60-odd YEARS of

MOSCOW MATHEMATICAL

OLYMPIADS

Edited by D. Leites

Compilation and solutions by G. Galperin and A. Tolpygo

with assistance of P. Grozman, A. Shapovalov and V. Prasolov

and with drawings by A. Fomenko

Translated from the Russian by D. Leites

Computer-drawn figures by

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Abstract

Nowadays, in the time when the level of teaching universally decreases and "pure" science does not appeal as it used to, this book can attract new students to mathematics.

The book can be useful to all teachers and instructors heading optional courses and mathematical groups.
It might interest university students or even scientists.

But it was primarily intended for high school students who like mathematics (even for those who, perhaps, are unaware of it yet) and to their teachers. The complete answers to all problems will facilitate the latter to coach the former.

The book also contains some history of Moscow Mathematical Olympiads and reflections on mathematical olympiads and mathematical education in the Soviet Union (the experience that might be of help to western teachers and students). A relation of some of the problems to "serious" mathematics is mentioned.

The book contains more than all the problems with complete solutions of Moscow Mathematical Olympiads starting from their beginning: some problems are solved under more general assumptions than planned during the Olympiad; there extensions are sometimes indicated. Besides, there are added about a hundred selected problems of mathematical circles (also with solutions) used for coaching before Olympiads.

The *Moscow Mathematical Olympiad* was less known outside Russia than the "*All-Union*" (i.e., National, the USSR), or the *International* Olympiad but the problems it offers are on the whole rather more difficult and, therefore, it was more prestigious to win at. In Russia, where sports and mathematics are taken seriously, more than 1,000,000 copies of an abridged version of a part of this book has been sold in one year.

This is the first book which contains *complete* solutions to all these problems (unless a hint is ample, in which case it is dutifully given).

The abriged Russian version of the book was complied by **Gregory Galperin**, one of the authors of a great part of the problems offered at Moscow Mathematical Olympiads (an expert in *setting* olympiad-type problems) and **Alexei Tolpygo**, a former winner of the Moscow, National and International Olympiads (an expert in *solving* mathematical problems). For this complete English edition **Pavel Grozman** and **Alexander Shapovalov** (a first and a third prize winners at the 1973 and 1972 International Mathematical Olympiads, respectively) wrote about 200 new solutions each.

The book is illustrated by **Anatoly Fomenko**, Corresponding Member of the Russian Academy of Sciences, Professor of Mathematics of Moscow University. Fomenko is very well known for his drawings and paintings illustrating the wonders of math.

Figures are sketched under supervision of **Victor Prasolov**, Reader at the Independent University of Moscow. He is well-known as the author of several amazingly popular books on planimetry and solid geometry for high-school students.

From I.M. Yaglom's *"Problems, Problems, Problems. History and Contemporaneity"*
(a review of *MOSCOW MATHEMATICAL OLYMPIADS*
compiled by G. Galperin and A. Tolpygo)

The oldest of the USSR Math Olympiads is the Leningrad High-school Olympiad launched in 1934 (the Moscow Math Olympiad runs since 1935). Still, for all these years the "most main" olympiad in the country was traditionally and actually the **Moscow Math Olympiad**. Visits of students from other towns started the expansion of the range of the Moscow Math Olympiad to the whole country, and, later, to the whole Earth: as International Olympiads.

More than half-a-century-long history of MMO is a good deal of the history of the Soviet high school, history of mathematical education and interactive work with students interested in mathematics. It is amazing to trace how the level of difficulty of the problems and even their nature changed with time: problems of the first Olympiads are of the "standard-schoolish" nature (cf. Problems 1.2.B.2, 2.2.1, 3.1.1 and 4.2.3) whereas even the plot of the problems of later olympiads is often a thriller with cops and robbers, wandering knights and dragons, apes and lions, alchemists and giants, lots of kids engaged in strange activities, with just few quadratics or standard problems with triangles.

Problems from the book compiled by Galperin and Tolpygo constitute a rare collection of the long work of a huge number of mathematicians of several generations; the creative potential of the (mainly anonymous) authors manifests itself in a live connection of many of the olympiads' problems with current ideas of modern

Mathematics. The abundance of problems associated with games people play, various schemes described by a finite set, or an array of numbers, or a plot, with only qualitative features being of importance, mirrors certain general trends of the modern mathematics.

Several problems in this book have paradoxical answers which contradict the "natural" expectations, cf. Problems 13.1.9-10.2, 24.1.8.2, 32.7.3, 38.1.10.5, 44.7.3, and Problems 32.9.4 and 38.2.9.19 (make notice also of auxiliary queries in Hints!).

Contents

Preface

I never liked Olympiads.

The reason is I am a bad sportsman: I hate to lose. Sorry to say, I realize that at any test there usually is someone who can pass the test better, be it a soccer match, an exam, or a competition for a promotion. Whatever the case, skill or actual knowledge of the subject in question often seem to be amazingly less important than self-assurance.

Another reason is that many of the winners in mathematical Olympiads that I know have, unfortunately, not been very successful as mathematicians when they grew up unless they continued to study like hell (which means that those who became good mathematicians were, perhaps, not very successful as human beings; however, those who did not work like hell were even less successful). Well, life is tough, but nevertheless it is sometimes very interesting to live and solve problems.

To business.

Regrettable as it is, an average student of an ordinary school and often, even the[1] teacher, has no idea that not all theorems have yet been discovered.

For better or worse, the shortest way for a kid to discover *mathematics as science*, not just a cook book for solving problems, is usually through an Olympiad: it is a spectacular event full of suspense, and a good place to advertise something really worth supporting like a math group or a specialized mathematical school. (Olympiads, like any sport, need sponsors. Science needs them much more but draws less.)

On the other hand, there are people who, though slow-witted at Olympiads, are good at solving problems that may take *years* to solve, and at inventing new theorems or even new *theories*.

One should also be aware of the fact that today's mathematical teaching all over the world is on the average at a very low level; the textbooks that students have to read and the problems they have to solve are very boring and remote from *reality*,whatever *that* might mean. As a Nobel prize winner and remarkable physicist Richard Feynman put it[2], most (school) textbooks are *universally lousy*.

This is why I tried to do my best to translate, edit and advertise this book — an exception from the pattern (for a list of a few very good books on elementary mathematics see Bibliography and paragraph H.5 of *Historical remarks*; regrettably, some of the most interesting books are in Russian).

If you like the illustrations in this book you might be interested in the whole collection of Acad.[3] A. Fomenko's drawings (A. Fomenko, *Mathematical impressions*, AMS, Providence, 1991) and the mathematics (together with works of Dali, Breughel and Esher) that inspired Fomenko to draw them.

* * *

This is the first complete compilation of the problems from Moscow Mathematical Olympiads with solutions of ALL problems. It is based on previous Russian selections: [SCY], [Le] and [GT]. The first two of these books contain selected problems of Olympiads 1–15 and 1–27, respectively, with painstakingly elaborated solutions. The book [GT] strives to collect formulations of **all** (cf. *Historical remarks*) problems of Olympiads 1–49 and solutions or hints to most of them.

For whom is this book? The success of its Russian counterpart [Le], [GT] with their 1,000,000 copies sold should not decieve us: a good deal of the success is due to the fact that the prices of books, especially text-books, were increadibly low (< 0.005 of the lowest salary.) Our audience will probably be more limited.

[1]We usually use a neutral "(s)he" to designate indiscriminately any homo, sapiens or otherwise, a Siamease twin of either sex, a bearer of any collection of X and Y chromosomes, etc. In one of the problems we used a "(s)he" speaking of a wise cockroach. *Hereafter editor's footnotes.*

[2]Feynman R. *Surely you're joking, Mr. Feynman.* Unwin Paperbacks, 1989.

[3]There were several scientific degrees one could get in the USSR: that of Candidate of Science is roughly equivalent to a Ph.D., that of Doctor of Science is about 10 times as scarce. Scarcer still were members of the USSR Academy of Sciences. Among mathematicians there were about 100 Corresponding Members — in what follows abbreviated to CMA — and about 20 Academicians; before the inflation of the 90's they were like gods. (This is why the soviet authors carefully indicate the scientists' ranks.)

However, we address it to ALL English-reading teachers of mathematics who could suggest the book to their students and libraries: we gave understandable solutions to ALL problems.

Do not ignore fine print, please. Though not as vital, perhaps, as contract clauses, *Remarks* and *Extensions*, i.e., generalizations of the problems, might be of no less interest than the main text.

Difficult problems are marked with an asterisk *.

Whatever the advertisements inviting people to participate in a Moscow Mathematical Olympiad say, some extra knowledge is essential and taken for granted. The compilers of [Le] and [SCY], not so pressed to save space, earmarked about half the volume to preparatory problems. We also provide sufficient Prerequisites. Most of the problems, nevertheless, do not require any special background.

The organizers of Olympiads had no time to polish formulations of problems. Sometimes the solutions they had in mind were wrong or trivial and the realization of the fact dawned at the last minute. It was the task of the "managers" (responsible for a certain grade) and the Vice Chairperson of the Organizing committee to be on the spot and clarify (sometimes considerably). Being unable to rescue the reader on the spot, I have had to alter some formulations, thus violating the Historical Truth in favour of clarity.

While editing, I tried to preserve the air of Moscow mathematical schools and circles of the period and, therefore, decided to season with historical reminiscences and clarifying footnotes. We also borrowed Acad. Kolmogorov's foreword to [GT] with its specific pompous style. One might think that political allusions are out of place here. However, the stagnation and oppression in politics and social life in the USSR was a reason that pushed many bright (at least in math) minds to mathematics.

The story *A little problem*[1] and *Historical remarks* describe those times. Nowadays the majority of them live or work in America or Europe. I hope that it is possible to borrow some experience and understand the driving forces that attracted children to study math (or, more generally, to mathematical schools, from where many future physicists, biologists, etc., or just millionaires, also emerged). It was partly the *way* they studied and later taught, that enabled them to collect a good number of professorial positions in leading Universities all over the world (or buy with cash a flat on Oxford street, London).

What is wrong with the educational system in the USA or Europe, that American or European students cannot (with few exceptions) successively compete with their piers from the USSR? This question is so interesting and important that *The Notices of American Mathematical Society* devoted the whole issue (v. 40, n.2, 1993) to this topic, see also the collection of reminiscences in: S. Zdravkovska, P. Duren (eds.), *The golden years of Moscow mathematics*, AMS, Providence, 1993.

There were several features that distinguished mathematical circles and mathematical olympiads. The better ones were almost free of official bureaucratic supervision: all circles, olympiads, even regular lectures at mathematical schools (a lot of hours!) were organized by volunteers who often worked "the second shift" gratis for weeks or years (sic!); their only reward being moral satisfaction. There was freedom of dress code, possibility for children to address the leader of a circle, a Professor, by the first name (unheard of at regular schools), and the possibility for students who ran the circles and olympiads to ridicule the governing Rules in problems, without endangering the whole enterprise, by sticking the head out too far.

One of the problems (32.2.9.4 on "democratic elections") was even published recently in a political magazin *Vek XX i mir* (*20-th Century and the World*, no. 10, 1991) with a discussion of its timelyness and realistic nature.

We should realize, however, that graduates of mathematical schools, though freer in thinking, were often handicapped by overestimation of professional (especially mathematical) skills of a person as opposed to humane qualities.

* * *

This compilation seemingly exhausts the topic: problems of the 70's are often more difficult than interesting; owing to the general lack of resources Moscow Mathematical Olympiads became less popular. About 15 years ago similar lack of enthusiasm gripped famous Moscow mathematical schools. A way to revitalize mathematical education was suggested by one of the principal organizers of Moscow mathematical schools, Nikolaj Nikolaevich (Kolya) Konstantinov. It was similar to the most effective modern scientific way of getting rid of stafillacocus in maternity wards in our learned times: burn down the whole damned house. Konstantinov organized several totally new mathematical schools and a so-called *Tournament of Towns* (as a rival to counterbalance the Olympiads). The tournament became an international event several years ago; for the first collection see [T].

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[1] This story was published during an abortive thaw in 60's; its author was unable to publish since.

I thank those who helped me: I. Bernstein, L. Makar-Limanova and Ch. Devchand; V. Pyasetsky, V. Prasolov and I. Shchepochkina. Pavel Grozman and Alexander Shapovalov had actually (re)written about 150 solutions each, Grozman made about a 1000 clarifying comments.

I also thank N. N. Konstantinov who introduced me to mathematics.

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Forewords

Mainly for the teacher. The problems collected in this book were originally designed for a competition, that is, to be solved in five hours time during an Olympiad. Many mathematicians in Russia were quite unhappy about this. They argued against this mixture of sport and science: many winners later did not achieve nearly so much in their studies as in this really very specific kind of "mathematical sport". Vice versa, many people who could never succeed under stress proved later to be among the most talented and productive. It is true also that real mathematics deals mostly with problems taking months and years, not hours, to make a step forward.

Still, for many schoolchildren, the idea of a competition is very attractive, and they can take part just for its sake and so discover how diverse and interesting Mathematics (not just math) can be. Afterwards one can find a lot of more productive mathematical activities than competitions: reading mathematical books is just one. But there should be the very first step, and Olympiads, as well as Olympiad style problems in school mathematical clubs and such, help to make it.

One can use this book as the source of problems to organize an Olympiad-like competition on one's own, or for the group or individual studies. In Moscow the same group of the University[1] professors and postgraduate students that launched the Olympiads (see *Historical Remarks*) also established a tradition of "mathematical circles" — weekly gatherings of schoolchildren at the University, where they can attend a lecture, solve some problems, report their progress and get advice. Many of the problems first proposed at the Olympiad later became the "circles' folklore" and taught several generations.

To use these problems in this way is probably much better, because it is up to a student to choose: either to compete with others for the number of problems solved, or just to besiege a single difficult one. Thus, different psychological types can be properly treated without hurting anybody. (A failure at the Olympiad can be a cause for a grave psychological disturbance in the whole future life.)

Some problems are tremendously difficult[2]; only few individuals could solve such problems. As you may learn from *Historical Remarks*, there were several problems with not a single correct solution presented to the Organizing Committee (while the Committee only knew a wrong solution). Therefore, never mind if you try to crack some of these hard nuts and fail: so did many others. Try it again later or look up Solutions: perhaps you just misunderstood the formulation. Just do not try a new problem on your pupils before examining it yourself properly: it may save a teacher a lot of trouble.

You may encounter some difficulties trying to explain solutions to your pupils due to the curriculum differences in the U.S. and S.U. You can find feeble consolation in the fact that your colleagues in Russia experience the same difficulties: three more or less radical reforms have passed since the first Olympiad, and the fourth catastrophe is in progress. However, the authors tried to use wherever possible only "elementary" mathematics in solutions, though throwing in a little Calculus could have made it much easier.

We hope that the spirit of the Moscow Mathematical Olympiads will remain the same and that for many years to come there will be ringing voices of teenagers in the rooms of Moscow University and questions will be asked again: "When will the next Olympiad be held?"[3]

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[1]M. V. Lomonosov Moscow University is, or rather was before the mass emigration of the '80s, for the USSR more than what Princeton and Harvard combined are for the USA, at least as regards mathematics. Mathematics was also well taught in some of Moscow Institutes but the study there was handicapped by the red tape and the general lack of the "air". At the moment the major part of Institutes in Moscow and larger cities are renamed into "universities", but still The University remains outstanding.

[2]Sometimes so much so that even after 9 years of editing and re-editing, nobody knew the answers; to a couple of problems we only knew a wrong answer. All this, together with the correct answers became clear when Pavel Grozman, a First prize winner at the 1973 International Mathematical Olympiad lent a hand. Several mistakes (with corrections) were discovered by A. Shapovalov, V. Prasolov and V. Pyasetsky.

[3]Or, rather, more usual "Will we be allowed to eat during the Olympiad?"

Mainly for students. This book may be useful for you in your studies and it may be an entertainment.

It may sound curious for those who know only usual text-books but a lot of students of your age get a lot of fun just solving mathematical problems. To feel this joy for the first time, one usually has to taste something very different from the common kind of school algebra and geometry.

The authors of the problems collected here tried their best not to be boring or scholastic; they prefered rather to be mocking and ridiculing.

There is a lot of good sense in these problems, too. School mathematics is usually formulated in a very specific "scientific" (pseudo-scientific far too often) way. You can recognize a school manual phrase in a hundred. But, in real life, nobody will prepare your problems for you in such a manner[1]: you will have to distill from an actual, vaguely put, problem a precise mathematical one yourself. So the stranger-looking problems teach you to recognize mathematics in the world around you.

Finally, while solving these problems you can get acquainted with many ideas and notions, quite common for mathematics of this century but still not popular enough with school curricula. Without bothering about strict terms, you will learn how to deal with many principles of the so-called "discrete" mathematics, which proved to be a universal language for all natural sciences.

The syllabus of mathematical studies at Soviet secondary schools has undergone in the course of history several radical changes. For instance, the translator of this book was taught complex numbers and trigonometry but not integrals whereas the next and previous generations enjoyed the opposite choice. Some of these changes were akin to smashing blows[2].

We have tried to solve the problems using elementary school mathematics; some of the solutions would, however, be too long if presented in elementary terms so we used some calculus. Ask your teacher in case of confusion and do not blame him/her if (s)he fails to solve some of the problems. In the awful case that a fault or misprint crept into the text please send a tip to the editor or compilers.

One of the points we'd like to make is that ability to solve Olympiad problems does not distinguish a good teacher nor a good mathematician; speaking mathematically this is neither a necessary nor sufficient condition. A good adult mathematician, however, can usually solve any Olympiad problem[3], at least by more advanced means.

You must know that some of the problems collected here are very complicated. Some even proved to be so difficult that in 5 hours of the Olympiad none of the students in the ten-million-city of Moscow succeeded in finding the correct solution. Such a problem can astound even Ph. D. holders[4]. So you should not consider yourself (or your schoolmate, or your professor) inadequate if you (or they) do not make progress even after a week-long struggle.

But you should not be scared off! Our advice: set a difficult problem aside for a while rather than rush immediately to *Solutions* after an unsuccessful first attack.

Now you can begin without further delay. Be sure to skim through *Prerequisites* first!

More advice: always put down your solutions to stew for a while. Discuss them, if possible, with your teacher and classmates. Afterwards, if you have found no faults in your proof, read the one proposed in this book. It may well differ from the one you invented yourself (and if it is similar to yours ... well, the greater chance for both to be correct).

Remember that an answer, such as "Yes", "Never" or "Five hundred and five", is not a solution, even if correct. *Proofs*[5] of all your statements are expected, and the word *clearly* should not litter it: try to explain everything. Be aware of the fact that when *we* wrote "clearly, ...", "it is easy to see", "it is not difficult to verify", etc., we *meant* that it is the inquisitive reader who will actually complete the proof. (Otherwise the book would have been twice its size and price.)

Use Part 2 (*Solutions*) to see if something you thought to be obvious can indeed be deduced from some much more evident facts.

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[1] This is a typical stand of an 'olympiadchik', but it is not altogether wrong.

[2] This might give to an optimistic student an idea that either of the above choices should be harmlessly sacrificed in favor of some other, seemingly healthier than math, activity.

[3] If has nothing better to do; a mathematician usually solves Olympiad-type problems around the clock, anyway; this is one's job and hobby.

[4] Some of the problems have been correctly solved for the first time here.

[5] An average student does NOT know what exactly this word means; neither do some (too many) school teachers. May this book help you to understand this.

Unlike exercise from Problem books, a real problem requires to be investigated: one has to find out at least a way to tackle it. Therefere, start with easier problems. Do not solve all the problems in a row; this is not a homework, choose the problems more interesting to you.

If you can not solve the problem, try to make a similar but easier problem and solve *it*. If you can not even that, read a hint. If it does not help either, try not to gulp the solution but read it slowly, as a detective story in which you try to guess the next turn of mind. Finally, look at the solution “in the large”: what are its main driving ideas, and, most important, how could one get to it.

If you managed to solve the problem, read its solution we offer, since it is instructive to compare different solutions (even if one of them is wrong).

To understand a solution deeper, ask yourself: at what stages of the proof we used different given data? Will the statement be true if we slacken or omit a condition? Is the converse statement true?

Important is not the quantity of problems solved but the deepness of understanding their solution, the new information acquired.

AT AN OLYMPYAD

1. Read all the problems offered and order them as you will solve them. Bare in mind that the order given is usually in accordance with their difficulty from the compilers’ point of view.

2. If the problem has a too easy solution, then, most probably, you misunderstood the formulation or made a mistake.

3. If you can not solve the problem, try to simplify it: make smaller numbers, consider particular cases, etc., or solve it “backwards”, by the rule of contraries, substitute indeterminates instead of given numbewrs or the other way round, etc.

4. If undecided whether a statement is true, try alternatevely to prove and to disprove it.

5. Do not stick to one problem for too long: from time to time make a break and estimate your position. If you managed to advance, continue, otherwise, leave the problem for a while.

6. If tired, relax immediately (look at the sky and contemplate the infinite or walk along the corridor).

7. Having solved the problem, immediately write it down in proper official style, not as a letter to a pal. This will help to verify the arguments and will free the old bean for other problems.

8. Each turn of idea should be documented even if it looks obvious. It is convenient, therefore, to express the solution as a series of statements (lemmas).

9. The student seldom rereads his/her own production *trying to put oneself into the jurys’ shoos: will anybody be able to understand anything you wrote?”*

Good luck and best ideas!

Acknowledgments. We deem it our pleasant duty to point out about 40 years of Sisyphus’ work on mathematical education performed by N. N. Konstantinov.

Konstantinov was (and still is) one of the principal organizers of the specialized Moscow’s mathematical schools, instrumental in arranging Moscow Olympiads and other mathematical contests (Tournament of Towns, etc.). He always was their soul.

Acad. A. N. Kolmogorov, who always actively participated in organizing Moscow Mathematical Olympiads from their start till his death, did much for the book [GT] as its editor and scientific consultant. We use this opportunity to express our warmest gratitude to him.

We are also obliged to all those who helped us in working on the book and preparing it for publication, and above all to V. V. Prasolov, V. M. Tikhomirov, N. B. Vasiliev, and A. M. Abramov, as well as A. P. Savin, S. M. Saakyan, A. L. Toom, E. A. Morozova, R. S. Cherkasov, and A. B. Khodulev.

We are grateful to V. G. Boltiansky and I. M. Yaglom, and to A. A. Leman for their kind permission to use parts of their article about the book [YB] and borrow from [Le], respectively.

G. Galperin and A. Tolpygo

Moscow–Bielefeld; Moscow–Kiev 1985–93

Academician A. N. Kolmogorov's foreword to [GT]

Our country needs many research mathematicians who are able to make discoveries in mathematics itself and to apply it in unusual ways that require great ingenuity. Usually, scientists who started to practice research-type activity while still at school were more successful later on. Many of them made serious discoveries when 17–19 years old. To postpone the involvement of young people in intense research is to irrevocably lose many of potentially very creative researchers.

Addressing school students who are seriously thinking of becoming real mathematicians, I will tell them the following. Just as in sports, practice requires plenty of a young mathematician's time. It will be profitable if you peruse this collection of problems on your own, choose a problem whose formulation seems interesting to you and start thinking it over without reading the solution.

Do not be afraid that you may waste many, many hours doing that. In this respect I recall the words of Boris Nikolaevich Deloné, one of the most remarkable Russian mathematician, who said that a great scientific discovery in mathematics differs from a tough Olympiad problem only in that the problem takes 5 hours to solve while an important research consumes 5 000 hours. Deloné liked to exaggerate; do not take these "5 000 hours" too literally. But it is typical of a mathematician who attacks a difficult problem to be able to ponder over it for days. If a problem proves a hard nut to crack it is reasonable to try another one. But it is also good to turn back to the first one after a while. It is sometimes useful even for mature mathematicians to put off a difficult problem for some time. It often happens that a solution suddenly emerges from the subconscious after a period of time.

It is only natural that one is delighted and even proud of his/her success at an Olympiad. But failure should not upset you too much or make you disappointed in your abilities in mathematics. The success at an Olympiad requires certain special talents which are not at all necessary for a successful research. The very fact of strict limitation of time allotted for solving problems during an Olympiad makes many people quite helpless. There are, however, mathematical problems whose solution can only be obtained as a result of a very long and calm contemplation and after moulding new concepts. Many problems of this kind were solved by Pavel Sergeevich Alexandrov who used to say that if there were mathematical olympiads in his time he may have never become a mathematician since his main accomplishments in mathematics resulted from a long and deep contemplation rather than a fast-working smartness.

I hope that our collection of problems will be of great help for all instructors of math clubs and for the organizers of local olympiads. I wish to make two comments for them.

The Moscow Mathematical Olympiads were originally addressed to 9–10 graders. Since 1940, however, 7-th and 8-th graders were also invited. I think this choice of age group is quite justified. It is at this age that the knack for mathematics becomes manifest. Certainly, one can organize olympiads for younger kids but one has to bear in mind that most of the boys and girls who distinguished themselves in problem-solving contests in 5–6 grades lose their special capabilities and even interest in mathematics as they grow up[1].

When organizing an olympiad for a particular group of students, it is very important to correctly estimate in advance the complexity of the problems to be offered. These should be planned so that the most capable participants could solve most of the problems but there should not be too many participants who failed to solve at least one problem. Some information about the problems which, unexpectedly, proved to be too difficult in practice can be found in reports on Olympiads published in the magazines *Matematika v shkole* and *Kvant*[2]. Regrettably, the level of difficulty was not always correct at some of Moscow Mathematical Olympiads. The content of the problem usually was, nevertheless, up to very high standards.

In *Historical remarks* the authors describe in detail the great experience of Moscow Mathematical Olympiads and how the process of devising olympiad-type problems went hand-in-hand with the work of mathematical clubs under the Moscow University's egid. The joint efforts of the leaders of the University's math circles in a great and outstanding job. It resulted in a book you are going to read now.

The job of the compilers, G. A. Galperin and A. C. Tolpygo, is wonderful and deserves deep gratitude.

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[1] This argument seems doubtful; more serious troubles are (a) the strain and stress of an Olympiad which is the real danger for students at the early age and (b) the difficulty for the organizers to devise reasonably tough and more or less meaningful problems at the level needed.

[2] It is a very good mathematical magazin and during its first 20 years it was a REMARKABLY GOOD magazin. Now a very close version to the Russian original is published in English as *Quantum.*

Part 1: Problems

Introduction

Prerequisites and notational conventions

The following prerequisites were largely assumed to be known to any participant of an Olympiad.[1] Lately it became clear that the gap between the standard school mathematics and that of an Olympiad should be bridged in order not to discriminate against an average student. For example, the collection[2] of preparatory problems for the jubilee 57-th Olympiad contained several very useful comments partly coinciding with ours. We borrowed some of them.

We expect that the reader of knows how to plot the graph of the function $y = ax^2 + bx + c$ given the coefficients a, b, c.

Various (good) books on elementary mathematics written in English use different notations, e.g., quadrangle — quadrilateral; cathetus — leg, etc. To augment the confusion the original problems for various Olympiads were compiled by different authors, each with the own style.

We edited the text in order to reduce such discrepancies but to please all was impossible. For example, the requirements of present AMS mathematics editors to style are sometimes at variance with Webster's dictionaries and differ from guidelines formerly advocated by AMS via Halmos's pamphlet"How to write mathematics", originally published in L'Enseignements Mathématiques t.XVI, fasc. 2, 123–152 and reprinted many times since then in many languages.

Problems are enumerated as follows: the *first* number is *the number of the Olympiad*, the *second* one is *the number of the tour* (if there was only one tour this number is skipped), the *third* number is that of *the grade*, and the *fourth* number is *the number of the problem itself*. There are natural modifications of these notations, e.g. **1.2.C.1** denotes Olympiad 1, tour 2, set C, Problem 1; **4.2.2** denotes Olympiad 4, tour 2, Problem 2; **10.2.7-8.3** denotes Olympiad 10, tour 2, grades 7-8, Problem 3; in **33.D.7.4** D is for Pythagoras' Day.

An asterisk marks a more difficult (heading of a) problem, e.g., 1.2.C.1 b)*.

The principles. Dirichlet's principle. *If n rabbits sit in k hutches, then there is a hutch with not less than $\frac{n}{k}$ rabbits and a hutch with not more than $\frac{n}{k}$ rabbits.*

Though this principle is obvious, it sometimes solves difficult problems: it is not always easy to select objects that play the role of rabbits and hutches.

The Dirichlet's principle applies to continuous quantities as well: *If n rabbits have eaten k kg of food, then there is a rabbit who has eaten not less than $\frac{n}{k}$ kg and a rabbit who has eaten not more than $\frac{n}{k}$ kg.*

The principle of mathematical induction is used to prove an infinite sequence of statements:

Consider a statement $S(n)$ that depends on a positive integer $n \geq n_0$. We believe $S(n)$ to be true for any positive integer $n \geq n_0$ if

1) *$S(n_0)$ holds for some n_0;*

2) *the validity of $S(l)$ for $n_0 \leq l \leq k$ implies $S(k+1)$.*

Heading 1) is called the *base of induction*; heading 2) is called the *inductive step* and the assumption we use in 2) *the inductive hypothesis*[3].

Example: *Find the sum* $1 + 3 + ... + (2n - 1)$.

Solution. Let us denote this sum by $S(n)$ and look at it for small n. We see that $S(1) = 1$, $S(2) = 4$, and $S(3) = 9$. An educated guess is: $S(n) = n^2$.

The base of induction is fulfilled for $n_0 = 1$.

Now the inductive step: $S(k+1) = S(k) + 2(k+1) - 1 =$ (by the inductive hypothesis) $= k^2 + 2k + 1 = (k+1)^2$. Q.E.D.

Sometimes the induction is used *backwards*, cf. Problem 20.2.10.5. Namely, *Consider a statement $S(n)$ that depends on a positive integer $n \geq n_0$. We believe $S(n)$ to be true for any positive integer $n \geq n_0$ if*

[1]Although every cultured person must know these facts, among other useless data, some people do not; therefore, we recommend to skim through this section before advancing further.

[2]Kanel-Belov A., Kovaldzhi A., Vasilev N. *Preparatory Problems to LVII Moscow Mathematical Olympiad*, Treade, Moscow, 1994.

[3]Some mathematicians with quite unorthodox minds doubt the universality of this principle. They only believe in the numbers we can *actually count*. Since the mathematics obtained this way is supposed to be very poor, these ideas are not popular. They did not die out, however, because the attempts to consider only *constructible* statements clarified some messy or nonconstructive proofs of interesting theorems.

1) *$S(n_0)$ holds for some n_0;*

2) *the validity of $S(l)$ for $k \geq l$ implies $S(k-1)$.*

Warning. It could happen that the inductive step is easy to perform but the conclusion is nevertheless wrong. This happens if the justification of the base of the induction is ignored and we are trying to prove a wrong statement.

Example: Let us "prove" that *The eyes of all people are of the same color.* Indeed, the eyes of one person are of the same color[1]. Now, assume that the statement is true for k people; then it is obviously true for $k+1$ people since any k of these $k+1$ persons have the eyes of the same color. The catch is that for $k=2$ the statement is generally false.

Games: selected ideas. 1) *Solution backwards*, cf. Problems ...

2) *Correspondence.* The presence of a lucky move can be justified by a symmetry, partition into pairs or a complement, cf. Problems ...

3) *Transfer of the move.* If we can use the opponent's strategy we are not worse off than the opponent. For example we win or draw if we can assume the opponent's winning position at will, cf. Problems...

4) "*Prepared homework*" (on an infinite field), cf. Problems

Selected ideas. *Reductio ad absurdum.* If we assume that the statement to prove is false and deduce a contradiction from this assumption, this will prove that the statement was true after all.

Estimates. We estimate a complicated quantity with a simpler one. The inequality between the mean arithmetic and the mean geometric is often used.

An *invariant.* A quantity is sometimes always even (odd) or just a constant. This implies that a situation in which this quantity is odd (even) or not a constant are impossible, cf. Problems ... Sometimes a quantity can be calculated (or estimated) in two ways and we compare the results, cf. Problems A residue can serve as an invariant and we only have to check the possibilities case-by-case, cf. Problems ...

Cycles or *periods* that arise in a *process* are examples of invariants, cf. Problems ...

The rule of an extreme element. Singular or *extreme* objects (the largest nummer, the nearest point, the vertex of a polygon, the degenerate circle, the limit case, etc.) often clarify the regular case. Cf. Problems ...

Standard common notations. In all problems on tournaments it is assumed that each participant competes with every other only once. In a chess tournament, a player gets 1 point for victory, half a point if the game ends in a draw, and 0 for loss; in soccer, all scores are twice as much. In basketball, tennis, etc., there are no draws.

The *main diagonal* (of a square array, or a table, or a matrix) is the one which connects the top left corner with the bottom right corner while the other longest diagonal is called the *side diagonal.* Dimensions $m \times n$ of a table show that it has m rows and n columns.

In all problems on *graph* or *checkered* paper or plane, we assume that all small squares or cells are uniform squares of side 1, and any *vertex* or *node* is the intersection of any two non-parallel lines of the grid, i.e., is a vertex of a square.

A *tableau* or just *table* is a rectangular piece of graph paper cut along the lines of the grid.

Space means $\mathbb{R}^3$, i.e., our usual (mathematically speaking, Euclidean) three-dimensional space in which for any two points the *distance* between them is defined.

In all problems involving light rays, billiard balls, etc., we assume that *the angle of reflection is equal to that of incidence.*

$\overline{ab\dots c}$ denotes the positive integer whose (decimal, usually) digits are a, b, $\dots$, c.

An expression of the form $\underbrace{aa\dots a}_{1993}\underbrace{bb\dots b}_{3991}$ means that a is repeated 1993 times and b is repeated 3991 times. Sometimes we write this explicitely if space permits.

$\emptyset$ denotes the *empty set*, i.e., , the set without elements; we assume that $\emptyset$ is a subset of any set.

$M \subset N$ denotes that every element of the set M belongs to N; we say that N is a *subset* of M; $s \in S$ denotes that the *element* s belongs to the set S; $A \setminus B = \{a \in A : a \notin B\}$ denotes the *set-theoretic difference* of sets A and B.

The *intersection* of the sets M and N is denoted by $M \cap N = \{x : x \in M \text{ and } x \in N\}$; the *union* of two sets M and N is denoted by $M \cup N = \{x : x \in M \text{ or (not exclusive) } x \in N\}$; a *disjoint* union is the union of nonintersecting sets; we often consider intersections and unions of several sets.

[1]Even this is sometimes wrong, but for the sake of argument we will consider such deviations as aliens, not humans.

A *partition* of a set is its representation as the disjoint union of its subsets. An example: *coloring* each element of the set in one color.

A (finite or infinite) family of sets A_1, A_2, ... is a *covering* of a set M if every point of M belongs to some A_i. One point of M can be *covered* several times (by different A_i.) A *tiling* is a covering (usually with identical sets) such that each point of M is covered exactly once.

Often (but not in this book) the description of a set $\{a_i\}_{i\in I}$ whose elements are indexed by the elements of a set I is to the confusion of the reader abbreviated to $\{a_i\}$ that, strictly speaking, denotes just the one-element set consisting of a_i.

The number of elements in a set S is called the *cardinality* of S and denoted by $\#(S)$ or card S.

The sets of all integer, nonnegative integer, natural, i.e., **positive** (in some books — not this one — nonnegative) integer, rational, real and complex numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$,$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively.

A 3-gon is called a *triangle*; a 4-gon is called a *quadrilateral*; a 5-gon is called a *pentagon* ; a 6-gon is called a *hexagon*; a 7-gon is called a *heptagon*; a 8-gon is called a *octagon*; [1]; a 10-gon is called a *decagon*; etc.

A triangle with nonequal sides is called a *scalene* one; a triangle with two equal sides is called an *isosceles* one. A *regular polygon* is a one with equal sides and equal angles but a triangle with equal sides (and, therefore, angles) is called *equilateral* and never *regular*.

In this book we often abbreviate a *straight line* to *line*; otherwise it is called a *curve*, unless otherwise specified. So a *border*line or an *air*line is generally a curve while a *broken* line consists of segments of lines. We use the term *segment* speaking of a *line segment*; when other segments appear, e.g., a *spherical segment*, we say so.

A *graph* is a collection of points (*vertices*) connected with curves (*edges*). It often happens that having constructed a graph we distill the objects under the discussion and relations between them. Several theorems of the graph theory are often encountered (*Euler's, Hall's, Menger's theorems*), cf. solutions to Problems .

A set of points in space are said to be in *general position* if no three of them lie on one line. A set of lines in space are said to be in *general position* if no three of them pass through one point and any two intersect (any three lines form a triangle).

The angle between two curves is the angle between the tangents to these curves at an intersection point. (Obviously, there is a choice among *two* angles; a choice of orientation and order of the curves fixes one of the angles. Often, however, it does not matter which of the angles we choose, i.e., if both angles are right ones.)

A *midperpendicular* to a segment is the line perpendicular to the segment and intersecting it at its center. The ray that bisects an angle, or part of the ray that lies inside the polygon under consideration, is called the *bisector* of the angle. (The term *perpendicular bisector* is often used instead of midperpendicular but not in this book.)

We say that a circle is *inscribed into* or *circumscribed about* a triangle if it is tangent to the triangle's three sides (from the inside of course) or passes through the triangle's vertices, respectively. A circle is called an *escribed about* a triangle if it is tangent, from the outside, to one side and the extensions of the triangle's two other sides.

Two lines (or their segments) in space are said to be *skew* if they do not intersect but are nonparallel; the *angle between skew lines* is the angle between two intersecting lines parallel to the skew lines.

A *dihedral angle* is a spatial angle between two intersecting planes π_1 and π_2; it is measured as the angle between two intersecting lines p_1 and p_2 perpendicular to the intersection line of the two planes and such that $p_1 \subset \pi_1$, $p_2 \subset \pi_2$.

A *trihedral angle* is any one of the 8 convex parts of the space between three intersecting planes (with no points of the planes inside it).

A *quadrant* is a quarter of the plane formed by coordinate axes. An *octant* is a trihedral angle with right angles at all its planar angles.

Two figures that can be identified after (1) a parallel translation, or (2) a rotation, or (3) a reflection through a plane or a line and (4) a composition of movements of types (1)– (3) are called *equal*[2].

A *great circle* on a sphere is a one obtained by intersection of the sphere with a plane passing through the sphere's center.

[1]It is funny, no medium-sized dictionary contains a word with a Greek or Latin root for a 9-gon. In this book 9-gons are discriminated, too: they never appear.

[2]Lately the books on elementary mathematics made the life of many a student and their instructors much more thrilling by introducing the terms *equivalent* for figures of equal area and *congruent* for figures that can be identified after an orientation-preserving transformation. The usage of the fancy term "congruent" would have been OK if it were not that much at variance with our every day usage of the language and selfcontradictory at times.

An angle with vertex at the center of a circle is often measured in *radians* or *degrees*, same as the arc it subtends on the circle, so the notation of the form $\angle A = \frac{1}{2}\smile BC$ makes sense.

$[a, b)$ denotes the set of real numbers x such that $a \leq x < b$; we similarly define $[a, b]$, (a, b), etc. We sometimes write $]a, b[$ for (a, b), $]a, b]$ for $(a, b]$, etc. Observe that either of a and b here might be equal to $-\infty$ or ∞.

$[x]$ denotes the *integer part* of x, i.e., the greater integer that does not exceed x, e.g. $[5] = 5$, $\left[1\frac{1}{2}\right] = 1$, $[3/4] = 0$, $[3/2] = 1$, $[\pi] = 3$, $[-1.5] = -2$, etc.

$\{x\} = x - [x]$ denotes the *fractional part* of x.

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$; this reads *factorial* (of) n. Clearly, $1! = 1$; we convene that $0! = 1$.

$(2n)!! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2) \cdot 2n$ denotes the product of n consecutive even numbers (reads *semi-factorial of* $2n$).

$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3) \cdot (2n-1)$ denotes the product of n consecutive odd numbers (reads *semi-factorial of* $2n-1$).

Clearly, $(2n)!! \cdot (2n-1)!! = (2n)!$

The inverse functions $\sin^{-1}$, $\cos^{-1}$, etc. were denoted in the USSR and in many older books by arcsin, arccos, etc. The oldfashioned notation has an advantage: no chance to confuse (except by accident) the value of the inverse function at a point with the reciprocal of the value of the function, e.g., generally, $\arcsin(x) \neq (\sin(x))^{-1}$.

Recall that

$$x = \arcsin y \Longleftrightarrow |y| \leq 1, \quad |x| \leq \frac{\pi}{2};\ y = \sin x$$

$$z = \arccos t \Longleftrightarrow |t| \leq 1, \quad |z| \in [0, \pi]; \quad t = \cos z.$$

$\lg x$ stands for $\log_{10} x$ and $\ln x$ or $\log x$ for the *natural* logarithm with base

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.71828\ldots.$$

A periodic decimal fraction is usually abbreviated to

$$a_n \ldots a_1 a_0.a_{-1} \ldots a_{-m}(a_{-m-1} \ldots a_{-m-t}),$$

where the digits in parentheses constitute the least period (of length t).

$a^{-n} = \dfrac{1}{a^n}$ for a positive integer n.

Two numbers a and b are called *incommensurable* if $\frac{a}{b} \notin \mathbb{Q}$.

Facts from algebra. An integer a is said to be *divisible* by an integer b if $a = bc$ for some integer c; in this case a is a *multiple of* b (and c) and b is a *divisor* of a; this is sometimes denoted as $a \vdots b$ or $b \mid a$, e.g. $a \vdots 2$ if and only if a is an even number. A *proper divisor* of a is an integer divisor b such that $1 < b < |a|$. A *prime* number is an integer $p > 1$ without proper divisors.

Let a and b be positive integers; we denote by $GCD(a, b)$ or just (a, b) for brevity their *greatest common divisor*, i.e., the maximal positive integer c such that both a and b are divisible by c. We denote by $LCM(a, b)$ their *least common multiple*, the least positive integer divisible by both a and b. The following property of GCD and LCM is often used to calculate LCM: $GCD(a, b) \times LCM(a, b) = ab$.

The above definition of $GCD(a, b)$ can be used to define $GCD(a, b)$ for $a, b \in \mathbb{Z}$ when $b \neq 0$; the above definition of $LCM(a, b)$ fits any nonzero integers; these generalized notions satisfy

$$GCD(a, b) \times LCM(a, b) = |ab|.$$

Numbers a and b such that $(a, b) = 1$ are called *relatively prime* or *coprime* if $GCD(a, b) = 1$; if a and b are relatively prime then $ac \vdots b$ implies $c \vdots b$.

* * *

We use divisibility theorems to prove by the rule of contraries the existence of irrational numbers, e.g., $\sqrt{2}$, $\sqrt{3}$ are irrational. (The same idea applies to $\sqrt{q}$ and $\sqrt{pq}$ for prime p, q.)

The fundamental theorem of arithmetic. *Every integer $n > 1$ is the product of its prime divisors defined uniquely up to a permutation. In particular, if $p_1 < p_2 < \ldots < p_s$ are all divisors of an integer n, then the representation*

$$n = p_1^{k_1} p_2^{k_2} \ldots p_s^{k_s}, \text{ where } k_1, \ldots, k_s \in \mathbb{Z}_+,$$

always exists and is unique.

An integer a is said to be *divisible* by a nonzero integer b with *remainder* r if

$$a = bq + r \quad \text{for some integers } q \text{ and } r, \quad 0 \le r < |b|.$$

We sometimes use the notation $r = r(a)$ for a given b. For a fixed b the possible values of $r(a)$ are 0, 1, $\dots$, $b-1$ and are called *residues modulo* b. Two numbers x and y are *congruent modulo* m if $(x-y)\,\vdots\, m$. We write $x \equiv y \pmod m$.

It is easy to demonstrate that if $a \equiv b \pmod n$ and $c \equiv d \pmod n$ then $a + c \equiv b + d \pmod n$ and $ac \equiv bd \pmod n$.

Similarly, given polynomials $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ and $g(x) = b_mx^m + b_{m-1}x^{m-1} + \dots + b_1x + b_0$, we say that $f(x)$ *is divisible by* $g(x)$ if $f(x) = g(x)q(x)$ for some polynomial $q(x)$.

Recall, that the *degree* of a polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ is the greatest power of its nonzero monomial; we write $\deg f(x) = n$. If $g(x) \neq 0$ it is always possible to uniquely represent $f(x)$ in the form

$$f(x) = g(x)q(x) + r(x), \quad \text{where } \deg r(x) < \deg g(x).$$

The above formula implies

Bezout' theorem: *For any number x_0 a polynomial $f(x)$ can be represented in the form*

$$f(x) = q(x)(x - x_0) + r(x), \quad \text{where } q(x) \text{ is a polynomial.}$$

Proof. In the displayed formula take $g(x) = x - x_0$, $\deg r(x) < \deg(x - x_0) = 1$; hence, $r(x)$ is a constant fumction: $r(x) = r(x_0)$, as was required. Q.E.D.

If $f(x_0) = 0$, then x_0 is a *root* or a *zero* of the polynomial f; this is true if and only if f is divisible by $x - x_0$.

The fundamental theorem of algebra. *Every nonconstant polynomial $f(x)$ with complex coefficients has* $(\deg f)$ *complex roots.*

Inequalities. *Cauchy's inequality*[1]

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \ge \sqrt[n]{a_1 \ldots a_n} \quad \text{for } a_1 \ge 0, a_2 \ge 0, \dots, a_n \ge 0$$

relates the *arithmetic mean* (the lhs) and the *geometric mean* (the rhs). One can prove it by induction (rather tedeous job). A particular case is the relation (prove it!):

$$\frac{a_1 + a_2}{2} \ge \sqrt{a_1 \cdot a_2} \ge \frac{2}{\frac{1}{a_1} + \frac{1}{a_2}},$$

where the last term is the *harmonic mean.*

Though we will not use it in this book, it is too tempting not to mention here the following fact (prove it yourself). Denote by $S_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}}$ for any non-negative real a, b and $p \neq 0$ *the p-th order mean of a and b.* On Fig. 1 the following proposition is illustrated:

If $a < b$ then $a \le S_p(a,b) \le S_q(a,b) \le b$ for any $p \le q$.

FIGURE 1. (N1)

FIGURE 2. (N2)

The means of order p and $-p$ are related: $S_p(a,b)S_{-p}(a,b) = ab$. One can prove (do it!) that $\lim\limits_{n\to\infty} S_{\frac{1}{n}}(a,b) = \sqrt{ab}$ and, therefore, it is natural to *define* $S_0(a,b)$ as $\sqrt{ab}$.

Progressions. An *arithmetic progression* is a sequence $\{x_n,$ where $n \in \mathbb{N}\}$ in which $x_n = x_{n-1} + d$. Hence, $x_n = x_0 + nd$.

Example. The progression $x_n = n$ (hence, $d = 1$) is often referred to as the *natural series.*

[1] Sometimes called *Schwarz' inequality* in Germany and *Bounyakovsky's inequality* in Russia.

For an arithmetic progression $\{x_n\}_{n\in\mathbb{N}}$ we have (add $x_0 + ... + x_n$ with $x_n + ... + x_0$ term-wise):

$$\sum_{k=0}^{n} x_k = \frac{x_0 + x_n}{2}(n+1).$$

A *geometric progression* is a sequence of nonzero terms $\{x_n\}_{n\in\mathbb{N}}$ in which $x_n = x_{n-1}q$. Hence, $x_n = q^n \cdot x_0$. An example: $x_n = q^n$. For a geometric progression $\{x_n\}_{n\in\mathbb{N}}$ with $q \neq 1$ we have:

$$\sum_{k=0}^{n} x_k = x_0 \frac{1 - q^{n+1}}{1 - q}.$$

If $|q| < 1$, then $|q^n|$ tends to 0 as $n \to \infty$; hence, we can define the infinite sum of all terms of the geometric progression to be $\sum\limits_{k=0}^{\infty} x_k = x_0 \frac{1}{1-q}$.

A *Fibonacci sequence* is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in which $x_n = x_{n-1} + x_{n-2}$. Sometimes n is allowed to run over $\mathbb{Z}$. Most often we encounter the sequence with $x_0 = 0$, $x_1 = 1$, i.e.,

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots.$$

The rule of sum and the rule of product. *Let A and B be finite sets, $n \geq 1$ and $m \geq 1$ be their cardinalities, respectively.*

S) If A and B have no common elements then there are exactly $n + m$ elements contained in the union of these sets.

P) There are exactly nm ordered pairs (a, b) with $a \in A$, $b \in B$.

The rule of product enables one to calculate C_n^k — the number of ways to choose k elements from n given (indistinguishable) ones. The answer: $C_n^k = \frac{n!}{k!(n-k)!}$. Another common notation for this number is $\binom{n}{k}$, reads: *n choose k*.

For any numbers a, b and a nonnegative integer n we have the *binomial formula*:

$$(a + b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}.$$

Observe that $C_n^k = C_n^{n-k}$ and deduce important identities: $\sum\limits_{k=0}^{n} C_n^k = (1 + 1)^n = 2^n$ and $\sum\limits_{k=0}^{n} (-1)^n C_n^k = 0$.

Vièta's theorem. *The roots x_1, x_2 of a quadratic equation $ax^2 + bx + c = 0$ satisfy the following relations: $x_1 + x_2 = -b$, $x_1 \cdot x_2 = c$.*

Facts from geometry. A *midline of a triangle* (the *midline of a trapezoid*) is the line segment connecting midpoints of two sides of the triangle (trapezoid). The midline's characteristic property (prove it!): *the midline is equal to a half the third side of the triangle (a half sum of the base and the upper side of the trapezoid).*

The *diameter* of a set in space (plane, line) is the maximum (more exactly, the least upper bound) of distances between every pair of its points.

A figure is called *convex* if together with any pair of its points it contains the segment that connects them. The *convex hull* of a set is the figure formed by segments that connect every pair of points of the given set.

Any *polygon* is assumed to be non-selfintersecting and convex unless otherwise specified.

For a triangle ABC with sides a, b, c opposite angles A, B, C, respectively, the *height*, the *bisector* and the *median* dropped from the vertex with angle A onto side a (or its continuation) is denoted by h_a, l_a and m_a. Similar notations are used for the other angles.

We often denote by r and R the (length of the) radii of the inscribed and the circumscribed circles, respectively.

The *inner* and the *outer* tangents to two circles on the plane are those of the form plotted on Fig. 2 and denoted by t_{in} and t_{out}, respectively.

The *orthocenter* of a triangle is the intersection point of the triangle's hights.

The law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

The law of cosines: $c^2 = a^2 + b^2 - 2ab\cos C$.

We will often denote the area of a polygon P by S_P.

Formulas for calculating the area of a triangle ABC:

$$S_{ABC} = \frac{1}{2}ah_a = \frac{1}{2}ab\sin C = \sqrt{p(p-a)(p-b)(p-c)},$$

where $p = \frac{1}{2}(a+b+c)$ (often denoted by s) is the *semiperimeter*. The last formula (with the square root) is called *Heron's formula*.

Thales' theorem. *On the legs of an angle parallel straight lines intercept the segments whose lengths satisfy*: $a : b : c = a' : b' : c'$, cf. Fig. 3.

Figure 3. (N3)

Figure 4. (N4)

Theorem on medians. *The three medians of a triangle meet at one point. This point divides every median into two segments with the ratio of their lengths* $2 : 1$ (counting from the corresponding vertex).

Theorem on bisectors. *All three bisectors of a triangle meet at one point — the center of the inscribed circle.*

Theorem on a bisector. *The bisector of the internal angle C of a triangle ABC divides the opposite side c into segments a' and b', adjacent to the sides a and b, respectively, so that $a' : b' = a : b$.*

Theorem on midperpendiculars. *The three midperpendiculars of a triangle meet at one point — the center of the circumscribed circle.*

Theorem on heights. *The three heights of a triangle meet at one point — the center of the circumscribed circle for the triangle on whose sides lie the vertices A, B, C and which are parallel to the corresponding sides of $\triangle ABC$*, see Fig. 4. The intersection point of heights is called the *orthocenter* of $\triangle ABC$.

Criteria for two triangles to be equal. *Two triangles ABC and $A'B'C'$ are equal if and only if any of the following is satisfied*:

1) $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $\angle A = \angle A'$;
2) $|AB| = |A'B'|$, $\angle A = \angle A$, and $\angle B = \angle B'$;
3) $a = a'$, $b = b'$, $c = c'$.

The measure of a central angle in a circle is equal to the measure of the base arc this angle intercepts. The measure of the inscribed angle in a circle is equal to half the measure of the base arc this angle intercepts, see Fig. 4.

Theorem on three perpendiculars. *Let k and l be two straight lines such that a plane Π contains k and the projection m of l on Π is a straight line. Then $(k \perp l)$ if and only if $k \perp m$.*

Instruments you can use to draw figures on a plane. *Calipers* allow one to measure the distance between any two points and find a point on a previously drawn line at a given distance from some point on that line. Unlike *compasses*, they do not let you draw a circle. A *compass* is used to draw a circle of any given radius around a fixed point on a plane and on the surface of a sphere. (The radius of the circle on the sphere is unknown.)

A *one-sided ruler* allows one to draw straight lines; a *two-sided ruler* enables us to draw parallel lines with the distance between them equal to the width of the ruler. These rulers are like a regular ruler but without marks.

A *protractor* is used to translate any given angle on a plane in such a way that one of the legs of the angle assumes any given position.

Miscellenea. Let i be the imaginary unit, i.e., $i^2 = -1$. *Euler's formula* holds:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi. \tag{E}$$

This remarkable formula is the only one worth memorizing in the whole of trigonometry: since for any complex numbers z and w we have $e^z e^w = e^{z+w}$, the reader will quickly learn to use (E) to derive in no time the facts like $\sin(a+b) = \sin a\cos b + \sin b\cos a$.

The *inner* (or *scalar*) *product* of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a}\cdot\mathbf{b}$ or by $(\mathbf{a},\mathbf{b})$, is defined as $|\mathbf{a}||\mathbf{b}|\cos\varphi$, where φ is the angle between $\mathbf{a}$ and $\mathbf{b}$. If (a_1, a_2, a_3) and (b_1, b_2, b_3) are Cartesian coordinates of $\mathbf{a}$ and $\mathbf{b}$, then $(\mathbf{a},\mathbf{b}) = a_1b_1 + a_2b_2 + a_3b_3$.

If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, we set $(\mathbf{a},\mathbf{b}) = 0$.

Solutions of a couple of problems require some topology. In the majority of problems, where some notions from topology seem to be needed, the answer can, actually, be guessed regardless, with the help of common sence, often called by mathematicians *physical considerations*. For example, when talking about a line segment, it is often inessential for the answer to be derived whether the segment's endpoints belong to it or not; we consider it not as a set, but as a structureless solid.

Still, in several problems it helps to know that any open interval (a, b), the union of any number of intervals and the intersection of any *finite* number of intervals is called an *open set* on the line. If a point P belongs to an open interval, this interval is called its *open neighborhood*. A point $P \in M$ is called an *inner* one for the set M if there is an open neighborhood of P that belongs to M. For example, every point of interval (a, b) is inner (prove it!). A point $P \in M$ is called an *outer* one for the set M if there is an open neighborhood of P that does not belong to M. A point that is neither inner nor outer is called a *boundary* one.

On the plane, the open discs $\{x, y \mid (x-a)^2 + (y-b)^2 < r^2\}$ play the role of open intervals in the above definition. In space the discs are replaced with open balls $\{x, y, z \mid (x-a)^2 + (y-b)^2 + (z-c)^2 < r^2\}$.

Observe that an open interval considered not on the line but on the plane or in space does not consist anymore of inner points: the open sets have changed.

The end of the proof is sometimes marked with a $\square$ or Q.E.D.

Selected lectures of mathmathematics circles

Dirichlet's principle
(Summary of Acad. I. M. Gelfand's lecture for 9-th –11-th graders)

First, Gelfand proposed the following

Problem: *An infinite number of narrow parallel ditches* $\sqrt{2}$ *apart have been dug out across a very long straight road*, see Fig. L1. *Prove that no matter how narrow these ditches are, a pedestrian with a step of length 1 m inevitably steps into a ditch.*

FIGURE 1. (Fig.L1)

FIGURE 2. (Fig.L2)

A short proof follows from *Dirichlet's* or *pigeonhole principle*.

Indeed, suppose we can 'wind' the road onto a reel of circumference $\sqrt{2}$ m, see Fig. L2. Then all ditches coincide and every step of the pedestrian is marked on the circle by an arc 1 m long. Let us mark the pedestrian's traces (points where the pedestrian touched the ground) after each step. We must prove that at least one of these traces belongs to the interior of a small arc on the circle representing the ditches no matter how short length h of this arc may be.

It is easy to see that if it is possible to find k and l such that the distance between the traces of the k-th step and $(k+l)$-th step along the circle is less than h, then the desired statement will be easy to prove.

FIGURE 3. (Fig.L3)

Indeed, after l more steps the $(k+2l)$-th trace moves again by a distance less than h, see Fig. L3; next we consider another l steps, and so on. Now, it is clear that after several groups of l steps we will inevitably discover a trace that falls in a ditch (since by hopping each time the same distance less than h it is impossible to hop over a ditch of width h).

Thus, we should find two traces on the circle with the distance between them less than h. This is where the rabbits (pigeons) come in handy.

Let us divide the circle into arcs of lengths less than h and call these arcs *hutches*. Suppose there are p of them. If we take more than p traces (observe that no two traces coincide since $\sqrt{2}$ is irrational) then by

Dirichlet's principle at least one of the hutches contains more than one trace (rabbit). The distance between two traces that belong to one arc (hutch) is less than h. This proves our statement.

As a second example of the same realm of ideas, consider the following

Problem. *Prove that there exists a power of* 2 *whose decimal expression begins with three nines, i.e.*, $2^n = 999\ldots$.

In other words, prove that there exist integers n and k such that

$$999 \cdot 10^k \leq 2^n < 10^{k+3}$$

or, equivalently,

$$k + \lg(999) \leq n \lg 2 < k + 3.$$

It is easy to see that this problem is quite similar to the initial one, the only difference being that here the length of the 'step' is equal to $\lg 2$ and that the distance between two neighboring 'ditches' of width $3 - \lg(999)$ is equal to 1.

In general, *if p is not a power of 10, then among the numbers p, p^2, p^3, ... we can find one whose decimal expression begins with any given combination of numbers.*

Further elaboration of the same argument leads to a number of interesting theorems of algebra and geometry. Here are some of them:

1) *Let l be a ray originating from a point on the x-axis,* $\tan\alpha$ *an irrational number, α the angle between the ray and the x-axis. Then l never crosses a point with integer coordinates but passes however close to some of such points.*

2) *There is a positive integer n such that* $\sin n < 10^{-10}$.

3) *If numbers α and β are incommensurable with π and with each other, then for any prescribed distance*[1] ε an n can be chosen so that[2] $|\sin(n\alpha) + \sin(n\beta) - 2| < \varepsilon$ although $\sin(n\alpha) + \sin(n\beta)$ is not equal to 2 for any n.

4) *If the radii of circles F and G are incommensurable* (i.e., their ratio is irrational) *then as circle F rolls without slipping along the fixed circle G any point of F traces a curve* (called *epicycloid*), see Fig. L4) *whose cusps are dense*[3] *on G.*

In conclusion of the lecture Gelfand discussed some qualitative estimates connected with Dirichlet's principle. For example, the problem on the pedestrian striding along the road with ditches was modified as follows: "*How often will the pedestrian step into a ditch*?"

Nondecimal number systems[4]
(Summary of A. M. Yaglom's lecture for 7-th and 8-th graders)

First, Yaglom challenged the students to play against him the game 'Nim'. This is a game played on the blackboard. Three pieces are placed on the 'chessboard' with three rows, see Fig L4. Each player can move any of the pieces to the right as far as (s)he likes. The winner is the one who makes the last move.

FIGURE 4. (Fig.L4)

Yaglom had prepared a number of winning positions and, using them, easily won several sets on the blackboard, the audience cheering the players. This experiment convinced the students that there existed winning and losing positions; then Yaglom led to the idea to practice playing Nim on a small chessboard.

Further on, Yaglom told the audience about *nondecimal number systems.* Fix a number q. Any positive integer x can be expressed in the form $x = a_n \cdot q^n + a_{n-1} \cdot q^{n-1} + \ldots + a_1 \cdot q + q_0$, where $0 \leq a_i < q$. If $q = 10$, we have the standard decimal representation, usually written in the abbreviated form as $x = \overline{a_n a_{n-1} \ldots a_1 a_0}$.

[1] ε usually stands among mathematicians for a small number; I wanted to show that it can be *very* small. *D.L.*

[2] You should make sure that you *understand* why this means that n can be chosen so that the distance between the points $\sin(n\alpha) + \sin(n\beta)$ and 2 is smaller than ε.

[3] i.e., any arc of G hosts infinitely many of the cusp points.

[4] For details see [YY].

If $q = 2$, we get the binary system widely used in activities related with computers and coding. With respect to this number system any number is expressed with the help of only two figures, 0 and 1, e.g. $1 = 1_2$, $2 = 10_2$, $4 = 100_2$, $8 = 1000_2$, $9 = 1001_2$, etc. (Here the subscript indicates "the base" of the number system). Fractions can also be written in the same fashion, e.g. $0.101_2 = 1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3}$.

Next, Yaglom said that, more generally, a *number system* is a method to express numbers with the help of a certain "basis" $u_1, u_2, \dots, u_k, \dots$ as follows:

$$N = a_1 \cdot u_1 + a_2 \cdot u_2 + \ldots + a_k \cdot u_k + \ldots , \qquad (*)$$

where the basis need not necessarily be the sequence of powers of a fixed number q and where the k-th "digit" a_k *does not exceed* a_k/a_{k-1}.

For instance, take for a basis the sequence of *factorials*, i.e., take $u_{k+1} = (k+1)u_k$, $u_0 = 1$. Then any number N is expressed in the form $(*)$, where the k-th "digit" a_k *does not exceed* $k = a_k/a_{k-1}$, for example

$$1000 = 1 \cdot 720 + 2 \cdot 120 + 1 \cdot 24 + 2 \cdot 6 + 2 \cdot 2 + 0 \cdot 1 = 121220_{Fa}.$$

The *Fibonacci number system* is another example. Its basis is of the form

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots \quad (\text{i.e., } u_{k+2} = u_{k+1} + u_k), \ldots$$

With respect to this system, any digit is either 0 or 1, as in the binary one, but here *no two 1's can stand in a row*[1], e.g.

$$100 = 1 \cdot 89 + 0 \cdot 55 + 0 \cdot 34 + 0 \cdot 21 + 0 \cdot 13 + 1 \cdot 8 + 0 \cdot 5 + 1 \cdot 3 + 0 \cdot 2 + 0 \cdot 1 = 1000010100_{Fib}.$$

Exercise. 1) *How to pass from one number system to another*?
2) *Write the multiplication table for the above systems.*
3)* *How to express fractions in the Fibonacci system*?

Then Yaglom used the binary system to discuss possible victories and defeats in Nim. Together with the audience they derived the rule:

A position (a, b, c) is a losing one if all sums of digits of all numbers a, b and c in the binary system corresponding to the same position are even.

The lecture ended with an analysis in terms of the Fibonacci system of another game, *zhì shí zi* (reads "tsin shi tsi") which in Chinese means *throwing stones*; its Western name is *Withoff's game.* Zhì shí zi differs from Nim in that its board has two rows; a player can shift with every move either one chip (to any place) or one can simultaneously shift both chips by the same distance, cf. Fig. L5.

Figure 5. (Fig.L5)

Yaglom proved that a position (a, b), where $a < b$, loses if a_{Fib} ends with an even number of zeros and $b_{Fib} = \overline{a0}_{Fib}$.

Here is a sketch of a proof.

Positions $([n\tau^2], [n\tau^2])$, where $n = 1, 2, \ldots$ and $\tau \approx 1.618\ldots$ is the positive root of the quadratic equation $x^2 = x + 1$, win.

Proof is based on the following lemma.

Lemma. *Let X, Y be positive irrational numbers such that*

$$\frac{1}{X} + \frac{1}{Y} = 1. \qquad (*)$$

Then every positive integer can be uniquely represented as either $[\frac{k}{X}]$ or $[\frac{l}{Y}]$ for some $k, l \in \mathbb{N}$.

Proof. Among numbers $1, 2, \ldots, n$ there are $[\frac{n}{X}]$ numbers of the form $[kX]$ and $[\frac{n}{Y}]$ numbers of the form $[lY]$. Since $\frac{1}{X} + \frac{1}{Y} = 1$, it follows that $\frac{n}{X} + \frac{n}{Y} = n \Longrightarrow [\frac{n}{X}] + [\frac{n}{Y}] = n - 1$. Similarly, among $1, 2, \ldots, n+1$ there are n such numbers. Therefore, between n and $n+1$ there is exactly one such number, Q.E.D.

Now, let us solve the problem. Since τ and τ^2 satisfy relation $(*)$, every natural number is of the form either $[n\tau]$ or $[m\tau^2]$. Moreover, it is clear that

$$[n\tau^2] - [n\tau] = n. \qquad (**)$$

Hence, the pairs $([m\tau^2], [m\tau])$ cover the whole natural series, and differences $(**)$ are distinct for distinct m. But this means exactly that we have found the set of loosing positions for the first player, Q.E.D.

Can you figure out how to explicitely incorporate the Fibonacci system in the proof?

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[1] It is rather difficult to explain why! For a solution see [YY].

Indefinite second-order equations
(Summary of CMA B. N. Deloné's lecture for 9-th – 10-th graders)

Deloné began with a short story about indefinite second-order equations for two integer unknowns. The most interesting among them is *Pell's equation*:

$$x^2 - my^2 = 1, \tag{$*$}$$

where m is a positive integer which is not a perfect square.

Theorem. *Equation* $(*)$ *has infinitely many solutions.*

To prove this let us take a rectangular coordinate system u, v and consider vectors $\mathbf{a} = (1,1)$ and $\mathbf{b} = (\sqrt{m}, -\sqrt{m})$. All points M such that $OM = x\mathbf{a} + y\mathbf{b}$, where x, y are integers, form a *lattice* closely related with the properties of equation $(*)$:

If M is one of the points of the lattice, then in coordinate system u, v the coordinates of M are

$$u = x + y\sqrt{m}, \quad v = x - y\sqrt{m}$$

and therefore $uv = x^2 - my^2$.

Thus, the proof of our Theorem reduces to the following

Problem: *Prove that the hyperbola* $uv = 1$ *contains infinitely many points of the lattice.* (The hyperbola is plotted by the dashed curve on Fig. L6.)

FIGURE 6. (Fig.L6) FIGURE 7. (Fig.L7)

One point of the lattice belonging to the hyperbola is obvious: the point M_0 with coordinates $u = v = 1$. The symmetric point M_0' $(u = v = -1)$ also belongs to the hyperbola. Suppose that in addition to these two points we have found one more point of the lattice, $M_1(u_1, v_1)$ such that $u_1v_1 = 1$.

Consider the transformation φ of the plane that sends an arbitrary point $A(u, v)$ into $A' = \varphi(A)$ with the coordinates $u' = uu_1$, $v' = vv_1$. Clearly, φ transforms the hyperbola $uv = 1$ into itself, i.e., the transformation moves the hyperbola along itself (and that is why mathematicians call such φ a *hyperbolic rotation*). Indeed,

$$u'v' = uu_1 \cdot vv_1 = uv \cdot u_1v_1 = 1.$$

Then, it is easy to verify that *the hyperbolic rotations* φ *map the points of the lattice into points of the lattice.*

Indeed, since M_1 is a point of the lattice, it follows that

$$u_1 = x_1 + y_1\sqrt{m}, \quad v_1 = x_1 - y_1\sqrt{m},$$

where x_1, y_1 are integers. Further on, if

$$M(u, v) = \big(x + y\sqrt{m}, x - y\sqrt{m}\big)$$

is one more point of the lattice, and x, y are integers, then

$$u' = uu_1 = \big(x + y\sqrt{m}\big)\big(x_1 + y_1\sqrt{m}\big) = (xx_1 + yy_1m) + (xy_1 + x_1y)\sqrt{m} = X + Y\sqrt{m};$$
$$v' = vv_1 = (x - y\sqrt{m})(x_1 - y_1\sqrt{m}) = (xx_1 + yy_1m) - (xy_1 + x_1y)\sqrt{m} = X - Y\sqrt{m},$$

i.e., point $M' = \varphi(M)$ withcoordinates (u', v') also belongs to the lattice.

The hyperbolic rotation φ transforms $M_0(1,1)$ into $M_1(u_1, v_1)$ and M_1 into $M_2 = \varphi(M_1)$, a new point of the lattice belonging to the hyperbola. The same rotation transforms M_2 into $M_3 = \varphi(M_2)$ that also belongs to the hyperbola, etc.

The *inverse rotation* φ^{-1} that transforms (u, v) to point $(u' = \frac{u}{u_1},\ v' = \frac{v}{v_1})$ sends M_0 into $M_{-1} = \varphi^{-1}(M_0)$; $M_{-2} = \varphi^{-1}(M_{-1})$, and so on. We get an *infinite* set of points of the lattice

$$\dots,\ M_{-2},\ M_{-1},\ M_0,\ M_1,\ M_2,\ \dots$$

which belong to the hyperbola and turn into each other under the hyperbolic rotation φ.

Thus, it suffices to find on the hyperbola at least one point M_1 different from M_0 and M_0'.

In order to do this let us move the segment connecting points $(1, 1)$ and $(1, -1)$ to the right along the u-axis until it meets a point N' of the lattice. If (u', v') are the coordinates of N', then $|u'| < 1$ and the rectangle G' with vertices at points $(\pm u', \pm v')$ contains only three points of the lattice: the origin O, the point N' and the point symmetric to N' with respect to O.

Now, let us move the right edge of the rectangle along the u-axis until we encounter a new point $N''(u'', v'')$ of the lattice. Then we may again move the right edge of the rectangle G'', now with vertices at points $(\pm u'', \pm v'')$, along the u-axis, etc. (see Fig. L7).

An elegant argument ascending to Herman Minkowski enables us to establish that the sequence of areas of the rectangles G', G'', ... (all these areas are integers) is bounded.

Therefore, among them, there are infinitely many rectangles with the same area. Hence, we can deduce that among the N', N'', ... there exist two points such that a hyperbolic rotation ψ sending one of them into another maps the lattice into itself. Therefore, ψ transforms M_0 into a different point of the lattice that belongs to the hyperbola $uv = 1$.

MOSCOW MATHEMATICAL OLYMPIADS 1 – 59

Olympiad 1 (1935)

Tour 1.1

Set 1.1.A

1.1.A.1. Find the ratio of two numbers if the ratio of their arithmetic mean to their geometric mean is 25 : 24.

1.1.A.2. Given the lengths of two sides of a triangle and that of the bisector of the angle between these sides, construct the triangle.

1.1.A.3. The base of a pyramid is an isosceles triangle with the vertex angle α. The pyramid's lateral edges are at angle φ to the base. Find the dihedral angle θ at the edge connecting the pyramid's vertex to that of angle α.

Set 1.1.B

1.1.B.1. A train passes an observer in t_1 sec. At the same speed the train crosses a bridge l m long. It takes the train t_2 sec to cross the bridge from the moment the locomotive drives onto the bridge until the last car leaves it. Find the length and speed of the train.

1.1.B.2. Given three parallel straight lines. Construct a square three of whose vertices belong to these lines.

1.1.B.3. The base of a right pyramid is a quadrilateral whose sides are each of length a. The planar angles at the vertex of the pyramid are equal to the angles between the lateral edges and the base. Find the volume of the pyramid.

Set 1.1.C

1.1.C.1. Find four consecutive terms a, b, c, d of an arithmetic progression and four consecutive terms a_1, b_1, c_1, d_1 of a geometric progression such that $a + a_1 = 27$, $b + b_1 = 27$, $c + c_1 = 39$, and $d + d_1 = 87$.

1.1.C.2. Prove that if the lengths of the sides of a triangle form an arithmetic progression, then the radius of the inscribed circle is one third of one of the heights of the triangle.

1.1.C.3. The height of a truncated cone is equal to the radius of its base. The perimeter of a regular hexagon circumscribing its top is equal to the perimeter of an equilateral triangle inscribed in its base. Find the angle φ between the cone's generating line and its base.

Set 1.1.D

1.1.D.1. Solve the system

$$\begin{cases} x^2 + y^2 - 2z^2 = 2a^2, \\ x + y + 2z = 4(a^2 + 1), \\ z^2 - xy = a^2. \end{cases}$$

1.1.D.2. In $\triangle ABC$, two straight lines drawn from an arbitrary point D on AB are parallel to AC and BC and intersect BC and AC at F and G, respectively. Prove that the sum of the circumferences of the circles circumscribed around $\triangle ADG$ and $\triangle BDF$ is equal to the circumference of the circle circumscribed around $\triangle ABC$.

1.1.D.3. The unfolding of the lateral surface of a cone is a sector of angle 120°. The angles at the base of a pyramid constitute an arithmetic progression with a difference of 15°. The pyramid is inscribed in the cone. Consider a lateral face of the pyramid with the smallest area. Find the angle α between the plane of this face and the base.

Tour 1.2

Set 1.2.A

1.2.A.1. The median, bisector, and height, all originate at the same vertex of a triangle. Given the intersection points of the median, bisector, and height with the circumscribed circle, construct the triangle.

1.2.A.2. Find the locus of points on the surface of a cube that serve as the vertex of the smallest angle that subtends the diagonal.

1.2.A.3. Triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ lie on different planes. Line AB intersects line A_1B_1; line BC intersects line B_1C_1 and line CA intersects line C_1A_1. Prove that either the three lines AA_1, BB_1, CC_1 meet at one point or that they are all parallel.

Set 1.2.B

1.2.B.1. How many real solutions does the following system have?

$$\begin{cases} x + y = 2, \\ xy - z^2 = 1. \end{cases}$$

1.2.B.2. Solve the system

$$\begin{cases} x^3 - y^3 = 2b, \\ x^2y - xy^2 = b. \end{cases}$$

1.2.B.3. Evaluate the sum: $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$.

Set 1.2.C

1.2.C.1. a) How many distinct ways are there are there of painting the faces of a cube six different colors? (Colorations are considered distinct if they do not coincide when the cube is rotated.)

b)* How many distinct ways are there are there of painting the faces of a dodecahedron 12 different colors? (Colorations are considered distinct if they do not coincide when the cube is rotated.)

1.2.C.2. How many ways are there of representing a positive integer n as the sum of three positive integers? Representations which differ only in the order of the summands are considered to be distinct.

1.2.C.3. Denote by $M(a, b, c, \ldots, k)$ the least common multiple and by $D(a, b, c, \ldots, k)$ the greatest common divisor of a, b, c, $\ldots$, k. Prove that:

a) $M(a, b)D(a, b) = ab$;

b) $\dfrac{M(a, b, c)D(a, b)D(b, c)D(a, c)}{D(a, b, c)} = abc$.

Olympiad 2 (1936)

Tour 2.1

2.1.1. Find a four-digit perfect square whose first digit is the same as the second, and the third is the same as the fourth.

2.1.2. All rectangles that can be inscribed in an isosceles triangle with two of their vertices on the triangle's base have the same perimeter. Construct the triangle.

2.1.3 (*P. Dirac's problem.*) Represent an arbitrary positive integer as an expression involving only 3 twos and any mathematical signs.

2.1.4. Consider a circle and a point P outside the circle. The angle of given measure with vertex at P subtends a diameter of the circle. Construct the circle's diameter with ruler and compass.

2.1.5. Find 4 consecutive positive integers whose product is 1680.

Tour 2.2

2.2.1. Solve the system:

$$\begin{cases} x + y = a, \\ x^5 + y^5 = b^5 \ . \end{cases}$$

2.2.2. Given an angle less than 180°, and a point M outside the angle. Draw a line through M so that the triangle, whose vertices are the vertex of the angle and the intersection points of its legs with the line drawn, has a given perimeter.

2.2.3. The lengths of a rectangle's sides and of its diagonal are integers. Prove that the area of the rectangle is an integer multiple of 12.

2.2.4. How many ways are there to represent 10^6 as the product of three factors? Factorizations which only differ in the order of the factors are considered to be distinct.

2.2.5. Given three planes and a ball in space. In space, find the number of different ways of placing another ball so that it would be tangent the three given planes and the given ball.

Olympiad 3 (1937)

Tour 3.1

3.1.1. Solve the system:

$$\begin{cases} x+y+z=a, \\ x^2+y^2+z^2=a^2, \\ x^3+y^3+z^3=a^3. \end{cases}$$

3.1.2*. On a plane two points A and B are on the same side of a line. Find point M on the line such that $MA+MB$ is equal to a given length.

3.1.3. Two segments slide along two skew lines. Consider the tetrahedron with vertices at the endpoints of the segments. Prove that the volume of the tetrahedron does not depend on the position of the segments.

Tour 3.2

3.2.1. Given three points that are not on the same straight line. Three circles pass through each pair of the points so that the tangents to the circles at their intersection points are perpendicular to each other. Construct the circles.

