x-f(y)) + 4xf(y)$

solution

Let $P(x,y)$ be the assertion $f(x+f(y)) = f(x-f(y)) + 4xf(y)$

$f(x) = 0\ \forall x$ is a solution and let us from now look for non allzero solutions. Let u such that $f(u) \neq 0$ Let $A = \{2f(x)\ \forall x \in \mathbb{R}\}$

$P(\frac{x}{8f(u)}, u) \implies x = 2f(\frac{x}{8f(u)} + f(u)) - 2f(\frac{x}{8f(u)} - f(u))$ So any $x \in \mathbb{R}$ may be written as $x = a - b$ where $a, b \in A$

Let then $g(x) = f(x) - x^2$ Let $a = 2f(y) \in A$ $P(x+f(y), y) \implies f(x+a) = f(x) + 2ax + a^2$ and so $g(x+a) = g(x)\ \forall x \in \mathbb{R},\ \forall a \in A$

So $g(x-b) = g(x)\ \forall x \in \mathbb{R},\ \forall b \in A$

So $g(x+a-b) = g(x-b) = g(x)\ \forall x \in \mathbb{R},\ \forall a, b \in A$

And since we already proved that any real may be written as $a - b$ with $a, b \in A$, we get $g(x+y) = g(x)\ \forall x, y \in \mathbb{R}$ and so $g(x) = c$

Hence the two solutions : $f(x) = 0\ \forall x$ $f(x) = x^2 + c\ \forall x$ and for any $c \in \mathbb{R}$, which indeed is a solution

145. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(a) + b$ divides $(f(b) + a)^2$ for all a, b positive integers.

solution

Let $P(x,y)$ be the assertion $f(x) + y | (f(y) + x)^2$

Let $x > 0$ and $p > f(x)$ prime. $P(p - f(x), x) \implies f(p - f(x)) + x | p^2$ and so $f(p - f(x)) \in \{p - x, p^2 - x\}$

Let $A_x = \{p$ prime integers $> f(x)$ such that $f(p - f(x)) = p^2 - x\}$

For $p \in A_x$: $P(p - f(x), y) \implies p^2 - x + y | (f(y) + p - f(x))^2$

And so (subtracting LHS from RHS) : $p^2 + y - x | x - y + (f(y) - f(x))(2p + f(y) - f(x))$

But, for p great enough, $|LHS| > |RHS|$ and RHS cant be zero for any y and any p and so impossibility

So A_x is upper bounded and $\exists N_x$ such that $\forall p > N_x$ $f(p - f(x)) = p - x$

Then, For $p > N_x$: $P(p - f(x), y) \implies p - x + y | (f(y) + p - f(x))^2$ And so (subtracting LHS^2 from RHS) : $p + y - x | (f(y) - f(x) - y + x)(2p + f(y) - f(x) + y - x)$ And (subtracting $2(f(y) - f(x) - y + x)LHS$ from RHS= : $p + y - x | (f(y) - f(x) - y + x)^2$

But, for p great enough, $|LHS| > |RHS|$ and so RHS must be zero for any y and so $f(y) - y = f(x) - x$

So $\boxed{f(x) = x + a}$ $\forall x$ and for any $a \in \mathbb{Z}_{\geq 0}$ which indeed is a solution

146. $\forall x, y \in \mathbb{Z}^+$ $f(f(x) + f(y)) = x + y$ find all $f : \mathbb{Z}^+ \to \mathbb{Z}^+$.

solution

If $f(x_1) = f(x_2)$, we get $x_1 = x_2$ and the function is injective

Then $f(f(x+1)+f(1)) = x+2 = f(f(x)+f(2))$ and, since injective, $f(x+1) = f(x)+f(2)-f(1)$

So $f(x) = (f(2)-f(1))x + 2f(1) - f(2) = ax + b$ for some integers a, b

because

Write : $f(2) = f(1)+f(2)-f(1)$ $f(3) = f(2)+f(2)-f(1)$ $f(4) = f(3)+f(2)-f(1)$ ... $f(x) = f(x-1)+f(2)-f(1)$

And add all these lines.

Plugging this back in original equation, we get $a = \pm 1$ and $b = 0$ and, since in $\mathbb{Z}^+$:

A unique solution $\boxed{f(x) = x}$ $\forall x$

147. $f^2(x) = f(x+y)f(x-y)$ find all $f : \mathbb{R} \to \mathbb{R}$ functions

solution

Let $P(x, y)$ be the assertion $f(x)^2 = f(x+y)f(x-y)$

if $f(u) = 0$ for some u, then $P(x, u-x) \implies f(x) = 0$ and we get the allzero solution.

So let us consider from now that $f(x) \neq 0$ $\forall x$

$P(\frac{x}{2}, \frac{x}{2}) \implies \frac{f(x)}{f(0)} = \frac{f(\frac{x}{2})^2}{f(0)^2}$ and so $\frac{f(x)}{f(0)} > 0$ $\forall x$

Let then $g(x) = \ln \frac{f(x)}{f(0)}$: we get the new assertion $Q(x, y)$: $2g(x) = g(x+y) + g(x-y)$ with $g(0) = 0$

$Q(x, x) \implies 2g(x) = g(2x)$ and so the equation is $g(2x) = g(x+y) + g(x-y)$

And so $g((x+y)+(x-y)) = g(x+y)+g(x-y)$ and so $g(x+y) = g(x)+g(y)$

And so $g(x)$ is any solution of Cauchy equation.

