USING COMPLEX NUMBER TO PROVE INEQUALITIES

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I.Theorem

Let a,b,a',b' be real numbers.

Let complex numbers z = $a+bi$ and z’ = $a'+b'i$. ($i^2=-1$)

We have $|z|+|z'|\geq|z+z'|$.

II.Application

Example 1: Let x,y be real numbers .Prove that

$$\sqrt{x^2-2x+2}+\sqrt{x^2+2x+17}\geq\sqrt{29}$$

Solution We have $\sqrt{x^2-2x+2}+\sqrt{x^2+2x+17}\geq\sqrt{29}$

$$\Leftrightarrow\sqrt{(x-1)^2+1}+\sqrt{(x+1)^2+16}\geq\sqrt{29}$$

Let complex numbers $z=x-1+i$, $z'=-x-1+4i$ and $z''=-2+5i$

We have: $z+z'=z''$

using inequality $|z|+|z'|\geq|z+z'|$, we have $\sqrt{(x-1)^2+1}+\sqrt{(x+1)^2+16}\geq\sqrt{29}$.

Example 2: Let a_1,a_2,b_1,b_2 be real numbers . Prove that:

$$\sqrt{(a_1+a_2)^2+(b_1+b_2)^2}\leq\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}$$

Solution

Let complex numbers $z=a_1+b_1i$ and $z'=a_2+b_2i$

We have $z+z'=(a_1+a_2)+(b_1+b_2)i$

using inequality $|z|+|z'|\geq|z+z'|$,we have:

$$\sqrt{(a_1+a_2)^2+(b_1+b_2)^2}\leq\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}\ .$$

Example 3: Let a,b,c be real numbers . Prove that:

$$\sqrt{a^2+ab+b^2}+\sqrt{a^2+ac+c^2}\geq\sqrt{b^2+bc+c^2}$$

Solution

We have

$$\sqrt{a^2+ab+b^2}+\sqrt{a^2+ac+c^2}\geq\sqrt{b^2+bc+c^2}$$

$$\Leftrightarrow\sqrt{\left(a+\frac{b}{2}\right)^2+\left(\frac{b\sqrt{3}}{2}\right)^2}+\sqrt{\left(a+\frac{c}{2}\right)^2+\left(\frac{c\sqrt{3}}{2}\right)^2}\geq\sqrt{\left(\frac{b}{2}-\frac{c}{2}\right)^2+\left(\frac{b\sqrt{3}}{2}+\frac{c\sqrt{3}}{2}\right)^2}$$

Let complex numbers $z=a+\frac{b}{2}+\frac{b\sqrt{3}}{2}i$, $z'=-a-\frac{c}{2}+\frac{c\sqrt{3}}{2}i$.

We have $z+z'=\frac{b}{2}-\frac{c}{2}+\left(\frac{b\sqrt{3}}{2}+\frac{c\sqrt{3}}{2}\right)i$.

using inequality $|z|+|z'|\ge|z+z'|$, we have:

$$\sqrt{\left(a+\frac{b}{2}\right)^2+\left(\frac{b\sqrt{3}}{2}\right)^2}+\sqrt{\left(a+\frac{c}{2}\right)^2+\left(\frac{c\sqrt{3}}{2}\right)^2}\ge\sqrt{\left(\frac{b}{2}-\frac{c}{2}\right)^2+\left(\frac{b\sqrt{3}}{2}+\frac{c\sqrt{3}}{2}\right)^2}\ .$$

Example 4: Let x, y, z be positive real numbers such that x + y + z = 3. Prove that: $\sqrt{x^2+xy+y^2}+\sqrt{y^2+yz+z^2}+\sqrt{x^2+xz+z^2}\ge 3\sqrt{3}$

Solution

Let $S=\sqrt{x^2+xy+y^2}+\sqrt{y^2+yz+z^2}+\sqrt{x^2+xz+z^2}$

We have $S=\sqrt{(x+\frac{y}{2})^2+\left(\frac{\sqrt{3}y}{2}\right)^2}+\sqrt{(y+\frac{z}{2})^2+\left(\frac{\sqrt{3}z}{2}\right)^2}+\sqrt{(z+\frac{x}{2})^2+\left(\frac{\sqrt{3}x}{2}\right)^2}$

Let complex numbers $z=x+\frac{y}{2}+\left(\frac{\sqrt{3}y}{2}\right)i$, $z'=y+\frac{z}{2}+\left(\frac{\sqrt{3}z}{2}\right)i$,