3.2.2*. Given a regular dodecahedron. Find how many ways are there to draw a plane through it so that its section of the dodecahedron is a regular hexagon?

3.2.3. Into how many parts can an n-gon be divided by its diagonals if no three diagonals meet at one point?

Olympiad 4 (1938)

Tour 4.1

4.1.? (See footnote 1 to *Historical remarks.*) In space 4 points are given. How many planes equidistant from these points are there? Consider separately (a) the generic case (the points given do not lie on a single plane) and (b) the degenerate cases.

Tour 4.2

4.2.1. The following operation is performed over points O_1, O_2, O_3 and A in space. The point A is reflected with respect to O_1, the resultant point A_1 is reflected through O_2, and the resultant point A_2 through O_3. We get some point A_3 that we will also consecutively reflect through O_1, O_2, O_3. Prove that the point obtained last coincides with A; see Fig. 1.

FIGURE 1. (Probl. 4.2.1)

4.2.2. What is the largest number of parts into which n planes can divide space?

4.2.3. Given the base, height and the difference between the angles at the base of a triangle, construct the triangle.

4.2.4. How many positive integers smaller than 1000 and not divisible by 5 and by 7 are there?

Olympiad 5 (1939)

Tour 5.1

5.1.1. Solve the system:

$$\begin{cases} 3xyz - x^3 - y^3 - z^3 = b^3, \\ x + y + z = 2b, \\ x^2 + y^2 - z^2 = b^2. \end{cases}$$

5.1.2. Prove that

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}.$$

5.1.3. Consider points A, B, C. Draw a line through A so that the sum of distances from B and C to this line is equal to the length of a given segment.

5.1.4. Solve the equation $\sqrt{a - \sqrt{a + x}} = x$ for x.

5.1.5. Prove that for any triangle the bisector lies between the median and the height drawn from the same vertex. (See Fig. 2.)

Figure 2. (Probl. 5.1.5)

Figure 3. (Probl. 5.2.3)

Tour 5.2

5.2.1. Factor $a^{10} + a^5 + 1$ into nonconstant polynomials with integer coefficients.

5.2.2. Let the product of two polynomials of a variable x with integer coefficients be a polynomial with even coefficients not all of which are divisible by 4. Prove that all the coefficients of one of the polynomials are even and that at least one of the coefficients of the other polynomial is odd.

5.2.3. Given two points A and B and a circle, find a point X on the circle so that points C and D at which lines AX and BX intersect the circle are the endpoints of the chord CD parallel to a given line MN. (See Fig. 3.)

5.2.4. Find the remainder after division of $10^{10} + 10^{10^2} + 10^{10^3} + \dots + 10^{10^{10}}$ by 7.

5.2.5. Consider a regular pyramid and a perpendicular to its base at an arbitrary point P. Prove that the sum of the lengths of the segments connecting P to the intersection points of the perpendicular with the planes of the pyramid's faces does not depend on the location of P.

5.2.6. What is the greatest number of parts that 5 spheres can divide the space into?

Olympiad 6 (1940)

Tour 6.1

Grades 7 – 8

6.1.7-8.1. Factor $(b-c)^3+(c-a)^3+(a-b)^3$.

6.1.7-8.2. It takes a steamer 5 days to go from Gorky to Astrakhan downstream the Volga river and 7 days upstream from Astrakhan to Gorky. How long will it take for a raft to float downstream from Gorky to Astrakhan?

6.1.7-8.3. How many zeros does 100! have at its end in the usual decimal representation?

6.1.7-8.4. Draw a circle that has a given radius R and is tangent to a given line and a given circle. How many solutions does this problem have?

Grades 9 – 10

6.1.9-10.1. Solve the system:

$$\begin{cases} (x^3+y^3)(x^2+y^2)=2b^5, \\ x+y=b. \end{cases}$$

6.1.9-10.2. Consider all positive integers written in a row:

$$123456789101112131415\dots.$$

Find the 206788-th digit from the left.

6.1.9-10.3. Construct a circle equidistant from four points on a plane. How many solutions are there?

6.1.9-10.4. Given two lines on a plane, find the locus of all points with the difference between the distance to one line and the distance to the other equal to the length of a given segment.

6.1.9-10.5. Find all 3-digit numbers $\overline{abc}$ such that $\overline{abc}=a!+b!+c!$.

Tour 6.2

Grades 7 – 8

6.2.7-8.1. See Problem 2.1.1.

6.2.7-8.2. Points A, B, C are vertices of an equilateral triangle inscribed in a circle. Point D lies on the shorter arc, $\smile AB$ (not $\smile ACB$); see Fig. 4. Prove that $AD+BD=DC$.

FIGURE 4. (Probl. 6.2.7-8.2)

6.2.7-8.3. How does one tile a plane, without gaps or overlappings, with the tiles equal to a given irregular quadrilateral?

6.2.7-8.4. How many pairs of integers x, y are there between 1 and 1000 such that x^2+y^2 is divisible by 49?

Grades 9 – 10

6.2.9-10.1*. Given an infinite cone. The measure of its unfolding's angle is equal to α. A curve on the cone is represented on any unfolding by the union of line segments. Find the number of the curve's self-intersections.

6.2.9-10.2. Which is greater: 300! or 100^{300}?

6.2.9-10.3. The center of the circle circumscribing $\triangle ABC$ is mirrored through each side of the triangle and three points are obtained: O_1, O_2, O_3. Reconstruct $\triangle ABC$ from O_1, O_2, O_3 if everything else is erased.

6.2.9-10.4. Let $a_1, \dots, a_n$ be positive numbers. Prove the inequality:

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n.$$

6.2.9-10.5. How many positive integers x less than 10 000 are there such that $2^x - x^2$ is divisible by 7 ?

Olympiad 7 (1941)

Tour 7.1

Grades 7 – 8

7.1.7-8.1. Construct a triangle given its height and median — both from the same vertex — and the radius of the circumscribed circle.

7.1.7-8.2. Find the number $\overline{523abc}$ divisible by 7, 8 and 9.

7.1.7-8.3. Given a quadrilateral, the midpoints A, B, C, D of its consecutive sides, and the midpoints of its diagonals, P and Q. Prove that $\triangle BCP = \triangle ADQ$.

7.1.7-8.4. A point P lies outside a circle. Consider all possible lines drawn through P so that they intersect the circle. Find the locus of the midpoints of the chords — segments the circle intercepts on these lines.

7.1.7-8.5. Prove that 1 plus the product of any four consecutive integers is a perfect square.

Grades 9 – 10

7.1.9-10.1. See Problem 7.1.7-8.2.

7.1.9-10.2. On the sides of a parallelogram, squares are constructed outwards. Prove that the centers of these squares are vertices of a square.

7.1.9-10.3. A polynomial $P(x)$ with integer coefficients takes odd values at $x = 0$ and $x = 1$. Prove that $P(x)$ has no integer roots.

7.1.9-10.4. Given points M and N, the bases of heights AM and BN of $\triangle ABC$ and the line to which the side AB belongs. Construct $\triangle ABC$.

7.1.9-10.5. Solve the equation:

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| = x + 2.$$

7.1.9-10.6. How many roots does equation $\sin x = \frac{x}{100}$ have?

Tour 7.2

Grades 7 – 8

7.2.7-8.1. Prove that it is impossible to divide a rectangle into five squares of distinct sizes. (Cf. Problem 7.2.9-10.1.)

7.2.7-8.2*. Given $\triangle ABC$, divide it into the minimal number of parts so that after being flipped over these parts can constitute the same $\triangle ABC$.

7.2.7-8.3. Consider $\triangle ABC$ and a point M inside it. We move M parallel to BC until M meets CA, then parallel to AB until it meets BC, then parallel to CA, and so on. Prove that M traverses a self-intersecting closed broken line and find the number of its straight segments.

7.2.7-8.4. Find an integer a for which $(x - a)(x - 10) + 1$ factors in the product $(x + b)(x + c)$ with integers b and c.

7.2.7-8.5. Prove that the remainder after division of the square of any prime $p > 3$ by 12 is equal to 1.

7.2.7-8.6. Given three points H_1, H_2, H_3 on a plane. The points are the reflections of the intersection point of the heights of the triangle $\triangle ABC$ through its sides. Construct $\triangle ABC$.

Grades 9 – 10

7.2.9-10.1. Prove that it is impossible to divide a rectangle into six squares of distinct sizes.

7.2.9-10.2. On a plane, several points are chosen so that a disc of radius 1 can cover every 3 of them. Prove that a disc of radius 1 can cover all the points.

7.2.9-10.3. Find nonzero and nonequal integers a, b, c so that $x(x-a)(x-b)(x-c)+1$ factors into the product of two polynomials with integer coefficients.

7.2.9-10.4. Solve in integers the equation

$$x + y = x^2 - xy + y^2.$$

7.2.9-10.5. Given two skew perpendicular lines in space, find the set of the midpoints of all segments of given length with the endpoints on these lines.

7.2.9-10.6. Construct a right triangle, given two medians drawn to its legs.

Olympiad 8 (1945)

Tour 8.1

Grades 7 – 8

8.1.7-8.1. Divide $a^{2^7} - b^{2^7}$ by $(a+b)(a^2+b^2)(a^4+b^4)\dots(a^{2^6}+b^{2^6})$. (Cf. Problem 8.1.9-10.1).

8.1.7-8.2. Prove that for any positive integer n the following inequality holds:

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{1}{2}.$$

8.1.7-8.3. Find all two-digit numbers $\overline{ab}$ such that $\overline{ab} + \overline{ba}$ is a perfect square.

8.1.7-8.4. Prove that it is impossible to divide a scalene triangle into two equal triangles.

8.1.7-8.5. Two circles are tangent externally at one point. Common external tangents are drawn to them and the tangent points are connected. Prove that the sum of the lengths of the opposite sides of the quadrilateral obtained are equal.

Grades 9 – 10

8.1.9-10.1. Divide $a^{2^k} - b^{2^k}$ by $(a+b)(a^2+b^2)(a^4+b^4)\dots(a^{2^{k-1}}+b^{2^{k-1}})$. (See Problem 8.1.7-8.2.)

8.1.9-10.2. Find three-digit numbers sucvh that any its positive integer power ends with the same three digits and in the same order.

8.1.9-10.3. The system

$$\begin{cases} x^2 - y^2 = 0, \\ (x-a)^2 + y^2 = 1 \end{cases}$$

generally has four solutions. For which a the number of solutions of the system is equal to three or two?

8.1.9-10.4. A right triangle ABC moves along the plane so that the vertices B and C of the triangle's acute angles slide along the sides of a given right angle. Prove that point A fills in a line segment and find its length.

Tour 8.2

Grades 7 – 8

8.2.7-8.1. Given the 6 digits: 0, 1, 2, 3, 4, 5. Find the sum of all even four-digit numbers which can be expressed with the help of these figures (the same figure can be repeated).

8.2.7-8.2. Suppose we have two identical cardboard polygons. We placed one polygon upon the other one and aligned. Then we pierced polygons with a pin at a point. Then we turned one of the polygons around this pin by $25^\circ 30'$. It turned out that the polygons coincided (aligned again). What is the minimal possible number of sides of the polygons?

8.2.7-8.3. The side AD of a parallelogram $ABCD$ is divided into n equal segments. The nearest to A division point P is connected with B. Prove that line BP intersects the diagonal AC at point Q such that $AQ = \dfrac{AC}{n+1}$; see Fig. 5.

8.2.7-8.4. Segments connect vertices A, B, C of $\triangle ABC$ with respective points A_1, B_1, C_1 on the opposite sides of the triangle. Prove that the midpoints of segments AA_1, BB_1, CC_1 do not belong to one straight line.

Grades 9 – 10

8.2.9-10.1. Solve in integers the equation

$$xy + 3x - 5y = -3.$$

Figure 5. (Probl. 8.2.7-8.3)

Figure 6. (Probl. 9.1.7-8.2)

8.2.9-10.2. The numbers $a_1, a_2, \dots, a_n$ are equal to 1 or -1. Prove that

$$2\sin\left(a_1 + \frac{a_1a_2}{2} + \frac{a_1a_2a_3}{4} + \cdots + \frac{a_1a_2\dots a_n}{2^{n-1}}\right)\frac{\pi}{4} = a_1\sqrt{2 + a_2\sqrt{2 + a_3\sqrt{2 + \cdots + a_n\sqrt{2}}}}.$$

In particular, for $a_1 = a_2 = \cdots = a_n = 1$ we have

$$2\sin\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}\right)\frac{\pi}{4} = 2\cos\frac{\pi}{2^{n+1}} = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}.$$

8.2.9-10.3. A circle rolls along a side of an equilateral triangle. The radius of the circle is equal to the height of the triangle. Prove that the measure of the arc intercepted by the sides of the triangle on this circle is equal to 60° at all times.

Olympiad 9 (1946)

Tour 9.1

Grades 7 – 8

9.1.7-8.1. What is the largest number of acute angles that a convex polygon can have?

9.1.7-8.2. Given points A, B, C on a line, equilateral triangles ABC_1 and BCA_1 constructed on segments AB and BC, and midpoints M and N of AA_1 and CC_1, respectively. Prove that $\triangle BMN$ is equilateral. (We assume that B lies between A and C, and points A_1 and C_1 lie on the same side of line AB, see Fig. 6.)

9.1.7-8.3. Find a four-digit number such that the remainders after its division by 131 and 132 are 112 and 98, respectively.

9.1.7-8.4. Solve the system of equations:

$$\begin{cases} x_1 + x_2 + x_3 = 6, \\ x_2 + x_3 + x_4 = 9, \\ x_3 + x_4 + x_5 = 3, \\ x_4 + x_5 + x_6 = -3, \\ x_5 + x_6 + x_7 = -9, \\ x_6 + x_7 + x_8 = -6, \\ x_7 + x_8 + x_1 = -2, \\ x_8 + x_1 + x_2 = 2. \end{cases}$$

9.1.7-8.5. Prove that after completing the multiplication and collecting the terms

$$(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + \cdots + x^{99} + x^{100})$$

has no monomials of odd degree.

Grades 9 – 10

9.1.9-10.1. Given two intersecting planes α and β and a point A on the line of their intersection. Prove that of all lines belonging to α and passing through A the line which is perpendicular to the intersection line of α and β forms the greatest angle with β.

9.1.9-10.2. Through a point M inside an angle a line is drawn. It cuts off this angle a triangle of the least possible area. Prove that M is the midpoint of the segment on this line that the angle intercepts.

9.1.9-10.3. Prove that $n^2 + 3n + 5$ is not divisible by 121 for any positive integer n.

9.1.9-10.4. Prove that for any positive integer n the following identity holds

$$\frac{(2n)!}{n!} = 2^n(2n-1)!!.$$

9.1.9-10.5. Prove that if α and β are acute angles and $\alpha < \beta$, then

$$\frac{\tan\alpha}{\alpha} < \frac{\tan\beta}{\beta}.$$

Tour 9.2

Grades 7 – 8

9.2.7-8.1. Two seventh graders and several eightth graders take part in a chess tournament. The two seventh graders together scored eight points. The scores of eightth graders are equal. How many eightth graders took part in the tournament?

9.2.7-8.2. Prove that for any integers x and y we have:

$$x^5 + 3x^4y - 5x^3y^2 - 15x^2y^3 + 4xy^4 + 12y^5 \neq 33.$$

9.2.7-8.3. On the legs of $\angle AOB$, the segments OA and OB lie; $OA > OB$. Points M and N on lines OA and OB, respectively, are such that $AM = BN = x$. Find x for which the length of MN is minimal.

9.2.7-8.4. Towns A_1, A_2, ..., A_{30} lie on line MN. The distances between the consecutive towns are equal. Each of the towns is the point of origin of a straight highway. The highways are on the same side of MN and form the following angles with it:

No.	1	2	3	4	5	6	7	8	9	10
	60°	30°	15°	20°	155°	45°	10°	35°	140°	50°
No.	11	12	13	14	15	16	17	18	19	20
	125°	65°	85°	86°	80°	75°	78°	115°	95°	25°
No.	21	22	23	24	25	26	27	28	29	30
	28°	158°	30°	25°	5°	15°	160°	170°	20°	158°

Thirty cars start simultaneously from these towns along the highway at the same constant speed. Each intersection has a gate. As soon as the first (in time, not in number) car passes the intersection the gate closes and blocks the way for all other cars approaching this intersection. Which cars will pass all intersections and which will be stopped?

9.2.7-8.5. A bus network is organized so that:
1) one can reach any stop from any other stop without changing buses;
2) every pair of routes has a single stop at which one can change buses;
3) each route has exactly three stops?
How many bus routes are there?

Grades 9 – 10

9.2.9-10.1. Ninth and tenth graders participated in a chess tournament. There were ten times as many tenth graders as ninth graders. The total score of tenth graders was 4.5 times that of the ninth graders. What was the ninth graders score?

9.2.9-10.2. Given the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \dots$, ascertain whether among its first 100 000 001 terms there is a number that ends with four zeros.

9.2.9-10.3. On the sides PQ, QR, RP of $\triangle PQR$ segments AB, CD, EF are drawn. Given a point S_0 inside triangle $\triangle PQR$, find the locus of points S for which the sum of the areas of triangles $\triangle SAB$, $\triangle SCD$ and $\triangle SEF$ is equal to the sum of the areas of triangles $\triangle S_0AB$, $\triangle S_0CD$, $\triangle S_0EF$.

Consider separately the case $\frac{AB}{PQ} = \frac{CD}{QR} = \frac{EF}{RP}$.

9.2.9-10.4. A town has 57 bus routes. How many stops does each route have if it is known that
1) one can reach any stop from any other stop without changing buses;
2) for every pair of routes there is a single stop where one can change buses;
3) each route has three or more stops?

9.2.9-10.5. See Problem 9.2.7-8.4.

Olympiad 10 (1947)

Tour 10.1

Grades 7 – 8

10.1.7-8.1. Find the remainder after division of the polynomial $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ by $x - 1$.

10.1.7-8.2. Prove that of 9 consecutive positive integers one that is relatively prime with the others can always be selected.

10.1.7-8.3. Find the coefficients of x^{17} and x^{18} after expansion and collecting the terms of $(1+x^5+x^7)^{20}$.

10.1.7-8.4. Given a convex pentagon $ABCDE$, prove that if an arbitrary point M inside the pentagon is connected by lines with all the pentagon's vertices, then either one or three or five of these lines cross the sides of the pentagon opposite the vertices they pass.

10.1.7-8.5. Point O is the intersection point of the heights of an acute triangle $\triangle ABC$. Prove that the three circles which pass: a) through O, A, B, b) through O, B, C, and c) through O, C, A, are equal. (See Fig. 7.)

FIGURE 7. (Probl. 10.1.7-8.5)

Grades 9 – 10

10.1.9-10.1. Find the coefficient of x^2 after expansion and collecting the terms of the following expression (there are k pairs of parentheses):

$$((\dots(((x-2)^2-2)^2-2)^2-\dots-2)^2-2)^2.$$

10.1.9-10.2. See Problem 10.1.7-8.2 for 16 consecutive numbers.

10.1.9-10.3. How many squares different in size or location can be drawn on an 8×8 chess board? Each square drawn must consist of whole chess board's squares.

10.1.9-10.4. Which of the polynomials, $(1+x^2-x^3)^{1000}$ or $(1-x^2+x^3)^{1000}$, has the greater coefficient of x^{20} after expansion and collecting the terms?

10.1.9-10.5. Calculate (without calculators, tables, etc.) with accuracy to 0.00001 the product

$$\left(1-\frac{1}{10}\right)\left(1-\frac{1}{10^2}\right)\dots\left(1-\frac{1}{10^{99}}\right).$$

10.1.9-10.6. Given line AB and point M. Find all lines in space passing through M at distance d.

Tour 10.2

Grades 7 – 8

10.2.7-8.1. Twenty cubes of the same size and appearance are made of either aluminum or of heavier duralumin. How can one find the number of duralumin cubes using not more than 11 weighings on a balance without weights? (We assume that all cubes can be made of aluminum, but not all of duralumin.)

10.2.7-8.2. How many digits are there in the decimal expression of 2^{100} ?

10.2.7-8.3. Given 5 points on a plane, no three of which lie on one line. Pprove that four of these points can be taken as vertices of a convex quadrilateral.

10.2.7-8.4. Prove that no convex 13-gon can be cut into parallelograms.

10.2.7-8.5. 101 numbers are selected from the set 1, 2, ..., 200. Prove that among the numbers selected there is a pair in which one number is divisible by the other.

Grades 9 – 10

10.2.9-10.1. In space, n wire triangles are situated so that any two of them have a common vertex and each vertex is the vertex of k triangles. Find all n and k for which this is possible.

10.2.9-10.2. In the numerical triangle

$$\begin{array}{c} 1 \\ 1\ 1\ 1 \\ 1\ 2\ 3\ 2\ 1 \\ 1\ 3\ 6\ 7\ 6\ 3\ 1 \\ \dots\dots\dots\dots\dots\dots \end{array}$$

each number is equal to the sum of the three nearest to it numbers from the row above it; if the number is at the beginning or at the end of a row then it is equal to the sum of its two nearest numbers or just to the nearest number above it (the lacking numbers above the given one are assumed to be zeros). Prove that each row, starting with the third one, contains an even number.

10.2.9-10.3. Inside a square, consider a convex quadrilateral and inside the quadrilateral, take a point A. It so happens that no three of the 9 points — the vertices of the square, of the quadrilateral and A — lie on one line. Prove that 5 of these points are vertices of a convex pentagon.

10.2.9-10.4. One number less than 16, and 99 other numbers are selected from the set 1, 2, ..., 200. Prove that among the selected numbers there are two such that one divides the other.

10.2.9-10.5. Prove that if the four faces of a tetrahedron are of the same area they are equal.

Olympiad 11 (1948)

Tour 11.1

Grades 7 – 8

11.1.7-8.1. The sum of the reciprocals of three positive integers is equal to 1. What are all the possible such triples?

11.1.7-8.2. Find all possible arrangements of 4 points on a plane, so that the distance between each pair of points is equal to either a or b. For what ratios of $a : b$ are such arrangements possible?

11.1.7-8.3. On a plane, n straight lines are drawn. Two domains are called *adjacent* if they border by a line segment. Prove that the domains into which the plane is divided by these lines can be painted two colors so that no two adjacent domains are of the same color.

Grades 9 – 10

11.1.9-10.1. Prove that if $\frac{2^n-2}{n}$ is an integer, then so is $\frac{2^{2^n-1}-2}{2^n-1}$.

11.1.9-10.2. Without tables and such (like calculators, virtually nonexistent in 1948) prove that

$$\frac{1}{\log_2 \pi} + \frac{1}{\log_5 \pi} > 2.$$

11.1.9-10.3. Consider two triangular pyramids $ABCD$ and $A'BCD$, with a common base BCD, and such that A' is inside $ABCD$. Prove that the sum of planar angles at vertex A' of pyramid $A'BCD$ is greater than the sum of planar angles at vertex A of pyramid $ABCD$.

11.1.9-10.4. Consider a circle and a point A outside it. We start moving from A along a closed broken line consisting of segments of tangents to the circle (the segment itself should not necessarily be tangent to the circle) and terminate back at A, as on Fig. 8. (On Fig. 8the links of the broken line are solid.) We label parts of the segments with a plus sign if we approach the circle and with a minus sign otherwise. Prove that the sum of the lengths of the segments of our path, with the signs given, is zero.

FIGURE 8. (Probl. 11.1.9-10.4)

FIGURE 9. (Probl. 11.2.7-8.2)

Tour 11.2

Grades 7 – 8

11.2.7-8.1. Find all positive integer solutions of the equation

$$x^y = y^x \quad (x \neq y).$$

11.2.7-8.2. Let R and r be the radii of the circles circumscribed and inscribed, respectively, in a triangle. Prove that $R \geq 2r$, and that $R = 2r$ only for an equilateral triangle. (See Fig. 9.)

11.2.7-8.3. Can a figure have a greater than 1 and finite number of centers of symmetry?

11.2.7-8.4. The distance between the midpoints of the opposite sides of a convex quadrilateral is equal to a half sum of lengths of the other two sides. Prove that the first pair of sides is parallel.

11.2.7-8.5. Two legs of an angle α on a plane are mirrors. Prove that after several reflections in the mirrors any ray leaves in the direction opposite the one from which it came if and only if $\alpha = \frac{90^\circ}{n}$ for an integer n. Find the number of reflections.

Grades 9 – 10

11.2.9-10.1. Find all positive rational solutions of the equation

$$x^y = y^x \quad (x \neq y).$$

11.2.9-10.2*. What is the radius of the largest possible circle inscribed into a cube with side a?

11.2.9-10.3. How many different integer solutions to the inequality $|x| + |y| < 100$ are there?

11.2.9-10.4. What is the greatest number of rays in space beginning at one point and forming pairwise obtuse angles?

11.2.9-10.5. Given three planar mirrors in space forming an octant (trihedral angle with right planar angles), prove that any ray of light coming into this mirrored octant leaves it, after several reflections in the mirrors, in the direction opposite to the one from which it came. Find the number of reflections. (Cf. Problem 11.2.7-8.5.)

Olympiad 12 (1949)

Tour 12.1

Grades 7 – 8

12.1.7-8.1. Prove that $27\,195^8 - 10\,887^8 + 10\,152^8$ is divisible by 26 460.

12.1.7-8.2. Prove that if a planar polygon has several axes of symmetry, then all of them intersect at one point.

12.1.7-8.3. Prove that $x^2 + y^2 + z^2 = 2xyz$ for integer x, y, z only if $x = y = z = 0$.

12.1.7-8.4. Consider a closed broken line of perimeter 1 on a plane. Prove that a disc of radius $\frac{1}{4}$ can cover this line.

12.1.7-8.5. Prove that for any triangle the circumscribed circle divides the line segment connecting the center of its inscribed circle with the center of one of the escribed circles in halves.

Grades 9 – 10

12.1.9-10.1. Find integers x, y, z, u such that

$$x^2 + y^2 + z^2 + u^2 = 2xyzu.$$

12.1.9-10.2. A finite solid body is symmetric about two distinct axes. Describe the position of the symmetry planes of the body.

12.1.9-10.3. Find the real roots of the equation

$$x^2 + 2ax + \frac{1}{16} = -a + \sqrt{a^2 + x - \frac{1}{16}} \qquad (0 < a < \tfrac{1}{4}).$$

12.1.9-10.4. Given a set of $4n$ positive numbers such that any distinct choice of ordered foursomes of these numbers constitutes a geometric progression. Prove that at least 4 numbers of the set are identical.

12.1.9-10.5. Prove that if opposite sides of a hexagon are parallel and the diagonals connecting opposite vertices have equal lengths, a circle can be circumscribed around the hexagon.

Tour 12.2

Grades 7 – 8

12.2.7-8.1. There are 12 points on a circle. Four checkers, one red, one yellow, one green and one blue sit at neighboring points. In one move any checker can be moved four points to the left or right, onto the fifth point, if it is empty. If after several moves the checkers appear again at the four original points, how might their order have changed?

12.2.7-8.2. Consider two triangles, ABC and DEF, and any point O. We take any point X in $\triangle ABC$ and any point Y in $\triangle DEF$ and draw a parallelogram $OXYZ$. See Fig. 10. Prove that the locus of all possible points Z form a polygon. How many sides can it have? Prove that its perimeter is equal to the sum of perimeters of the original triangles.

12.2.7-8.3. Consider 13 weights of integer mass (in grams). It is known that any 6 of them may be placed onto two pans of a balance achieving equilibrium. Prove that all the weights are of equal mass.

12.2.7-8.4. The midpoints of alternative sides of a hexagon are connected by line segments. Prove that the intersection points of the medians of the two triangles obtained coincide.

Figure 10. (Probl. 12.2.7-8.2)

Figure 11. (Probl. 12.2.9-10.2)

12.2.7-8.5. Prove that some (or one) of any 100 integers can always be chosen so that the sum of the chosen integers is divisible by 100.

Grades 9 – 10

12.2.9-10.1. See Problem 12.2.7-8.1.

12.2.9-10.2. Construct a convex polyhedron of equal "bricks" shown in Fig. 11.

12.2.9-10.3. What is a centrally symmetric polygon of greatest area one can inscribe in a given triangle?

12.2.9-10.4*. Prove that a number of the form 2^n for a positive integer n may begin with any given combination of digits.

12.2.9-10.5. Two squares are said to be *juxtaposed* if their intersection is a point or a segment. Prove that it is impossible to juxtapose to a square more than eight non-overlapping squares of the same size.

Olympiad 13 (1950)

Tour 13.1

Grades 7 – 8

13.1.7-8.1. On a chess board, the boundaries of the squares are assumed to be black. Draw a circle of the greatest possible radius lying entirely on the black squares.

13.1.7-8.2. Given 555 weights: of 1 g, 2 g, 3 g, ... , 555 g, divide them into three piles of equal mass.

13.1.7-8.3. See Problem 13.1.9-10.5 below for $n = 3$ circles.

13.1.7-8.4. Let a, b, c be the lengths of the sides of a triangle and A, B, C, the opposite angles. Prove that

$$Aa + Bb + Cc > \frac{Ab + Ac + Ba + Bc + Ca + Cb}{2}.$$

13.1.7-8.5. In a country, one can get from some point A to any other point either by walking, or by calling a cab, waiting for it, and then being driven. Every citizen always chooses the method of transportation that requires the least time. It turns out that the distances and the traveling times are as follows: 1 km takes 10 min; 2 km takes 15 min; 3 km takes 17.5 min. We assume that the speeds of the pedestrian and the cab, and the time spent waiting for cabs, are all constants. How long does it take to reach a point which is 6 km from A?

Grades 9 – 10

13.1.9-10.1. Let A be an arbitrary angle; let B and C be acute angles. Is there an angle x such that

$$\sin x = \frac{\sin B \cdot \sin C}{1 - \cos B \cdot \cos C \cdot \cos A}?$$

13.1.9-10.2. Two triangular pyramids have common base. One pyramid contains the other. Can the sum of the lengths of the edges of the inner pyramid be longer than that of the outer one?

13.1.9-10.3. Arrange 81 weights of 1^2, 2^2, ..., 81^2 (all in grams) into three piles of equal mass.

13.1.9-10.4. Solve the equation

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1.$$

13.1.9-10.5. We are given n circles O_1, O_2, ..., O_n, passing through one point O. Let A_1, ..., A_n denote the second intersection points of O_1 with O_2, O_2 with O_3, etc., O_n with O_1, respectively. We choose an arbitrary point B_1 on O_1 and draw a line segment through A_1 and B_1 to the second intersection with O_2 at B_2, then draw a line segment through A_2 and B_2 to the second intersection with O_3 at B_3, etc., until we get a point B_n on O_n. We draw the line segment through B_n and A_n to the second intersection with O_1 at B_{n+1}. If B_k and A_k coincide for some k, we draw the tangent to O_k through A_k until this tangent intersects O_{k+1} at B_{k+1}. Prove that B_{n+1} coincides with B_1.

Tour 13.2

Grades 7 − 8

13.2.7-8.1. In a convex 13-gon all diagonals are drawn, dividing it into smaller polygons. What is the greatest number of sides can these polygons have? (Cf. Problem 13.2.9-10.1.)

13.2.7-8.2. Prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\dots\cdot\frac{99}{100}<\frac{1}{10}.$$

13.2.7-8.3. A circle is inscribed in a triangle and a square is circumscribed around this circle so that no side of the square is parallel to any side of the triangle. Prove that less than half of the square's perimeter lies outside the triangle.

13.2.7-8.4*. On a circle, 20 points are chosen. Ten non-intersecting chords without mutual endpoints connect some of the points chosen. How many distinct such arrangements are there?

Grades 9 − 10

13.2.9-10.1. In a convex 1950-gon all diagonals are drawn, dividing it into smaller polygons. What is the greatest number of sides can these polygons have? (Cf. Problem 13.2.7-8.1.)

13.2.9-10.2. The numbers 1, 2, 3, ... , 101 are written in a row in some order. Prove that it is always possible to erase 90 of the numbers so that the remaining 11 numbers remain arranged in either increasing or decreasing order.

13.2.9-10.3. A spatial quadrilateral is circumscribed around a sphere. Prove that all the tangent points lie in one plane.

13.2.9-10.4. Is it possible to draw 10 bus routes with stops such that for any 8 routes there is a stop that does not belong to any of the routes, but any 9 routes pass through all the stops?

Olympiad 14 (1951)

Tour 14.1

Grades 7 − 8

14.1.7-8.1. Prove that $x^{12}-x^9+x^4-x+1>0$ for all x.

14.1.7-8.2. Let $ABCD$ and $A'B'C'D'$ be two convex quadrilaterals whose corresponding sides are equal, i.e., $AB=A'B'$, $BC=B'C'$, etc. Prove that if $\angle A>\angle A'$, then $\angle B<\angle B'$, $\angle C>\angle C'$, $\angle D<\angle D'$.

14.1.7-8.3. Which number is greater:

$$\frac{2.00\,000\,000\,004}{(1.00\,000\,000\,004)^2+2.00\,000\,000\,004}\quad\text{or}\quad\frac{2.00\,000\,000\,002}{(1.00\,000\,000\,002)^2+2.00\,000\,000\,002}?$$

14.1.7-8.4. Given an isosceles trapezoid $ABCD$ and a point P. Prove that a quadrilateral can be constructed from segments PA, PB, PC, PD.

14.1.7-8.5. Given a chain of 60 links each weighing 1 g. Find the smallest number of links that need to be broken if we want to be able to get from the obtained parts all weights 1 g, 2 g, ... , 59 g, 60 g? A broken link also weighs 1 g. (Cf. Problem 14.1.9-10.4.)

Grades 9 – 10

14.1.9-10.1. Find the first three figures after the decimal point in the decimal expression of the number

$$\frac{0.123456789101112\dots495051}{0.515049\dots121110987654321}.$$

14.1.9-10.2. One side of a convex polygon is equal to a, the sum of exterior angles at the vertices not adjacent to this side are equal to 120°. Among such polygons, find the polygon of the largest area.

14.1.9-10.3. We have two concentric circles. A polygon is circumscribed around the smaller circle and is contained entirely inside the greater circle. Perpendiculars from the common center of the circles to the sides of the polygon are extended till they intersect the greater circle. Each of the points obtained is connected with the endpoints of the corresponding side of the polygon (Fig. 12). When is the resulting star-shaped polygon the unfolding of a pyramid?

FIGURE 12. (Probl. 14.1.9-10.3)

FIGURE 13. (Probl. 14.1.9-10.5)

14.1.9-10.4. Given a chain of 150 links each weighing 1 g. Find the smallest number of links that need to be broken if we want to be able to get from the obtained parts all weights 1 g, 2 g, ... , 149 g, 150 g? A broken link also weighs 1 g. (Cf. Problem 14.1.7-8.5.)

14.1.9-10.5. Given three equidistant parallel lines. Express by points of the corresponding lines the values of the resistance, voltage and current in a conductor so as to obtain the voltage $V = I \cdot R$ by connecting with a ruler the points denoting the resistance R and the current I. (Each point of each scale denotes only one number). See Fig. 13.

Tour 14.2

Grades 7 – 8

14.2.7-8.1. Prove that the number $1\underbrace{00\dots00}_{49\text{ zeroes}}5\underbrace{00\dots00}_{99\text{ zeroes}}1$ is not the cube of any integer.

14.2.7-8.2*. On a plane, given points A, B, C and angles $\angle D$, $\angle E$, $\angle F$ each less than 180° and the sum equal to 360°, construct with the help of ruler and protractor a point O such that $\angle AOB = \angle D$, $\angle BOC = \angle E$ and $\angle COA = \angle F$.

14.2.7-8.3. Prove that the sum $1^3 + 2^3 + \dots + n^3$ is a perfect square for all n.

14.2.7-8.4. What figure can the central projection of a triangle be? (The center of the projection does not lie on the plane of the triangle.)

14.2.7-8.5. To prepare for an Olympiad 20 students went to a coach. The coach gave them 20 problems and it turned out that (a) each of the students solved two problems and (b) each problem was solved by two students. Prove that it is possible to organize the coaching so that each student would discuss one of the problems that (s)he had solved, and so that all problems would be discussed.

14.2.7-8.6. Dividing $x^{1951} - 1$ by $P(x) = x^4 + x^3 + 2x^2 + x + 1$ one gets a quotient and a remainder. Find the coefficient of x^{14} in the quotient.

Grades 9 – 10

14.2.9-10.1. A sphere is inscribed in an n-angled pyramid. Prove that if we align all side faces of the pyramid with the base plane, flipping them around the corresponding edges of the base, then (1) all tangent points of these faces to the sphere would coincide with one point, H, and (2) the vertices of the faces would lie on a circle centered at H.

14.2.9-10.2*. Given several numbers each of which is less than 1951 and the least common multiple of any two of which is greater than 1951. Prove that the sum of their reciprocals is less than 2.

14.2.9-10.3. Among all orthogonal projections of a regular tetrahedron to all possible planes, find the projection of the greatest area.

14.2.9-10.4. Consider a curve with the following property: *inside the curve one can move an inscribed equilateral triangle so that each vertex of the triangle moves along the curve and draws the whole curve.* Clearly, every circle possesses the property. Find a closed planar curve without self-intersections, that has the property but is not a circle.

14.2.9-10.5*. A bus route has 14 stops (counting the first and the last). A bus cannot carry more than 25 passengers. We assume that a passenger *takes a bus from A to B* if (s)he enters the bus at A and gets off at B. Prove that for any bus route

a) there are 8 distinct stops A_1, B_1, A_2, B_2, A_3, B_3, A_4, B_4 such that

$$\text{no passenger rides from } A_k \text{ to } B_k \text{ for all } k = 1, 2, 3, 4; \qquad (*)$$

b) there might not exist 10 distinct stops A_1, B_1, …, A_5, B_5 with property $(*)$.

Olympiad 15 (1952)

Tour 15.1

Grade 7

15.1.7.1. The circle is inscribed in $\triangle ABC$. Let L, M, N be the tangent points of the circle with sides AB, AC, BC, respectively. Prove that $\angle MLN$ is always an acute angle.

15.1.7.2. Prove the identity:

$$(ax + by + cz)^2 + (bx + cy + az)^2 + (cx + ay + bz)^2 = (cx + by + az)^2 + (bx + ay + cz)^2 + (ax + cy + bz)^2.$$

15.1.7.3. Prove that if all faces of a parallelepiped are equal parallelograms, they are rhombuses.

15.1.7.4. See Problem 15.1.8.2 below. When should the girl C leave N for A and B to arrive simultaneously in N?

Grade 8

15.1.8.1. Prove that if the orthocenter divides all hights of a triangle in the same proportion, the triangle is equilateral.

15.1.8.2. Two men, A and B, set out from town M to town N, which is 15 km away. Their walking speed is 6 km/hr. They also have a bicycle which they can ride at 15 km/hr. Both A and B start simultaneously, A walking and B riding a bicycle until B meets a pedestrian girl, C, going from N to M. Then B lends his bicycle to C and proceeds on foot; C rides the bicycle until she meets A and gives A the bicycle which A rides until he reaches N. The speed of C is the same as that of A and B. The time spent by A and B on their trip is measured from the moment they started from M until the arrival of the last of them at N. When should C leave N to minimize this time?

15.1.8.3. Prove the identity:

$$\begin{aligned}(ax + by + cz + du)^2 &+ (bx + cy + dz + au)^2 + \\ &(cx + dy + az + bu)^2 + (dx + ay + bz + cu)^2 = \\ (dx + cy + bz + au)^2 &+ (cx + by + az + du)^2 + \\ &(bx + ay + dz + cu)^2 + (ax + dy + cz + bu)^2.\end{aligned}$$

15.1.8.4. See Problem 15.1.7.3.

Grade 9

15.1.9.1. Given a geometric progression whose denominator q is an integer not equal to 0 or -1, prove that the sum of two or more terms in this progression cannot equal any other term in it.

15.1.9.2. Prove that if $|x| < 1$ and $|y| < 1$, then $\left|\frac{x - y}{1 - xy}\right| < 1$.

15.1.9.3. $\triangle ABC$ is divided by a straight line BD into two triangles. Prove that the sum of the radii of circles inscribed in triangles ABD and DBC is greater than the radius of the circle inscribed in $\triangle ABC$. (See Fig. 14.)

FIGURE 14. (Probl. 15.1.9.3)

FIGURE 15. (Probl. 15.2.8.2)

15.1.9.4. A sequence of integers is constructed as follows: a_1 is an arbitrary three-digit number, a_2 is the sum of squares of the digits of a_1, a_3 is the sum of squares of the digits of a_2, etc. Prove that either 1 or 4 must occur in the sequence $a_1, a_2, a_3, \dots$.

15.1.9.5. See Problem 15.1.10.5 below.

Grade 10

15.1.10.1. How $\arcsin(\cos(\arcsin x))$ and $\arccos(\sin(\arccos x))$ are related with each other?

15.1.10.2. Prove that $(1-x)^n + (1+x)^n < 2^n$ for an integer $n \geq 2$ and $|x| < 1$.

15.1.10.3. A sphere with center at O is inscribed in a trihedral angle with vertex S. Prove that the plane passing through the three tangent points is perpendicular to OS.

15.1.10.4. Prove that if for any positive p all roots of the equation

$$ax^2 + bx + c + p = 0$$

are real and positive then $a = 0$.

15.1.10.5. Given three skew lines. Prove that they are pair-wise perpendicular to their pair-wise perpendiculars.

Tour 15.2

Grade 7

15.2.7.1. Solve the system of equations

$$\begin{cases} 1 - x_1x_2 = 0, \\ 1 - x_2x_3 = 0, \\ \dots\dots\dots\dots\dots \\ 1 - x_{14}x_{15} = 0, \\ 1 - x_{15}x_1 = 0. \end{cases}$$

(Cf. Problem 15.2.9.1 below.)

15.2.7.2. In a convex quadrilateral $ABCD$, let $AB + CD = BC + AD$. Prove that the circle inscribed in $\triangle ABC$ is tangent to the circle inscribed in $\triangle ACD$.

15.2.7.3. Prove that if the square of a number begins with 0.9...9 (100 nines), then the number itself begins with 0.9...9 (not less than 100 nines). (Cf. Problem 15.2.8.1 below).

15.2.7.4. Given a line segment AB, find the set of vertices C that form an acute triangle ABC.

Grade 8

15.2.8.1. Calculate $\sqrt{0.9...9}$ (60 nines) to 60 decimal places.

15.2.8.2. From a point C, tangents CA and CB are drawn to a circle O. From an arbitrary point N on the circle, perpendiculars ND, NE, NF are dropped to AB, CA and CB, respectively. Prove that the length of ND is the mean proportional of the lengths of NE and NF. (See Fig. 15).

15.2.8.3. Seven chips are numbered 1, 2, 3, 4, 5, 6, 7. Prove that none of the seven-digit numbers formed by these chips is divisible by any other of these seven-digit numbers.

15.2.8.4. 99 straight lines divide a plane into n parts. Find all possible values of n less than 199.

Grade 9

15.2.9.1. Solve the system of equations

$$\begin{cases} 1 - x_1x_2 = 0, \\ 1 - x_2x_3 = 0, \\ \dots\dots\dots\dots\dots \\ 1 - x_{n-1}x_n = 0, \\ 1 - x_nx_1 = 0. \end{cases}$$

How does the solution vary for distinct values of n?

15.2.9.2. How to arrange three right circular cylinders of diameter $\frac{a}{2}$ and height a into an empty cube with side a so that the cilinders could not change position inside the cube? Each cylinder can, however, rotate about its axis of symmetry.

15.2.9.3. See Problem 15.2.8.3.

15.2.9.4. In an isosceles triangle $\triangle ABC$, $\angle ABC = 20^\circ$ and $BC = AB$. Points P and Q are chosen on sides BC and AB, respectively, so that $\angle PAC = 50^\circ$ and $\angle QCA = 60^\circ$. Prove that $\angle PQC = 30^\circ$. (See Fig. 16).

FIGURE 16. (Probl. 15.2.9.4)

FIGURE 17. (Probl. 16.1.8.1)

15.2.9.5. 200 soldiers occupy in a rectangle (military call it a *square* and educated military a *carrée*): 20 men (per row) times 10 men (per column).

In each row, we consider the tallest man (if some are of equal height, choose any of them) and of the 10 men considered we select the shortest (if some are of equal height, choose any of them). Call him A.

Next the soldiers assume their initial positions and in each column the shortest soldier is selected; of these 20, the tallest is chosen. Call him B.

Two colonels bet on which of the two soldiers chosen by these two distinct procedures is taller: A or B. Which colonel wins the bet?

Grade 10

15.2.10.1. Prove that for arbitrary fixed $a_1, a_2, \dots, a_{31}$ the sum

$$\cos 32x + a_{31} \cos 31x + \dots + a_2 \cos 2x + a_1 \cos x$$

can take both positive and negative values as x varies.

15.2.10.2. See Problem 15.2.9.2.

15.2.10.3. Prove that for any integer a the polynomial $3x^{2n}+ax^n+2$ cannot be divided by $2x^{2m}+ax^m+3$ without a remainder.

15.2.10.4. See Problem 15.2.9.4.

15.2.10.5. See Problem 15.2.9.5.

Olympiad 16 (1953)

Tour 16.1

Grade 7

16.1.7.1. Prove that the sum of angles at the longer base of a trapezoid is less than the sum of angles at the shorter base.

16.1.7.2. Find the smallest number of the form 1...1 in its decimal expression which is divisible by 3...3 (100 three's).

16.1.7.3. Divide a segment in halves using a right triangle. (With a right triangle one can draw straight lines and erect perpendiculars but cannot drop perpendiculars.)

16.1.7.4. Prove that $n^2 + 8n + 15$ is not divisible by $n + 4$ for any positive integer n.

Grade 8

16.1.8.1. Three circles are pair-wise tangent to each other. Prove that the circle passing through the three tangent points is perpendicular to each of the initial three circles; see Fig. 17.

16.1.8.2. Prove that if in the following fraction we have n radicals in the numerator and $n - 1$ in the denominator, then

$$\frac{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} > \frac{1}{4}.$$

16.1.8.3. See Problem 16.1.7.2.

16.1.8.4. See Problem 16.1.7.3.

Grade 9

16.1.9.1. On the plane find the locus of points whose coordinates satisfy $\sin(x + y) = 0$.

16.1.9.2. Let AB and A_1B_1 be two skew segments, O and O_1 their respective midpoints. Prove that OO_1 is shorter than a half sum of AA_1 and BB_1.

16.1.9.3. Prove that the polynomial $x^{200} \cdot y^{200} + 1$ cannot be represented in the form $f(x) \cdot g(y)$, where f and g are polynomials of only x and y, respectively.

16.1.9.4. Let A be a vertex of a regular star-shaped pentagon, the angle at A being less than 180° and the broken line $AA_1BB_1CC_1DD_1EE_1$ being its contour. Lines AB and DE meet at F. Prove that polygon $ABB_1CC_1DED_1$ has the same area as the quadrilateral AD_1EF.

16.1.9.5. See Problem 16.1.8.2

Grade 10

16.1.10.1. See Problem 16.1.9.1.

16.1.10.2. Given a right circular cone and a point A. Find the set of vertices of cones equal to the given one, with axes parallel to that of the given one, and with A inside them.

16.1.10.3. See Problem 16.1.9.3.

16.1.10.4. See Problem 16.1.9.4.

16.1.10.5. See Problem 16.1.8.2.

Tour 16.2

Grade 7

16.2.7.1. Prove that $GCD(a + b,\ LCM(a,\ b)) = GCD(a,\ b)$ for any a, b.

16.2.7.2. A quadrilateral is circumscribed around a circle. Its diagonals intersect at the center of the circle. Prove that the quadrilateral is a rhombus.

16.2.7.3. On a plane, 11 gears are arranged so that the teeth of the first gear mesh with the teeth of the second gear, the teeth of the second gear with those of the third gear, etc., and the teeth of the last gear mesh with those of the first gear. Can the gears rotate? (See Problem 16.2.8.4 below.)

16.2.7.4. Inside a convex 1000-gon, 500 points are selected so that no three of the 1500 points — the ones selected and the vertices of the polygon — lie on the same straight line. This 1000-gon is then divided into triangles so that all 1500 points are vertices of the triangles, and so that these triangles have no other vertices. How many triangles will there be?

16.2.7.5. Solve the system

$$\begin{cases} x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 = 1, \\ x_1 + 3x_2 + 4x_3 + 4x_4 + 4x_5 = 2, \\ x_1 + 3x_2 + 5x_3 + 6x_4 + 6x_5 = 3, \\ x_1 + 3x_2 + 5x_3 + 7x_4 + 8x_5 = 4, \\ x_1 + 3x_2 + 5x_3 + 7x_4 + 9x_5 = 5. \end{cases}$$

(See Problem 16.2.8.5 below.)

Grade 8

16.2.8.1. Let a, b, c, d be the lengths of consecutive sides of a quadrilateral, and S its area. Prove that $S \le \frac{(a+c)(b+d)}{4}$.

16.2.8.2. Somebody wrote 1953 digits on a circle. The 1953-digit number obtained by reading these figures clockwise, beginning at a certain point, is divisible by 27. Prove that if one begins reading the figures at any other place, one gets another 1953-digit number also divisible by 27.

16.2.8.3. On a circle, distinct points $A_1, \dots, A_n$ are chosen. Consider all possible convex polygons all of whose vertices are among $A_1, \dots, A_n$. These polygons are divided into 2 groups, the first group comprising all polygons with A_1 as a vertex, the second group comprising the remaining polygons. Which group is more numerous?

16.2.8.4. On a plane, n gears are arranged so that the teeth of the first gear mesh with the teeth of the second gear, the teeth of the second gear with those of the third gear, etc., and the teeth of the last gear mesh with those of the first gear. (See Fig. 18.) Can the gears rotate?

FIGURE 18. (Probl. 16.2.8.4)

16.2.8.5. Let $n = 100$. Solve the system

$$\begin{cases} x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + \dots + 2x_n = 1, \\ x_1 + 3x_2 + 4x_3 + 4x_4 + 4x_5 + \dots + 4x_n = 2, \\ x_1 + 3x_2 + 5x_3 + 6x_4 + 6x_5 + \dots + 6x_n = 3, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_1 + 3x_2 + 5x_3 + 7x_4 + 9x_5 + \dots + (2n-1)x_n = n. \end{cases}$$

Grade 9

16.2.9.1. See Problem 16.2.8.2.

16.2.9.2. Given triangle $\triangle A_1A_2A_3$ and a straight line l outside it. The angles between the lines A_1A_2 and A_2A_3, A_1A_2 and A_2A_3, A_2A_3 and A_3A_1 are equal to α_3, α_1 and α_2, respectively. The straight lines are drawn through points A_1, A_2, A_3 forming with l angles of $\pi - \alpha_1$, $\pi - \alpha_2$, $\pi - \alpha_3$, respectively. All angles are counted in the same direction from l. Prove that these new lines meet at one point.

16.2.9.3. Given the equations

$$ax^2 + bx + c = 0 \tag{1}$$

$$-ax^2 + bx + c = 0 \tag{2}$$

prove that if x_1 and x_2 are some roots of equations (1) and (2), respectively, then there is a root x_3 of the equation $\frac{a}{2}x^2 + bx + c = 0$ such that either $x_1 \le x_3 \le x_2$ or $x_1 \ge x_3 \ge x_2$.

16.2.9.4. Given a 101×200 sheet of graph paper, we start moving from a corner square in the direction of the square's diagonal (not the sheet's diagonal) to the border of the sheet, then change direction obeying the laws of light's reflection. Will we ever reach a corner square?

16.2.9.5. Divide a cube into three equal pyramids.

Grade 10

16.2.10.1. Find roots of the equation

$$1 - \frac{x}{1} + \frac{x(x-1)}{1 \cdot 2} - \dots + \frac{(-1)^n x(x-1)\dots(x-n+1)}{n!} = 0.$$

16.2.10.2. See Problem 16.2.9.2.

16.2.10.3. Let $x_0 = 10^9$, $x_n = \frac{x_{n-1}^2+2}{2x_{n-1}}$ for $n > 0$. Prove that $0 < x_{36} - \sqrt{2} < 10^{-9}$.

16.2.10.4. See Problem 16.2.9.5.

16.2.10.5. A knight stands on an infinite chess board. Find all places it can reach in exactly $2n$ moves.

Olympiad 17 (1954)

Tour 17.1

Grade 7

17.1.7.1. A regular star-shaped hexagon is split into 4 parts. Construct from them a convex polygon.

17.1.7.2. Given two convex polygons, $A_1A_2\dots A_n$ and $B_1B_2\dots B_n$ such that $A_1A_2 = B_1B_2$, $A_2A_3 = B_2B_3$, … , $A_nA_1 = B_nB_1$ and $n-3$ angles of one polygon are equal to the respective angles of the other. Find whether these polygons are equal.

17.1.7.3. Find a four-digit number whose division by two given distinct one-digit numbers goes along the following lines:

FIGURE 19. (Probl. 17.1.7.3)

17.1.7.4. Are there integers m and n such that $m^2 + 1954 = n^2$?

17.1.7.5. Define the maximal value of the ratio of a three-digit number to the sum of its digits.

Grade 8

17.1.8.1*. Cut out of a 3×3 square an unfolding of the cube with edge 1.

17.1.8.2. From an arbitrary point O inside a convex n-gon, perpendiculars are dropped to the (extensions of the) sides of the n-gon. Along each perpendicular a vector is constructed, starting from O, directed towards the side onto which the perpendicular is dropped, and of length equal to half the length of the corresponding side; see Fig. 20. Find the sum of these vectors.

17.1.8.3. See Problem 17.1.7.3.

17.1.8.4. Find all solutions of the system consisting of 3 equations:

$$x\left(1 - \frac{1}{2n}\right) + y\left(1 - \frac{1}{2n+1}\right) + z\left(1 - \frac{1}{2n+2}\right) = 0 \text{ for } n = 1, 2, 3.$$

FIGURE 20. (Probl. 17.1.8.2)

17.1.8.5. See Problem 17.1.7.4.

Grade 9

17.1.9.1. Prove that if

$$x_0^4 + a_1x_0^3 + a_2x_0^2 + a_3x_0 + a_4 = 0 \quad \text{and} \quad 4x_0^3 + 3a_1x_0^2 + 2a_2x_0 + a_3 = 0,$$

then

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \vdots (x - x_0)^2.$$

17.1.9.2. Delete 100 digits from the number 1234567891011 ... 9899100 so that the remaining number were as big as possible.

17.1.9.3. Given 100 numbers $a_1, \dots, a_{100}$ such that

$$\begin{cases} a_1 - 3a_2 + 2a_3 \geq 0, \\ a_2 - 3a_3 + 2a_4 \geq 0, \\ \dots\dots\dots\dots\dots\dots \\ a_{99} - 3a_{100} + 2a_1 \geq 0, \\ a_{100} - 3a_1 + 2a_2 \geq 0, \end{cases}$$

prove that the numbers are equal.

17.1.9.4. Consider $\triangle ABC$ and a point S inside it. Let A_1, B_1, C_1 be the intersection points of AS, BS, CS with BC, AC, AB, respectively. Prove that at least in one of the resulting quadrilaterals AB_1SC_1, C_1SA_1B, A_1SB_1C both angles at either C_1 and B_1, or C_1 and A_1, or A_1 and B_1 are not acute.

17.1.9.5. Do there exist points A, B, C, D in space, such that $AB = CD = 8$, $AC = BD = 10$, and $AD = BC = 13$?

Grade 10

17.1.10.1. Find all real solutions of the equation $x^2 + 2x \cdot \sin(xy) + 1 = 0$.

17.1.10.2. See Problem 17.1.9.2.

17.1.10.3. Given numbers $a_1 = 1$, $a_2, \dots, a_{100}$ such that

$$\begin{aligned} &a_i - 4a_{i+1} + 3a_{i+2} \geq 0 \quad \text{for all } i = 1, 2, 3, \dots, 98, \\ &a_{99} - 4a_{100} + 3a_1 \geq 0, \\ &a_{100} - 4a_1 + 3a_2 \geq 0. \end{aligned}$$

Find $a_2, a_3, \dots, a_{100}$. (cf. Problem 17.1.9.3.)

17.1.10.4. See Problem 17.1.9.4.

17.1.10.5. See Problem 17.1.9.5.

Tour 17.2

Grade 7

17.2.7.1. Given a piece of graph paper with a letter assigned to each vertex of every square such that on every segment connecting two vertices that have the same letter and are on the same line of the mesh, there is at least one vertex with another letter. What is the least number of distinct letters needed to plot such a picture?

17.2.7.2*. Solve the system

$$\begin{cases} 10x_1 + 3x_2 + 4x_3 + x_4 + x_5 = 0, \\ 11x_2 + 2x_3 + 2x_4 + 3x_5 + x_6 = 0, \\ 15x_3 + 4x_4 + 5x_5 + 4x_6 + x_7 = 0, \\ 2x_1 + x_2 - 3x_3 + 12x_4 - 3x_5 + x_6 + x_7 = 0, \\ 6x_1 - 5x_2 + 3x_3 - x_4 + 17x_5 + x_6 = 0, \\ 3x_1 + 2x_2 - 3x_3 + 4x_4 + x_5 - 16x_6 + 2x_7 = 0, \\ 4x_1 - 8x_2 + x_3 + x_4 + 3x_5 + 19x_7 = 0. \end{cases}$$

17.2.7.3. How many axes of symmetry can a heptagon have?

17.2.7.4. Let 1, 2, 3, 5, 6, 7, 10, ... , N be all the divisors of

$$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$$

(the product of primes 2 to 31) written in increasing order. Below this series of divisors, write the following series of 1's or -1's: write 1 below any number that factors into an even number of prime factors and below a 1; write -1 below the remaining numbers. Prove that the sum of the series of 1's and -1's is equal to 0. (Cf. Problem 17.2.8.5.)

17.2.7.5. The map of a town shows a plane divided into equal equilateral triangles. The sides of these triangles are streets and their vertices are intersections; 6 streets meet at each junction. Two cars start simultaneously in the same direction and at the same speed from points A and B situated on the same street (the same side of a triangle). After any intersection an admissible route for each car is either to proceed in its initial direction or turn through 120° to the right or to the left; see Fig. 21. Can these cars meet? (Either prove that these cars won't meet or describe a route by which they will meet.)

Figure 21. (Probl. 17.2.7.5)

Grade 8

17.2.8.1. A 17×17 square is cut out of a sheet of graph paper. Each cell of this square has one of the numbers from 1 to 70. Prove that there are 4 distinct squares whose centers A, B, C, D are the vertices of a parallelogramsuch that $AB \parallel CD$, moreover, the sum of the numbers in the squares with centers A and C is equal to that in the squares with centers B and D.

17.2.8.2. Given four straight lines, m_1, m_2, m_3, m_4, intersecting at O and numbered clockwise with O as the center of the clock, we draw a line through an arbitrary point A_1 on m_1 parallel to m_4 until the line meets m_2 at A_2. We draw a line through A_2 parallel to m_1 until it meets m_3 at A_3. We also draw a line through A_3 parallel to m_2 until it meets m_4 at A_4. Now, we draw a line through A_4 parallel to m_3 until it meets m_1 at B. Prove that $OB \leq \frac{OA_1}{2}$. (See Fig. 22.)

17.2.8.3. See Problem 17.2.7.2.

FIGURE 22. (Probl. 17.2.8.2)

17.2.8.4. See Problem 17.2.7.3.

17.2.8.5. Let 1, 2, 3, 5, 6, 7, 10, ... , N be all the divisors of

$$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37$$

(the product of primes 2 to 37) written in increasing order. Below this series of divisors, write the following series of 1's or -1's: write 1 below any number that factors into an even number of prime factors and below a 1; write -1 below the remaining numbers. Prove that the sum of the series of 1's and -1's is equal to 0. (Cf. Problem 17.2.7.4.)

Grade 9

17.2.9.1. Rays l_1 and l_2 pass through a point O. Segments OA_1 and OB_1 on l_1, and OA_2 and OB_2 on l_2, are drawn so that $\frac{OA_1}{OA_2} \neq \frac{OB_1}{OB_2}$. Find the set of all intersection points of lines A_1A_2 and B_1B_2 as l_2 rotates around O while l_1 is fixed.

17.2.9.2. See Problem 17.2.8.2; prove that $OB \leq \frac{1}{4}OA_1$. (See Fig. 22.)

17.2.9.3*. Positive numbers $x_1, x_2, \dots, x_{100}$ satisfy the system

$$\begin{cases} x_1^2 + x_2^2 + \dots + x_{100}^2 > 10\,000, \\ x_1 + x_2 + \dots + x_{100} < 300. \end{cases}$$

Prove that among these numbers there are three whose sum is greater than 100.

17.2.9.4. Given a sequence of numbers a_1, a_2, ... , a_{15}, one can always construct a new sequence b_1, b_2, ... , b_{15}, where b_i is equal to the number of terms in the sequence $\{a_k\}_{k=1}^{15}$ less than a_i $(i = 1, 2, \dots, 15)$. Is there a sequence $\{a_k\}_{k=1}^{15}$ for which the sequence $\{b_k\}_{k=1}^{15}$ is 1, 0, 3, 6, 9, 4, 7, 2, 5, 8, 8, 5, 10, 13, 13?

17.2.9.5. Consider five segments AB_1, AB_2, AB_3, AB_4, AB_5. From each point B_i there can exit either 5 segments or no segments at all, so that the endpoints of any two segments of the resulting graph (system of segments) do not coincide. (See Fig. 23.) Can the number of free endpoints of the segments thus constructed be equal to 1001? (A *free endpoint* is an endpoint from which no segment begins.)

FIGURE 23. (Probl. 17.2.9.5)

Grade 10

17.2.10.1. How many planes of symmetry can a triangular pyramid have?

17.2.10.2. See Problem 17.2.9.2.

17.2.10.3. See Problem 17.2.9.3.

17.2.10.4. The absolute values of all roots of the quadratic equation $x^2+Ax+B=0$ and $x^2+Cx+D=0$ are less then 1. Prove that so are absolute values of the roots of the quadratic equation

$$x^2 + \frac{A+C}{2}x + \frac{B+D}{2} = 0.$$

17.2.10.5. Consider the set of all 10-digit numbers expressible with the help of figures 1 and 2 only. Divide it into two subsets so that the sum of any two numbers of the same subset is a number which is written with not less than two 3's.

Olympiad 18 (1955)

Tour 18.1

Grade 7

18.1.7.1. The numbers $1, 2, \dots, 49$ are arranged in a square table as follows:

$$\begin{array}{cccc} 1 & 2 & \dots & 7 \\ 8 & 9 & \dots & 14 \\ \dots & \dots & \dots & \dots \\ 43 & 44 & \dots & 49 \end{array}$$

Among these numbers we select an arbitrary number and delete from the table the row and the column which contain this number. We do the same with the remaining table of 36 numbers, etc., 7 times. Find the sum of the numbers selected. (See Problem 18.1.9.1 below.)

18.1.7.2. We are given a right triangle ABC and the median BD drawn from the vertex B of the right angle. Let the circle inscribed in $\triangle ABD$ be tangent to side AD at K. Find the angles of $\triangle ABC$ if K divides AD in halves.

18.1.7.3. Consider an equilateral triangle $\triangle ABC$ and points D and E on the sides AB and BC such that $AE = CD$. Find the locus of intersection points of AE with CD as points D and E vary.

18.1.7.4. Is there an integer n such that n^2+n+1 is divisible by 1955?

18.1.7.5. Find all rectangles that can be cut into 13 equal squares.

Grade 8

18.1.8.1. Let a, b, n be positive integers, $b < 10$ and $2^n = 10a + b$. Prove that if $n > 3$, then 6 divides ab.

18.1.8.2. Consider a quadrilateral $ABCD$ and points K, L, M, N on sides AB, BC, CD and AD, respectively, such that $KB = BL = a$, $MD = DN = b$ and $KL \nparallel MN$. Find the set of all the intersection points of KL with MN as a and b vary.

18.1.8.3. A square table with 49 small squares is filled with numbers 1 to 7 so that in each row and in each column all numbers from 1 to 7 are present. Let the table be symmetric through the main diagonal. Prove that on this diagonal all the numbers 1, 2, 3, … , 7 are present. (See Problem 18.1.10.1 below.)

18.1.8.4. Which convex domains on a plane can contain an entire straight line?

18.1.8.5. There are four points A, B, C, D on a circle. Circles are drawn through each pair of neighboring points. Denote the intersection points of neighboring circles by A_1, B_1, C_1, D_1. (Some of these points may coincide with previously given ones.) Prove that points A_1, B_1, C_1, D_1 lie on one circle; see Fig. 24.

Grade 9

18.1.9.1. The numbers $1, 2, \dots, k^2$ are arranged in a square table as follows:

$$\begin{array}{cccc} 1 & 2 & \dots & k \\ k+1 & k+2 & \dots & 2k \\ \dots & \dots & \dots & \dots \\ (k-1)k+1 & (k-1)k+2 & \dots & k^2 \end{array}$$

FIGURE 24. (Probl. 18.1.8.5)

Among these numbers we select an arbitrary number and delete from the table the row and the column which contain this number. We do the same with the remaining table of $(k-1)^2$ numbers, etc., k times. Find the sum of the numbers selected.

18.1.9.2. Given two distinct nonintersecting circles none of which is inside the other, see Fig. 25. Find the locus of the midpoints of all segments whose endpoints lie on the circles.

FIGURE 25. (Probl. 18.1.9.2)

18.1.9.3. Find all real solutions of the system:

$$\begin{cases} x^3 + y^3 = 1, \\ x^4 + y^4 = 1. \end{cases}$$

18.1.9.4. Suppose that primes $a_1, a_2, \dots, a_p$ form an increasing arithmetic progression and $a_1 > p$. Prove that if p is a prime, then the difference of the progression is divisible by p.

18.1.9.5. Inside $\triangle ABC$, there is fixed a point D such that $AC - DA > 1$ and $BC - BD > 1$. Prove that $EC - ED > 1$ for any point E on segment AB; see Fig. 26.

Grade 10

18.1.10.1. A square table with n^2 small squares is filled with numbers 1 to n so that in each row and in each column all numbers from 1 to n are present. Let n be odd and the table be symmetric through the main diagonal. Prove that on this diagonal all the numbers 1, 2, 3, … , n are present.

18.1.10.2. See Problem 18.1.9.3.

18.1.10.3. See Problem 18.1.9.5.

18.1.10.4. Given a trihedral angle with vertex O. Find whether there is a planar section ABC such that the angles $\angle OAB$, $\angle OBA$, $\angle OBC$, $\angle OCB$, $\angle OAC$, $\angle OCA$ are acute?

FIGURE 26. (Probl. 18.1.9.5)

Tour 18.2

Grade 7

18.2.7.1. Find integer solutions of the equation

$$x^3 - 2y^3 - 4z^3 = 0.$$

18.2.7.2. The quadratic expression ax^2+bx+c is the 4-th power (of an integer) for any integer x. Prove that $a = b = 0$.

18.2.7.3. The centers O_1, O_2 and O_3 of circles escribed about $\triangle ABC$ are connected. Prove that $\triangle O_1O_2O_3$ is an acute-angled one.

18.2.7.4. 25 chess players are going to participate in a chess tournament. All are on distinct skill levels, and of the two players the one who plays better always wins. What is the least number of games needed to select the two best players?

18.2.7.5. Cut a rectangle into 18 rectangles so that no two adjacent ones form a rectangle.

Grade 8

18.2.8.1*. The quadratic expression $ax^2 + bx + c$ is a square (of an integer) for any integer x. Prove that $ax^2 + bx + c = (dx + e)^2$ for some integers d and e.

18.2.8.2*. Two circles are tangent to each other externally, and to a third one from the inside. Two common tangents to the first two circles are drawn, one outer and one inner. Prove that the inner tangent divides in halves the arc intercepted by the outer tangent on the third circle. (Cf. Problem 20.2.9.5.)

18.2.8.3. A point O inside a convex n-gon $A_1A_2\dots A_n$ is connected with segments to its vertices. The sides of this n-gon are numbered 1 to n (distinct sides have distinct numbers). The segments $OA_1, OA_2, \dots, OA_n$ are similarly numbered.

a) For $n = 9$ find a numeration such that the sum of the sides' numbers is the same for all triangles A_1OA_2, A_2OA_3, ... , A_nOA_1.

b) Prove that for $n = 10$ there is no such numeration.

18.2.8.4. Let the inequality

$$Aa(Bb + Cc) + Bb(Aa + Cc) + Cc(Aa + Bb) > \frac{ABc^2 + BCa^2 + CAb^2}{2}$$

with given $a > 0$, $b > 0$, $c > 0$ hold for all $A > 0$, $B > 0$, $C > 0$. Is it possible to construct a triangle with sides of lengths a, b, c?

18.2.8.5. Find all numbers a such that (1) for a fixed positive integer N all numbers $[a]$, $[2a]$, ... , $[Na]$ are distinct and (2) all numbers $\left[\frac{1}{a}\right]$, $\left[\frac{2}{a}\right]$, ... , $\left[\frac{N}{a}\right]$, are distinct.

Grade 9

18.2.9.1. Given $\triangle ABC$, points C_1, A_1, B_1 on sides AB, BC, CA, respectively, such that

$$\frac{AC_1}{C_1B} = \frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{1}{n}$$

and points C_2, A_2, B_2 on sides A_1B_1, B_1C_1, C_1A_1 of $\triangle A_1B_1C_1$, respectively, such that

$$\frac{A_1C_2}{C_2B_1} = \frac{B_1A_2}{A_2C_1} = \frac{C_1B_2}{B_2A_1} = n.$$

Prove that $A_2C_2 \parallel AC$, $C_2B_2 \parallel CB$, $B_2A_2 \parallel BA$.

18.2.9.2. On the numerical line, arrange a system of closed segments of length 1 without common points (endpoints included) so that any infinite arithmetic progression with any difference and any first term has a common point with a segment of the system.

18.2.9.3. Prove that the equation

$$x^n - a_1x^{n-1} - a_2x^{n-2} - \cdots - a_{n-1}x - a_n = 0, \quad \text{where } a_1 \geq 0,\, a_2 \geq 0,\, \dots,\, a_n \geq 0,$$

cannot have two positive roots.

18.2.9.4. See Problem 18.2.8.2.

18.2.9.5. Five men play several sets of dominoes (two against two) so that each player has each other player once as a partner and two times as an opponent. Find the number of sets and all ways to arrange the players.

Grade 10

18.2.10.1. Prove that if $\frac{p}{q}$ is an irreducible rational number that serves as a root of the polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

with integer coefficients, then $p - kq$ is a divisor of $f(k)$ for any integer k.

18.2.10.2. See Problem 18.2.9.2.

18.2.10.3. A right circular cone stands on plane P. The radius of the cone's base is r, its height is h. A source of light is placed at distance H from the plane, and distance 1 from the axis of the cone. What is the illuminated part of the disc of radius R, that belongs to P and is concentric with the disc forming the base of the cone?

18.2.10.4. What greatest number of triples of points can be selected from 1955 given points, so that each two triples have one common point?

18.2.10.5. Consider $\triangle A_0B_0C_0$ and points C_1, A_1, B_1 on its sides A_0B_0, B_0C_0, C_0A_0, points C_2, A_2, B_2 on the sides A_1B_1, B_1C_1, C_1A_1 of $\triangle A_1B_1C_1$, respectively, etc., so that

$$\frac{A_0B_1}{B_1C_0} = \frac{B_0C_1}{C_1A_0} = \frac{C_0A_1}{A_1B_0} = k, \quad \frac{A_1B_2}{B_2C_1} = \frac{B_1C_2}{C_2A_1} = \frac{C_1A_2}{A_2B_1} = \frac{1}{k^2}$$

and, in general,

$$\frac{A_nB_{n+1}}{B_{n+1}C_n} = \frac{B_nC_{n+1}}{C_{n+1}A_n} = \frac{C_nA_{n+1}}{A_{n+1}B_n} = \begin{cases} k^{2^n} & \text{for } n \text{ even} \\ \dfrac{1}{k^{2^n}} & \text{for } n \text{ odd.} \end{cases}$$

Prove that $\triangle ABC$ formed by lines A_0A_1, B_0B_1, C_0C_1 is contained in $\triangle A_nB_nC_n$ for any n.

Olympiad 19 (1956)

Tour 19.1

Grade 7

19.1.7.1. Prove that there are no four points A, B, C, D on a plane such that all triangles $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, $\triangle DAB$ are acute ones.

19.1.7.2. Find all two-digit numbers x the sum of whose digits is the same as that of $2x$, $3x$, etc., $9x$.

19.1.7.3. A closed self-intersecting broken line intersects each of its segments once. Prove that the number of its segments is even.

19.1.7.4. Find all integers that can divide both the numerator and denominator of the ratio $\frac{5l+6}{8l+7}$ for an integer l.

19.1.7.5. What is the least number of points that can be chosen on a circle of length 1956, so that for each of these points there is exactly one chosen point at distance 1, and exactly one chosen point at distance 2 (distances are measured along the circle)?

Grade 8

19.1.8.1. On sides AB and CB of $\triangle ABC$ there are drawn equal segments, AD and CE, respectively, of arbitrary length (but shorter than $\min(AB, BC)$). Find the locus of midpoints of all possible segments DE.

19.1.8.2. In the decimal expression of a positive number, a, all decimals beginning with the third after the decimal point, are deleted (i.e., we take an approximation of a with accuracy to 0.01 with deficiency). The number obtained is divided by a and the quotient is similarly approximated with the same accuracy by a number b. What numbers b can be thus obtained? Write all their possible values. (Cf. Problem 19.1.9.2, 19.1.10.2.)

19.1.8.3. On a circle of length 15 there are selected n points such that for each of them there is exactly one selected point at distance 1 from it, and exactly one is selected point at distance 2 from it. (All distances are measured along the circle.) Prove that n is divisible by 10. (Cf. Problem 19.1.7.5.)

19.1.8.4. Let a, b, c, d, l be integers. Prove that if the numerator and denominator of the ratio $\frac{al+b}{cl+d}$ are both divisible by k, then so is $ad-bc$. (Cf. Problem 19.1.7.4.)

19.1.8.5. On an infinite sheet of graph paper a table is drawn so that in each square of the table stands a number equal to the arithmetic mean of the four adjacent numbers. Out of the table a piece is cut along the lines of the graph paper. Prove that the largest number on the piece always occurs at an edge; see Fig. 27, where $x = \frac{1}{4}(a+b+c+d)$.

FIGURE 27. (Probl. 19.1.8.5)

FIGURE 28. (Probl. 19.1.10.4)

Grade 9

19.1.9.1. In a convex quadrilateral $ABCD$, consider quadrilateral $KLMN$ formed by the centers of mass of triangles ABC, BCD, DBA, CDA. Prove that the straight lines connecting the midpoints of the opposite sides of quadrilateral $ABCD$ meet at the same point as the straight lines connecting the midpoints of the opposite sides of $KLMN$.

19.1.9.2. In the decimal expression of a positive number, a, all decimals beginning with the third after the decimal point, are deleted (i.e., we take an approximation of a rounding off to 0.001 with deficiency). The number obtained is divided by a and the quotient is similarly approximated with the same accuracy by a number b. What numbers b can be thus obtained? Write all their possible values. (Cf. Problems 19.1.8.2, 19.1.10.2.)

19.1.9.3. See Problem 19.1.8.5.

19.1.9.4. Consider positive numbers h, s_1, s_2, and a spatial triangle $\triangle ABC$. How many ways are there to select a point D so that the height of tetrahedron $ABCD$ dropped from D equals h, and the areas of faces ACD and BCD equal s_1 and s_2, respectively?

19.1.9.5. See Problem 19.1.8.4.

Grade 10

19.1.10.1. A square of side a is inscribed in a triangle so that two of the square's vertices lie on the base, and the other two lie on the sides of the triangle. Prove that if r is the radius of the circle inscribed in the triangle, then $r\sqrt{2} < a < 2r$.

19.1.10.2. In the decimal expression of a positive number, a, all decimals beginning with the third after the decimal point, are deleted (i.e., we take an approximation of a with accuracy to 0.0001 with deficiency). The number obtained is divided by a and the quotient is similarly approximated with the same accuracy by a number b. What numbers b can be thus obtained? Write all their possible values. (Cf. Problems 19.1.8.2, 19.1.9.2.)

19.1.10.3. See Problem 19.1.8.4.

19.1.10.4. Given a closed broken line $A_1A_2A_3 \dots A_n$ in space and a plane intersecting all its segments, A_1A_2 at B_1, A_2A_3 at B_2, … , A_nA_1 at B_n, see Fig. 28, prove that

$$\frac{A_1B_1}{B_1A_2} \cdot \frac{A_2B_2}{B_2A_3} \cdot \frac{A_3B_3}{B_3A_4} \cdot \dots \cdot \frac{A_nB_n}{B_nA_1} = 1. \qquad (*)$$

19.1.10.5. Prove that the system of equations

$$\begin{cases} x_1 - x_2 = a, \\ x_3 - x_4 = b, \\ x_1 + x_2 + x_3 + x_4 = 1 \end{cases}$$

has at least one solution in positive numbers if and only if $|a| + |b| < 1$.