[u][b]Hence the solutions [/b][/u]:

$\boxed{f(x) = a \cdot e^{h(x)}}$ $\forall x$ and for any real a and any $h(x)$ solution of Cauchy equation, which indeed is a solution

Notice that $a = 0$ gives the allzero solution

148. $\forall x \in \mathbb{Q}^+$ find all f functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$

$a) f(x+1) = f(x) + 1$

$b) f(x^2) = f(x)^2$

solution

From a) we get $f(x+n) = f(x) + n$

From b) we get $f((\frac{p}{q} + q)^2) = f(\frac{p}{q} + q)^2$

And so $f(\frac{p^2}{q^2} + 2p + q^2) = (f(\frac{p}{q}) + q)^2 = f(\frac{p}{q})^2 + 2qf(\frac{p}{q}) + q^2$

But $LHS = f(\frac{p^2}{q^2}) + 2p + q^2 = f(\frac{p}{q})^2 + 2p + q^2$ and so $p = qf(\frac{p}{q})$ andso $f(\frac{p}{q}) = \frac{p}{q}$

Hence the solution : $\boxed{f(x) = x}$ $\forall x \in \mathbb{Q}^+$ which indeed is a solution.

149. $\forall x, y \in \mathbb{Z}^+$ 1. $f(2) = 2$ 2.$f(mn) = f(m)f(n)$ 3.$f(n+1) \geq f(n)$ find all $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ functions.

solution

Using $m = n = 1$ in 2, we get $f(1) = 1$

Let $u > 1 \in \mathbb{N}$. Let $a, b, c, d \in \mathbb{N}$ such that $\frac{a}{b} \geq \frac{\ln u}{\ln 2} \geq \frac{c}{d}$

This implies $2^a \geq u^b$ and $u^d \geq 2^c$ and and so, using 3 : $f(2^a) \geq f(u^b)$ and $f(u^d) \geq f(2^c)$ and so, using 1 and 2 : $2^a \geq f(u)^b$ and $f(u)^d \geq 2^c$ and so : $\frac{a}{b} \geq \frac{\ln(f(u))}{\ln(2)} \geq \frac{c}{d}$

So $\frac{a}{b} \geq \frac{\ln u}{\ln 2} \geq \frac{c}{d}$ implies $\frac{a}{b} \geq \frac{\ln(f(u))}{\ln(2)} \geq \frac{c}{d}$

So $\frac{\ln(f(u))}{\ln 2} = \frac{\ln u}{\ln 2}$

So $f(u) = u$

Hence the result : $\boxed{f(n) = n}$ $\forall n \in \mathbb{N}$, which indeed is a solution.

150. Determine all functions f from the nonnegative integers to the nonnegative integers such that $f(1) \neq 0$ and, for all x and y in the nonnegative integers: $f(x)^2 + f(y)^2 = f(x^2 + y^2)$.

solution

Let $P(x, y)$ be the assertion $f(x)^2 + f(y)^2 = f(x^2 + y^2)$

1) $f(x) = x$ $\forall$ integer $x \in [0, 9]$ ==== $P(0,0) \implies f(0) = 0$ $P(1,0) \implies f(1) = 1$ $P(1,1) \implies f(2) = 2$ $P(2,0) \implies f(4) = 4$ $P(2,1) \implies f(5) = 5$ $P(5,0) \implies f(25) = 25$ $P(5,5) \implies f(50) = 50$ $P(3,4) \implies f(3) = 3$ $P(7,1) \implies f(7) = 7$ $P(2,2) \implies f(8) = 8$ $P(3,0) \implies f(9) = 9$ $P(9,2) \implies f(85) = 85$ $P(6,7) \implies f(6) = 6$ Q.E.D.

2) $f(x) = x$ $\forall x$ === Let $x \geq 4$ $P(2x+1, x-2) \implies f(2x+1)^2 + f(x-2)^2 = f(5x^2+5)$ $P(2x-1, x+2) \implies f(2x-1)^2 + f(x+2)^2 = f(5x^2+5)$ and so $f(2x+1)^2 = f(2x-1)^2 + f(x+2)^2 - f(x-2)^2$

$P(2x+2, x-4) \implies f(2x+2)^2 + f(x-4)^2 = f(5x^2+20)$ $P(2-2, x+4) \implies f(2x-2)^2 + f(x+4)^2 = f(5x^2+20)$ And so $f(2x+2)^2 = f(2x-2)^2 + f(x+4)^2 - f(x-4)^2$

And so knowledge of $f(n)$ up to $2x \geq 8$ gives unique knowledge of $f(2x+1)$ and $f(2x+2)$

And since $f(x)$ is quite defined up to $f(9)$, there is at most one solution $f(x)$

And since $f(x) = x$ $\forall x$ is obviously a solution, this is the unique one.

151. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 + y + f(y)) = 2y + (f(x))^2$ for every $x, y \in \mathbb{R}$.

solution

Let $P(x, y)$ be the assertion $f(x^2 + y + f(y)) = 2y + f(x)^2$

1) $f(x) = 0 \iff x = 0$ === $P(0, -\frac{1}{2}f(0)^2) \implies f(\text{something}) = 0$ and so $\exists u$ such that $f(u) = 0$

Let u such that $f(u) = 0$, then, comparing $P(u, 0)$ and $P(-u, 0)$, we get that $f(u) = f(-u) = 0$ and so : $P(0, u) \implies 0 = 2u + f(0)^2$ $P(0, -u) \implies 0 = -2u + f(0)^2$ And so $u = 0$ Q.E.D.

2) $f(x)$ is injective === $P(0, -\frac{1}{2}f(x)^2) \implies f(x^2 - \frac{1}{2}f(x)^2 + f(-\frac{1}{2}f(x)^2)) = 0$ And so, using 1) above : $x^2 - \frac{1}{2}f(x)^2 + f(-\frac{1}{2}f(x)^2) = 0$ Then $f(x_1) = f(x_2)$ implies $|x_1| = |x_2|$

Comparing $P(x, y)$ and $P(-x, y)$, we get $f(-x) = \pm f(x)$ Let then t such that $f(-t) = f(t)$ $P(0, t) \implies f(t + f(t)) = 2t$ and so $P(t + f(t), 0) \implies f((t + f(t))^2) = 4t^2$ $P(0, -t) \implies f(-t + f(t)) = -2t$ and so $P(-t + f(t), 0) \implies f((-t + f(t))^2) = 4t^2$

So $f((t + f(t))^2) = f((-t + f(t))^2)$ and so (see some lines above) $|(t + f(t))^2| = |(-t + f(t))^2|$ Which implies $tf(t) = 0$ and so $t = 0$ (using 1) above)

So $f(-x) = -f(x)$ $\forall x$

And then "$f(x_1) = f(x_2)$ implies $|x_1| = |x_2|$" becomes "$f(x_1) = f(x_2)$ implies $x_1 = x_2$" (using again 1) above) Q.E.D.

3) $x + f(x)$ is surjective === $P(0, \frac{1}{2}f(x)) \implies f(\frac{1}{2}f(x) + f(\frac{1}{2}f(x))) = f(x)$

And so, since injective, $\frac{1}{2}f(x) + f(\frac{1}{2}f(x)) = x$ Q.E.D.

4) $f(x) = x$ $\forall x$ === $P(x, 0) \implies f(x^2) = f(x)^2$ $P(0, y) \implies f(y + f(y)) = 2y$ So $P(x, y)$ becomes $f(x^2 + y + f(y)) = f(x^2) + f(y + f(y))$

And since $x + f(x)$ is surjective, this becomes $f(x + y) = f(x) + f(y)$ $\forall x \geq 0$, $\forall y$ Since $f(-x) = -f(x)$, this implies $f(x + y) = f(x) + f(y)$ $\forall x, y$ And since $f(x^2) = f(x)^2$, we get that $f(x) \geq 0$ $\forall x \geq 0$ and so $f(x + y) = f(x) + f(y)$ implies that $f(x)$ is non decreasing.

So, as a monotonous solution of Cauchy's equation, $f(x) = ax$ $\forall x$ Plugging this back in original equation, we get $a = 1$

And so the unique solution $\boxed{f(x) = x}$ $\forall x$

152. Determine all such funtions f, g, h from R^+ to itself, that $f(g(h(x)) + y) + h(z + f(y)) = g(y) + h(y + f(z)) + x$.

solution

I supposed that the domain of functional equation is the same than domain of functions (better to indicate both domains).

Let $P(x,y,z)$ be the assertion $f(g(h(x))+y)+h(z+f(y)) = g(y)+h(y+f(z))+x$ $P(x,y,y) \implies f(g(h(x))+y) = g(y)+x$ and so $h(x)$ is injective

Subtracting $P(x,y,y)$ from $P(x,y,z)$, we get $h(z+f(y)) = h(y+f(z))$ and so, since $h(x)$ is injective : $z+f(y) = y+f(z)$ and so $f(x) = x+a$ for some $a \geq 0$

Plugging this in $P(1,x,x)$, we get $g(h(1)) + x + a = g(x) + 1$ and so $g(x) = x + b$ for some $b \geq 0$

Plugging $f(x) = x + a$ and $g(x) = x + b$ in original equation, we get $h(x) = x - a$ and so $a = 0$

Hence the solutions : $\boxed{(f,g,h) = (x, x+b, x)}$ for any real $b \geq 0$

153. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, y in $\mathbb{R}$, $xf(x+xy) = xf(x) + f(x^2).f(y)$

solution

Let $P(x,y)$ be the assertion $xf(x+xy) = xf(x) + f(x^2)f(y)$

$P(0,0) \implies f(0) = 0$ If $f(1) = 0$, then $P(1, x-1) \implies f(x) = 0$ which indeed is a solution Let us from now consider that $f(1) = a \neq 0$

If $a \neq 1$, $P(1,x) \implies f(x+1) = af(x)+a$ and we easily get $f(n) = a\frac{a^n-1}{a-1}$ $\forall n \in \mathbb{N}$ Plugging this expression in $P(m,n)$, we see that this is not a solution (rather ugly, I think).

