# The Technique of Conjugate Ratio for Geometric Theorem Proving

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# Abstract

∗ Vector ratio and conjugate ratio are proposed for geometric theorem proving based on complex numbers.

∗ Several new algebraic formulations based on these concepts, including two kinds of straight line equations, parallelism and perpendicularity, and the conjugate ratio of a straight line tangent to a circle are proposed.

∗ ∗These formulations are implemented into a MATHEMATICA program, and are used to generate readable proofs for geometric theorems like the Simson Theorem, Feuerbach Theorem, etc.

# Vector ratio

$*$Let the angle from $\overrightarrow{CD}$ to $\overrightarrow{AB}$ be $\theta$. The vector ratio of $\overrightarrow{CD}$ and $\overrightarrow{AB}$ is defined as $\frac{\overrightarrow{AB}}{\overrightarrow{CD}}$.

$*$It is clear that $\frac{\overrightarrow{AB}}{\overrightarrow{CD}} = \frac{AB}{CD}e^{i\theta}$. $*$If $\frac{\overrightarrow{AB}}{\overrightarrow{AC}} = \frac{\overrightarrow{A'B'}}{\overrightarrow{A'C'}}$ , then $\triangle ABC \sim \triangle A'B'C'$.

$*$From $\theta = \arg \frac{\overrightarrow{AB}}{\overrightarrow{CD}}$,it is easy to deduce that $\theta = \arg \frac{\overrightarrow{\bar{C}\bar{D}}}{\overrightarrow{\bar{A}\bar{B}}}$.

# fixed vector ratio formulas

*Let $\frac{AB}{CD} = \lambda$, $e^{i\theta} = v$,then $\frac{\overrightarrow{AB}}{\overrightarrow{AC}} = \lambda v$.

*If point B coincides with D, we have $\frac{\overrightarrow{AB}}{\overrightarrow{CB}} = \frac{b-a}{b-c} = \lambda v$ .

$$b = \frac{a - \lambda v c}{1 - \lambda v} \text{ and } \bar{b} = \frac{\bar{a} v - \lambda \bar{c}}{v - \lambda}.$$

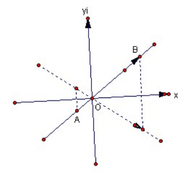
* The interior formulas

$$b = \frac{a + \lambda c}{1 + \lambda} \text{ and } \bar{b} = \frac{\bar{a} + \bar{c}}{1 + \lambda}.$$

*exterior fixed ratio formulas

$$b = \frac{a - \lambda c}{1 - \lambda} \text{ and } \quad \bar{b} = \frac{\bar{a} - \bar{c}}{1 - \lambda}.$$

# conjugate ratio




∗ A, B are two points on line $l$,then $\frac{\overrightarrow{AB}}{\overline{\overrightarrow{AB}}} = \frac{b-a}{\bar{b}-\bar{a}} = e^{2i\theta}$

∗ Denote k=$e^{2i\theta}$ . We call $k$ the conjugate ratio of $l$. Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be on $l$ and have the same direction, then

$ABCD = \frac{\overrightarrow{AB}\,\overrightarrow{CD}}{k} = \overrightarrow{AB}\,\overline{\overrightarrow{CD}} = (b-a)(\bar{d}-\bar{c})$

∗ If l∥real axis:k=1, ∥imaginary axis, k=-1.

$*$Suppose that a straight line $l$'s conjugate ratio is $k$ and point $Z_0$ lies on $l$, then the equation of $l$ is

$$z - k\bar{z} = z_0 - k\bar{z}_0.$$

The equation of the line that passes through points A and B is

$$\begin{vmatrix} z & \bar{z} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} = 0$$

where a and b are the corresponding complex numbers of points A and B.

The sufficient and necessary condition for equation

$$az + b\bar{z} + c = 0 \tag{1}$$

to represent a straight line is that a, b and c are not zero, and

$$\frac{a}{\bar{b}} = \frac{b}{\bar{a}} = \frac{c}{\bar{c}} \text{ or } c = 0 \text{ and } \frac{a}{\bar{b}} = \frac{b}{\bar{a}}$$

Let r is a real number,then

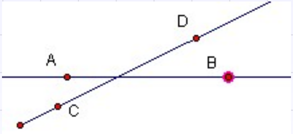
$$\bar{z}_0 z + z_0 \bar{z} = r.$$

is called the standard form of a straight line.