Tour 19.2

Grade 7

19.2.7.1. Let O be the center of the circle circumscribed around $\triangle ABC$, let A_1, B_1, C_1 be symmetric to O through respective sides of $\triangle ABC$. Prove that all hights of $\triangle A_1B_1C_1$ pass through O, and all hights of $\triangle ABC$ pass through the center of the circle circumscribed around $\triangle A_1B_1C_1$.

19.2.7.2. Points A_1, A_2, A_3, A_4, A_5, A_6 divide a circle of radius 1 into six equal arcs. Ray l_1 from A_1 connects A_1 with A_2; ray l_2 from A_2 connects A_2 with A_3, and so on, ray l_6 from A_6 connects A_6 with A_1. From a point B_1 on l_1 the perpendicular is dropped to l_6; from the foot of this perpendicular another perpendicular is dropped to l_5, and so on. Let the foot of the 6-th perpendicular coincide with B_1. Find the length of segment A_1B_1. (Cf. Problem 19.2.9.5.)

19.2.7.3. 100 numbers (some positive, some negative) are written in a row. All of the following three types of numbers are underlined: 1) every positive number, 2) every number whose sum with the number following it is positive, 3) every number whose sum with the two numbers following it is positive. Can the sum of all underlined numbers be (a) negative? (b) equal to zero?

19.2.7.4. 64 non-negative numbers whose sum equals 1956 are arranged in a square table, eight numbers in each row and each column. The sum of the numbers on the two longest diagonals is equal to 112. The numbers situated symmetrically with respect to any of the longest diagonals are equal. Prove that the sum of numbers in any column is less than 1035. (Cf. Problem 19.2.8.2.)

19.2.7.5*. Assume that the number of a tree's leaves is a multiple of 15. Neglecting the shade of the trunk and branches prove that one can rip off the tree $\frac{7}{15}$ of its leaves so that not less than $\frac{8}{15}$ of its shade remains.

Grade 8

19.2.8.1*. A shipment of 13.5 tons is packed in a number of weightless containers. Each loaded container weighs not more than 350 kg. Prove that 11 trucks each of which is capable of carrying ≤ 1.5 ton can carry this load.

19.2.8.2. 64 non-negative numbers whose sum equals 1956 are arranged in a square table, eight numbers in each row and each column. The sum of the numbers on the two longest diagonals is equal to 112. The numbers situated symmetrically with respect to any of the longest diagonals are equal. Prove that the sum of numbers in any row is less than 518. (Cf. Problem 19.2.7.4.)

19.2.8.3. Find the union of all projections of a given line segment AB to all lines passing through a given point O.

19.2.8.4. See Problem 19.2.7.3.

19.2.8.5*. In a rectangle of area 5 sq. units, 9 rectangles of area 1 are arranged. Prove that the area of the overlap of some two of these rectangles is $\ge \frac{1}{9}$. (Cf. Problem 19.210.2.)

Grade 9

19.2.9.1. See Problem 19.2.8.1.

19.2.9.2. 1956 points are chosen in a cube with edge 13. Is it possible to fit inside the cube a cube with edge 1 that would not contain any of the selected points? (See Fig. 29.)

19.2.9.3. Given three numbers x, y, z denote the absolute values of the differences of each pair by x_1, y_1, z_1. From x_1, y_1, z_1 form in the same fashion the numbers x_2, y_2, z_2, etc. It is known that $x_n = x$, $y_n = y$, $z_n = z$ for some n. Find y and z if $x = 1$.

FIGURE 29. (Probl. 19.2.9.2)

19.2.9.4. A quadrilateral is circumscribed around a circle. Prove that the straight lines connecting neighboring tangent points either meet on the extension of a diagonal of the quadrilateral or are parallel to it. (See Fig. 30.)

FIGURE 30. (Probl. 19.2.9.4)

19.2.9.5*. Let A, B, C be three nodes of a graph paper. Prove that if $\triangle ABC$ is an acute one, then there is at least one more node either inside $\triangle ABC$ or on one of its sides.

Grade 10

19.2.10.1. n numbers (some positive and some negative) are written in a row. Each positive number and each number whose sum with several of the numbers following it is positive is underlined. Prove that the sum of all underlined numbers is positive. (Cf. Problem 19.2.8.4.)

19.2.10.2. In a rectangle of area 5 sq. units, lie 9 arbitrary polygons each of area 1. Prove that the area of the overlap of some two of these rectangles is $\geq \frac{1}{9}$. (Cf. Problem 19.2.8.5.)

19.2.10.3. See Problem 19.2.9.3.

19.2.10.4*. Prove that if the trihedral angles at each of the vertices of a triangular pyramid are formed by the identical planar angles, then all faces of this pyramid are equal.

19.2.10.5. Find points B_1, B_2, ... , B_n on the extensions of sides A_1A_2, A_2A_3, ... , A_nA_1 of a regular n-gon $A_1A_2 \dots A_n$ such that $B_1B_2 \perp A_1A_2$, $B_2B_3 \perp A_2A_3$, ... , $B_nB_1 \perp A_nA_1$.

Olympiad 20 (1957)

Tour 20.1

Grade 7

20.1.7.1. Find all isosceles trapezoids that are divided into 2 isosceles triangles by a diagonal.

20.1.7.2. Let $ax^3 + bx^2 + cx + d$ be divisible by 5 for given positive integers a, b, c, d and any integer x. Prove that a, b, c and d are all divisible by 5.

20.1.7.3. A snail crawls over a table at a constant speed. Every 15 minutes it turns by 90°, and in-between these turns it crawls along a straight line. Prove that it can return to the starting point only in an integer number of hours.

20.1.7.4. See Problem 20.1.8.4.

20.1.7.5. The distance between towns A and B is 999 km. At every kilometer of the road that connects A and B a sign shows the distances to A and B as follows:

$$\boxed{0|999} \quad \boxed{1|998} \quad \boxed{2|997} \quad \dots \quad \boxed{998|1} \quad \boxed{999|0}$$

How many signs are there, with both distances written with the help of only two distinct digits?

Grade 8

20.1.8.1. Given two concentric circles and a pair of parallel lines. Find the locus of the fourth vertices of all rectangles with three vertices on the concentric circles, two vertices on one circle and the third on the other and with sides parallel to the given lines. (See Fig. 31.)

FIGURE 31. (Probl. 20.1.8.1)

FIGURE 32. (Probl. 20.1.10.2)

20.1.8.2. See Problem 20.1.7.3.

20.1.8.3. Of all parallelograms of a given area find the one with the shortest possible longer diagonal.

20.1.8.4. For any column and any row in a rectangular numerical table, the product of the sum of the numbers in a column by the sum of the numbers in a row is equal to the number at the intersection of the column and the row. Prove that either the sum of all the numbers in the table is equal to 1 or all the numbers are equal to 0.

20.1.8.5. Let $ax^4 + bx^3 + cx^2 + dx + e$ be divisible by 7 for given positive integers a, b, c, d, e and all integers x. Prove that a, b, c, d and e are all divisible by 7. (Cf. Problem 20.1.7.2.)

Grade 9

20.1.9.1. See Problem 20.1.8.4.

20.1.9.2. Solve the equation $x^3 - [x] = 3$.

20.1.9.3. In a quadrilateral $ABCD$ points M and N are the midpoints of the diagonals AC and BD, respectively. The line through M and N meets AB and CD at M' and N', respectively. Prove that if $MM' = NN'$, then $AD \parallel BC$.

20.1.9.4. A student takes a subway to an Olympiad, pays one ruble and gets his change. Prove that if he takes a tram (street car) on his way home, he will have enough coins to pay the fare without change.

Note: In 1957, the price of a subway ticket was 50 kopeks, that of a tram ticket 30 kopeks, the denominations of the coins were 1, 2, 3, 5, 10, 15, and 20 kopeks. (1 rouble = 100 kopeks.)

20.1.9.5. See Problem 20.1.10.5.

Grade 10

20.1.10.1. For which integer n is $N = 20^n + 16^n - 3^n - 1$ divisible by 323?

20.1.10.2. The segments of a closed broken line in space are of equal length, and each three consecutive segments are mutually perpendicular. Prove that the number of segments is divisible by 6. (Cf. Problem 20.1.7.3.) See Fig. 32.

20.1.10.3. See Problem 20.1.9.3.

20.1.10.4. A student is going to a club. (S)he takes a tram, pays one ruble and gets the change. Prove that on the way back by a tram (s)he will be able to pay the fare without any need to change. (See Note to Problem 20.1.9.4.)

20.1.10.5. A planar polygon $A_1A_2A_3 \ldots A_{n-1}A_n$ $(n > 4)$ is made of rigid rods that are connected by hinges. Is it possible to bend the polygon (at hinges only!) into a triangle? (See Fig. 33.)

FIGURE 33. (Probl. 20.1.10.5)

Tour 20.2

Grade 7

20.2.7.1. Straight lines OA and OB are perpendicular. Find the locus of endpoints M of all broken lines OM of length l, which intersect each line parallel to OA or OB at not more than one point.

20.2.7.2. A radio lamp has a 7-contact plug, with the contacts arranged in a circle. The plug is inserted into a socket with 7 holes. Is it possible to number the contacts and the holes so that for any insertion at least one contact would match the hole with the same number? (Cf. Problem Problem 20.2.9.3.)

20.2.7.3. The lengths, a and b, of two sides of a triangle are known. What length should the third side be, in order for the largest angle of the triangle to be of the least possible value?

20.2.7.4. A circle is inscribed in a triangle. The tangent points are the vertices of a second triangle in which another circle is inscribed; its tangency points are the vertices of a third triangle; the angles of this triangle are identical to those of the first triangle. Find these angles.

20.2.7.5. Eight consecutive numbers are chosen from the Fibonacci sequence 1, 2, 3, 5, 8, 13, 21, ... Prove that the sequence does not contain the sum of chosen numbers.

Grade 8

20.2.8.1. The lengths, a and b, of two sides of a triangle are known. What length should the third side be in order for the smallest angle of the triangle to be of the greatest possible value? (Cf. Problem 20.2.7.3.)

20.2.8.2. Prove that the number of all digits in the sequence $1, 2, 3, \ldots, 10^8$ is equal to the number of all zeros in the sequence $1, 2, 3, \ldots, 10^9$. (Cf. Problem 20.2.10.4.)

20.2.8.3. Given a point O inside an equilateral triangle $\triangle ABC$. Line OG connects O with the center of mass G of the triangle and intersects the sides of the triangle, or their continuations, at points A', B', C' (See Fig. 34.). Prove that

$$\frac{A'O}{A'G} + \frac{B'O}{B'G} + \frac{C'O}{C'G} = 3.$$

FIGURE 34. (Probl. 20.2.8.3)

20.2.8.4. Solve the system:

$$\begin{cases} \dfrac{2x_1^2}{1+x_1^2} = x_2, \\ \dfrac{2x_2^2}{1+x_2^2} = x_3, \\ \dfrac{2x_3^2}{1+x_3^2} = x_1. \end{cases}$$

20.2.8.5. A circle is inscribed in a scalene triangle. The tangent points are vertices of another triangle, in which a circle is inscribed whose tangent points are vertices of a third triangle, in which a third circle is inscribed, etc. Prove that the resulting sequence does not contain a pair of similar triangles. (Cf. Problem 20.2.7.4.)

Grade 9

20.2.9.1. Two rectangles on a plane intersect at eight points. Consider every other intersection point; they are connected with line segments; these segments form a quadrilateral. Prove that the area of this quadrilateral does not vary under translations of one of the rectangles.

20.2.9.2. Find all real solutions of the system :

$$\begin{cases} 1 - x_1^2 = x_2, \\ 1 - x_2^2 = x_3, \\ \dots\dots\dots\dots\dots \\ 1 - x_{98}^2 = x_{99}, \\ 1 - x_{99}^2 = x_1. \end{cases}$$

(Cf. Problem 20.2.10.2.)

20.2.9.3. A radio lamp has a 20-contact plug, with the contacts arranged in a circle. The plug is inserted into a socket with 20 holes. Let the contacts in the plug and the socket be already numbered. Is it always possible to insert the plug so that none of the contacts matches its socket? (Cf. Problem 20.2.7.2.)

20.2.9.4. Represent 1957 as the sum of 12 positive integer summands $a_1, a_2, \dots, a_{12}$ for which the number $a_1! \cdot a_2! \cdot a_3! \cdot \dots \cdot a_{12}!$ is minimal.

20.2.9.5*. Three equal circles are tangent to each other externally and to the fourth circle internally. Tangent lines are drawn to the circles from an arbitrary point on the fourth circle. Prove that the sum of the lengths of two tangent lines equals the length of the third tangent. (Cf. Problem 20.2.8.2.)

Grade 10

20.2.10.1. Given quadrilateral $ABCD$ and the directions of its sides. Inscribe a rectangle in $ABCD$.

20.2.10.2*. Find all real solutions of the system :

$$\begin{cases} 1 - x_1^2 = x_2, \\ 1 - x_2^2 = x_3, \\ \dots\dots\dots\dots\dots \\ 1 - x_{n-1}^2 = x_n, \\ 1 - x_n^2 = x_1. \end{cases}$$

20.2.10.3. Point G is the center of the sphere inscribed in a regular tetrahedron $ABCD$. Straight line OG connecting G with a point O inside the tetrahedron intersects the faces at points A', B', C', D'. Prove that

$$\frac{A'O}{A'G} + \frac{B'O}{B'G} + \frac{C'O}{C'G} + \frac{D'O}{D'G} = 4.$$

(Cf. Problem 20.2.8.3.)

20.2.10.4. Prove that the number of all digits in the sequence 1, 2, 3, ... , 10^k is equal to the number of all zeros in the sequence 1, 2, 3, ... , 10^{k+1}.

20.2.10.5. Given n integers $a_1 = 1$, a_2, ... , a_n such that

$$a_i \leq a_{i+1} \leq 2a_i \quad (i = 1, 2, 3, \ldots, n-1)$$

and whose sum is even, find whether it is possible to divide them into two groups so that the sum of numbers in one group is equal to the sum of numbers in the other group.

Olympiad 21 (1958)

Tour 21.1

Grade 7

21.1.7.1. In the following system :

$$\begin{cases} *x + *y + *z = 0, \\ *x + *y + *z = 0, \\ *x + *y + *z = 0, \end{cases}$$

two players replace the asterisks with numbers doing so in turns, one number each. Prove that the one who starts can always get a system with a nonzero solution.

21.1.7.2. Consider two diameters AB and CD of a circle. Prove that if M is an arbitrary point on the circle, and P and Q are its projections to these diameters, then the length of PQ does not depend on the location of M. (See Fig. 35.)

FIGURE 35. (Probl. 21.1.7.2)

21.1.7.3. How many four-digit numbers from 0000 to 9999 (we complete a one-, two-, or three-digit number to a four-digit one by writing zeros in front of it) are there such that the sum of their first two digits is equal to the sum of their last two digits?

21.1.7.4. Given two points A and B on a plane. Construct a square with A and B on its sides and with the least possible sum of distances of A to the vertices of the square.

21.1.7.5. In the following triangular table

$$\begin{array}{ccccccccc} 0 & & 1 & & 2 & \dots\dots\dots\dots & 1957 & & 1958 \\ & 1 & & 3 & & 5 \quad \dots\dots\dots & & 3915 & \\ & & & & & \dots\dots\dots & & & \end{array}$$

each number (except for those in the upper row) is equal to the sum of the two nearest numbers in the row above. Prove that the lowest number is divisible by 1958.

Grade 8

21.1.8.1. Consider a point O inside $\triangle ABC$ and three vectors of length 1 on rays OA, OB, OC. Prove that the sum of the lengths of these vectors is < 1.

21.1.8.2. Prove that if one root of the following system with integer coefficients is not an integer, then $p_1 = p_2$, $q_1 = q_2$:

$$\begin{cases} x^2 + p_1x + q_1 = 0, \\ x^2 + p_2x + q_2 = 0. \end{cases}$$

21.1.8.3. On a circular clearing of radius R grow three pines of the same diameter. The centers of the pines' trunks are the vertices of an equilateral triangle, each at distance $\frac{R}{2}$ from the center of the clearing. Two men are looking for one another. They go around the clearing along its border, starting from diametrically opposite points. They move at the same speed and in the same direction, and cannot see each other.

Can three men see one another if they go around the clearing starting from the points situated at the vertices of an equilateral triangle inscribed in this clearing?

21.1.8.4. See Problem 21.1.9.3 for $n = 1958$.

21.1.8.5*. The length of the projections of a polygon to the OX-axis, the bisector of the first and third coordinate angles, the OY-axis, and the bisector of the second and fourth coordinate angles are equal to 4, $3\sqrt{2}$, 5 and $4\sqrt{2}$, respectively. Prove that the area S of the polygon is ≤ 17.5.

Grade 9

21.1.9.1. An infinite broken line $A_0A_1\dots A_n\dots$ on a plane, with right angles between its segments, begins at point A_0 with coordinates $x = 0$, $y = 1$, and circumvents the origin O clockwise.

The first segment of this broken line is of length 2 and is parallel to the bisector of the fourth coordinate angle. Each of the subsequent segments intersects one of the coordinate axes, and has an integer length which is the least length sufficient to intersect the axis. Denote the lengths of OA_n by r_n; let the sum of the lengths of the first n segments of the broken line be S_n. Prove that there exists an n for which $\frac{S_n}{r_n} > 1958$.

21.1.9.2. What is the greatest number of axes of symmetry that a figure in space might have, if the figure is formed of three straight lines no two of which are parallel or identical?

21.1.9.3. Solve in positive integers

$$1 - \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \dots \cfrac{1}{(n-1) + \cfrac{1}{n}}}}} = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \dots \cfrac{1}{x_{n-1} + \cfrac{1}{x_n}}}}}$$

21.1.9.4. A segment of length 3^n is split into three equal parts. The first and third parts are fixed. Each of the fixed segments is split into three equal parts the first and third of which are fixed again, and so on, until we get segments of length 1. The endpoints of all fixed segments are called *fixed points*. Prove that for any integer k such that $1 \leq k \leq 3^n$ there are two fixed points the distance between which is equal to k.

Grade 10

21.1.10.1*. See Problem 21.1.8.5. Prove that the area S of the polygon is ≥ 10.

21.1.10.2. Prove that $1155^{1958} + 34^{1958} \neq n^2$ for any integer n.

21.1.10.3. See Problem 21.1.9.2.

21.1.10.4. On a table lies a regular 100-gon whose vertices are numbered consecutively: 1, 2, … , 100. These numbers were then rewritten, in increasing order, according to the distance of the corresponding vertex from the front edge of the table. If vertices are at the same distance from the edge, the left number is written first, and then the right one. All possible sets of numbers corresponding to different positions of the 100-gon are written out. Calculate the sum of the numbers in the 13-th position from the left in all these sets.

21.1.10.5* (*J.Littelwood*'s problem.) Of four straight lines on a plane no two are parallel and no three meet at one point. Along each line a pedestrian walks at a constant speed. It is known that the first pedestrian meets the second, third and fourth ones, and the second pedestrian meets the third and fourth ones. Prove that the third pedestrian meets the fourth one.

Tour 21.2

Grade 7

21.2.7.1. Prove that on a plane it is impossible to arrange more than 4 convex polygons so that each two of them have a common side.

21.2.7.2. There are two rows of 1's and −1's, each containing 1958 numbers. At each step one is allowed to change the sign of any 11 numbers of the first row. Prove that after a finite number of steps one could change the first row into the second one.

21.2.7.3. Each face of a cube is pasted over with two equal right triangles with a common hypotenuse, one of them white and the other black. (See Fig. 36.) Is it possible to arrange these triangles so that the sum of the white angles at each vertex of the cube be equal to the sum of the black angles at the same vertex?

21.2.7.4. Prove that $(n!)^2 > n^n$ for $n > 2$.

21.2.7.5. On a piece of graph paper with squares of side 1, an $m \times n$ rectangle is drawn along the lines of the graph. Is it possible to draw inside the rectangle, along the lines of the graph, a broken line passing each vertex of the graph inside or on the boundary of the rectangle exactly once? If this is possible, what is the length of the brocken line?

FIGURE 36. (Probl. 21.2.7.3)

Grade 8

21.2.8.1. A polygon (not necessarily convex) is cut out of paper. Through two points on the boundary of the polygon a straight line is drawn. The polygon is folded along this straight line and the two pieces of paper are glued to form a new polygon. Prove that the perimeter of the new polygon does not exceed that of the initial polygon. (See Fig. 37.)

FIGURE 37. (Probl. 21.2.8.1)

21.2.8.2. Prove that for any nonnegative a_1 and a_2 such that $a_1 + a_2 = 1$ there exist nonnegative b_1 and b_2 such that $b_1 + b_2 = 1$ and $(1.25 - a_1)b_1 + 3(1.25 - a_2)b_2 > 1$.

21.2.8.3. Inside $\angle AOB$, a point C is taken. From C perpendiculars are dropped: CD to OA and CE to OB. From D and E, perpendiculars are also dropped: DN to OB and EM to OA. Prove that $OC \perp MN$.

21.2.8.4. Prove that $1^1 \cdot 2^2 \cdot 3^3 \cdot \dots \cdot n^n < n^{n(n+1)/2}$ for $n > 1$.

21.2.8.5. Let a be the greatest number of nonintersecting discs of diameter 1 whose centers are inside a polygon M, and let b be the least number of discs of radius 1 that entirely cover M. Which is greater, a or b?

Grade 9

21.2.9.1. See Problem 21.2.10.1 below.

21.2.9.2. From a point O draw n rays on a plane so that the sum of all angles formed by pairs of rays (their total is $\frac{1}{2}n(n-1)$) is the greatest possible.

21.2.9.3. A playboard is shaped like a rhombus with an angle of 60°. Each side of the rhombus is divided into 9 parts. Straight lines parallel to the sides and to the smaller diagonal of the rhombus are drawn through the division points thus splitting the playboard into triangular cells. If a chip stands in a cell, we draw three straight lines through the center of this cell parallel to the sides and to the smaller diagonal of the rhombus. We say that the chip *wins* all the cells that these three lines intersect. What is the least number of chips needed to win all cells on the chessboard?

21.2.9.4. Let a be the least number of discs of radius 1 which completely cover a polygon M, and let b be the greatest number of nonintersecting discs of radius 1 with centers inside M. Which is greater, a or b?

21.2.9.5. A circuit of several resistors connects clamps A and B. Each resistor has an input and an output clamp. What is the least number of resistors needed and what should the principal circuit design be for the circuit not to be short or open if any 9 resistors between A and B break? (A resistor is *broken* if it executes a short or open circuit.)

Grade 10

21.2.10.1. Solve in positive integers

$$x^{2y} + (x+1)^{2y} = (x+2)^{2y}.$$

21.2.10.2. In a polygon, there are points A and B such that the length of any broken line connecting them and passing inside or along the boundary of the polygon is > 1. Prove that the perimeter of the polygon is > 2.

21.2.10.3. A school curriculum has $2n$ subjects. All students get only A's and B's. We will say for the sake of argument that one student is *better* than another if (s)he is not worse than the other in all subjects and better in some subjects. Suppose that no two students get the same grades and it is impossible to say which of any two students is better. Prove that the number of students in this school does not exceed $\binom{2n}{n}$.

21.2.10.4. The lengths of a parallelogram's sides are equal to a and b. Find the ratio of the volumes of bodies obtained by rotating the parallelogram around side a and around side b.

21.2.10.5. We are given n cards with numbers written on them, one number on each side: 0 and 1 on the 1-st, 1 and 2 on the 2-nd, etc., $n-1$ and n on the n-th card. One person takes several cards and shows to his/her partner one side of these cards. Indicate all the cases in which the second person can determine the number written on the other side of the last card shown to him.

Olympiad 22 (1959)

Tour 22.1

Grade 7

22.1.7.1. Let a and b be integers. Let us fill in two columns as follows. Write a and b in the first row. In the second row write a number a_1 equal to $a/2$ if a is even and $(a-1)/2$ if a is odd and $b_1 = 2b$. In the third row write a number a_2 equal to $a_1/2$ if a_1 is even and $(a_1-1)/2$ if a_1 is odd and $b_2 = 4b$. Continue until you get a 1 in the left column.

Prove that the sum of the b_i for which a_i is odd is equal to ab.

22.1.7.2. Prove that $2^{2^{1959}} - 1$ is divisible by 3.

22.1.7.3*. Is it possible to arrange in a sequence all three-digit numbers that do not end in zeros so that the last digit of each number is equal to the first digit of the number following it?

22.1.7.4. How should a rook move on a chessboard to pass each square once and with the least number of turning points?

22.1.7.5. Given a square of side 1, find the set of points the sum of whose distances to the sides of this square (or their extensions) equals 4.

Grade 8

22.1.8.1. Consider two barrels of sufficient capacity. Find if it is possible to pour exactly 1 liter from one barrel into the other using two containers that can hold $2-\sqrt{2}$ and $\sqrt{2}$ liters?

22.1.8.2. On a piece of paper, write figures 0 to 9. Observe that if we turn the paper through 180° the 0's, 1's (written as a vertical line segment, not as in the typed texts) and 8's turn into themselves, the 6's and 9's interchange, and the other figures become meaningless.

How many 9-digit numbers are there which turn into themselves when a piece of paper on which they are written is turned by 180°?

22.1.8.3. Consider a convex quadrangle $ABCD$. Denote the midpoints of AB and CD by K and M, respectively; denote the intersection point of AM and DK by O and that of BM and CK by P. Prove that the area of quadrangle $MOKP$ is equal to the sum of the areas of $\triangle BPC$ and $\triangle AOD$.

22.1.8.4. See Problem 22.1.7.4.

22.1.8.5. Two circles with centers at O_1 and O_2 do not intersect. Let a_1 and a_2 be the inner tangents and a_3 and a_4 the outer tangents to these circles. Further, let a_5 and a_6 be the tangents to the circle with center at O_1 drawn from O_2; let a_7 and a_8 be the tangents to the circle with center at O_2 drawn from O_1. Denote the intersection point of a_1 with a_2 by O.

Prove that it is possible to draw two circles with centers at O so that the first one is tangent to a_3 and a_4 and the second one is tangent to a_5, a_6, a_7, a_8, and so that the radius of the second circle is half that of the first one.

Grade 9

22.1.9.1. Consider 1959 positive numbers $a_1, a_2, \dots, a_{1959}$ whose sum is equal to 1. Consider all different combinations (subsets) of 1 000 of these numbers. Two combinations are assumed to be identical if they differ only in the order of their elements. For each combination we formed the product of its elements. Prove that the sum of all these products is < 1.

22.1.9.2. See Problem 22.1.8.2.

22.1.9.3*. Given a circle and two points. Construct a circle passing through the given points and intercepting a chord of given length on the given circle.

22.1.9.4. Consider a sheet of graph paper with squares of side 1, let p_k be the number of all broken lines of length k beginning at a fixed known node O of the graph (all broken lines are constituted by segments of the graph). Prove that $p_k < 2 \cdot 3^k$ for any k.

22.1.9.5*. Prove that there is no tetrahedron such that each its edge is a leg of an obtuse planar angle.

Grade 10

22.1.10.1. Prove that there are no integers x, y, z such that $x^k + y^k = z^k$ for an integer $k > 0$ provided $z > 0$, $0 < x < k$, $0 < y < k$.

22.1.10.2. See Problem 22.1.8.3.

22.1.10.3. Can there be a tetrahedron each edge of which is a side of an obtuse planar angle? (Cf. Problem 22.1.9.5.)

22.1.10.4. In a square $N \times N$ table, the numbers 1 to N^2 are written in the following way: 1 can stand at any place, 2 can occupy the row with the same index as that of the column containing 1, number 3 can occupy the row with the same number as that of the column containing 2, etc. What is the difference between the sum of the numbers in the row containing 1 and the sum of the numbers in the column containing N^2?

22.1.10.5. Consider a sequence $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$ of positive numbers such that

$$a_1 = \frac{1}{2k}; \quad a_1 + a_2 + \dots + a_n + \dots = 1.$$

Prove that there are k numbers in the sequence such that the least of these k number is greater than half the greatest.

Tour 22.2

Grade 7

22.2.7.1. For $a_1 > a_2 > \dots > a_n$ and $b_1 > b_2 > \dots > b_n$ prove that

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n > a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1.$$

22.2.7.2. Given $\triangle ABC$, find a point whose reflection through any side of the triangle lies on the circumscribed circle.

22.2.7.3. What should 999 999 999 be multiplied by to get a number whose decimal expression contains only 1's?

22.2.7.4. Prove that the digits of any six-digit number can be permuted so that the difference between the sum of the first and the last three digits of the new number is less than 10.

22.2.7.5. Consider n numbers $x_1, \dots, x_n$ each equal to 1 or -1. Prove that if

$$x_1 x_2 + x_2 x_3 + \dots + x_{n-1} x_n + x_n x_1 = 0,$$

then n is divisible by 4.

Grade 8

22.2.8.1. See Problem 22.2.7.5. This problem can be reformulated in a "romantic" way: some of n knights sitting at a round table are enemies. The number of knights whose left neighbors are their friends is equal to the number of knights whose left neighbors are their enemies. Prove that $n \vdots 4$.

22.2.8.2*. Consider 12 numbers $a_1, \dots, a_{12}$ such that

$$a_2(a_1 - a_2 + a_3) < 0,$$
$$a_3(a_2 - a_3 + a_4) < 0,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{11}(a_{10} - a_{11} + a_{12}) < 0.$$

Prove that at least three of these numbers are positive and at least three are negative.

22.2.8.3. Given $\triangle ABC$ and its escribed circles O_1, O_2, O_3. For each pair of these circles we draw the second common outer tangent (one such tangent is already drawn: it is a side of $\triangle ABC$). The three outer tangents form a triangle. Find its angles if the angles of $\triangle ABC$ are known.

22.2.8.4. Given two intersecting line segments AB and CD of length 1. Prove that at least one of the sides of quadrilateral $ABCD$ is not less than $\sqrt{2}/2$.

22.2.8.5. Prove that a knight cannot pass each square of a 4×4 chessboard exactly once.

Grade 9

22.2.9.1. Given 100 numbers $x_1, \dots, x_{100}$ whose sum is equal to 1 and such that $|x_{k+1} - x_k| < \frac{1}{50}$ for all k. Prove that of these 100 numbers, 50 numbers may be selected so that their sum would differ from $\frac{1}{2}$ by not more than $\frac{1}{100}$.

22.2.9.2. n segments of length 1 meet at one point. Prove that at least one side of the $2n$-gon whose vertices are the endpoints of the given segments is not shorter than a side of a regular $2n$-gon inscribed in a circle of diameter 1.

22.2.9.3. Prove that a tetrahedron has not more than one vertex such that the sum of any two planar angles at this vertex is greater than $180°$.

22.2.9.4. Prove that there are infinitely many integers that cannot be expressed as the sum of three cubes.

22.2.9.5. Two white knights stand at the upper corners of a 3×3 chessboard, and two black knights at the lower corners. In one move any knight can go to any unoccupied place in accordance with chess rules. We want to shift the white knights to the lower corners and the black knights to the upper corners. Prove that this requires at least 16 moves.

Grade 10

22.2.10.1. See Problem 22.2.9.4.

22.2.10.2. $ABCD$ is a spatial quadrilateral. Points K_1 and K_2 divide AB and DC, respectively, into segments with ratio α; and K_3 and K_4 divide BC and AD, respectively, into segments with ratio β. Prove that K_1K_2 and K_3K_4 intersect. (For the position of the segments α and β see Fig. 38, where $\dfrac{AK_1}{K_1B} = \dfrac{DK_2}{K_2C} = \alpha$, $\dfrac{BK_3}{K_3C} = \dfrac{AK_4}{K_4D} = \beta$.)

FIGURE 38. (Probl. 22.2.10.2)

FIGURE 39. (Probl. 23.1.7.4)

22.2.10.3. Given several intersecting discs covering an area of 1 on a plane, prove that it is possible to select from these discs several *non*intersecting discs covering an area of not less than $\frac{1}{9}$.

22.2.10.4. Given n complex numbers $c_1, \dots, c_n$ that being represented as points on a complex line are the vertices of a convex n-gon. Prove that if

$$\frac{1}{z-c_1}+\frac{1}{z-c_2}+\dots+\frac{1}{z-c_n}=0,$$

then the point z is inside this n-gon.

22.2.10.5. Two discs of different diameters are divided into $2n$ equal sectors each, and each sector is painted white or black so that each disc has n white sectors and n black sectors. If the two discs are fixed by a pin piercing their centers, it turns out that the circle bounding the smaller disc is painted twice: on the inside (as part of the small disc) and on the outside (as part of the large disc). Thus, some parts of the circle are painted different colors, and the other parts are of the same color on both sides.

Prove that it is possible to rotate the smaller disc so that the parts painted differently will constitute no less than half of the circle's length.

Olympiad 23 (1960)

Tour 23.1

Grade 7

23.1.7.1. Indicate all amounts of roubles that may be changed with the help of both an even and an odd number of bills.

Remark. We assume that, as it was in reality in 1960, the bills are of denominations of 1, 3, 5, 10, 25, 50 and 100 roubles.

23.1.7.2. Three equal circles with centers O_1, O_2, O_3 intersect at a given point, let A_1, A_2, A_3 be the other intersection points. Prove that $\triangle O_1O_2O_3 = \triangle A_1A_2A_3$.

23.1.7.3. 30 undergraduates from 1-st through 5-th year took part in compiling 40 problems for an Olympiad. Any 2 students of the same year brought about the same number of problems. Any two undergraduates of different years suggested distinct number of problems. How many undergraduates suggested one problem each?

23.1.7.4. Two circles with centers O_1 and O_2 intersect at points M and N. Line O_1M intersects the first circle at A_1, and the second one at A_2. Line O_2M intersects the first circle at B_1, and the second one at B_2. Prove that A_1B_1, A_2B_2, and MN intersect at one point. (See Fig. 39.)

23.1.7.5. Prove that an integer n cannot have more than $2\sqrt{n}$ divisors.

Grade 8

23.1.8.1. Prove that a number whose decimal expression contains 300 digits 1, all other digits being zeros, is not a perfect square.

23.1.8.2. In a tournament, each chessplayer got half of his (her) final score in matches with participants who occupied three last places. How many persons participated in the tournament?

23.1.8.3. Draw a straight line through a given vertex A of a convex quadrilateral $ABCD$ so that it divides $ABCD$ into parts of equal area.

23.1.8.4. There are given segments AB, CD and a point O such that no three of the points A, B, C, D, O are on one straight line. The endpoint of a segment is *marked* if the straight line passing through it and O does not intersect another segment. How many marked endpoints are there?

23.1.8.5*. Prove that there are infinitely many positive integers not representable as $p + n^{2k}$ for any prime p and positive integers n and k.

Grade 9

23.1.9.1. Prove that any proper fraction can be represented as a (finite) sum of the reciprocals of distinct integers.

23.1.9.2. See Problem 23.1.8.5.

23.1.9.3. Given a convex polygon and a point O inside it such that any straight line through O divides the polygon's area in halves. Prove that the polygon is symmetric with respect to O.

23.1.9.4. iven a circle and a point inside it. Find the locus of fourth vertices of rectangles, two of whose vertices lie on the given circle and a third vertex is the given point.

Grade 10

23.1.10.1. Two equal equilateral triangular laminas are arranged in space on parallel planes P_1 and P_2 so that the segment connecting their centers is perpendicular to their planes. Find the locus of midpoints of all segments connecting points of one lamina with the points of the other.

23.1.10.2. Prove that if the fraction $\frac{a^n+b^n}{a+b}$ is an integer for positive integers a, b, n, and both the numerator and the denominator of this fraction are divisible by n, then so is the fraction itself.

23.1.10.3. See Problem 23.1.9.4.

23.1.10.4. In the decimal expression of an integer A all digits except the first and the last are zeros; the first and the last are not zeros; and the number of digits is not less than three. Prove that A is not a perfect square.

23.1.10.5. Given numbers $a_1, a_2, \dots, a_k$ such that $a_1^n + a_2^n + \cdots + a_k^n = 0$ for any odd n, prove that nonzero of the numbers $a_1, \dots, a_k$ can be combined in pairs consisting of two opposite numbers, i.e., a and $-a$.

Tour 23.2

Grade 7

23.2.7.1. Given four points, A, B, C, D on a plane. Find a point O such that the sum of the distances from O to the given points is the least possible.

23.2.7.2. Prove that a trapezoid can be constructed from the sides of any quadrilateral.

23.2.7.3. Prove that any nonselfintersecting pentagon is situated on one side of at least one of its edges.

23.2.7.4. One year a Sunday never fell on a certain date in any month. Find this date. (A *date* here is a number $n, 1 \le n \le 31$).

Grade 8

23.2.8.1. For what smallest n can n points be arranged on a plane so that every 3 of them are the vertices of a right triangle?

23.2.8.2. On an infinite chessboard, denote by (a, b) the square at the intersection of the a-th row and the b-th column. A piece may move from square (a, b) to any of the 8 squares $(a \pm m, b \pm n)$ or $(a \pm n, b \pm m)$, where m and n are fixed numbers. We know that the piece returns to its starting point after x moves. Prove that x is even.

23.2.8.3. See Problem 23.2.7.2.

23.2.8.4*. A snail crawls along a straight line, always forward, at a variable speed. Several observers in succession follow its movements during 6 minutes. Each person begins to observe before the preceding observer finishes the observation and observes the snail for exactly one minute. Each observer noticed that during his (her) minute of observation the snail has crawled exactly 1 meter. Prove that during 6 minutes the snail could have crawled at most 10 meters.

23.2.8.5. Given pentagon $ABCDE$ in which $AB = BC = CD = DE$ and $\angle B = \angle D = 90^\circ$. Prove that a plane may be tiled with such pentagons without gaps or overlaps.

Grade 9

23.2.9.1. We are given m points; each of them are connected with line segments to l points. What values can l take?

23.2.9.2. We are given an arbitrary centrally-symmetric hexagon on whose sides equilateral triangles are constructed outward. Prove that the midpoints of the segments connecting the vertices of neighboring triangles are vertices of a regular hexagon.

23.2.9.3. Prove that on any rectangular chessboard 4 squares wide a knight cannot pass each square exactly once and return in the last move to its starting position.

23.2.9.4. Find the locus of the centers of all rectangles circumscribed around a given acute triangle.

23.2.9.5*. In a square of side 100, N circles of radius 1 are arranged so that any segment of length 10 lying inside the square intersects at least one circle. Prove that $N \ge 400$.

Grade 10

23.2.10.1. The number A is divisible by 2, 3, …, 9. Prove that if $2A$ is represented as the sum $2A = a_1 + a_2 + \cdots + a_k$ of positive integers each less than 10, then it is possible to select from $a_1, a_2, \dots, a_k$ certain numbers so that the sum of numbers selected is equal to A.

23.2.10.2. A $6n$-digit number is divisible by 7. The last digit is moved to the beginning of the decimal expression. Prove that the number thus obtained is also divisible by 7.

23.2.10.3. At a gathering of n people, every two persons have two common acquaintances, and every two acquaintances have no common acquaintances. Prove that each of persons present has the same number of acquaintances.

23.2.10.4. See Problem 23.2.9.4.

23.2.10.5. A snail has to crawl $2n$ units along the lines of a piece of graph paper, starting and finishing at a given crossing. Prove that the number of possible routes the snail can take is equal to $\binom{2n}{n}^2$.

Olympiad 24 (1961)

Tour 24.1

Grade 7

24.1.7.1. See Problem 24.1.9.3 below for an even n.

24.1.7.2. Given a 3-digit number $\overline{abc}$. We take the number $\overline{cba}$, and subtract the smaller from the greater to get the number $\overline{a_1b_1c_1}$; we perform the same operation with it, and so on (the case $a_1 = 0$ is allowed). Prove that at some step we get either 495 or 0.

24.1.7.3. Given an acute triangle $\triangle A_0B_0C_0$ let points A_1, B_1, C_1 be the centers of squares constructed on sides B_0C_0, C_0A_0, A_0B_0 outwards. We take triangle $\triangle A_1B_1C_1$, perform the same operation with it and get $\triangle A_2B_2C_2$, etc. Prove that $\triangle A_nB_nC_n$ and $\triangle A_{n+1}B_{n+1}C_{n+1}$ intersect in exactly 6 points.

24.1.7.4. Consider 100 points on a plane such that (1) the distance between any two of them does not exceed 1 and (2) if A, B, C are any three of these points, then $\triangle ABC$ is obtuse. Prove that there is a circle of radius 1/2 such that all given points are either inside it or on it.

24.1.7.5*. On a chessboard, two squares of the same color are selected. Prove that a rook can traverse all squares, starting from one of those selected, and visiting each square exactly once except for the other selected square which the rook must visit twice.

Grade 8

24.1.8.1. Consider $\triangle ABC$ and a point O, denote by M_1, M_2, M_3 the centers of mass of $\triangle OAB$, $\triangle OBC$, $\triangle OCA$, respectively. Prove that $S_{M_1M_2M_3} = \frac{1}{9}S_{ABC}$.

24.1.8.2. One of two players selects a set of one-digit numbers $x_1, \dots, x_n$ (either all positive or all negative). The second player can ask what is the value of $a_1x_1 + a_2x_2 + \dots + a_nx_n$, where $a_1, \dots, a_n$ are any numbers the second player wishes. What is the least number of questions the second player can use to guess what is the selected set of x's?

24.1.8.3. See Problem 24.1.7.3.

24.1.8.4. Prove that a rook can pass all squares of a rectangular chessboard visiting each square exactly once and return to the first square only if the number of squares is even.

24.1.8.5. A set of consequtive positive integers a, $a+1$, ... , $a+k$ is called a *segment* (of the natural series). Two segments of length 1961 are written one below the other. Prove that it is possible to arrange the numbers of each segment so that by adding digits which stand one below the other we get another segment.

Grade 9

24.1.9.1. See Problem 24.1.7.1.

24.1.9.2. See Problem 24.1.8.2.

24.1.9.3. Prove that it is possible to arrange the numbers from 1 to n^2 in an $n \times n$ table so that the sums of numbers in each column are equal.

24.1.9.4. See Problem 24.1.10.3 below assuming that k is divisible by 4.

24.1.9.5. On a plane there are n points such that if A, B, C are any three of them, no other point is inside $\triangle ABC$. Prove that these points may be numbered so that the polygon $A_1A_2 \dots A_n$ is convex.

Grade 10

24.1.10.1. Given the Fibonacci sequence 1, 1, 2, 3, 5, $\dots, u_k, \dots$ Prove that u_{5k} is divisible by 5 for any $k = 1, 2, 3, \dots$.

24.1.10.2. On a plane several strips of different width are drawn so that no two of them are parallel. How should the strips be transported parallel with themselves to maximize the area of their intersection F? (See Fig. 40.)

FIGURE 40. (Probl. 24.1.10.2)

FIGURE 41. (Probl. 24.2.7.1)

24.1.10.3. k persons took a bus without a conductor. They had only coins of denomination 10, 15, or 20 kopecks. Each person paid his/her fare and got the change from other passengers. Prove that the least number of coins needed for this operation is equal to $k+\left[\frac{k+3}{4}\right]$.

Remark. Recall that the machines that sold tickets in the public transport in Moscow were self service. They had receptacles (cash-boxes) for any amount of money but could not give any change. The bus fare was 5 kopecks. So if you had just a 10 kopeck coin you had to ask someone in need of a ticket to give you his/her 5 kopecks, insert your 10 kopecks and take 2 tickets. So the passangers had to help each other or risk a fine[1].

24.1.10.4. A circle S and a point O outside it are both on the same plane. Consider an arbitrary sphere through S and the cone with vertex at O tangent to this sphere. Find the locus of the centers of all circles along which such cones are tangent to such spheres.

24.1.10.5. Given n nonzero complex numbers z_i, $i=1,\dots,n$, such that $z_1+z_2+\dots+z_n=0$, prove that among them there are two numbers with the difference between their arguments $\geq 120^\circ$.

Tour 24.2

Grade 7

24.2.7.1. The sides of an arbitrary convex polygon are painted on the outside. Consider several diagonals; let each of them be similarly painted on one side. Prove that at least one of the polygons into which the initial one is divided by the diagonals is painted completely on the outside. (We allow the paint to leak inside a polygon at its vertices.) See Fig. 41.

24.2.7.2. On sides AB, BC, CD and AD of square $ABCD$ points P, Q, R, S, respectively, are selected so that $PQRS$ is a rectangle. Prove that either $PQRS$ is a square or its sides are parallel to the respective diagonals of $ABCD$.

24.2.7.3. Prove that among any 39 consecutive positive integers there is at least one the sum of whose digits is divisible by 11.

24.2.7.4. Given a 4×4 table. Show that it is possible to arrange 7 asterisks in the table's squares so that if we strike out any two rows and any two columns the remaining squares still contain at least one asterisk. Prove that if there are fewer than 7 asterisks it is always possible to strike out two rows and two columns with no asterisks remaining.

24.2.7.5. Prove that the following system has no integer solutions for a, b, c, d

$$\begin{cases} abcd-a &= 1961,\\ abcd-b &= 961,\\ abcd-c &= 61,\\ abcd-d &= 1.\end{cases}$$

[1]One could actually take as many tickets as there were in the machine; it was only your conscious and, perhaps, the presence of other passangers, if any, that prevented you from abuse. Miraculously, this seldom happened.

Grade 8

24.2.8.1. Given a figure of 16 segments, see Fig. 42. Prove that it is impossible to draw a broken line intersecting each of the segments exactly once. The broken line may be open and selfintersecting but its vertices may not lie on the segments and its links may not pass through the common endpoints of the given segments.

FIGURE 42. (Probl. 24.2.8.1)

24.2.8.2*. The length of a diagonal of a rectangle is equal to d. The rectangle's vertices are the centers of 4 circles of radii r_1, r_2, r_3, r_4 such that $r_1 + r_3 = r_2 + r_4 < d$. Two pairs of outer tangents to circles 1, 3 and 2, 4, are drawn. Prove that a circle can be inscribed in the quadrilateral formed by these four tangents. (See Fig. 43.)

FIGURE 43. (Probl. 24.2.8.2)

24.2.8.3. The sum of digits of integers k and $k + l$ is divisible by 11 and there is no number with similar properties between them. What is the greatest value of l? (Cf. Problem 24.2.7.3.)

24.2.8.4. See Problem 24.2.7.4.

24.2.8.5. Given four numbers, a, b, c, d, we construct another four numbers: ab, bc, cd, da (each number is multiplied by the next one and the fourth number is multiplied by the first one). From these four numbers a third foursome is obtained by the same rule, etc. Prove that in the resulting sequence of foursomes we never encounter the initial one except for the case $a = b = c = d = 1$.

Grade 9

24.2.9.1. Points A and B move uniformly with equal angle velocities clockwise along circles O_1 and O_2, respectively. Prove that vertex C of equilateral triangle $\triangle ABC$ also moves uniformly along a circle.

24.2.9.2. An $m \times n$ table is filled with certain numbers. It is allowed to simultaneously change the sign of all numbers in a column or a row. Prove that by applying this operation several times, any given table may be altered so that the sum of the numbers in any one of its columns or rows will be nonnegative.

24.2.9.3. n points are connected by segments so that each point is connected to any other by a "route", and no two points are connected by more than one such "route". Prove that there are $n - 1$ segments altogether.

24.2.9.4. a, b, p are integers. Prove that there exist relatively prime integers k, l such that $ak + bl$ is divisible by p.

24.2.9.5. Nick and Pete divide between themselves $2n+1$ nuts, $n \geq 2$, and each tries to get the greater share, naturally. According to The Rule there are three ways to divide the nuts. Each way consists of three steps and the 1-st and 2-nd steps are common for all three ways.

1-st step: Pete divides all nuts into two piles, each containing no less than two nuts.

2-nd step: Nick divides both piles into two, each new pile containing no less than one nut.

3-rd step: Nick sticks to either of the following methods:

a) Nick takes either the biggest and the smallest pile, or

b) Nick takes both medium-sized piles, or

c) Nick choses either the biggest and the smallest or the medium-sized piles, but pays Pete one nut for the choice.

Find the most profitable and the least profitable of methods a) – c) for Nick to divide the nuts.

Grade 10

24.2.10.1. Prove that for any three infinite sequences of positive integers

$$a_1, a_2, \dots, a_n, \dots;$$
$$b_1, b_2, \dots, b_n, \dots;$$
$$c_1, c_2, \dots, c_n, \dots.$$

there exist p and q such that $a_p \geq a_q$, $b_p \geq b_q$, $c_p \geq c_q$.

24.2.10.2. 120 squares of side 1 are tossed onto a 20×25 rectangle. Prove that a disc of diameter 1 can be placed in the rectangle so that the disc does not intersect any of the squares.

24.2.10.3. See Problem 24.2.9.2.

24.2.10.4. On a plane, the distance from a fixed point P to two vertices, A and B, of an equilateral $\triangle ABC$ is 2 and 3 units, respectively. Find the maximal possible length of PC.

24.2.10.5. From an arbitrary sequence of 2^k numbers 1 and -1 we get a new sequence by the following operation: each number is multiplied by the one following it, and the last 2^k-th number is multiplied by the 1-st one. We perform the same operation with the sequence obtained, and so on. Prove that eventually we get a sequence consisting entirely of 1's.

Olympiad 25 (1962)

Tour 25.1

Grade 7

25.1.7.1. Given a straight line l perpendicular to and intersecting segment AB. For any point M on l we can find a point N such that $\angle NAB = 2\angle MAB$ and $\angle NBA = 2\angle MBA$. Prove that the absolute value $|AN - BN|$ does not depend on M. (See Fig. 44.)

Figure 44. (Probl. 25.1.7.1)

25.1.7.2. We reflect an equilateral triangle with one marked side through one of its sides. Then we similarly reflect the resulting triangle, etc., until at a certain step the triangle returns to its initial position. Prove that the marked side also returns to its initial position.

25.1.7.3. Let a, b, c, d be the sides of a quadrilateral that is not a rhombus. Prove that from the segments a, b, c, d one can construct a self-intersecting quadrilateral.

25.1.7.4. Denote by $S(a)$ the sum of digits of a number a. Prove that if $S(a) = S(2a)$, then a is divisible by 9.

25.1.7.5. On each side of n given cards one of the numbers $1, 2, \dots, n$ is written so that each number occurs exactly twice. Prove that the cards may be arranged on a table so that all numbers $1, 2, \dots, n$ face upward.

Grade 8

25.1.8.1. On sides AB, BC, CA of an equilateral triangle $\triangle ABC$ find points X, Y, Z, respectively, so that the area of the triangle formed by lines CX, BZ, AY is one-fourth of the area of $\triangle ABC$ and so that

$$\frac{AX}{XB} = \frac{BY}{YC} = \frac{CZ}{ZA}.$$

25.1.8.2. See Problem 25.1.7.2.

25.1.8.3. Prove that for any integer d there exist integers m and n such that

$$d = \frac{n - 2m + 1}{m^2 - n}.$$

25.1.8.4. See Problem 25.1.7.4.

25.1.8.5. See Problem 25.1.7.5.

Grade 9

25.1.9.1. Given two intersecting segments AA_1 and BB_1 on which lie points M and N, respectively, so that $AM = BN$. Find positions of M and N for which the length of MN is the shortest. (Cf. Problem 25.1.9.2.7-8.3).

25.1.9.2. A chessman that crosses n squares in one move diagonally and 1 square up (or the other way round) is called a *Boo*. A Boo stands on a square of an infinite chessboard. What n is required for the Boo to reach any given square? For what n this is impossible?

25.1.9.3. See Problem 25.1.7.4.

25.1.9.4. Given the system of equations:

$$\begin{cases} x_1x_2x_3\dots x_{1961}x_{1962} = 1, \\ x_1 - x_2x_3\dots x_{1961}x_{1962} = 1, \\ x_1x_2 - x_3\dots x_{1961}x_{1962} = 1, \\ \\ x_1x_2x_3\dots x_{1961} - x_{1962} = 1, \end{cases}$$

find what values x_{25} can take.

25.1.9.5. Prove that in a rectangle of area 1 nonintersecting circles can be arranged so that the sum of their radii is equal to 1962.

Grade 10

25.1.10.1. See Problem 25.1.9.1, the segments being replaced with intersecting rays. (See Fig. 45.)

FIGURE 45. (Probl. 25.1.10.1) FIGURE 46. (Probl. 25.1.10.2)

25.1.10.2. The sides of a square are the bases of equal acute isosceles triangles constructed outward. Prove that the figure obtained cannot be divided into parallelograms. (See Fig. 46.)

25.1.10.3. Prove that any positive integer can be represented as the sum of several distinct terms of the Fibonacci sequence 1, 2, 3, 5, 8, 13, … .

25.1.10.4. See Problem 25.1.9.4.

25.1.10.5. See Problem 25.1.7.5.

Tour 25.2

Grade 7

25.2.7.1. A ball rests at the side of a billiard table shaped in the form of a regular $2n$-gon without pockets. How should the ball be shot so that after reflections through all sides[1] *(except the initial one) exactly once* it returns to the same point? Prove that *for the pathes that consequtively reflect themselves through neighboring sides* the length of the ball's path does not depend on the starting point.

25.2.7.2*. Let $\triangle ABC$ be an isosceles triangle, $AB = BC$, BH its height, M the midpoint of AB, and K the other intersection point of BH with the circle drawn through B, M and C. Prove that $BK = 3R/2$, where R is the radius of the circle circumscribed around $\triangle ABC$.

25.2.7.3. An L-shaped figure (see Fig. 47) is constructed of three squares with side 1. Prove that a) it is impossible to split a rectangle of size 1961×1963 into such figures but b) it is possible to do so with a rectangle of size 1963×1965.

Figure 47. (Probl. 25.2.7.3)

Figure 48. (Probl. 25.2.8.5)

25.2.7.4. Prove that the number 100...01 with 1961 zeros between the 1's is not a prime.

25.2.7.5. Given 25 points on a plane such that from any three points we can choose two points that are less than 1 unit of length apart. Prove that 13 of the given points lie on a unit disc.

Grade 8

25.2.8.1. Several diagonals in a convex polygon satisfy the following condition: no two of them intersect except at an endpoint identical with a vertex. Prove that no diagonal come out of at least 2 vertices of this polygon.

25.2.8.2. How should one arrange the numbers $1, 2, \dots, 1962$ in a sequence $a_1, a_2, \dots, a_{1962}$ in order to obtain the greatest possible value of the sum

$$|a_1 - a_2| + |a_2 - a_3| + \dots + |a_{1961} - a_{1962}| + |a_{1962} - a_1| \ ?$$

25.2.8.3. An irregular n-gon is inscribed in a circle. After a rotation of the circle around its center through an angle of $\alpha \neq 2\pi$ the n-gon coincides with itself. Prove that n is not prime.

25.2.8.4*. From the numbers x_1, x_2, x_3, x_4, x_5 ten sums are composed each having as summands two of these numbers. Denote the sums by $a_1, a_2, \dots, a_{10}$; we do not know what summands constitute them. Prove that given $a_1, a_2, \dots, a_{10}$, one can find $x_1, x_2, \dots, x_5$.

25.2.8.5. Two circles, O_1 and O_2, intersect at M and P. Denote by MA the chord of O_1 tangent to O_2 at M, and by MB the chord of O_2 tangent to O_1 at M. On line MP, segment PH equal to MP is constructed. Prove that quadrilateral $MAHB$ can be inscribed in a circle. (See Fig. 48.)

[1] The words in italics are added to the original formulation to make the problem correct.

Grade 9

25.2.9.1. During every period of 7 consecutive days throughout the school year a student must solve exactly 25 problems. The time required to solve any (one) problem does not vary during a day but does vary during the year according to a Rule known to the student. This time is always less than 45 minutes. The student wants to spend as little time as possible on solving all problems. Prove that to this end (s)he can choose a certain day every week and solve all 25 problems during this day.

Remark. We disregard here the fact that unless the student is looking for trouble at school and at home (s)he can be engaged in solving the problems only on Sundays, for about 18 hours in a row.

25.2.9.2. See Problem 25.2.8.2, where 25 arbitrary different numbers replace 1, 2, ..., 1962.

25.2.9.3. The sides of a convex polygon whose perimeter is equal to 12 are moved a distance of $d = 1$ outward and their extensions form a new polygon. Prove that the area of the new polygon is at least 15 square units greater than the area of the original polygon.

25.2.9.4. See Problem 25.2.8.4.

25.2.9.5. Given 2^n finite sequences of 0's and 1's such that none of them is the beginning of another, prove that the sum of the lengths of these sequences is not less than $n2^n$.

Grade 10

25.2.10.1. A point C is fixed on a given straight line l passing through the center O of a given circle. Points A and A' lie on the circle on one side of l, so that the angles formed by lines AC and $A'C$ with l are equal. Lines AA' and l meet at B. Prove that the location of B does not depend on that of A. (See Fig. 49.)

FIGURE 49. (Probl. 25.2.10.1)

FIGURE 50. (Probl. 25.2.10.4)

25.2.10.2. See Problem 25.2.9.2.

25.2.10.3. See Problem 25.2.9.3.

25.2.10.4. How should a right parallelepiped be placed in space so that the area of its projections to the horizontal plane is the greatest possible? (See Fig. 50.)

25.2.10.5. In a chess tournament, each participant played one game with each other. Prove that the participants may be so numbered, that none of them loses to the one with the next number.

Olympiad 26 (1963)

Tour 26.1

Grade 7

26.1.7.1. From vertex B of an arbitrary $\triangle ABC$, straight lines BM and BN are drawn outside the triangle so that $\angle ABM = \angle CBN$. Points A' and C' are symmetric to A and C through BM and BN, respectively. Prove that $AC' = A'C$. (See Fig. 51.)

26.1.7.2. Let a, b, c be three numbers such that $a + b + c = 0$. Prove that $ab + bc + ca \leq 0$.

Figure 51. (Probl. 26.1.7.1)

26.1.7.3. We have a 4×100 sheet of graph paper. The Rule allows to divide it into 200 rectangular cards of size 1×2 each consisting of 2 cells of the paper and write 1 on one cell of the card and -1 on the other. Is it possible to ensure that the products of the numbers in each column and each row of the table obtained are positive? (Cf. Problem 26.1.8.5 below.)

26.1.7.4. See Problem 26.1.8.4 below.

26.1.7.5. Is it possible to draw a straight line on a 20×30 piece of graph paper so that it would intersect 50 squares? (Cf. Problem 26.1.10.3 below.)

Grade 8

26.1.8.1. Let $a_1, \dots, a_n$ be numbers such that $a_1 + a_2 + \dots + a_n = 0$. Let S be the sum of all products $a_i a_j$ for $i \neq j$. Prove that $S \leq 0$. (Cf. Problem 26.1.7.2.)

26.1.8.2. Given a convex quadrilateral $ABCD$ of area S, a point M inside it and points E, F, G, H symmetric to M through the midpoints of the sides of the quadrilateral $ABCD$, respectively, find the area of quadrilateral $EFGH$. (See Fig. 52.)

Figure 52. (Probl. 26.1.8.2)

Figure 53. (Probl. 26.1.9.5)

26.1.8.3. Solve in integers the equation

$$\frac{xy}{z} + \frac{xz}{y} + \frac{yz}{x} = 3.$$

26.1.8.4. Given 7 lines on a plane, no two of which are parallel, prove that two of them meet at an angle $< 26^\circ$.

26.1.8.5. A $5 \times n$ piece of graph paper is divided into rectangular 1×2 cards of two cells of the paper each. We write a 1 on one cell of the card and a -1 on the other cell. It is known that the product of the numbers in each row and each column of the resulting table is positive. For which n this is possible? (Cf. Problem 26.17.3.)

Grade 9

26.1.9.1. The first term and the difference of an arithmetic progression are integers. Prove that there exists a term in this progression whose decimal expression contains figure 9.

26.1.9.2. See Problem 26.1.8.5.

26.1.9.3. Let a, b, c be some positive numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}.$$

26.1.9.4. Prove that of any four points on a plane, no three of which are on the same line, three points may be selected so that the triangle with vertices at these points has at least one angle $\leq 45°$. (Cf. Problem 26.1.10.2 below.)

26.1.9.5. Is it possible to inscribe in a rectangle with the ratio of sides $9:16$ another rectangle, with the ratio of sides $4:7$, so that on each side of the first rectangle there is a vertex of the second one? (See Fig. 53.)

Grade 10

26.1.10.1. See Problem 26.1.9.1.

26.1.10.2. Prove that of any six points in a plane, no three of which are on the same line, three points may be chosen so that the triangle with vertices at these points has at least one angle that is not greater than 30°.

26.1.10.3. What is the greatest number of squares that a line drawn on an $m\times n$ piece of graph paper can intersect?

26.1.10.4. Given numbers a, b, c such that $abc>0$ and $a+b+c>0$, prove that $a^n+b^n+c^n>0$ for any positive integer n.

26.1.10.5. Given an arbitrary $\triangle ABC$, find the locus of points M such that the perpendiculars to lines AM, BM, CM dropped from points A, B, C, respectively, meet at one point.

Grade 11

26.1.11.1. Prove that $x+y+z>xyz$ if $x,y,z>0$ and

$$\arctan x+\arctan y+\arctan z<\pi.$$

26.1.11.2. Consider a system of 25 distinct segments with a common endpoint at point A and other endpoints lying on a line l not passing through A. Prove that there does not exist a closed 25-angled broken line each of whose segments is parallel and equal to one of the segments from the system considered.

26.1.11.3. See Problem 26.1.10.5.

26.1.11.4. Prove that the sum of all possible 7-digit numbers in whose decimal expression each of the figures 1, 2, 3, 4, 5, 6, 7 is used exactly once is divisible by 9.

26.1.11.5. Each edge of a regular tetrahedron is divided into three equal parts. Through each division point two planes are drawn parallel to the two faces of the tetrahedron that do not pass through this point. Into how many parts do these planes divide the tetrahedron?

Tour 26.2

Grade 7

26.2.7.1. A factory produces rattles shaped in the form of a ring with 3 red and 7 blue spherical beads on it. Two rattles are said to be *of the same type* if one can be obtained from the other one by moving a bead along the ring or by flipping the ring over in space. How many different types of rattles can be manufactured?

26.2.7.2. See Problem 26.2.9.2.

26.2.7.3. Given $\triangle ABC$. Consider straight line intersecting sides AB and AC of the triangle so that the distance from the line to point A is equal to the sum of the distances from the line to points B and C. Prove that all such lines pass through one point.

26.2.7.4. What greatest number of elements can be selected from the set of numbers 1, 2, ... , 1963 so that the sum of any two of the selected numbers is divisible by 26?

26.2.7.5. A system of segments is called *connected* if from the endpoints of any segment any of endpoints of any other segment can be reached by moving along the segments. We assume that it is impossible to pass from one segment to another one at intersection points other than those of connection. Is it possible to connect five points by segments into a connected system so that after erasing any of its segments one gets exactly two connected systems of segments, disconnected from each other?

Grade 8

26.2.8.1. Let $a_1, \dots, a_n$ be arbitrary positive integers. Denote by b_k the number of integers that satisfy $a_i \geq k$. Prove that

$$a_1 + a_2 + \dots + a_n = b_1 + b_2 + \dots .$$

26.2.8.2. An 8×8 table contains all integers from 1 to 64. The numbers are called *adjacent* if the squares they are written upon have a common side. Prove that there exist two adjacent numbers whose difference is not less than 5.

26.2.8.3. Find the set of the centers of mass of all acute triangles inscribed in a given circle.

26.2.8.4. What greatest number of integers can be selected from the set 1, 2, ... , 1963 so that no sum of any two selected numbers were divisible by their difference?

26.2.8.5*. Three gentlemen walk along a path 100 meters long at a constant speed of 1, 2, and 3 km/hr, respectively. Reaching the end of the path each of them turns and goes back at the same speed. Prove that there is an interval of 1 minute during which all three gentlemen walk in the same direction.

Grade 9

26.2.9.1*. Given an arbitrary $\triangle ABC$, its medians AM, BN, CQ, and a point X outside it. Prove that the area of one of the triangles $\triangle XAM$, $\triangle XBN$, or $\triangle XCQ$ is equal to the sum of areas of the other two.

26.2.9.2. A closed 14-angled broken line is drawn along the lines of a piece of mesh paper. No line of the graph contains more than two links of the broken line and no two links can go in succession along one horizontal or vertical line. What is the greatest number of self-intersection points that the brocken line can have?

26.2.9.3. We drew all diagonals in a regular decagon. How many nonsimilar triangles is it possible to form from all sides and diagonals of the decagon?

26.2.9.4. A 9×9 table contains all integers from 1 to 81. Prove that there exist two adjacent numbers whose difference is not less than 6. (Cf. Problem 26.2.8.2.)

26.2.9.5. See Problem 26.2.7.5.

Grade 10

26.2.10.1. Prove that the equation $x^n + y^n = z^n$ cannot have integer solutions if $x + y$ is prime and n is an odd number > 1.

26.2.10.2. We drew a mesh of n horizontal and n vertical straight lines on a sheet of paper. How many distinct closed broken, perhaps, self-intersecting, lines of $2n$ segments each can one draw along the lines of the mesh so that each broken line traverses along all horizontal and vertical lines?

26.2.10.3. In a regular 25-gon we drew vectors from the center to all the vertices. How to select several of these 25 vectors for the sum of the selected vectors to be the longest?

26.2.10.4. Let A', B', C', D', E' be the midpoints of the sides of convex pentagon $ABCDE$. Prove that $2S_{A'B'C'D'E'} \geq S_{ABCDE}$.

26.2.10.5*. Consider the sequence $a_1 = a_2 = 1$, $a_n = \frac{a_{n-1}^2+2}{a_{n-2}}$ for $n \geq 3$. Prove that the a_n are integers.

Grade 11

26.2.11.1. Prove that there are no distinct positive integers x, y, z, t such that

$$x^x + y^y = z^z + t^t.$$

26.2.11.2. Prove that of 11 arbitrary infinite decimal fractions one can select two fractions with the difference between them having either an infinite number of zeros or an infinite number of nines in the decimal expression.

26.2.11.3. Find all polynomials $P(x)$ satisfying the identity

$$xP(x-1) = (x-26)P(x) \quad \text{for all } x.$$

26.2.11.4. See Problem 26.2.10.4.

26.2.11.5*. Prove that on a sphere it is impossible to arrange three arcs of great circles of measure 300° each so that no two of them have any common points (endpoints included). (See Fig. 54.)

FIGURE 54. (Probl. 26.2.11.5)

Olympiad 27 (1964)

Tour 27.1

Grade 7

27.1.7.1. In $\triangle ABC$, the heights dropped to sides AB and BC are not shorter than the respective sides. Find the angles of the triangle.

27.1.7.2. On a given circle, there are selected two diametrically opposite points A and B and a third point, C. The tangent to this circle at B meets line AC at M. Prove that the tangent drawn to this circle at C divides BM in halves.

27.1.7.3. Prove that the sum of the digits in the decimal expression of a perfect square cannot be equal to 5.

27.1.7.4. We drew 11 horizontal and 11 vertical intersecting straight lines on a sheet of paper. We call a segment of one of the straight line drawn that connects two neighboring intersections a "link". What least number of links must we erase in order for each intersection to be a junction of not more than 3 links?

27.1.7.5. Consider the sequence $a_0 = a_1 = 1$; $a_{n+1} = a_{n-1}a_n + 1$ for $n = 1, 2, \dots$. Prove that a_{1964} is not divisible by 4.

Grade 8

27.1.8.1. See Problem 27.1.7.1.

27.1.8.2. Find all positive integers n such that $(n-1)!$ is not divisible by n^2.

27.1.8.3. Solve in integers for unknowns x, y and z:

$$\sqrt{x + \sqrt{x + \dots + \sqrt{x}}} = z \quad (y\text{-many square roots}).$$

27.1.8.4. See Problem 27.1.9.4 a) below.

27.1.8.5. Take the sums of digits of all numbers from 1 to 1 000 000. Next, take the sums of digits of the numbers obtained, etc., until you get 1 000 000 one-digit numbers. Which number is more numerous among them: 1 or 2?

Grade 9

27.1.9.1. Solve the system in positive numbers:

$$\begin{cases} x^y = z, \\ y^z = x, \\ z^x = y. \end{cases}$$

27.1.9.2. Prove that the product of two consecutive positive integers is not a power of any integer.

27.1.9.3. Given that $a - k^3 \vdots 27 - k$ for any integer k, except $k = 27$, find a.

27.1.9.4. a) Prove that if all angles of a hexagon are equal, then its sides satisfy the following relations:

$$a_1 - a_4 = a_5 - a_2 = a_3 - a_6.$$

b) Prove that if the lengths of segments $a_1, \dots, a_6$ satisfy the above relations, then one can construct from them an equiangular hexagon.

27.1.9.5. In quadrilateral $ABCD$, We drop perpendiculars from vertices A and C to diagonal BD, and from vertices B and D, to AC. Let M, N, P, Q be the bases of the perpendiculars. Prove that quadrilaterals $ABCD$ and $MNPQ$ are similar. (See Fig. 55.)

FIGURE 55. (Probl. 27.1.9.5)

FIGURE 56. (Probl. 27.2.7.1)

Grades 10 – 11

27.1.10-11.1. A number N is a perfect square and does not end with a zero. After erasing its two last digits, one gets another perfect square. Find the greatest N with this property.

27.1.10-11.2. See Problem 27.1.8.3.

27.1.10-11.3. It is known that for any integer $k \neq 27$ the number $a - k^{1964}$ is divisible by $27 - k$. Find a. (Cf. Problem 27.1.9.3.)

27.1.10-11.4. See Problem 27.1.8.4.

27.1.10-11.5. What is the least number of nonintersecting tetrahedrons into which a cube can be divided?

Tour 27.2

Grade 7

27.2.7.1. We select an arbitrary point B on segment AC. Segments AB, BC, and AC are diameters of circles T_1, T_2 and T_3, respectively. Consider a straight line through B; let it intersect T_3 at P and Q, and let it intersect T_1 and T_2 at R and S, respectively. Prove that $PR = QS$. (See Fig. 56.)

27.2.7.2. $2n$ persons attanded a party. Everyone was acquainted with at least n guests. Prove that it is possible to select 4 of the guests and seat them at a round table so that each sits next to his or her acquaintances.

27.2.7.3. 102 points, no three of which are on the same straight line, are chosen in a square with side 1. Prove that there exists a triangle with vertices at these points and of area less than $\frac{1}{100}$.

27.2.7.4. Through opposite vertices A and C of quadrilateral $ABCD$ a circle is drawn intersecting AB, BC, CD and AD at M, N, P and Q, respectively. Suppose $DP = DQ = BM = BN = R$, where R is the radius of the circle. Prove that $\angle ABC + \angle ADC = 120°$.

27.2.7.5. For what positive integers a the equation $x^2 + y^2 = axy$ has a solution for x and y in positive integers?

Grade 8

27.2.8.1. Each of n glasses of sufficient capacity contains the same amount of water as the other glasses do. At one step we may pour as much water from any glass into any other as the recepting glass already contains. For what n is it possible to empty all glasses into one glass in a finite number of steps?

27.2.8.2. Consider three points A, B, C on the same straight line and one point, O, not on it. Denote by O_1, O_2, O_3 the centers of circles circuscribed around triangles $\triangle OAB$, $\triangle OBC$, $\triangle OAC$. Prove that the points O_1, O_2, O_3 and O are all on one circle, see Fig. 57.

FIGURE 57. (Probl. 27.2.8.2)

27.2.8.3. Two players sit at a 99×99 tic-tac-toe board. The first player draws a "$\times$" in the central square. Then the second player may draw a "O" in any of the eight squares adjacent to the $\times$. Now, the first player draws a $\times$ in any of the squares adjacent to those already occupied, and so on. The first player wins if (s)he can draw his/her $\times$ in any corner square. Prove that the first player can always win.

27.2.8.4. Inside an equilateral (not necessarily regular) heptagon $A_1A_2A_3A_4A_5A_6A_7$ an arbitrary point O is chosen. Denote by H_1, H_2, H_3, H_4, H_5, H_6, H_7 the bases of the perpendiculars dropped from O to A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_7, respectively. It is known that points H_1, H_2, H_3, H_4, H_5, H_6, H_7 belong to the sides themselves, not to their extensions. Prove that

$$\begin{aligned}&A_1H_1 + A_2H_2 + A_3H_3 + A_4H_4 + A_5H_5 + A_6H_6 + A_7H_7 \\ &\qquad = H_1A_2 + H_2A_3 + H_3A_4 + H_4A_5 + H_5A_6 + H_6A_7 + H_7A_1.\end{aligned}$$

27.2.8.5*. 101 distinct points are chosen at random in a square of side 1 (not necessarily inside it, some points might lie on the sides), so that no three of the points belong to one straight line. Prove that there is a triangle with vertices at some of the fixed points whose area does not exceed 0.01.

Grade 9

27.2.9.1. See Problem 27.2.8.1.

27.2.9.2. See Problem 27.2.8.4.

27.2.9.3. Prove that any non-negative even number $2n$ can be uniquely represented in the form $2n = (x+y)^2 + 3x + y$, where x and y are nonnegative integers.

27.2.9.4. In $\triangle ABC$, side BC is equal to a halfsum of the other two sides ($AB \neq AC$). Prove that the bisector of $\angle BAC$ is perpendicular to the segment connecting the centers of the inscribed and circumscribed circles.

27.2.9.5*. On a graph paper consider a closed broken line whose vertices are in the nodes of the grid and all segments of the broken line are equal. Prove that the number of the segments of such a broken line is even.

Grade 10

27.2.10.1. n beakers contain n distinct liquids, there is also an empty beaker. We assume that each beaker is continuously graded so that we can measure the volume of liquid inside it. Is it possible to compose uniform mixtures in each beaker inside a finite length of time? In other words, is it possible to arrange so that each of the n beakers contains exactly $\frac{1}{n}$ of the initial quantity of each liquid and one beaker is empty?

27.2.10.2. We have a system of n points on a plane such that for any two points there is a movement of the plane sending the first point to the second one and the whole system into itself. Prove that all points of such a system belong to a circle.

27.2.10.3. In $\triangle ABC$, side BC is equal to a halfsum of the other two sides. Prove that vertex A, the midpoints of AB and AC and the centers of the inscribed and circumscribed circles belong to one circle.

27.2.10.4. See Problem 27.2.9.5.

27.2.10.5*. Several positive integers are written on each of infinitely many cards so that for any n there is exactly n cards on which the divisors of n are written. Prove that every positive integer is encountered on at least one card.

Grade 11

27.2.11.1. Several vectors begin from point O on a plane; the sum of their lengths is equal to 4. Prove that it is possible to select several of these vectors (perhaps, one) the length of whose sum (whose length) is greater than 1.

27.2.11.2. See Problem 27.2.8.3.

27.2.11.3. In $\triangle ABC$, sides AB and AC are of different length, side BC is equal to their halfsum.[1] Consider the circle through A and the midpoints of AB and AC. Consider the tangents to the circle pass through the triangle's center of mass. Prove that a) one of the tangent points is the center of the circle inscribed in $\triangle ABC$, b) the straight line through the intersection point of medians and the intersection point of bisectors of $\triangle ABC$ is tangent to this circle. (Cf. Problem 27.2.8.4 and 27.2.10.3.)

27.2.11.4*. A pie is of the form of a regular n-gon inscribed in a circle of radius 1. One straight cut of length 1 is made from the midpoint of each side. Prove that in this way we always cut off a piece of the pie (even if we'd rather not).

27.2.11.5*. Once upon a time there were $2n$ knights at King Arthur's court; each of the knights had not more than $n-1$ enemies among the knights present. Prove that Merlin, King Arthur's counsellor, can place the knights at the Round Table so that no knight will have his enemy as a neighbor.

Olympiad 28 (1965)

Tour 28.1

Grade 8

28.1.8.1. Given circle S, straight line a intersecting S, and a point M. Draw a line b through M so that the part of b inside S is bisected by a. (See Fig. 58.)

FIGURE 58. (Probl. 28.1.8.1)

28.1.8.2. Prove the validity of the following test of divisibility by 37. *Divide the decimal expresesion of n into groups of 3 digits from right to left. If the sum of the resulting three-digit numbers is divisible by 37, then* $n \vdots 37$. (These three-digit numbers may begin with zeros and, therefore, be actually two-digit or one-digit numbers; e.g., the left-most group can be so.)

28.1.8.3. Given straight line a and two nonparallel segments AB and CD on one side of it. Find a point M on a such that $\triangle ABC$ and $\triangle CDM$ have equal areas.

28.1.8.4. 30 teams participate in a soccer tournament. Prove that during the tournament there always exist two teams which have played the same number of games.

Grade 9

28.1.9.1. A six-digit number is divisible by 37. All its digits are different. Prove that one can form from the same digits another six-digit number divisible by 37.

28.1.9.2. Inside a given triangle ABC, find a point O such that the ratio of areas of triangles $\triangle AOB$, $\triangle BOC$ and $\triangle COA$ is equal to $1:2:3$.