So $a = 1$ and $P(1,x) \implies f(x+1) = f(x) + 1$ and so $f(n) = n$ and $f(x+n) = f(x) + n$

$P(x,-1) \implies f(x^2) = xf(x)$ Plugging this in $P(x,y)$, we get $xf(x(y+1)) = xf(x)(f(y)+1) = xf(x)f(y+1)$

And so $f(xy) = f(x)f(y)$

$P(x,y)$ becomes then $xf(x)f(y+1) = xf(x)+f(x)^2f(y) \iff xf(x)(f(y)+1) = xf(x) + f(x)^2f(y)$

And so, setting $y = 1$: $f(x)(f(x) - x) = 0$ and so $\forall x$, either $f(x) = 0$, either $f(x) = x$ But, if for some $x \neq 0$, we have $f(x) = 0$, then $f(x+1) = f(x)+1$ implies $f(x+1) = 1$ which is impossible since either $f(x+1) = x+1 \neq 1$, either $f(x+1) = 0 \neq 1$

So $f(x) = x$ $\forall x$, which indeed is a solution.

[u][b]Hence the answer [/b][/u]: $f(x) = 0$ $\forall x$ $f(x) = x$ $\forall x$[

154. Find all polynomials $P(x) \in \mathbb{R}[X]$, $degP = 3$ with the property that $P(x^2) = -P(x)P(-x)$

solution

$P(x)$ is obviously monic and may be written $x^3 + ax^2 + bx + c$ and the equation is :

$x^6 + ax^4 + bx^2 + c = (x^3 + bx + ax^2 + c)(x^3 + bx - (ax^2 + c) = = x^2(x^2 + b)^2 - (ax^2 + c)^2$ and so :

$a = 2b - a^2$ $b = b^2 - 2ac$ $c = -c^2$ and so $c = 0$ or $c = -1$

$c = 0$ gives $b = b^2$ and so $b = 0$ or $b = 1$ $c = 0$ and $b = 0$ gives $a = 0$ or $a = -1$ and so two solutions x^3 and $x^3 - x^2$ $c = 0$ and $b = 1$ gives $a = 1$ or $a = -2$ and so two solutions $x^3 + x^2 + x$ and $x^3 - 2x^2 + x$

$c = -1$ implies $a^2 + a - 2b = 0$ and $b^2 - b + 2a = 0$ and so two solutions $x^3 - 1$ and $x^3 - 3x^2 + 3x - 1$

[u][b]Hence the six solutions [/b][/u]: $P(x) = x^3$ $P(x) = x^2(x-1)$ $P(x) = x(x^2 + x + 1)$ $P(x) = x(x-1)^2$ $P(x) = x^3 - 1$ $P(x) = (x-1)^3$

155. Find all monotonic functions $u : \mathbb{R} \to \mathbb{R}$ which have the property that there exists a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) = f(x)u(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

solution

Let $P(x, y)$ be the assertion $f(x + y) = f(x)u(x) + f(y)$

Subtracting $P(x, 0)$ from $P(x, y)$, we get $f(x + y) = f(x) + f(y) - f(0)$ and so, since strictly increasing, $f(x) = ax + b$ with $a > 0$

And so $x = (x + \frac{b}{a})u(x)$

Setting $x = -\frac{b}{a}$, we get $b = 0$ and so the solution :

$u(x) = 1$ $\forall x \neq 0$ and $u(0) = c$ any rea

156. Give all functions $f : R- > R$ such that $f(y)f(z) + f(x)f(x + y + z) = f(x + y)f(x + z)$ for all x, y, z real.

solution

Is it a real olympiad exercise ? With no forgotten constraint (like continuity, for example) ? In what contest did you get this problem ?

It's easy to show that the functional equation is equivalent to $f(x)^2 - f(y)^2 = f(x + y)f(x - y)$

And this equation has infinitely many solutions. For example : $f(x) =$ any solution of additive Cauchy equation $f(x) = a\sin(g(x))$ where $g(x)$ is any solution of additive Cauchy equation $f(x) = a\sinh(g(x))$ where $g(x)$ is any solution of additive Cauchy equation

And I dont know if these are the only solutions.

157. Find all $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$f(x - f(y) + y) = f(x) - f(y)$

all real numbers x, y.

solution

Let $g(x) = f(x) - x$ and the equation becomes assertion $P(x, y)$: $g(x - g(y)) = g(x) - y$

This implies that $g(x)$ is a bijection. So $\exists u$ suvh that $g(u) = 0$. $P(x, u)$ implies then $u = 0$

$P(g(x), x) \implies g(g(x)) = x$ $P(x, g(y)) \implies g(x - y) = g(x) - g(y)$ So $g(x)$ is any involutive solution of Cauchy's equation.

And it'simmediate to verify that this is indeed a solution.