# The angle between lines

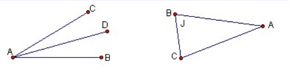
$*$Let $0 \leq \theta_1, \theta_2 < \pi$ , $k_1 = e^{i\theta_1}, k_2 = e^{i\theta_2}$, then the necessary and sufficient condition for $\theta_1 = \theta_2$ is $k_1 = k_2$.

$*$If $\theta$ is the angle from line CD to AB, then $e^{2i\theta} = \dfrac{k_{AB}}{k_{CD}}$.

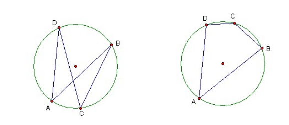



$*$The angle from the straight line a to b is equal to the angle from line c to d, then

$$\frac{k_b}{k_a} = \frac{k_d}{k_c}.$$




∗The sufficient and necessary condition that AD bisects $\angle ABC$ equally or $\angle B = \angle C$ is $k_{AD}^2 = k_{AB}k_{AC}$ or $k_{BC}^2 = k_{AB}k_{AC}$.




∗The sufficient and necessary condition that A, B, C,and D lie on the same circle is $\dfrac{k_{AB}}{k_{AD}} = \dfrac{k_{CB}}{k_{CD}}$.

$$AB \parallel CD \Leftrightarrow k_{AB} = k_{CD} \text{ and } AB \perp CD \Leftrightarrow k_{AB} = -k_{CD}.$$

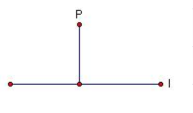
# Distance

$*$Distance from point P to line $l : az + b\bar{z} + c = 0$ . Let H be P's perpendicular projection on $l$, then

$$h = \frac{ap - b\bar{p} - c}{2a} = \frac{1}{2}(p + \frac{-b\bar{p} - c}{a})$$

$$\overrightarrow{HP} = \frac{ap + b\bar{p} + c}{2a}$$

$$|HP| = \frac{|ap + b\bar{p} + c|}{|2a|}.$$



Proof

$*$Let P's foot on line $l$ be H,then $k_{HP} = -\frac{b}{a}$.

$*$Then $HP$'s equation is:

$z + \frac{b}{a}\bar{z} = p - \frac{b}{a}\bar{p}$.

$*$ From $l$ and $HP$'s equations it is easy to show that

$$h = \frac{ap - b\bar{p} - c}{2a} = \frac{1}{2}(p + \frac{-b\bar{p} - c}{a})$$

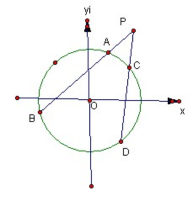
.

$*$ We can have,

$$\overrightarrow{HP} = p - h = \frac{ap + b\bar{p} + c}{2a}$$

$$|HP| = \frac{|ap + b\bar{p} + c|}{|2a|}.$$

# Secant line's conjugate ratio




$*$Points A and B lies on a circle $\odot Z_0 : (z - z_0)(\bar{z} - \bar{z}_0) = r^2$, then

$$k_{AB} = -\frac{a - z_0}{\bar{b} - \bar{z}_0} = -\frac{b - z_0}{\bar{a} - \bar{z}_0} = -\frac{(a - z_0)(b - z_0)}{r^2}.$$

Especially, if the center of the circle is at the origin, then

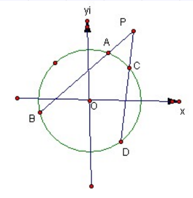
$$k_{AB} = -\frac{a}{\bar{b}} = -\frac{b}{\bar{a}} = -\frac{ab}{r^2}.$$

proof

∗From the circle's equation,we have

$$k_{AB} = \frac{a-b}{\bar{a}-\bar{b}} = \frac{a-b}{\frac{r^2}{a-z_0} - \frac{r^2}{b-z_0}}$$

$$= -\frac{(a-z_0)(b-z_0)}{r^2} = -\frac{a-z_0}{\bar{b}-\bar{z}_0} = -\frac{b-z_0}{\bar{a}-\bar{z}_0}$$