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[1] The original formulation was vague. It did not state that $AB \neq BC$. But if $AB = BC$ the circle's center of mass lies on the circle causing a degeneracy. It was also unclear whether the center of mass will automatically be outside the circle constructed, or to have it outside is an extra condition.

28.1.9.3. Consider $\triangle ABC$ with $AB > BC$ and bisectors AK and CM, where K is on BC and M on AB. Prove that $AM > MK > KC$.

28.1.9.4. In Illiria, some pairs of towns are connected by direct airlines. Prove that there exist two towns in Illiria that are connected with the same number of other towns. (Cf. Problem 28.1.8.4.)

28.1.9.5. An elderly woman decides to reduce noise from the flat below by placing along her (rectangular) corridor rectangular mats of the same width as the corridor. The mats cover the entire floor and even overlap so that certain portions of the floor are covered by several layers.

Prove that it is always possible to remove several mats, perhaps taking them from underneath and leaving the others in their original positions, so that the floor will remain completely covered and the combined length of the remaining mats will be less than twice the length of the corridor.

Grade 10

28.1.10.1. The circles O_1 and O_2 are inside $\triangle ABC$. They are tangent to each other externally; moreover, O_1 is tangent to AB and BC, and O_2 is tangent to AB and AC. Prove that the sum of the radii of these circles is greater than the radius of the circle inscribed in ABC. (See Fig. 59.)

FIGURE 59. (Probl. 28.1.10.1)

FIGURE 60. (Probl. 28.1.10.3)

28.1.10.2. See Problem 28.1.9.1.

28.1.10.3. The endpoints of a segment of fixed length slide along the legs of a given angle. The perpendicular to the segment is erected from its midpoint. Prove that the distance from the base of the sliding perpendicular to the point where it meets the bisector of the angle is a constant. (See Fig. 60.)

28.1.10.4. Let $x > 2$. Somebody writes on cards the numbers 1, x, x^2, x^3, $\dots$, x^k (a number per card). Then Somebody puts some of the cards in her right pocket, some in her left pocket, and throws away the rest.

Prove that the sum of the numbers in Somebody's right pocket cannot be equal to the sum of the numbers in her left pocket. (Cf. Problem 28.1.11.1.)

28.1.10.5. A paper square has 1965 perforations. No three of the 1969 points — the union of the perforation points with the square's vertices — lie on the same straight line. We cut along several nonintersecting line segments with endpoints at perforations or vertices on the square. It turns out that the cuts divide the square into triangles inside which there are no perforations. How many cuts were made and how many triangles were obtained?

Grade 11

28.1.11.1. Each coefficient of a polynomial $f(x)$ is equal to 1, 0 or -1. Prove that all real roots (if any) of the polynomial lie on the segment $[-2, 2]$.

28.1.11.2. Given three points on a plane, construct three circles tangent to one another at these points. Consider all possible cases.

28.1.11.3. In the quadratic equation $x^2 + px + q = 0$, the coefficients p and q independently take on all values from segment $[-1, 1]$. Find the set of real roots of these quadratic equations.

28.1.11.4. Given circle O, a point A on it, the perpendicular erected at A to the plane on which O lies, and a point B on this perpendicular. Find the locus of bases of perpendiculars drawn from A to the straight line through B and any point on circle O.

28.1.11.5*. Given 20 cards with each of the figures $0, \dots, 9$ written on two of these cards. Find whether it is possible to arrange the cards in a row so that the zeros are next to one another, the 1's have one card between them, the twos have two cards between them, etc., and the nines have nine cards between them.

Tour 28.2

Grade 8

28.2.8.1*. Given an infinite in both ways sequence

$$\dots,\ a_{-n},\ \dots,\ a_{-1}, a_0, a_1,\ \dots,\ a_n, \dots, \qquad \text{where } a_n = \frac{1}{4}(a_{n-1} + a_{n+1}),$$

prove that if some two of its terms (not necessarily adjacent) are equal, then the sequence contains an infinite number of pairs of equal terms.

28.2.8.2. We place a rectangular billiard table of size 26×1965 so that two of its longer sides are oriented North-South and its shorter sides are oriented East-West. The pockets are only at the vertices of the rectangle. A ball is shot from the lower left (SW) pocket of the billiard at an angle of 45°. Prove that after several rebounds from the sides the ball will reach the upper left (NW) pocket.

28.2.8.3. We divide two paper discs of different size into 1965 equal sectors. On each of the discs we select at random 200 sectors and paint them red. We put the smaller disc on top of the bigger one, so that their centers coincide and the sectors of one lie just over the sectors of the other. A *position* is the new relation between discs that they assume after we rotate the smaller disc through all angles that are multiples of $\frac{2\pi}{1965}$, while the bigger disc is fixed. Prove that in at least 60 positions not more than 20 red sectors of both discs coincide.

28.2.8.4*. In a fairyland, a row of houses, with square foundations of side a, stands between two parallel streets. The distance between the streets is $3a$, and the distance between two neighboring houses is $2a$. One street is patrolled by cops who stroll at a distance of $9a$ from one another, at a constant speed v no matter what. When the first cop passes the middle of a certain house, a robber appears, exactly opposite the cop, on the other street, see Fig. 61.

FIGURE 61. (Probl. 28.2.8.4)

The robber is doomed to move with a constant speed; thanks to a Good Fairy the robber can reach any value of speed, without any acceleration, instantaneously. At what constant speed and in which direction should the robber move along that street so that no cop spots him?

Grade 9

28.2.9.1. See Problem 28.2.10.1.

28.2.9.2. We shot a ball from a vertex of a rectangular billiard table with pockets at its vertices at an angle of at 45° to the side. If the ball reaches a pocket, it fells into it. After a while the ball reached the midpoint of a certain side. Prove that it could not have already touched the midpoint of the opposite side.

28.2.9.3. See Problem 28.2.8.1.

28.2.9.4. See Problem 28.2.10.2.

28.2.9.5. Find the locus of the centers of equilateral triangles circumscribed around an arbitrary given triangle.

Grade 10

28.2.10.1. We have 11 sacks of coins and a balance with two pans and a hand dial that indicates which pan contains a heavier load and what is the difference in their weights. We can weigh any number of coins from any sack. We know that all coins in one sack are counterfeit, and all other coins are genuine. All genuine coins are of weight x, whereas all counterfeit coins are of weight y, where neither x nor y are known. What is the least number of weighings needed to determine which sack has counterfeit coins?

28.2.10.2. On a $n \times n$ piece of graph paper, we arrange black and white cubes so that each cube stands on exactly one 1×1 square formed by the paper's mesh. We had formed the first layer of n^2 cubes when The Rule was issued: *two cubes are called neighboring to each other if they have a common face; each black cube must have an even number of neighboring white cubes, and each white cube must have an odd number of neighboring black cubes.*

So we arranged the second layer of cubes in such a way that all cubes of the first layer obeyed The Rule. If all cubes of the second layer also satisfy The Rule, we are done. If this is not the case, we have to fill in the third layer so that all cubes of the second layer satisfy The Rule, etc. Does there exist an arrangement of cubes in the first layer for which this process is infinite?

28.2.10.3. Let p and q be odd integers. A $p \times 2q$ rectangular billiard table has pockets at each vertex and in the midpoints of sides of length $2q$. A ball is shot from a vertex at an angle of 45° to the sides. Prove that the ball will wind up in one of the middle pockets. (Cf. Problem 28.2.9.2.)

28.2.10.4. All integers 1 to $2n$ are written in a row in an arbitrary order. Then to each integer the number of its place in the row is added. Prove that among the sums obtained there are at least two that have the same remainders after division by $2n$.

28.2.10.5*. In a box there are two smaller boxes, in each of which there are two more boxes, etc. There are 2^n smallest boxes, each contains a coin. Some of these coins are heads up, some tails up. In one move, any box may be turned upside down, together with everything it contains. Prove that in not more than n moves the boxes may be so arranged that the number of coins with heads up is equal to the number of coins with tails up.

Grade 11

28.2.11.1. Find all primes of the form $p^p + 1$ and of not more than 19 decimal digits, where p is a positive integer.

28.2.11.2. Prove that the last digits of numbers of the form n^n, where n is a positive integer, constitute a periodic sequence.

28.2.11.3*. Given plane P and two points A and B on either side of it, construct a sphere through these points that cuts in P a disc of the smallest possible area.

28.2.11.4. Consider a non-convex and non-selfintersecting polygon on a plane. Let D be the union of points on those diagonals of the polygon that do not go outside its limits (i.e., are either entirely inside it or partly inside and partly on its boundary; the endpoints of these diagonals should also belong to D). Prove that any two points of D may be connected by a broken line contained entirely within D.

28.2.11.5. Each square of an $M \times M$ table contains nonnegative integers so that if a 0 is at the intersection of a row and a column, then the sum of the numbers in this row and this column is not less than M. Prove that the sum of all numbers in the table is not less than $\frac{M^2}{2}$.

Olympiad 29 (1966)

Tour 29.1

Grade 8

29.1.8.1. Find the locus of the centers of all rectangles inscribed in a given $\triangle ABC$ with one side of the rectangles on AB.

29.1.8.2. Find all two-digit numbers that being multiplied by an integer yield a product whose penultimate digit is 5.

29.1.8.3. See Problem 29.1.9-11.1.

29.1.8.4. See Problem 29.1.9-11.5.

29.1.8.5*. From a complete set of 28 dominoes, we remove all dominoes that have 6 dots (on any half, not in sum). Is it possible to arrange the remaining dominoes in a chain?

Grades 9 – 11

29.1.9-11.1. Solve in positive integers the system of equations

$$\begin{cases} x + y = zt, \\ z + t = xy. \end{cases}$$

29.1.9-11.2. For what value of k is the expression $A_k = \frac{19^k + 66^k}{k!}$ the greatest?

29.1.9-11.3. We place a convex pentagon inside a circle, so that its vertices are either on the circle or inside it. Prove that at least one of the pentagon's sides is not longer than the side of a regular pentagon inscribed in this circle.

29.1.9-11.4. Prove that the positive integers k, for which $k^k + 1$ is divisible by 30, constitute an arithmetic progression and describe that progression.

29.1.9-11.5. In checkers, what is the greatest number of kings that may be arranged on the black squares of an 8×8 checker-board, so that each king may be jumped by at least one other king?

Tour 29.2

Grade 8

29.2.8.1. Divide a line segment into six equal parts with a ruler and compass constructing not more than eight curves (straight lines or arcs).

29.2.8.2*. Let $a_1 = 1$ and for $k > 1$ define $a_k = [\sqrt{a_1 + a_2 + \dots + a_{k-1}}\,]$, where $[x]$ denotes the integer part of x. Find a_{1000}.

29.2.8.3*. There is a test that for any set of balls can determine whether the set contains any radioactive balls, but but it cannot tell how many of the balls are radioactive. We know that two of the given 19 balls are radioactive. Find both the radioactive balls after 8 tests. (Cf. Problem 29.2.9-11.3 below).

29.2.8.4. A subway system has not more than four stations along each line, not more than three of which are intersections with the other lines. Moreover, not more than two lines meet at any of the intersections. What greatest number of lines can such a system have if it is possible to get from any station to any other station with not more than two train changes?

29.2.8.5*. Prove that there exists k such that the first 4 digits of $k!$ are 1966.

Grades 9 – 11

29.2.9-11.1. See Problem 29.2.8.1.

29.2.9-11.2*. See Problem 29.2.8.2, where $a_1 = 1966$, and find a_{1966}.

29.2.9-11.3*. There is a test that for any set of balls can determine whether the set contains any radioactive balls, but but it cannot tell how many of the balls are radioactive. We know that two of the given 11 balls are radioactive. Prove that fewer than seven tests do not guarantee the discovery of both radioactive balls, whereas one can determine them by seven tests.

29.2.9-11.4. Given a collection of weights 1, 2, ..., 26 g. Select 6 weights so that it is impossible to compose with the help of (some or all of) these 6 weights two piles of equal weight. Prove that it is impossible to select 7 weights with the same property.

29.2.9-11.5. On an 11×11 checker-board 22 squares are marked so that exactly two of the marked squares lie in each column and each row. Two arrangements of marked squares are considered *equivalent* if in any number of permutations of the columns and/or (independent) permutations of rows one arrangement can be obtained from the other. How many nonequivalent arrangements of marked squares are there?

Olympiad 30 (1967)

Tour 30.1

Grade 8

30.1.8.1. Do there exist two consecutive positive integers such that the sum of the digits of each of them is divisible by 125? Either find the smallest such pair of numbers or prove that they do not exist.

30.1.8.2. Given $\triangle ABC$, find a point M on side AB or its extension so that the sum of the radii of the circles circumscribed abound $\triangle ACM$ and $\triangle BCM$ is minimal.

30.1.8.3. A spy must cipher his (her) message. For this (s)he wants to divide all decimal "words" — sets of ten signs, each either a dot or a dash — into two groups, so that any two words of the same group differ in not less than three places. Either describe such a division or prove that the spy's assignment is hopeless.

30.1.8.4. Given $\triangle ABC$, find the locus of all points M for which $\triangle ABM$ and $\triangle BCM$ are isosceles.

30.1.8.5. In the city of Fuchs, Ostap Bender organized a distribution of elephants to people. There were present 28 trades union members and 37 non-members. Ostap distributed the elephants so that the shares of all trades union members were equal, and the shares of non-members were also equal. It turned out that there existed only one such distribution (of all elephants). What greatest number of elephants could O. Bender have had?

Remark. O. Bender is a main character of a satiric dilogy by I. Il'f and E. Petrov. It became immensely popular since it had been first published in the late '20s. A part of it, translated into English in the '30s under the title *The Little Golden Calf*, is a series of adventures of Jeff Peters and Andy Tuckers type. Among O. Bender's rackets were paid popular lectures and prophesies. An announcement written in the spirit of 'Royal Nonsuch' from Huck Finn's adventures said that after the lecture O. Bender was to distribute elephants. At the time of Bender's adventures the majority of the audience — hicks and red-necks — had no hope to get 'elephants'; all of them, even the "privileged" trades union members, were primarily interested not even in future-telling but in the immediate *now* and prone to ask Bender "Why there is no butter on sale?" or "Are you a Jew?" (Miraculously, this book is even more timely now, in late '90s.)

Grade 9

30.1.9.1. A maze consists of n circles, all tangent to straight line L at M. All circles are on the same side of L and their lengths form a geometric progression with denominator 2. Two pedestrians enter the maze at different moments. Their speeds are equal but the directions of their trajectories are different. Each of them circumvents all circles, starting with the smaller, in increasing order and, having circumvent the greatest, enters the smallest one again. Prove that the pedestrians will meet each other. (See Fig. 62.)

FIGURE 62. (Probl. 30.1.9.1)

FIGURE 63. (Probl. 30.1.9.2)

30.1.9.2. Is it possible to cut a square pie into 9 pieces of equal area by choosing two points inside the square and connecting each of them by straight cuts with all vertices of the square? If it is possible, how can the two points be found? (See Fig. 63.)

30.1.9.3. See Problem 30.1.8.2.

30.1.9.4. Consider integers with the sum of their digits divisible by 7. What is the greatest difference between two consecutive such integers?

30.1.9.5. We transpose the first 12 digits of a 120-digit number in all possible ways, and out of the 120-digit numbers obtained we randomly choose 120 numbers. Prove that the sum of the chosen numbers is divisible by 120.

Grade 10

30.1.10.1. Inside a square consider k points ($k > 2$). Into what least number of triangles must we divide the square for each triangle to contain not more than one point?

30.1.10.2. Prove that in a circle of radius 1 there may be not more than 5 points such that the distance between any two of them is greater than 1.

30.1.10.3. Prove that the equation $19x^3 - 17y^3 = 50$ has no integer solutions.

30.1.10.4. An infinite pie occupying all space has raisins of diameter 0.1 with centers at the points with integer coordinates. Finitely many planes cut the pie. Prove that there still exists an uncut raisin.

30.1.10.5. Of the first k primes 2, 3, 5, $\dots$, p_k $(k > 4)$ we compose all possible products every prime entering the product not more than once, e.g. $3 \cdot 5$, $3 \cdot 5 \cdot \dots \cdot p_k$, $11 \cdot 13$, 7, etc. Let S be the sum of all such products. Prove that $S + 1$ is the product of more than $2k$ prime factors.

Tour 30.2

Grade 7

30.2.7.1. In $\triangle ABC$, consider heights AE, BM and CP. It turns out that $EM \parallel AB$ and $EP \parallel AC$. Prove that $MP \parallel BC$.

30.2.7.2. Four electric bulbs must be installed above a square skating-rink in order to illuminate it completely. At what least height may the lamps be hung, if each lamp illuminates a disc of a radius equal to the lamps' height from the floor?

30.2.7.3. Prove that there exists an integer q such that the decimal expression of $q \cdot 2^{1000}$ contains no zeros. Cf. Problem 30.2.8.1.

30.2.7.4. A number y is obtained from a positive integer x by a permutation of its digits. Prove that $x + y \neq 99...99$ (1967 nines).

30.2.7.5. Spotlights, each of which illuminates a right angle, are placed at four given points on a plane. The sides of the illuminated angles may be directed only to the north, south, west or east. Prove that the spotlights may be so directed that they illuminate the entire plane; see Fig. 64.

FIGURE 64. (Probl. 30.2.7.5)

Grade 8

30.2.8.1. See Problem 30.2.7.3 for $q \cdot 2^{1967}$.

30.2.8.2. Denote by $d(N)$ the number of divisors of N (1 and N are also considered as divisors of N). Find all N such that $N/d(N) = p$ is a prime.

30.2.8.3. A square is constructed on each side of a right triangle and the entire figure is inscribed in a circle. For what right triangles is this possible? Cf. Problem 30.2.9.3.

30.2.8.4. A black king and 499 white rooks stand on a 1000×1000 chess-board. Black and white pieces move in turn. Prove that whatever the strategy of the whites, the king may always commit suicide after several moves (i.e., get in the way of a rook).

30.2.8.5. Seven children decided to visit seven movie theaters one day. At each movie theater the shows started at 9.00, 10.40, 12.20, 14.00, 15.40, 17.20, 19.00 and 20.40 (altogether 8 shows). Six children went together to each show, and each time the seventh kid (not necessarily the same person each time) decided to be independent and went to another movie theater. By night each kid had been to each of the seven theaters chosen. Prove that there was a show in each movie that none of the children saw.

Grade 9

30.2.9.1. A number y is obtained from a positive integer x by a permutation of its digits and $x + y = 1\underbrace{00\dots00}_{200 \text{ zeros}}$. Prove that x is divisible by 50.

30.2.9.2. Given a sequence of positive integers $x_1, x_2, \dots, x_n$, each not greater than M, and with $x_k = |x_{k-1} - x_{k-2}|$ for all $k > 2$. Determine the greatest possible length of this sequence.

30.2.9.3. We construct a square outwards on each side of a right $\triangle ABC$. It turns out that all vertices of the squares distinct from A, B, C lie on one circle. Prove that $\triangle ABC$ is isosceles. Cf. Problem 30.2.8.3.

30.2.9.4*. Let $\overleftarrow{N}$ be the number N written in reverse order, e.g., $\overleftarrow{1967} = 7691$, $\overleftarrow{450} = 54$. For any positive integer N divisible by K the number $\overleftarrow{N}$ is also divisible by K. Prove that K is a divisor of 99.

30.2.9.5. A king of Spain decided to rearrange the portraits of his predecessors that hung in a circular tower of his castle. He ruled, however, that only two adjacent portraits be interchanged in one day and that, moreover, they could not be the portraits of the kings one of whom immediately succeeded the other. Two distinct arrangements that could have been obtained from each other, if the castle could rotate, were ordered to be considered as *identical*. Prove that, following this Rule, the king can always find any new arrangement of the portraits regardless of their initial positions.

Grade 10

30.2.10.1*. Let m and $n-k$ be relatively prime given numbers. We are given an $n \times n$ table filled in with numbers as follows: numbers 1, 2, ... , n are written in the first row; if some row contains the numbers

$$a_1, \dots, a_k, a_{k+1}, \dots, a_m, a_{m+1}, \dots, a_n$$

then the next row contains the same numbers but in the following order:

$$a_{m+1}, \dots, a_n, a_{k+1}, \dots, a_m, a_1, \dots, a_k.$$

Prove that, after the table is filled, each column contains all numbers 1 to n.

30.2.10.2. See Problem 30.2.9.3.

30.2.10.3. Is it possible to arrange the numbers 1, 2, ... , 12 on a circle so that the difference between any two adjacent numbers is 3, 4 or 5?

30.2.10.4. Eight spotlights are situated at eight given points in space, each spotlight illuminates a trihedral angle with mutually perpendicular faces. Prove that the spotlights may be turned so as to illuminate the entire space. (Cf. Problem 30.2.7.5.)

30.2.10.5. Consider all possible n-digit numbers, $n \geq 2$, composed of figures 1, 2 and 3. At the end of each of these n-digit numbers we write a 1, 2 or 3 in such a way that if two numbers differ in all the corresponding digits, then we write additional different digits at their ends (one digit each). Prove that there exists an n-digit number which contains only one 1 and at whose end a 1 is written.

Olympiad 31 (1968)

Tour 31.1

Grade 7

31.1.7.1. Number 4 has the following property: when divided by q^2, for any q, the remainder is less than $\frac{q^2}{2}$. Find all numbers with the same property.

31.1.7.2. Arrange 16 numbers in a 4×4 table so that their sum along any vertical, horizontal or diagonal line is equal to zero. We assume that the table has 14 diagonals altogether.

31.1.7.3. Prove that for any three given numbers, each $< 1\,000\,000$, there is a number < 100 that is relatively prime to every one of the given numbers.

31.1.7.4. How may 50 cities be connected by the least possible number of airlines so that from any city one could get to any other by changing airplanes not more than once (i.e., using two planes)?

Grade 8

31.1.8.1. 12 people took part in a chess tournament. After the end of the tournament every participant made 12 lists. The *first list* consisted of the author; the *second list* – of the author and of those (s)he has beaten; and so on; the *12-th list* consisted of all the people on the 11-th list and those they have beaten. It is known that the 12-th list of every participant contains a person who is not on the participant's 11-th list. How many games ended in a draw?

31.1.8.2. Given numbers 4, 14, 24, ... , 94, 104, prove that it is impossible to strike out first one number, then another two, then another three, and then another four, so that after each striking out the sum of the remaining numbers is divisible by 11.

31.1.8.3. Is it possible to inscribe a convex heptagon $A_1A_2A_3A_4A_5A_6A_7$ with angles $\angle A_1 = 140°$, $\angle A_2 = 120°$, $\angle A_3 = 130°$, $\angle A_4 = 120°$, $\angle A_5 = 130°$, $\angle A_6 = 110°$, $\angle A_7 = 150°$ in a circle?

31.1.8.4. Find 100 numbers ssuch that

$$\begin{aligned}&x_1 = 1;\\&0 \le x_2 \le 2x_1;\\&0 \le x_3 \le 2x_2;\\&\dots\dots\dots\dots\dots\\&0 \le x_{99} \le 2x_{98};\\&0 \le x_{100} \le 2x_{99};\end{aligned}$$

so that the expression $S = x_1 - x_2 + x_3 - x_4 + \dots + x_{99} - x_{100}$ is the greatest possible.

31.1.8.5. Is it possible to arrange 1000 segments on a plane so that the endpoints of every segment are on other segments but not at their endpoints?

Grade 9

31.1.9.1. Is there a quadrilateral $ABCD$ of area 1 such that for any point O inside it the area of at least one of the triangles $\triangle OAB$, $\triangle OBC$, $\triangle OCD$, or $\triangle OAD$ is an irrational number?

31.1.9.2. Cf. Problem 31.1.8.5 for 1968 segments.

31.1.9.3. A corridor 100 meters long is covered with 20 rugs of the same width as the corridor and of a combined length of 1000 meters. What greatest number of uncovered parts may the corridor have?

31.1.9.4. Is it possible to select $100\,000$ telephone numbers consisting of 6 digits each so that if we simultaneously strike out the k-th digit ($k = 1, 2, 3, 4, 5, 6$) of every number, we get all numbers $00\,000$ to $99\,999$?

31.1.9.5. Prove that if p and q are primes and $q = p + 2$, then $p^q + q^p \,\vdots\, p + q$.

Grade 10

31.1.10.1*. 100 airplanes (one in the lead, 99 following) take off simultaneously from the same airport. A plane with a full tank of fuel can cover a distance of 1000 km. During a flight, fuel may be transferred from one plane to another. A plane that gave all its fuel to the other planes makes a gliding landing. How should the flight be organized for the leading plane to fly as far as possible?

31.1.10.2. Two people are playing a game. There are two piles containing 33 and 35 candies. A player eats up one of the piles and divides the second one into two (not necessarily equal) parts. If (s)he cannot divide a pile because it only has one candy, (s)he eats the candy and wins. Moves are made in turn. Who will win the game, the one who starts or the other party, and how should they play to win?

31.1.10.3. The Rule states: integers m and n *belong to the same subset* if one can be obtained from the other by striking out two of its adjacent identical digits or two identical groups of digits (for example, the numbers 7, 9 339 337, 93 223 393 447, 932 239 447 belong to the same subset). Is it possible to divide the set of all non-negative integers into 1968 subsets, with at least one number in each, so that the Rule is fulfilled?

31.1.10.4. Using a given sequence of positive numbers $q_1, q_2, \dots, q_n, \dots$, a sequence of polynomials is constructed in the following way:

$$f_0(x) = 1, \quad f_1(x) = x \text{ and } f_{n+1}(x) = (1 + q_n) \cdot x f_n(x) - q_n f_{n-1}(x) \quad \text{for } n \geq 1.$$

Prove that all real roots of these polynomials belong to $[-1, 1]$.

31.1.10.5. Given 4 lines l_1, l_2, l_3, l_4 in space, each pair of them skew and no three of them parallel to one plane. Draw a plane, P, such that the intersection points A_1, A_2, A_3, A_4 of these lines with P make a parallelogram. How many solutions are there?

Tour 31.2

Grade 7

31.2.7.1. The vertices of a regular 1968-gon are marked on a plane. Two players, in turn, connect two vertices of the polygon by a segment, obeying the following Rule: *two points may not be connected if one of them is already connected to a point, and segments already drawn may not be intersected by others.* The player who may not make a move, according to the Rule, loses. How should one play to win? Who wins if both play optimally?

31.2.7.2. On a plane, there are given three points. We select any two of them, draw the perpendicular through the midpoint of the segment connecting them, and reflect all 3 points through this perpendicular. Then we again select two points among all the points, the original ones and their reflections, and repeat the procedure *ad infinitum*. Prove that there exists a straight line on the plane such that all points obtained in the end lie on one side of it.

31.2.7.3. Two painters paint a long straight fence consisting of 100 parts. They come every other day, alternately, painting a fence around one plot red or green. The first painter is color-blind and mixes up the colors; (s)he remembers what part of the fence (s)he has painted and what color (s)he has used. (S)he can also feel the fresh paint left after the second painter, but can not tell its color. The first painter tries to make the number of places where green borders red the greatest possible. What maximal number of such places can (s)he get, whatever the second painter does?

31.2.7.4. Let x and y be unknown digits. The 200-digit number $89\,252\,525 \dots 2\,525$ is multiplied by the number $\overline{444\,x18\,y27}$. It turns out that the 53-rd digit from the right of the product is 1, and the 54-th digit is 0. Find x and y.

31.2.7.5*. A cowboy Jimmy bets with his friends that he can shoot through all the four blades of his fan with one bullet. His fan is constructed so that it can not effectively work as a fan but suits Jimmy fine as a target, see Fig. 65:

FIGURE 65. (Probl. 31.2.7.5)

Each of the four blades is a half-disc. The blades sit on a shaft perpendicularly to it; the distances between the planes of the blades are equal. The diameters bounding the half-discs are slanted with respect to one another. The shaft rotates at the rate of 50 revolutions per second. Jimmy, as a true cowboy, can shoot whenever needed and his bullet may have any (but constant) speed he wants. Prove that Jimmy can win the bet.

Grade 8

31.2.8.1. Let us divide all positive integers into groups so that there is one number in the first group, two numbers in the second, three numbers in the third and so on. Is it possible to do this so that the sum of elements in every group is the 7-th power of an integer?

31.2.8.2*. Two straight lines on a plane meet at an angle of α. A flea sits on one of the lines. Every second it jumps from the line it sits on to the other line. (The intersection point is considered to lie on both lines.) It is known that the length of each jump is equal to 1 and that the flea never returns to the place where it was just before. After a while the flea returns to its initial point. Prove that α has a rational number of degrees; see Fig. 66.

FIGURE 66. (Probl. 31.2.8.2)

FIGURE 67. (Probl. 31.2.8.3)

31.2.8.3*. A round pie is cut by a special cutter that cuts off a fixed sector of the angle measure α, turns this sector upside down, and then inserts back; after that the whole pie is rotated through an angle of β.

Given $\beta < \alpha < 180°$, prove that after a finite number of such operations (the beginning of the first and the second operations are shown on Fig. 67) every point of the pie will return to its initial place.

31.2.8.4. Consider a paper scroll of bus tickets numbered 000 000 to 999 999. The tickets with the sum of the digits in the even places equal to the sum of the digits in the odd places are marked blue. What is the greatest difference between the numbers on two consequitive blue tickets?

31.2.8.5. The land Farra lies on 1 000 000 000 islands. Boats ply the routes between certain islands every day. Boat routes are organized so that one can get to any island from any other island (it could take a few days). The timetable allows a spy and Major Pronin[1] only one passage per day and there is no other way to get from one island to another except via regular boats. The spy never boards a boat on the 13-th of a month, but Major Pronin is not superstitious and, besides, informers always tell Major Pronin where the spy is. According to the Rule Major Pronin *catches* the spy if they are both on the same island. Prove that Major Pronin will catch the spy.

Grade 9

31.2.9.1. Consider a regular pentagon $A_1A_2A_3A_4A_5$ on a plane. Is there a set of points on a plane with the following property: through any point outside the pentagon it is possible to draw a segment whose endpoints belong to the set and it is impossible to do so through points inside the pentagon.

31.2.9.2. We mark point O_1 on a unit circle and, using O_1 as the center, we mark (by means of a compass) we mark point O_2 on the circle (clockwise starting with O_1). Using point O_2 as a new center, we repeat the procedure in the same direction with the same radius; and so on. After we had marked point O_{1968} a circle is cut through each mark so we get 1968 arcs. How many different arc lengths can we thus procure?

[1] A notorious hero of Soviet spycatchers.

31.2.9.3. The following game with chess pieces is played. Two kings stand in the opposite corners of the chessboard: the white king on square $a1$, the black king on square $h8$. Players move in turns (a white begins). A player may move his/her king to any adjacent square, if it is vacant, according to the following Rule: Theleast number of king's moves needed to get from one square to another is called *the distance between the squares*; thus, at the beginning of the game the distance between the kings was 7 moves. It is not allowed to increase the distance between the kings.

To *win* is to get one's king to the opposite side of the chessboard (the white king to the vertical h or the eighth horizontal, the black king to the vertical a or the first horizontal). How should one play to win? Who wins if plays optimally?

31.2.9.4. Prove that if $a^n - b^n \,\vdots\, n$, where a, b, n are positive integers, $a \neq b$, then $\frac{a^n-b^n}{a-b} \,\vdots\, n$.

31.2.9.5*. Let N be a positive integer. We perform with N the following operation: we write every digit of N on a separate card (we may also add, or strike out, any number of cards on which a digit 0 is written), and then divide these cards into two piles. In each pile, we arbitrarily arrange the cards in a row and let N_1 be the sum of the two numbers obtained by reading these rows of digits. We perform the same operation with N_1, and so on. Prove that it is possible to obtain a one-digit number in ≤ 15 steps.

Grade 10

31.2.10.1. It is known that moving a unit segment of length 1 as a solid rod inside a convex polygon M we can turn the segment by any angle. Prove that a disk of radius $\frac{1}{3}$ can be placed inside M.

31.2.10.2. Some numbers are written in a 10×10 table A. Denote the sum of the numbers in the first row by s_1, the sum of the numbers in the second row by s_2, and so on. Similarly, the sum of the numbers in the first column is denoted by t_1, in the second column by t_2, and so on. A new 10×10 table B is filled in by the following Rule: the lesser of the numbers s_i and t_j is written in the j-th square of the i-th row. It turns out that one can index the squares of table B from 1 to 100 so that the number in the k-th square is $\leq k$. What is the greatest possible value of the sum of all the numbers in table A?

31.2.10.3. Prove that for some k the system

$$\begin{cases} x_1 + x_2 + \cdots + x_k = 0, \\ x_1^3 + x_2^3 + \cdots + x_k^3 = 0, \\ x_1^5 + x_2^5 + \cdots + x_k^5 = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_1^{17} + x_2^{17} + \cdots + x_k^{17} = 0, \\ x_1^{19} + x_2^{19} + \cdots + x_k^{19} = 0, \\ x_1^{21} + x_2^{21} + \cdots + x_k^{21} = 1, \end{cases}$$

has a real solution.

31.2.10.4. An equilateral triangle ABC is divided into N convex polygons so that any straight line intersects not more than 40 of them. (A line *intersects* a polygon if the line and the polygon have a common point, for example, a vertex of the polygon.) Can N be greater than one million?

31.2.10.5. On the surface of a cube 100 distinct points are marked with chalk. Prove that it is possible to place the cube onto the same place of a black desk in two ways so that the chalk imprints on the desk would be different. (We assume that a marked point on an edge or vertex of the cube also leaves an imprint.)

Olympiad 32 (1969)

Tour 32.1

Grade 7

32.1.7.1. A white rook is chasing a black bishop across a 3×1969 chessboard (they move in turns according to common rules). How should the rook play to jump the bishop if the white makes the first move? (Cf. Problem 32.1.8.3.)

32.1.7.2. Once upon a time a castle was fortified with a triangular wall. Every side of the triangle was trisected and towers E, F, K, L, M, N (listed here as we tour the wall clockwise) were built at the points of trisection and in addition to towers at the vertices A, B, C of the triangle. Since then all the walls and towers, except towers E, K, M, perished. How to recover the location of towers A, B, C from the remaining towers?

32.1.7.3. An international soccer tournament took place in Chile in February. The home team "Colo-Colo" won the first place with 8 points. "Dynamo" Moscow was second with 1 point less. A Brazilian team "Corinthians" was the third with 4 points. The fourth was a Yugoslavian team, "Crvena Zvezda", also with four points. Prove that from these data it is possible to exactly reconstruct how many other teams participated in the tournament and how many points they got.

32.1.7.4. Prove that no power of 2 can end with four identical digits.

32.1.7.5. 1000 regular wooden 100-gons are nailed to the floor. We stretch a rope around the entire system using the nails. Prove that the polygon formed by the rope has more than 99 vertices. (Cf. Problem 32.1.8.2.)

Grade 8

32.1.8.1. See Problem 32.1.7.4.

32.1.8.2. 57 regular wooden 57-gons are nailed to the floor. We stretch a rope around the entire system using the nails. Prove that the polygon formed by the rope has more than 56 vertices. (Cf. Problem 32.1.7.4.)

32.1.8.3. A white rook is chasing a black knight across a 3×1969 chessboard (they move in turns according to common rules). How should the rook play to jump the bishop if the white makes the first move? (Cf. Problem 32.1.7.1.)

32.1.8.4. Given segment AB. Find the locus of points C such that $m_b = h_a$ in $\triangle ABC$ (see Fig. 68).

Figure 68. (Probl. 32.1.8.4)

32.1.8.5. Is it possible to write 20 numbers in a row so that the sum of any three consecutive numbers is strictly positive, and the sum of all 20 numbers is strictly negative? (Cf. Problem 32.1.9.3.)

Grade 9

32.1.9.1. Find all positive integers x such that we can subtract the same nonzero digit a from each digit of x (this means that every digit of x is not less than a) and get the number $(x-a)^2$.

32.1.9.2. The Tolpygo Island is of the form of a polygon. There are several countries are on the island. Each country is of the form of a triangle and every two countries bordering along (parts of) their sides have an entire side in common, i.e., a vertex of one triangle never lies on the side of another triangle (except at a vertex). Prove that it is possible to paint the map of the island three colors, one color for each country and so that any two bordering countries are painted different colors.

32.1.9.3. Is it possible to write 50 numbers in a row so that the sum of any 17 consecutive numbers is strictly positive, and the sum of any 10 consecutive numbers is strictly negative? (Cf. Problem 32.1.8.5.)

32.1.9.4. See Problem 32.1.7.4.

32.1.9.5. There are 500 towns in Tsar Dodon's kingdom, each in the form of a regular 37-angled star, with towers at the vertices. Tsar Dodon decides to wall the towers in a convex wall so that every segment of the wall connects two towers. Prove that the wall will consist of not less than 37 segments, provided we count segments on the same straight line only once.

Grade 10

32.1.10.1. Particles emitted by a betatron move along a straight line through two identical thin hoops situated in perpendicular planes so that each hoop passes through the center of the other. Along what straight line should the particles move so as to be as far from the hoops as possible, i.e., so that the shortest distance between the particle and the hoops were the longest possible?

32.1.10.2. An infinite sequence of numbers $a_1, \dots, a_n, \dots$ is periodical, with period 100, i.e., $a_1 = a_{101}$, $a_2 = a_{102}, \dots$. It is known that $a_1 \geq 0$, $a_1 + a_2 \leq 0$, $a_1 + a_2 + a_3 \geq 0$, and, generally, the sums $a_1 + a_2 + \dots + a_n$ are alternately non-negative if n is odd or non-positive if n is even. Prove that $|a_{99}| \geq |a_{100}|$.

32.1.10.3. A pack of cards with their backs down is arranged in a row. If two cards of the same suit are next to one another, or have just one card between them, then the Rule allows us to remove the extreme left one. Besides, the Rule allows us to add any number of cards from other packs to the right hand side of the row. Prove that it is possible to add or remove cards so that in the end only 4 cards are left.

32.1.10.4. Is there a real number h such that $[h \cdot 1969^n]$ is not divisible by $[h \cdot 1969^{n-1}]$ for any positive integer n?

32.1.10.5. Given square $ABCD$, find the locus of points M such that $\angle AMB = \angle CMD$.

Tour 32.2

Grade 7

32.2.7.1. m and n are two positive integers. All different divisors of m — the numbers $a, b, \dots, k$ — and all different divisors of n — the numbers $s, t, \dots, z$ — are written out (m, and n, and 1 are included). It turns out that

$$a + b + \dots + k = s + t + \dots + z$$

and

$$\frac{1}{a} + \frac{1}{b} + \dots + \frac{1}{k} = \frac{1}{s} + \frac{1}{t} + \dots + \frac{1}{z}.$$

Prove that $m = n$.

32.2.7.2. We strike out two consecutive digits a and b (a preceding b) of the number

$$N = 123456789101112\dots9989991000$$

and replace them with the number $a + 2b$; the number a may be an unwritten zero if b is the first digit of N. (Clearly, there are many ways to perform this operation.)

The same operation is repeated with the numbers obtained, and so on. (For example, in one step the numbers 218307, 38307, 117307, 111407, 11837, 118314 may be obtained from 118307.) Prove that several application of this operation turn the given number into 1.

32.2.7.3. A crook acquired a square lot, fenced it in and got permission from the credulous president of his collective farm to perform a few times the following operation: draw a straight line through any two points of the fence, destroy the part of the fence between these two points on one side of the line, and build a new part of the fence symmetric to the destroyed part with respect to the line. Can the crook increase the area of his patch with such manipulations? (See Fig. 69)

FIGURE 69. (Probl. 32.2.7.3)

32.2.7.4. Two players play the following game. Every player, in turn, chooses 9 numbers in the sequence 1, 2, 3, … , 100, 101 and strikes them out. After eleven turns there are two numbers left. The second player then pays the first one the difference between the two numbers in roubles. Prove that the first player can always win at least 55 roubles, no matter how the second one plays.

Note. Students who play this game will be fired from the school.

32.2.7.5. A pearl of radius 3 mm is baked inside a round pudding of radius 10 cm. The Rule allows us to cut the pudding along a straight line with a sharp knife into two (equal or unequal) parts. If the pearl is not found in one cut, (does not occur under the knife), one of the parts may be cut again; if this does not help, it is allowed to cut one of the three obtained parts and so on. Prove that it is possible not to find the pearl after 32 cuts, no matter how they are made. Prove that it is possible to make 33 cuts so that the pearl will be found, no matter where it is.

Grade 8

32.2.8.1. See Problem 32.2.7.2.

32.2.8.2. A white knight is on square $a1$ of a chessboard. Two players take turns daubbing one square of the chessboard at a time with bauxite glue. They must do this in such a way that the knight could move according to usual rules onto any clean square without getting stuck. The *loser* is the player who cannot make a move. Who wins provided both play optimally?

32.2.8.3*. Two regular pentagons have one common vertex. The vertices of both pentagons are numbered clockwise 1 to 5, the number of the common vertex being 1. The vertices with the same numbers are connected by straight lines. Prove that these four lines meet at one point. (See Fig. 70)

FIGURE 70. (Probl. 32.2.8.3)

32.2.8.4. Finite sequences of positive integers are composed so that every next number is greater than the square of the preceding one, and the last number of each sequence is equal to 1969 (sequences may have different lengths). Prove that there are fewer than 1969 different such sequences.

32.2.8.5*. 100 cubes are arranged in a row, 77 black and 23 white among them. They are arranged *approximately uniformly*, i.e., if a string of consecutive cubes is marked at one place of the row and the same length is then marked at another place (the strings can intersect), then the number of black cubes in the first string differs from the number of black cubes in the second string by not more than 1; and if the first string begins on the left end of the row, then the number of black cubes in it is *not greater* than that of the second string; and if it terminates the row, then the number of black cubes in it is *not less* than that of the second string. Prove that if another string of 77 black cubes and 23 white cubes satisfies the same conditions, then the white cubes in it occupy the same places as in the first string.

Grade 9

32.2.9.1. Two players play the following game: taking turns, they strike out one number each from the set $\{1, 2, 3, \dots, 27\}$, until there are only two numbers left. If the sum of these numbers is divisible by 5, then the first player wins, otherwise the second one wins. Who wins an optimally played game (the player who begins or the second one)?

32.2.9.2. On a plane, the circle may be traced around a coin. The Rule allows us to use this coin to draw a circle through one or two given points that are sufficiently close to one another. Three points are given on the plane; they can be covered by the coin, and they are not all on one straight line or on the circle equal to the outer circle of the coin. Using the coin construct a fourth point, such that all four points are vertices of a parallelogram.

32.2.9.3*. There are 2^{n-1} different sequences of length n built of 0's and 1's so that for any three sequences there exists a number p such that the p-th digit in all three sequences is 1. Prove that there is exactly one place in each sequence that a 1 occupies.

32.2.9.4*. The time of a new presidential election is approaching in the country Anchuria, where President Miraflores now rules. There are 20 million voters in the country but only 1 percent supports Miraflores (the Anchurian military). Naturally, Miraflores wants to be reelected but he wants the election to look democratic. A "democratic vote", according to Miraflores, is like this: all voters are divided into equal groups; each of these groups is again divided into some number of equal groups (the groups of voters may be subdivided at distinct stages of the election into distinct numbers of smaller subgroups); then these groups are divided once again, and so on. A representative from each smallest group — an elector — is elected for voting within the greater group; electors of this greater group elect a new elector for voting in the group which is greater than this one, and so on. And, finally, representatives from the greatest groups elect the President. Acording to the constitution, Miraflores has the right to divide all voters into the groups as he chooses and he instructs his supporters as to how they are to vote. Is it possible for him to organize "democratic" elections so as to be elected?

32.2.9.5. Consider a regular 1000-gon. Its nonintersecting diagonals divide it into triangles. Prove that among these diagonals there are at least 8 of different lengths.

Grade 10

32.2.10.1. Two wizards play the following game. Numbers 0, 1, 2, ... , 1024 are written out. The first wizard strikes out 512 numbers of (s)he chooses. Then the second wizard strikes out 256 of the remaining numbers. Then the first wizard strikes out 128 of the remaining numbers, and so on. The second wizard strikes out one number during the tenth move; two numbers remain. After that the second wizard pays the first wizard the absolute value of the difference between these numbers in roubles as the fee for the instruction in the exciting game.

How should the first wizard play to gain as much as possible? How should the second wizard play to lose less? How much will the second wizard have to pay the first wizard if both play optimally? (Cf. Problems 32.2.7.4 and 32.2.9.1).

32.2.10.2. A rigid wire is bent to form an equilateral triangle, and its endpoints are soldered. The Rule allows to bend a piece of the wire between any two of its points in such a way that the bent piece is symmetrical to the original one, with respect to the straight line through these two points (if these points coincide, then any line through them will do). This operation may be repeated. Is it possible to obtain a regular hexagon with the same perimeter in several such operations? (Cf. Problem 32.2.7.3, See Fig. 71)

FIGURE 71. (Probl. 32.2.10.2)

32.2.10.3. See Problem 32.2.7.2 with the circle replaced with a sphere of radius 20 cm, and the numbers of cuts — 32 and 33 — replaced with 65 and 66, respectively.

32.2.10.4. Numbers whose sum equals zero are written on the squares of an 8×8 chessboard. Every square is then divided by a vertical and a horizontal line into four square cells. Is it possible to write numbers in these cells so that

a) there are zeros in all the cells along the sides of the chessboard;

b) the sums of the numbers in the four cells of each square are equal to the number written in the square before;

c) the sums of the numbers in the four cells at every vertex of any original square is zero?

32.2.10.5*. Arrange 1969 cubes in a row, some of them (any number between 0 and 1969) white and the rest black, so that the colors are distributed *approximately uniformly* (see Problem 32.2.8.5). Prove that there exist at least 1970 different arrangements of the cubes which meet this requirement.

Olympiad 33 (1970)

Tour 33.1

Grade 7

33.1.7.1. Two black checkers are positioned on two neighboring black squares on the diagonal of an infinite (in two perpendicular directions) chessboard. Is it possible to place several black checkers and a white one on the chessboard so that the white checker could jump all black checkers in one move?

33.1.7.2. We number 99 cards 1 through 99. Then we shuffle the cards, lay them out with the blank sides up and number the blank sides 1 through 99. We sum the two numbers of every card and multiply the 99 sums. Prove that the product is an even number.

33.1.7.3. Point O lies inside an equilateral triangle ABC. It is known that $\angle AOB = 113^\circ$, $\angle BOC = 123^\circ$. Find the angles of the triangle whose sides are equal to segments OA, OB, OC; see Fig. Probl. 33.1.7.3.

Figure 72. (Probl. 33.1.7.3)

33.1.7.4. A set has 100 weights, the difference between every two of them is ≤ 20 g. Prove that it is possible to put the weights on the pans of a balance, 50 weights on each pan, so that the difference between the weights of the pans is ≤ 20 g.

33.1.7.5. There are 1000 cottages in a town X; just one person occupies each cottage. One day, every man moves to another cottage and every cottage has again one occupant. Prove that it is possible to paint all 1000 cottages blue, green or red so that, for every person, the color of his/her new new cottage is distinct from the color of the old cottage.

Grade 8

33.1.8.1. See Problem 33.1.7.2.

33.1.8.2. The circle is inscribed in pentagon $ABCDE$ whose sides are integer numbers and $AB = CD = 1$. Find the length of the segment BK, where K is the tangency point of BC with the circle.

33.1.8.3. There are 16 black points on a rectangular piece of paper. We connect a pair of points by the segment. Consider a rectangle one of whose diagonals is this segment and whose sides are parallel to the sides of the paper. We paint the rectangle red (black points are visible through the red paint). We do so with every pair of points and get a painted figure on the paper.

How many sides can the figure have for various positions of the points?

33.1.8.4. Each pan of a balance has k weights, numbered from 1 to k. The left pan is heavier. It turns out that if we interchange the places of any two weights with the same number, then either the right pan becomes heavier or the two pans reach an equilibrium. For what k this is possible?

33.1.8.5. 12 players took part in a tennis tournament. It is known that every two of them played with one another only once and that there was no player who was always beaten. Prove that among these 12 there are players A, B, and C such that B was beaten by A, C by B, and A by C.

Grade 9

33.1.9.1. 113 kings lived each in his own palace along a straight road. Every morning one of the kings gave a reception which all the others attended, and every evening the servants transported the kings back home. In this way they lived for a year without doing anything else and lieve of absence. Prove that during this year one of the kings who lived at one of the road's ends collected the biggest milage.

33.1.9.2. What is the greatest number of black checkers that one can place on on black squares of an 8×8 checker-board so that a white checker can jump all of them in one move without becoming a king?

33.1.9.3. A given 999-digit number is such that erasing all but any 50 of its successive digits yields a number (that may begin with zeroes or just be zero) divisible by 2^{50}. Prove that the given number is divisible by 2^{999}.

33.1.9.4. Construct triangle $\triangle ABC$ given the radius of the circumscribed circle and the bisector of angle $\angle A$, and knowing that $\angle B - \angle C = 90°$.

33.1.9.5. A wise cockroach who cannot see farther than 1 cm decided to find the Truth. The latter is located at a point D cm away from the cockroach. The cockroach can move step by step, each step not longer than 1 cm, and after each step the cockroach is told whether (s)he is closer to the Truth or not. The cockroach remembers everything, in particular, the directions of his/her steps. Prove that (s)he can find the Truth taking not more than $\frac{3D}{2} + 7$ steps.

Grade 10

33.1.10.1. Given 19 weights, each of an integer mass (in grams) that does not exceed 70 grams, prove that it is impossible to compose more than 1230 different masses of these weights.

33.1.10.2. Two non-intersecting circles O_1 and O_2 are inscribed into angle ABC. Denote by M the tangent point of O_1 and BA, and by P the tangent point of O_2 and BC. Prove that the chords that circles O_1 and O_2 intercept on straight line MP are of equal length. (See Fig. 73)

FIGURE 73. (Probl. 33.1.10.2)

33.1.10.3. We strike out the first digit of the number 2^{1970} and add it to the obtained number. We perform the same operation with the resulting number, and so on, until we get a 10-digit number. Prove that this 10-digit number has two identical digits.

33.1.10.4. Given 200 points on a plane, no three of which are on the same straight line, find whether it is possible to number these points 1 to 200 so that every two of the hundred straight lines through points 1 and 101, 2 and 102, ... , 100 and 200 intersect.

33.1.10.5. There are crosses in some of the squares of a 100×100 table. It is known that there is at least one cross in every row and in every column. Prove that it is possible to mark 10 rows and 10 columns so that if we erase all crosses in the marked rows and columns, at least one cross will still be left in every unmarked row and column.

Tour 33.2

Grade 7

33.2.7.1. Prove that if a positive integer k is divisible by 999 999 999, then there are more than 8 non-zero digits in its decimal expression.

33.2.7.2. 100 points are marked on a circle of radius 1. Prove that it is possible to find a point on the circle such that the sum of the distances from it to all the other marked points is greater then 100.

33.2.7.3. In a park, 6 narrow alleys of equal length are arranged as the sides and medians of a square. A boy Kolya is running away from his Mother and Father along these alleys. All three can see each other at all times. Can the parents catch the boy if he runs three times faster than any of his parent?

33.2.7.4. A straight cut divides a square piece of paper into two parts. Another straight cut divides one of the parts into two parts. One of the three pieces of paper obtained is again cut into two parts along a line, and so on. What least number of cuts must one do in order to obtain 73 various (perhaps, equal) 30-gons? (Cf.Problem 33.29.4.)

33.2.7.5. King Louis distrusted some of his courtiers. He made a full list of his courtiers and told every one of them to keep an eye on one of the rest. The first one was to spy on the courtier who was spying on the second, the second one was to spy on the one who was spying on the third, and so on, the penultimate one was spying on the courtier who was spying on the last, and the last was spying on the one who was spying on the first. Prove that King Louis had an odd number of courtiers.

Grade 8

33.2.8.1. There are n points inside a circle of radius 1 m. Prove that there exists a point inside the circle or on its perimeter such that the sum of the distances between it and all the other points is not less than n m. (Cf. Problem 33.2.7.2).

33.2.8.2. A monkey ran away from its cage in a small zoo. Two guards are trying to catch it. Both of the guards and the monkey obey The Rule and run only along the paths. There are 6 straight paths in the zoo: 3 long paths form an equilateral triangle, 3 shorter ones connect the midpoints of its sides. Every moment the monkey and the guards can see each other. At the beginning the guards are at one vertex of the triangle and the monkey at another one. Can the guards catch the monkey if the monkey runs three times faster than the guards? (Cf. Problem 33.2.7.3.)

33.2.8.3. In a park grow 10 000 trees. They had been square-cluster[1] planted in 100 rows with 100 trees in each row. What maximum number of these trees can one cut down under the following Rule: standing on any stump, one should be unable to see any other stump behind the trees? The trees are considered to be sufficiently thin.

33.2.8.4. On a roll of paper tape there are written 80 non-zero digits. We cut the tape across into several strips so that there is more than one digit on each strip. Then we add the numbers formed by the digits on each strip. Prove that there exist two distinct ways of cutting the tape to get equal sums.

33.2.8.5. A flat corridor of width 1 m is of the shape of letter Γ and infinite in both directions. There is a flat piece of rigid wire of the form of a nonclosed brocken line. Prove that if the distance between the endpoints of the wire is $> 2 + 2\sqrt{2}$ m, then it is impossible to carry the wire along the whole length of the corridor without tilting. Cf. Problem 33.2.9.2.

Grade 9

33.2.9.1. A toy railroad has n components each in the form of a quarter of a circle with radius 10 cm. Their endpoints are joined in succession so that they fit to form a smooth track. Prove that it is impossible to construct a railroad that would begin and end in the same place, with its first and the last components forming an angle of 0, as shown on Fig. 74.

33.2.9.2. A flat infinite L-shaped corridor is of width 1 m. What is the greatest possible distance between the endpoints of a length of rigid wire (not necessarily straight but flat) such that it is possible to pull the wire through the corridor without tilting? (See Fig. 75)

33.2.9.3. There are plus signs in all squares of a 100×100 table. The Rule permits to simultaneously change the signs in all squares of any one row or column. Is it possible to get 1970 minus signs under the Rule?

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[1] A method of planting advocated by the then ex-leader of the Soviet Union, N. Khrushchev, as the most advanced and best suited to overtake America in agricultural production. The method was abolished. (Perhaps, unwisely?)

FIGURE 74. (Probl. 33.2.9.1)

FIGURE 75. (Probl. 33.2.9.2)

33.2.9.4. A straight line cuts a square piece of paper into two parts. Another straight line cuts one of the parts into two parts. One of the three pieces of paper obtained is again cut into two parts along a line, and so on. What least number of lines must be drawn in order to obtain 100 various (perhaps, (perhaps, identical) 20-gons? Cf. Problem 33.2.7.4.

33.2.9.5. Three spiders and a wingless fly are crawling along the edges of a wire cube. The top speed of the fly is three times that of the spiders. At the beginning, all spiders sat at one vertex of the cube and the fly at the opposite vertex. Can the spiders catch the fly? (The spiders and the fly see each other at all times.)

Grade 10

33.2.10.1*. A 19-hedron is circumscribed around a sphere of radius 10. Prove that on the surface of the polyhedron there are two points with the distance between them ≥ 21.

33.2.10.2. Prove that if an integer K is divisible by $10\,101\,010\,101$, then there are at least 6 non-zero digits in the decimal expression of K.

33.2.10.3*. See Problem 33.2.9.5, where two spiders are chasing a fly and all have the same top speed.

33.2.10.4. Given an integer $n > 1970$, prove that the sum of the remainders after division of 2^n by 2, 3, 4, $\dots$, n is greater than $2n$.

33.2.10.5. Merlin has two 100×100 tables; one of them is blank, and on the other table some magic numbers are written. The blank table is nailed to a rock at the entrance to his cave, and the magic one is nailed to a wall inside the cave. You may outline any square (1×1, 2×2, $\dots$, or 100×100) on the blank table, at any place on the table but only along the lines, and for a shilling Merlin will tell you the sum of the numbers of the corresponding square in the magic table. What is the least amount of money one needs to learn the sum of the numbers on the main diagonal of the magic table?

Additional set (Pythagoras' Day)

Grade 7

33.D.7.1. We multiply the number $1234567\dots1000$ (juxtaposed are all natural numbers 1 to 1000) by a number from 1 to 9, and strike out all 1's in the product. We multiply the number obtained by a nonzero one-digit number once again, and strike out the 1's, and so on, many times over. What is the least number one can obtain in this manner?

33.D.7.2. A 13×13 m^2 hall is divided into squares with sides of 1 m. The Rule requires that rectangular rugs of arbitrary sizes be placed on the floor so that their sides lie on the sides of the squares; in particular, along the side of the hall. Any rug may be partially or even completely covered by other rugs but no single rug may completely cover, or lie under, another rug (even if there are several layers between them). What greatest number of rugs may cover the hall under this Rule?

33.D.7.3. In an ordinary game of dominoes the difference between the numbers on adjacent displayed tiles is equal to 0. Is it possible to arrange all 28 tiles in a closed chain so that the difference throughout the chain would be equal to ± 1?

33.D.7.4. Is it possible to divide the numbers 1, 2, 3, $\dots$, 33 into 11 groups, three numbers in each group, so that in any group one of the numbers is equal to the sum of the other two?

33.D.7.5. Ali Baba tries to enter the cave. At the entrance to the cave there is a drum with four holes in its sides. Inside the drum, next to each hole, there is a switch which has two positions, "up" and "down". The Rule permits Ali Baba to stick his fingers into any two holes, learn the position of their switches (by touch) and flip them as he pleases (for example not to flip at all). Then the drum is rotated very quickly so that after it stops it is impossible to ascertain which switches were flipped or touched last. Ali Baba may repeat the operation up to 10 times. The door to the cave opens the moment all the switches are in the same position. Prove that Ali Baba can get into the cave.

33.D.7.6. It is known that objects A and B cannot both fit into the picture taken by a camera at point O if $\angle AOB > 179°$. There are 1000 such cameras on a plane. All cameras simultaneously take a picture each. Prove that among these pictures there is one photo that shows ≤ 998 cameras.

Olympiad 34 (1971)

Tour 34.1

Grade 8

34.1.8.1. A town is walled in a 1000-gon (not necessarily convex but with nonintersecting sides). A guard stands at every vertex outside the wall. Prove that there is a guard who can see < 500 other guards. (The guards standing at the endpoints of one side of the 1000-gon can see each other.)

34.1.8.2. A circle intersects convex pentagon $ABCDE$ at points A_1, A_2, B_1, B_2, $\dots$, E_1, E_2; see Fig. 76. Knowing that $AA_1 = AA_2$, $BB_1 = BB_2$, $CC_1 = CC_2$, $DD_1 = DD_2$, prove that $EE_1 = EE_2$.

FIGURE 76. (Probl. 34.1.8.2)

FIGURE 77. (Probl. 34.1.9.2)

34.1.8.3. 25 teams took part in a national soccer tournament. In the end it turned out that no team scored more than four goals in any game. What lowest place could the team from Tbilisi have gotten, if overall it scored more goals, and was scored less goals against, than any other team?

34.1.8.4. A 100×100 square is drawn on a graph paper. There is a red or blue point in every square (of the grid) so that in every column and in every row there are 50 blue and 50 red points. Let us connect every pair of red points in adjacent squares (squares with a common side) with a red segment, and every pair of blue points in adjacent squares with a blue segment. Prove that the number of red segments equals the number of blue segments.

34.1.8.5. Prove that $k(5^{1\ 090\ 701}) - k(2^{1\ 090\ 701}) \vdots 2$, where $k(A)$ is the number of digits in the decimal expression of A.

Grade 9

34.1.9.1. Numbers $a_1, a_2, a_3, \dots, a_{25}$, where $a_1 = a_2 = \dots = a_{13} = 1$, and $a_{14} = a_{15} = \dots = a_{25} = -1$ are written at the vertices of a regular 25-gon. Set $b_1 = a_1 + a_2$, $b_2 = a_2 + a_3$, etc., $b_{24} = a_{24} + a_{25}$, $b_{25} = a_{25} + a_1$, and replace $a_1, a_2, \dots, a_{25}$, with $b_1, b_2, \dots, b_{25}$, respectively. This operation is then repeated 100 times. Prove that one of the numbers obtained in the operation is greater than 10^{20}.

34.1.9.2. The perimeter of a convex k-gon P $(k > 6)$ is equal to 2. Construct a new convex k-gon M with vertices at the midpoints of the sides of the k-gon P and prove that the perimeter of M is greater than 1; see Fig. 77.

34.1.9.3. Consider n straight lines ($n > 2$) on a plane. No two lines are parallel and no three of them meet. It is possible to rotate the plane about some point O through an angle of $\alpha < 180°$ so that each of the lines drawn gets identical with another of lines drawn on the fixed copy of the plane. Find all values of n for which such a system of lines exists.

34.1.9.4. Prove that no integer obtained by permutation of the digits in the decimal expression of 2^k ($k > 3$), is equal to 2^n for $n > k$. (Obviously, n and k are integers here.)

34.1.9.5. Prove that there are infinitely many non-primes among the numbers $[2^k \cdot \sqrt{2}]$, $k = 1, 2, \dots$.

Grade 10

34.1.10.1. Consider a closed broken line $A_1A_2 \dots A_nA_1$ in space such that every segment of it intersects a fixed sphere at two points, and all vertices of the line are located outside the sphere. The intersection points divide the broken line into $3n$ segments so that the segments at the vertex A_1 are equal and the same holds true for the vertices A_2, A_3, ... , A_{n-1}. Prove that the segments at A_n are also equal. (Cf. Problem 34.1.8.2).

34.1.10.2. Peter has a set "Young tiler" of tiles arranged in a rectangular box so that they completely cover the bottom of the box in one layer. Every tile has an area of 3 cm and is either a rectangle or an L-shaped figure, see Fig. 78.

FIGURE 78. (Probl. 34.1.10.2)

Peter says that he lost an L-shaped tile, made a new rectangular tile instead of it, and arranged all tiles in the box in one layer. Can one be certain that Peter is lying?

34.1.10.3. The terms a sequence $x_1, x_2, \dots, x_n, \dots$ satisfy the equation $3x_n - x_{n-1} = n$ for any $n > 1$ and $|x_1| < 1971$. Find x_{1971} to the nearest millionth.

34.1.10.4. All vertices of a convex n-gon and k more points inside it are marked. It turns out that any three of these $n+k$ points are not on the same straight line and are the vertices of an isosceles nondegenerate triangle. What value may k take?

34.1.10.5. There is a pile of 10 million matches. Two players play a game, taking turns. A player may take p^n matches, where p is a prime and $n = 0, 1, 2, 3, \dots$ (for example, the first takes 25 matches, the second takes 8, the first 1, the second 5, the first 49, and so on). The player who takes the last match is the winner. Who wins if both play optimally?

Tour 34.2

Grade 7

34.2.7.1. Is there a number whose square begins with the digits 123456789 and ends with the digits 987654321?

34.2.7.2. Consider square $ABCD$, a point O inside it and perpendiculars AH_1, BH_2, CH_3, DH_4 dropped from points A, B, C, D to segments BO, CO, DO, AO, respectively. Prove that the straight lines on which these perpendiculars lie meet at one point.

34.2.7.3. A colony of n bacteria lived in a beaker. Once, a virus got into the beaker. In the first minute the virus destroyed one bacterium and immediately after that both the virus and the remaining bacteria split in halves. In the second minute the two viruses destroyed two new bacteria, and then the viruses and the remaining bacteria again split in halves, and so on. Will a moment come when no bacteria are left?

34.2.7.4. There is a mesh of 1×1 squares. Its every node is painted one of four given colors so that the nodes of any 1×1 square are differently colored. Prove that there is a straight line of the grid such that the nodes lying on it are painted only two colors.

34.2.7.5*. On a plane stand 7 point-size searchlights. Every searchlight illuminates an angle of 90°. If there is a searchlight in a quadrant illuminated by another searchlight, then the first one casts a shadow, a dark infinite ray. Prove that it is possible to arrange these 7 searchlights so that every one of them will cast a shadow of 7 km long; see Fig. 79.

FIGURE 79. (Probl. 34.2.7.5)

Grade 8

34.2.8.1. Consider a 29-digit number $X = \overline{a_1a_2...a_{28}a_{29}}$ such that for every k the digit a_k occurs a_{30-k} times in the expression of X. (For example, if $a_{10} = 7$, then the digit a_{20} occurs 7 times.) Find the sum of the digits of X.

34.2.8.2. We cut a cardboard (perhaps, non-convex) 1000-gon along a straight line. This cut brakes it into several new polygons. What is the greatest possible number of triangles among the new polygons?

34.2.8.3. Prove that the sum of the digits of a positive integer K is not more than 8 times the sum of the digits of the number $8K$.

34.2.8.4. Take any number consisting of zeros and fours in its decimal expression. Now, we can either divide it by 2, 3 or 5 if this division is possible without a remainder, or insert 0's or 4's between the digits of this number, or write a 4 at the beginning or at the end, or write a 0 at the end. With the number obtained we can repeat the same operations, and so on. Is it possible to obtain in this way any positive integer?

34.2.8.5. See Problem 34.2.7.2.

Grade 9

34.2.9.1. A convex 1971-gon is such that for every vertex A, every side that does not pass through A subtends equal angles with the angle's vertex in A. Prove that the polygon is a regular one.

34.2.9.2. See Problem 34.2.8.1.

34.2.9.3. Is it possible to divide every side of a square into 100 parts so that it would be impossible to contour with these 400 segments any rectangle other then the initial square?

34.2.9.4. A circle is divided into n equal parts, and the numbers $x_1, x_2, \dots, x_n$ equal to either 1 or -1 are written at the division points so that if one turns the circle through an angle of $k \cdot \dfrac{360^\circ}{n}$ and multiplies the numbers at points coinciding before and after the rotation, the sum of n products thus obtained is equal to 0 for any $k = 1, \dots, n-1$. Prove that n is a perfect square. (Cf. Problem 34.2.10.1.)

34.2.9.5. Prove that it is possible to write non-zero real numbers $x_1, x_2, \dots, x_n$ at the vertices of a regular n-gon so that for any regular k-gon all of whose vertices are the vertices of the original n-gon, the sum of the numbers at its vertices is equal to 0.

Grade 10

34.2.10.1. See Problem 34.2.9.4 with additional question: what might number n be?

34.2.10.2. Given numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$, arrange the numbers a_k in the increasing order and numbers b_k in the decreasing order. We get sets $a_1^* \le \dots \le a_n^*$ and $b_1^* \ge \dots \ge b_n^*$. Prove that

$$\max(a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \ge \max(a_1^* + b_1^*, a_2^* + b_2^*, \dots, a_n^* + b_n^*).$$

34.2.10.3. Banker and Gambler play the following hazardous game. Banker names a 1000-digit number, A_1. Upon learning this number, Gambler suggests to Banker an arbitrary number B_1. This ends the first move.

Next, Banker either subtracts the smaller number from the greater, or adds one to the other, as he chooses, and tells the result — a number A_2 — to the Gambler. Then Gambler suggests to Banker the next number, B_2. Banker repeats the same operation with numbers A_2 and B_2, and so on.

The game ends when Banker gets one of the following numbers: 1, 10, 100, 1000, ... Prove that Gambler can always end the game suggesting not more than 20 of his numbers to Banker.

34.2.10.4. A point O and n straight lines, no two of which are parallel, are given in space. We take the projections of O to all given straight lines. Each of the points obtained is projected to all straight lines again, and so on. Is there a sphere containing inside it all points obtained in such a way?

34.2.10.5*. Prove that the sum of the digits of an integer N is not more than five times the sum of the digits of $5^5 \cdot N$. (Cf. Problem 34.2.8.3).

Olympiad 35 (1972)

Tour 35.1

Grade 7

35.1.7.1. Prove that if positive integers $a_1, a_2, \dots, a_{17}$ satisfy

$$a_1^{a_2} = a_2^{a_3} = a_3^{a_4} = \cdots = a_{16}^{a_{17}} = a_{17}^{a_1},$$

then $a_1 = a_2 = \cdots = a_{17}$.

35.1.7.2. 1000 delegates from various countries came to a Congress. Every delegate could speak several languages and it was known that any three delegates could have a common conversation without assistance. (A delegate could serve as a translator for a pair of his colleagues.) Prove that it was possible to distribute all delegates in 500 rooms, so that in every room there were 2 delegates and they can understand each other.

35.1.7.3. Every vertex of a regular 13-gon is painted either black or white. Prove that there exist three points of the same color which are the vertices of an isosceles triangle.

35.1.7.4. Let AD and BE be medians in triangle ABC; let the angles CAD and CBE be equal to 30°. Prove that $AB = BC$. (See Problem 35.1.8.5.)

Grade 8

35.1.8.1. There are asterisks in some of the squares of an $n \times n$ graph paper. It is known that if we strike out any set of rows (but not all of them), a column with exactly one asterisk will remain (if one does not strike out any row there still remains a column with exactly one asterisk). Prove that if one strikes out any number of columns (but not all of them), a row with exactly one asterisk will remain.

35.1.8.2. Given two identical L-shaped figures on a plane. Denote the endpoints of their shorter sides by A and A' and divide their longer sides into n equal parts by points $a_1, \dots, a_{n-1}$; $a'_1 \dots, a'_{n-1}$. (We number these dividing points beginning at the free endpoints of the longer sides.) Draw the straight lines Aa_1, Aa_2, $\dots$, Aa_{n-1} and $A'a'_1$, $A'a'_2$, $\dots$, $A'a'_{n-1}$ and denote the intersection point of lines Aa_1 and $A'a'_1$ by X_1; of lines Aa_2 and $A'a'_2$ by X_2, and so on. Prove that points $X_1, X_2, \dots, X_{n-2}$ are vertices of a convex polygon.

35.1.8.3. A pawn got a tip that out of 1000 coins the robber brought him, 0, 1 or 2 are counterfeit. It is known that all counterfeit coins are of the same weight different from the weight of genuine coins. Is it possible to determine (a) whether there are counterfeit coins in this set and (b) whether their weight is greater or less than the weight of genuine coins by weighing groups of coins three times on a balance without using weights? (It is not necessary to determine how many counterfeit coins are there.)

35.1.8.4. Given a set of positive integers with the sum of any seven of them less than 15 and the sum of all the numbers in the set equal to 100, determine the least possible number of elements in this set.

35.1.8.5. Let AD and BE be medians in triangle $\triangle ABC$; let each of the angles $\angle CAD$ and $\angle CBE$ be equal to 30°. Prove that triangle $\triangle ABC$ is an equilateral one.

Grade 9

35.1.9.1. Angle $\angle C$ in triangle ABC is obtuse. Points E and H are marked on side AB and points K and M on sides AC and BC, respectively. It turns out that $AH = AC$, $EB = BC$, $AE = AK$, $BH = BM$. Prove that points E, H, K, M lie on the same circle.

35.1.9.2. There are numbers in all squares of an $n \times n$ chessboard: number a_{km} stands in the intersection of the k-th row with the m-th column. Suppose that for any arrangement of n rooks on this chessboard such that none can be jumped by another, the sum of the numbers covered by the rooks is equal to 1972. Prove that there exist two sets of numbers $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ that for every k and m satisfy the equation: $a_{km} = x_k + y_m$; cf. Problem 18.1.9.1.

35.1.9.3. The distance between any two trees in a forest is not greater than the difference between their heights. None of the trees is higher than 100 m. Prove that it is possible to fence the forest with a fence 200 m long.

35.1.9.4*. Positive integers m and n are relatively prime and $n < m$. Which number is greater: $\left[1 \cdot \frac{m}{n}\right] + \left[2 \cdot \frac{m}{n}\right] + \dots + \left[n \cdot \frac{m}{n}\right]$ or $\left[1 \cdot \frac{n}{m}\right] + \left[2 \cdot \frac{n}{m}\right] + \dots + \left[m \cdot \frac{n}{m}\right]$?

35.1.9.5. In town X, ten infinite parallel avenues cross perpendicular streets at equal intervals. Two cops moving along the avenues and streets try to find a robber who, according to the Rule, can not shelter in a house and is hiding behind the houses. If the robber turns up on an avenue or street with a cop, he is *found*. The robber's speed is not more than 10 times that of a cop and an informer tipped the cops that the distance between them and the robber at the beginning of the chase was not greater than 100 blocks. Prove that the cops can find the robber.

Grade 10

35.1.10.1. There are n inhabitants in town Variety. Every two of them are either friends or enemies. Every day not more than 1 inhabitant may turn a new leaf: quarrel with all his friends and befriend all his enemies. The Rule of Variety says: if A is a friend of B and B is a friend of C, then A is also a friend of C. Prove that all inhabitants of the town can become friends.

35.1.10.2. Given an infinite sequence $a_1, a_2, \dots, a_n, \dots$, where a_1 is an arbitrary 10-digit number and each subsequent number is obtained from the preceding one by writing any digits but 9 after it, prove that there are no fewer than two non-prime numbers in the sequence.

35.1.10.3. In tetrahedron $ABCD$ all dihedral angles are acute and all opposite edges are equal. Find the sum of the cosines of all dihedral angles of the tetrahedron.

35.1.10.4. Consider a non-selfintersecting non-convex n-gon P and the locus T of points inside P from which one can see all the vertices of P. Prove that if T is nonempty and does not lie on one straight line, then T is a convex k-gon with $k \le n$.

35.1.10.5. See Problem 35.1.9.5.

Tour 35.2

Grade 7

35.2.7.1. Consider a convex quadrilateral $ABCD$ and point O where its diagonals meet. The perimeters of triangles $\triangle ABO$, $\triangle BCO$, $\triangle CDO$, $\triangle ADO$ are equal. Prove that $ABCD$ is a rhombus.

35.2.7.2. Four straight lines a, b, c, d are drawn on a plane. No two of them are parallel and no three of them meet at one point. It is known that a is parallel to one of the medians of the triangle formed by lines b, c, and d. Prove that b is parallel to a median of the triangle formed by lines a, c, and d.

35.2.7.3. Given twelve consecutive positive integers. Prove that at least one of them is smaller than the sum of its proper divisors.

35.2.7.4*. There are several castles in country Mara and three roads lead from every castle. A knight leaves his castle. Traveling around the country he leaves every new castle via the road that is either to the right or to the left of the one by which he arrived. According to The Rule the knight never takes the same direction (right or left) twice in a row. Prove that some day he will return to his own castle.

35.2.7.5. A straight line intersects sides AB and BC of triangle ABC at points M and K, respectively. Knowing that the area of triangle MBK is equal to the area of quadrilateral $AMKC$, prove that $\frac{MB + BK}{AM + CA + KC} \ge \frac{1}{3}$.

Grade 8

35.2.8.1. See Problem 35.2.7.1.

35.2.8.2. Numbers a, b, c, d, e and f are positive integers such that $\frac{a}{b} > \frac{c}{d} > \frac{e}{f}$ and $af - be = 1$. Prove that $d \ge b + f$.

35.2.8.3. A town of Nikitovka had only two-way traffic. Repairs of all its streets took two years. During the first year some of the streets were turned into one-way streets. The next year the two-way traffic was reestablished on these roads whereas all other roads became one-way roads. The repairs were made under strict adherence to the following Rule: *one should be able to drive from any point of the town to any other at all times during the repairs.* Prove that it is possible to introduce a one-way traffic throughout Nikitovka so that one could still drive from any point of the town to any other point.

35.2.8.4. Let $I(x)$ be the number of irreducible fractions $\frac{a}{b}$, where both a and b are positive integers such that $a \leq x$ and $b \leq x$. For example, $I\left(\frac{5}{2}\right) = 3$ and the corresponding fractions are $\frac{1}{2}$; 1; 2. Find the sum:

$$I(100) + I\left(\frac{100}{2}\right) + I\left(\frac{100}{3}\right) + \cdots + I\left(\frac{100}{99}\right) + I\left(\frac{100}{100}\right).$$

35.2.8.5. See Problem 35.2.9.5 for 300 straight lines and 100 triangles.

Grade 9

35.2.9.1. All sides of a pentagon are of the same length and each of its angles is less than 120°. Prove that all its angles are obtuse.

35.2.9.2. See problem 35.2.8.2.

35.2.9.3*. The streets of town M form a regular square net of 20×20 blocks. There are subway stations at some corners. It is known that one can get to a subway station from any point in the town passing not more than two blocks along the streets. What is the least number of subway stations in the town?

35.2.9.4*. Are there any rational numbers a, b, c, d satisfying for a positive integer n the equation

$$(a + b\sqrt{2})^{2n} + (c + d\sqrt{2})^{2n} = 5 + 4\sqrt{2}\ \ ?$$

35.2.9.5*. 3000 straight lines are drawn on a plane, no two of them are parallel, and no three of them meet at the same point. These lines divide the plane into several parts. Prove that among these parts there are at least a) 1000 triangles; b*) 2000 triangles.

Grade 10

35.2.10.1*. Consider plane P and triangle ABC, not on this plane, see Fig. 80. Triangle $A_1B_1C_1$ is a perpendicular projection of triangle ABC to P. Prove that triangle $A_1B_1C_1$ can be completely covered by a triangle equal to triangle ABC.