$z''=z+\frac{x}{2}+\left(\frac{\sqrt{3}x}{2}\right)i$

We have $z+z'+z''=\frac{3}{2}(x+y+z)+\frac{\sqrt{3}}{2}(x+y+z)i$

using inequality $|z|+|z'|+|z''|\ge|z+z'+z''|$, we have:

$$\sqrt{(x+\frac{y}{2})^2+\left(\frac{\sqrt{3}y}{2}\right)^2}+\sqrt{(y+\frac{z}{2})^2+\left(\frac{\sqrt{3}z}{2}\right)^2}+\sqrt{(z+\frac{x}{2})^2+\left(\frac{\sqrt{3}x}{2}\right)^2}\ge\sqrt{\frac{9}{4}(x+y+z)^2+\frac{3}{4}(x+y+z)^2}$$

$$\Leftrightarrow\sqrt{(x+\frac{y}{2})^2+\left(\frac{\sqrt{3}y}{2}\right)^2}+\sqrt{(y+\frac{z}{2})^2+\left(\frac{\sqrt{3}z}{2}\right)^2}+\sqrt{(z+\frac{x}{2})^2+\left(\frac{\sqrt{3}x}{2}\right)^2}\ge\sqrt{\frac{9}{4}.9+\frac{3}{4}.9}=\sqrt{27}$$

Thus $\sqrt{(x+\frac{y}{2})^2+\left(\frac{\sqrt{3}y}{2}\right)^2}+\sqrt{(y+\frac{z}{2})^2+\left(\frac{\sqrt{3}z}{2}\right)^2}+\sqrt{(z+\frac{x}{2})^2+\left(\frac{\sqrt{3}x}{2}\right)^2}\ge 3\sqrt{3}$

Example5: Let a, b, c be positive real numbers such that ab + bc +ca = abc.
Prove that:

$$\frac{\sqrt{a^2+2b^2}}{ab}+\frac{\sqrt{b^2+2c^2}}{bc}+\frac{\sqrt{c^2+2a^2}}{ca}\ge\sqrt{3}$$

Solution

Let $x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}$, we have: x, y , z > 0 and x + y + z =1.

LHS = $\frac{\sqrt{a^2+2b^2}}{ab}+\frac{\sqrt{b^2+2c^2}}{bc}+\frac{\sqrt{c^2+2a^2}}{ca}$ = $\sqrt{x^2+2y^2}+\sqrt{y^2+2z^2}+\sqrt{z^2+2x^2}$

Let complex numbers $z=x+\sqrt{2}yi$, $z'=y+\sqrt{2}zi$, $z''=z+\sqrt{2}xi$

We have $z+z'+z''=(x+y+z)+\sqrt{2}(x+y+z)i$

using inequality $|z|+|z'|+|z''|\ge|z+z'+z''|$, we have:

$$\sqrt{x^2+2y^2}+\sqrt{y^2+2z^2}+\sqrt{z^2+2x^2}\ge\sqrt{\left(x+y+z\right)^2+2\left(x+y+z\right)^2}$$

$\Leftrightarrow\sqrt{x^2+2y^2}+\sqrt{y^2+2z^2}+\sqrt{z^2+2x^2}\ge\sqrt{3}$ (because x + y + z = 1)

Thus $\frac{\sqrt{a^2+2b^2}}{ab}+\frac{\sqrt{b^2+2c^2}}{bc}+\frac{\sqrt{c^2+2a^2}}{ca}\ge\sqrt{3}$

<u>Example6</u>: Let a, b, c be positive real numbers such that x + y + z ≤ 1. Prove that $\sqrt{x^2+\frac{1}{x^2}}+\sqrt{y^2+\frac{1}{y^2}}+\sqrt{z^2+\frac{1}{z^2}}\ge\sqrt{82}$

<u>Solution</u>

Let complex numbers $z=x+\frac{1}{x}i$, $z'=y+\frac{1}{y}i$, $z''=z+\frac{1}{z}i$

We have $z+z'+z''=(x+y+z)+(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})i$

using inenquality $|z|+|z'|+|z''|\ge|z+z'+z''|$, we have:

$$S=\sqrt{x^2+\frac{1}{x^2}}+\sqrt{y^2+\frac{1}{y^2}}+\sqrt{z^2+\frac{1}{z^2}}\ge\sqrt{(x+y+z)^2+(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})^2}$$

On the other hand, we have

$$(x+y+z)^2+(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})^2=\left[81(x+y+z)^2+(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})^2\right]-80(x+y+z)^2$$

We will use the AM-GM inequality, we have

$$81(x+y+z)^2+(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})^2\ge 2.\sqrt{81(x+y+z)^2(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})^2}\ge 18.9$$

thus

$$(x+y+z)^2+(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})^2=\left[81(x+y+z)^2+(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})^2\right]-80(x+y+z)^2\ge 18.9-80.1=82$$

then $LHS=\sqrt{x^2+\frac{1}{x^2}}+\sqrt{y^2+\frac{1}{y^2}}+\sqrt{z^2+\frac{1}{z^2}}\ge\sqrt{82}$

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