∗Furthermore, if r=1, then $k_{AB} = -ab$ , and line AB's equation is $z + ab\bar{z} = a + b$.
∗A, B, C and D lie on an unit circle with its centered at the origin, then the intersection point of AB and CD is

$$p = -\frac{(a+b)cd - ab(c+d)}{ab - cd}, \bar{p} = \frac{(a+b) - (c+d)}{ab - cd}.$$

# Tangent line's conjugate ratio

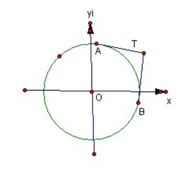
∗If the straight line $l$ is tangent to the circle $\odot Z_0 : (z - z_0)(\bar{z} - \bar{z}_0) = r^2$ at A, then

$$k_l = -\frac{a - z_0}{\bar{a} - \bar{z}_0} = -\frac{(a - z_0)^2}{r^2}$$

∗if the center of the circle is the origin, then

$$k_l = -\frac{a}{\bar{a}} = -\frac{a^2}{r^2}$$

∗If r=1,then $k_l = -a^2$ and line $l$'s equation is $z + a^2\bar{z} = 2a$.




∗Let the straight lines TA and TB be tangent to circle $\odot O : z\bar{z} = 1$ at A and B respectively, then

$$t = \frac{2ab}{a+b}, \bar{t} = \frac{2}{a+b}.$$

# The Complex Algorithm

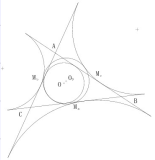
∗ Step1. We determine those free and half-free points, lines, circles,etc.Complex parameters related to them are used as basic variants in algebraic expressions.

∗ Step2. According to geometric conditions,some points are constructed.We could obtain these points' complex numbers and their conjugates by using complex expressions related geometric conditions.

∗ Step3.We transform the geometric conclusions into complex expressions.If the left of the conclusion is equal to the right,we could decide that the conclusion is true.

# Feuerbach Theorem

∗ The 9-point circle of a triangle is tangent to its inscribed circle and the three escribed circles.



∗Let the origin be at the center of the inscribed circle or one of the three escribed circles, and let this circle be of unit radius. Then sides AB, BC and AC are tangent to the circle at points D,E,F, and Mc, Ma and Mb are midpoints of AB, BC and CA respectively. Let $O_9$ be the center of the nine-point circle.

$a = \frac{2df}{d+f}; a' = \frac{2}{d+f}; b = \frac{2de}{d+e}; b' = \frac{2}{d+e}; c = \frac{2ef}{e+f}; c' = \frac{2}{e+f};$
$\mathrm{Mc} = \frac{a+b}{2}; \mathrm{Mc}' = \frac{a'+b'}{2}; \mathrm{Mb} = \frac{a+c}{2}; \mathrm{Mb}' = \frac{a'+c'}{2}; \mathrm{Ma} = \frac{b+c}{2}; \mathrm{Ma}' = \frac{b'+c'}{2};$

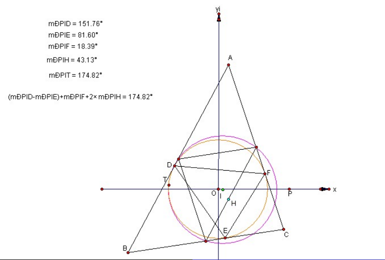
$f[\mathrm{z\_}, \mathrm{o\_}] := (z-o)(z'-o')$; (*a distance's square*)
$\mathrm{Simplify}[\{1, \mathrm{o9}-\mathrm{Mc}, \mathrm{o9}'-\mathrm{Mc}', 2, f[\mathrm{o9}, \mathrm{Mc}]\}/.$
$\mathrm{Solve}[f[\mathrm{o9}, \mathrm{Mc}] == f[\mathrm{o9}, \mathrm{Mb}] == f[\mathrm{o9}, \mathrm{Ma}], \{\mathrm{o9}, \mathrm{o9}'\}]]$

$\mathrm{Simplify}[\mathrm{Solve}[\{f[\mathrm{o9}, \mathrm{Mc}] == f[\mathrm{o9}, \mathrm{Mb}] == f[\mathrm{o9}, \mathrm{Ma}], f[z, \mathrm{o9}] == f[\mathrm{o9}, \mathrm{Mb}],$
$zz' == 1\}, \{\mathrm{o9}, \mathrm{o9}', z, z'\}$