FIGURE 80. (Probl. 35.2.10.1)

35.2.10.2. Given two sets of numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ such that

$$x_1 > x_2 > \ \ldots\ > x_n > 0, \quad y_1 > y_2 > \ \ldots\ > y_n > 0;$$

and

$$x_1 > y_1, \quad x_1 + x_2 > y_1 + y_2, \ \ldots, \quad x_1 + x_2 + \ \ldots\ x_n > y_1 + y_2 + \ \ldots\ + y_n,$$

prove that for any positive integer k we have

$$x_1^k + x_2^k + \cdots + x_n^k > y_1^k + y_2^k + \cdots + y_n^k.$$

35.2.10.3. Numbers 1, 2, 3, ... , 400 are written on 400 cards, one on each card. Players A and B play the following game. Player A selects any 200 cards (the first set) and gives the rest (the second set) to B. Then B takes 100 cards from each set and gives the rest to A. Thus, both players once again have 200 cards. The end of the first move.

Then A again takes 100 cards from each set and gives the rest to B, and so on. After B has moved for the 200-th time, both players count the sum of the numbers on their cards, $S(A)$ and $S(B)$; and A pays B the difference $S(B) - S(A)$. What greatest amount of money can B get if both play optimally?

35.2.10.4. Arrange all rational numbers between zero and one whose denominators do not exceed n in increasing order. Let irreducible fractions $\frac{a}{b}$ and $\frac{c}{d}$ be two consecutive such numbers. Prove that $|bc-ad| = 1$ whatever n.

35.2.10.5*. There is a positive integer in every square of an 8×8 chessboard. The Rule allows one to take any square of size 3×3 or 4×4 and increase all numbers in it by 1 to ensure that the numbers in all squares are divisible by 10. Is this always possible after several such operations?

Olympiad 36 (1973)

Tour 36.1

Grade 8

36.1.8.1. There are several countries on a square island. Is it possible to divide these countries into smaller ones without creating new intersection points of their borders, and so that the map of the island could be painted two colors?

36.1.8.2. Can a number whose decimal expression consists of 600 copies of figure 6 and several zeros be the square of a positive integer?

36.1.8.3. Consider five points in a plane, no three of which lie on the same straight line and no four of which are on the same circle. Prove that two of these points may be selected so that they lie on both sides of the circle passing through three other points.

36.1.8.4. Prove that the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$, where x, y are positive integers, has exactly 3 solutions if p is a prime and the number of solutions is greater than three if $p > 1$ is not a prime. We consider solutions (a, b) and (b, a) for $a \neq b$ as distinct.

36.1.8.5. On a plane, in three vertices of a square sit three grasshoppers. At some moment the grasshoppers begin playing a game of leap-frog according to the following Rule: they leap over each other so that if grasshopper A leaps over grasshopper B, then after the jump it is at the same distance from B as before and on the same line. Is it possible for any of the grasshoppers to reach the fourth vertex of the square after a few jumps?

Grade 9

36.1.9.1. The area of a quadrilateral with vertices on the sides of a parallelogram is equal to half the area of the parallelogram. Prove that at least one of the quadrilateral's diagonals is parallel to a side of the parallelogram.

36.1.9.2. A square is divided into convex polygons. Prove that it is possible to divide them into smaller convex polygons so that each of these new ones has an odd number of adjacent polygons (with a common side).

36.1.9.3. The value of a polynomial $P(x)$ with integer coefficients is equal to 2 at three integer points. Prove that there exists no integer point at which the polynomial is equal to 3.

36.1.9.4. In the city of X one can get to any subway station from any other. Prove that it is possible to close one of the stations for repairs and not let trains pass through it but still enable people to get to any of the remaining stations from any other.

36.1.9.5. The faces of a cube are numbered 1, 2, ... , 6 so that the sum of the numbers on every pair of opposite faces is equal to 7. There is a 50×50 chessboard whose squares are equal to the faces of the cube. The cube rolls from the lower left corner of the chessboard to its upper right corner. The Rule allows it to move only to the right or upwards (not to the left or downwards). The cube prints the numbers painted on its faces in every square of the chessboard that a face touches as the cube rolls. What is the greatest possible sum of the numbers printed and what is the least possible one? (The figure 6 printed upside down still counts as 6, not 9!)

Grade 10

36.1.10.1. We factor a positive integer k into its prime factors: $k = p_1 \cdot p_2 \cdot \dots \cdot p_{n-1} \cdot p_n$ and set $f(k) = p_1 + p_2 + \dots + p_{n-1} + p_n + 1$. We perform the same operation f with $f(k)$, and so on. Prove that the resulting sequence of numbers k, $f(k)$, $f(f(k))$, ... eventually becomes a periodic one.

36.1.10.2. See Problem 36.1.9.1.

36.1.10.3. At some integer points a polynomial $P(x)$ with integer coefficients takes values 1, 2 and 3. Prove that there exists not more than one integer at which the polynomial is equal to 5.

36.1.10.4. Prove that every convex polyhedron has two faces with the same number of sides.

36.1.10.5. A control panel of N switches and a board with N bulbs are on the sides of a "black box". By switching consecutively all possible combinations of the switches we consecutively light all possible combinations of the bulbs. The state of the panel of lights directly depends on the state of the switches on the control panel. It is known that when one switch is flipped, exactly one bulb lights up or goes off. Prove that the state of each bulb depends on exactly one switch (for every bulb its own switch).

Tour 36.2

Grade 7

36.2.7.1. A four-digit number is subtracted from a number composed of the same digits in reverse order. Can the difference be equal to 1008?

36.2.7.2. Consider an acute triangle ABC and discs centered at the vertices of the triangle with their radii equal to heights dropped from respective vertices. Prove that every point of the triangle is covered by at least one disc.

36.2.7.3. A 100×100 piece of graph paper is painted 100 different colors. Every unit square of the grid is either painted one of the colors or not painted at all. A coloring will be called *regular* if no column and no row has two squares of the same color. Is it possible to paint this piece of paper regularly so that all squares are painted if initially there are a) $100^2 - 1$; or b) $100^2 - 2$; or c) 100 regularly painted squares?

36.2.7.4. See Problem 36.2.8.3 a) below.

Grade 8

36.2.8.1. There is an ink blot on a piece of paper. For every point of the blot consider its minimal and the maximal distance to the boundary of the blot. The greatest of all minimal distances and the least of all maximal distances are selected and compared. What is the shape of the blot if these two numbers are equal? (See Fig. 81)

FIGURE 81. (Probl. 36.2.8.1)

36.2.8.2. See Problem 36.2.7.3 replacing 100 with an arbitrary n.

36.2.8.3. At the center of a square stands a cop and at one of the square's vertices stands a robber. The Rule allows the cop to run anywhere in the square and even digress outside its limits, while the robber can only move along the square's sides. For each of the following ratios of the cop's top speed to that of the robber a) 1/2; b) 0.49; c) 0.34; d) 1/3, prove that the cop can run so as to be on the same side as the robber at some moment.

36.2.8.4. Prove that it is possible to place an equilateral triangle into a convex equilateral (but not necessarily regular) pentagon, with sides equal to the sides of the triangle, so that they have one side in common and the entire triangle is inside the pentagon.

Grade 9

36.2.9.1. The decimal expression of a 100-digit number consists of 1's and 2's. The Rule allows one to select arbitrarily 10 consecutive digits of which the first five may change places with the second five. Two numbers will be called *similar* if one can be obtained from the other one in several such operations. What greatest number of such 100-digit numbers can be selected so no two of them are similar?

36.2.9.2. A closed non-selfintersecting broken line is drawn on an infinite chessboard along the sides of its squares. There are K black squares inside the broken line. What is the greatest area of the figure bounded by the broken line?

36.2.9.3. See Problem 36.2.10.1 for $m = 5$.

36.2.9.4*. Two 1's are situated at the endpoints of a line segment. The first move is to insert their sum — the number 2 — between them. Next move is to insert between every two adjacent numbers their sum, and so on, 1 000 000 times; see Fig. 82. How many times will the number 1973 be written during this process?

Figure 82. (Probl. 36.2.9.4)

36.2.9.5*. See Problem 36.2.8.3. Let the robber's top speed be 2.9 times that of the cop. Is it possible for the cop to arrive on the same side with the robber?

Grade 10

36.2.10.1. Let m and n be positive integers ≥ 2. Prove that there is a positive integer k such that

$$\left(\frac{n+\sqrt{n^2-4}}{2}\right)^m = \frac{k+\sqrt{k^2-4}}{2}.$$

36.2.10.2. Prove that the angles between every two bisectors of planar angles of a trihedral angle are either all acute, or all obtuse, or all right ones.

36.2.10.3. 12 painters live in a commune of 12 red and white houses along a ring-shaped road. Every month one of the painters leaves his or her house with red and white paints and goes clockwise along the road. When (s)he sees a red house (s)he paints it white and goes further, and when (s)he sees a white house (s)he paints it red and then goes home to wash his or her brush. Every painter only works once a year. Prove that if at the beginning of the year there is at least one red house then at the end of a year every house will be painted its initial color.

36.2.10.4. See Problem 36.2.9.1.

36.2.10.5. A lion runs over a circular circus ring of radius 10 m. Moving along a broken line he covers 30 km. Prove that the sum of the angles of all of the lion's turns is not less than 2998 radians.

Olympiad 37 (1974)

Tour 37.1

Grade 9

37.1.9.1. Prove that the number 100...001 with $2^{1974} + 2^{1000} - 1$ zeros is not a prime.

37.1.9.2. Prove that it is impossible to place two triangles, each of area greater than 1, into a disc of radius 1 so that they would not overlap.

37.1.9.3. Two identical gears have 32 teeth each. One of the gears was placed atop the other one so that their teeth aligned. Then 6 pairs of corresponding teeth were sawed off from both gears. Prove that it is possible to rotate one gear relative the other one so that in the places where teeth of one gear are missing there will be teeth of the other gear. (Cf. Problem 37.1.10.3.)

37.1.9.4. Prove that if it is possible to construct a triangle from segments of lengths a, b and c, it is also possible to construct a triangle from segments of lengths $\frac{1}{a+c}$, $\frac{1}{b+c}$, $\frac{1}{a+b}$.

37.1.9.5. A convex polygon has the following property: if all straight lines on which its sides lie are moved outwards by a distance of 1, then the straight lines in their new positions form a new polygon similar to the original one, with the proportional parallel sides. Prove that it is possible to inscribe a circle into the original polygon.

Grade 10

37.1.10.1. See Problem 37.1.9.4.

37.1.10.2. Prove that for any 13-gon there exists a straight line which contains exactly one of its sides but for any $n > 13$ there exists such an n-gon for which this does not hold.

37.1.10.3. Two identical gears have 92 teeth each. One of the gears was placed atop the other one so that their teeth aligned. Then 10 pairs of corresponding teeth were sawed off from both gears. Prove that it is possible to rotate one gear relative the other one so that in the places where teeth of one gear are missing there will be teeth of the other gear. (Cf. Problem 37.1.9.3.)

37.1.10.4. Suppose we mark all vertices and centers of the faces of a cube and draw all diagonals of its faces. Is it possible, moving along the diagonals, to pass every marked point only once?

37.1.10.5. See Problem 37.1.9.5.

Tour 37.2

Grade 7

37.2.7.1. Point M inside a regular hexagon with side 1 is connected with all vertices of the hexagon thus dividing the hexagon into triangles. Prove that among the triangles there are two whose sides are not shorter than 1.

37.2.7.2. On a straight line 100 points are fixed. Let us mark the midpoints of all segments with both endpoints among the fixed points. What is the minimal number of marked points? (Cf. Problem 37.2.8.2.)

37.2.7.3. How many sides can a convex polygon have if its diagonals are of equal length?

37.2.7.4. A few marbles are distributed into three piles. A boy who has an access to an unlimited stock of marbles may take one marble from every pile or add to one of the piles as many marbles from his stock as there are already in the pile. Prove that in a few such operations the boy can make it so that there are no marbles left in every pile.

Grade 8

37.2.8.1. See Problem 37.2.7.3.

37.2.8.2. On a straight line n points are fixed. Let us mark the midpoints of all segments with both endpoints among the fixed points. What is the minimal number of marked points? (Cf. Problem 37.2.7.2.)

37.2.8.3. Positive integers fill in a rectangular table of 8 rows and 5 columns. In one move we may double all numbers in one row or subtract 1 from every number in one column. Prove that it is possible to make all the numbers in the table equal to 0 in finitely many moves.

37.2.8.4. Prove that a convex pentagon with all angles obtuse has two diagonals such that discs constructed on them as on diameters completely cover the pentagon.

37.2.8.5. The sum of 100 positive integers, each not greater than 100, is equal to 200. Prove that from these integers one can select several so that their sum is equal to 100.

Grade 9

37.2.9.1. Is there a sequence of positive integers such that one can uniquely express any positive integer $1, 2, 3, \dots,$ as the difference of two numbers of the sequence?

37.2.9.2. Prove that in an arbitrary $2n$-gon there exists a diagonal not parallel to any of its sides.

37.2.9.3*. There are several weights of (positive) integer masses. It is known that they can be divided into K groups of equal mass. Prove that in not less than K ways one can take away a weight so that it is impossible to divide the remaining weights into K groups of equal mass.

37.2.9.4. Given triangle ABC with $AB > BC$ and its bisectors AK and CM, prove that $AM > MK > KC$. (See Solution to Problem 28.1.9.3.)

37.2.9.5. An $a \times b$ piece of paper is cut into rectangular strips, each one with a side of 1 cm. The cuts are parallel to the edges of the paper. Prove that at least one of the numbers a or b is an integer.

Grade 10

37.2.10.1. See Problem 37.2.9.1.

37.2.10.2. Prove that the decimal expressions of the numbers $2^n + 1974^n$ and 1974^n have the same number of digits.

37.2.10.3. A spherical planet is surrounded by 37 point-size asteroids. An asteroid on the horizon is invisible. Prove that at any moment of time there is a point on the surface of the planet from which an astronomer cannot see more than 17 asteroids.

37.2.10.4. Scientists, some of whom are acquainted, come to a congress. It turns out that no two scientists with the same number of acquaintances have any acquaintances in common. Prove that there is a scientist who has exactly one acquaintance among the participants of the congress.

37.2.10.5. See Problem 37.2.9.5.

Olympiad 38 (1975)

Tour 38.1

Grade 10

38.1.10.1. Solve in real numbers

$$x^2 + y^2 + z^2 + t^2 = x(y + z + t).$$

38.1.10.2. The distance between the center of a disc of radius 1 cm and a point A is 50 cm. We can symmetrically reflect point A through any straight line intersecting the disc; any point obtained may also be reflected symmetrically through any straight line intersecting the disc, and so on. Prove that a) it is possible to herd point A inside the disc in 25 reflections; b) it is impossible to do so in 24 reflections.

38.1.10.3. Positive integers a, b, c are such that the numbers $p = b^c + a$, $q = a^b + c$, and $r = c^a + b$ are primes. Prove that two of the numbers p, q, r are equal.

38.1.10.4. The centers of the squares of an 8×8 chessboard — 64 points — are marked. Is it possible to separate every marked point from the rest by drawing 13 straight lines that do not intersect these points?

38.1.10.5*. Is it possible to arrange 4 lead balls and a point source of light in space so that every ray of light from the source would end in at least one of the balls?

Tour 38.2

Grade 7

38.2.7.1. See Problem 38.2.8.1 a) where $n = 100$.

38.2.7.2. A convex heptagon is inscribed in a circle. It is known that three of the heptagon's angles are equal to 120°. Prove that two of the heptagon's sides are of the same length.

38.2.7.3. Kolya and Vitya play the following game. There is a pile of 31 stones on the table. The boys take turns making moves and Kolya begins. In one turn a player divides every pile which has more than one stone into two lesser ones. The player who after his turn leaves all piles with only one stone in each wins. Can Kolya win no matter how Vitya plays?

38.2.7.4. In the sequence 19752... every digit beginning with the fifth one is equal to the last digit of the sum of the preceding four digits. Is it possible to find in the sequence a) strings of consecutive digits 1234? 3269? b) a second string 1975?

Grade 8

38.2.8.1. Which of the two numbers is greater:

a) $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (n many 2's) or $3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}$ ($n-1$ many 3's)?

b) $3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}$ (n many 3's) or $4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}$ ($n-1$ many 4's)?

38.2.8.2. See Problem 38.2.7.2.

38.2.8.3. See Problem 38.2.7.4 with the addition: c) the set 8197?

38.2.8.4. There are two countries: Ourland and the Behind the Looking Glass, or just the Behindland. Every town in Ourland has its "double" in the Behindland and vice versa. If some two towns A and B are connected by a railroad in Ourland, then their doubles A' and B' are not connected in the Behindland, but the doubles of two unconnected towns of Ourland are connected by a railroad in the Behindland. A girl Alice from Ourland cannot reach town B from town A changing trains fewer than two times. Prove that her double, Ecila, in the Behindland can get from one town to any other changing trains not more than twice.

38.2.8.5. In a soccer tournament n teams take part. Every team plays with the other one only once. What can the greatest difference between the final scores of the team with neighboring final positions be?

Grade 9

38.2.9.1. See Problem 38.2.8.1.

38.2.9.2. See Problem 38.2.7.2.

38.2.9.3. See Problem 38.2.8.5.

38.2.9.4. In the land Mantissa towns are connected by roads. The length of any road is less than 500 km, and it is possible to get from any town to any other one driving less than 500 km. When one of the roads was closed for repairs it turned out that it was still possible to get from any town to any other one. Prove that in this case one can find a road between any two towns not longer than 1500 km.

38.2.9.5*. Is it possible to cut a convex polygon into a finite number of non-convex quadrilaterals? (See Fig. 83.)

FIGURE 83. (Probl. 38.2.9.5)

Grade 10

38.2.10.1. See Problem 38.2.8.1.

38.2.10.2. See Problem 38.2.7.3 and replace 31 with 100.

38.2.10.3. See Problem 38.2.9.4.

38.2.10.4*. Several ($n > 0$) distinct spotlights illuminate a circus ring in the form of a disc. Every spotlight illuminates some convex lamina on the ring. It is known that if any of spotlights is turned off the ring is still fully illuminated, and if 2 arbitrary spotlights are turned off the ring is not fully illuminated. For what n this is possible?

38.2.10.5. See Problem 38.2.9.5.

Olympiad 39 (1976)

Tour 39.1

Grade 10

39.1.10.1. Find all positive solutions of the system of equations:

$$\begin{cases} x_1 + x_2 = x_3^2, \\ x_2 + x_3 = x_4^2, \\ x_3 + x_4 = x_5^2, \\ x_4 + x_5 = x_1^2, \\ x_5 + x_1 = x_2^2. \end{cases}$$

39.1.10.2. We drew median AM, bisector BK and height CH in an acute triangle $\triangle ABC$. Let $\triangle M'H'K'$ be the triangle formed by the intersection points of the three segments drawn. Can it be so that $S_{\triangle M'H'K'} > 0.499 \cdot S_{\triangle ABC}$? (See Fig. 84.)

Figure 84. (Probl. 39.1.10.2)

39.1.10.3. In the decimal expression of $1^1 + 2^2 + 3^3 + \cdots + 999^{999} + 1000^{1000}$, what are its a) first three digits from the left? b) first four digits?

39.1.10.4. An astronomical searchlight can illuminate an octant. The searchlight stands at the center of a cube. Is it possible to turn the searchlight so that it will not illuminate any of the cube's vertices?

39.1.10.5. Domino tiles 1×2 are placed on an infinite graph paper composed of unit squares. The tiles cover all squares. Can it be so that any straight line of the graph of the paper intersects only a finite number of tiles?

Tour 39.2

Grade 7

39.2.7.1. There are four balls, identical in appearance, of mass 101 g, 102 g, 103 g, and 104 g. The Rule allows you to use a balance with two pans and an arrow (indicating weight on a continuous scale). The balance can measure any weight. Find the mass of every ball in only two weighings.

39.2.7.2. Can a convex non-regular pentagon have exactly 4 sides of equal length and exactly 4 diagonals of equal length?

39.2.7.3. Is there a positive integer n such that the sum of the digits of the number n^2 is equal to 100?

39.2.7.4. Is it possible to fix finitely many points on a plane so that every fixed point has exactly 3 nearest neighboring points? (Cf. Problem 32.2.10.4.)

39.2.7.5. There are 200 distinct numbers arranged in a 10×20 table. The two greatest numbers of every row are marked red and the two greatest numbers of every column are marked blue. Prove that at least three (skoljko tochno?) numbers are marked both red and blue. (Cf. Problem 32.2.10.3.)

Grade 8

39.2.8.1. See Problem 39.2.7.3.

39.2.8.2. The length of the side of square $ABCD$ is an integer. Line segments parallel to the square's sides divide it into smaller squares; the lengths of the sides of the smaller squares are also integers. Prove that the sum of the lengths of all segments is divisible by 4.

39.2.8.3. See Problem 39.2.9.2.

39.2.8.4. See Problem 39.2.7.5.

39.2.8.5. See Problem 39.2.10.4.

Grade 9

39.2.9.1. Is there an integer n such that $n!$ terminates with the digits 1976000...000 (the number of zeroes is not specified)? (I.e., if you find such an n for any number of zeros, you have answered in affirmative, otherwise you have to prove that whatever number of zeros, there is no such n.)

39.2.9.2. On the spherical Sun finitely many circular spots are discovered. Each spot covers less than half of the Sun's surface. The spots are considered to be *closed* (i.e., a spot's boundary belongs to it) and they neither intersect nor touch one another. Prove that on the Sun there are two diametrically opposite points not covered by the spots. (See Fig. 85.)

FIGURE 85. (Probl. 39.2.9.2)

39.2.9.3*. Prove that there exists a positive integer n greater than 1000 such that the sum of the digits in the decimal expression of 2^n is greater than same of 2^{n+1}.

39.2.9.4. There are no zeros in the decimal expression of a given number N. If two identical digits or two identical two-digit numbers neighbor in the decimal expression of N, we may strike them out. Besides, we are allowed to insert two identical neighboring digits or two identical neighboring two-digit numbers into any place in the decimal expression of N. Prove that with such operations we can obtain from N a number less than 10^9.

39.2.9.5*. On a table, there is a vast piece of graph paper (the side of each square of the grid being 1 cm). There is also an unlimited number of 5-kopek coins of radius 1.3 cm. Prove that it is possible to put the coins on the paper so that they cover all nodes of the graph but do not overlap.

Grade 10

39.2.10.1. Is there a positive integer A such that $\overline{AA}$ is a perfect square?

39.2.10.2. Is there a convex 1976-hedron such that for an arbitrary arrangement of arrows, one on each edge, the sum of the vectors the arrows represent is not equal to $\vec{0}$?

39.2.10.3. There are 200 different numbers arranged in a 10×20 table. The three greatest numbers of each row are marked red, and the three greatest numbers of each column are marked blue. Prove that at least 9 numbers are marked both red and blue. (Cf. Problem 39.2.7.5.)

39.2.10.4. On a plane, there are fixed several (finitely many) points. For every fixed point A consider the shortest distance r from A to any other fixed point; a fixed point at distance r from A is called a *neighbor* of A. Prove that there is a fixed point with not more than three neighbors. (Cf. Problem 39.2.7.4.)

39.2.10.5. Every point in space is painted one of five given colors, and there are fixed 5 points painted different colors. Prove that there exists a straight line all whose points are painted not less than three colors, and a plane all whose points are painted not less than four colors.

Olympiad 40 (1977)

Tour 40.1

Grade 10

40.1.10.1. A sequence is determined by recurrence: $x_1 = 2$, $x_{n+1} = \left[\frac{3}{2}x_n\right]$ for $n > 1$. Prove that the sequence has an infinite set of a) odd numbers; b) even numbers.

40.1.10.2. On a table, n cardboard squares and n plastic squares are arranged. No two cardboard squares have a common point (boundary points included). The same holds for the plastic squares. It turns out that the set of vertices of the cardboard squares coincides with the set of vertices of the plastic squares. Must then every cardboard square coincide with some plastic square?

40.1.10.3*. a) Twelve thin solid wires of length 1 each are joined to form the frame of a unit cube. Is it possible to make in a plane a hole of area ≤ 0.01, not cutting the plane into several parts, so that the whole frame can be pulled through the hole?

b) The same question for the frame of a tetrahedron with edge of length 1.

40.1.10.4. On the real line every point with integer coordinate is painted either red or blue. Prove that either red or blue the following property: for every positive integer K there is an infinite number of points of this color whose coordinates are divisible by K.

Tour 40.2

Grade 7

40.2.7.1. In every vertex of a convex n-gon lies a hunter with a laser gun. All hunters simultaneously fire at a rabbit sitting in a point O inside this n-gon. At the moment of the shot the rabbit lies down and all hunters get killed[1]. Prove that, apart from O, there is no other point with the same property.

40.2.7.2. A $3 \times 3 \times 3$ cube is made of 14 white and 13 black smaller cubes with edge 1. A *stack* is a collection of three smaller cubes standing in a row in one direction: width, length or height. Could there be an odd number of (a) white cubes or (b) black cubes in every stack?

40.2.7.3. Prove that there are more than 1000 three-tuples of positive integers (a, b, c) satisfying $a^{15} + b^{15} = c^{16}$.

40.2.7.4. 1977 nails stick out of a board. Two players make moves taking turns. In one move a player connects two nails with a wire. Two nails previously connected may not be connected again. If a move results in a closed chain, the player who made the move wins. Who wins if both play optimally — the first player or the second one?

40.2.7.5. Find the minimal n such that any convex 100-gon can be obtained as the intersection of n triangles. Prove that for a smaller n not every convex 100-gon can be obtained in this way.

Grade 8

40.2.8.1. See Problem 40.2.7.1.

40.2.8.2. See Problem 40.2.7.2.

40.2.8.3. See Problem 40.2.7.3.

40.2.8.4. See Problem 40.2.9.3 a).

40.2.8.5. See Problem 40.2.7.5.

Grade 9

40.2.9.1. In space there are n segments no three of which are parallel to one plane. For any two of them a straight line connecting their midpoints is perpendicular to both of them. For what greatest n is this possible?

40.2.9.2. a) Are there 6 different positive integers such that $(a + b) \vdots (a - b)$ for any two of them, a and b?

b) The same question for 1000 numbers.

40.2.9.3. a) At the end of a volleyball tournament it turned out that for any two teams there was a third one which had beaten both of them. Prove that the number of teams in the tournament was ≥ 7.

b) In another volleyball tournament for any three teams there was a team which had beaten all three. Prove that the number of teams in this tournament was ≥ 15.

40.2.9.4. The vertices of a convex polyhedron in space are all situated at integral points (i.e., all three coordinates of every vertex are integers). There are no other integral points either inside the polyhedron or on its faces and edges. Prove that the polyhedron has not more than 8 vertices.

40.2.9.5*. Consider a polynomial $P(x)$ with integer coefficients such that $P(n) > n$ for any positive integer n and such that for every positive integer N the sequence

$$x_1 = 1, \quad x_2 = P(x_1), \ \ldots, \quad x_n = P(x_{n-1}), \ldots$$

has a term divisible by N. Prove that $P(x) = x + 1$.

[1] Which serves them right: don't get involved into such a problem.

Grade 10

40.2.10.1. Is it possible to place an infinite set of identical discs on a plane so that any straight line on this plane intersects not more than two discs?

40.2.10.2. See Problem 40.2.9.2 for 15 numbers.

40.2.10.3. See Problem 40.2.9.3 b).

40.2.10.4. Considr the recurrence: $x_1 = 2$, $x_{n+1} = \left[\frac{3}{2}x_n\right]$ for $n > 1$. Prove that the sequence $\{y_n = (-1)^{x_n}\}_{n\in\mathbb{N}}$ is non-periodic.

40.2.10.5. See Problem 40.2.9.5.

Olympiad 41 (1978)

Grade 7

41.7.1. Solve in positive integers $3 \cdot 2^x + 1 = y^2$.

41.7.2*. On a plane lies a plastic triangle. If it is rolled over and over its sides and at some moment intersects its initial position, then we know that it simply coincides with its initial position. For what triangles this is true? Indicate all types of such triangles.

41.7.3. Prove that it is possible to arrange dominoes of size 1×2 in two layers on an $n \times 2m$ rectangle $(m, n \in \mathbb{N})$ so that each layer fully covers the rectangle and so that no two dominoes of different layers coincide.

41.7.4. See Problem 41.10.2 a).

Grade 8

41.8.1. See Problem 41.9.1.

41.8.2. See Problem 41.7.2.

41.8.3. See Problem 41.7.3.

41.8.4. See Problem 41.10.2 a).

41.8.5. A 1000-digit natural number A has the following remarkable property. Any 10 of its consecutive digits form a number divisible by 2^{10}. Prove that A is divisible by 2^{1000}.

Grade 9

41.9.1. Several points inside an n-gon are situated in such a way that inside any triangle formed by three vertices of the n-gon there lies at least one of the points. What is the least possible number of these points?

41.9.2. Is there a finite number of vectors $\overrightarrow{a_1}$, $\overrightarrow{a_2}$, ... , $\overrightarrow{a_n}$ on a plane such that for any pair of distinct vectors of this set there is another pair of vectors of the set whose sum is equal to that of the first pair?

41.9.3. See Problem 41.10.2 below.

41.9.4. In plane, consider several (finitely many) straight lines and points. Prove that there exists a point A on the plane, which does not coincide with any of the given points, and with distance to any given point greater than the distance to any of the given straight line.

41.9.5. There are 100 gossips in a town. Every gossip has 3 friends, also gossipy. A gossip learns some interesting news on the first of January and tells the news to his or her three friends. On the second of January the friends tell the news to every one of their friends, and so on. Is it possible that by the 5-th of March not all gossips have learned the news, but that all of them will have learned it by the 19-th of March?

Grade 10

41.10.1. A white sphere has 12% of its area painted red. Prove that it is possible to inscribe a parallelepiped into the sphere so that all its vertices are white.

41.10.2. A square town has 6 streets: 4 streets are the sides of the square and two are its medians. A cop is chasing a robber in this town. If the cop and the robber arrive at the same street simultaneously, then the robber gives in. Prove that the cop can catch the robber if the cop's top speed is a) 3 times that of the robber; b*) 2.1 times that.

41.10.3. See Problem 41.9.4.

41.10.4*. Prove that there exists a) a positive integer, b) an infinite set of positive integers n such that several consequtive last digits of 2^n in its decimal expression form the number n.

41.10.5*. Given 8 real numbers: a, b, c, d, e, f, g, h, prove that at least one of the six numbers $ac+bd$, $ae+bf$, $ag+bh$, $ce+df$, $cg+dh$, $eg+fh$ is non-negative.

Olympiad 42 (1979)

Grade 7

42.7.1. On a plane point O is marked. Is it possible to place on the plane a) 5, b) 4 discs that do not cover O so that any ray originating in O intersects at least two discs?

42.7.2. There are several weights with total mass of 1 kg. The weights are numbered 1, 2, 3, … . Prove that there is n such that the mass of the n-th weight is greater than 2^{-n} kg.

42.7.3. A square is cut into rectangles. Prove that the sum of areas of the discs circumscribed around the rectangles is not less than the area of the disc circumscribed around the square. (See Fig. 86.)

FIGURE 86. (Probl. 42.7.3)

42.7.4. Kolya and Vitya play the following game on an infinite graph paper. Kolya begins and taking turns they mark nodes of the paper, one node each per move. Both must mark so that after a move all points marked would be the vertices of a convex polygon (beginning with Kolya's second move). The player who cannot make such a move loses. Who wins if both play optimally?

Grade 8

42.8.1. A point O is marked on a plane. Is it possible to place on the plane a) 7 discs, b) 6 discs, that do not cover point O, so that any ray beginning from O intersects at least three discs? (Cf.Problem 42.7.1).

42.8.2. See Problem 42.7.2.

42.8.3. A quadrilateral $ABCD$ is inscribed in a circle with center O. Diagonals AC and BD are perpendicular. Prove that the length of perpendicular OH dropped from the center of the circle to side AD is equal to half the length of side BC. (See Fig. 87.)

FIGURE 87. (Probl. 42.8.3)

42.8.4. See Problem 42.7.3.

42.8.5. k scientists — chemists and alchemists — take part in a conference on chemistry, There are more chemists than alchemists among the scientists. It is known that chemists always tell the truth, no matter what they are asked, and that alchemists sometimes tell the truth and sometimes do not (lie).

A mathematician wants to know about every scientist whether the person in question is a chemist or alchemist. The Rule allows the mathematician ask any scientist the question: "What is such and such: chemist or alchemist?" (referring to any scientist, including the one questioned). Prove that the mathematician can learn what (s)he wants to know in a) $4k$ questions; b) $2k-2$ questions.

Grade 9

42.9.1. Given a collection of stones. The mass of each stone is ≤ 2 kg and their total mass is equal to 100 kg. We selected a set of stones whose total mass differs from 10 kg by the least possible for this set number d. What is the greatest value of d for every admissible collection of stones?

42.9.2*. Is it possible to represent the whole space as the union of an infinite number of pairwise skew lines?

42.9.3*. a) Does there exist a sequence of positive integers $a_1, a_2, a_3, \dots$ such that none of its elements is equal to the sum of some other ones, and $a_n \leq n^{10}$ for every n?

b) The same question with $a_n \leq n\sqrt{n}$ for every n.

42.9.4. See Problem 42.8.3.

42.9.5*. See Problem 42.8.5 with a new heading: c) $2k-3$ questions.

Grade 10

42.10.1. See Problem 42.9.1.

42.10.2. On a segment of length 1 several intervals are marked. It is known that the distance between any two points from the same or different marked intervals is not equal to 0.1. Prove that the sum of lengths of the marked intervals is not greater than 0.5.

42.10.3. A function $y = f(x)$ is defined and is twice differentiable on segment $[0, 1]$. Moreover, $f(0) = f(1) = 0$ and $|f''(x)| \leq 1$ on the whole segment. What is greatest value of $\max\limits_{x\in[0,1]} f(x)$ have for all such functions?

42.10.4. The union of several discs has an area of 1. Prove that it is possible to find several non-intersecting discs among them with the total area $> \frac{1}{9}$.

42.10.5*. See Problem 42.9.5.

Olympiad 43 (1980)

Grade 7

43.7.1. Find the greatest five-digit number A in which the fourth digit is greater than the fifth; the third greater than the sum of the fourth and fifth; the second greater than the sum of the third, fourth and fifth; and the first greater than the sum of the other digits.

43.7.2. In every square of a rectangular graph paper stands 1 or -1. The number of 1's is not less than two and the number of -1's is not less than two. Prove that there are two rows and two columns such that the sum of the four numbers in the squares at their intersections is equal to 0.

43.7.3. Consider a convex 100-gon. Prove that the greatest number of sides of a convex polygon, whose sides lie on diagonals of the 100-gon, is ≤ 100.

43.7.4. Three straight corridors of equal length l form a figure shown in Fig. 88. A cop and a robber are running along the corridors. The top speed of the cop is two times that of the robber. The cop is shortsighted and can only recognize the robber when the distance between them is $\leq r$. Prove that the cop will always catch the robber if a) $r > \frac{l}{3}$; b) $r > \frac{l}{4}$. (See Problem 437.4.)

FIGURE 88. (Probl. 43.7.4)

43.7.5. Ten vertices of a regular 20-gon $A_1A_2A_3 \dots A_{20}$ are painted black, and 10 are painted white. Consider the set consisting of diagonal A_1A_4 and all the other diagonals of the same length. Prove that in this set the number of diagonals with two black endpoints is equal to the number of diagonals with two white endpoints.

Grade 8

43.8.1. Prove that if $a_1 \le a_2 \le a_3 \le \cdots \le a_{10}$, then

$$\frac{a_1 + \cdots + a_6}{6} \le \frac{a_1 + \cdots + a_{10}}{10}.$$

43.8.2. See Problem 43.7.2.

43.8.3. * A point C is on a chord AB of circle K with center at O. Let D be the second intersection point of K with the circle circumscribed around $\triangle ACO$. Prove that $CD = CB$.

43.8.4. See Problem 43.7.4.

43.8.5. See Problem 43.7.5.

Grade 9

43.9.1. Let $a_1 < a_2 < a_3 < \dots$ be an increasing sequence of positive integers such that $a_{n+1} \le 10a_n$ for any $n \in \mathbb{N}$. Prove that the infinite decimal fraction $0.a_1a_2a_3\dots$ obtained by writing these numbers one after another is non-periodic.

43.9.2. There are several push buttons on a panel that controls lamps on a desk. Pressing any button turns some lamps on the desk on or off (every button governs its own set of lamps and the sets may intersect). Prove that the number of all possible states of the desk is equal to a power of 2.

43.9.3. On an $m \times n$ rectangular piece of graph paper there are several squares whose sides are on the vertical and horizontal lines of the paper. It is known that no two squares coincide and no square is situated inside another one. What is the maximal number of such squares?

43.9.4. See Problem 43.7.4 for a) $r > \frac{l}{5}$; b*) $r > \frac{l}{7}$.

43.9.5*. See Problem 43.8.3.

Grade 10

43.10.1. See Problem 43.9.1.

43.10.2. See Problem 43.9.2.

43.10.3. See Problem 43.9.4.

43.10.4. One of the numbers -1, 0 or 1 is written in every square of a 1980×1980 table. The sum of all numbers is equal to 0. Prove that there exist two rows and two columns such that the sum of the four numbers written in the squares of their intersections is equal to 0.

43.10.5. On a unit sphere, there are given several arcs of great circles. The sum of the length of all these arcs is less than π. Prove that there is a plane passing through the center of the sphere and not intersecting any of the arcs; see Fig. 89.

Figure 89. (Probl. 43.10.5)

Olympiad 44 (1981)

Grade 7

44.7.1. The remainders after divisions of a positive integer A by 1981 and 1982 are both equal to 35. What is the remainder after division of A by 14?

44.7.2. See Problem 44.9.1 below for a 13-digit number.

44.7.3. A painter drew two identical dragons on two identical paper discs so that the first dragon's eye is at the center of the first disc and the second dragon's eye is not at the center of the second disc. Prove that it is possible to cut the second disc into two parts so that they can be put together again and the same disc with the same dragon can be obtained but this time the dragon's eye will be at the center.

44.7.4. Recall that $[x]$ is the integer part of x. For a number x greater than 1, is it necessary that $\left[\sqrt{[\sqrt{x}]}\right] = \left[\sqrt{\sqrt{x}}\right]$?

44.7.5. There are 5 identically looking weights. Their masses are 1000 g, 1001 g, 1002 g, 1004 g, and 1007 g but we do not know which mass is which. Given a balance with an arrow that shows mass in grams, how to find the weight with mass 1000 g in three weighings?

Grade 8

44.8.1. In a pentagon, all diagonals are drawn. Which 7 angles between the diagonals or between the diagonals and the sides should be marked so that if the angles marked are equal it would follow that the pentagon is regular?

44.8.2. See Problem 44.7.2.

44.8.3. See Problem 44.7.3.

44.8.4. See Problem 44.7.4.

44.8.5. Given 10 positive integers $a_1 < a_2 < a_3 < \dots < a_{10}$, prove that their least common multiple is not less than $10a_1$.

Grade 9

44.9.1. A number is expressed with an odd number of digits. Prove that it is possible to strike out one of its digits so that in the number obtained, there are as many 7's in even places as in odd places.

44.9.2. Positive integers $a_1, a_2, \dots, a_n$ are such that each of them is not greater than its index (i.e., $a_k \leq k$), and the sum of all numbers is even. Prove that one of the sums $a_1 \pm a_2 \pm a_3 \pm \dots \pm a_n$ is equal to zero.

44.9.3. X and Y are two convex polygons, X lies inside Y. Let $S(X)$ and $S(Y)$ be the areas of the polygons, and $P(X)$ and $P(Y)$ be their perimeters. Prove that $\dfrac{S(X)}{P(X)} < 2\dfrac{S(Y)}{P(Y)}$.

44.9.4*. Is it possible to divide the set of positive integers into an infinite number of infinite subsets, so that each subset can be obtained from any other one by adding a fixed integer element-wise?

44.9.5*. 64 vertices of a regular 1981-gon are marked. Prove that there exists a trapezoid with vertices in marked points.

Grade 10

44.10.1. A function $y = f(x)$ is defined on the whole real line and satisfies the relation $f(x+k)(1 - f(x)) = 1 + f(x)$ for some $k \neq 0$. Prove that $f(x)$ is a periodic function.

44.10.2. Given a positive integer p and a polynomial $P(x)$ of degree n with leading coefficient 1 and such that if y is an integer, then $P(y)$ is an integer divisible by p. Prove that $n!$ is divisible by p. (Cf. Problems 20.1.7.2 and 20.1.8.5.)

44.10.3. Prove that the sequence $x_n = \sin(n^2)$ does not tend to 0 as $n \longrightarrow \infty$.

44.10.4. Inside a unit square lies a non-selfintersecting broken line of length ≥ 200. Prove that there is a straight line parallel to one of the sides of the square that intersects the broken line in no fewer than 101 points.

44.10.5. Consider a triangle. The radius of the inscribed circle is equal to $\frac{4}{3}$; the lengths of the triangle's heights are integers whose sum is equal to 13. Find the lengths of the triangle's sides.

44.10.6*. n people sit at a round table. Any two neighbors may change places. What is the least number of times that people must change places so that in the end they all have their initial neighbors but in the reverse order?

Olympiad 45 (1982)

Grade 7

45.7.1. At *Turing Machines* store Pete bought a calculator that performs the following operations: it can calculate $x+y$ and $x-y$ for any numbers x and y and $\frac{1}{x}$ for $x \neq 0$. Pete says that he can find the square of any positive number in not more than 6 operations on his calculator. a) If you also can, explain how. b) Can you, moreover, multiply any two positive integers in not more than 20 operations if you are allowed to write down intermediate results and use them during your calculations many times?

45.7.2. There are 5 points inside square $ABCD$. Prove that the distance between some two of them is not greater than $\frac{AC}{2}$.

45.7.3. At *Turing Machines* store Pete bought a paid calculating machine that for 5 kopeks multiplies any number punched into it by 3 and for 2 kopeks adds 4 to any number. Pete wants to obtain the number 1981 for the least amount of money and begins with 1 which may be punched in for free. How much will his calculations cost Pete's parents? Same question if he wants to obtain 1982.

45.7.4. What least number of points on a plane must be selected so that among all distances between pairs of points there should be 1, 2, 4, 8, 16, 32, 64?

Grade 8

45.8.1*. Simplify the expression:

$$\frac{2}{\sqrt{4-3\sqrt[4]{5}+2\sqrt{5}-\sqrt[4]{125}}}.$$

45.8.2. A rectangle is cut into 5 rectangles. Prove that there is a pair of these 5 rectangles one of which fits completely inside the other.

45.8.3. The squares of 1, 2, 3, … , 1982 are juxtaposed in some order to form a number. Can the number obtained be the square of an integer?

45.8.4. All diagonals of a convex pentagon are parallel to the opposite sides. Prove that the ratio of every diagonal to the opposite side is equal to $\frac{\sqrt{5}+1}{2}$.

45.8.5. a) Knowing that (one can easily prove this by induction)

$$1^3+2^3+\dots+n^3=\left(\frac{n(n+1)}{2}\right)^2,$$

prove that for distinct positive integers $a_1, a_2, \dots, a_n$ the following inequality holds:

$$(a_1^7+a_2^7+\dots+a_n^7)+(a_1^5+a_2^5+\dots+a_n^5)\geq 2(a_1^3+a_2^3+\dots+a_n^3)^2.$$

b) Are there some distinct positive integers $a_1, a_2, \dots, a_n$ for which the equality is attained?

Grade 9

45.9.1. Find all integers n for which the number $n2^n+1$ is divisible by 3.

45.9.2. On a plane find a point such that the sum of the distances from it to four given points is minimal.

45.9.3. On a plane the points with integral coordinates are marked. Prove that there exists a circle with exactly 1982 marked points inside it.

45.9.4. The number

$$A = 0.1+0.02+0.003+\dots+n\cdot 10^{-n}+\dots$$

is written in the form of an infinite decimal fraction. Prove that the digits 1982 in succession do not appear in this decimal.

45.9.5. Two sides of a convex quadrilateral are of length 1 and two other sides and both diagonals are not longer than 1. What is the longest possible perimeter of the quadrilateral?

Grade 10

45.10.1. a) Prove that if all edges of a regular tetrahedron subtend equal angles with a common vertex inside the tetrahedron, then this vertex is the center of the sphere circumscribed around the tetrahedron.

b) Can the vertices of equal angles subtending the tetrahedron's edges be outside the tetrahedron?

Note: If the vertex lies on an edge or its extension, we say that the edge subtends an angle of π or 0, respectively.

45.10.2. a) Let a, b, c be the lengths of a triangle's sides. Prove that

$$a^4+b^4+c^4-2(a^2b^2+a^2c^2+b^2c^2)+a^2bc+b^2ac+c^2ab \geq 0.$$

b) Prove that the inequality in a) holds for any $a, b, c \geq 0$.

45.10.3. Pete bought a useful calculator at the *Turing Machines* store: it can find $xy+x+y+1$ for any real numbers x and y but cannot perform any other operations. Pete wants to write a "program" to compute the polynomial $1+x+x^2+\dots\dots+x^{1982}$. He regards his "program" to be the sequence of polynomials $f_1(x), \dots, f_n(x)$ such that

$$f_1(x)=x; \quad f_n(x)=1+x+\dots+x^{1982};$$

$f_i(x)$ for $1<i<n$ is either $\begin{cases}\text{a constant } c_i \text{ that Pete can choose or}\\ f_j(x)\cdot f_k(x)+f_k(x)+f_j(x)+1, \text{ where } j,k<i \text{ for each } i=2,\dots,n.\end{cases}$

a) Write Pete's "program".

b) Can one write a "program" for the calculator that can only perform the following operation: $x, y \mapsto xy+x+y$?

45.10.4. Find all positive integers n for which both $\frac{1}{n}$ and $\frac{1}{n+1}$ are finite decimal fractions.

45.10.5. A regular hexagon with side a is inside another regular hexagon with side $2a$. Prove that the center of the larger hexagon is inside the smaller one.

Olympiad 46 (1983)

Grade 7

46.7.1. Find all pairs of integers (x, y) satisfying the equation

$$x^2=y^2+2y+13.$$

46.7.2. A white plane is stained with black Indian ink. Prove that for any l there exists a line segment of length l whose both endpoints are of the same color.

46.7.3. A positive integer begins with a 4. If this digit 4 is transplanted to the end of the number, the resulting number is $\frac{1}{4}$ of the original one. Find the smallest such number.

46.7.4. Two friends want to reach a nearby town. They have a bicycle for one person only. The Rule allows any of them to leave the bicycle for the other friend at any place. Their speeds as pedestrians are u_1 and u_2, their speeds on bicycles are v_1 and v_2, respectively, and the distance between the towns is S. What is the least least time the friends need to reach the town?

46.7.5. Is there a pentagon with sides 3, 4, 9, 11 and 13 cm, into which a circle can be inscribed?

Grade 8

46.8.1. Prove that $x^4-x^3y+x^2y^2-xy^3+y^4>x^2+y^2$ for any $x>\sqrt{2}$ and $y>\sqrt{2}$.

46.8.2. Equilateral triangles ABC_1, BCA_1 and CAB_1 are constructed outwards on the sides of triangle ABC. Prove that $\overrightarrow{AA_1}+\overrightarrow{BB_1}+\overrightarrow{CC_1}=\vec{0}$. (See Fig. 90.)

46.8.3. Can the square of a positive integer begin with 1983 nines in a row?

46.8.4. The numbers 1, 2, ... , 1983 stand at the vertices of a regular 1983-gon. Any of the axes of symmetry of the 1983-gon divides the numbers which do not stand at the vertices through which the axis passes (if any) into two sets: on either side of the axis. Let us call an arrangement of numbers *good* with respect to a given axis of symmetry if every number of one set is greater than the number symmetrical to it. Is there an arrangement good with respect to any axis of symmetry?

46.8.5. Given five points on a circle: A_1, A_2, A_3, A_4, H. Denote the distance between H and straight line A_iA_j by h_{ij}. Prove that $h_{12}\cdot h_{34}=h_{14}\cdot h_{23}$.

FIGURE 90. (Probl. 46.8.2)

Grade 9

46.9.1. Prove that
$$\frac{1}{2} < x^{2n} \pm x^{2n-1} + x^{2n-2} \pm x^{2n-3} + \cdots + x^4 \pm x^3 + x^2 \pm x + 1$$
for any signs of odd powers of a real x.

46.9.2. Three circles of radii 3, 4, 5 are externally tangent to one another. The common tangent to the first two circles is drawn through the point at which they are tangent to each other. Find the length of this tangent contained inside the circle of radius 5.

46.9.3. Prove that $1^{1983} + 2^{1983} + \cdots + 1983^{1983}$ is divisible by $1 + \cdots + 1983$.

46.9.4. Twenty towns are connected by 172 airlines; not more than one airline connects two towns. Prove that using these airlines one can fly from any town to any other (perhaps changing lines).

Grade 10

46.10.1. Let A_1, B_1, C_1 be the points where the circle inscribed into triangle ABC is tangent to sides BC, AC and AB, respectively. It is known that $AA_1 = BB_1 = CC_1$. Prove that triangle ABC is equilateral.

46.10.2. Prove that $4^m - 4^n \,\vdots\, 3^{k+1}$ if and only if $m - n \,\vdots\, 3^k$, where a) $k = 1, 2, 3$; b) $k \in \mathbb{N}$.

46.10.3. After classes, the following inscription was left on a blackboard (instead of the erased numbers we write $***$ in this book):

"Find
$$t(0) - t\left(\frac{\pi}{5}\right) + t\left(\frac{2\pi}{5}\right) - t\left(\frac{3\pi}{5}\right) + \cdots + t\left(\frac{8\pi}{5}\right) - t\left(\frac{9\pi}{5}\right), \qquad (*)$$
where
$$t(x) = \cos 5x + *** \cos 4x + *** \cos 3x + *** \cos 2x + *** \cos x + *** \cos 0 \qquad (**)\text{"}$$
A student told his girlfriend that he could find the sum $(*)$ even without knowing the coefficients erased from the blackboard in $(**)$. Is he just boasting?

46.10.4. Consider eight points in space such that no four of them lie on the same plane, and 17 segments with both endpoints in given points. Prove that the segments form a) at least one triangle; b)* ≥ 4 triangles.

46.10.5. 13 knights from k towns ($1 < k < 13$) are sitting at a round table. Every knight holds a gold or a silver goblet in his hand, and the number of gold goblets is also equal to k. Prince tells every knight to pass his goblet to the neighbor on his right and to repeat this *until a pair of knights from the same town gets golden goblets*. Prove that eventually Prince's wish will be fulfilled and the knights will be able to pass to refreshments.

Olympiad 47 (1984)

Grade 7

47.7.1. Some people call a bus ticket *lucky* if the sum of digits in its number is divisible by 7. Is it possible for two tickets with consecutive numbers to be lucky?

Note. In 1984 bus tickets in Moscow were numbered 000000 to 999999.

47.7.2. Paths in a zoo form an equilateral triangle with the midpoints of its sides connected. A monkey has run away from its cage and two guards are trying to catch it. Can they catch the monkey if all three run only along the paths, the speed of the monkey and that of the guards are equal and they all can see one another at all times? (Cf. Problem 33.2.8.2).

47.7.3. A customer bought some goods worth 10 roubles and gave a 25-rouble note to the salesman. The salesman did not have change at the moment and so he asked his neighbor to change the note. After they got even and the customer had gone, the neighbor discovered that the note was counterfeit. The salesman returned 25 roubles to his neighbor and pondered: how much money did he lose? Same question to you.

47.7.4. A parallelogram is cut out of a paper triangle. Prove that the area of the parallelogram is not greater than half the area of the triangle. (See Fig. 91.)

FIGURE 91. (Probl. 47.7.4)

47.7.5. There are 10 rooks and a king on a 20×20 chessboard. The king is not in check and moves along the diagonal from the lower left corner to the upper right corner. The pieces move taking turns as follows: first the king, then one of the rooks.

Prove that no matter what the initial position of the rooks is or how they move, the king will either be in check or bump into a rook.

Grade 8

47.8.1. Solve the equation $\dfrac{x^3}{\sqrt{4-x^2}} + x^2 - 4 = 0$.

47.8.2. Every two of six computers are to be connected by one colored cable. Choose one color out of five for each cable so that cables of five different colors would come out of each computer.

47.8.3. Prove that the sum of distances from the center of a regular heptagon to all its vertices is less than that from any other point.

47.8.4. The sum of five non-negative numbers is equal to 1. Prove that it is possible to arrange them in a circle so that the sum of all five products of pairs of neighboring numbers is not greater than $\frac{1}{5}$.

47.8.5. Cut a square into 8 acute triangles. (Cf. Problem 47.10.5 below.)

47.8.6. Is the number of all 64-digit positive integers without zeros in their decimal expression and are divisible by 101 even or is it odd?

Grade 9

47.9.1. In a triangular pyramid 3 lateral edges are equal to one another, and the areas of three lateral faces are equal to one another. Prove that the base of the pyramid is an isosceles triangle.

47.9.2. Is it possible to connect 13 computers in pairs with cables of twelve different colors so that 12 cables of different colors come out of each computer? (Cf. Problem 47.8.2.)

47.9.3. What is the least possible width of an infinite strip from which any triangle of area 1 can be cut out?

47.9.4. On a circle, there are arranged n non-negative numbers whose sum is equal to 1. Prove that the sum S_n of n products of two neighboring numbers is not greater than $\frac{1}{4}$. (Cf. Problem 47.8.4).

47.9.5*. Given 4 points inside a 3×4 rectangle. Prove that there are two among the given points that are not farther than $\frac{25}{8}$ apart.

47.9.6. Do there exist three non-zero digits with which the squares of an infinite number of different integers can be expressed?

Grade 10

47.10.1. Prove (without using calculators, tables and such) that $\sin 1 < \log_3 \sqrt{7}$.

47.10.2. At the Olympiad 6 problems were offered. The Olympiad jury decided to assign to every participant a positive integer according to his/her results in the Olympiad so that it would be possible to reconstruct unambiguously the score every participant got for every problem and so that for every two participants the greater number would be assigned to the one with the greater sum of scores. How could the jury enumerate the participants?

47.10.3. Solve in integers $19x^3 - 84y^2 = 1984$.

47.10.4. Let $n_1 < n_2 < n_3 < n_4 < \dots$ be an infinite sequence of positive integers. In a kingdom there was minted an infinite number of coins of denominations $n_1, n_2, n_3, n_4, \dots$ kopeks.

Prove that it is possible to break the sequence at some point N so that any amount of money which can be paid without need for change with all coins minted can in fact be paid with the coins of denominations of $n_1, n_2, n_3, \dots, n_N$ kopeks only.

47.10.5. A square is cut into acute triangles. Prove that there are ≥ 8 such triangles. (Cf. Problem 47.8.5).

47.10.6*. A triangle section of a cube is tangent to the sphere inscribed in the cube. Prove that the area of the section is less than half the area of the cube's face.

Olympiad 48 (1985)

Grade 7

48.7.1. Solve the equation $xy + 1 = x + y$.

48.7.2. Given five distinct positive numbers. They can be divided into two groups so that the sums of the numbers in these groups are equal. In how many ways can this be done?

48.7.3. The lengths a, b, c, d of four segments satisfy the inequalities $0 < a \leq b \leq c < d$ and $d < a+b+c$. Is it possible to construct a trapezoidal from these segments?

48.7.4. A rabbit is sitting in the center of a square and 4 wolves are sitting in the four vertices. Is it possible for the rabbit to run out of the square if the wolves can only run along the sides and the wolf's top speed is 1.4 times higher than that of the rabbit?

48.7.5. A tank of milk was brought to a store. The salesman has a balance and pans but no weights. However, milk cans can be put on a pan and there are three identical milk cans in the store, two of which are empty, and the third one has 1 liter of milk in it. A can holds not more than 85 l. By a *weighing* we mean putting a can with milk on one balance pan and an empty can on the other pan whereupon milk is added to the empty can until the balance is in equilibrium. How can the salesman pour 85 l of milk into one can weighing not more than 8 times?

Grade 8

48.8.1. Solve the equation $(x - y + z)^2 = x^2 - y^2 + z^2$.

48.8.2. The numbers $a_1, a_2, \dots, a_{1985}$ are the numbers $1, 2, 3, \dots, 1985$ arranged in some order. Prove that $\max\limits_k k \cdot a_k \geq 993^2$.

48.8.3. A paper square Q is placed on a piece P of graph paper; the area of Q is four times that of a little square q of the graph paper. Let a *node* be an intersection of lines on the paper; a node on the boundary of Q is considered to be *covered*. What is the least number of nodes that Q can cover? (See Fig. 92.)

48.8.4. An infinite number of knights lined up in a row in front of Wizard. Prove that Wizard can tell some of them to stand out of line, so that there would still be an infinite number of knights left in line, and so that all knights in line would stand ordered with respect to their height in increasing or decreasing order.

48.8.5. Prove that if the length of every one of the three bisectors of a triangle is greater than 1, then its area is greater than $\dfrac{1}{\sqrt{3}}$.

Grade 9

48.9.1. Solve the eqation $\sqrt{x - y + z} = \sqrt{x} - \sqrt{y} + \sqrt{z}$.

FIGURE 92. (Probl. 48.8.3)

48.9.2. In some country there are 1985 airports. Consider the Earth to be a plane, the air routes to be straight lines, and all pairs of distances between the airports to be distinct. From every airport an airplane departs and lands at the airport fartherest from the place of its departure. Is it possible that as a result all 1985 airplanes arrived in 50 airports?

48.9.3. Under notations of Problem 48.8.3, suppose we know that a 2×2 square covers ≥ 7 nodes of the graph plane. How many nodes (exactly) can a 2×2 square cover?

48.9.4. Prove that it is possible to select two people from a group of 12, and then choose five more people from the remaining 10 so that each of these five people satisfies the following condition: (s)he is either a friend of both or of neither of the people in the pair chosen first.

48.9.5* (*Leonard Euler*'s problem)**.** . Prove that any number 2^n for $n \geq 3$ can be expressed as $2^n = 7x^2 + y^2$, where x and y are odd.

Grade 10

48.10.1. Solve the equation

$$\frac{x-49}{50} + \frac{x-50}{49} = \frac{49}{x-50} + \frac{50}{x-49}.$$

48.10.2. See Problem 48.7.3.

48.10.3. Let the "complexity" of a given number be the least possible length of a numerical sequence (if there is one) which begins with a 0 and ends with this number, each next term being either equal to half the preceding one or its sum with the preceding term being equal to 1. (The length of the empty sequence is assumed to be equal to 0.) Find the number with the greatest "complexity" among all numbers of the form $\frac{m}{2^{50}}$, where $m = 1, 3, 5, \dots, 2^{50} - 1$.

48.10.4. We have 1985 sets. Each of the sets has 45 elements, the union of any two sets has exactly 89 elements. How many elements has the union of all these 1985 sets?

48.10.5. Prove that if the distances between skew edges of a tetrahedron are equal to h_1, h_2, h_3, respectively, then the volume of the tetrahedron is $\geq \frac{1}{3}h_1h_2h_3$.

Olympiad 49 (1986)

Grade 7

49.7.1. A quadrilateral is drawn on a transparent piece of paper. How should the paper be folded (perhaps more than once) in order to ascertain whether the quadrilateral is a rhombus? (Cf. Problem 49.8.1.)

49.7.2. Prove that there are no numbers x, y, z satisfying the system

$$\begin{cases} |x| < |y - z|, \\ |y| < [z - x|, \\ |z| < |x - y| \end{cases}$$

49.7.3. Three dwarfs live in different houses on a plane and walk with speeds 1, 2 and 3 km/h, respectively. What place for their everyday meetings should they choose to minimize the sum of the times it takes them to walk from their houses to this place (each walks along a straight line)?

49.7.4. The product of some 1986 positive integers has exactly 1985 different prime divisors. Prove that either one of these integers or the product of some of them is a perfect square. (Cf. Problem 49.9.4.)

49.7.5. A code lock has three buttons with numbers 1, 2, 3. The code is a three-digit number, and the lock opens only if you press all three buttons in succession in the right order. What least number of times must Houdini press the buttons to unlock the lock?

Grade 8

49.8.1. A quadrilateral is drawn on a transparent piece of paper. How should the paper be folded (perhaps more than once) in order to ascertain whether the quadrilateral is a square? (Cf. Problem 49.7.1.)

49.8.2. Find all positive integers which cannot be expressed as the difference of the squares of some positive integers.

49.8.3. Prove that if $a_1 = 1$, $a_n = \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}}$ for $n = 2, 3, \dots, 10$, then $0 < a_{10} - \sqrt{2} < 10^{-370}$.

49.8.4. A square field is divided into 100 identical square plots, nine of which become overgrown with weeds. It is known that every next year weeds begin to grow on the plots which are adjacent (have a common side) to at least two plots overgrown with weeds the year before and only on these plots. Prove that the whole field will never become overgrown with weeds.

49.8.5. Prove that there are no solutions to the system

$$\begin{cases} |x| > |y - z + t|, \\ |y| > |x - z + t|, \\ |z| > |x - y + t|, \\ |t| > |x - y + z|. \end{cases}$$

Grade 9

49.9.1. Points A, B, C, D are marked on a piece of paper. A detecting device can perform two types of operations: (a) measure the distance between two given points in centimeters; (b) compare two given numbers. What least number of operations must be performed to ascertain whether quadrilateral $ABCD$ is a rectangle?

49.9.2. An ant moves at a constant speed starting from point M on a plane. Its path is a spiral that winds around a point O and is homothetic to some part of itself with respect to this point. Is it possible for the ant to cover its entire pass in a finite time?

49.9.3. Solve the system:

$$\begin{cases} |x| < |y - z + t|, \\ |y| < |x - z + t|, \\ |z| < |x - y + t|, \\ |t| < |x - y + z|. \end{cases}$$

49.9.4. A product of some 48 positive integers has exactly 10 different prime divisors. Prove that the product of some four of these integers is a perfect square. (Cf. Problem 49.7.4.)

49.9.5. Discs of radius $\frac{1}{14}$ and with centers at every point with integer coordinates are drawn on the coordinate plane. Prove that any circle of radius 100 intersects at least one of the discs drawn.

Grade 10

49.10.1. See Problem 49.9.1 with *rectangle* replaced with *square* in the quastion.

49.10.2. The bisector of angle A of triangle ABC is extended until it meets (at point D) the circumscribed circle. (See Fig. 93.) Prove that $AD > \frac{1}{2}(AB + AC)$.

49.10.3. Solve the equation $x^{x^4} = 4$ for $x > 0$.

49.10.4. Prove that there are no vector solution to the system:

$$\begin{cases} \sqrt{3}|\mathbf{a}| < |\mathbf{b} - \mathbf{c}|, \\ \sqrt{3}|\mathbf{b}| < |\mathbf{c} - \mathbf{a}|, \\ \sqrt{3}|\mathbf{c}| < |\mathbf{a} - \mathbf{b}|. \end{cases}$$

49.10.5. For $y(x) = |\cos x + \alpha \cos 2x + \beta \cos 3x|$ find $\min\limits_{\alpha,\beta} \max\limits_{x} y(x)$.

Figure 93. (Probl. 49.10.2)

Olympiad 50 (1987)

Grade 7

50.7.1. In March the math club held 11 meetings. Prove that if there were no meetings on weekends, then in March there were three days in a row during which no meetings were held.

50.7.2. Prove that among any 27 different positive integers less than 100 each there are two not relatively prime ones.

50.7.3. On a meadow shaped in the form of an equilateral triangle with side 100 m a wolf is running. A hunter can hit the wolf if (s)he shoots from a distance not greater than 30 m. Prove that the hunter can hit the wolf no matter how quickly it runs.

Figure 94. (Probl. 50.7.3)

Figure 95. (Probl. 50.8.4)

50.7.4. Let AB be the base of trapezoid $ABCD$. Prove that if $AC + BC = AD + BD$ then $ABCD$ is an isosceles trapezoid.

50.7.5. Ali-Baba and 40 thieves have to split a treasure of 1987 gold coins among themselves according to the following Rule: the first thief splits the whole treasure into two parts; then the second thief divides one of these parts into two parts, etc. After the fortieth division, the first thief takes the greatest of the parts; then the second thief takes the greatest of the remaining parts, etc. The last, fortyfirst, part goes to Ali-Baba.

What is the greatest number of coins each thief can get under this Rule regardless of the other thieves' actions?

Grade 8

50.8.1. Prove that $\frac{1}{2}\left(\frac{x}{a} + \frac{y}{b}\right) > \frac{x+y}{a+b}$ for $a > b > 0$ and $\frac{x}{a} < \frac{y}{b}$.

50.8.2. A boy decided to cut out of a $2n \times 2n$ piece of paper the greatest possible number of $1 \times (n+1)$ rectangles. What is this number if: a) $n < 3$; b) $n = 3$; c) $n > 3$?

50.8.3. A teacher organizes a tug-of-war tournament and decides that all possible teams that can be made from students of her class (obviously not counting the whole class as a team) should participate exactly once. Prove that each team will compete with the team made up of the remaining students.

50.8.4. In pentagon $ABCDE$, $\angle ABC$ and $\angle CDE$ are right angles, $\angle BCA = \angle DCE$, and M is the midpoint of side AE. Prove that $MB = MD$ (See Fig. 95.)

50.8.5. Is there a set of positive integers such that for any positive integer n at least one of the numbers n, $n + 50$ belongs to the set, and at least one of the numbers n or $n + 1987$ does not?

Grade 9

50.9.1. Given a set of 7 different integers from 0 to 9. Prove that for any positive integer n there exists a pair of integers from the set whose sum ends with the same digit as n does.

50.9.2. Given k vertices of a regular pentagon, find the remaining vertices using a two-sided ruler for a) $k = 4$, b) $k = 3$.

50.9.3. Find 50 positive integers such that none of them is divisible by another, and the product of any two is divisible by any of the rest.

50.9.4. Prove that if $n = 1987$, then

$$\frac{(a_1 + \dots + a_n)^2}{b_1 + \dots + b_n} \le \frac{a_1^2}{b_1} + \dots + \frac{a_n^2}{b_n}$$

for any $a_1, a_2, \dots, a_n$ and positive $b_1, b_2, \dots b_n$.

50.9.5. Tanya dropped a ball into a huge rectangular pool. She wants to rescue it using 30 narrow planks, each 1 m long to make a bridge so that each plank is supported by either the edges of the pool or by the planks already settled, and so that ultimately one of the planks is right over the ball. Prove that Tanya will not be able to do this if the distance from the sides of the pool to the ball exceeds 2 m. (See Fig. 96.)

FIGURE 96. (Probl. 50.9.5)

Grade 10

50.10.1. a) Prove that of three positive numbers it is always possible to select two, say, x and y, so that $0 \le \dfrac{x - y}{1 + xy} \le 1$.

b) Is it possible to select such numbers from any 4 (not necessarily positive) numbers?

50.10.2. The measures of the angles between a plane in space and the sides of an equilateral spatial triangle are equal to α, β, γ. Prove that one of the numbers $\sin\alpha$, $\sin\beta$, $\sin\gamma$ is equal to the sum of the other two.

50.10.3. On a piece of graph paper, 17 squares with side 1 are shaded. Prove that they can be covered by rectangles, the sum of whose perimeters is less than 100, so that the distance between any two points on distinct rectangles is $\ge \sqrt{2}$.

50.10.4. Is it possible to divide the set of integers into 3 subsets so that for any integer n the numbers n, $n - 50$, $n + 1987$ would belong to different subsets?

50.10.5. The side of a square shaped kingdom is 2 km. The king of this kingdom decides to summon all his subjects to a ball at 7 p.m. At noon he sends a messenger who may give any orders to any citizen who, in turn, is empowered to give any order to any other citizen, etc. The whereabouts (home) of each citizen are known and every citizen can move at a speed of 3 km/h in any direction. Prove that the king can organize the transmission of messages so that all his loyal subjects can reach the court in time for the opening of the ball.

Olympiad 51 (1988)

Grade 7

51.7.1. Prove that for any prime $p > 7$ the number $p^4 - 1$ is divisible by 240.

51.7.2. Points M and P are the midpoints of two edges of a cube. On the surface of the cube, find the locus of points equidistant from M and P. The distance between two points of the surface is calculated as the length of the shortest broken line lying on the surface.

51.7.3. Using only a ruler and calipers draw the straight line through a given point and parallel to a given line.

51.7.4. Colored wires connect 20 phones so that each wire connects two phones, not more than one wire connects each pair of phones and not more than two wires lead from each phone. By the Rule we should select the colors of the wires so that every two wires leading from the same phone have different colors. What is the least number of wire's colors needed for such a connection? (Cf. Problem 51.9.5.)

Grade 8

51.8.1. Four numbers: 1, 9, 8, 8 are written in line. We apply to them the following operation: between each two numbers a and b we write their difference $b-a$. Then the same operation is applied to the resulting line, and so on, 100 times. What is the sum of all numbers in the final line?

51.8.2. Find the midpoint of a given segment using only a ruler without marks on it and calipers.

51.8.3. Prove that the equation $3x^4 + 5y^4 + 7z^4 = 11t^4$ has no solution in natural numbers.

51.8.4. There are four coins and a spring balance with a single pan. It is known that some of the coins may be forged and a real coin weighs 10 g while a forged one only 9 g. How many times has one to weigh the coins to find out for sure which of them are forged?

Grade 9

51.9.1. Consider a convex quadrilateral. Its diagonals divide it into four triangles of integer area. Prove that the product of these four integers cannot end with digits 1988.

51.9.2. Prove that $p_1^2 + p_2^2 + \ldots + p_{24}^2 \vdots 24$ for any primes $p_1, p_2, \ldots, p_{24} \geq 5$.

51.9.3. Two perpendicular straight lines lie on a plane. Using only calipers find three points on the plane that represent vertices of an equilateral triangle.

51.9.4. Let $f(x, y) = \frac{1}{2}(x + y - 1)(x + y - 2)$ be a function of two positive integers. Prove that for any positive integer z there exists a single pair x, y such that $f(x, y) = z$.

51.9.5. Colored wires connect 20 phones so that each wire connects two phones, not more than one wire connects each pair of phones and not more than three wires lead from each phone. One is asked to select the colors of the wires so that every two wires leading from the same phone have different colors. What is the least number of wires' colors needed to establish any such connection?

Grade 10

51.10.1. A calculator can add, subtract, divide, multiply and take the square root. Find a formula to calculate the minimum of two numbers using the calculator.

51.10.2. Is there a straight line on the coordinate plane such that the graph of the function $y = 2^x$ is symmetric with respect to this line?

51.10.3. Can one intersect any parallelepiped with a plane so that the section is a rectangle?

51.10.4. One has a one-sided ruler, a pencil and a length standard allowing one to find on a previously drawn straight line a point at fixed distance from some other point on the same line. Draw a perpendicular to a given straight line using only these instruments.

51.10.5. One selects a pair of positive integers and performs the following *operation*: the greater number of the pair (the first one it they are equal) is divided by the other number, and the pair: (the quotient, the remainder) replace the original pair. Then the operation is repeated until the smaller number becomes 0. We start with numbers not greater than 1988. Prove that not more than 6 operations can be performed.

Olympiad 52 (1989)

Grade 7

52.7.1. We cut a square into 16 smaller equal squares. How to place each of the letters A, B, C, and D in the squares in four ways so that no horizontal, no vertical and none of the two greater diagonals would contain the same letters.

52.7.2. Given a fixed line l and passing through a given point not on l. With the help of a ruler and compass draw a straight line parallel to l and passing through the given point by drawing the least possible number of curves (circles and straight lines).

52.7.3. There are 4 pairs of socks of two different sizes and of two colors lying pell-mell on a shelf in a dark room. What is the minimal number of socks from the shelf that we should put into a bag, without leaving the room for inventory of the bag, in order to have in the bag two socks of the same size and color?

52.7.4. A tourist left a tourist lounge in a boat at 10:15. (S)he promised to come back not later than at 1:00 p.m. the same day. The speed of the river's current is known to be 1.4 km/h and the top speed of the boat in still water is 3 km/h. What is the greatest distance from the lounge that the tourist can cover if (s)he rests for 15 minutes after every 30 minutes of rowing without mooring and may turn back only after a rest?

52.7.5. Find all positive integers x satisfying the following condition: the product of the digits of x is equal to $44x - 86\,868$ and their sum is equal to a cube of a positive integer.

Grade 8

52.8.1. Solve the equation $(x^2 + x)^2 + \sqrt{x^2 - 1} = 0$.

52.8.2. Some randomly chosen squares of an infinite graph paper are red and the rest are white. A grasshopper jumps on red squares and a flea on white ones and each jump can be made over any distance vertically or horizontally. Prove that the grasshopper and the flea can find themselves side by side after at most three jumps.

52.8.3. Construct with the help of a ruler and compass the perpendicular to the given straight line passing through the given point (a) not in this line and (b) on this line. You may only draw the least possible number of curves (circles and straight lines).

52.8.4. A subset X of the set of all two-digit "numbers" 00, 01, ... , 98, 99 is such that any infinite sequence of digits contains two neighboring digits that form a number from X. What is the least cardinality of X?

52.8.5. Prove that a party of scouts can be always divided into two teams so that the cardinality of the set of pairs of friends in the same team is less than that of the set of pairs of friends who found themselves in distinct teams.

52.8.6. If $|ax^2 + bx + c| \le 1$ for $x \in [0, 1]$ what can the greatest possible value of $|a| + |b| + |c|$ be?

Grade 9

52.9.1. There are 4 different straight lines in space. Two lines are red and two are blue, any red line is perpendicular to any blue line. Prove that either red lines are parallel or blue lines are parallel.

52.9.2. Points M, K, and L are selected on sides AB, BC, and AC, respectively, of $\triangle ABC$ so that $MK \parallel AC$ and $ML \parallel BC$. Segment BL meets MK at P while AK meets ML at Q. Prove that segments $PQ \parallel AB$.

52.9.3. The numbers A_1, A_2, ... form a geometric progression, and so do B_1, B_2, ... We form a new sequence by adding the progressions term-wise: $A_1 + B_1$, $A_2 + B_2$, ... , etc. Can you determine the fifth term of the new sequence if you know the first four of its terms?

52.9.4. The streets of a city are represented on a map as straight lines that divide a square into 25 smaller squares of side 1. (The borderline of the city is considered to be the union of 4 streets.) There is a snow plow at the bottom right corner of the bottom left square. Find the length of the shortest path for the plow to pass through all streets and come back to its starting point.

52.9.5. Find all positive numbers $x_1, x_2, \dots, x_n$ that satisfy the system of n equations:

$$(x_1 + x_2 + \ldots + x_k)(x_k + x_{k+1} + \ldots + x_n) = 1, \quad k = 1, 2, \dots, n$$

if a) $n = 3$, b) $n = 4$, c) $n = 10$, d) n is an arbitrary integer.

Grade 10

52.10.1. Solve the equation $\lg(x-2) = 2x - x^2 + 3$.

52.10.2. Is there a function whose graph on the coordinate plane has a common point with any straight line?

52.10.3. Is it possible to put down crosses and noughts on a sheet of graph paper of an arbitrary (or infinite) size so that no three signs in a row would be the same on any vertical, horizontal, or diagonal line?

52.10.4. Consider n distinct natural numbers. Prove that any infinite arithmetic progression whose first term does not exceed its difference, d, contains 3 or 4 of the numbers considered if a) $n = 5$, b) $n = 1989$.

52.10.5. Calculate with an accuracy to 2.0 the least total length of the cuts that must be made to recut a unit square into a rectangle with diagonal of length 100.

52.10.6. We select a point on every edge of an arbitrary tetrahedron. We draw a plane through every three points that belong to edges with a common vertex. Prove that if three of the four planes thus drawn are tangent to the sphere inscribed into the tetrahedron, the fourth plane is also tangent to it.

Olympiad 53 (1990)

Grade 8

53.8.1. Prove that if $0 < a_1 < a_2 < \ldots < a_9$, then
$$\frac{a_1 + a_2 + \ldots + a_9}{a_3 + a_6 + a_9} < 3.$$

53.8.2. Let $M = m(n+9)(m+2n^2+3)$. What is the least number of distinct prime divisors the number M can have?

53.8.3. 11 winners of grades 8, 9, 10 and 11 were invited to pass a selection test to an Olympiad. Can they be arranged at a round table so that among any five successive students there are representatives of all four grades?

53.8.4. Quadrilateral $ABCD$ is inscribed in a circle; $AB = BC$. Let diagonals meet at O, let E be the other intersection point of CD with the circle that passes through B, C and O. Prove that $AD = DE$.

53.8.5. A display board composed of 64 bulbs is controlled by 64 buttons, each bulb being switched on/off by a separate button. Any set of buttons can be pushed simultaneously. This was done and the bulbs that lighted as a result were marked. What is the least number of switchings that allows one to find out which button controls which bulb?

Grade 9

53.9.1. 7 boys got together and each of them has three brothers among the other present. Prove that all seven boys are brothers.

