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Two Pairs of Archimedean Circles in the Arbelos

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Abstract. We construct four circles congruent to the Archimedean twin circles in the arbelos.

Consider an arbelos formed by semicircles (O_1), (O_2), and (O) of radii a, b, and $a+b$. The famous Archimedean twin circles associated in the arbelos have equal radii $\frac{ab}{a+b}$ (see [2, 3]).

Let CD be the dividing line of the smaller semicircles, and extend their common tangent PQ to intersect (O) at T_a and T_b.

Theorem 1. *Let A' and B' be the orthogonal projections of D on the tangents to (O) at T_a and T_b respectively. The circles with diameters DA' and DB' are congruent to the Archimedean twin circles.*

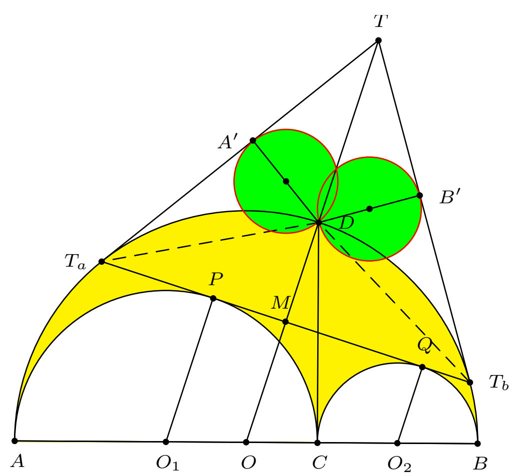

Figure 1

Proof. Let the tangents at T_a and T_b intersect at T. Since OT is the perpendicular bisector of T_aT_b, it intersects the semicircle (O) at the midpoint D of the arc T_aT_b (see [3, §5.2.1]). Since O_1P, OM and O_2Q are parallel, and $O_1P = OO_2 = a$, $O_2Q = O_1O = b$,

$$OM = \frac{a}{a+b}\cdot O_1P + \frac{b}{a+b}\cdot O_2Q = \frac{a^2+b^2}{a+b} \implies DM = OD - OM = \frac{2ab}{a+b}.$$

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Now, $\angle DT_aT = \angle DT_bT_a = \angle DT_aT_b$. Therefore, T_aD bisects angle TT_aT_b. Similarly, T_bD bisects angle TT_bT_a, and D is the incenter of triangle TT_aT_b. It follows that $DA' = DB' = DM$, and the circles with DA' and DB' are congruent to the Archimedean twin circles. □

Remark. The circle with DM as diameter is the Archimedean circle (A_3) in [2] (or (W_4) in [1]).

Theorem 2. *Let A_1A_2 and B_1B_2 be tangents to the smaller semicircles with A_1, B_1 on the line AB and $A_1A_2 = a$, $B_1B_2 = b$. If H and K are the midpoints of the semicircles (O_1) and (O_2) respectively, and $A'' = CH \cap A_1B_2$, $B'' = CK \cap B_1A_2$, then the circles through C with centers A'' and B'' are congruent to the Archimedean twin circles.*

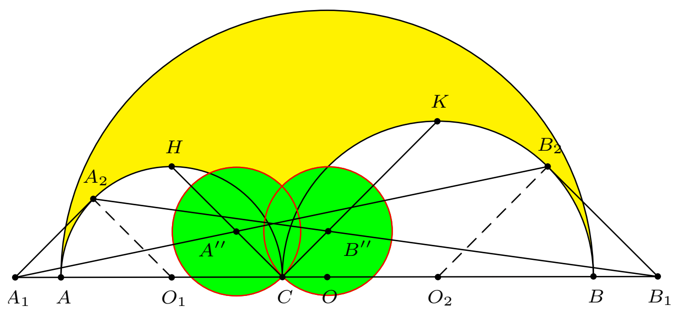

Figure 2

Proof. Clearly, $\angle A''CA_1 = \angle HCO_1 = 45°$. Since $B_1B_2 = O_2B_2 = b$, $\angle B_2B_1O_2 = 45°$, the lines CA'' and B_1B_2 are parallel. Also, $B_1O_2 = \sqrt{2}b$. Similarly, $A_1O_1 = \sqrt{2}a$, and $A_1B_1 = (\sqrt{2}+1)(a+b)$. Therefore,

$$CA'' = B_1B_2 \cdot \frac{A_1C}{A_1B_1} = b \cdot \frac{(\sqrt{2}+1)a}{(\sqrt{2}+1)(a+b)} = \frac{ab}{a+b}.$$

Similarly, $CB'' = \frac{ab}{a+b}$. Therefore, the circles through C with centers A'' and B'' are congruent to the Archimedean twin circles. □

References

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