[u][b]Hence the answer [/b][/u]: $f(x) = x + g(x)$ where $g(x)$ is any involutive solution of Cauchy's equation

Notice that we have infinitely many solutions. The only continuous solutions are $f(x) = 0$ $\forall x$ and $f(x) = 2x$ $\forall x$

Notice that the general solution for "involutive solutions of Cauchy's equation" may also be written as :

Let A, B two supplementary subvectorspaces of the $\mathbb{Q}$-vectorspace $\mathbb{R}$ Let $a(x)$ and $b(x)$ the projections of x in A and B so that $x = a(x) + b(x)$ with $a(x) \in A$ and $b(x) \in B$

Then $g(x) = a(x) - b(x)$

1) proof that any such $g(x)$ is an involutive solution of Cauchy's equation and so this is a solution $===$

$a(x)$ and $b(x)$ are additive and so $g(x)$ is solution of Cauchy's equation. $a(a(x)) = a(x)$ and $a(b(x)) = 0$ and $a(a(x) - b(x)) = a(x)$ $b(a(x)) = 0)$ and $b(b(x)) = b(x)$ and $b(a(x) - b(x)) = -b(x)$ And so $g(g(x)) = a(x) + b(x) = x$ Q.E.D.

2) proof that any solution may be written in this form and so it's a general solution $====$

Let $A = \{x \text{ such that } g(x) = x\}$ Let $B = \{x \text{ such that } g(x) = -x\}$ Obviously, since $g(x)$ is additive, A, B are subvectorspaces of the $\mathbb{Q}$-vectorspace $\mathbb{R}$ $A \cap B = \{0\}$

Since $g(g(x)) = x$, we get that $g(x + g(x)) = x + g(x)$ and so $a(x) = \frac{x+g(x)}{2} \in A$ Since $g(g(x)) = x$, we get that $g(x - g(x)) = g(x) - x$ and so $b(x) = \frac{x-g(x)}{2} \in B$

And since $a(x) + b(x) = x$, we conclude that A, B are supplementary subvectorspaces.

And we clearly have $g(x) = a(x) - b(x)$ Q.E.D.

158. Find all function $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition:

$$f(y + f(x)) = f(x)f(y) + f(f(x)) + f(y) - xy$$

solution

Let $P(x, y)$ be the assertion $f(y + f(x)) = f(x)f(y) + f(f(x)) + f(y) - xy$

$f(x)=-1\ \forall x$ is not a solution and so let v such that $f(v)\neq -1$ $P(v,0)$ $\implies f(0)(f(v)+1)=0$ and so $f(0)=0$

$f(x)=0\ \forall x$ is not a solution and so let u such that $f(u)\neq 0$

$P(x,f(u)) \implies f(f(x)+f(u))=f(x)f(f(u))+f(f(x))+f(f(u))-xf(u)$
$P(u,f(x)) \implies f(f(x)+f(u))=f(u)f(f(x))+f(f(x))+f(f(u))-uf(x)$
Subtracting, we get $f(f(x))+x=f(x)\frac{f(f(u))+u}{f(u)}$

and so $f(f(x))=af(x)-x$ for some $a\in\mathbb{R}$

So we can rewrite $P(x,y)$ as new assertion $Q(x,y)$: $f(y+f(x))=f(x)f(y)+af(x)-x+f(y)-xy$

$Q(y,-1) \implies f(f(y)-1)=f(y)(f(-1)+a)+f(-1)=cf(y)+d$

$Q(x,f(y)-1) \implies f(f(x)+f(y)-1)=f(x)(cf(y)+d)+af(x)-x+cf(y)+d-x(f(y)-1)$ and so :

$f(f(x)+f(y)-1)=cf(x)f(y)+(a+d)f(x)+(c-x)f(y)+d$ Swapping x,y, we get $f(f(x)+f(y)-1)=cf(x)f(y)+(a+d)f(y)+(c-y)f(x)+d$
Subtracting : $(a+d-c+y)f(x)=(a+d-c+x)f(y)$

Setting $y=0$ in this line, we get $a+d-c=0$ and so $yf(x)=xf(y)\ \forall x,y$

Setting $y=1$ in this expression, we get $f(x)=xf(1)$

Plugging in original equation, we get $f(1)=\pm1$

[u][b]And so the two solutions [/b][/u]: $f(x)=x\ \forall x$ $f(x)=-x\ \forall x$

159. Find all functions $f:R^+\to R^+$ such that $f(xyz)+f(x)+f(y)+f(z)=f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for positive reals x,y,z and also $f(x)<f(y)$ for $1\le x<y$

solution

Let $P(x,y)$ be the assertion $f(xyz)+f(x)+f(y)+f(z)=f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$

$P(1,1,1) \implies 4f(1)=f(1)^3$ and so $f(1)=2$

$P(x^2,1,1) \implies f(x^2)=f(x)^2-2$

$P(x^2,y^2,1) \implies f(x^2y^2)+f(x^2)+f(y^2)+2=f(xy)f(x)f(y)$ And so, using $f(x^2)=f(x)^2-2$ for x^2y^2,x^2 and y^2 :

$f(xy)^2-f(xy)f(x)f(y)+f(x)^2+f(y)^2-4=0$

The discriminant of this quadratic in $f(xy)$ is $(f(x)^2-4)(f(y)^2-4)$ And since we now that $f(x)>2\ \forall x>1$, we get that $f(x)\ge 2\ \forall x>0$

Let then $u(x)\ge 1$ such that $f(x)=u(x)+\frac{1}{u(x)}$ (which always exists since $f(x)\ge 2$)

The above quadratic implies $u(xy)=u(x)u(y)$ or $u(xy)=\frac{u(x)}{u(y)}$ or $u(xy)=\frac{u(y)}{u(x)}$