53.9.2. Prove that among any 53 distinct natural numbers whose sum does not exceed 1990 there are two numbers whose sum is equal to 53.

53.9.3. Inside a circle of radius 1 point A is marked. We drew various chords through A and then drew a circle of radius 2 through the endpoints of each chord. Prove that all such circles for various points A are tangent to a certain fixed circle.

53.9.4. There are two counterfeit coins among 8 coins that look alike. One of the counterfeits is lighter and the other is heavier that a genuine coin. Can one find out in three weighings on scales without weights whether the two counterfeit coins together are heavier, lighter or of the same weight as two genuine coins?

53.9.5. The decimal representation of a rational number A is a periodic fraction with the period of length n. What is the longest length of the period of A^2 as A varies?

Grade 10

53.10.1. Can one cut a square into three pairwise non-equal and pairwise similar rectangles?

53.10.2. Find all primes p, q, r that satisfy $p^q + q^p = r$.

53.10.3. Prove that for all values of parameters a, b, c there is a number x such that
$$a\cos x + b\cos 3x + c\cos 9x \geq \frac{1}{2}(|a| + |b| + |c|).$$

53.10.4. How should four points in a disc be arranged so as to have the greatest product of all pairwise distances between them?

53.10.5. Points A, B, C, D in space are positioned so that segment BD subtends angles $\angle A$ and $\angle C$ of measure α and AC subtends angles $\angle B$ and $\angle D$ of measure β. Find the ratio $AC : BD$ if $AB \neq CD$.

Grade 11

53.11.1. Find $\max\limits_{x,y}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$.

53.11.2. Prove that if a function $f(x)$ is continuous on $[0,1]$ and satisfies the identity $f(f(x)) = x^2$ for all x, then $x^2 < f(x) < x$ for any $x \in (0,1)$. Give an example of such a function.

53.11.3. In triangle ABC, consider median BD and bisector BE. Can it happen that BD is a bisector in $\triangle ABE$ and BE a median in $\triangle BCD$?

53.11.4. Prove that there is a multiple of any odd n, whose decimal representation contains only odd digits.

53.11.5. Four points are projections of a point to four faces of a tetrahedron. How are the points arranged in space?

Olympiad 54 (1991)

Grade 8

54.8.1. Prove that if $a > b > c$, then $a^2(b-c)+b^2(c-a)+c^2(a-b) > 0$.

54.8.2. Given points A and B on a plane, construct a point C on ray AB, such that $AC = 2AB$. Is it possible to do it using a compass with a fixed span r if a) $AB < 2r$, b) $AB \geq 2r$?

54.8.3. To guard a military installation around the clock, a day shift and a night shift are required. A sentry guard may take either a day, or a night shift, or work around the clock. In these cases the guard is given a leave of absence of not less than 1, 1.5 or 2.5 full days, respectively. What is the least number of guards necessary to ensure the security of the installation?

54.8.4. Given 6 seemingly indistinguishable weights of 1, 2, 3, 4, 5 and 6 g, respectively, a drunken workman painted them at random “1 g”, ... , “6 g”. How can you check whether the labels match the weights using only two weighings on a balance without any other weights except the given ones?

54.8.5. An air line was established between two countries so that any two cities, one from each country, are connected by precisely one flight which is a one-way flight and one can fly somewhere from each city. Prove that there are cities A, B, C, D, which can be visited by flying directly from A to B, from B to C, from C to D and from D to A.

Grade 9

54.9.1. Solve the equation:

$$(1+x+x^2)(1+x+\ldots+x^{10}) = (1+x+\ldots+x^6)^2.$$

54.9.2. A conjurer divided a deck of a) 36, b) 54 cards into several piles and wrote a number equal to the number of cards in the pile on each card from every pile. Then he mixed the cards in a special way, divided them into piles once again and wrote another number equal to the number of cards in the new pile on each card to the right of the first number. Could the conjurer do this so that there are no equal pairs among the pairs of numbers on the cards and for every pair a, b there is a “symmetric” pair b, a? (A pair a, a is assumed to be symmetric to itself.) Cf. Problem 54.10.5.

54.9.3. Prove that in a regular 12-gon $A_1A_2\ldots A_{12}$ the diagonals A_1A_5, A_2A_6, A_3A_8 and A_4A_{11} meet at one point.

54.9.4. After the graph of the function $y = \frac{1}{x}$ for positive x was drawn the coordinate axes were erased and their directions forgotten. How to recover the erased axes using a ruler and compass?

54.9.5. Cells of a 15×15 table contain nonzero numbers such that each of them is equal to the product of all neighboring numbers. (Two numbers are said to be neighboring if their cells have a common side.) Prove that all numbers in the table are positive.

Grade 10

54.10.1. A function f satisfies $f(x) + \left(x+\frac{1}{2}\right)f(1-x) = 1$ for any $x \in (-\infty, \infty)$. Find a) $f(0)$ and $f(1)$; b) all such functions f.

54.10.2. What is the number n of identical billiard balls that can be arranged in space so that each ball is tangent to exactly three other balls? List all possible values of n.

54.10.3. Two nonintersecting circles are inscribed in a given angle. An isosceles triangle ABC is placed between the circles so that its vertices are on the sides of the angle and the equal sides AB and AC are tangent to the corresponding circles. Prove that the sum of the radii of the circles is equal to the height of the triangle drawn from vertex A.

54.10.4. We constructed a cube of size $10 \times 10 \times 10$ of 500 black and 500 white small identical cubes so that the cubes adjacent to each other were of different colors. Several of small cubes were removed from the cube so that exactly 1 small cube was missing in each of 300 rows or columns of size $1 \times 1 \times 10$ parallel to an edge of the cube. Prove that the number of black cubes removed is divisible by 4.

54.10.5. A conjurer divided a deck of 54 cards into several piles. A spectator writes the number equal to the number of cards in the corresponding pile on each card. Then the conjurer mixes the cards in a special way, divides them into piles again and the spectator writes another number equal to the number of cards in the new pile to the right of the first number on each card. They repeat this process several times. What is the least number of deals required for the conjurer to make different cards have different sets of numbers (whatever their position on the cards)?

Grade 11

54.11.1. Between which digits of the number $1\,99...99\,1$ with 1991-many nines, one should insert a) $+$ (the summation sign) to get the least possible number; b) $\times$ (the multiplication sign) to get the greatest possible number?

54.11.2. Fig. 97shows an orthogonal projection of the Earth (which is supposed to be an ideal ball) and its equator, A and B being the common points of the projection of the equator and the circle — the projection of the Earth).

FIGURE 97. (Probl. 54.11.2)

How can the projection of the North pole be found with the help of a ruler and compass?

54.11.3. Prove that in a regular 54-gon there are 4 diagonals that do not pass through the center and meet at one point.

54.11.4. A Parliament of 2000 MPs decided to ratify the state budget of 200 expenditure items. Each MP prepared a draft budget with what (s)he thinks the maximum possible allocation for each item so that the total expenditure does not exceed a given ceiling, S. For each item, the Parliament approves the maximum expenditure approved by not less than k MPs. What is the least value of k to ensure that the approved total does not exceed S?

54.11.5. On a rectangular screen of size $m \times n$ divided into unit cells more than $(m-1)\cdot(n-1)$ cells are lighted. If in a 2×2 square 3 cells are not lighted, then the fourth cell switches itself off after a while. Prove that at least one cell of the screen is lighted at all times.

Olympiad 55[1] (1992)

Grade 8

55.8.1 (Se). Prove that if $a+b+c+d>0$, $a>c$, $b>d$, then $|a+b|>|c+d|$.

[1]The authorship of all problems of this olympiad is indicated after the number of the problem by an abbreviation boldfaced: **A. G**alochkin, **S. G**ashkov, B. **Ku**kushkin, I. **Se**rgeev, I. **Sh**arygin, A. **Sk**openkov, A. **Sp**ivak, S. **To**karev.

55.8.2 (To). Can it happen during a game of chess that on each of 30 diagonals of the chess-board there stands an odd number of chips (each own number for each diagonal; some of these numbers may be equal)?

55.8.3 (To). An Olympiad lasted two days. Each participant solved during the first day as many problems as all other participants together during the second day. Prove that all participants of the Olympiad solved equal number of problems.

55.8.4 (To). What is the least number of weights in a set which can be divided into either 3, or 4, or 5 piles of equal mass?

55.8.5 (SG). Prove that in a right triangle the length of the bisector of the right angle does not exceed a half of the projection of the hypotenuse to the line perpendicular to the bisector.

55.8.6 (To). Are there four arrangements of 9 people at a round table such that no two of these people sit beside each other more than once? (Cf. Problem 55.9.6.)

Grade 9

55.9.1 (To). Each participant of a chess tournament won, as white, as many games as all remaining players together when they played as black. Prove that all participants won the same number of games each. (Cf. Problem 55.8.3.)

55.9.2 (AG). Which odd positive integers $n < 10000$ are more numerous: those for which the number formed by the four last digits of n^9 is greater than n or those for which it is smaller than n?

55.9.3 (To). At the center of a square pie sits a raisin (of point size). A triangular piece can be cut off the pie along the line which intersects two neighboring sides of the square at the points different from vertices of the square; another triangular piece can be cut off the remaining part in the same manner, etc. Is it possible to cut the raisin off, i.e., to get a piece of the pie with the raisin? (Cf. Problem 55.10.2.)

55.9.4 (Sp). In a 9×9 square table, 9 cells are marked: those at the intersection of the 2-nd, 5-th and 8-th rows with the 2-nd, 5-th and 8-th column. In how many ways can one get from the lower left cell to the upper right one moving only upwards and to the right without entering marked cells?

55.9.5 (Sh). Diagonal AC of trapezoid $ABCD$ is equal to the lateral side CD. The line symmetric to BD with respect to AD intersects AC at point E. Prove that line AB divides DE in halves.

55.9.6 (To). Is it possible to place $2n+1$ people at a round table n times so that no two sit beside each other more than once if (a) $n = 5$, (b) $n = 10$? (Cf. Problem 55.8.6.)

Grade 10

55.10.1 (AG). Prove that if the sum of cosines of the angles of a quadrilateral is equal to 0 then it is either a parallelogram, or a trapezoid, or an inscribed quadrilateral.

55.10.2 (To). A triangular piece can be cut off a pie of the form of a convex pentagon along the line that meets two neighboring sides at points distinct from the vertices; another piece can be cut off the remaining part in the same way, etc.

What are points on the surface of the pie should one stick a candle into so that it were impossible to get a piece of pie with the candle? (Cf. Problem 55.9.3.)

55.10.3 (AG). A white chip is placed in the bottom left corner of an $m \times n$ rectangular board, a black one is placed in the top right corner. Two players move their chips in turn along the horizontals or verticals 1 cell per move; the white can only move to the right or upwards. The white begins. The winner is the one who places his (her) chip on the cell occupied by the other player. Who can ensure the success: the white or the black?

55.10.4 (To). What is the least number of weights in the set that can be divided into either 4, 5, or 6 piles of equal mass? (Cf. Problem 55.8.4.)

55.10.5 (SG). Consider a convex centrally symmetric polygon. Prove that a rhombus of half the polygon's area can be placed inside the polygon.

55.10.6 (To). Each face of a convex polyhedron is a polygon with an even number of sides. Is it always possible to paint the edges of the polyhedron 2 colors so that each face has equal numbers of differently colored edges?

Grade 11

55.11.1 (To). It is required to place numbers into each cell of a $n \times n$ square table so that the sum of the numbers on each of $4n-2$ diagonals were equal to 1. Is it possible to do this for (a) $n = 55$; (b) $n = 1992$?

55.11.2 (Ku). Find the angles of a convex quadrilateral $ABCD$ in which $\angle BAC = 30°$, $\angle ACD = 40°$, $\angle ADB = 50°$, $\angle CBD = 60°$ and $\angle ABC + \angle ADC > 180°$.

55.11.3 (Sk). Aladdin visited every point of equator moving sometimes to the west, sometimes to the east and sometimes being instantaneously transported by genies to the diametrically opposite point on the Earth. Prove that there was a period of time during which the difference of distances moved by Aladdin to the west and to the east was not less than half the length of equator.

55.11.4 (Sp). Inside a tetrahedron a triangle is placed whose projections to the faces of tetrahedron are of area P_1, P_2, P_3, P_4, respectively. Prove that

a) in a regular tetrahedron $P_1 \leq P_2 + P_3 + P_4$;

b) if S_1, S_2, S_3, S_4 are the areas of the corresponding faces of tetrahedron, then $P_1S_1 \leq P_2S_2 + P_3S_3 + P_4S_4$.

55.11.5 (To). Is it always possible to paint the edges of a convex polyhedron two colors so that for each face the number of edges painted one color would differ from the number of edges painted the other color by not more than 1?

55.11.6 (Se). A calculator can compare $\log_a b$ and $\log_c d$, where $a, b, c, d > 1$. It works according to the following rules:

if $b > a$ and $d > c$ the calculator passes to comparing $\log_a \frac{b}{a}$ with $\log_c \frac{d}{c}$;

if $b < a$ and $d < c$ the calculator passes to comparing $\log_c d$ with $\log_b a$;

if $(b - a)(d - c) \leq 0$ it prints the answer.

a) Show how the calculator compares $\log_{25} 75$ with $\log_{65} 260$. b) Prove that the calculator can compare two nonequal logarithms after finitely many steps.

Olympiad 56[1] (1993)

Grade 8

56.8.1 (Ku). Denote by $s(x)$ the sum of the digits of a positive integer x. Solve:

a) $x + s(x) + s(s(x)) = 1993$

b) $x + s(x) + s(s(x)) + s(s(s(x))) = 1993$

56.8.2 (Bo). Knowing that n is the sum of squares of three positive integers, prove that n^2 is also the sum of squares of three positive integers.

56.8.3 (Sl). On a straight line stand two chips, a red to the left of a blue. The Rule allows the following two operations: (a) to insert two chips of one color in a row at any place on the line and (b) to delete any two neighboring chips of one color. Is it possible to leave after finitely many operations only two chips on the line: a red to the right of a blue?

56.8.4 (Be). At the court of Tsar Gorokh, the tsar's astrologist declares a moment of time *favorable* if on a watch with a centrally placed second hand the minute hand occurs after the hour hand and before the second one (counting clockwise). Does the whole day (24 h) contain more favorable time than unfavorable?

56.8.5 (Sp). Is there a finite word composed of the letters of Russian alphabet (32 letters) that has no two identical neighboring subwords but such subwords appear if one ascribes any letter (of the same alphabet) in front or at the back of this word?

56.8.6 (Ak). A circle centered at D passes through points A, B, and the center O of the escribed circle of triangle $\triangle ABC$ tangent to side BC and the extensions of sides AB and AC. Prove that points A, B, C, and D lie on one circle.

Grade 9

56.9.1 (Sh). For distinct points A and B on a plane, find the locus of points C such that triangle $\triangle ABC$ is acute and the value of its angle $\angle A$ is intermediate among the triangle's angles.

56.9.2 (Ko). Let $x_1 = 4$, $x_2 = 6$ and define x_n for $n \geq 3$ to be the least non-prime greater than $2x_{n-1} - x_{n-2}$. Find x_{1000}.

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[1] The authorship of all problems of this olympiad is indicated after the number of the problem by an abbreviation boldfaced: I. F. **Ak**ulich, A. **An**dzhans, A. Ya. **Be**lov, A. I. **Gal**ochkin, **G. G**alperin, S. B. **Ga**shkov, **GZ-B**: S. M. Gusein-Zade, A. Ya. Belov, G. **Kon**dakov, S. V. **Ko**nyagin, B. N. **Ku**kushkin, D. **Bo**tin, **Sl**itinsky, A. W. **Sp**ivak, I. F. **Sh**arygin, S. I. **To**karev, **VI**: A. Vladimirov, R. Ismailov, **VT**: M. Vyaly, D. Tereshin.

56.9.3 (Gal). A paper triangle with angles of 20°, 20°, 140° is cut along one of its bisectors into two triangles; one of these triangles is also cut along one of its bisectors, etc. Can we obtain a triangle similar to the initial one after several cuts?

56.9.4 (To). In Pete's class there 28 students beside him. Each two of these 28 have distinct number of friends among the classmates. How many friends does Pete have in this class?

56.9.5 (GG). To every pair of numbers x, y the Rule assigns a number $x * y$. Find $1993 * 1935$ if it is known that

$$x * x = 0, \text{ and } x * (y * z) = (x * y) + z \text{ for any } x, y, z.$$

56.9.6 (Sh). Given a convex quadrilateral $ABMC$ with $AB = BC$, $\angle BAM = 30^\circ$, $\angle ACM = 150^\circ$, prove that AM is the bisector of $\angle BMC$.

Grade 10

56.10.1 (Ga). In the representation of numbers A and B as decimal fractions the lengths of their minimal periods are equal to 6 and 12, respectively. What might the length of the minimal period in the similar representation of $A + B$ be? Find all answers.

56.10.2 (Ga). The grandfather of Baron K. F. I. von Münchhausen constructed a castle with a square in the horizontal cross-section. He divided the castle into 9 equal square ball rooms and placed the arsenal in the middle one. Baron's father divided each of the remaining 8 ball rooms into 9 equal square halls and organized winter gardens in all central halls. Baron himself divided each of the 64 empty halls into 9 equal square rooms and placed a swimming pool in each of the central rooms. Baron furnished the other rooms and made a door between every pair of neighboring furnished rooms. Baron shut all the other temporary doors.

Baron boasts that he once managed to go over his furnished rooms visiting each just once and returning in the initial one. We know Baron as a gentleman with a name for honesty won by his truthful stories, but still wonder: is he telling the truth in this instance?

56.10.3 (Kon). A river connects two circular lakes of radius 10 km each; the banks of the river and the lakes are segments of either straight lines or circles. From any point on any of the river's banks one can take a boat and reach the other bank by swimming not longer than 1 km. Assuming that the boat is a point is it possible for a pilot to lead the boat along the river in order to be at the distance of not more than (a) 700 m (b) 800 m away from each of the banks?

56.10.4 (VI). For every pair of real numbers a and b consider the sequence[1] $p_n = [2\{an + b\}]$. Any k successive terms of this sequence is called a *word*. Is it true that any ordered set of 0's and 1's of length k can be a word of the sequence determined by certain a and b for (a) $k = 4$, (b) $k = 5$?

56.10.5 (VT). In a botanical classifier a plant is determined by 100 features. Each of the features can either be present or absent. A classifier is considered to be *good* if any two plants have less than half of the features in common. Prove that a good classifier can not describe more than 50 plants.

56.10.6 (Sh). On side AB of triangle ABC the square is constructed outwards, its center is O. Points M and N are the midpoints of AC and BC; the lengths of these sides are equal to a and b, respectively. Find the maximum of the sum $OM + ON$ as the angle $\angle ACB$ varies.

Grade 11

56.11.1 (Be). Knowing that $\tan\alpha + \tan\beta = p$ and $\cot\alpha + \cot\beta = q$ find $\tan(\alpha + \beta)$.

56.11.2 (Be). The unit square is divided into finitely many smaller squares (of, perhaps, distinct sizes). Consider the squares whose intersection with the main diagonal is nonempty. Is it possible for the sum of perimeters of the squares be greater than 1993?

56.11.3 (An). Given n points on a plane no three of which lie on one line. A straight line passes through every pair of the points. What is the least number of pair-wise non-parallel lines among these lines?

56.11.4 (GZB). Stones lie in several boxes. The Rule allows us in one move: to select a number n; to unite the stones in each box in groups of n and a residue of less than n stones in it; to leave in each group a stone and the whole residue; it also allows us to pocket the rest of the stones. Is it possible to ensure in 5 moves that each box contains one stone if initially there were not more than (a) 460 stones, (b) 461 stones in each box?

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[1]Recall that $\{x\}$ and $[x]$ denotes the fractional and the integer part of x, respectively.

56.11.5 (Be). It is known that the domain of definition of a function f is segment $[-1,1]$, and $f(f(x)) = -x$ for all x; the graph of f is the union of finitely many points and intervals.

Is it possible to draw the graph of f if the domain of f is a) $]-1,1[$? b) the whole real line?

56.11.6 (Sh). A fly flies inside a regular tetrahedron with edge a. What is the shortest length of the flight the fly should take to visit every face and return to the initial spot?

Olympiad 57[1] (1994)

Grade 6

57.6.1. Can there be four people among which no three have identical first name, patronimic (middle name) and the last name but any pair of these people has identical either first, or middle, or last name?

57.6.2. Find a) the 6-th, b) the 1994-th number in the sequence 2, 6, 12, 20, 30, ...

57.6.3. Several teams of guards of social property, manned by identical number of guards each, slept more nights during their vigil than there are guards in the team but less than there are teams. How many guards are there in the team if all guards from the team together slept 1001 man-night?

57.6.4. Construct a $3\times3\times3$ cube of $1\times1\times1$ red, green and yellow cubes so that in any $3\times1\times1$ layer there are cubes of all three colors.

57.6.5. Cut a square into three parts from which it is possible to construct a nonright scalane triangle.

57.6.6. Kate's family drank coffee. Each member of the family drank out a full cup of coffee with milk and Kate drank a quarter of the milk and a sixth of the coffee. How many people are there in Kate's family?

57.6.7. Among any 9 of 60 kids three are from the same grade. Is it necessary that there are a) 15, b) 16 kids from the same grade?

57.6.8. A pedestrian walked along (across?) 6 streets of a town in a row passing each street exactly twice; however long he contemplated over the map he could not find a route so as to pass along any street just once during one stroll. Is there such a route?

Grade 7

57.7.1. During the past two years a factory lowered the volume of the products it manufactured by 51%. Each year the volume diminished by the same number of percents. What is this number? (5 points)

57.7.2. Each staircase of a house has the same number of floors; the same number of appartments on each floor. There are more floors than the number of appartments on the floor; more appartments on the floor than there are staircases and there is more than one staircase. How many floors are there in the house if the total number of its appartments is 105?

a) Find at least one solution. (2 points)

b) Find all solutions and prove that there are no more. (4 points)

57.7.3. When the committee asked Neznajka (Master Ignoramus) to contribute with a problem for a Math Olympiad in the Sunny Town he wrote the following head-twister, where different letters replace different figures:

$$\begin{array}{r} +\begin{array}{r} ABC \\ DEF \end{array} \\ \hline GHKL \end{array}$$

Is it possible to solve it? (5 points)

57.7.4. There are plenty of red, green and yellow cubes of size $1\times1\times1$. Is it possible to compose of them a $3\times3\times3$ cube so that each $3\times1\times1$ layer has all three colors? (6 points)

57.7.5. On a 4×6 board there stand two Ivan's black chips and two Sergey's white chips (as on Fig. 98. a)).

Each player, in turn, moves any of his chips one step along the vertical. Ivan, though plays black, starts. If after somebody's move a black chip ocuurs among two white ones along either a horisontal or a diagonal (as on Fig. 98. b)) it is considered *killed* and should be removed from the board. Ivan's goal is to lead his chips from the top row to the bottom one. Can Sergey prevent Ivan from getting his goal? (8 points)

[1]Problems for the 6-th and 7-th grades were selected by a committee headed by D. Bochina, S. Dorichenko, A. Kovaldzhi and I. Yashchenko; the authors of the other problems are: A. Kovaldzhi (8.2, 8.4, 10.1), D. Botin (8.5), Yu. Chekanov (9.2), G. Galperin (11.4), A. Galochkin (9.5), K. Ignatiev (9.6), A. Kovaldzhi, G. Kondakov (11.1), O. Kryzhanovski (8.6, 10.6), S. Markelov (11.5), I. Nagel (9.4), V. Proizvolov (10.5), G. Shabat (10.2), I. Sharygin (10.4), N. Vasiliev (9.3, 11.3, 11.6); the authors of several problems are anonimous.

Figure 98. (Probl. 57.7.5)

57.7.6. In a school the astronomical circle gathered 20 times. Each time there were exactly 5 listeneres and no 2 students met during the circle's getherings more than once. Prove that at least 20 different students attended the circle. (12 points)

Grade 8

57.8.1. A cooperative enterprize gets apple and grape juice in identical cans and produces a mixed drink in equal jars. One can of apple juice suffices for exactly 6 jars of the drink; one can of grape juice suffices for exactly 10 jars of the drink. When they changed the recipe one can of apple juice became sufficient for exactly 5 jars of the drink only. For how many jars of the drink will now suffice one can of the grape juice? (The drink is a pure mixture not diluted with water or preservatives, etc.)

57.8.2. A student did not notice a multiplication sign between two three-digit numbers and wrote one 6-digit number that happened to be 7 times greater than the product of the two three-digit numbers. Find the factors.

57.8.3. In a triangle ABC the bisectors of angles A and C are drawn. Points P and Q are the bases of the perpendiculars dropped from vertex B to these bisectors. Prove that $PQ \parallel AC$.

57.8.4. Four grasshoppers sit at the vertices of a square. Each minute one of them hops into the point symmetric with respect to another grasshopper. Prove that it it impossible for the grasshoppers to sit at some moment at the vertices of a larger square.

57.8.5. The royal astrologer considers a moment of time *favorable* if the hour, minute and second hands of the clock are on one side of the dial's diameter. All other time is considered *unfavorable*. The hands turn around a common axis uniformly, without jumps. Which kind of time prevails during the full day (24 hours), favorable or unfavorable?

57.8.6. Two play a game on a 19×94 checkered board. Each in turn marks a square (of any possible size) along the lines of the mesh and shades it. The one who shades the last cell wins. It is forbidded to shade a cell twice. Who wins if played optimally and what should the strategy be?

Grade 9

57.9.1. Is there a nonconvex pentagon no two of whose five diagonals have a common point apart from a vertex?

57.9.2. Kolya has a line segment of length k, Leo has another one, of length l. First, Kolya divides his segment into three parts; then Leo divides his segment into three parts. If it is possible to build two triangles from the six segments obtained, Leo wins; otherwise Kolya wins. Depending on the ratio $\frac{k}{l}$, who, Kolya or Leo, can assure victory and what should the winning strategy be?

57.9.3. Prove that the equation

$$x^2 + y^2 + z^2 = x^3 + y^3 + z^3$$

has infinitely many solutions in integers.

57.9.4. Two circles intersect at points A and B. To both circles tangents are drawn through A. The tangents intersect the circles at points M and N. The straight lines intersect the circles again at points P and Q (P lies on BM, Q lies on BN). Prove that $MP = NQ$.

57.9.5. Find the maximal natural number not ending with a 0 such that if we strike out one (not the first) of its figures we get a divisor of the initial number.

57.9.6. During dull lessons students sometimes play “marine battle”. In a 10×10 square of checkered paper one should place ships — rectangles — of sizes: one 1×4, two 1×3, three 1×2 and four 1×1. The ships should not have common points (even vertices) but can have common points (even edges) with the sides of the square. Prove that

a) if one places the ships as listed above (starting with the largest), one can always squeeze all the ships in the square even if one lives the running moment at all times and places each ship without thinking about the other ships' future;

b) if one places the ships in the opposit order (starting with smaller ships), a situation might arize when it is impossible to squeeze in the next ship. (Give an example.)

Grade 10

57.10.1. A student did not notice the multiplication sign between two 7-digit numbers and wrote one 14-digit number which turned out to be 3 times the would be product. What are the initial numbers?

57.10.2. An infinite sequence of numbers x_n is determined by the formula

$$x_{n+1} = 1 - |1 - 2x_n|, \quad 0 \leq x_1 \leq 1.$$

Prove that the sequance is periodic starting from a certian place a) if and b) only if $x_1 \in \mathbb{Q}$.

57.10.3. Each of the 1994 Parliament members slapped exactly one of his/her colleagues on the face. Prove that it is possible to compose a Parliament Committee of 665 members none of whom settles disputes with the colleagues in this way.

57.10.4. Let D be a point on side BC of $\triangle ABC$. Circles are drawn inside $\triangle ABD$ and $\triangle ACD$; a common outer tangent (distinct from BC) is drawn to the circles; it intersects AD at K. Prove that the length of AK does not depend on the position of D on BC.

57.10.5. Consider an arbitrary polygon, not even necessarily convex one. Recall that a *chord* of a polygon is a line segment whose endpoints belong to the polygon's contour while the segment itself lies entirely inside the polygon, the contour included.

a) Is there always a chord of the polygon that divides it into parts of equal area?

b) Prove that any polygon can be divided by a chord into parts the area of each of them not less than $\frac{1}{3}$ of the total area of the polygon. (We always assume that a chord divides the polygon into two parts: the part that splits into several pieces is, nevertheless, considered as one part.)

57.10.6. Is there a polynomial $P(x)$ with a negative coefficient while all the coefficients of any power $P^n(x)$ are positive for $n > 1$?

Grade 11

57.11.1. Devise a polyhedron with no three faces having the same number of edges.

57.11.2. See Probl. 57.10.2

57.11.3. In a round goblet whose section is the graph of the function $y = x^4$ a cherry — a ball of radius r — is dropped. What is the largest r for which the ball can touch the lowest point of the bottom? (In plain math words: what is the maximal radius of the disc lying in the domain $y \geq x^4$ and containning the origin?)

57.11.4. A convex polyhedron has 9 vertices, one of which is A. Parallel translations that send A into each of the other vertices form 8 equal polyhedra. Prove that at least two of these 8 polyhedra have an inner point in intersection.

57.11.5. Extensions of the sides AB and CD of a convex polygon $ABCD$ intersect at point P; extensions of the sides BC and AD intersect at point Q. Prove that if each of the Consider three pairs of bisectors: the outer angles of the quadrilateral at vertices A and C; the outer angles at vertices B and D; and the outer angles at vertices P and Q of triangles $\triangle QAB$ and $\triangle PBC$, respectively. Prove that if each of the three pairs of bisectors intersects, the intersection points lie on one straight line.

57.11.6. Prove that for any $k > 1$ there exists a power of 2 such that among its k last digits the nines constitute not less than one half. For example: $2^{12} = 4096$, $2^{53} = ...992$

Olympiad 58[1] (1995)

Grade 8

58.8.1. M. V. Lomonosov spent one denezhka a day for a loaf of bread and kvas. When prices went up 20%, he bought half a loaf of bread and kvas for the same denezhka. Will a denezhka be enough to buy at least kvas if the prices will again rise 20%?

58.8.2. Prove that the numbers of the form 10017, 100117, 1001117,... are divisible by 53.

58.8.3. Consider a convex quadrilateral and a point O inside it such that $\angle AOB = \angle COD = 120°$, $AO = OB$ and $CO = OD$. Let K, L and M be the midpoints of sides AB, BC and CD, respectively. Prove that a) $KL = LM$; b) triangle KLM is an equilateral one.

58.8.4. To manufacture a parallelepipedal closed box of volume at least 1995 units we have a)962, b) 960, c) 958 square units of material. Assuming our production is wasteless, is the stock sufficient?

58.8.5. Several villages are connected with a town; there is no direct communication between villages. A truck with goods for all villages starts from the town. The cost of the truck's trip is equal to the product of the total weight of the load by the distance. Suppose that the weight of each item in the load is equal in some units to the distance from the town to the item's destination. Prove that the cost of the delivery does not depend on the order in which the goods are delivered.

58.8.6. A straight line cuts off a regular quadrilateral $ABCDEF$ triangle AKN such that $AK + AN = AB$. Find the sum of the angles with vertices in the vertices of the quadrilateral that subtend segment KN.

Grade 9

58.9.1. Prove that if we insert any number of digits 3 between the zeroes of the number 12008, we get a number divisible by 19.

58.9.2. Consider an isosceles triangle ABC. For an arbitrary point P inside the triangle consider intersection points A' and C' of straight lines AP with BC and CP with BA, respectively. Find the locus of points P for which segments AA' and CC' are equal.

58.9.3. Let us refer to a rectangular of size$1 \times k$ for any natural k a *strip*. For what integer n can one cut a $1995 \times n$ rectangle into pairwise different strips?

58.9.4. Consider a quadruple of natural numbers a, b, c and d such that $ab = cd$. Can $a + b + c + d$ be a prime?

58.9.5. We start with four identical right triangles. In one move we can cut one of the triangles along the hight from the right angle into two triangles; so we get 5 right triangles. Prove that after any number of moves there are two identical triangles among the whole lot.

58.9.6. Geologists took 80 cans with preserved food for a trip. The weights of cans are known and pairwise distinct (there is an inventory). After a while the labells became unreadable and only the cook knows which can contains what. She can prove it beyond any doubt without opening the cans and using only the list of inventory and a balance with two pans and a hand that shows the difference of weight in the pans. Prove that to this end a) 4 weighings suffice while b) 3 do not.

58.9.1. The number $\sin a$ is known. What is the largest number of different values that a) $\sin \frac{a}{2}$? b) $\sin \frac{a}{3}$ can take?

Grade 10

58.10.2. See Probl. 58.9.2.

58.10.3. Consider trapezoid $ABCD$. We construct circles with the lateral sides of the trapezoid as diameters. Suppose that the diagonals of $ABCD$ meet at point K not on these circles. Prove that the lengths of the tangents to these circles from point K are equal.

58.10.4. See Probl. 58.9.5.

58.10.5. Prove that if a, b and c are integers and, moreover, $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $\frac{a}{c} + \frac{c}{b} + \frac{b}{a}$ are integers, then $a = b = c$.

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[1]The authors of the problems are: A. Belov (10.6, 11.5, 11.7), D. Botin(8.4), Yu. Chekanov (9.3), A. Galochkin (8.2, 9.1, 11.1, 11.2), A. Gribalko (10.5), G. Kondakov (11.4), W. K. Kovaldzhi (8.1, 8.5), S. Markelov (8.3, 9.2, 10.1, 10.3), A. Shapovalov (9.5), V. Senderov (10.6, 11.5, 11.7), I. Sharygin (11.3), V. Proizvolov (8.6), A. Tolpygo (9.6).

58.10.6. On a board, several bulbs are on. There are several buttons on the control panell. Pressing a button changes the state of the bulbs it is connected with. It is known that for any collection of bulbs there is a bulb connected with an odd number of bulbs from this set. Prove that by pressing buttons on can switch off all the bulbs.

Grade 11

58.11.1. Prove that $|x+y+z| \le |x+y-z|+|x-y+z|+|-x+y+z|$, where x, y, z are real numbers.

58.11.2. Is it possible to paint the edges of n-angled prizm 3 colors so that each face had the boundary painted all 3 colors and each vertex was the intersection point of edges of different colors if a) $n = 1995$, b) $n = 1996$?

58.11.3. Consider triangle ABC, its median AM, bisector AL and a point K on AM such that $KL \parallel AC$. Prove that $AL \perp KC$.

58.11.4. Divide segment $[-1, 1]$ into black and white subsegments so that the integral of any a) linear function, b) quadratic polynomial along black segments was equal to that along white ones.

58.11.5. Consider two infinite in both ways sequences A of period 1995 and B which is either nonperiodic or the length of its period is $\neq 1995$. Let any segment of sequence B not longer than n be contained in A. What is the largest n for which such sequences exist?

58.11.6. Prove that there exist infinitely many nonprime n's such that $3^{n-1} - 2^{n-1} \vdots n$.

58.11.7. Is there a polygon and a point outside it such that from this point non of its vertices is visible?

Olympiad 59[1] (1996)

Grade 8

59.8.1. It is known that $a + b^2/a = b + a^2/b$. Is it true that $a = b$? (*R Fedorov*)

59.8.2. Along a circle stand 10 iron weighs. Between every two weighs there is a brass ball. Mass of each ball is equal to the difference of masses of its neighboring weighs. Prove that it is possible to divide the balls among two pans, so as to make the balance in equilibrium. (*V. Proizvolov*)

59.8.3. At nodes of graph paper gardeners live; flowers grow everywhere around them. Each flower is to be taken care of by the three nearest to it gardeners. One of the gardeners wishes to know what is the flower (s)he has to take care of. Sketch the plot of these gardeners. (*I. F. Sharygin*)

59.8.4. Consider an equilateral triangle $\triangle ABC$. The points K and L divide the leg BC into three equal parts, the point M divides the leg AC in ratio $1 : 2$ counting from the vertex A. Prove that $\angle AKM + \angle ALM = 30^\circ$. (*V. Proizvolov*)

59.8.5. A rook stands in a corner of an $n \times n$ chess board. For what n, moving alternately along horizontals and verticals, the rook can visit all the cells of the board and return to the initial corner after n^2 moves? (A cell is *visited* only if the rook stops on it, those that the rook "flew over" during the move are not counted as visited.) (*A. Spivak*)

59.8.6. Eight students solved 8 problems. a) It turned out that each problem was solved by 5 students. Prove that there are two students such that each problem is solved by at least one of them.

b) If it turned out that each problem was solved by 4 students, it can happen that there is no pair of students such that each problem is solved by at least one of them. (Give an example.) (*S. Tokarev*)

Grade 9

59.9.1. Numbers a, b and c satisfy inequalities $|a-b| \ge |c|$, $|b-c| \ge |a|$, $|c-a| \ge |b|$. Prove that one of the numbers a, b or c is equal to the sum of the other two numbers. (*A. Galochkin*)

59.9.2. The circle is circumscribed about $\triangle ABC$; through the points A and B tangents are drawn, they meet at M. The point N lies on the leg BC, and $MN \parallel AC$. Prove that $AN = NC$. (*I. F. Sharygin*)

59.9.3. Integers 1 to n are written in a row. Under them, the same numbers are written in some other order. Could it happen that the sum of each number with the one under it is a perfect square? Consider a) $n = 9$, b) $n = 11$, c)$n = 1996$? (*P. Filevich*)

59.9.4. Let A and B be points on a circle. They divide the circle into two parts. Find the locus of the midpoints of all chords whose endpoints lie on different arcs $\smile AB$. (*I. F. Sharygin*)

59.9.5. Ali-Baba and a robber divide a treasure consisting of 100 golden coins. The treasure is split into 10 piles of 10 coins. Ali-Baba chooses 4 piles, places a mug beside each pile, and puts several coins (not less than 1, but not the whole pile) from the respective pile into each mug. The robber must rearrange the mugs by altering their initial attribution to piles, after which the coins are taken out from each mug and added to the newly attributed pile.

Then Ali-Baba again selects 4 piles of 10, places mugs beside the piles, etc.

At any moment Ali-Baba can quit and go away with any 3 mugs he chooses. The remaining coins will be the robber's share. What is the greatest number of coins Ali-Baba can collect, if the robber is no altruist either? (*A. Ja. Belov*)

Grade 10

59.10.1. Positive numbers a, b and c satisfy $a^2+b^2-ab=c^2$. Prove that $(a-c)(b-c)\le 0$. (*A. Egorov, V. Bugaenko*)

59.10.2. In a 10×10 square drawn on a graph paper along its lines, the centers of all unit squares (100 points altogether) are marked. What is the least number of straight lines non parallel to the sides of big square and passing through all the points marked? (*A. Shapovalov*)

59.10.3. The points P_1, P_2, ..., P_{n-1} divide the side BC of an equilateral triangle $\triangle ABC$ into n equal segments: $BP_1=P_1P_2=\dots=P_{n-1}C$. The point M on the side AC is such that $AM=BP_1$. Prove that $\angle AP_1M+\angle AP_2M+\dots+\angle AP_{n-1}M=30^\circ$, if a) $n=3$; b) n is an arbitrary positive integer. (*V. Proizvolov*)

59.10.4. In a corner of an $m\times n$ chessboard stands a bishop. Two play in turns; they alternately move the bishop horizontally or vertically any distance; the Rule forbids the bishop to stop on the field over which it had been already moved or at which it had already stoped. The one who is stuck is the looser. Which player can assure victory for him/herself: the one who starts or the other one and now should (s)he move? (*B. Begun*)

59.10.5. In a country, the houses of the inhabitants being represented by points on the plane, two Laws act:

1) A person can play basketbol only if (s)he is taller the majority of his/her neighbors.

2) A person has the right for free usage of the public transport only if (s)he is shorter the majority of his/her neighbors.

According to Law, the person's *neighbors* are the inhabitants living in side the circle centered at the person's house. The humane Law lets each person to chose the radii for each section of the Law. Can not less than 90% of the population play basketbol and not less than 90% have the right for free usage of the public transport? (*N. N. Konstantinov*)

59.10.6. Prove that for any nth degree polynomial $P(x)$ with natural coefficients there exists a k such that the numbers $P(k)$, $P(k+1)$, ..., $P(k+1996)$ are not prime ones, if a) $n=1$;) n is an arbitray positive integer. (*V. A. Senderov*)

Grade 11

59.11.1. Positive numbers a, b and c satisfy equation $a^2+b^2-ab=c^2$. Prove that $(a-c)(b-c)\le 0$. (*A. Egorov, V. Bugaenko*)

59.11.2. Find a polynomial with integer coefficients whose roots are $\sqrt[5]{2+\sqrt{3}}+\sqrt[5]{2-\sqrt{3}}$. (*B. Kukushkin*)

59.11.3. In space, consider 8 parallel planes such that??? the distances between each two neighboring ones are equal. A point is selected on each of the planes. Can the points selected be vertices of a cube? (*V. Proizvolov*)

59.11.4. Prove that there are infinitely many natural numbers n such that n is representable as the sum of squares of two natuaral numbers, while $n-1$ and $n+1$ are not. (*V. A. Senderov*)

59.11.5. Point X outside of nonintersecting circles, ω_1 and ω_2, is such that the segments of the tangents drawn from X to ω_1 and ω_2 are equal. Prove that the intersection point of the diagonals of the quadrilateral, determined by the tangent points, coinsides with the intersection point of the common inner tangents to ω_1 and ω_2. (*S. Markelov*)

59.11.6. A $2^n\times n$ table consists of all possible lines of length n composed from numbers 1 and -1. Part of the numbers was replaced with zeros. Prove that one can choose several lines whose sum (if we consider each line as a number) is zero. (*G. Kondakov*)

Olympiad 60[1] (1997)

Grade 8

60.8.1. In certain cells of the chess board stand some figures. It si known that on each horizontal line stands at least one figure and on different horizontals a different number of figures stand. Prove that it is possible to mark 8 figures so that on each horizontal and each vertical stands exactly one marked figure. (*V. Proizvolov*)

60.8.2. From a volcano observatory to the top of Stromboly volcano one has to take a road and then a passway, each takes 4 hours. There are two craters on the top. The first crater erupts for 1 hour and then is silent for 17 hours, next all over again, it erupts for 1 hour and then is silent for 17 hours, etc. The second crater erupts for 1 hour and then is silent for 9 hours, then it erupts for 1 hour, then is silent for 17 hours, etc. During the eruption of the first crater it is dangerous to take both the passway and the road, during the eruption of the second crater it is dangerous to take the passway only. At noon scout Vanya saw that both the craters simultaneously started to erupt. Will it be ever possible for him to mount the top of the volcano without risking his life? (*I. Yashchenko*)

60.8.3. Inside of the acute angle $\angle XOY$ points M and N are taken so that $\angle XON = \angle YOM$. On the segment OX a point Q is taken so that $\angle NQO = \angle MQX$; on segment OY a point P is taken so that $\angle NPO = \angle MPY$. Prove that the lengths of the broken lines MPN and MQN are equal. (*V. Proizvolov*)

60.8.4. Prove that there exists a positive non-prime integer such that if any three of its neighboring digits are replaced with any given triple of digits the number remains on-prime. Does there exist a 1997-digit such number? (*A. Shapovalov*)

60.8.5. In the rhombus $ABCD$ the measure of $\angle B = 40°$, E is the midpoint of BC, F is the base of the perpendicular dropped from A on DE. Find the measure of $\angle DFC$. (*M. Volchkevich*)

60.8.6. Banker learned that among similarly looking golden coins one is counterfeit (of less weight). Banker asked an expert to determine the coin by means of a balance without weights and demanded that each coin should participate in not more than two weighings (otherwise it will get too worn out and loose its market value). What largest number of coins should Banker have had to ensure the fulfilment of the expert's task? (*A. Shapovalov*)

Grade 9

60.9.1. In a triangle one side is 3 times shorter than the sum of the other two. Prove that the angle opposite the said side is the smallest of the triangle's angles. (*A. Tolpygo*)

60.9.2. On a plate lie 9 different pieces of cheese. Is it always possible to cut one of them into two parts so that the 10 pieces obtained were divisible into two portions of equal mass of 5 pieses each? (*V. Dolnikov*)

60.9.3. A convex octagon $AC_1BA_1CB_1$ satisfies: $AB_1 = AC_1$, $BC_1 = BA_1$, $CA_1 = CB_1$ and $\angle A + \angle B + \angle C = \angle A_1 + \angle B_1 + \angle C_1$. Prove that the area of $\triangle ABC$ is equal to a halv area of the octagon. (*V. Proizvolov*)

60.9.4. Along a circular railroad n trains circulate in the same direction and at equal distances between them. Stations A, B and C on this railroad (denoted as the trains pass them) form an equilateral triangle. Ira enters station A at the same time as Alex enters station B in order to take the nearest train. It is knows that if they enter the stations at the same moment of time as the driver Roma passes a forest, then Ira takes her train earlier than Alex; otherwise Alex takes the train earlier than or simultaneously with Ira. What part of the railroad goes through the forest? (*V. Proizvolov*)

60.9.5. $2n$ sportsmen twice met at a circle tournament. Prove that if the sum of points of each altered not less by n (during the second tournament), it altered by exactly n. (*V. Proizvolov*)

60.9.5. Let $1 + x + x^2 + \cdots + x^{n-1} = F(x)G(x)$, where $n > 1$ and where F and G are polynomials, whose coefficients are zeroes and units. Prove that one of the polynomials F and G can be represented in the form $(1 + x + x^2 + \cdots + x^{k-1})T(x)$, where $k > 1$ and where T is also a polynomial whose coefficients are zeroes and units. (*V. Senderov, M. Vyaly*)

Grade 1

0

60.1.1. Is there a convex body distinct from ball whose three orthogonal projections on three pairwise perpendicular planes are discs? (*A. Kanel-Belov*)

60.1.2. Prove that among the quadrilaterals with given lengths of the diagonals and the angle between them the parallelogram has the least perimeter. (*Folklore*)

60.1.3. Consider a quadrileteral. a) As the quadrileteral was circumwent clockwise, each side of the quadrileteral was extended by its length in the direction of the movement. It turned out that the endpoints of the segments constructed are the vertices of a square. Prove that the initial quadrilateral is a square.

b) Prove that if as a result of the procedure similar to the above-discribed is applicable to an n-gon we get a regular n-gon, than the initial n-gon is a regular one. (*M. Evdokimov*)

60.1.4. Given real numbers $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$ such that

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3,$$
$$a_1a_2 + a_2a_3 + a_1a_3 = b_1b_2 + b_2b_3 + b_1b_3.$$

Prove that if $a_1 \leq b_1$ and $a_3 \leq b_3$. (*Folklore*)

60.1.5. In a circle tournament with more than two participants the *coefficient* of each participant was defined to be the sum of points scored by those defeated by the sportsman considered. It turned out that the coefficients of all participants are equal. Prove that all the partiipants scored equal number of points. (*B. Frenkin*)

60.1.6. Consider the powers of 5: 1, 5, 25, 125, 625, ... Consider the sequesnce formed by their first digits: 1, 5, 2, 1, 6, ... Prove that any segment of this sequence written in reverse order will be encountered in the sequence of the first digits of the powers of 2: 1, 2, 4, 8, 1, 3, 6, 1, ... (*A. Kanel-Belov*)

Grade 1

1

60.1.1. On sides AB, BC and CA of $\triangle ABC$ points C', A' and B', respectively, are marked. Prove that the area of $\triangle A'B'C'$ is equal to

$$\frac{AB' \cdot BC' \cdot CA' + AC' \cdot CB' \cdot BA'}{4R},$$

where R is the radius of hte circumscribed circle of $\triangle ABC$. (*A. Zaslavsky*)

60.1.2. Compute

$$\int_0^{\pi/2} \big(\cos^2(\cos x) + \sin^2(\sin x)\big)\, dx.$$

where R is the radius of hte circumscribed circle of $\triangle ABC$. (*M. Vyaly*)

60.1.3. Consider three functions:

$$f_1(x) = x + \frac{1}{x}, \quad f_2(x) = x^2, \quad f_3(x) = (x-1)^2.$$

The Rule lets you to add subtract and multiply these functions (in particular, you can square and raise to higher powers, etc.), multiply by an arbitrary number, add an arbitrary number to your result and perform the above described operations with the expressions obtained. Get in this way $\frac{1}{x}$. Prove that if one of the functions f_1, f_2 or f_3 is taken out of the consideration, then it is impossible to get $\frac{1}{x}$ in the way described. (*M. Evdokimov*)

60.1.4. Is it possible to divide a regular tetrahedron with edge 1 into regular tetrahedrons and octahedrons with the lengths of their edges less than $\frac{1}{100}$? (*V. Proizvolov*)

60.1.5. Positive numbers a, b and c are such that $abc01$. Prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq 1.$$

(*G. Galperin*)

60.1.6. On the plane, consider a finite number of strips with the sum of their widths equal to 100 and a disc of radius 1. Prove that it is possible to translate parallelly each strip so that the totality of translated strips will cover the disc. (*M. Smurov*)

Selected problems of Moscow mathematical circles

The following are problems we find most interesting among those offered to the participants of mathematical clubs, to the winners of the Moscow Olympiads when they were coached to International Olympiads and also some problems from the archives of the Moscow Olympiad jury which were not used in any of the tournaments, and, therefore, are not well known. The grade for which the problem was intended is given in parentheses.

1.(7-9). a) Find the sum of the digits of the number $123456789101112\ldots999998999999$.

b) How many digits 7 are there in this number?

2.(8-10). Find the least positive integer $\overline{ab...c}$ without any zeroes in its decimal notation, such that its sum with itself written in reverse order (i.e. the sum with the number $\overline{c...ba}$), is a number whose digits can be obtained by a permutation of the digits of the original number.

3.(9-10). Are there irrational numbers x and y for which x^y is a rational number?

4.(8-10). We place 1's and -1's at the vertices of a cube, one number per vertex. In the center of each face we put the product of the numbers at the vertices of this face. Can the sum of the 14 numbers obtained be equal to 0? Can it be equal to 7?

5.(8-10). a) There is a finite number of stars in space. The number of and directions to visible stars can be determined from an observation post. No single observation, however, determines the exact number of stars, as some might be hidden behind the others. It is only possible to say, after several observations, that the number of stars in the sky is not less than the greatest of the numbers of stars visible from observation posts. Can the exact number of stars in the sky be still determined after several observations? If so, what is the least number of observation posts needed to ascertain the exact number of stars in space?

b) Solve the same problem on a Flatland, the planar Universe.

6.(7-9). Consider n identical cars on a circular highway. The total quantity of fuel in all these cars is enough for one of them to cover the whole circle. Is it possible to find a car that can drive around the entire circle by borrowing fuel from other cars along the way for any arrangement of cars and distribution of fuel among them?

7.(8-10). Find non-negative integer solutions of the equation:

$$x^4 + 2x^3 + x^2 - 11x + 11 = y^2.$$

8.(8-10). On the numerical line, paint red all points that correspond to positive integers of the form $81x + 100y$, where x and y are positive integers; paint the remaining integers blue. Find a point on the line such that any two points symmetrical with respect to it are painted different colors.

9.(9-10). Integers $x_1, x_2, \ldots, x_n;\ y_1, \ldots, y_m$ satisfy the inequalities

$$1 < x_1 < x_2 < \ldots < x_n < y_1 < \ldots < y_m \quad \text{and} \quad x_1 + \ldots + x_n > y_1 + \ldots + y_m.$$

Prove that $x_1x_2\ldots x_n > y_1y_2\ldots y_m$.

10.(10). Prove that it is possible to cut any two polyhedrons of equal volume into several tetrahedrons of pair-wise equal volumes.

11.(8-9). Consider a square $ABCD$ and point O inside it. Prove that

$$135^\circ < \angle OAB + \angle OBC + \angle OCD + \angle ODA < 225^\circ.$$

12.(10). A) Given a finite set of n points not in the same straight line. For any two pairs of given points belonging to two different lines the intersection point of these lines also belongs to the set of given points. Prove that all points of the set but one lie on the same line.

B) Is it possible to draw n straight lines through point O in space so that *for any two of these lines there is a third straight line from the same set, which is perpendicular to the two lines* for (a) $n = 99$ or (b) $n = 100$?

(c) Point out all n for which there exists an arrangement of n lines satisfying the condition from heading B) and describe all possible arrangements of these lines in space.

13.(7-10). A pie is of the form of a square lamina. Two perpendicular straight lines cut the pie into four parts. Three of these parts weigh 200 g each. What is the weight of the pie?

14. (10). There are n point-size searchlights that illuminate angles (the vertex and the legs included) $\alpha_1, \alpha_2, \ldots, \alpha_n$ on a plane. If these searchlights were placed at one point they would have illuminated the whole plane. Prove that for any n it is possible to permute the locations of searchlights (without rotating searchlights themselves) so that they would still illuminate the entire plane.

15.(8-9). Consider convex quadrilateral $ABCD$ such that $AC = BD$, $\angle B = 2\angle C$, $\angle C + \angle D = 90^\circ$. Find angles $\angle B$ and $\angle D$ of the quadrilateral.

16.(9-10). A) There are 9 points on the surface of a cube with edge 1. Prove that two of these 9 points are not farther than $\frac{\sqrt{3}}{2}$ from each other.

B) Can the surface of the cube with edge 1 have (a) 8 points and (b) 7 points so that the distance between any two of them is > 1?

17.(9-10). a) The projections of a solid to two planes in space are circles. Prove that these circles are equal.

b) The projections of a convex n-gon to two non-parallel planes in space are regular n-gons. Prove that these projections are equal n-gons.

18.(8-9). The sum of the digits in the decimal expression of 5^n is less than 10^{100}. Is the set of such positive integers n finite or infinite?

19.(7-9). Prove that no digit is repeated 5 000 001 times in a row in the first 10 million digits of the decimal representation of $\sqrt{2}$ (the integer part included).

20.(7-9). There are 1 000 airports in the land Shvambrania, and the distances between every two airports are distinct. Suppose an airplane departs from each airport and flies to the nearest airport. What greatest number of airplanes can land in an arbitrary airport if Shvambrania is a) a plane? b) a sphere?

21.(9-10). Several spherical holes are made in a cheese cube. Prove that it is possible to cut the cheese into convex polyhedrons so that there is exactly one hole inside each of the polyhedrons.

22.(10). Let $\sin\alpha = \frac{3}{5}$. Prove that $\sin 25\alpha = \frac{n}{5^{25}}$, where n is an integer not divisible by 5.

23.(7-10). Three bulbs — one blue, one green, and one red — are somehow connected by wires to n switches. Each switch can be in one of three positions. For any position of all the switches exactly one bulb is turned on, but if all n switches are simultaneously flipped (each by its own of the 2 possible ways), another bulb is turned on. Prove that the color of the bulb which is turned on is determined by one fixed switch and does not depend in any way on the other switches.

24.(8-10). A grasshopper hops on an infinite chessboard with squares of side 1 moving with each hop a distance of α to the right and β upwards. Prove that if numbers α, β and $\frac{\alpha}{\beta}$ are irrational, then the grasshopper will necessarily reach a black square.

25.(9-10). Prove that $\tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \sqrt{11}$.

26.(8-10). Solve in positive integers:

$$520(xyzt + xy + xz + zt + 1) = 577(yzt + y + z).$$

27.(7-8). Prove that if at all times at least one of ten uniformly functioning alarm clocks shows correct time, then at least one of them always shows correct time.

28.(9-10). The space is divided into identical and identically *oriented* parallelepipeds.

a) Prove that for each parallelepiped at least 14 of the parallelepipeds have a common point with it.

b) What is the least number of parallelepipeds that have a common point with a given parallelepiped if the parallelepipeds are still identical but not equally oriented?

29.(8-10). A triangular lamina of area 1 is cut into 4 parts (three triangles and 1 quadrilateral) by two straight cuts. Three parts have the same area. Find the area of every part.

30.(8-9). Prove that if the arithmetic mean of the first $10^{10^{10}}$ digits in the decimal expression of $2-\sqrt{2}$ is between $4\frac{1}{3}$ and $4\frac{2}{3}$, then the same is true for $\sqrt{2}-1$.

31.(8-10). Prove that at any given moment there is a point on the surface of the Sun (considered as a sphere) from which one can see not more than 3 planets (out of 9 known ones).

32.(7-9). There are two containers: the first one has $1l$ of water in it, the second one is empty. We pore half of the water from the first container into the second one; then we pore one third of the water from the second container back into the first container; then we pore one fourth of the water from the first container into the second container, and so on. How much water is there in the first container after 12345 refills?

33.(9-10)*. Prove that it is possible to arrange infinitely many squares with sides $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\dots$, $\frac{1}{n}$, $\dots$ inside a square with side $\frac{5}{6}$ so that they do not overlap, but it is impossible to do this in a square of a smaller side.

34.(8-10). Given a wooden ball, a compass, and a piece of paper, draw on the paper a circle of radius equal to that of the ball. (It is allowed to draw circles on the ball.)

35.(9-10). A square is divided in two ways into 100 parts of equal area. Prove that it is possible to select 100 points such that after each partition there is exactly one point in every part.

36.(8). Prove that the number

$$\frac{\left(2^4+\frac{1}{4}\right)\left(4^4+\frac{1}{4}\right)\left(6^4+\frac{1}{4}\right)\left(8^4+\frac{1}{4}\right)\left(10^4+\frac{1}{4}\right)\left(12^4+\frac{1}{4}\right)}{\left(1^4+\frac{1}{4}\right)\left(3^4+\frac{1}{4}\right)\left(5^4+\frac{1}{4}\right)\left(7^4+\frac{1}{4}\right)\left(9^4+\frac{1}{4}\right)\left(11^4+\frac{1}{4}\right)}$$

is an integer, and find the number by simplification without actual calculations.

37.(8-10). A table is entirely covered with 100 square tablecloths, A round hole burnt through the cover damageing each of the tablecloths. Each tablecloth would have covered the table but for the hole. Prove that some three of the tablecloths completely cover the table.

38.(8-10). Given 40 dice with the sum of the numbers on the opposite faces of each die equal to 7. Place the dice on one another to form a parallelepiped. Is it possible to rotate each die about its vertical axis so that the sums of the numbers on four lateral sides of the parallelepiped were equal?

39.(8-10). Concider two concentric circles, two parallel chords l and m tangent to the inner circle, point A on the outer circle between l and m; tangents to the inner circle through A, and their intersection points C and D with the chords. Prove that the product $AC \cdot AD$ does not depend on the position of A.

40.(8-10). Prove that the difference between the numbers

$$1-\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\dots\cfrac{1}{(n-1)+\cfrac{1}{n-1}}}}} \quad\text{and}\quad 1-\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\dots\cfrac{1}{(n-1)+\cfrac{1}{n-1+\cfrac{1}{n}}}}}}$$

is less than $\frac{1}{(n-1)!n!}$.

41.(8-9). Points A, B and C move uniformly along three circles in the same direction with the same angular velocity. How does the center of mass of triangle ABC move?

42.(8-9). Numbers 1, 2, 3, ... , 1974 are written on a blackboard. In one move we can erase any two numbers of the set and write in their place the absolute value of their difference. After 1973 moves one number is left. What number can it be?

43.(8-9). A and B are the tangent points of straight lines a and b and a circle. We selected point C on line a, and point D on line b. Segment AB meets segment CD at point M. Prove that $CA/CM = DB/DM$.

44.(8-10). Given $p+1$ distinct positive integers for a prime p, prove that among them there is a pair of numbers x and y such that the quotient after the division of the greater of these two numbers by $GCD(x, y)$ is not less than $p+1$.

45.(9-10). Prove that $\sum\limits_{n=1}^{k} a_n \cos nx \geq -1$ for any x, then $a_1 + \ldots + a_k \leq k$.

46.(9-10). Two people play a game in which one thinks a five-digit number consisting of 0's and 1's, and the other must guess it.

Guesser names a five-digit number consisting of 0's and 1's, and Thinker tells Guesser at how many places the digits of this number coincide with the corresponding digits of the one Thinker has in mind. Is it possible to guess the number in 3 guesses?

47.(9-10). A closed broken line is situated on the surface of a cube with edge 1. On each face of the cube there is at least one segment of the line. Prove that the length of the broken line is not less than $3\sqrt{2}$.

48.(9-10). Prove that for any integer $n \geq 2$ we have

$$n\left(\sqrt[n]{n+1}-1\right) \leq 1+\frac{1}{2}+\ldots+\frac{1}{n} \leq 1+n\left(1-\frac{1}{\sqrt[n]{n}}\right).$$

49.(7-10). On a white plane there sit a man and a black cat. The man is superstitious and does not want to cross the cat's path; the cat, full of spite, wants to move along a closed non-selfintersecting path so as to avoid the man and not give him a possibility to avoid the cat's path. Is it possible for the cat to circumvent the man within a finite length of time if its top speed is $\lambda > 1$ times that of the man? (The cat and the man may not be at the same point simultaneously.)

50.(8-10). There are 650 distinct points inside a disc of radius 16. Prove that there is an annulus with inner radius 2 and outer radius 3 on which lie at least 10 of the given points.

51.(8-10). Is there a positive integer n for which any rational number between 0 and 1 can be expressed in the form of the sum of n reciprocals of positive integers?

52.(8-10). A regular $2n$-gon is inscribed in a regular $2k$-gon, i.e., each vertex of the $2n$-gon lies on the boundary of the $2k$-gon. Prove that $2k$ is divisible by n.

53.(9-10). A cube contains a convex polyhedron M whose projection to each face of the cube covers the entire face. Prove that the volume of the polyhedron M is not less than one third of the volume of the cube.

54.(9-10). A city of the form of a square with side 10 km is divided into n^2 identical square blocks. The blocks are numbered from 1 to n^2 so that two blocks with consecutive numbers have a common side. Prove that a cyclist can find any block (s)he needs by riding not more than 100 km.

55.(8-10). In convex pentagon P_1 we drew all the diagonals. As a result P_1 split into 10 triangles and one pentagon, P_2. Let S be the difference between the sum of the areas of the triangles adjacent to the sides of P_1 and the area of P_2.

Let us perform the above operation (draw diagonals, etc.) with the inner pentagon P_2; let P_3 be its inner pentagon. Let s be the difference between the sum of the areas of the triangles adjacent to the sides of P_2 and the area of P_3. Prove that $S > s$.

56.(8). Find the greatest number of vertices of a non-convex non-selfintersecting n-gon from which no inner diagonal can be drawn.

57.(8). At every integer point of the numerical line a positive integer is written. Between every two neighboring numbers we write their arithmetic mean and then erase the original numbers. We repeat this operation many times. It turns out that all numbers obtained after each step are positive integers. Is this sufficient to conclude that after some step all numbers will be equal?

58.(9-10). A $3 \times 3 \times 3$ cube is constructed of 27 cubic blocks with side 1. Each block is either white or black. Every hour a painter comes and white-washes all blocks with an even number of black neighbors, and paints black all the other blocks. Prove that eventually all blocks will be painted white.

59.(9-10). In space, there are n distinct points of equal mass. Consider sphere S_1 of radius 1 and with center at one of the given points. Let S_2 be the sphere of radius 1 (perhaps identical to S_1) with center at the center of mass of all the given points that are inside of S_1. Let S_3 be the sphere of radius 1 (perhaps identical to S_2) with center at the center of mass of all the given points that are inside S_2, etc. Prove that $S_k = S_{k+1} = ...$ for some k.

60.(8). You are allowed to make two operations with ordered n-tuples of 0's and 1's: to change the first (left) digit and also to change the digit following the first (from the left) 1. Prove that such operations can turn any set into any other set.

61.(8). There are four equal circles O_1, O_2, O_3, O_4 inside a triangle such that circle O_1 is tangent to two sides of the triangle; circle O_2 is tangent to another pair of sides of the triangle; circle O_2 to the third pair of sides; and circle O_4 is tangent to the first three circles.

Prove that the center of O_4 lies on the same straight line as the centers of the circles inscribed in and circumscribed around the triangle.

62.(8-10)*. Two infinite (in both ways) non-selfintersecting broken lines are drawn on an infinite piece of graph paper. The segments of the broken lines are on the lines of the paper, and each broken line passes through all intersections of the grid of the paper. Must the broken lines have common segments?

63.(8-10). Denote the sum of the first n primes by S_n. Prove that there is a perfect square between S_n and S_{n+1}.

64.(8-10)**. Consider square and an equilateral triangle are drawn on a plane. Prove that one of the distances between a vertex of the square and a vertex of the triangle is irrational.

65.(9-10)**. A square town is divided into n^2 square blocks. The streets inside the town are two-way ones and the street skirting the town is a one-way one. A cyclist moves in the town in accordance with the following traffic rules: (s)he moves only along the right side of any street and does not turn left at intersections; on the one-way street that surrounds the town, (s)he moves so that all houses are on his/her right.

For what n can the cyclist ride through the whole town passing once each side of each street (and once the only side of the one-way street around the town)? Try to find the greatest possible set of such values of n.

66.(8). For an inscribed octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ we have $A_1A_2 \parallel A_5A_6$, $A_2A_3 \parallel A_6A_7$, and $A_3A_4 \parallel A_7A_8$. Prove that $A_4A_5 = A_1A_8$.

67.(8-9). A square with side n is divided into n^2 square cells with sides 1. Can n^2 different numbers be written in the cells so that in any square whose sides coincide with the sides of the given $n \times n$ square or with the lines that divide it, the product of the numbers along one longest diagonal is equal to the product of the numbers along the other longest diagonal?

68.(8-10). A road is straight but not flat. Is it possible for three people to walk from points whose distances from the beginning of the road are 0, 1 and 2, respectively, to points whose distances from the beginning of the road are 1000, 1001, and 1002, respectively, without passing one another and so that the last person sees the first one all the time and does not see the second person for a single moment? The heights of the people do not matter, i.e., a short one can see through a tall one.

69.(8). Denote the sum of the digits of a number N by $S(N)$. Prove that there is an infinite number of N's that have no zeroes in their decimal expression and such that a) N is divisible by $S(N)$ or b) N is divisible by $S(N)+1$.

70.(8-10). Prove that it is possible to construct a convex equiangular 1980-gon from segments of lengths 1, 2, 3, ... , 1980. Is the same true for a 1981-gon?

71.(9-10). Functions f and g defined on the real line are such that the equality

$$f(x-y)+f(x+y)=2f(x)g(y)$$

holds for all x and y. Prove that if f is not identically equal to zero, then the values of g are not less than -1 for all y.

72.(9-10).[1] We have a *complete* graph: n points, every pair of which are connected by a segment. Each segment is painted either red or blue, and from any point one can get to any other point both along blue lines only and along red lines only.

Prove that there are four points among these n points such that the complete graph of these four points and the segments connecting them has the same property: one can get from any point to any other one both along blue lines only and along red lines only.

73.(9-10). Polynomial $P(x)$ is non-negative for all real x. Are there two polynomials $Q(x)$ and $R(x)$ such that $P(x)=Q(x)^2+R(x)^2$?

74.(7-10). Find all integer solutions:

$$x^2-2y^2=6t^2-3z^2.$$

75.(10). A plane flew from town Γ_1 to town Γ_2. During its entire flight it was seen from observation posts A and B hidden somewhere on segment $\Gamma_1\Gamma_2$. Prove that

a) there was one second during which the plane moved from some point X to a point Y of its trajectory in such a way that $\angle XAY=\angle XBY$;

b) statement a) is false if at least one of the observation posts is out of segment $\Gamma_1\Gamma_2$, no matter how close points A and B are from points Γ_1 and Γ_2, respectively.

Convention. We assume that the plane moves in space at a variable speed; the time of its flight is more than 1 sec; towns T_1 and T_2, observation posts A and B, and the airplane are all points.

76.(9-10). A swimming pool has the form of a convex quadrilateral with trees growing in its vertices. Each tree casts a circular shadow with its center in respective vertex. It is known that the swimming pool is entirely in the shade. Prove that the shadow of some 3 trees entirely covers the triangle in whose vertices these trees grow.

77.(8-10). A colony of a finite number of bacteria lives on a straight line. Some bacteria may die at moments 1, 2, 3, ...; no new bacteria are ever born. Those and only those bacteria die for which there are no bacteria at a distance of 1 on their left and $\sqrt{2}$ on their right.

Can such a colony of bacteria live forever?

78.(9-10). Is there a finite set of points on a plane such that for each of the points there are at least 1000 other points of this set the distance between which and this point is exactly equal to 1?

79.(10). A dodecahedron with its vertices painted red is rolled over its edges on a plane, its vertices leaving red point marks. Prove that for any disc of any radius on the plane, it is possible to roll the dodecahedron so that some vertex leaves a red mark inside the disc.

80.(7-8). There are n boxes, some of which have n boxes inside them, some of which again have n boxes inside them, etc. There are altogether k boxes with other boxes inside them. What is the total number of boxes?

81.(8-10). Prove that if $x>1$, $y>1$, and $x^y+y^x=x^x+y^y$, then $x=y$.

82.(9-10). Prove that if a, b, c are the lengths of the lateral edges of a triangular pyramid and α, β, γ are the angles between the edges, then the volume of the pyramid is equal to

$$V=\frac{abc}{3}\sqrt{\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\alpha-\beta+\gamma}{2}\sin\frac{\alpha+\beta-\gamma}{2}\sin\frac{-\alpha+\beta+\gamma}{2}}.$$

83.(10). a) Any section of a solid by a plane is a disc. Prove that the solid is a ball.

b)*** Any section of a solid is a polygon. Prove that the solid is a polyhedron.

[1]Suggested by V.Gurvich for the selection competition to ??? in 1971.

84.(7-10). An organization committee of a Math Olympiad consists of 11 members. The problems for the Olympiad are kept in a strongbox. How many locks must the strongbox have and how many keys should every member of the committee have so that any six members can open the strongbox whereas no fewer group can do it?

85.(7-9). The hands of a clock are fixed but the dial can rotate. Prove that it is possible to turn the dial so that the clock shows a correct time between 12:00 p.m. and 1:00 p.m.

86.(8-10). Prove that if $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \frac{\pi}{2}$, then

$$\tan\alpha_1 < \frac{\sin\alpha_1 + \ldots + \sin\alpha_n}{\cos\alpha_1 + \ldots + \cos\alpha_n} < \tan\alpha_n.$$

87.(7-8). Prove that if

$$\frac{1}{x+y+z} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

then two of the numbers x, y, z are equal in absolute value but have opposite signs.

88.(7-8). For which $n \in \mathbb{N}$ do there exist positive integers $k_1 < k_2 < \ldots < k_n$ such that

$$\frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n} = 1\ ?$$

89.(7-10). Prove that if the tips of the legs of a table are vertices of a square, then it is possible to place the table on an uneven floor so that the table does not rock, i.e. so that all four tips of the legs touch the floor.

90.(8-10). For $a, b, c > 0$ solve the system for unknowns x, y, z:

$$\begin{cases} \frac{a}{x} - \frac{b}{y} &= c - xy, \\ \frac{b}{z} - \frac{c}{x} &= a - xz, \\ \frac{c}{y} - \frac{a}{z} &= b - yz. \end{cases}$$

91.(10). A sphere with center O is inscribed in tetrahedron $ABCD$. Prove that if $\angle ODC = 90°$, then planes AOD and BOD are perpendicular.

92.(8-10). We write parentheses in the expression $x_1 : x_2 : x_3 : \ldots : x_n$ with distinct x_i's to indicate the order in which the numbers should be divided. The result is written in the form of the following fraction:

$$\frac{x_{i_1} x_{i_2} \ldots x_{i_k}}{x_{j_1} x_{j_2} \ldots x_{j_{n-k}}}.$$

How many distinct fractions of this kind is it possible to derive from the given expression by different arrangements of parentheses therein?

93.(7-9). Three soccer teams played the same number of matches with one another. Is it possible that the winner won the least number of matches while the team that took the last place won a maximal number of games?

94.(7-10). Prove that from the edges of an arbitrary tetrahedron it is possible to construct two triangles so that each edge is a side of one of the triangles.

95.(9-10). Consider three straight lines in space, each two of them skew and not all parallel to a plane. How many straight lines can intersect all three given lines?

96.(8-9). Twelve laces are used to make a net in the form of a cube with side of 10 cm. Inside the net is a spherical balloon. It is inflated so that the net fits tight on its surface. Find the radius of the inflated balloon.

97.(7-10). An entire rectangular map of Moscow lies on top of another similar map of a larger scale (the sides of the maps are not necessarily parallel). Prove that it is possible to puncture both maps with a pin so that the point of the puncture denotes the same point of the city on both maps.

98.(8-10). Is $2222^{5555} + 5555^{2222}$ divisible by 7?

99.(9-10). Three rods of equal lengths are used to construct a rigid spatial structure in which the rods do not touch one another but are connected by non-elastic threads fastened to their ends.

a) What least number of threads is necessary for this?

b)* What ratios of rod lengths and thread lengths make such a construction possible?

Hints to selected problems of Moscow mathematical circles

2. It is easy to see that there are no 1- and 2-digit examples. There are no 3-digit examples (for any base, not only decimal) either: indeed, it is clear that in the sum $\overline{abc}+\overline{cba}$ neither the first nor the last figure can be equal to either a or c.

A case-by-case checking shows that the least answer contains 5 digits.

REMARK. If the base were not 10, but, say, 9, there would have been 4-digit examples, say, $2\,563_9 + 3\,652_9 = 6\,325_9$. Similar examples exist for any base divisible by 3 starting with 6.

4. Prove that the sum of these 14 numbers either is equal to 14 (if all vertices are labeled by 1's) or differs from 14 by a multiple of 4, i.e. can be $10, 6, 2, -2, \dots$. It suffices to prove that by changing the sign of one of the units at a vertex we alter the sum by a multiple of 4.

9. Make use of the fact that if $1 < z < t$ then $z + t < zt$.

10. First, divide both polyhedrons into arbitrary tetrahedrons (e.g., one can first divide a polyhedron into pyramids by connecting its inner point with the vertices). Then, selecting the smallest of the tetrahedrons obtained, one should cut a tetrahedron of the same size from one of the remaining tetrahedrons. Further, apply the induction on the number of tetrahedrons.

11. Let M be the center of square $ABCD$ and let O lie, for example, in the triangle ABM. Now, prove that $\angle OAB + \angle OCD \leq 90°$. The inequality $\angle ODA < 45°$ being taken into account, it suffices to prove the estimate from above.

The estimate from below is derived in the same way from $\angle OBC + \angle ODA \geq 90°$.

18. If this set is infinite, the decimal representation of 5^n would contain too long sequences of zeros for a large n.

21. Define the distance from a point outside a sphere to the sphere as the length of the tangent from the point to this sphere. For each spherical hole, consider the set of all points "distanced" from the sphere of the hole not farther than from any other hole. It is easy to prove that for two spherical holes we thus get two half-spaces. In the general case, each of the sets obtained is a convex polyhedron containing a sphere. All polyhedrons adjoining each other entirely fill up the cube.

24. First, find an integer m such that the numbers[1] $\{m\alpha\}$ and $\{m\beta\}$ are both small (say, smaller than 0.01) but their ratio is not less than 2.

27. If the alarm-clock ticks uniformly but generally shows wrong time, then during, say, an hour there is only a finite number of moments when it indicates a right time.

28. a) It is easy to demonstrate that the parallelepipedal lattice can be replaced by that of cubes and the solution of the problem will not change.

Circumscribe the ball round each of 8 vertices of the given cube (from cubic lattice) centered at each vertex; let the volume of each ball be equal to 1. The sum of the balls' volumes is equal to 8 and the volume of the balls inside the cube is equal to $8 \times \frac{1}{8} = 1$. So the remaining $8 - 1 = 7$ volume units fall on the neighboring cubes. Let us prove that each neighboring cube has not more than $\frac{1}{2}$ of the volume unit, and, therefore, there number is not less than 14.

Indeed, if the vertex of the given cube lies inside the face of the neighboring cube, this volume is $\frac{1}{2}$ of the volume unit (the neighboring cube does not touch other balls).

If the vertex of the cube lies on an edge of the neighboring cube, one more vertex can lie on its edges and the total volume does not exceed $2 \times \frac{1}{4} = \frac{1}{2}$ of the volume unit.

Finally, if the vertex of our cube is also the vertex of the neighboring cube, this cube may touch maximum 4 more vertices of our cube and this gives a volume of $4 \times \frac{1}{8} = \frac{1}{2}$ of the volume unit.

b)[2] See Fig. 99.

32. After any odd number of refills both containers have the same amount of water.

36. Make use of the identity $\left(n^4 + \frac{1}{4}\right) = \left(n^2 + n + \frac{1}{2}\right)\left(n^2 - n + \frac{1}{2}\right)$.

37. Apply Helly's theorem: *If any* 3 *of* n *given convex figures (e.g., discs) on the plane have a common point, then this point belongs to all of them.*

39. Prove that $AC \cdot AD = R^2$, where R is the radius of the exterior circle. This follows from the fact that $\triangle OAC \sim \triangle OAD$.

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[1] Here $\{x\}$ is the fractional part of x.