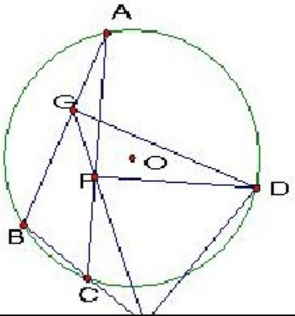
$$\left\{\left\{1, \frac{e^2f^2}{(d+e)(d+f)(e+f)}, \frac{d^2}{(d+e)(d+f)(e+f)}, 2, \frac{d^2e^2f^2}{(d+e)^2(d+f)^2(e+f)^2}\right\}\right\}$$
$$\left\{\left\{z \to \frac{ef+d(e+f)}{d+e+f}, z' \to \frac{d+e+f}{ef+d(e+f)}, \mathrm{o9} \to \frac{(ef+d(e+f))^2}{(d+e)(d+f)(e+f)}, \mathrm{o9}' \to \frac{(d+e+f)^2}{(d+e)(d+f)(e+f)}\right\},\right.$$
$$\left.\left\{z \to \frac{ef+d(e+f)}{d+e+f}, z' \to \frac{d+e+f}{ef+d(e+f)}, \mathrm{o9} \to \frac{(ef+d(e+f))^2}{(d+e)(d+f)(e+f)}, \mathrm{o9}' \to \frac{(d+e+f)^2}{(d+e)(d+f)(e+f)}\right\}\right\}$$

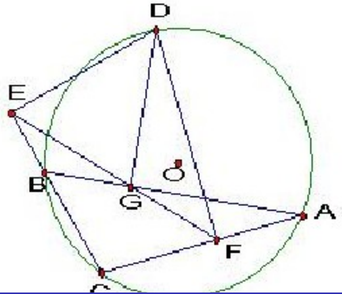
# Find a new Way for Feuerbach Point

Let $\alpha, \beta, \gamma$ be the angles of the complex numbers d,e,f.Let G be the barycenter of $\triangle DEF$, let the angle from vector $\overrightarrow{OG}$ to vector $\overrightarrow{O\bar{G}}$ be $\theta$, then the intersection of the circle and the positive real axis is the Feuerbach point after being rotated with angle $\alpha + \beta + \gamma + \theta$. It is very amazing that there is no "-" in the result.



*Example 5.* [Simson Theorem] Let D be a point on the circumscribed circle of triangle ABC.From D three perpendiculars are drawn to the three sides BC,AC,and AB of triangle ABC.Let E,F,G be three foot respectively.Show that E,F and G are collinear.



Let O be the origin,circle O be an unit circle. Mathematica Program could be designed.

$a' = 1/a; b' = 1/b; c' = 1/c; d' = 1/d;$

$k[\mathrm{a\_}, \mathrm{b\_}] := \frac{a-b}{a'-b'};$

$\mathrm{Foot}[\mathrm{p\_}, \mathrm{a\_}, \mathrm{b\_}] := \frac{1}{2}(p - abp' + a + b);$

$\mathrm{Foot}'[\mathrm{p\_}, \mathrm{a\_}, \mathrm{b\_}] := \frac{1}{2}(p' - a'b'p + a' + b');$

$e = \mathrm{Foot}[d, b, c]; e' = \mathrm{Foot}'[d, b, c];$

$f = \mathrm{Foot}[d, a, c]; f' = \mathrm{Foot}'[d, a, c];$

$g = \mathrm{Foot}[d, b, a]; g' = \mathrm{Foot}'[d, b, a];$

$\mathrm{Simplify}[\{e, f, g, k[e, f], k[f, g]\}]$

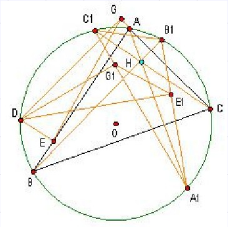
$$\left\{\frac{1}{2}\left(b + c - \frac{bc}{d} + d\right), \frac{1}{2}\left(a + c - \frac{ac}{d} + d\right), \frac{1}{2}\left(a + b - \frac{ab}{d} + d\right), \frac{abc}{d}, \frac{abc}{d}\right\}$$

*This proof is more concise than that generated by full angle method[5],area method[2],vector method[6].From the result,it is obvious that

$$2arg\frac{\overrightarrow{GE}}{\overrightarrow{AB}} + arg\frac{d}{c} = \pi \text{ or } 2arg\frac{\overrightarrow{GE}}{\overrightarrow{AB}} + arg\frac{d}{c} = 3\pi.$$

Let two triangles lie on a circle,it is easy to show that the angle between two Simson lines for any points for them is a constant.