Using the fact that $f(x)$ is increasing for $x\ge 1$ and so $u(x)$ is increasing too, we get that $u(xy)=u(x)u(y)\ \forall x,y\ge 1$

So $u(x) = x^a$ with $a > 0$ $\forall x \geq 1$

Plugging this back in original equation, we get that any real $a > 0$ fits and so $f(x) = x^a + x^{-a}$ $\forall x \geq 1$

$P(x, \frac{1}{x}, 1) \implies f(x^2) + f(\frac{1}{x^2}) + 4 = 2f(x)f(\frac{1}{x})$ And so, using $f(x^2) = f(x)^2 - 2$ for x^2 and $\frac{1}{x^2}$:

$(f(x) - f(\frac{1}{x}))^2 = 0$ and so $f(\frac{1}{x}) = f(x)$

So $\boxed{f(x) = x^a + x^{-a}}$ $\forall x$ and for any real $a \neq 0$ which indeed is a solution

160. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ so that : $f(f(x)) = f(x) + x$, $\forall x \in \mathbb{R}$

solution

Let $x \in \mathbb{R}$ and the sequence $a_0 = x$ and $a_{n+1} = f(a_n)$ We get $a_0 = x$ and $a_1 = f(x)$ and $a_{n+2} = a_{n+1} + a_n$.

Let $r_1 < r_2$ be the two real roots of equation $x^2 - x - 1 = 0$. We get $a_n = \frac{(f(x)-r_2x)r_1^n-(f(x)-r_1x)r_2^n}{r_1-r_2}$

$f(x)$ is injective. It's easy to see that $f(x)$ is neither upper bounded, neither lower bounded and so $f(x)$ is a bijection from $\mathbb{R} \to \mathbb{R}$

So the equality $a_n = \frac{(f(x)-r_2x)r_1^n-(f(x)-r_1x)r_2^n}{r_1-r_2}$ is true also for $n < 0$

Setting $x = 0$ in the equation, we get $f(f(0)) = f(0)$ and so $f(0) = 0$, since injective. $f(x)$ is injective and continuous, and so monotonous and so $\frac{f(x)-f(0)}{x-0}$ has a constant sign and so $\frac{a_{n+1}}{a_n}$ has a constant sign.

So $\frac{(f(x)-r_2x)r_1^{n+1}-(f(x)-r_1x)r_2^{n+1}}{(f(x)-r_2x)r_1^n-(f(x)-r_1x)r_2^n}$ has a constant sign.

If $f(x)$ is decreasing and $f(x)-r_1x \neq 0$, then the above quantity has limit $r_2 > 0$ when $n \to +\infty$, in contradiction with the fact $f(x)$ decreasing. So the only continuous decreasing solution may be $f(x) = r_1x$ which indeed is a solution.

If $f(x)$ is increasing and $f(x)-r_2x \neq 0$, then the above quantity has limit $r_1 < 0$ when $n \to -\infty$, in contradiction with the fact $f(x)$ increasing. So the only continuous increasing solution may be $f(x) = r_2x$ which indeed is a solution.

Hence the only solutions : $f(x) = \frac{1+\sqrt{5}}{2}x$

$f(x) = -\frac{\sqrt{5}-1}{2}x$

161. Let $f : \mathbb{N}-> \mathbb{N}$ be a function satisfying: $f(f(n)) = 4n-3$ $(2^n) = 2^{n+1}-1$, for all natural n Find $f(1993)$, can you find expliciety the value $f(2007)$? what values can $f(1997)$ take?

solution

I suppose that third line must be read $f(2^n) = 2^{n+1} - 1$

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ Let $g(n)$ from $\mathbb{N}_0 \to \mathbb{N}_0$ defined as $g(n) = f(n+1) - 1$. The equation is then $g(g(n)) = 4n$ whose general solution is :

Let A, B two equinumerous sets whose intersection is empty and whose union is the set of all natural numbers not divisible by 4. Let $h(x)$ any bijection from $A \to B$ and $h^{-1}(x)$ it's inverse function.

Then $g(x)$ may be defined as : $g(0) = 0$ $\forall x \in A : g(x) = h(x)$ $\forall x \in B :$ $g(x) = 4h^{-1}(x)$ $\forall x \in \mathbb{N} \setminus (A \cup B) : g(x) = 4^{v_4(x)} g(x 4^{-v_4(x)})$

The constraint $f(2^n) = 2^{n+1} - 1$ becomes $g(2^n - 1) = 2^{n+1} - 2$ and so we just have to add to the previous general solution the constraints : $2^n - 1 \in A$ $\forall n \in \mathbb{N}$ $2^n - 2 \in B$ $\forall n > 1 \in \mathbb{N}$ $h(2^n - 1) = 2^{n+1} - 2$

Then $f(1993) = g(1992) + 1 = 4g(498) + 1$ and since 498 is not divisible by 4 and is not in the form $2^n - 1$ neither $2^{n+1} - 1$, we get nearly no constraint for $g(498)$: We can put 498 in A and then $g(498) \in B$ may be any value not divisible by 4 and not in the form $2^{n+1} - 2$ We can put 498 in B and then $g(498) = 4u$ where u is any number not divisible by 4 and not in the form $2^n - 1$