[2] We thank M. Urakov who suggested this extension.

FIGURE 99. (Hint A28)

42. Apply induction with the following hypothesis: the procedure described in the problem allows one to get from $1, 2, \dots, 2k$ any number 0 to $2k$ whose parity coincides with that of k.

47. The length of the projection of the given broken line to any edge is > 2. Therefore, the sum of these projections is ≥ 6. Now, prove that every chain of the broken line is not more than $\sqrt{2}$ times shorter than the sum of its projections. Make use of the fact that one of the projections of any chain is always equal to 0.

48. Make use of the Cauchy inequality for the arithmetic mean and the geometric mean. For the left inequality make use of the fact that

$$\sqrt[n]{n+1} = \sqrt[n]{\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{n+1}{n}} < \frac{1}{n}\left(\frac{2}{1} + \frac{3}{2} + \ldots + \frac{n+1}{n}\right).$$

Find on your own a similar inequality for the right side of the problem.

52. Prove first that the centers of both polygons coincide.

54. The cyclist should act as follows: First, (s)he should divide the square 10×10 into 4 squares with the midlines and find out which of the squares thus obtained contains the city block (s)he is looking for (assuming that the numbers of adjacent blocks differ by 1); then divide this quarter of the square in a similar way and select the 16-th part (2-*nd order square*) which contains the block (s)he wants, etc., see Fig. 100.

To evaluate in each particular case the longest path the cyclist takes requires to sum the number of stages to the appropriate square of the highest order.

The first stage takes 25 km (or a bit more if n is odd); each next stage takes one half of the preceding one.

56. Prove that of two neighboring vertices, only one can satisfy the condition of the problem.

57. First, prove that if not all numbers are equal and m is the smallest of them (it may occur not once but many, even an infinite, number of times) then eventually all numbers on any segment of finite length become greater than m.

58. Let us write 1 on each white cube and -1 on each black cube. The painter's performance is an operation that replaces every number by the product of its neighboring ones. It is easy to deduce that it suffices to consider the case of one black and 26 white cubes; the general case is obtained from this one by the above "multiplication". It remains to consider 4 variants (deal with them on your own): when the black cube is in the corner; on the edge; in the center on the face; in the center of the cube.

59. Consider the function $f(X) = \sum\left(1 - |A_iX|^2\right)$, where the sum runs over the terms with A_i contained in the ball of radius 1 centered at X. Prove that the value of f increases when we replace X with the center of mass of A_i's.

60. Prove by induction on n that any set can be reduced by the described operations to any of the following two forms: 000...00 or 000...01.

61. First, notice that the center O_4 of the circle tangent to the given circles with centers at O_1, O_2, O_3 is the same as that of the circle circumscribed about triangle $O_1O_2O_3$ and their radii differ by the radius of one of the given circles, O_1, O_2, or O_3. The intersection point of the bisectors of $\triangle ABC$ is the same as that of $\triangle O_1O_2O_3$ (the latter are continuations of the former), i.e., point O is the center of the circles inscribed into $\triangle ABC$ and $\triangle O_1O_2O_3$. At the same time, O is the center of a homothety for the homothetic triangles $\triangle ABC$ and $\triangle O_1O_2O_3$ (since $AB \parallel O_1O_2$, $BC \parallel O_2O_3$, $AC \parallel O_1O_3$).

FIGURE 100. (Hint A54)

FIGURE 101. (Hint A56)

Let point O' be the center of the circle circumscribed about $\triangle ABC$. It is easy to see that the homothety with the center at O which sends $\triangle O_1O_2O_3$ into $\triangle ABC$ sends O_4 to O'. Hence, O_4 belongs to $O'O$, Q.E.D.

63. Let p_n be the n-th prime. Make use of the two facts: that S_n is (for $n > 4$) less than the sum of the first n odd numbers (which is equal to n^2) and $p_{n+1} > p_n + 2$.

67. First, prove that the condition of the problem will be satisfied if and only if the product of the numbers in two opposite corners of any square is equal to the product of the numbers in the other two corners. The simplest way to proceed now is to place the number p_iq_j in the square at the intersection point of the i-th row and the j-th column, where p_1 and $p_2, \dots, p_n$, $q_1, \dots, q_n$ are numbers arbitrary except that all products p_iq_j are distinct, e.g. take $2n$ distinct primes.

70. Make sure first that an equiangular hexagon can be composed of segments of length $k+1$, $k+2, \dots,$ $k+6$ (for any k), then the 1980-gon is composed of several hexagons. (Fig. 102 shows how it can be done for a 12-gon.)

This construction is possible due to the fact that $\cos 60^\circ = \frac{1}{2}$ is a rational number. Since $\cos \dfrac{360^\circ}{n}$ is irrational for $n = 1981$ and for the nontrivial divisors of 1981, the problem cannot be solved for the 1981-gon.

72. Use induction on n.

FIGURE 102. (Hint A70)

91. Draw the plane π perpendicular to OD through the center O of the sphere. Then make use of the facts that DC is parallel to π and π intersects lines parallel to DC on the planes of the faces ADC and BDC.

92. Prove by induction that the result may be any fraction with x_1 in the numerator and x_2 in the denominator.

The induction should start with $n = 3$. It should be born in mind that if $(x_n : x_{n+1})$ is substituted for x_n in the expression $x_1 : x_2 : \ldots : x_n$, the final result will be that x_{n+1} is in the numerator if x_n was in the denominator and vice versa. But if $(P : x_n)$ is replaced by $((P : x_n) : x_{n+1})$ in the expression $(x_1 : \ldots : (P : x_n))$, where P is a bracket or just a letter x_{n-1}, then x_{n+1} takes the same place as x_n.

95. We draw all kinds of planes through skew line No. 2, each of which intersects skew lines No. 1 and No. 3 at two points. By connecting the latter we get a straight line l that intersects all given lines Nos. 1, 2 and 3. (It is not difficult to show that only one of such lines can be parallel to line No. 2.)

Answers to selected problems of Moscow mathematical circles

1. a) $(1+2+\ldots+9)\cdot(6\cdot 10^5)=27\cdot 10^6$; b) $6\cdot 10^5$.

2. $12\,897+79\,821=92\,718$.

3. There are.

4. No. It can not.

6. Yes.

7. $(x,y)=(1,2)$ or $(2,5)$.

8. $\frac{1}{2}(8100+181)$.

13. 800 g.

15. $\angle B=60^\circ$, $\angle D=150^\circ$.

16. A) ??
B) 7 points can be arranged in the way required but 8 points can not.

18. This set is finite.

29. There are two cases depicted on Fig. 103: in case b) $S_1=S_2=S_3=\frac{1}{6}$, $S_4=\frac{1}{2}$; in case c) $S_1=S_2=S_3=\frac{\sqrt{5}-1}{4}$, $S_4=\frac{7-3\sqrt{5}}{4}$.

Figure 103. (Answ. A29)

36. 313.

38. No, this is not always possible.

42. Any odd number between 1 and 1973.

46. No.

49. Yes.

51. No, such n does not exist.

56. $\left[\frac{n}{2}\right]$; for an example see Fig. 104.

57. No.

62. No, see Fig. 105.

65. For any n.

67. Yes.

FIGURE 104. (Answ. A56)

FIGURE 105. (Answ. A62)

68. Yes, it is possible, see Fig. 106. The three persons should change their respective coordinates in the following manner:

a) $2 \longrightarrow 1002$ (visible from 0);
b) $0 \longrightarrow 0.5$;
c) $1 \longrightarrow 1001$ (not visible from 0.5);
d) $0.5 \longrightarrow 1000$ (visible from 1002, not visible from 1001).

FIGURE 106. (Answ. A68)

73. Certainly.

77. No.

78. Yes.

80. $(k+1)n$ boxes.

84. The number of locks is $\binom{11}{5} = 462$ and the number of keys for each member of the organizing committee is $\binom{10}{5} = 252$.

88. Any $n \geq 3$.

92. 2^{n-2}.

93. Yes; an example is shown in Table:

	I	II	III	Total	Score
I	XXXXX	$+1 =6 -0$	$+2 =3 -2$	$+3 =9 -2$	15
II	$+0 =6 -1$	XXXXX	$+4 =0 -3$	$+4 =6 -4$	14
III	$+2 =3 -2$	$+3 =0 -4$	XXXXX	$+5 =3 -6$	13

95. Infinitely many.

98. Yes.

Solutions to selected problems of Moscow mathematical circles

1. The given number is "built" of numbers 1 to 999999. The sum of digits, as well as the number of digits 7, does not vary if we insert several zeroes, say, as follows : 000000 000001 000002 … 999999. Now, it is clear that we have written $6 \cdot 10^6$ digits which constitute all possible combinations of 6 digits. Therefore, all digits are encountered the same number of times, namely $(6 \cdot 10^6)710$. this immediately implies the answer.

3. If $\left(\sqrt{2}\right)^{\sqrt{2}}$ is rational, take $y = x = \sqrt{2}$. Otherwise take $x = \left(\sqrt{2}\right)^{\sqrt{2}}$, $y = \sqrt{2}$, then $x^y = \left(\left(\sqrt{2}\right)^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$.

4. Let all vertices be labelled by 1; then the sum equals 14. By changing the number at a vertex we change three more numbers (on thre adjacent faces), so the sum diminishes by 8 and becomes 6.

Now, let us change the numbers at one more vertex. Again, this induces a simultaneous change of 3 more numbers and one easily seas that the sum differs by $\pm 8, \pm 4$ or 0. Namely, if four 1's turn into -1's, then the sum diminishes by 8; if three 1's and one -1 change, the sum diminishes by 4, and so on.

Therefore, the sum may differ from 14 by a multiple of -4, i.e., it can be 10, 6, 2, -2, ... So the sum can not be equal to 0 or 7.

5. In this problem it is tacitly assumed that a star and an observation post are just points.

Observe that the answer for the 3-dimensional space differs from that for the 2-dimensional space (plane): *two* observation points are enough for the former and *three* for the latter.

a) Since one observation point is obviously insufficient, choose point A at random and consider all rays, starting from A, on which lie all stars visible from A. ("Consider all … " means "plot the rays and indicate their position in space relative to a preselected system of coordinates".)

Now, consider all possible planes $\pi_1, \pi_2, \dots$, drawn through each pair of the rays. Let B be an arbitrary point that does not belong to any of the planes $\pi_1, \pi_2, \dots$. Let us prove that B is the required observation post since ALL stars in the sky can be seen from there, i.e., the stars visible from B do not hide behind each other.

Indeed, all stars lie on straight lines that connect pairs of stars. Thus, all stars are in planes $\pi_1, \pi_2, \dots$. Since point B does not belong to any of these planes by construction, B does not belong to any of the lines connecting the stars. But the stars can hide behind each other only from an observer located on these lines. Thus, two posts suffice.

b) The reasoning in a) does not apply since there is no point B outside the plane. Two observation posts are not enough because for any chosen observation post B stars may happen to be on a ray coming from point B and crossing the rays coming from the first point, A, to the stars; their intersection points may happen to be stars.

So, select at random two points A and B from which not all stars may be visible. But all stars are sure to lie at some of the points where rays connecting A with stars intersect the rays connecting B with stars.

It remains to draw all possible lines through all pairs of the intersection points of the two bunches of rays with vertices at A and B, respectively, and to select a point, C, not belonging to either of these lines or any of the rays from the bunches starting from A or B. As is clear from this construction, the point C is the desired one.

6. Proof: by induction. If there is only one car, there is nothing to prove.

Suppose the statement is already proved for $n-1$ cars and consider n cars. Clearly, at least one of the cars (call it A) has enough gas to drive to the next car, B. Remove B from the road and add its petrol to A. Now there are $n-1$ cars on the road with the same quantity of gas and by the inductive hypothesis there exists a car C which can run the whole length of the road. Notice that the same car C can run the whole length of the road also in the initial situation when car B is present on the road.

7. For $x = 3$ there are no solutions, since the left hand side is equal to 122 which is not a perfect square.

Let us show that for $x > 4$ there are no solutions either. Observe that to find the minimal positive value of $z = R^2 - t^2$ for a fixed R and variable integer t, one has to look at the graph of this function to deduce that the minimum is attained at $t = R - 1$.

Let us now rewrite the equation in the form

$$(x^2 + x)^2 - y^2 = 11(x-1). \tag{$*$}$$

By the observation above, the left hand side of $(*)$ takes the least positive values for a fixed x if $y = (x^2+x)-1$. But then it is equal to $2x^2 + 2x - 1$ which is greater than $11(x-1)$ for $x > 4$ as is easy to verify by setting

$t = x - 1$. Therefore, if the left hand side of $(*)$ is positive, then it is greater than the right hand side. Therefore, the solutions indicated above are the only ones.

8. First, let us prove that all numbers greater than 8100 are painted red. Indeed, let $8100 < A = 81k + 100l$. By Euclid's algorithm, any integer can be represented in this form with integer, though not necessarily positive, k and l. At least one of the numbers k and l, say, k, is positive. From all representations of A in the above form select the one for which k is the least positive. Then $k < 100$ (otherwise there would have existed a representation $(k - 100, l + 81)$), therefore, $81k < 8100$. But then $100l > 0$ and $l > 0$, as was required.

As is not difficult to figure out, the number 8100 is painted blue; hence, this is the right-most of the blue points. On the other hand, it is clear that the left-most of the red points is 181. Therefore, it is clear that the point whose existence is claimed in the problem should equal to $\frac{1}{2}(8100 + 181)$ and numbers A and B are symmetric with respect to it if $A + B = 8281$.

It remains to prove that of two such numbers one is necessarily red and another one blue. Let us write again $A = 81k + 100l$ and $B = 81m + 100n$. Assume that k and m are the least positive numbers for which such a representation exists, i.e., $0 < k, m < 100$. Then

$$8281 = 81(k + m) + 100(l + n)$$

and $0 < k + m \leq 200$.

But it is not difficult to verify that there exists only two representations of 8281 in the form $81x + 100y$ so that $0 < x < 200$. These representations are: $x = 1$, $y = 82$ and $x = 101$, $y = 1$.

In the first case $k + m = x = 1$ which contradicts the condition $k, m > 0$. Thus, the second case holds. But then $l + n = 1$ and therefore, one of these numbers is positive and the other one is not. This directly implies that one of the numbers A, B is red. We leave it to the reader to establish that the other number is blue.

12. A) Draw all possible straight lines through all pairs of points in the set. Denote straight line l containing points A and B by AB. We can now demonstrate that at least three points of the set lie on at least one of the drawn lines.

Indeed, if A, B, C, and D are four points of the set which do not coincide, and M — the intersection point of AB and CD — also belongs to the set, then A, B, C are either on the same straight line and then M coincides with C (see Fig. 107 a)) or three points A, B, M and, respectively, C, D, M already lie on straight lines AB and CD; see Fig. 107 b).

So let us choose a line l containing at least three points of the set and prove that all points of the set but one lie on that line. Assume the contrary: let A and B lie outside l. Let us prove then that the set contains an infinite number of points: contradiction.

To this end denote the intersection point of l with AB by C_1. By the hypothesis it belongs to the set. (In what follows we will remember that all intersection points of the lines under consideration belong to the set.) Line l was said to have also points C_2 and C_3 of the set, see Fig. 107 c). Denote the intersection point of AC_3 with BC_2 by X_1 (AC_3 and BC_2 are not parallel), X_1 not lying in l. Lines AC_2 and C_1X_1 meet at point X_2, also outside l. Then BX_2 intersects l at point C_4 which does not coincide with either C_1, C_2, or C_3 (since BC_4 does not coincide with either BC_1 or BC_3).

Further on, AC_4 meets C_1X_1 at X_3 and then BX_3 meets l at point C_5 distinct from $C_1, \dots, C_4$.

Let us continue the same operation: if $C_k \in l$ is already plotted, lines AC_k and C_1X_1 meet at a point X_{k-1} distinct from $X_1, \dots, X_{k-2}$ and then BX_{k-1} intersects l at point C_{k+1} distinct from $C_1, \dots, C_k$. The process may go on *ad infinitum* and an infinite number of points from the set will appear on l (and we did not yet consider other intersection points of the lines!). Since the given set is a finite one, the contradiction proves that our assumption was wrong.

FIGURE 107. (Sol. A12)

B) Let us start solution with heading (c). We will give two solutions related to different branches of geometry (see Remark below).

First solution. Draw plane π which does not pass through point O and is not parallel to either of the straight lines $l_1, \dots, l_n$ of the set (a "generic" plane). Denote n intersection points of these lines with π by $A_1, \dots, A_n$, respectively. Hereafter we will denote the plane containing the straight lines l_i and l_j by l_il_j and denote the line through points A_i and A_j by A_iA_j.

Now, we can prove that the set of points $A_1, \dots, A_n \in \pi$ satisfies the condition of Part A) of the solution. Indeed, let A_iA_j and A_kA_p be two distinct straight lines. We can prove that they meet at one of the points A_s of the set (generally, A_s may coincide with either of the points A_i, A_j, A_k, and A_p).

To this end consider planes l_il_j and l_kl_p. These planes intersect because they contain point O. Denote by l the straight line of their intersection. Let us prove that *l is a straight line from our set*. Indeed, denote a line from the set perpendicular to l_i and l_j (there is such a line by hypothesis) by L_1 and a line from the set perpendicular to l_k and l_p by L_2. Then l is perpendicular to plane L_1L_2 because both L_1 and L_2 are perpendicular to l.

On the other hand, by the hypothesis a straight line perpendicular to plane L_1L_2 must belong to the set $l_1, \dots, l_n$. Thus, l is a straight line from the set, Q.E.D.

If $l = l_s$, then point A_s which belongs to l_s must also belong to line A_iA_j (since l belongs to l_il_j) and to line A_kA_p (because l belongs to plane l_kl_p). Thereby we have proved that lines A_iA_j and A_kA_p meet at point A_s from the set.

But then it follows from heading A) that $n-1$ points of the set lie on the same line (let them be $A_1, \dots, A_{n-1}$) and the remaining point A_n lies outside this line.

Consequently, the lines $l_1, \dots, l_{n-1}$ lie on the same plane P while l_n is outside it. How then is l_n arranged relative to plane P?

We see that $l_n \perp P$, since there is a line from the set perpendicular to the lines l_1 and l_2, and l_n is the only line from the set perpendicular to the plane l_1l_2.

Finally, taking l_n and an arbitrary line l_i, $l_i \subset P$, we deduce from the hypothesis that among the remaining lines of the set lying in plane P there exists a line l_j perpendicular to both l_i and l_n. Hence, for each line L from the set, $L \subset P$, there exists a line M from the set such that $M \subset P$ and $M \perp L$. Thus, all lines $l_1, \dots, l_n \subset P$ can be divided into pairs of pair-wise perpendicular lines. Hence, $n-1$ is even; thus, n is odd ($n \geq 3$).

But for any odd n the desired arrangement of the lines exists and can be uniquely described as the union of a line, l, with $n-1$ lines lying in the plane perpendicular to l; moreover, the $n-1$ lines consist of $\frac{n-1}{2}$ pairs of mutually perpendicular lines.

Therefore, the answers to (a) and (b) are: 99 straight lines can be drawn in the way required and 100 lines can not.

Second solution. Let us prove that all straight lines save one lie on one plane. (The remaining part of this solution does not differ from the first solution). For $n = 3$ the statement is clear. For $n > 3$ it is impossible that all lines are pair-wise perpendicular. Take two non-perpendicular lines, x and y. Let their common perpendicular be the z-axis. Two more lines should lie in xy-plane: the common perpendiculars to the z-axis and the original lines.

Let us prove that all other lines lie in xy-plane. Assume the contrary and among all lines select the one, l, forming the least angle with the z-axis. Of 4 lines lying on xy-plane, select one, m, not in plane zl and not perpendicular to zl-plane. Since lines m and l are not perpendicular, a line p lying on ml-plane and perpendicular to m also belongs to our set.

It remains to observe that line p is the image of the orthogonal projection of the z-axis to ml-plane; hence, the angle between z and p is smaller than the angle between z and any other line lying in ml-plane. In particular, it is smaller than the angle between z and l. Contradiction. Q.E.D.

Remark. Problems A) and B) are statements of the so-called "geometry of order" or "geometry of position" (another name is "descriptive geometry") where the main idea is that of a "position between" (e.g. a segment is a set of points lying between two given points, etc.) and the idea of an "arrangement in a certain order". The solution of A) and the first solution of B) are in the spirit of this geometry. The second solution uses the notion of distance (angle) and is related to the metric geometry.

13. To demonstrate that the total weight is equal to 800 g, let us prove that both straight lines pass through the center of the square. If the lines do not meet at the center, let us translate them parallelly so that they would. After this translation one of the pieces increases in weight and the opposite piece decreases and all four pieces become equal figures. Contradiction.

14. This problem generalizes Problem 30.2.7.5. We will describe a generalization of the three-dimensional analogue of this problem (cf. Problem 30.2.10.4) at the end of the solution below.

First of all, we outline the idea of the solution. Let us randomly distribute the searchlights over given points $M_1, M_2, \dots, M_n$. If not the whole plane is illuminated we will construct another, "improved", arrangement of the searchlights, the "quality" of the arrangement being evaluated with a numerical function f.

Since the number of the arrangements of searchlights over the points $M_1, M_2, \dots, M_n$ is finite, we will automatically light up the entire plane by taking the arrangement for which the value of f is *maximal*. Indeed, if not the whole plane were illuminated in this case we could still improve the arrangement: contradiction.

FIGURE 108. (Sol. A14)

Let us start carrying out this plan. Let us transport all searchlights to one point O and draw a convex n-gon $A_1A_2\dots A_n$ whose sides are lighted up by the searchlights[1], see Fig. 108; let $\angle A_iOA_{i+1} = \alpha_i$, $i = 1, \dots, n$. Drop the perpendiculars OH_i from O to the sides of the polygon or to their extensions. Let $|OH_i| = h_i$. Consider the vectors

$$\vec{e}_i = \frac{1}{h_i^2} \cdot \overrightarrow{OH_i} \qquad \text{for} \quad i = 1, \dots, n.$$

Clearly, $|\vec{e}_i| = \frac{1}{h_i}$, see Fig. 108 b).

Let us attach each vector $\vec{e}_i$ to its respective searchlight α_i. Then the arrangement of the searchlights over the points $M_1, \dots, M_n$ corresponds to the distribution of the vectors $\vec{e}_i, \dots, \vec{e}_n$ over these points (we arrange the searchlights together with the vectors as solid bodies).

To justify the appearance of the strange vectors $\vec{e}_i$ we need the following geometric fact:

Lemma 1. *Suppose the searchlight α_p placed at point M illuminates a point N while the searchlight α_q placed at M does not illuminate N. Then $\overrightarrow{MN} \cdot \vec{e}_p > \overrightarrow{MN} \cdot \vec{e}_q$, see* Fig. 108 c).

(Hereafter a "$\cdot$" means the inner product of vectors; the i-th searchlight is denoted by α_i — the angle it illuminates indexed by its number).

PROOF. Let $\overrightarrow{MN} = \vec{v}$. Draw the vector $\overrightarrow{OP} = \vec{v}$ with O as its initial point, see Fig. 108 d).

The ray $[OP)$ intersects the p-th side of the polygon $A_1 \dots A_n$ at a point K and does not intersect the q-th side because the searchlight α_p lights up N while α_q does not.

There are two possibilities:

1) $[OP)$ does not intersect the straight line on which the q-th side lies;

2) $[OP)$ intersects this line at a point R.

In case 1), Lemma 1 is obvious since $\vec{v} \cdot \vec{e}_p > 0$ and $\vec{v} \cdot \vec{e}_q \le 0$.

In case 2), the inequality $|OK| < |OR|$ is satisfied and, therefore,

$$\begin{aligned} \vec{v} \cdot \vec{e}_p &= |\overrightarrow{OP}|\,|\vec{e}_p| \cos(\vec{v}, \vec{e}_p) = |\overrightarrow{OP}| \frac{1}{|\overrightarrow{OH}_p|} \cos(\overrightarrow{OH}_p, \overrightarrow{OK}) \\ &= \frac{|\overrightarrow{OP}|}{|\overrightarrow{OK}|} > \frac{|\overrightarrow{OP}|}{|\overrightarrow{OR}|} = |\overrightarrow{OP}| \frac{1}{|\overrightarrow{OH}_q|} \cos(\overrightarrow{OH}_{qp}, \overrightarrow{OK}) = |\overrightarrow{OP}|\,|\vec{e}_p| \cos(\vec{v}, \vec{e}_q) = \cos(\vec{v}, \vec{e}_q). \quad \square \end{aligned}$$

Now, let us introduce a function $f_\sigma(N)$ that depends on the point N of the plane and on the distribution σ of searchlights over the points $M_1, \dots, M_n$:

$$f_\sigma(N) = \overrightarrow{M_1N} \cdot \vec{e}_{\sigma(1)} + \overrightarrow{M_2N} \cdot \vec{e}_{\sigma(2)} + \dots + \overrightarrow{M_nN} \cdot \vec{e}_{\sigma(n)}, \qquad (*)$$

---

[1] If we leave out the 3-dimensional generalization for better days, the following solution can be considerably simplified by assuming that all OA_i are of unit length.

where $\sigma(i)$ is the number of the searchlight placed at point M_i.

Lemma 2. *If point N is not lighted up under the arrangement σ of the searchlights, there is an arrangement τ of searchlights such that $f_\tau(N) > f_\sigma(N)$.*

PROOF. First, we give an algorithm to improve the arrangement σ.

ALGORITHM. Let all searchlights α_i first stand at points M_i with the same index. Let α_1 not light up N. Remove it for a while and place at point M_1 the searchlight which does illuminate N. (Explain yourselves why such a searchlight will certainly be found.)

Let it be α_2. Now, there is nothing at point M_2, where α_2 used to be. Out of the remaining searchlights, place at M_2 the one which will light N. Let it be α_3. Then we place the searchlight α_4 at M_3 to light N, the searchlight α_5 at M_4, and so on, until we have a cycle, see Fig. 109 e).

FIGURE 109. (Sol. A14)

This means that the searchlight α_{k+1} is moved over to M_k, the searchlight α_{k+2} is moved to M_{k+1}, etc., and α_k to M_s, where $s > k$. Bring back all searchlights $\alpha_1, \dots, \alpha_{k-1}$ which did not get into the cycle to their initial points $M_1, M_2, \dots, M_{k-1}$.

The result of the application of the algorithm is that N is lighted up by *each* searchlight $\alpha_k, \alpha_{k+1}, \dots \alpha_s$, where $s \geq k+1$, because N was not lighted up before.

Let us see how $f(N)$ changed under the application of the algorithm. The summands with the index of M_i not equal to $k, k+1, \dots, s$ in the sum $(*)$ did not change, whereas, as follows from Lemma 1, the summands with the indices $k, k+1, \dots, s$ increased. Q.E.D.

However, f_σ cannot be directly used yet to evaluate the "quality" of the arrangement σ since it depends on N. The situation is saved by the wonderful

Lemma 3. *For any two points N_1 and N_2, the difference $f_\sigma(N_1) - f_\sigma(N_2)$ does not depend on the arrangement of searchlights.*

PROOF. Observe that the sum $\sum\limits_{i=1}^{n} \overrightarrow{e_{\sigma(i)}}$ does not depend on the choice of an arrangement; therefore,

$$\begin{aligned} f_\sigma(N_1) - f_\sigma(N_2) &= \sum_{i=1}^{n} \overrightarrow{M_{\sigma(i)}N_1} \cdot \overrightarrow{e_{\sigma(i)}} - \sum_{i=1}^{n} \overrightarrow{M_{\sigma(i)}N_2} \cdot \overrightarrow{e_{\sigma(i)}} \\ &= \sum_{i=1}^{n} \left(\overrightarrow{M_{\sigma(i)}N_1} - \overrightarrow{M_{\sigma(i)}N_2}\right) \cdot \overrightarrow{e_{\sigma(i)}} = \sum_{i=1}^{n} \overrightarrow{N_2N_1} \cdot \overrightarrow{e_{\sigma(i)}} = \overrightarrow{N_2N_1} \cdot \sum_{i=1}^{n} \overrightarrow{e_{\sigma(i)}} = const. \quad \text{Q.E.D.} \end{aligned}$$

Let us prove that the function

$$f_\sigma = f_\sigma(N) + \overrightarrow{NN_0} \cdot \sum_{i=1}^{n} \overrightarrow{e_{\sigma(i)}} = \sum_{i=1}^{n} \overrightarrow{M_{\sigma(i)}N_0} \cdot \overrightarrow{e_{\sigma(i)}},$$

where N_0 is an arbitrary fixed point in the plane, is the desired one.

Indeed, the value of f_σ does not depend on N and increases under any rearrangement described in the algorithm.

Thus, we have found the "quality function", f_σ, which only depends on the arrangement σ of the searchlights and demonstrated how to increase its value if a point in the plane is not lighted up. Since f_ω as the function of its index, the arrangement (for a fixed arrangement it is a constant as we just proved), attains its maximum for an arrangement ω, the entire plane is lighted up in this case.

EXTENSION. (THREE-DIMENSIONAL GENERALIZATION). Let there be given a convex polyhedron with faces $\Gamma_1, \dots, \Gamma_n$ and a point O inside, as well as n arbitrary points $M_1, \dots, M_n$ in space. Each face Γ_i

defines a polyhedral angle α_i at which this face is seen from O, see Fig. 109 f). There are n searchlights at O, the i-th searchlight lighting up the polyhedral angle α_i $(1 < i < n)$; thus the entire space is lighted up from O.

Prove that the searchlights can be moved over to points $M_1, \dots, M_n$ so that the whole space will be still lighted up. The solution of the problem almost literally repeats that of the flat version.

15. Denote: $\angle ABD = \alpha$, then $\angle ACD = \frac{\alpha}{2}$, $\angle ADB = 90° - \frac{\alpha}{2}$, see Fig. 110.

FIGURE 110. (Sol. A15)

From $\triangle ABD$ we have
$$\angle BAD = 180° - (\alpha + 90° - \tfrac{\alpha}{2}) = 90° - \tfrac{\alpha}{2},$$
that is $\angle BAD = \angle BDA$ and so $\triangle ABD$ is an isosceles triangle: $AB = BD = r$.

Draw a circle centered at B with radius r. Since $\angle ACD = \frac{\alpha}{2}$, point C lies on the circle (analyze this situation on your own). Consequently, $BC = r$. We have $AB = BC = AC$, i.e., $\triangle ABC$ is equilateral; $\angle ABC = 60°$ and $\angle BDC = \angle BCD = 60° + \frac{\alpha}{2}$. Hence, $\angle ADC = (90° - \frac{\alpha}{2}) + (60° + \frac{\alpha}{2}) = 150°$.

16. A) Draw three mutually perpendicular planes through the center of the cube. They divide the cube into 8 smaller cubes with edge $\frac{1}{2}$. Since there are 9 points, two of the points can be found in at least one of the 8 cubes. The distance between the points does not exceed the length of the diagonal of a smaller cube, i.e., it does not exceed $\frac{\sqrt{3}}{2}$. Therefore, the two points are the desired ones.

B) Consider two subcases:

a) Draw three mutually perpendicular planes through the center of the cube. They divide the cube into 8 smaller cubes of edge 1. If some two of the 8 points are in one small cube, we are in the situation solved in A) and everything is proved.

Therefore, assume that each cube has exactly one of the 8 points. Suppose now that the distance between any two of the points is greater than 1.

Denote the distances from each of the points to the vertex of the small cube *nearest* to this point by $d_1 \geq d_2 \geq \dots \geq d_8$. Choose a smaller cube such that its point A corresponds to the distance d_1. We can assume without loss of generality that A is within the black triangle of the upper face in the right-hand cube nearest to us, see Fig. 111 a).

Consider its neighboring non-black cube on Fig. 111 a) and let the point B on this cube correspond to the distance to the vertex V, which is equal to d_i.

Draw the ball of radius d_i centered at V and its intersections with the faces of this cube. We will produce three quarter-circles of radius d_i on the faces. It is on one of them that the point B lies. But then it is easy to prove (there are many ways of doing this, see below) that the distance from A to any point of an arbitrary quarter-circle centered at V does not exceed 1.

Here is one of such proofs. The most difficult case is the one when B is on the rear face (invisible to us) of the neighboring cube. (Analyze the two other cases on your own.)

We introduce coordinate axes at the upper face of the front cube (the one that contains point A) and at the rear face of the neighboring cube (the one that contains point B) as shown in Fig. 111 a). The coordinates (z, t) of A satisfy the condition
$$\frac{1}{2} \geq z \geq t \qquad (1)$$
and the coordinates (x, y) of B satisfy the condition
$$x^2 + y^2 \leq z^2 + t^2. \qquad (2)$$

The squared distance between A and B is

$$|AB|^2 = (1-z)^2 + x^2 + (t-y)^2,$$

which follows from triangles $\triangle AMB$ and $\triangle MBH$, since $AM \perp MH$ and $BH \perp MH$. After simplification and taking (2) into account we get:

$$|AB|^2 = 1 - 2z + (z^2+t^2) + (x^2+y^2) - 2yt \leq 1 - 2z + 2(z^2+t^2) - 2yt.$$

Since $z^2 + t^2 \leq 2z^2 \leq z \leq z + yt$ (we have made use of (1)), we have:

$$-2z + 2(z^2+t^2) - 2yt \leq 2(z^2+t^2-(z+yt)) \leq 0$$

and, therefore, $|AB|^2 < 1$.

This contradiction proves the statement of the problem.

b) Let us place the cube on its vertex V so that one of its great diagonals is perpendicular to the horizontal plane. Put a point in each of the other vertices and number them 1 to 7, as shown in Fig. 111 b).

FIGURE 111. (Sol. A16)

Move points 1, 2, 3 over a short distance ε along the edges $1V$, $2V$ and $3V$; we get points $1', 2', 3'$. Then choose a number δ many times smaller than ε, e.g. 100 times smaller, and move points 4, 5, 6 to V over the distance δ along the diagonals of the squares shown in Fig. 111 a) and denote the new points by $4', 5', 6'$. Point 7 will not be moved.

It is easy to check that the distance between any two of 7 "hatched" (see *Prerequisites* on Dirichlet's principle) points is strictly greater than 1 (verify it yourself).

17. a) If the planes are parallel, the statement of the problem is obvious and so it suffices to consider the case when they are not parallel. Project the solid body under consideration to the intersection line l of the given planes. We get segment AB.

On the other hand, the projection of the body to l can be obtained by projecting the body first to any of the given planes and then by projecting the projection thus obtained to l. The result is that both circles — projections of the body to our planes — are projected *onto* the segment AB whose length coincides, therefore, with the length of the diameter of each circle. Hence, the circles are identical.

REMARK. The body is not necessarily a ball: it can be of a complex shape, e.g. neither convex nor flat. This body lies in the intersection of two identical infinite cylinders perpendicular to the planes.

b) Just as in a), we can prove that the projections of the vertices of each polygon on l, the intersection line of the planes, are the same points $A_1, \ldots, A_n$. In addition, the centers of both regular n-gons are projected into the same point O on l (to prove this use the theorem on three perpendiculars). Denote the radii of the circles circumscribed around the regular n-gons by R and R'; it is obviously sufficient to prove that $R = R'$. So let us find the sum $OA_1^2 + \ldots + OA_n^2$:

$$\sum_{k=1}^{n} OA_k^2 = \sum_{k=0}^{n-1} \left[R\cos\left(\alpha + \frac{2\pi k}{n}\right)\right]^2 = R^2 \sum_{k=0}^{n-1} \frac{\cos 2\left(\alpha + \frac{2\pi k}{n}\right) + 1}{2} = \frac{nR^2}{2}.$$

Here we made use of the fact that

$$\sum_{k=0}^{n-1} \cos 2\left(\alpha + \frac{2\pi k}{n}\right) = 0;$$

this equality follows from the fact that *the sum of vectors drawn from the center of a regular polygon to its vertices is zero.*

In the same way, $\sum\limits_{k=1}^{n} OA_k^2 = \frac{nR'^2}{2}$; hence, $R = R'$ and the two n-gons are equal, Q.E.D.

19. Let us assume the contrary and let, for example,

$$\sqrt{2} = 1.\underbrace{\dots}_{k}\underbrace{77\dots7}_{5000001}\dots, \quad k \le 4\,999\,998.$$

Then $A = \frac{r}{10^k} + \frac{7}{9\times 10^k}$, where r consists of the integer part and the first k digits of the decimal representation of $\sqrt{2}$, approximates $\sqrt{2}$ with an accuracy to $10^{-k-5000001}$ and $|a^2-2| = (a+\sqrt{2})|A-\sqrt{2}| \le 3\cdot 10^{-k-500001} < 10^{-k-5000000}$.

But on the other hand, $|A^2-2|$ is a rational number with denominator $(9\cdot 10^k)^2$; hence, $|A^2-2| \ge \frac{1}{(9\cdot 10^k)^2} > 10^{-2k-2}$. Therefore, $-k-5000000 < -2k-2$ or $k > 4999998$. Contradiction.

20. a) If n airplanes flew to the airfield O from the airfields $A_1, \dots, A_n$, this means that in each triangle $\triangle A_1OA_{i+1}$ the sides OA_i and OA_{i+1} are smaller than A_iA_{i+1}. Hence, $\angle A_iOA_{i+1} > 60^\circ$ for any $i = 1, \dots, n-1$.

But then the sum of these angles is greater than $n \times 60^\circ$ and, on the other hand, it is 360°, implying $n < 6$, i.e., $n \le 5$.

b) For both the spherical and the flat triangles A_iOA_{i+1} it is also true that the greater angle subtends the longer side. Therefore the above reasoning can be applied literally to the case of the sphere. (The sum of the angles of a spherical triangle is $> 180^\circ$.)

22. Obviously, $\cos\alpha = \pm\frac{4}{5}$. Let for example, $\cos\alpha = \frac{4}{5}$.

Set $\sin k\alpha = \frac{a_k}{5^k}$, $\cos k\alpha = \frac{b_k}{5^k}$. Then the trigonometric formulas for the sine and cosine of the sum yield

$$a_{k+1} = 4a_k + 3b_k, \qquad b_{k+1} = -3a_k + 4b_k.$$

This immediately implies that a_k and b_k are integer for all k. It remains to prove that they are not divisible by 5. The easiest way to do this is to notice (and prove by induction) that $a_k \equiv 3^k \not\equiv 0 \pmod 5$ and $b_k \equiv 3^{k-1}4 \not\equiv 0 \pmod 5$.

23. Note that if the three positions of the switch are labeled by $1, 2, 3$ and the colors of the bulbs are also labeled by $1, 2, 3$ then the situation described in the problem coincides with that of Problem 30.2.10.5 (its solution is given in Part 2).

25. Multiply the equation by $\cos\left(\frac{3\pi}{11}\right)$ and express $\sin\left(\frac{k\pi}{11}\right)\cdot\cos\left(\frac{3\pi}{11}\right)$ and $\cos^2\left(\frac{3\pi}{11}\right)$ as a sum. We get

$$\sin\left(\frac{3\pi}{11}\right) + 2\sin\left(\frac{5\pi}{11}\right) - 2\sin\left(\frac{\pi}{11}\right) = \sqrt{11}\cos\left(\frac{3\pi}{11}\right).$$

Squaring this equation and replacing again all products in terms of sums, and the squares in terms of the doubled angles we get after simplification

$$1 + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) = 0.$$

which is obviously equivalent to the initial equation.

The latter identity can be also obtained geometrically. Namely, put the center of the right 11-gon in the origin, align the 11-gon symmetrically with respect to the x-axis and consider the projections of all vectors from the origin into the vertices to the x-axis, see Fig. 112.

The sum of the projections is clearly equal to the left hand side of the above equation. But since the sum of these vectors is obviously equal to $\vec{0}$, so is the sum of their projections, as required.

26. Let us rewrite the given equation in the form

$$520(x-1)(yzt+y+z) + 520(zt+1) = 57(yzt+y+z)$$

Now, it is clear that if $x > 1$, then the left hand side is greater than the right hand side and there are no solutions.

FIGURE 112. (Sol. A25)

Thus, $x = 1$, and the equation takes the form

$$57(yzt + y + z) = 520(zt + 1)$$

or

$$57z = (520 - 57y)(zt + 1).$$

Since $0 < z < zt + 1$, we should have $57 > 520 - 57y > 0$ which is only possible for $y = 9$. Substituting $y = 9$ we get $57z = 7(zt + 1)$.

Applying similar trick for the third time we finally get $\overline{xyzt} = 1978$. (The problem was suggested in 1978.)

ANOTHER SOLUTION. If you can notice that the equation can be rewritten in the form:

$$x + \cfrac{1}{y + \cfrac{1}{z + \cfrac{1}{t}}} = 1 + \cfrac{1}{9 + \cfrac{1}{7 + \cfrac{1}{8}}}$$

you immediately get

$$x = 1, \quad y = 9, \quad z = 7, \quad t = 8.$$

Indeed, if two numbers, e.g. continued fractions, are equal, then their integer and fractional parts are equal, too. This implies the uniqueness of expression of a number as a finite continued fraction.

29. A) The areas of triangles OAH, OAB and OBM cannot be equal. Indeed, if we assume the opposite and the areas of the triangles OAH, OAB and OBM are equal, see Fig. 113 a), then the fact that $S_{OAB} = S_{OBM}$ implies that $OA = OM$ (the heights of these triangles dropped from the common vertex B to the bases coincide) and the fact that $S_{OAB} = S_{OAM}$ implies that $OB = OH$. Thus, BH and BM are divided in halves at O. Hence, $ABMH$ is a parallelogram and so BM and AH are parallel. This contradicts the fact that they meet at C.

Consequently, if some three areas of four considered are the same and are equal to S, then one of the parts with this area is quadrilateral $HOMC$. The following cases are possible:

B) See Fig. 113 b). In this case the second solution (given below) is even simpler than arguments in case C) and the answer is $S = \frac{1}{6}$.

C) Suppose that triangles AOB and BOM have the same areas S and the area of $\triangle AOH$ is $s = 1 - 3S$. Connect H and M with a segment, see Fig. 113 b).

Let $BC = a$, $BM = x$, $AC = b$, $CH = y$. The ratio $\dfrac{BM}{MC}$ is equal to the ratio of the areas of the triangles with the bases BM and MC and a common vertex A, i.e., $\frac{x}{a - x} = \frac{2S}{S + s}$.

Similarly, having considered $\triangle ABH$ and $\triangle CBH$ we get $\frac{y}{b - y} = \frac{2S}{S + s}$. Consequently, $\frac{x}{a - x} = \frac{y}{b - y}$, whence

$$x(b - y) = y(a - x) \Longleftrightarrow bx = ay \Longleftrightarrow y = \frac{xb}{a}.$$

Denote: $|AB| = c$. From $\triangle ABM$ we have

$$S = \tfrac{1}{2} \cdot S_{ABM} = \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot xc \sin B = \tfrac{1}{4} bx \sin C$$

because $c \sin B = b \sin C$ by the law of sines.

To define the areas of all four parts into which $\triangle ABC$ is divided, we can calculate s by two methods.

On the one hand, s is the area of $\triangle AHO$:

$$s = 1 - 3S = 1 - \tfrac{3}{2} S_{\triangle ABM} = \tfrac{1}{2} ab \sin C - \tfrac{3}{4} bx \sin C = \tfrac{1}{2} b \left(a - \tfrac{3}{2} x\right) \sin C. \tag{1}$$

On the other hand, $S - s$ is the area of $\triangle HMC$, i.e., it is equal to $\frac{1}{2} y(a-x)\sin C$, whence $s = S - \frac{1}{2} y(a-x)\sin C$. By substituting $S = \frac{1}{4} bx \sin C$ and $y = \frac{b}{a} x$ into (1) we get

$$s = \tfrac{1}{4} bx \sin C - \frac{1}{2}\frac{b}{a} x(a-x)\sin C = \tfrac{1}{2} bx \left(\tfrac{1}{2} - \tfrac{1}{a}(a-x)\right)\sin C. \tag{2}$$

We equate the right-hand sides of (1) and (2):

$$\tfrac{1}{2} b \left(a - \tfrac{3}{2}\right) \sin C = \tfrac{1}{2} bx \left(\tfrac{1}{2} - \tfrac{1}{a}(a-x)\right) \sin C \iff \left(a - \tfrac{3}{2}\right) = x\left(\tfrac{x}{a} - \tfrac{1}{2}\right)$$

therefrom we get the following quadratic equation for x:

$$x^2 + ax - a^2 = 0.$$

Hence,

$$x = a\frac{\sqrt{5}-1}{2} = \tau a, \qquad y = \frac{xb}{a} = b\frac{\sqrt{5}-1}{2} = \tau b,$$

where $\tau = \frac{\sqrt{5}-1}{2}$. It remains to calculate S and s:

$$S = \tfrac{1}{4} bx \sin C = \tfrac{1}{2} ab \sin C \frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{4} = \frac{\tau}{2}$$

(since $\frac{1}{2} ab \sin C = 1$ by the hypothesis);

$$s = 1 - 3S = 1 - \tfrac{3}{2}\tau = \frac{7 - 3\sqrt{5}}{4}.$$

FIGURE 113. (Sol. A29)

ANOTHER SOLUTION. The same problem has a simpler solution if we use allow to use *affine transformations* of the plane — a composition of (a) a parallel translation and (b) a rotation in space with subsequent projection to original plane and (c) a homothety. Such a transformation does not change the ratio of areas of any two figures. Let us make use of the affine transformation which turns the original triangle into an equilateral one and solve the problem set for the triangle obtained.

Despite of the fact that the solution seems to satisfy only the particular case of an equilateral triangle, it nevertheless is a solution for all triangles because of the affine nature of the problem.

The solution is based on the same reasoning as above but is much simpler. Indeed, case b) becomes completely obvious and in case c)

$$x = y, \qquad \frac{1}{2}(a-x)\frac{\sqrt{3}}{2} = S - s, \qquad s + 3S = \frac{a\sqrt{3}}{4}, \qquad \frac{1}{2}\frac{\sqrt{3}}{2} a = 2S$$

(a is the length of the side of the regular triangle) therefrom it is easy to get the same quadratic equation $x^2 + ax - a^2 = 0$ with the positive root $x = \tau a$.

The number $\tau = \frac{\sqrt{5}-1}{2}$, the positive root of the equation $x^2 + x - 1 = 0$, is a remarkable number in Mathematics and even has a special name: the "golden section" or "Mister Tau." It has many interesting and beautiful properties, one of which is that its continued fraction expansion is one of the simplest possible:

$$\tau = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

The golden section first appeared in geometry during the search for "golden" rectangles which remain similar to themselves after squares are cut off from them. Let the longer side of a golden rectangle be 1 and the shorter side be τ; the "goldness" property is then $\frac{\tau}{1} = \frac{1-\tau}{\tau}$ equivalent to $\tau^2 + \tau - 1 = 0$.

Mr. τ appears in many other geometric problems, e.g., in a problem on a regular pentagon. Let $ABCDE$ be a regular pentagon, see Fig. 113 d). Then its side is τ times smaller than the diagonal ($AB = \tau BD$) and the diagonals divide each other in the ratio of $\tau : 1$ ($BT = \tau BD, DK = \tau AD$). You can work out a proof yourself or look up in [Cox].

But why did Mr. τ appear when we solved our problem of cutting a triangle? Could we anticipate it and perceive Mr. τ directly without calculations?

Since, as was mentioned above, the problem is of affine nature, it suffices to replace from the very beginning an arbitrary triangle with a special one for which everything is clear. We tried an equilateral triangle as a most natural simple example but it proved to be only a trifle simpler than the original one and we still had to calculate. So for this problem the obvious idea of simplicity does not fit. What we need is the triangle ABD composed of the diagonals BD, AD and the side AB of the regular pentagon $ABCD$ and the intercepts AT and BK in it, see Fig. 113 d).

Indeed, $S_{\triangle AOB} = S_{\triangle BOT}$ since $AO = OT$ (the diagonal AT of parallelogram $ABTE$ is divided in halves by the other diagonal BE). In addition, $S_{\triangle BOT} = S_{KOTD}$ because $\triangle BTO$ and $KOTD$ are equally composed, the former consisting of triangle BMH and rectangle $MHTO$ and the latter of respectively equal figures: triangle TRD and rectangle $OTRK$.

Finally, we get

$$S_{\triangle AOB} = S_{\triangle BOT} = S_{KOTD}.$$

The diagonals of a regular pentagon have a remarkable property:

$$BT = \tau BD \qquad \text{and} \qquad KD = \tau AD;$$

hence,

$$S_{\triangle AOB} = S_{\triangle BOT} = S_{KOTD}$$

should be true for the original triangle.

You may be interested to plot the straight lines AM and BH using a ruler and compass. For this it suffices to learn how to plot lengths τ having a scale unit.

It can be done, for example, as follows, see Fig. 113 e). Construct square $ABCD$ with side 1, find the midpoint K of AD and then draw the circle centered at K with radius KC using your compass. Then the intersection point C' of this circle with an extension of AD is such that $DC' = \tau$. Indeed,

$$KC = \sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5}}{2} = KC', \qquad DC' = KC' - KD = \frac{\sqrt{5}}{2} - \frac{1}{2} = \tau.$$

Now to plot a segment of length τa from the segment a it suffices to perform the construction shown on Fig. 113 b) and well-known from school textbooks.

Plotting the segments of lengths $x = \tau a$ and $y = \tau b$, plot points M and H on AC and BC, see Fig. 113 b), so that $BM = x$, $CH = y$ and then draw the required line segments AM and BH. The construction is completed.

30. If the i-th digit of the number $2 - \sqrt{2}$ is equal to a, then the i-th digit of the number $\sqrt{2} - 1$ is equal to $9 - a$. If $4\frac{1}{3} < a < 4\frac{2}{3}$, then $4\frac{1}{3} < 9 - a < 4\frac{2}{3}$. It remains to generalize this fact to arithmetic means.

31. Select two planets and draw the plane through them and the center of the Sun. Let this plane be the equator of the Sun. Then from the northern and southern poles of the Sun not more than 7 planets can be seen (the planets which belong to the equatorial plane are invisible) and from at least one of the poles ≤ 3 planets are seen. Therefore, the pole from which ≤ 3 planets are seen is the point desired.

33. It is easy to see that even the squares of size $\frac{1}{2}$ and $\frac{1}{3}$ cannot be squeezed inside any square whose side is less than $\frac{5}{6}$. In the square with side $\frac{5}{6}$ all squares might be put, since their total area is equal to $\frac{\pi^2}{6} - 1 < \left(\frac{5}{6}\right)^2$. (We used the fact that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$; for its proof see any good text-book on Calculus.) This inequality certainly does not prove that they can actually be squeezed into a $\frac{5}{6} \times \frac{5}{6}$ square, one should guess an ingenious way to arrange them; look at Fig. 114. The bubbles on Fig. indicate the sized of the squares placed in the square indicated in the same way as the squares depicted on Fig. are placed in the given $\frac{5}{6} \times \frac{5}{6}$ squares.

FIGURE 114. (Sol. A33)

FIGURE 115. (Sol. A34)

34. The construction required is carried out as follows. Fix two distinct points, A and B, not diametrically opposite, on the surface of the ball. Draw two circles centered at A and B on the ball; let C and D be the intersection points of the circles. (It is important here that A and B are not the endpoints of the same diameter, because otherwise the circles drawn would have either merged or have an empty intersection.)

Then draw two more intersecting circles on the surface of the ball centered at C and D. Let M and N be their intersection points. It is easy to see that all four points — A, M, N, B — lie on the same great circle, see Fig. 115.

REMARKS. 1) To draw the great circle on a *sheet of paper*, construct, for example, triangle $\triangle AMB$ (or $\triangle AMN$, or $\triangle ANB$). This can be done by the standard method of transferring segments of size AM, AB and MB with the help of a compass to the plane (of the paper). The circle circumscribed around $\triangle AMB$ is the required one.

2) It is easy to draw this circle also on the ball itself, not just on the plane, if the legs of the compass can bend.

35. Let us numerate parts of both partitions 1 to 100: $A_1, \dots, A_{100}$ and $B_1, \dots, B_{100}$. Let S_{ij} be the area of $A_i \cap B_j$. Make an array of size 100×100, where the (i, j)-th number is S_{ij}. Since the parts A_i and B_j of the partitions are of the same area, we have:

$$\sum_{i=1}^{n} S_{ij} = \sum_{j=1}^{n} S_{ij} = \frac{1}{100}.$$

Define a "snake" to be a set of 100 numbers S_{ij} in distinct rows and distinct columns. If all numbers S_{ij} constituting the "snake" are nonzero, the required 100 points should be placed in the respective intersections $A_i \cap B_j$. Thus, it remains to prove the existence of a snake built of nonzero numbers; such a snake is briefly called a *nonzero* snake.

Observe (it is quite obvious) that for any k rows selected there are k columns in whose intersection points with the rows selected there are at least k nonzero numbers S_{ij}.

Indeed, if the sets $A_{i_1}, \dots, A_{i_k}$ have a nonempty intersection with l elements $B_{j_1}, \dots, B_{j_l}$ of the second system, then $A_{i_1} \cup \ldots \cup A_{j_k} \subset B_{j_1} \cup \ldots \cup B_{j_l}$. Hence, $k \le l$. Contradiction.

Let us prove the existence of a nonzero snake by induction on size of a minor[1]. Namely, let us prove that for any $k \times k$ minor with a nonzero snake and an arbitrary $(k+1)$-th row there exists the $(k+1)$-st column such that the $(k+1) \times (k+1)$ minor thus obtained also has a nonzero snake.

---

[1]A $k \times k$ *minor* is a set of k^2 squares — the table formed by the points of intersection of k rows and k columns.

The base of the induction: Take any nonzero element (a 1×1 minor) and add any nonzero element from another row. Adding also two more elements which are in the same rows and columns we get a 2×2 minor with a nonzero snake.

The inductive step: Let the statement be true for all $k \times k$ minors. We can assume without loss of generality that this minor is in the top left-hand corner of the array. Add another row. Without loss of generality, let it be the $(k+1)$-st row. We will prove that it is possible to select a column such that the respective minor contains a nonzero snake.

By observation above, in one of the first $k+1$ rows there exists a nonzero element A_{il} standing to the right of the minor, i.e., $i \leq k+1$, $l > k$. If A_{il} stands in the $(k+1)$-st row, let just augment the minor with the l-th column and mark the element $A_{k+1,l}$ in it. We get a "nonzero snake" of size $(k+1) \times (k+1)$, see Fig. 116 a).

FIGURE 116. (Sol. A35)

If $i \leq k$, then the i-th row contains a marked element, A_{ij} with $j \leq k$. Let us delete from the minor the i-th row and the j-th column. We get a $(k-1) \times (k-1)$ minor with a marked nonzero snake. By the inductive hypothesis it can be augmented with the $(k+1)$-st row and a column to a $k \times k$ minor with a nonzero snake. Let us augment the minor again with the i-th row and the j-th column (with the element $A_{ij} \neq 0$ marked); if the j-th column is already added, let us augment the minor with the l-th column and mark $A_{il} \neq 0$. We get a $(k+1) \times (k+1)$ minor with a marked nonzero snake.

38. For example, one die is placed so that along its lateral sides stand 2, 3, 5, and 4, the other 39 dice being placed so that along there lateral sides stand 1, 2, 6, and 5. Denote the number of 1's, 2's, 6's, and 5's over the 2 of the first die by x, y, z, t, respectively. Let us try to find a position for which the sums of dots on the lateral sides were equal:

$$2 + 1 \cdot x + 2 \cdot y + 6 \cdot z + 5 \cdot t = 3 + 2 \cdot x + 6 \cdot y + 5 \cdot z + 1 \cdot t = 5 + 6 \cdot x + 5 \cdot y + 1 \cdot z + 2 \cdot t.$$

The first and second equalities yield, respectively:

$$(x - z) + 4(y - t) + 1 = 0, \quad 4(x - z) - (y - t) + 2 = 0,$$

therefrom we can determine $x - z$ and $y - t$ and see that they are not integer.

40. Let

$$a_k = k + \cfrac{1}{(k+1) + \ldots + \cfrac{1}{n-1}} \qquad \text{and} \qquad b_k = k + \cfrac{1}{(k+1) + \ldots + \cfrac{1}{(n-1) + \cfrac{1}{n}}}$$

be the "tails" of the given continued fraction beginning with the integer k and $h_k = |a_k - b_k|$. Then

$$a_{k-1} - b_{k-1} = \frac{1}{a_k} - \frac{1}{b_k} = \frac{b_k - a_k}{a_k b_k}, \qquad k = 2, \ldots, n-1,$$

so $h_{k-1} = \frac{h_k}{a_k b_k} < \frac{h_k}{k^2}$ (because $a_k > k$ and $b_k > k$) and $h_{n-1} = \frac{1}{n}$. Hence, the difference we are interested in is estimated as follows:

$$h_1 < \frac{h_2}{2^2} < \frac{h_3}{2^2 3^2} < \ldots < \frac{h_{n-1}}{2^2 3^2 \ldots (n-1)^2} = \frac{1}{(n-1)!n!}, \quad Q.E.D.$$

41. Let points A, B, C move along the circles centered at O_1, O_2, O_3 in the same direction at an angular speed ω; let $A(t)$, $B(t)$, $C(t)$ be their positions at the moment of time t. We assume that the plane under consideration is a complex line and O is the origin. Then the centers of the circles are complex numbers c_1, c_2, c_3, the original points are numbers z_1, z_2, z_3, and the points $A(t)$, $B(t)$, $C(t)$ are numbers:

$$A(t) = c_1 + (z_1 - c_1)e^{i\omega t}; \qquad B(t) = c_2 + (z_2 - c_2)e^{i\omega t}; \qquad C(t) = c_3 + (z_3 - c_3)e^{i\omega t}.$$

The radii of the circles are $|z_1 - c_1|$, $|z_2 - c_2|$ and $|z_3 - c_3|$, respectively.

The center of mass of $\triangle A(t)B(t)C(t)$ is the complex number

$$Z(t) = \frac{1}{3}(A(t) + B(t) + C(t)) = \frac{1}{3}(c_1 + c_2 + c_3) + \left(\frac{1}{3}(z_1 + z_2 + z_3) - \frac{1}{3}(c_1 + c_2 + c_3)\right) e^{i\omega t}.$$

Consequently, the center of mass of $\triangle A(t)B(t)C(t)$ is moving at an angular speed ω in the same direction as the original points A, B, C over the circle centered at the point $\frac{1}{3}(c_1 + c_2 + c_3)$ and with radius $\frac{1}{3}|(z_1 + z_2 + z_3) - (c_1 + c_2 + c_3)|$. The center of this circle is the center of mass of $\triangle O_1 O_2 O_3$ and its radius R is the length of the segment connecting the centers of mass of $\triangle ABC$ and $\triangle O_1 O_2 O_3$.

43. It is easy to see from Fig. 117 that $\beta = \frac{\smile AnB}{2}$, $\gamma = \frac{\smile BmA}{2}$, therefrom $\beta + \gamma = \frac{1}{2}(\smile AnB + \smile BmA) = \frac{360^\circ}{2} = 180^\circ$, $\gamma = 180^\circ - \beta$. Hence, $\sin\gamma = \sin\beta$.

Figure 117. (Sol. A43)

By the law of sines, for $\triangle AMC$ we have

$$\frac{CA}{\sin\alpha} = \frac{CM}{\sin\beta} \Longrightarrow \frac{CA}{CM} = \frac{\sin\alpha}{\sin\beta}$$

and for $\triangle BMD$

$$\frac{DB}{\sin\alpha} = \frac{DM}{\sin\gamma} \Longrightarrow \frac{DB}{DM} = \frac{\sin\alpha}{\sin\gamma} = \frac{\sin\alpha}{\sin\beta}.$$

Hence, $\frac{CA}{CM} = \frac{DB}{DM}$, Q.E.D.

44. Obviously, we can assume that not all given numbers are divisible by p. On the other hand, among the given numbers there are two, a and b, that yield the same remainder when divided by p, i.e., $a-b=lp$.

Suppose that a and b are not divisible by p. Clearly, $d=(a,b)$, their GCD, is also the GCD of a and $a-b$. Then $\frac{a}{d} > \frac{a-b}{d} = p\frac{l}{d} \geq p$, as required.

Analyze on your own the case when a and b are divisible by p.

45. Denote the function $\sum\limits_{n=1}^{k} a_n \cos nx$ by $P(x)$. We see that $P(x) \geq -1$ for all x under the condition of the problem.

Let us prove that $\sum\limits_{l=0}^{k} P(\frac{2\pi l}{k+1}) = 0$. For this it suffices to prove that

$$\begin{aligned}
\sum_{l=0}^{k} a_1 \cos\left(\frac{2\pi l}{k+1}\right) &= 0,\\
\sum_{l=0}^{k} a_2 \cos 2\left(\frac{2\pi l}{k+1}\right) &= 0,\\
\dots\dots\dots\dots\dots\dots\dots &\\
\sum_{l=0}^{k} a_k \cos k\left(\frac{2\pi l}{k+1}\right) &= 0.
\end{aligned}$$

But each of these sums is the sum of the real parts of complex numbers lying at the vertices of a regular polygon, i.e., it is actually equal to 0 and the required equality is proved.

It follows, therefore, that

$$P(0) = -\sum_{l=1}^{k} P\left(\frac{2\pi l}{k+1}\right)$$

and since $P\left(\frac{2\pi l}{k+1}\right) \geq -1$ for all $l=1,\dots,k$, we have

$$P(0) \leq -\sum_{l=1}^{k}(-1) = k.$$

46. Clearly, the first prompt of Guesser is *arbitrary.* It divides 16 possible numbers into groups $1+4+6+4+1$ (if the prompt is: "11 111", then the i-th group has numbers with i-many 1's).

If we are unlucky as Guesser and got in the middle group we will be unable to guess.

Indeed, at the second prompt only the number of 1's is important, not their order; 10 001 divides 6 numbers into groups $3+3$ and 10 011 divides 6 numbers into groups $1+4+1$. Therefore, it is clear that the third prompt can not divide a group of 4 (since no prompt divides into four groups; if we do not prompt a number of type 10 011, we will be even unable to divide the group of 3).

Remark. One can interpret the problem differently: since there stands a "five-digit number" in the formulation, we may think that it is assumed that the first number is necessarily a 1.

49. The simplest way to solve this problem is to regard the white plane as a complex line, and consider the locations of the man and the cat as complex numbers $z_m = z_m(t)$ and $z_c = z_c(t)$, respectively, that depend on time t (t is a real number). Assume that the cat is at point O at the moment $t=0$.

The cat encircles the man in three stages.

Stage 1. Denote: $w = \frac{z_c}{z_m}$ (observe that $z_m \neq 0$: the man will not go to the origin *because the cat already was there*). At the first stage, the cat runs along a straight line until the absolute value of w becomes $|w| = 1 + \frac{\varepsilon}{2}$, where $\varepsilon = \lambda - 1 > 0$.

Stage 2. The cat moves in a special manner: so that $|w| = 1+\frac{\varepsilon}{2}$ at all times. At the beginning of the second stage, $\arg z_c = 0$, $0 < \arg z_m < 2\pi$, $\arg w = -\arg z_m < 0$; all the angles (arg's) depend continuously on t. If the cat wanted to make $w = const$, it would suffice for it to move at a speed of $1+\frac{\varepsilon}{2}$. Therefore, it has an excess of speed $\frac{\varepsilon}{2}$ which it can use to change the argument of w. The cat runs over a distance not greater than $t(1+\varepsilon)$ over the time interval t, so $|z_c| \leq t(1+\varepsilon)$ and the cat can ensure that the rate of variation of $\arg w$ is at least

$$\frac{\frac{\varepsilon}{2}}{|z|} \geq \frac{\frac{\varepsilon}{2}}{t(1+\varepsilon)}$$

Since $\int\limits_c^\infty \frac{dt}{t} = \infty$, the variation of $\arg w$ can be infinitely large. At stage 2, therefore, the cat increases $\arg w$ at the maximum possible rate. The second stage terminates when $\arg w$ becomes positive.

Stage 3. Finally, the cat moves so that $|z_c|$ does not increase, $|w| \le 1 + \frac{\varepsilon}{2}$ and $\arg z_c$ increases at a rate limited from below by a constant. But the man cannot run away and the path of the cat closes after a finite time.

50. Consider one of the given points and all annuli in which it is contained. The centers of these annuli constitute an annulus with the inner radius 2 and the outer one 3. The area of this annulus equals $9\pi - 4\pi = 5\pi$.

Let us construct 650 of such annuli for each given point; all of them are contained in the disc of radius 19. Suppose that the statement of the problem is false. Then no point of the plane is contained simultaneously in 10 of the annuli constructed. Therefore, the area of the union of the annuli is greater than $650 \cdot 5 \cdot \frac{\pi}{9} = \pi \cdot 361.11...$

But this is greater than the area of the disc of radius 19 in which, as we have already shown, all of the annuli are contained. Contradiction.

51. Let us prove by induction that *for any number x and an integer n there exists $\varepsilon > 0$ such that no number from the segment $]x-\varepsilon, x[$ can be represented as the sum of n numbers, each inverse to a positive integer.*

The *base of induction*, $n = 0$ is obvious: take $\varepsilon = x$. The *inductive step*: Let the statement be true for an n; let us prove that it holds for $n+1$.

For every integer $q \leq 2\frac{n+1}{x}$ there exists by the inductive hypothesis a number ε_q such that no number from the segment $]x-\varepsilon_q-\frac{1}{q}, x-\frac{1}{q}[$ can be represented as the sum $\frac{1}{a_1}+\dots+\frac{1}{a_n}$ for positive integers a_i, $i = 1, 2, \dots, n$. Hence, no number from the segment $]x-\varepsilon_q, x[$ can be represented as $\frac{1}{a_1}+\dots+\frac{1}{a_n}+\frac{1}{q}$.

Let $\varepsilon = \min\limits_{1\leq q\leq 2\cdot\frac{n+1}{x}} \varepsilon_q$. Then no number from the segment $]x-\varepsilon, x[$ can be represented as $\frac{1}{a_1}+\dots+\frac{1}{a_n}+\frac{1}{a_{n+1}}$, where $a_i \leq 2\cdot\frac{n+1}{x}$ for $i = 1, 2, \dots, n+1$ (the a_i's can be permuted and renumbered).

If $a_i > 2\cdot\frac{n+1}{x}$ for all i, then $\frac{1}{a_1}+\dots+\frac{1}{a_n}+\frac{1}{a_{n+1}} < \frac{1}{2}x$ which for $\varepsilon < \frac{1}{2}x$ does not belong to the segment either. Q.E.D.

53. Each edge of the cube has a point of the polyhedron because otherwise the projection of the polyhedron along this edge would not coincide with the face. Take one point of the polyhedron on each edge of the cube and consider the new convex polyhedron with vertices at these points. Since the new polyhedron is a part of the original one, it suffices to prove that its volume is not less than one-third of the volume of the cube.

We can assume that the length of the edge of the cube is 1. The new polyhedron is produced by cutting off tetrahedrons from trihedral angles at the vertices of the cube. Let us prove that the sum of the volumes of two tetrahedrons corresponding to the vertices belonging to one edge of the cube does not exceed $\frac{1}{6}$. This sum is $\frac{1}{3}S_1h_1+\frac{1}{3}S_2h_2$, where h_1 and h_2 are heights dropped to the opposite faces of the cube from the vertex of the polyhedron, which belongs to the given edge of the cube, and where S_1 and S_2 are areas of the respective faces of the tetrahedron. It remains to observe that $S_1 \leq \frac{1}{2}$, $S_2 < \frac{1}{2}$, and $h_1+h_2 = 1$.

Four parallel edges of the cube set a partition of the cube's vertices into 4 pairs. Therefore, the volume of all cut-off tetrahedrons does not exceed $\frac{4}{6} = \frac{2}{3}$, i.e., the volume of the remaining part is $\geq \frac{1}{3}$. This estimate is a precise one: for cube $ABCDA_1B_1C_1D_1$ the equality is attained for tetrahedrons AB_1CD_1 and A_1BC_1D.

FIGURE 118. (Sol. A54)

FIGURE 119. (Sol. A55)

55. First, prove the following

Lemma: *for an arbitrary triangle ABC and for arbitrary points B_1 on side AC and C_1 on AB the area of $\triangle CBM$ is greater than that of $\triangle C_1B_1M$, where M is the intersection point of BB_1 and CC_1*, see Fig. 119 a).

Now, observe that if $ABCDE$ is the original pentagon and $A_1B_1C_1D_1E_1$ is the smaller pentagon, then the number s calculated for the smaller pentagon is equal to

$$S_{A_1B_1C_1}+S_{B_1C_1D_1}+S_{C_1D_1E_1}+S_{D_1E_1A_1}-S_{A_1B_1C_1D_1E_1}.$$

(The same is true, of course, for the greater pentagon but we do not need this fact.) Thus, it is easy to see that the difference $S-s$ is equal to the sum of five differences: $S_{AA_1B}-S_{A_1B_1E_1}$, etc. (see Fig. 119 b)) each of which is positive by Lemma. Q.E.D.

57. To construct a counterexample, suppose that at the first step two units stand aside; hence, at the second step the number between them will also be a 1. Let us depict this as

$$\begin{array}{ccc} 1 & & 1 \\ & 1 & \end{array}$$

Further, suppose that at 3-rd step under the lowest 1 two 2's will be obtained and, therefore, at 4-th step we get

$$\begin{array}{ccc} 1 & & 1 \\ & 1 & \\ 2 & & 2 \\ & 2 & \end{array}$$

Such an arrangement can be obtained from the triangle

$$\begin{array}{ccccccc} 5 & & 1 & & 1 & & 5 \\ & 3 & & 1 & & 3 & \\ & & 2 & & 2 & & \\ & & & 2 & & & \end{array}$$

At the next step we can construct, e.g., the triangle

$$\begin{array}{ccccccccccccccc} 13 & & 5 & & 1 & & 1 & & 5 & & 13 & & & & \\ & 9 & & 3 & & 1 & & 3 & & 9 & & & & & \\ & & 6 & & & 2 & & 2 & & & & & 6 & & \\ & & & 4 & & & & 2 & & & & & & & 4 \\ & & & & & & 3 & & 3 & & & & & & \\ & & & & & & & 3 & & & & & & & \end{array}$$

And so on. The method indicated makes it clear that it is always possible to obtain at the $2k$-th step the number k under the initial 1. On the other hand, if at some moment all the numbers become equal, they remain equal to the same constant later, while our construction puts at the center the number k at the $2k$-th step, i.e., the middle number grows monotonously. this is the counterexample desired.

Another description of the same solution. Let the numbers in the first row be given by the formula $a_n = 2n^2 - 2n + 1$ for $n \in \mathbb{Z}$; therefore, the first row is of the form

$$\dots \quad 25 \quad 13 \quad 5 \quad 1 \quad 1 \quad 5 \quad 13 \quad 25 \quad \dots$$

Make sure on your own that after the first step all the numbers remain integers and after the second one each number accrues by 1. This easily implies that (a) the numbers will always remain positive integers, (b) they will never become equal.

59. Note that the total number of distinct spheres is finite: there are finitely many given points and, therefore, there are finitely many various centers of mass of distinct subsets of our points. Therefore, we immediately deduce that after a while we have a cycle of spheres. A sphere will eventually get another number and then the spheres will start to be counted cyclicly.

It remains to prove that the length of this cycle is equal to 1.

Lemma. *The function* $\sum\limits_{i=1}^{n} |x - x_i|^2$ *in the k-dimensional space attains its minimum if and only if x is the center of mass of points $x_1, \dots, x_n$.*

Proof. Indeed, let O be the center of mass of the points x_i, i.e. $\sum\limits_{i=1}^{n} \vec{x}_i = 0$ (recall that the points are of equal mass). Then

$$S = \sum_{i=1}^{n} |x - x_i|^2 = \sum_{i=1}^{n}\sum_{j=1}^{k} \left(x^{(j)} - x_i^{(j)}\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{k} \left(\left(x^{(j)}\right)^2 + \left(x_i^{(j)}\right)^2 - 2x^{(j)}x_i^{(j)}\right).$$

Since $\sum\limits_{i=1}^{n} \vec{x}_i = 0$, we deduce that $\sum\limits_{i=1}^{n} x_i^{(j)} = 0$; hence, $\sum\limits_{j=1}^{k}\sum\limits_{i=1}^{n} x_i^{(j)} = 0$ and $S \geq \sum\limits_{j=1}^{k}\sum\limits_{i=1}^{n} \left(x_i^{(j)}\right)^2$, where the equality occurs only in case $x^{(1)} = \dots = x^{(k)} = 0$, as required.

Now, consider the function

$$S(X) = \sum_{i\colon |XA_i| \leq 1} \left(1 - |XA_i|^2\right),$$

where the summation runs over the points A_i from our set lying inside the unit sphere centered at X. Let $X' \neq X$ be the center of mass of these points. Let us show that $S(X') > S(X)$.

Indeed, if under the shift of the center of the sphere from X to X' the content of points A_i does not vary, the inequality follows from Lemma. Adding new points causes addition of new positive summands to $S(X')$, while the elimination of old points causes deletion of negative summands. Therefore, a change of the content of points increases the value of $S(X')$. Hence, the value of $S(X)$ increases under each change of the center of mass; this eliminates cycles of length > 1.

64. Extension. Moreover, they(who??) can not be placed so that the *squares of distances* between the vertices were rational.

First, let us prove that *a triangle with vertices at nodes of the square lattice can not be equilateral.*

Assume the contrary; let there exist such a triangle and let a be the length of its side. Assume that the grid of the lattice is integer; then a^2 is an integer and the area of the triangle $S = \frac{\sqrt{3}}{4}a^2$ is an irrational number. But the area of the triangle with vertices at nodes of the lattice is a multiple of $\frac{1}{2}$ (see ....). Contradiction, Q.E.D.

Let us return to the solution of the problem. Let such a position exists. By zooming the whole picture the common denominator of all distances (or of all squares of distances if we are solving the extension of the problem) times we make all distances into integers. Choose a coordinate system so that the coordinates of the vertices of the square were $(c, 0)$, $(0, c)$, $(-c, 0)$, $(0, -c)$; let (x, y) be one of the vertices of the triangle. Since the distances from (x, y) to both $(c, 0)$ and $(-c, 0)$ are integer, it follows that $(x - c)^2 + y^2$ and $(x + c)^2 + y^2$ are integers; hence, so is their difference, $4cx$. Similarly, $4cy$ is an integer. Therefore, all vertices of the triangle should be situated at nodes of the square lattice with mesh of $\frac{1}{4c}$. But we have already proved that this is impossible. Contradiction.

65. The routes are designed as follows: we shall *always* turn right on the boundary of the town; while inside the town we shall only do so at the *marked* turning points. Until no turning points are marked, the tentative route is divided into $2n - 3$ rings (for $n \geq 2$); namely, one route around the town along the penultimate roads, $n-2$ vertical and $n-2$ horizontal ovals, see Fig. 120 a). If 4 rings meet at an intersection, we will unite the rings into one ring by marking the intersection point (with a "Turn Right" poster). In this way, by marking the intersection with coordinates $(2, 1)$ we will adjoin to the first ring two horizontal rings and a vertical one. Next, marking the intersection with coordinates $(3, 3)$ we will adjoin to the previous route a horizontal ring and two vertical ones, etc.

This method works if $n - 2$ is divisible by 3. In other cases we have to select several turning points and then proceed as above. The following turning points should be marked:

$$\begin{array}{lll} n = 3k+2 & \geq 2: & (2,1),\ (3,3),\ (5,4),\ (6,6), \dots (n-2, n-2) \\ n = 3k+1 & \geq 4: & (2,1),\ (1,2),\ (4,3),\ (5,5),\ (7,6), \dots (n-2, n-2) \\ n = 3k & \geq 6: & (2,1),\ (1,2),\ (3,2),\ (4,4),\ (6,5),\ (7,7), \dots (n-2, n-2) \end{array}$$

A singular case: $n = 3$. A case-by-case verification shows that in this case there exists no closed routes; but it is not difficult to construct a route with the beginning point and endpoint on the opposite sides of one road; see Fig. 120 b).

Figure 120. (Sol. A65)

Figure 121. (Sol. A66)

66. If the midpoints of the parallel sides in an octagon are connected with straight lines, these lines pass through the center O of the circle. Introduce the following notations, see Fig. 121:

$$x = \frac{1}{2}\smile A_1A_2, \qquad z = \frac{1}{2}\smile A_2A_3, \qquad v = \frac{1}{2}\smile A_3A_4,$$
$$y = \frac{1}{2}\smile A_5A_6, \qquad t = \frac{1}{2}\smile A_6A_7, \qquad w = \frac{1}{2}\smile A_7A_8.$$

Since $x + z = t + y$ and $z + v = t + w$ (two pairs of vertical angles as shown on Fig. 121), it follows that $x - v = y - w$ and $x + w = y + v$. Therefore,

$$t + 2w + 2x + z = t + 2y + 2v + z,$$

i.e., line l connecting the midpoints of $\smile A_2A_3$ and $\smile A_6A_7$ cuts off the same sum of the arcs on the half-circles.