# Orthocentre and Simson line




$*$Let $A_1, B_1$ and $C_1$ be intersections of three altitudes and the circle.

$*$While $i \geq 1$,let $A_{i+1}$ be the intersection of the side $B_iC_i'$s altitude and the circle,we have

$a_{i+1} = -\frac{b_i c_i}{a_i}, b_{i+1} = -\frac{a_i c_i}{b_i}, c_{i+1} = -\frac{b_i a_i}{c_i}$.

$$a_1 = -\frac{cb}{a}, b_1 = -\frac{ac}{b}, c_1 = -\frac{ab}{c}, a_2 = \frac{a^3}{bc}, b_2 = \frac{b^3}{ac}, c_2 = \frac{c^3}{ab},$$

$$a_3 = -\frac{b^3c^3}{a^5}, b_3 = -\frac{a^3c^3}{b^5}, c_3 = -\frac{a^3b^3}{c^5}, a_4 = \frac{a^{11}}{b^5c^5}, b_4 = \frac{b^{11}}{a^5c^5}, c_4 = \frac{c^{11}}{a^5b^5}$$

$$a_5 = -\frac{b^{11}c^{11}}{a^{21}}, b_5 = -\frac{a^{11}c^{11}}{b^{21}}, c_5 = -\frac{a^{11}b^{11}}{c^{21}}, a_6 = \frac{a^{43}}{b^{21}c^{21}}, b_6 = \frac{b^{43}}{a^{21}c^{21}}, c$$

Obviously,the Simson line $E_iG_i \perp E_{i+1}G_{i+1}, E_iG_i || E_{i+2}G_{i+2}$.

# Simson-Zhang Curve

∗Let Z be a point on the complex plane,lines AZ,BZ,CZ intersect the circumscribed circle of triangle ABC at $A_1$, $B_1$, $C_1$,D lie on this circle,the origin at circumcentre.Then
$k_{aa_1} = \frac{z-a_1}{\bar{z}-\bar{a}_1} = -aa_1, k_{bb_1} = \frac{z-b_1}{\bar{z}-\bar{b}_1} = -bb_1, k_{cc_1} = \frac{z-c_1}{\bar{z}-\bar{c}_1} = -cc_1$

∗If the two Simson lines for D for $\triangle ABC$ and $\triangle A_1B_1C_1$ are perpendicular ,according to the conclusion in this example,then Z will lies on the curve

$$Simson - Zhang \text{ Curve} : \frac{(z-a)(z-b)(z-c)}{(\bar{z}-\bar{a})(\bar{z}-\bar{b})(\bar{z}-\bar{c})} = (abc)^2 \qquad (2)$$

∗Simson-Zhang Curve could be changed to

$$\sin(\alpha+\beta+\gamma)x^3+\cos(\alpha+\beta+\gamma)y^3-3\cos(\alpha+\beta+\gamma)x^2y-3\sin(\alpha+\beta+\gamma)xy^2-[\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)](x^2-y^2)+2[\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)]xy+(\sin\alpha+\sin\beta+\sin\gamma)x-(\cos\alpha+\cos\beta+\cos\gamma)y=0$$

∗According to the former conclusions,it is easy to show that the orthocentre,incentre, three escentres and circumcentre are on Simson-Zhang curve.

# A Equilateral or Right Triangle's Simson-Zhang Curve

$*$Let $\triangle ABC$ be a equilateral triangle,$a = 1, b = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, b = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$,then its Simson-Zhang Curve is
$z^3 - \bar{z}^3 = 0$,or $y = 0, y = \sqrt{3}x, y = -\sqrt{3}x$.

Let $\triangle ABC$ be a right triangle,$a = 1, b = i, c = -i$,then its Simson-Zhang Curve is
$(z^3 - \bar{z}^3) - (z^2 - \bar{z}^2) + (z - \bar{z}) = 0$,or $y = 0$ and $3x^2 - y^2 - 2x + 1 = 0$.

# Conclusion

After the vector ratio and conjugate ratio is proposed,the technique based on complex number has been fully developed.The expressions is more readable and concise that generated by coordinates.