And the same conclusions may be obtained for $g(2006)$ and $g(1996)$

162. Find all function $f \colon \mathbb{R} \cdot \mathbb{R} \to \mathbb{R}$ such that $f(f(x,z), f(z,y)) = f(x,y) + z$ for all real numbers x,y and z

solution

Let $P(x,y,z)$ be the assertion $f(f(x,z), f(z,y)) = f(x,y) + z$

Let $s(x) = f(x,x)$ where "s" stands for "same" Let $r(x) = f(0,x)$ where "r" stands for "right" Let $l(x) = f(x,0)$ xhere "l" stands for "left"

$P(x,x,x) \implies s(s(x)) = s(x) + x$ and so $s(x)$ is injective $P(0,0,0) \implies$ $s(s(0)) = s(0)$ and so, since injective : $s(0) = 0$ and so $r(0) = l(0) = 0$ and $f(0,0) = 0$

$P(x,0,0) \implies l(l(x)) = l(x)$ $P(0,x,0) \implies r(r(x)) = r(x)$

$P(x,y,0) \implies f(l(x), r(y)) = f(x,y)$ Then, $l(l(x)) = l(x) \implies f(x,y) = f(l(x), y)$ Same, $r(r(y)) = r(y) \implies f(x,y) = f(x, r(y))$

$P(0,0,x) \implies f(r(x), l(x)) = x$

Suppose $\exists u, v$ such that $l(u) = d(v) = a$. Then : $l(l(u)) = l(u)$ and so $l(a) = a$ $r(r(v)) = r(v)$ and so $r(a) = a$ $a = f(r(a), l(a))$ and so $f(a,a) = a$ and so $s(a) = a$ $P(a,a,a) \implies s(s(a)) = s(a) + a$ and so $a = 2a$ and $a = 0$ So $l(\mathbb{R}) \cap r(\mathbb{R}) = \{0\}$

Suppose now $\exists u, v$ such that $f(u,v) = 0$ $P(u,v,u) \implies f(f(u,u), f(u,v)) = f(u,v) + u \implies l(s(u)) = u \implies u \in l(\mathbb{R})$ $P(u,v,v) \implies f(f(u,v), f(v,v)) = f(u,v) + v \implies r(s(v)) = v \implies v \in r(\mathbb{R})$ $l(u) = f(u,0) = f(u, f(u,v)) = f(f(r(u), l(u)), f(l(u), v)) = f(r(u), v) + l(u) \implies f(r(u), v) = 0 \implies r(u) \in l(\mathbb{R})$ $r(v) = f(0,v) = f(f(u,v), v) = f(f(u, r(y)), f(r(v), l(v))) = f(u, l(v)) + r(v) \implies f(u, l(v)) = 0 \implies l(v) \in r(\mathbb{R})$ So $r(u) \in l(\mathbb{R}) \cap r(\mathbb{R})$

and so $r(u) = 0$ and so $u = f(r(u), l(u)) = r(l(u))$ and $r(u) = r(r(l(u))) = r(l(u)) = u$ and so $u = 0$ Same : $l(v)) \in l(\mathbb{R}) \cap r(\mathbb{R})$ and so $l(v) = 0$ and so $v = f(r(v), l(v)) = l(r(v))$ and so $l(v) = l(l(r(v))) = l(r(v)) = v$ and so $v = 0$

So $f(x, y) = 0 \iff x = y = 0$

Then $P(x, y, -f(x, y)) \implies f(f(x, -f(x, y)), f(-f(x, y), y)) = 0$ and so $f(x, -f(x, y)) = f(-f(x, y), y) = 0$ and so $x = y = 0$, impossible

So no solution for this equation

163. Let $f : R \to R$ satisfy $f(4^x) = f(2^x) + x$. Find all functions f with the given property.

solution

Obviously $f(x)$ can take any value we want for $x \leq 0$ For $x > 0$, let us write $f(x) = g(\ln x) + \log_2 x$ and the equation becomes $g(2x) = g(x)$ which is very classical with solution : $g(x) = u(\{\log_2 x\}$ for any $x > 0$ where $u(x)$ is any function defined over $[0, 1)$ $g(0) = a$ where a is any real we want $g(x) = v(\{\log_2 -x\}$ for any $x < 0$ where $v(x)$ is any function defined over $[0, 1)$

[u][b]Hence a general solution of required equation [/b][/u]:

$\forall x \leq 0 :$ $f(x)$ is any function we want $\forall x \in (0, 1) :$ $f(x) = \log_2 x + v(\{\log_2 |\ln x|\})$ where $v(x)$ is any function defined over $[0, 1)$ $f(1) = a$ where a is any real we want $\forall x > 1 :$ $f(x) = \log_2 x + u(\{\log_2 \ln x\})$ where $u(x)$ is any function defined over $[0, 1)$

164. $f : \mathbb{R} \to \mathbb{R}$ $f(x+y) + f(y+z) + f(z+x) ? f(x+2y+3z)$ for any real x, y, z

solution

There are obviously infinitely many solutions and I wonder how we can find a general formula for these. Some examples : $f(x) = 1$

$f(x) = 2\pi + \arctan(x)$

$f(x) = \frac{x^2+2}{x^2+1}$

$f(x) = 5 + q(x)$ with q(x)=x-floorfunction(x)

165. Find all $f : \mathbb{Q} \to \mathbb{Q}$ satisfy : $f(f(x) + y) = x + f(y), \forall x, y \in \mathbb{Q}$

solution

Let $P(x, y)$ be the assertion $f(f(x) + y) = x + f(y)$

$P(x, 0) \implies f(f(x)) = x + f(0)$ $P(f(x), y) \implies f(x + y + f(0)) = f(x) + f(y)$

Writing $f(x) = g(x + f(0))$, this becomes $g((x + f(0)) + (y + f(0))) = g(x + f(0)) + g(y + f(0))$ So $g(x + y) = g(x) + g(y)$ and $g(x) = g(1)x$ $\forall x \in \mathbb{Q}$ and so $f(x) = g(1)(x + f(0))$

So $f(x) = ax + b$ and, plugging this in original equation, we get $b = 0$ and $a^2 = 1$

Hence the two solutions : $f(x) = x\ \forall x\ f(x) = -x\ \forall x$

166. Find all $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfy: $\begin{cases} f(x+1) = f(x) + 1 \\ f(x^2) = f^2(x) \end{cases} \quad \forall x \in \mathbb{Q}^+$

solution

The first equation implies $f(x+n) = f(x) + n\ \forall x \in \mathbb{Q}^+,\ \forall n \in \mathbb{N}$

Then $f((\frac{p}{q}+q)^2) = f(\frac{p^2}{q^2} + 2p + q^2) = f(\frac{p^2}{q^2}) + 2p + q^2 = f^2(\frac{p}{q}) + 2p + q^2$

But $f((\frac{p}{q}+q)^2) = f^2(\frac{p}{q}+q) = (f(\frac{p}{q}) + q)^2 = f^2(\frac{p}{q}) + 2qf(\frac{p}{q}) + q^2$

And so $2p = 2qf(\frac{p}{q})$ and $f(\frac{p}{q}) = \frac{p}{q}$

Hence the unique solution $\boxed{f(x) = x}$ $\forall x \in \mathbb{Q}^+$, which indeed is a solution

167. Find all contiuous function $f : R+ -> R+$ satisfying: $f(x+\frac{1}{x}) + f(y+\frac{1}{y}) = f(x+\frac{1}{y}) + f(y+\frac{1}{x})$ for every x,y from R+

solution

Consider then $a, b > 0$ such that $a \neq b$ and $ab \geq 4$

Consider the system : $x \geq \sqrt{\frac{a}{b}}$ and $y \geq \sqrt{\frac{b}{a}}$ $x + \frac{1}{y} = a$ $y + \frac{1}{x} = b$ Ths system always have a unique real solution

Let then $u = x + \frac{1}{x}$ and $v = y + \frac{1}{y}$ It's easy to see that : $f(a) + f(b) = f(u) + f(v)$ $a + b = u + v$ $|u - v| < |a - b|$ $u \neq v$ and $uv \geq 4$

And so we can create a sequence $(a, b) \to (u, v)$, repeating the process It's easy to see that the two numbers have their difference tending towards 0 and so have the same limit $\frac{a+b}{2}$

and so, since continuous, $f(a) + f(b) = 2f(\frac{a+b}{2})\ \forall a, b > 0$ such that $a \neq b$ and $ab \geq 4$

This is a classical functional equation which implies easily (continuity again) $f(x) = cx + d\ \forall x \geq 2$

Using then the functonal equation with for example $y \geq \frac{1}{2}$, we get $x + \frac{1}{x}, y + \frac{1}{y}, x + \frac{1}{y} \geq 2$ and so $f(y + \frac{1}{x}) = c(y + \frac{1}{x}) + d$ and so $f(x) = cx + d$ $\forall x > \frac{1}{2}$

And it's easy to use similar steps as many times as we want to get $f(x) = cx + d\ \forall x > 0$

And this indeed is a solution as soon as $c \geq 0$ and $d \geq 0$ or $c = 0$ and $d > 0$

Hence the answer : $\boxed{f(x) = ax + b}$ $\forall x > 0$ and for any ($a > 0$ and $b \geq 0$) or ($a = 0$ and $b > 0$)

168. Find the $f : \mathbb{R} \to \mathbb{R}$ such that f is a continuous function and satisfy : $f(x+y) = f(x) + f(y) + 2xy, \forall x, y \in \mathbb{R}$

olution

Let $f(x) = x^2 + g(x)$ and the equation becomes $g(x+y) = g(x) + g(y)$ and so $g(x) = ax$, since continuous and $\boxed{f(x) = x^2 + ax}$ which indeed is a solution. $\frac{x^2}{2} \to x^2$

169. Let f be a contiuous and injective function $R -> R$; $f(1) = 1$; $f(2x - f(x)) = x$. Prove that $f(x) = x$.

solution

So $f(x)$ is strictly monotonous. If $f(x)$ is decreasing, then $2x - f(x)$ is increasing and $f(2x - f(x))$ is decreasing, which is wrong.

So $f(x)$ is increasing.

If $f(a) > a$, then $2a - f(a) < a$ and $f(2a - f(a)) < f(a)$ and so $f(a) > a$, impossible If $f(a) < a$, then $2a - f(a) > a$ and $f(2a - f(a)) > f(a)$ and so $f(a) < a$, impossible

So $\boxed{f(x) = x}$ $\forall x$, which indeed is a solution