Since $\smile A_1A_8$ and $\smile A_4A_5$ supplement these sums to half-circles, $\smile A_1A_8 = \smile A_4A_5$ implying $A_1A_8 = A_4A_5$, Q.E.D.

69. a) For example, take for such numbers all n-digit numbers of the form 11...11, where n is a power of 3.

b) Let $S(N) = 2^n - 1$ and let the number consisting of n last digits of N be divisible by 2^n; then N is also divisible by 2^n. For example, $N = 92112$ is divisible by 16.

71. Let $f(x) \neq 0$. Then

$$f(x - 2y) + f(x) = 2f(x - y)g(y), \qquad f(x) + f(x + 2y) = 2f(x + y)g(y)$$

implying

$$f(x - 2y) + f(x + 2y) = -2f(x) + 2g(y)(f(x - y) + f(x + y)) = f(x)(-2 + 4g(y)^2).$$

On the other hand,

$$f(x - 2y) + f(x + 2y) = 2f(x)g(2y).$$

Dividing by $f(x)$ we get $g(2y) = -1 + 2g(y)^2 > -1$. Since $2y$ is an arbitrary number, we are done.

Remark. For example, the functions $f(x) = \sin x$, $g(y) = \cos y$ possess the property. Accordingly, we have $\cos y > -1$.

73. First, suppose that all roots of the polynomial P can be divided into pairs of complex-conjugate non-real numbers, z_i and $\bar{z}_i$, as follows:

$$P(x) = [(x - z_1) \dots (x - z_n)][(x - \bar{z}_1) \dots (x - \bar{z}_n)] = P_1(x)P_2(x),$$

where $P_1(x) = (x - z_1) \dots (x - z_n)$; $P_2(x) = (x - \bar{z}_1) \dots (x - \bar{z}_n)$.

Then $P_2(x) = \overline{P_1(x)}$. Denote: $P_1(x) = Q(x) + iR(x)$, $P_2(x) = Q(x) - iR(x)$, where Q and R are polynomials with real coefficients. We get

$$P(x) = P_1(x)P_2(x) = Q(x)^2 + R(x)^2.$$

Now, let us turn to the general case and remember that any real polynomial can be expressed in the form

$$P(x) = a(x - x_1)^{k_1} \dots (x - x_s)^{k_s} \tilde{P}(x),$$

where $a \neq 0$ is the coefficient of the highest term, $x_1, \dots, x_k$ all distinct real roots of $P(x)$, $\tilde{P}(x)$ a polynomial without real roots, i.e., $\tilde{P}(x)$ is the product of quadratic polynomials $(x-z)(x-\bar{z}) = x^2+px+q$. If $P(x) \geq 0$ for some $x \geq 0$ (e.g. for $x > x_j$, $1 \leq j \leq s$) then $a > 0$.

Then, $a = b^2$, $b > 0$ and by the already proven $\tilde{P}(x) = \tilde{Q}(x)^2 + \tilde{R}(x)^2$. Suppose all k_j are even. Then $P(x) = b^2Q_1^2(x)\tilde{P}(x) = (bQ_1\tilde{Q})(x)^2 + (bQ_1\tilde{R})(x)^2$ and we are done. Now it suffices to prove that all k_j are indeed even.

It remains, moreover, to consider only one case: $P(x) = (x - x_1) \dots (x - x_s)$ with distinct roots. To conclude the proof show that $P(x_0) < 0$ for a real x_0.

74. Let x be not divisible by 3. Then $x^2 \equiv 1 \pmod 3$ implying $2y^2 \equiv 1 \pmod 3$. But this means that $y^2 \equiv 2 \pmod 3$ which is impossible. Therefore, x is divisible by 3, hence so is y. But then the left hand side is divisible by 9 and dividing by 3 we see that $2t^2 - z^2$ is divisible by 3. Hence, both t and z should be divisible by 3 by the same reasons. Thus, any solution (x, y, z, t) has a common divisor 3. This easily implies that the only solution is

$$x = y = z = t = 0.$$

75. a) Denote the position of the airplane at the moment t by $X(t)$ and let the whole time of the flight be T sec. Then $X(0) = \Gamma_1$, $X(T) = \Gamma_2$, where $0 \leq t \leq T$, $T > 1$.

Denote:

$$\alpha(t) = \angle X(t)AX(t+1), \qquad \beta(t) = \angle X(t)BX(t+1).$$

Since the path of the plane is continuous, so are the functions $\alpha(t)$ and $\beta(t)$ defined for $0 \leq t \leq T-1$.

Let us draw a plane through $\Gamma_1\Gamma_2$ and $X(1)$. From Fig. 122 a) we see that $\alpha(0) > \beta(0)$ because $\alpha(0)$ is an exterior angle and $\beta(0)$ is an interior angle of $\triangle AX(1)B$.

Similarly, by drawing the plane through $\Gamma_1\Gamma_2$ and $X(T-1)$, we deduce from $\triangle AX(1)B$ that $\alpha(T-1) < \beta(T-1)$. Then the continuity of $\alpha(t)$ and $\beta(t)$ implies the existence of a moment t_0 such that $\alpha(t_0) = \beta(t_0)$, Q.E.D.

b) Fig. 122 b) shows a half-circle of diameter $\Gamma_1 B$ (the whole picture is two-dimensional); Γ_2 lies on the half-circle near B and A belongs to the diameter near Γ_1. Then $\beta = \frac{1}{2}\smile XY$ and $\alpha > \frac{1}{2}\smile XY$, since $\alpha = \frac{1}{2}\smile XY + \smile X'Y'$, where X' and Y' are the intersection points of XA and YA, respectively, with the second part of the circle. Therefore, $\alpha(t) > \beta(t)$ for all t.

FIGURE 122. (Sol. A75)

76. Let us assume that the shades are *closed*, i.e., the boundary of a shade is a part of the shade. (If the shades are open the same proof is applicable with slight modifications.)

Assume that the statement of the problem is false and quadrilateral $ABCD$ with discs K_A, K_B, K_C, K_D is a counterexample. We may assume that the discs are in *general position*, i.e., are not tangent to each other, no 3 of them intersect at one point and the pair-wise intersection points do not lie on the sides and diagonals of the quadrilateral; otherwise — i.e., for a *singular* position — we can slightly enlarge the discs without violating the nature of the counterexample.

Let us prove that the system of discs indicated should possess the following properties:

1°. *No three discs have a common point.* Indeed, if, say, K_A, K_B and K_C have a common point, they cover $\triangle ABC$.

2°. *The points of pair-wise intersection of circles (the boundaries of discs $K_A, \dots, K_D$) cannot lie inside $ABCD$.* Indeed, if such a point would have existed, it would have been covered by one more disc (in order to exclude the possibility of a lighted spot in its neighborhood). This contradicts property 1°.

3°. *None of the discs can intersect two other discs.* Indeed, if, say, K_A intersects K_B and K_C, then the lighted (after K_D is deleted) part of $\triangle ABC$ can not intersect segments AB and AC and can not have vertices inside $\triangle ABC$ (by property 2°). It is not difficult to observe that this is impossible.

The discs with centers at the vertices of the quadrilateral possessing property 3° can not, however, cover the whole quadrilateral. The contradiction obtained proves the statement of the problem.

77. On the line, introduce a coordinate system so that at the Beginning of Time, simply called initial moment, one bacterium, call it F for first, were in the origin. It survives at the next moment only if there is a bacterium in one of the points: (with coordinates) -1 or $\sqrt{2}$. These bacteria, in their turn, survive only if at the initial moment there were bacteria at points -2, $-1+\sqrt{2}$ or $2\sqrt{2}$. And so on.

Consider all the bacteria that at the initial momets occupied points with coordinates $-n + m\sqrt{2}$ for positive integer n, m. Since the total number of all bacteria is finite, there finitely many such chosen bacteria. So there is one bacterium with the maximal m. If there are several such species, take the left-most of them, corresponding to the greatest value of n; call it B.

It is clear now that there are no bacteria at points $-(n+1) + m\sqrt{2}$ and $-n + (m+1)\sqrt{2}$ and bacterium B is doomed to inevitable death.

Now it is not difficult to see that all the bacteria living at points with coordinates $-k + m\sqrt{2}$ for all k will eventually die out, too.

Now apply the inverse induction on m to see that in the end bacterium F will also perish. Since it was chosen at random (we could have shifted the origin), the statement is proved.

78. Let a *k-configuration* $C(k)$ be an arrangement of a finite number of points such that there exists at least k points at distance 1 from any of the points.

Take a k-configuration and move it a a solid body by a vector of length 1 so that none of the points thus moved coincide with any of the initial points of the configuration. The new set of points, the union of the initial ones and the moved ones, clearly constitute a $(k+1)$-configuration. The endpoints of a segment of length 1 is a 1-configuration; a 1000-configuration contains 2^{1000} points — the projections of the vertices of a 1000-dimensional unit cube onto a two-dimensional plane.

Let us try to find a configuration of points with the number of points $< 2^{1000}$. Denote by $A+B$ *the vector sum of A and B* — the set of all pairwise sums of points from A and B (each point being identified here with the endpoints of the vector equal to the sum of the two vectors corresponding to the summands).

Clearly, if A has m points and B has n points then $A+B$ has no more than $m \cdot n$ points.

Let $|C(k)|$ be the minimal possible number of points in a k-configuration $C(k)$. Let us prove that $|C(m+n)| \leq |C(m)| \cdot |C(n)|$.

Consider any point $a_i + b_j \in C(m) + C(n)$; using point a_i we can find all the points $a_{i_1}, \dots, a_{i_m} \in C(m)$ at distance 1 from a_i and using point b_j we can find all points $b_{j_1}, \dots, b_{j_n} \in C(n)$ at distance 1 from b_j. (We have to have all points $b_j + a_{i_k}$ distinct from any of the points $a_i + b_{j_l}$.) Then the distance between any of the points $b_j + a_{i_1}, \dots, b_j + a_{i_m}$ or any of the points $a_i + b_{j_1}, \dots, a_i + b_{j_n}$ is equal to 1:

$$|(b_j + a_{i_s}) - (b_j + a_{i_t})| = |a_{i_s} - a_{i_t}| = 1, \qquad |(a_i + b_{j_s}) - (a_i + b_{j_t})| = |b_{j_t} - b_{j_s}| = 1.$$

But there is a total of $|C(m)| \cdot |C(n)|$ points in the set $C(m) + C(n)$ (since there are that many pairs (a_i, b_j)) and our statement is proved. Therefore, we have $|C(2)| = 3$ (a triangle) and:

$$\begin{aligned} |C(1000)| &\leq |C(2)| \cdot |C(998)| \leq |C(2)|^2 \cdot |C(996)| \\ &\leq |C(2)|^3 \cdot |C(994)| \leq \ldots \leq |C(2)|^{500} = 3^{500} < 2^{1000}. \end{aligned}$$

Extension. For a 1000-configuration in space the same argument yields $|C(3)| = 4$ (a tetrahedron) and $|C(1000)| \leq |C(3)|^{\frac{999}{3}} \cdot |C(1)| = 4^{333} \cdot 2$.

79. The problem is equivalent to rolling a regular pentagon $ABCDE$ over its sides. Let us prove that prints of one point (not necessarily the vertex) eventually produce an *everywhere dense set*, i.e., in any disc of any radius there will be a print.

First, we roll pentagon twice but so that its vertex A does not move. The point C will produce prints C^* and C^{**}. It is easy to see that C^{**} is made by C after rotation of the vector $\overrightarrow{AC}$ through an angle of $2\pi - 2\frac{3\pi}{5} = \frac{4\pi}{5}$.

Denote the rotation through the angle of ϕ about A by R_A^ϕ. In these notations $R_A^{4\pi/5}(C) = C^{**}$. We can demonstrate that the superposition (successive performance) of the rotations $R_A^{4\pi/5}$ and $R_B^{-4\pi/5}$ in any order amounts to the parallel translation by a vector whose length may be taken to be equal to 1.

It is also easy to deduce that by rolling the pentagon one can produce both rotations through angles $\frac{k\pi}{5}, k \in \mathbb{Z}$, and parallel translations by vectors with the angles between them equal also to an integer multiple of $\frac{\pi}{5}$.

Let us consider four such translations by vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, $\overrightarrow{e_3}$, $\overrightarrow{e_4}$ with the angle between the neighboring vectors being $\frac{\pi}{5}$. We see that the vector $\overrightarrow{f_2} = \overrightarrow{e_1} + \overrightarrow{e_3}$ is collinear with $\overrightarrow{e_2}$ and is of length $2\cos\frac{\pi}{5}$; the vector $\overrightarrow{f_3} = \overrightarrow{e_2} + \overrightarrow{e_4}$ is collinear with $\overrightarrow{e_3}$ and is of the same length. Since $2\cos\frac{\pi}{5}$ is an irrational number, the set of prints produced on the straight lines by $\overrightarrow{f_2}$ and $\overrightarrow{f_3}$ is everywhere dense on the plane, Q.E.D.

Remark. The required statement can also be proved with the help of the following lemma: *the square is the only regular polygon with all its vertices at the nodes of a square lattice.*

80. Put all boxes into one large empty box. When we put n boxes into an empty box, the number of empty boxes increases by $n-1$ and the number of filled ones by 1.

So, if $k+1$ boxes are filled (k boxes by hypothesis and 1 box added at the beginning), then $(k+1)(n-1)+1$ boxes are empty. We have a total of $(k+1)(n-1)+1+(k+1) = (k+1)n+1$ boxes and should remove the large box that we added.

81. By the condition $x^x - x^y = y^x - y^y$. Hence,

$$(y^{x-y} - 1)y^y = x^y(x^{x-y} - 1), \qquad \frac{x^y}{y^y} = \frac{y^{x-y} - 1}{x^{x-y} - 1}. \tag{$*$}$$

Let $x > y > 1$. We have: $x^y > y^y$ and $y^{x-y} - 1 < x^{x-y} - 1$ and, therefore, $\frac{x^y}{y^y} > 1 > \frac{y^{x-y}-1}{x^{x-y}-1}$. Contradiction with $(*)$. Consequently, $x = y$.

82. With the help of the formula for the product of sines the formula to be proved takes the form

$$\frac{abc}{6}\sqrt{2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^{\beta} - \cos^2\gamma + 1}.$$

Let us prove the latter formula. Let us take the coordinate system so that the vertex O of tetrahedron $OABC$ were at the origin, edge a along the Ox-axis, b on the plane Oxy; see Fig. 123.

FIGURE 123. (Sol. A82)

Then the coordinates of A and B are $(a, 0, 0)$ and $(b\cos\gamma, b\sin\gamma, 0)$, respectively, and coordinates (x, y, z) of point C can be found from the equations

$$|\vec{c}|^2 = x^2 + y^2 + z^2 = c^2; \quad (\vec{a}, \vec{c}) = ax = ac\cos\beta; \quad (\vec{b}, \vec{c}) = b\cos\gamma \cdot x + b\sin\gamma \cdot y = bc\cos\alpha$$

implying

$$x = c\cos\beta; \quad y\frac{1}{\sin\gamma}(c\cos\alpha - c\cos\beta\cos\gamma)$$

$$z = \sqrt{c^2 - x^2 - y^2} = c\sqrt{1 - \cos^2 s\beta - \frac{1}{\sin^2\gamma}(\cos^2\alpha - 2\cos\alpha\cos\beta\cos\gamma + \cos^2\beta\cos^2\gamma)}.$$

the volume of tetrahedron is equal to

$$\frac{1}{6}ab\sin\gamma \cdot z = \frac{1}{6}abc\sqrt{\sin^2\gamma - \cos^2\beta\sin^2\gamma - \cos^{\alpha} + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\beta\cos^2\gamma}.$$

Now, it remains to replace $\sin^2\gamma$ with $1 - \cos^2\gamma$ and $-\cos^2\beta\sin^2\gamma - \cos^2\beta\cos^2\gamma$ with $-\cos^2\beta(\sin^2\gamma + \cos^2\gamma) = -\cos^2\beta$.

83. a) Consider an arbitrary section of the body. It is a disc D_0 centered at O. Draw line l through O at a right angle to the plane of the disc. Then draw an arbitrary plane P through l. The section of the body by P is a disc and so a segment of l intercepted by this section — denote it AB — is the diameter of this disc. Indeed, let CD be the segment along which two mutually perpendicular sections intersect. Since AB meets CD at point O, the midpoint of CD, and $AB \perp CD$, we are done.

Finally, drawing all possible sections of the body passing through l we see that all of them are discs with diameter AB. And since all these sections fill the entire body, the latter is a ball with diameter AB, Q.E.D.

b) We do not know a solution elementary enough to fit in this book.[1]

[1] Well, actually, we do not know a non-elementary one either.

84. 1°. Each key should be in ≥ 6 copies (otherwise in the absence of ≤ 5 keyholders the strongbox is impossible to open);

2°. For any 6 people there is a key which only these persons possess (to unable the others to open the strongbox); but this key, as any other one, is in ≥ 6 copies; hence, only these 6 persons are holders of this key.

85. Let the angle between hands of the clock be equal to α. For the clock to show some minutes past twelve the hour hand must form an angle of x with the ray passing from the center to 12 and so $0 < x < \frac{360^\circ}{12}$ and the minute hand must form an angle of $x + \alpha$ on one side and $12x$ on the other side with the same ray.

From the equation $12x = x + \alpha$ we find: $x = \frac{\alpha}{12}$.

86. The following lemma can be directly verified.

Lemma. *If a, b, c, d are positive and $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.*

Now, we have $0 < \tan\alpha_1 < \tan\alpha_2 < \ldots < \tan\alpha_n$ and so

$$\begin{aligned}\tan\alpha_1 = \frac{\sin\alpha_1}{\sin\alpha_2} &< \frac{\sin\alpha_1 + \sin\alpha_2}{\cos\alpha_1 + \cos\alpha_2} < \frac{\sin\alpha_2}{\cos\alpha_2} = \tan\alpha_2 < \tan\alpha_3 = \frac{\sin\alpha_3}{\cos\alpha_3} \\ &\Longrightarrow \frac{\sin\alpha_1 + \sin\alpha_2}{\cos\alpha_1 + \cos\alpha_2} < \frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}{\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3} < \tan\alpha_3 < \tan\alpha_4 = \frac{\sin\alpha_4}{\cos\alpha_4} \\ &\Longrightarrow \frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}{\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3} < \frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3 + \sin\alpha_4}{\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3 + \cos\alpha_4} < \tan\alpha_4 < \tan\alpha_5, \text{ etc.}\end{aligned}$$

We get what we want by applying Lemma n times.

87. Let $S = x + y + z$. Subtract $\frac{1}{x}$ from this equation. We get $-\frac{y+z}{xS} = \frac{y+z}{yz}$. Hence, $y + z = 0$ or $xS = -yz$.

If we assume the contrary to the desired, then $y + z \neq 0$ and, therefore, $xS = -yz$. Similarly, we have $yS = -xz$, and $zS = -yx$. Dividing $xS = -yz$ by $yS = -xz$ we get $\frac{x}{y} = \frac{y}{x}$. In a similar way we get $\frac{x}{z} = \frac{z}{x}$, and $\frac{y}{z} = \frac{z}{y}$. By the hypothesis, $y \neq -z \neq x \neq -y$, so we conclude that $x = y = z$ and $S = 3x$, $xS = 3x^2 = -x^2$. Hence, $x = 0 = y = z$. But the denominators are nonzero. Contradiction.

88. Observe that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

and so:

$$1 = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) = \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 6} \quad \text{for} \quad n = 4,$$

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{4 \cdot 2} + \frac{1}{4 \cdot 3} + \frac{1}{4 \cdot 6} \quad \text{for} \quad n = 5$$

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8 \cdot 2} + \frac{1}{8 \cdot 3} + \frac{1}{8 \cdot 6} \quad \text{for} \quad n = 6$$

and so on.

Extension. Can all denominators be distinct odd numbers?

It turns out they can for all odd $n \geq 9$. Try to prove this on your own. Start with two examples:

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{35} + \frac{1}{45} + \frac{1}{231} \quad \text{for} \quad n = 9$$

$$\begin{aligned}&1 = \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{231} = \\ &\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{35} + \frac{1}{45} + \frac{1}{5 \cdot 77} + \frac{1}{9 \cdot 77} + \frac{1}{45 \cdot 77} \quad \text{for} \quad n = 11\end{aligned}$$

Further on, at each step we replace the least of the fractions $\frac{1}{3m}$ (with $m = 15 \cdot 77$ for $n = 11$) with the sum $\frac{1}{5m} + \frac{1}{9m} + \frac{1}{4m}$ by increasing the number of summands by 2 each time.

89. Let us place the table on an uneven floor first so that the tips A and C of its opposite legs are on the floor while the tips B and D are above the floor at the same distance from it (we will denote this distance by $h > 0$, $ABCD$ being a square). Now, press on the table so that A and C puncture the floor and stop at a distance $-h$ from it (i.e. at the distance h below the floor). Then B and D are on the floor.

We can obtain the same position of the table by rotating it about the center through 90°, points A and C being tangent to the floor at all times until they arrive at their final positions D and B, respectively. We see that the distance $h(t)$ from B and D to the floor (at time t) varies continuously in t from h to $-h$ as t

varies from $0°$ to $90°$, and A and C belonging to the floor at all times. So all four points A, B, C, D belong to the floor for some t and at this moment $h(t) = 0$.

EXTENSION. (We do not know a solution of these more general problems.) Consider the problem for a *rectangle* $ABCD$. An even more general problem (suggested by S. Tabachnikov) is the one when $ABCD$ is a *quadrilateral inscribed into a circle* (the table must be inscribed so as not to shake, for example, on a spherical floor of a sufficiently large radius).

90. It is rather difficult to find x, y, z directly. Let us solve the system for a, b, c with x, y, z as parameters. We have a linear system in a, b, c and, since $x \neq 0$, $y \neq 0$, $z \neq 0$, it has a solution (perhaps, non-unique if the discriminant of the matrix, equal to $1 - \frac{1}{x^2} - \frac{1}{y^2} - \frac{1}{z^2}$, vanishes). But $a = xz$, $b = yz$, $c = xy$ is a solution of the system (with or without the help of linear algebra). Therefore, the system is compatible and there are no other solutions.

So $xyz = \sqrt{abc}$; hence, $x = \sqrt{\frac{ac}{b}}$, $y = \sqrt{\frac{bc}{a}}$, $z = \sqrt{\frac{ab}{c}}$.

94. Denote the lengths of the tetrahedron's edges at the base by x, y, z and those of the edges coming out of the vertex by a, b, c. Let $a \geq \max(b, c)$.

If $a < b + c$, then two triangles can be constructed from the edges a, b, c and x, y, z.

Let now $a \geq b + c$. Since $a < x + c$ and $a < b + z$, it follows that $2a < x + z + b + c \leq x + z + a$ whence $a < x + z$ and two triangles are constructed from the edges a, x, z and b, c, y.

Finally, take an edge of maximal length (a) and two adjacent edges (b, c) or (x, z); the triangle with side a can be constructed in at least one of the cases; the remaining three edges now form a face of the tetrahedron.

95. Solutions different from that hinted at in Hint can be obtained with the help of canonical equations of the straight lines:

$$\frac{x - x_i}{a_i} = \frac{y - y_i}{b_i} = \frac{z - z_i}{c_i}, \qquad i = 1, 2, 3.$$

96. It is easy to see that after the balloon is inflated each of the net's strings (laces) can be seen from the center of the balloon at the same angle as an edge of the standard $1 \times 1 \times 1$ cube is seen from its center. This angle is equal to the angle at the vertex of an isosceles triangle with base 1 and side $\frac{\sqrt{3}}{2}$ (half the diagonal of the cube): $\alpha = 2 \arcsin \frac{1}{\sqrt{3}}$.

Now, the radius R of the balloon can be found from the equation: $10 = R\alpha$, see Fig. 124; hence,

$$R = \frac{5}{\arcsin\left(\frac{1}{\sqrt{3}}\right)}.$$

FIGURE 124. (Sol. A96)

FIGURE 125. (Sol. A97)

97. Consider a mapping f that converts the map[1] K_0 with a larger scale into the map map K_1 with the smaller scale: each point representing some spot in K_0 (by considering only the part of the town depicted on K_0 we assume that $K_0 \supset K_1$) is placed upon the point representing the same spot in K_1. Denote the image of K_1 under the same mapping by K_2, see Fig. 125.

[1]The assumption that the map of Moscow is continuous was quite unjustified before the time of "glasnost" and probably still is: large portions of Moscow were "classified", i.e., restricted for "security reasons" and quite a number of scientists were busy producing more or less plausibly distorted maps without "white spots". But, kidding aside, this assumption is also dubious mathematically if we take into account the *fractal* theory, see e.g., K. Falconer *Fractal Geometry*, Wiley, Chichester ea, 1990 and refs therein (esp. on p. xxii).

Further, set $K_n = f(K_{n-1})$, $n = 1, 2, \ldots$. The rectangles K_0, K_1, $K_2, \ldots, K_n, \ldots$ have just one common point, x, since the sizes of the rectangles tend to zero.

The point x is precisely the point for pearcing. Indeed, it follows from $x \in K_{n-1}$ that $f(x) \in K_n$. Thus the point $f(x)$ also belongs to all rectangles but there is only one such point and so $x = f(x)$.

REMARK. In general, the following theorem is true:

Any continuous map of a rectangle onto itself has a fixed point.

So the statement of the problem remains true even if one map is crumpled before being placed on top of the other map and pierced.

98. Since $2222^{5555} \equiv 3^{5555} \equiv (-4)^{5555} \equiv -4^{5555} \pmod 7$ and $5555^{2222} \equiv 4^{2222} \pmod 7$, we have

$$\begin{aligned} 2222^{5555} + 5555^{2222} &\equiv 4^{2222} - 4^{5555} \equiv 4^{2222}(1 - 4^{3333}) \\ &\equiv 4^{2222}(1 - 64^{1111}) \equiv 4^{2222}(1 - 1^{1111}) \equiv 0 \pmod 7. \end{aligned}$$

Therefore, $2222^{5555} + 5555^{2222}$ is a multiple of 7.

ANOTHER SOLUTION. Note that $4^3 = 64 \equiv 1 \pmod 7$. Hence

$$2222^{5555} \equiv 3^{5555} \equiv (-4)^{5555} \equiv (-4)^5 \equiv (-4)^2 \equiv -2 \pmod 7.$$

Similarly,

$$5555^{2222} \equiv 4^{2222} \equiv 4^2 \equiv 2 \pmod 7,$$

and we are done.

FIGURE 126. (Sol. A99)

99. A sketch of the required construction with nine threads is shown in Fig. 126.

A convenient way of making it is to take 3 pencils.

If all threads have the same length l and the rods the same length d, then, for the construction to be rigid, it is necessary to have $d = l\sqrt{1 + \frac{2}{\sqrt{3}}}$.

The endpoints of the rods in our construction are vertices of two equilateral triangles arranged in parallel planes which are perpendicular to the straight line connecting the centers of triangles and rotated through an angle relative to one another. The rods themselves lie on straight lines which cross pairwise. The connection of the rods and threads is the same as for an octahedron.

It is extremely difficult to prove the sufficiency of the above condition.

REMARK. This connection was invented in the 1960s by an architect, B. Fuller. Many different designs of this type have appeared since then.

In this section we generously borrow from Introduction to [Le], [BY] and from [GT].

Historical remarks

H.1. On this book. The book contains problems of all mathematical Olympiads held in Moscow since the first one, in 1935. For the first time all of them are provided with solutions or at least hints and answers.

This book is not the first compendium of problems of Moscow Mathematical Olympiads. Selected problems of Moscow Olympiads 1–15 were collected in [SCY]; those from Olympiads 1–27 were compiled almost completely and published in Russian in [Le]. The compiler, A. A. Leman, and all who helped him, did a tremendous job to put through that edition. Written with care and with easy to understand solutions, the collection [Le] has become a rarity long ago. Both [SCY] and [Le], however, only contained solutions to selected problems.

A critical review of these collections revealed a number of omissions and errors which we corrected as best as we could[1]. We generously borrowed from Introduction of [Le] and from the book [GT], where most of the problems from Olympiads 1–49 are supplied with at least an answer or a hint.

In this book, we offer the reader for the first time all (see footnote) problems of the first 27 Olympiads and all problems of the Olympiads after the 49-th. (Problems from Olympiad 50 can be found in Chinese [GT*]. The book was published in precopyright era without anybody's consent[2].) No complete (and correct) solutions of ALL problems had ever been published yet. A new generation of high school students will be able to get acquainted with a great number of interesting and beautiful ideas contained in more than 2 000 problems of the Moscow Mathematical Olympiads and learn the history of these happenings.

The problems for the Olympiads were put together and composed by many generations of graduate, undergraduate and post graduate students (mainly) from the Moscow University. The preparatory problems, those of the Olympiads themselves, and the problems told to school pupils at consultation sessions and lectures constitute a very valuable material for study; a sort of the mathematical folklore.

Enthusiastic university students pester undergraduates and professors offering their problems, fiercely criticizing others' problems and demanding to create more and more of new problems to the pool. Sometimes the discussions are very heated; sometimes a problem is discussed in whispers and the speakers look around like conspirators. It means that they discuss a problem that has a chance to be accepted for the Olympiad. More often than not a problem is so transformed during the discussion that the author can hardly recognize his creation. Thus, the preparatory problems and those of the Olympiad are mainly the result of a collective brain storm.

Unfortunately, this most valuable folklore is lost to a great extent and partly beyond recover. It is only with great difficulties that we managed to restore some of the problems of Moscow Olympiads and sometimes even complete original solutions that seemed to have been lost.

H.2. On necessity of Olympiads. In the mid-1930s many Soviet mathematicians pondered about the need for cooperation with the high school to bring up the next mathematical generation. The training of future mathematicians should begin in their childhood, the earlier the better. Nobody is surprised to see a ballet dancer or a musician starting their career at the age of 8 or 6 or earlier. The explanation is that it is impossible for a teenager to master all intricacies of the dancing art or of music, without specialized training when a child, to develop the ear and the feeling of rhythm, the flexibility of knuckles or agility of

[1]Concerning completeness, this goal seem to be out of reach. Apart from Olympiads held during WW II, several problems of one more Olympiad seem to have sunk in Lethe. As I. M. Yaglom writes in *Problems, Problems, Problems. History and Contemporaneity*, Matematika v shkole (Mathematics in School), No. 5, 1989, pp. 143–148 : " ... I can not figure out why neither Leman nor Galperin and Tolpygo addressed the participants of that Olympiad ... for instance, me.

The 2-nd set of Olympiad 4 contained 5 problems, not 4. Of the problems of the 1-st set I remember the one which I found the most difficult at that time. I hope that *you* will not find it difficult *now*, but bear in mind that it was the 1-st set:

In space (not on one plane) 4 *points are given; how many planes equidistant from these points are there?* I do not bet that in the formulation of this 49-years old problem the parenthetical restriction was explicit. Perhaps, it was required to consider separately the other case as well."

Now Isaak Moiseevich Yaglom is dead. So nobody will, probably, provide us with the lacking problems.

[2]Which is a pity: the compilers would have supplied with corrections. For free.

fingers, etc. And each year lost in childhood might only be compensated later with many years of incredibly hard work.

It would be wrong to think that the situation is different in science, particularly in mathematics. Just like in dancing or music, the years lost in childhood are difficult to compensate later on. The trouble is that mathematics requires some agility of mind, the ability to think in abstract terms, and some logic culture which are impossible to pick up even by hard work at the university. To be sure, all qualities making up what is called "mathematical abilities" can develop at an ordinary school without any special training of a teenager. This is a spontaneous process for born mathematicians which has taken place in all countries and at all times. For example, **S. Ramanujan** (1887–1920), a very famous Indian mathematician had practically no mathematical education.

The mathematical talent, however, like that of a musician, manifests itself quite early, as a rule. Moreover, when developed properly, a mathematician can make his most important discoveries when still quite young. For example, **Evariste Galois** (1811–1832), a French mathematician, had time in his short life to create an algebraic theory remarkable in its depth, which greatly advanced the development of mathematics. **Carl F. Gauss** (1777–1855) published his classical studies of constructions with a ruler and compass when 19 years old; several years later he presented the book "*Disquisitiones arithmeticae*" which has few equals in the history of mathematical science.

Participation of the mathematicians in school education is extremely essential. This was the case before the Revolution; this work was reinitiated after the Revolution only in 30s; in Leningrad by CMA B. N. Deloné and Prof. V. A. Tartakovsky , and in Moscow by CMA L. G. Shnirelman, and Prof. L. A. Lusternik (later a CMA). The first mathematical Olympiad for schoolchildren in the USSR was held in Leningrad in the spring of 1934.

H.3. On Moscow Mathematical Olympiads. In the spring of 1935, the Board of the Moscow Mathematical Society, following the example of Leningrad, decided to organize the 1-st Moscow Mathematical Olympiad. The organizing committee included all professors of mathematics from the Moscow University and was headed by P. S. Alexandrov, the then President of the Moscow Mathematical Society. The purpose of the Olympiad was to find the most talented students, to attract attention of the young people at large to some of the most important problems and methods of modern mathematics, and to show the kids, least partly, what Soviet mathematicians are working on, what progress they have made and what challenges they have.

314 high school students participated in the Olympiad, 120 of them took part in Set 2 (the final). Three students were awarded first prizes, five got second prizes and, in addition, 44 kids were given honorary prizes. A place at the top of the Olympiad determined for many their future scientific career.

It is interesting how problems in Set 2 were selected. Three series of problems were offered and they were designated A, B and C. A. N. Kolmogorov told us that it was done by his initiative in order to enable students with different mathematical mentalities — geometrical, computational (algorithmic), or combinatorially logical — to develop. (Details see in [Ko]). It is according to these types of thinking that the series of problems were selected for the first Olympiad.

H.4. Mathematical circles. The success of the 1-st Moscow Mathematical Olympiads helped to restructure the relations of the researchers with schoolchildren. This eventually brought about the School Mathematical Circle attached to Moscow University. It was organized by L. A. Lusternik, L. G. Shnirelman and I. M. Gelfand (a member of the US National Academy, the French Academy, and almost all other Academies; at the dawn of perestrojka he was finally elected member of the USSR Academy of Sciences). The Circle had two types of activities: lectures on all kinds of subjects and meetings of their members. The lectures were attended first by dozens and then by hundreds of boys and girls from all over Moscow. Initially, the lectures were addressed to 8-th to 10-th graders; later (since 1940) the lectures were delivered also for 7-th and 8-th graders. The lectures set forth in a popular form serious mathematical results, including the latest scientific achievements.

The subjects of the lectures were quite diverse. Here are some examples of the lectures delivered at different times of the circle's existence:

L. S. Pontryagin. *What is topology?*
A. G. Kurosh. *What is algebra?*
N. A. Glagolev. *Construction using only the ruler and without compass.*
A. A. Lyapunov. *Thinking machines.*
A. I. Markushevich. *Areas and logarithms.*
L. G. Shnirelman. *Multidimensional geometry.*
B. N. Deloné. *Derivation of the seven crystal systems.*
A. N. Kolmogorov. *The fundamental theorem of algebra.*

S. L. Sobolev. *What is mathematical physics?*
I. R. Shafarevich. *Solutions of equations in radicals.*
L. A. Lusternik. *Convex figures.*
P. S. Alexandrov. *Transfinite numbers.*
S. A. Yanovskaya. *What does it mean to solve a problem?*
Ya. S. Dubnov. *Errors in geometric proofs.*
I. A. Kibel. *Mathematical methods of weather forecasting.*
V. V. Golubev. *Why does an airplane fly?*
I. M. Gelfand. *Dirichlet's principle.*
V. A. Efremovich. *Non-Euclidean geometry.*
B. V. Gnedenko. *How science studies random phenomena.*
N. K. Bari. *Arithmetics of the infinite.*
G. E. Shilov. *About a derivative.*
R. L. Dobrushin. *Mathematical methods of linguistics.*
V. G. Boltyansky. *Continued fractions and the musical scale.*
I. M. Yaglom. *How can we measure information?*
O. A. Oleinik. *Helly's theorem.*

This list is far from complete, of course: many hundreds of lectures have been delivered for schoolchildren during the circle's existence.

We can say that the mathematical circle of Moscow University helped to revise considerably the term "elementary mathematics" (when it implies the body of mathematical knowledge that could be made fully understandable to school pupils).

Thus, for example, in his lecture "The fundamental theorem of algebra" made in 1937, Acad. A. N. Kolmogorov set forth an essentially full proof of the theorem on the existence of a complex root in any algebraic equation. This proof (called in the mathematical circle "The Lady with a lap dog" after a story by Chekhov) was published later in exactly the same form in the book "What Is Mathematics?" by R. Courant and H. Robbins, [CR].

The same year, L. G. Shnirelman in his lecture "The group theory and its application to solving 3rd order equations" brought up the group theory considerations, which actually go back to Galois, to obtain an explicit formula for a solution of 3rd order equations.

B. N. Delоné in his lecture "The geometry of continued fractions" delivered in 1947 not only proved a subtle theorem on the best rational approximations of irrational numbers but also described an elegant results obtained by Hurvitz about the irrationalities worst approximated by rational numbers.

Acad. S. I. Sobolev delivered in 1940 a lecture "What is mathematical physics?" and very skillfully brought the description (at the level understandable to teenagers) up to the 2-nd order partial differential equations indicating qualitative differences in the behavior of their solutions.

Sometimes the lectures were accompanied by problems to be solved at home or on the spot. I. M. Gelfand offered particularly numerous problems to his audience. The boys and girls who knew well his manner of lecturing often preferred to sit as far as possible from the speaker not to be called to the blackboard to solve a problem.

Ya. S. Dubnov's lectures were interestingly arranged. Sometimes he delivered a series of two lectures, the first lecture offering some problems whose solutions (partly found by the listeners) were discussed at the second one (in two weeks' time).

While delivering a lecture on deduction arithmetics and Boolean algebra, A. N. Kolmogorov drew a plot of an electric circuit shown in Fig. H1 to place two switches near the door and over a bed, each switch being able to turn on and off a lamp in the room irrespective of the position of the other switch. At the end of the first hour he challenged the listeners to find during the break the circuit which would enable one to switch the light on and off from n places in the room.

FIGURE 1. (Sol. AH1)

During preparations for the lectures, the lecturers often found new elegant proofs of well-known theorems, obtained new generalizations of some facts that they knew earlier, and even made small mathematical discoveries. Unfortunately, most part of this very valuable material is lost forever.

H.5. How to run mathematical circles (after Shklyarsky). Along with the lectures, there were regular meetings of sections of the circle. They were conducted generally by senior and post graduate students from "mekh-mat", i.e., the Department of Mechanics[1] and Mathematics of Moscow University

[1]A more adequate translation would be "Mechanical engineering" instead of the conventional "Mechanics".

(except two sections in 1936 and 1937: the section of *geometric methods of the number theory* was headed by L. G. Shnirelman and the section of *qualitative geometry* was headed by A. N. Kolmogorov).

At first, the reports at the meetings were made by schoolchildren themselves; but soon it was found that this form of work was non-productive. The trouble was that most of the reports were of little interest and boring for all members of the circle (with an exception, perhaps, of the reporter himself). After all, it is not enough to understand everything what is said in the mathematical text given by the section head to make a good report.

A well-made report must arouse the interest of the audience and make the listeners think over and over what they have heard; it should contain a clear presentation of the problems to be discussed, the main ideas of the solution should be emphasized, the beautiful and original parts of proofs should be vividly depicted, and so on. Besides, a lecture can rarely be good if the lecturer knows the subject only within the limits of the lecture. Therefore, a teenager's report is usually far inferior to that of an experienced teacher.

The radical change in the work of the sections was associated with the name of David Shklyarsky, a talented mathematician and brilliant teacher, who headed the circles 1938 till 1941 while still a student. (D. O. Shklyarsky was killed in a guerrilla combat in 1942 at the age of 23 during WW II.)

Reports of the school pupils at the meetings were practically abolished. Instead, the head of a section delivered a short lecture that contained as a rule a complete description of a small mathematical theory. Then, the members described their solutions of the problems given at the previous meeting. Problems of varying complexities enabled Shklyarsky to involve virtually all section members into active work and by repeating a member's solution he had two objectives in mind: the audience understood better the solution exposed by an expert while the author of the solution had a lesson: how to lucidly present a mathematical proof.

This system has born a wonderful fruit: in 1938, at the 4-th Olympiad, members of Shklyarsky's section took away half of the prizes (12 out of 24), including all 4 first prizes! The results of the 4-th Olympiad astonished the heads of the other sections and next year practically all of them followed this example. Since then the form of the work of the circle found by D. O. Shklyarsky became predominant.

From the very beginning a tradition was established to issue annually a small collection of preparatory problems for the next Olympiad, which was given to the circle members and to all who came to the Olympiad.

H.6. Examples of programs and syllabus of specialized sections of the circle.

The Geometric Probabilities series

The problem on a meeting: *Two persons agreed to meet at a certain place, each has to come to the place between 10 and 11 o'clock and waits for the other one for exactly 15 minutes. What is the probability of the meeting?* The basic geometric idea is that the probability depends on the area or volume of the figure formed in the space of the events by points corresponding to favorable events.

The problem of constructing a triangle given three segments. (All kinds of varieties of the basic problem: *a stick is randomly cut into three pieces; what is the probability that these pieces can form a triangle?*)

The Buffon problem on throwing a needle for experimental determination of π. *Throwing a closed convex curve on a piece of paper ruled with parallel lines find the probability of the curve crossing a line.*

Barbier's theorem on the length of curves of constant width (as a corollary of the above or of Buffon's theorem).

The "area" (measure) of a set of straight lines crossing a given arc.

Crofton 's theorem and basic ideas of integral geometry.

Geometric Maximum and Minimum Problems

The rectification method as applied to problems on inscribed polygons of minimum perimeter. (The typical problem is to find a point the sum of whose distances to the vertices of a triangle is minimal.)

An isoperimetric problem for n-gons ($n = 3$, 4 and the general case). Polygons of the greatest perimeter inscribed into a circle; escribed polygons of the least perimeter.

An isoperimetric problem for arbitrary lines. Steiner's four-hinge method and its critique. The problem whether there exists a solution of the minimum or the maximum problem.

Blaschke's theorem on the existence of a converging subsequence of convex figures. Substantiation of Steiner's method. Other examples of application of Blaschke's theorem.

Variational methods including the search for maximal and minimal figures. (The typical problem is to draw a straight line through a point inside an angle so as to cut off a triangle of the least area; solution of the problem using the method of geometric differentiation.)

The section of algebra

(Subtitled "Generalization of the notion of the number")

Natural numbers (or, as they are more often called in science, **positive integers**) were the main building blocks for further constructions. A quotation from L. Kronecker: "Natural numbers were created by God; the rest was done by humans."

The solvability of the equations $x + a = b$; subtraction. Generalization of the set of numbers in order to make subtraction always possible. The integers as material for sufficiently meaningful constructions; the theory of numbers. Examples of the number theory problems.

The solvability of the linear equations $ax + b = 0$; rational numbers. The number as a result of measurement; the number axis. The possibility of using only rational numbers in problems concerning measurements of geometric and physical values.

The solvability of quadratic equations. The insolvability of the quadratic equation $x^2 - 2 = 0$. The solvability of linear equations — the existence of points where the x-axis crosses the straight lines $ax + b = y$ (with rational coefficients); the absence (within the given stock of numbers) of the crossing point of the x-axis with the parabola $y = x^2 - 2$. Quadratic radicals. The solvability of all quadratic equations with real roots (the existence of the crossing points of the x-axis with the parabolas $y = ax^2 + bx + c$, where a, b, c are numbers from the given stock).

Construction of the point $x = \sqrt{2}$ (the diagonal of a unit square). Segments that can be constructed with a ruler and compass. The proof of insolvability of the problem on duplication of a cube — the parabola $y = x^3 - 2$ "slips through the known points of the x-axis". Further generalization of our stock of numbers; the real radicals of any order.

The problem of solution of cubic equations. The proof of the fact that the cubic parabola $y = x^3 - 4x - 2$ "slips through the points of the x-axis" (the irreducible case of a solution of a cubic equation.) The need to enlarge yet the stock of numbers.

The problem on solution of cubic equations again. **Complex numbers**. Geometric interpretations of complex numbers — the complex numbers as points in a plane; complex numbers as operators of rotational dilation. De Moivre's formula and problems on complex numbers.

The problem on solution of cubic equations once again. Cardano's formula. Another extension of the set of (real) numbers — combinations of complex radicals. The problem of solution of 4-th order equations.

The 5-th order equations. The proof that the parabola $y = x^5 - 4x - 2$ "slips through the known points of the x-axis." The need for a new extension of the set of numbers. All kinds of roots of all kinds of algebraic equations.

Another approach to "numbers" — infinite decimal fractions. A new extension of the set of numbers proving that the number

$$0.110001000000000000000000010000\ldots\ (= \sum_{n\geq 1} 10^{-n!})$$

is not a root of any algebraic equation.

A different extension of the notion of numbers: quaternions as operators of rotational dilation in plane and in space. The geometric theory of quaternions. The quaternions and vectors; operation of vector algebra in space; problems. Frobenius' theorem.

Another extension of the notion of numbers — complex numbers, dual numbers ($\mathbb{Z}[x]/(x^2)$) and double numbers ($\mathbb{Z}[x]/(x^2 - 1)$). Geometric interpretation of dual numbers (as directed straight lines of a plane.) Geometric applications of double numbers. The idea of Hurvitz' theorem and of its generalizations.

(Of course, even in this extremely "theoretical" orientation of the circle, quite a few materials were left for exercises of the members on their own.)

H.7. Books for mathematical circles. In 1950, "Gostekhizdat" (The State Technical Publishers)[1] began publication of a special series of booklets under the heading "Popular Lectures on Mathematics", most of which were brought about as a specially prepared edition of lectures delivered at the mathematical circle of Moscow University. Some of the lectures were also published in the collection "Mathematical Education". A brief summary of some unpublished lectures is given in *Selected Lectures*. An extensive bibliography is given in [Le].

One of the main virtures of these books was, undoubtedly, their low cost.

H.8. How Moscow Mathematical Olympiads were arranged. The procedure has not practically changed since the 1-st Olympiad held in 1935.

The first 36 Olympiads (1935 through 1973) were held in two sets on Sundays at the end of March and early April (because of the kids' vacations).

Set 1 was selective, every participant being given 4 to 6 relatively easy problems (but with a catch!) and informed that it would be enough to solve two of the problems to get through to set 2.

A week after the first set, the problems were reviewed, different solutions and typical errors were pointed out, and the results were announced. Set 2 was held a week after that and all those who were successful in set 1 were invited (they made usually 30% to 50% of the total); sometimes teenagers who failed at set 1 were also allowed to participate in set 2 as well as those who missed set 1 for some reason[2] (generally, because of an illness; these were placed in special rooms.) The problems of the second set were considerably more difficult than those of set 1. In each set 5 or 6 hours were allotted to solve problems.

Finally, a week after, the problems of set 2 were discussed. Usually famous mathematicians were invited to review the problems conceived mostly by under and post-graduate students. The purpose was to combine descriptions of problem solutions with indications of broader perspectives in "big mathematics". Thus, one of us saw A. N. Kolmogorov in front of an audience that consisted mainly of several hundred of 8-th graders who started from two problems of the 38-th Olympiad (Problems

[1]Later renamed "Fizmatgiz" (The Phys. and Math. State Publishers) and now called "Nauka" (Science). Whatever the name, the books were increadibly cheap *Où sont les naiges d'autrefois? Où sont elles...*

[2]This happened usually on recommendation of a member of the organizing committee who had noticed the originality of a teenager's thinking, and sometimes by personal request from a teenager; the objective was to *attract* kids to science, not just to gauge their sports skills.

38.8.3 and 38.8.4) and went over to discussion of modern issues of discrete mathematics (the graph theory and the information theory).

Then there was an awarding ceremony. The winners were given prizes: piles of mathematical books (1 m high) with dedicatory inscriptions. There were on the average about 10 first prizes (for different grades), twice that many second prizes and the thrice many third prizes. In addition, there were certificates of merit, of degrees 1 and 2. The results in set 1 were usually not taken into consideration in summing up the Olympiad and awarding the winners.

There was a three years' interval during the World War II from 1942 through 1944; during these years Moscow mathematicians held a number of olympiads in Ashkhabad and Kazan instead. Regrettably, we could not find any records of these events.

During the first five Olympiads all students were offered the same problems; beginning with Olympiad 6 the problems (and the students) were divided into two streams: for 7–8-th graders and for senior students.

Starting with the 15-th Olympiad (1952), the contest was held separately for each grade although some more interesting problems were given in parallel to several grades.

From the very beginning, a great assistance in organizing the Olympiads was rendered by the Moscow City Department of Public Education and the Lenin Advanced Training Institute for Teachers (abbreviated in Russian as MGPI, the Moscow Pedagogical Inst.). The staff of the latter together with experienced teachers and university mathematicians began to hold district olympiads in 1949. Their problems are given in ref. [SCY2]. They allowed to involve in mathematics a greater number of school pupils and not only senior graders but pupils of 5-th to 7-th grades as well.

While the mathematical circle for high school students had been the predominant form of extra-curricular mathematical activities for about a quarter of a century, the Moscow Olympiads focusing all lines of those activities, their forms have been noticeably diversified for the last 20 years. Specialized mathematical schools were set up and in 1963 many circles were combined in a new structure called "evening mathematical schools" to be followed a year later by "correspondence (extra-mural) mathematical schools". Following the example of Moscow University, other higher educational establishments in Moscow started to hold mathematical olympiads of their own, and along with city olympiads there appeared a system of Republican, National, and, finally, International olympiads. However, the Moscow Olympiads continued to be something special for many years since their awards were considered to be a great honor and the standards of their problems were much higher than those of all other mathematical olympiads.

In 1961, teams from regions and Union Republics were invited to attend the 24-th Moscow Olympiad and so the unified multistage mathematical olympiad was began on the national scale. The first National Mathematical Olympiad was held on April 16, 1967 in Tbilisi. Since the second set of the 30-th Moscow Olympiad took place on the same day, the Moscow team had to be made up on the basis of the results of the first set of problems. Later on, the periods of Moscow Olympiads were shifted back from April to March (and sometimes to February) in order to have time to select a team for the National Olympiad held in mid-April.

At first, the Moscow team was selected directly from the results of a Moscow Olympiad. Later on, additional "qualification sets" were arranged where 15 to 20 people were invited, including those who had been awarded first, second and sometimes third prizes. The qualification problems were selected, as a rule, out of those which had been considered for the olympiad but were rejected as too difficult or because of unclear formulation (it is much easier to make a problem clearer for 15 participants in the qualification contest than at an olympiad with many hundreds of participating teenagers of various levels), or simply because they were too many. But the qualification contest has never been regarded as an additional (final) set of an olympiad. Certain problems from these competitions are given in *Selected problems*.

H.9. An accident that cased an important innovation. Once, an additional set was held: at the 33-rd Moscow Olympiad for 7-graders (1970). It was called "Pythagoras' Day". That year a disaster happened: The VC lost the briefcase with all papers of the 7-th grade. So the organizing committee decided to run another set. But it deviated from the established tradition. First, the teenagers were given three problems. Two hours later their papers were collected and a break for half an hour was announced after which three more problems were given.

Regrettably, it is impossible to repeat this procedure at a regular olympiad since it proved to be very difficult to collect even 7-th graders after the half-hour break and what could have happened if not a hundred but several thousands schoolchildren were set loose for half an hour to run about? It sometimes happened at Olympiads that teenagers stormed the unmanned locker room leaving behind a small pile of buttons torn off in the process ...

Another innovation — pity, this did not become a regular practice — was of greater interest: one of the problems on the Pythagorus' Day (33.D.7.2) was suggested by the organizing committee with no definite solution known. The participants were told that it was a research problem and they should try to advance as far as possible in solving it.

H.10. Two exceptional first prizes. There have been occasions when the first prize (no less!) was given to those who had not completely solve any (!) problem.

• At the 9-th Olympiad. Erik Balash, a 10-th grader, spent the entire time of the Olympiad trying to solve just one problem (9.2.9-10.2): *For the Fibonacci sequence* 0, 1, 1, 2, 3, 5, 8, ... *find whether among the first* 100 000 001 *terms of the series there is a number ending with four zeros.*

The organizers thought that the students would try to solve this problem by means of relatively simple considerations related to Dirichlet's principle. But Balash approached the problem from quite a different angle. He decided to give a full investigation, i.e., to indicate the numbers *of all terms in the series*, which end in four zeros. For this purpose, he conducted an arithmetic investigations which he failed (or had no time) to complete. Erik pointed out correctly that the first term ending in four zeros is the one numbered 7501 and found the law of recurrence of such terms further on. The solution was marked ($\pm$!) and Balash got the first prize although he did not even start to solve the remaining problems.

• At the 8-th Olympiad. The organizers believed that the following Problem 8.2.7-8.4 was relatively easy.

Vertices A, B, and C of triangle ABC are connected with points A', B', and C' lying on the opposite sides, but not in the vertices, see Fig. H2. Prove that the midpoints segments AA', BB', and CC' do not lie on the same straight line.

FIGURE 2. (Sol. AH2)

Indeed, the midpoints M, N, and P of the pointing out segments belong to the midlines DE, EF, and FD of ABC. Hence, the assertion required, since *no line passing through the vertices of the triangle can cross all the sides in their inner points.*

The organizers believed the statement italicized is obvious. However, a participant, Yulik Dobrushin from the 8-th grade (now the world-famous mathematician, Roland Lvovich Dobrushin, Dr.Sc.), reached this stage in the solution and added: "For a long time I have tried to prove that a straight line cannot cross all three sides of a triangle at their inner points but failed to do so. I am horrified to realize that I do not know what a straight line is!"

Dobrushin was crowned with the first prize for this frank recognition of his failure. The members of the organizing committee might have understood the meaning of Dobrushin's phrase better than its author himself. The point is that in modern geometry the answer to the question *what is a straight line* is given only by listing the line's properties among which the impossibility to cross all three sides of a triangle (or an equivalent property) is usually included.

H.11. The rise and fall of the Olympiads. The main mathematical forces in the USSR had been concentrated at the "mekh-mat" of Moscow University and at the V. A. Steklov Institute of Mathematics of the USSR Academy of Sciences until the 1960s. Later on many young mathematicians appeared also in other educational and research institutes[1]. They were very enthusiastic about preparing problems and holding olympiads: to preserve the spirit of science. Some institutions of higher education in bigger towns started to arrange their own olympiads and, in addition, the level of district olympiads was raised.

It was decided that Moscow Olympiads should (1) be held by Moscow University jointly with MGPI (Moscow Federal Teachers Traning Institute) and the Moscow Institute of Railway Engineers (MIIT)[2], and (2) have only one set for junior grades.

The grades were divided among the Institutes. The Moscow University was to hold the olympiads for the 7-th and 10-th grades while MGPI took the 8-th graders and MIIT the 9-th graders.

The olympiads for the 7-th grade has been conducted by the Department of Computational Mathematics and Cybernetics of Moscow University since 1981 (and now it runs the olympiads also for 9-th graders). The organizing committee meets to discuss problems, to sum up an olympiad and, for other matters held jointly, at the Moscow University, as a rule. The review of the results and the awarding ceremony also took place there.

The first set of the 37-th Olympiad (1974) was held only for pupils of grades 9 and 10 while at the 38-th to 40-th Olympiads it was provided only for 10-th graders. The results of the first set were taken into account in the general review and the participants who had solved the problems of the first set received a prize or a certificate of merit one degree higher than they would have been entitled to simply from the results of the second set.

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[1]Thanks to the state antisemitism: they were not admitted to the principal Universities.

[2]This was done in 1975.

Starting with the 41-st Olympiad (1978), it was decided to hold only **one** (final) set for all school grades since the role of the first set was played by the district mathematical olympiads whose winners were allowed to participate in the final of the Moscow Olympiad.

We will not dwell here on matters pertaining to the work of the organizing committee. Suffice it to say that the holding of an olympiad, publication of the collection of preparatory problems, compilation of a long list of new problems and selection of variants of the olympiad's problems requires a tremendous effort whose magnitude the majority of the participants do not even approximately appreciate and which would have been impossible without enthusiasm of, mostly undergraduate, mathematics-major students and post-graduates and without help from the Moscow Mathematical Society and other organizations.

When an olympiad is held, the working day of the organizing committee members may last from early morning till late at night. The papers are checked and rechecked several times and quite a few teenagers got their certificates due to attentive members of the organizing committee, who were not lazy to reread carefully the solutions to find the grain of rationality in them.

The teachers and school pupils most often ask if a pupil who did not attend the circle has any chance to win an Olympiad?

There is only one answer to this question: yes, of course one has. (But the point is that this question is a "wrong" one: we would like to teach to value mathematics, rather than the accompanying sports.) Among the winners there always were kids who had not been members of any mathematical circle. Many of the participants and winners of an Olympiad came to the circle on the subsequent academic year and then took part in Olympiads (or, sometimes, willingly refrained from participating; having made a choice between sport and science). Of course, the systematic studies in the circle, the mathematical culture and skills in solving mathematical problems acquired there came in very handy for participation in an Olympiad.

While the circle involved several hundreds of Moscow teenagers in systematic work, the number of participants in a Moscow Olympiad was always considerably greater and was as high as several thousands. For example, in 1964 there were over 4000 participants, the 1966 Olympiad was attended by about 5000 boys and girls while in 1974 their number reached 6000(!).

True, this figure decreased later but still the count was in thousands; a thousand school students came to the jubilee Olympiad in 1985. all rooms at the University were overcrowded in those years and some of the participants had to be placed in laboratories of the physical, chemical and biological departments.

H.11. Tournament of towns.

H.12. On relation of olympiad problems with the "big" mathematics. As for the olympiad problems, there are stringent requirements: the problems should be diverse in form and in ideas they are based on but their solution should not go beyond the limits of the existing school curriculum. Two in five or six problems are generally simple; algebraic and text problems alternate with geometric ones while their complexity usually grows as their number increases in the assignment list.

Notice that the problems given at olympiads are non-standard. Their novelty and attractiveness can be explained to a great extent by the fact that they are inspired by fresh ideas of modern mathematics and *every one of them* is a small investigation opening up new horizons for the person who tries to solve it. Quite a few olympiad problems are related to "serious" mathematics. Here are some examples of such problems:

Problem **9.2.7-8.4** came from crystallography and is related to growth of crystals. When crystals start to grow in a solution, a crystal stops growing if it comes up against another crystal (in the problem "a car finds itself in front of a road block").

Problems **9.2.7-8.5** and **9.2.9-10.5** are related to the theory of projective planes over finite fields.

Problem **13.1.9-10.1** was taken from the "*Imaginary Geometry*", the famous book by N. I. Lobachevsky, one of the discoverers of the non-Euclidean geometry and the one who described its *theory.*

Problem **15.2.10.1** is associated with *Lagrange's problem* in celestial mechanics.

The concept of an "attracting" point and a "repulsing" point in the *iteration method* was reflected in Problem **20.2.10.2** (all equations of the problem have the same form "iterating", so to say, the first equation).

The solutions of Problems **21.1.10.5** and **31.2.7.5** use the concept of the *world line* in time and space.

26.2.8.1 is a problem on *Young tableaux* used in the representation theory of symmetric groups.

29.2.8.3 and **29.2.9-10.3** are typical problems of the *information theory.*

Problem **30.2.10.1** is "the exchange transformation" from *ergodic theory.*

31.1.9.4 is the first problem in the *coding theory* ("the check for evenness" by Hamming); cf. also Problem **30.1.8.3**.

The question raised in Problem **47.10.2** is related to the one of the ways of *tight packing of information* in computer memory while Problems **45.7.1, 45.10.3** and **48.10.3** are indirectly connected with the *theory of algorithms and computations.*

Problem **48.9.5** was taken from the note-books of one of the greatest mathematician, Leonard Euler, and is connected with the *ideal theory.*

The fast speed of convergence to a fixed point (to $\sqrt{2}$) in Problem **49.8.3** was occasioned in the general case by *Newton's method* for finding roots of the arbitrary function $f(x)$.

Problem **49.10.5** is related to the theory of approximations of functions.

The list of examples can be extended further (e.g., to indicate some problems from the number theory) but unfortunately it is impossible to explain the idea in more detail if we want to remain on the high school level.

The school curricula have been changed several times for the last 50 years and new trends in the curricula immediately affected problems of Moscow Olympiads. So in certain years there were given problems on complex numbers, problems with a derivative, etc. Our solutions and hints correspond to the present school curriculum although it is worth saying that a different (often more cumbersome for the lack of an adequate language) solution was expected from the participants in some cases. For example, in Problem 15.2.10.1 we made use of the properties of the integral, well-known to today's school pupils, and gave a solution taking up just a few lines (in contrast to the two-page solution of this problem in [SCY].)

H.12. What makes Olympiads run. Despite the great help of enthusiasts, the compilation and selection of an olympiad's problems is one of the most arduous tasks in the work of the organizing committee. It is the subject for debate at a number of meetings that last for many hours and where the organizing committee members argue till they are blue in the face fighting for some problems and rejecting others. Problems may change beyond recognition before one's eyes; sometimes several seemingly quite different ideas are integrated into one problem but sometimes, on the contrary, one problem disintegrates into two or three others that may be from different mathematical disciplines.

When problems are selected for an olympiad, they have to be kept in secret, on the one hand, but, on the other hand, attempts are made to find out whether heads of circles (who never are members of the organizing committee) have ever given the same or similar problem to their disciples. This was always a delicate matter since a problem unknown before could be made public by chance and so spread widely among students. So the final selection of problems have always been quite difficult.

As the olympiad approaches, the 'problem rush' increases more and more. More often than not, the final list of problems is approved one or two days before the start but it has also happened that the list was typed during the night on the eve of an olympiad. So one should not blame the organizing committees of olympiads for a rushed work in this respect since it was the compilation of final variants literally on the eve of contests that made it possible to keep the problems in secret and also to take advantage of lucky discoveries made, as usual, at the last moment.

The complexity of problems at olympiads varied noticeably in different years. The most difficult problems of early olympiads which were solved by just a few participants, now look nothing out of the ordinary. The inquisitive reader will notice that the style itself of later problems has changed substantially as compared with that of the first olympiads. However, the complexity of the olympiad in each particular year has always been very high. Sometimes it was impossible to make variants easier however hard the authors worked on it. There have been some particularly difficult olympiads, including the 27-th (1964), 29-th (1966), 31-st (1968), and 35-th (1972). Nobody solved some of the problems in these olympiads (we can cite as an example Problems 29.2.8.5, 31.2.8.2, 31.2.8.3, and 35.2.9.3) and sometimes only one participant succeeded (e.g., Problem 35.2.9.1).

The jubilee 48-th Olympiad (50 years of Olympiads) can not be called very difficult; still, it had problems that none of the kids in the respective grade could solve (48.7.4, 48.8.5, and 48.9.5). But all this is exception rather than a rule; every problem at most olympiads was solved by at least one participant and there were difficult problems solved by many participants.

The spirit and nature of an olympiad, and the content and complexity of its problems were affected to a great extent by the professors of mekh-mat, who headed the organizing committee in different years and who were entrusted with this task by the Board of the Moscow Mathematical Society.

Here is the list of the Chairpersons of the Organizing Committee:

Olympiad	Year	Chairperson	Olympiad	Year	Chairperson
1	1935	P. S. Alexandrov	23	1960	I. R. Shafarevich
2	1936	N. A. Glagolev	24	1961	V. A. Efremovich
3	1937	A. N. Kolmogorov	25	1962	N. V. Efimov
4	1938	A. G. Kurosh	26	1963	A. N. Kolmogorov
5	1939	L. A. Lusternik	27	1964	I. R. Shafarevich
6	1940	L. S. Pontryagin	28	1965	N. V. Efimov
7	1941	A. O. Gelfond	29	1966	A. A. Kronrod
8	1945	I. M. Gelfand	30	1967	V. V. Nemytsky
9	1946	S. A. Galpern	31	1968	N. S. Bakhvalov
10	1947	I. G. Petrovsky	32	1969	V. A. Efremovich
11	1948	V. V. Nemytsky	33	1970	V. M. Alekseev
12	1949	A. I. Markushevich	34	1971	I. R. Shafarevich
13	1950	M. A. Kreines	35	1972	B. P. Demidovich
14	1951	B. N. Delone	36	1973	A. A. Kirillov
15	1952	P. K. Rashevsky	37	1974	V. I. Arnold
16	1953	D. E. Menshov	38	1975	A. N. Kolmogorov
17	1954	S. V. Bakhvalov	39	1976	A. V. Arkhangelsky
18	1955	G. E. Shilov	40	1977	V. A. Uspensky
19	1956	E. B. Dynkin	41	1978	Yu. I. Manin
20	1957	O. A. Oleinik	42	1979	V. M. Tikhomirov
21	1958	V. G. Boltyansky	43	1980	A. S. Mishchenko
22	1959	E. M. Landis	44	1981-??	O. B. Lupanov

After 1981, the prime of "stagnation period", and by inertia after it the Chairman was the Dean of mekh-mat, Prof. O. B. Lupanov.

The Chairpersons of the organizing committee usually did not overwork. The Vice-Chair, on the other hand, not only worked hard to arrange the event but also headed the numerous meetings of their organizing committees, etc.

The VC had several main assistants, including heads of school grades. Among VC 's numerous responsibilities were arrangements for lectures to be delivered to the school students, putting up posters of the olympiad, the run of the olympiad itself, publication of the collection of preparatory problems and blank forms for certificates to be awarded to winners, arrangements for the rooms where the olympiad is to be held, to be followed by reviews of problems and the awarding ceremony, etc. In addition, the VC had to participate in discussions of problems and help to conceive them. One of the authors of this book served as VC and can testify how difficult this job is.

The grade's managers are also very busy; in addition to helping the VC in organization, they must also select the problems they like best for their respective grades to be used later by the organizing committee in the final variant of problems for a particular school grade. They should also supervise the progress of the olympiad in their grade, go round the rooms and answer questions of participants (to an extent specified by the organizing committee in advance); organize the checking of the papers in their grades, and find the best papers presenting all ir decisions to regular meetings of the organizing committee.

When an olympiad is in progress, it is served by many university students who help the school pupils to find their rooms, sit in the rooms (answering questions if necessary to an extent specified by the organizing committee in advance), see to it that the corridors and toilets are not turned into discussion clubs by the participants, and collect papers at the end. Then heads of grades distribute the collected papers among the university students for checking and grading.

H.13. How grading is being done. The grading is the most important part of the job; it often happens that an interesting paper is read by several members of the organizing committee and those nominated for an award by all members. The diversity of demands placed on papers makes the opinion of each individual member of the organizing committee rather subjective and it is only the collective discussion makes the final decision correct and objective.

The papers are evaluated not in marks as at school but using a more flexible system of pluses and minuses. The marks that a solution may get are:

(0) there was no attempt to solve the problem;

($-$) the problem was not solved or solved incorrectly;

($-$?) the solution is wrong and contains very bad errors;

($\overset{-}{.}$) the problem was not solved but there are some reasonable considerations in the draft or in the clean copy;

($\mp$) the problem was solved incompletely but the approach is correct;

($\pm$) the problem was not solved completely;

$\overset{+}{(-)}$ the problem was solved but the solution contains small omissions or defects;

($+$) the problem was solved completely;

(!) the solution contains unexpected (and sometimes even unforeseen by the organizing committee) bright ideas.

Other marks are also used sometimes (e.g., $1/2$, ε, etc.). It should be noted that the mark (!) does not mean that the problem was solved. For example, there are often marks $-$!, $\cdot$!, $\mp$!, etc. However, even ($-$!) increases considerably the chance to get a prize.

Everything is taken into account when prizes are awarded, including the correctness of a solution, the clearness of the mathematical thought, originality of the solution, the completeness and fullness of the investigation, the nature of the description of fine points, etc. However, the handwriting and the tidiness in the arrangement of the material, as well as the general appearance of a paper are never considered, unlike the regular procedure in a usual school. The greatest importance is attached to non-standard reasoning, unexpected solutions, and the original interpretation of the conditions of a problem.

H.14. Anecdotes from the history of Moscow Mathematical Olympiads[1]. Olympiad is a great event for schoolchildren who are interested in mathematics. The faculty members and the students at the Department of Mechanics and Mathematics of Moscow University are barely able to cope with the multitude of questions fired at them by excited teenagers and sometimes by no less excited teachers:

"When will lectures be provided for participants in the Olympiad?"

"Are any consultations planned?"

"Is it only for the best pupils or for all?"

"Where can we get problems for practicing and how many of them suffice to be solved?"

"Will a boy be permitted to participate if he is only from the 6-th grade?"

"Can we bring textbooks with us?"

The stream of similar questions never stops.

• The desire to be as objective as possible and the great awareness of the organizing committee members of their duty sometimes resulted in curious situations. Consider just one such case. At the 21st Olympiad, a paper by Misha Khazen who had solved four problems out of five was nominated for the first prize. Unfortunately for him, Lida Khazen, Misha's sister, was a member of the organizing committee. She stated with assurance that Misha had known a solution of one problem before the Olympiad (although he solved it himself), that he was not going to apply to the Department of Mechanics and Mathematics anyway, and so he should not be awarded the first prize. The members of the organizing committee spent a lot of time trying to prove to Lida that the inclusion of a known problem into the Olympiad was the fault of the organizers but not Misha's, that the accidental relationship of Misha with one of the organizing committee members allowing them to learn what he knew and what he did not know put him in more difficult conditions as compared with the others, that the question of entering the University was of no importance, that, in general, they discussed the paper but not its author, and so on. Nothing helped. The poor girl was on the verge of crying and only the democratic procedure of voting (perhaps an hour or an hour and half after the debate has started!) made Lida agree.

• For a long while it was a custom to include in the final list of problems the one whose answer is the year of the current Olympiad. The following solution of one of the participants of the 33-rd Olympiad put an end to it: "At every Olympiad there was a problem whose answer was the year the Olympiad was held. In this Olympiad the problem I am solving is the only such problem. By the induction, the answer: 1970". (Cf. Problem 33.2.7.4.)

H.15. Who did what (very incomplete). Several generations of outstanding mathematicians have worked on the main material of this book — the problems — and some of the problems are really nice. To find the authors of most of the problems is impossible. Besides, part of interesting problems are results of brainstorms held at the meetings of the organizing committee, and so they have a collective author.

However, the most beautiful (in our opinion) and original problems were devised by individual authors and we are sorry that can not mention all of them. Such problems were widely spread first among the organizing committee, and, after the olympiad, became a mathematical folklore. Experts recognize them at once by their nicknames. Here are some authors of such problems (this list is incomplete in every sense; we hope that the authors not mentioned will not be offended):

N. N. Konstantinov: 17.2.7.5 (Triangular City) and 23.2.8.4 (Snail); S. A. Eliseev: 38.2.9.5 (Non-convex Cutting); D. B. Fuchs: 24.1.8.2 (Scalar Product), 27.2.11.5 and 31.2.9.2 (Fuchs' Arcs); G. A. Galperin: 33.2.7.6, 34.2.7.5, 38.2.10.4,

[1]Some of these stories might sound strange for the Westerner, more used (and sometimes prone) to esteem the law.

33.2.7.6 (Courtiers of King Louis), 38.2.7-10.1 (The 2s, 3s and 4s), 39.2.9.5 (Nickels) and 48.9.2 (The Airfields); S. B. Gashkov: 46.10.3 (The Turing Machine), 48.7.4 (Wolves and a Hare) and 48.10.3 (Complexity); B. D. Ginzburg: 23.2.9.3 (Knight's Way); A. I. Gruntal: 36.1.10.4 (The Polyhedron); V. Gurvich (a complete graph); V. G. Kac: 30.2.9.5 (A King of Spain); M. P. Kovtun: 34.10.5 (Matches); A. V. Klimov: 37.2.10.3 (Asteroids); S. V. Konyagin: 42.8-10.5 (Chemists & Alchemists); A. G. Kushnirenko: 33.2.10.1 (The Orange); O. V. Lyashko: 35.2.7.4 (The Knight-Errand); A. P. Savin: 33.1.9.1 (Extreme Kings); I. N. Sergeev: 48.8.4(Uncle Chernomor[2]); A. S. Shvarts: 24.2.9.2 (Shvarts' Matrix); A. C. Tolpygo: 30.2.7.3 ($q \cdot 2^{1000}$), 30.2.8.5 (moovies), 31.1.9.4 (telephones), 32.2.7.3 (a trickster) and 32.2.10.1 (wizards); A. L. Toom: 35.1.9.3 (The Forest); N. B. Vasilyev: 25.2.10.4 (The Box) and 26.2.8.5 (Gentlemen); A. V. Zelevinsky: 34.2.10.3 (Banker and Gambler).

One of the problems of the Pythagorus' Day (33.D.7.5), as it turned out, suddenly became quite popular outside the USSR. "Mathematical Gardner" [Kl] contains its generalization for the case of a "many-handed Ali-Baba" given in the section entitled "Entertaining Table-Turning". It said there that the problem visited first the pages of Scientific American in 1979, where it was published by Martin Gardner, a famous popularizer of mathematics, well-known to Soviet readers from a number of books and articles (see refs. [G1]–[G14]). However, Gardner admitted that he had got this problem from "Robert Tappey who believed that the problem had come to us from the Soviet Union." ([Kl]). Thus, Problem 33.D.7.5 has come a long way before returning home (anonymously), albeit in a generalized form.

The authors of the latest Olympiads, held in the copyright era, were mentioned explicitely on various liflets issued on the occasion, so we dutifully reproduce the information.

The list of authors is easy to extend but almost impossible to complete (let alone the fact that the above mentioned authors suggested far more problems from this book than we mentioned); some authors donated many problems without bothering for stacke claim (like Joseph Bernstein, who in his time solved all problems offered in Olympiads he participated). We apologize to all authors of problems for Moscow Mathematical Olympiads who are not mentioned.

[2]The Russian reformulation after a fairy-tale hero.

A little problem

I. Zverev

To Mark Scheinberg, a much honored student of the
9-th grade and the winner of many a mathematical olympiad,
with the highest respect from the author who can hardly
remember the multiplication table.

“Well, that’s it,” Leo said. “Settled and signed.”

“Who by?” Mashka asked.

“By myself,” said Leo solidly and looked at her in a severe schoolteacher’s way, his eyes like goldfishes behind his glasses. “Not enough for you?”

“And also by me,” Yura Fonarev added.

“O.K., count me in then,” Mashka sighed. ”I agree.”

“What do you mean ’you agree’? Nobody’s forcing you.” The boys looked at her with indignation.

“It would be too much if you forced me.” Now Mashka got angry. “Too much, really.”

“There, there,” Leo said soothingly. “You won’t regret it. You’ll be grateful. Do you know what kind of school it is?”

Mashka knew no less than they did. They had been there together on the Open-Day.

A week earlier, Ádochka (this meant Ariádna Nikoláyevna, their math teach), had informed Leo, their class genius, that there would be such an occasion and so she would advise him to . . . .

The three of them went together. The school was really amazing. The classrooms were not called classrooms but *auditoriums*, one of them even had a computer[1]. The lessons were called *lectures* and they were given not by schoolteachers but by professors from the University, among them even one full professor with the doctoral degree.

Of course, the students there might have stretched a bit but prodigies there really seemed to fulfill the freshmen’s and partially sophomore’s curriculum in the 9-th grade. So it would have been ridiculous for those who were not straight A students to even think of getting in.

Yura had two B’s, Leo had no B’s, but he had an uncomfortable C in German. Mashka, of course, had lots of B’s.

When the Open-Day was almost over and the boys were about to leave, there appeared the principal of this special mathematical school. He was a rather strange man, cross-eyed, with a big nose and wild hair, just like Leo’s but grey. He said right away that those who were not entirely straight A students, but talented nevertheless, shouldn’t give up; for it was the results of the Mathematical Olympiad that would count most of all.

It was, therefore, settled and signed that all three should go to the Olympiad on Sunday and take a chance.

Mashka didn’t think it was such a great idea to try and enter this mathematical school. She had other plans. She didn’t know exactly what, but no doubt they had nothing to do with mathematics. It wouldn’t have been fair, however, to leave the boys alone at such an important moment, so she would go and flunk, of course, but still give their morales a boost.

“Are you going too?” asked the surprised Ariadna Nikolayevna and immediately blushed. Perhaps she was afraid Mashka might get offended.

Adochka was very kind, and when she accidentally hurt somebody always suffered terribly. She started to worry and tried to soothe the offended.

“I’ll do it just to keep the boys company,” Mashka comforted her.

“Oh, no!” Adochka cried excitedly. “I’ve always said you are talented . . . just a bit lazy . . . But if you try and organize yourself you may gain . . . I mean, achieve . . . .”

She said nothing more because she was honest and knew well enough that Mashka could never gain anything in mathematics, let alone achieve.

Mashka’s father was also surprised. He said, “Oh, my!”

But since he held a doctorate in philosophy he felt he had to philosophize a little. So he told Mashka’s mother how wonderful it was that their daughter had chosen such a nice field of activity, where everything is simple and clear, directives are definite and not subject to frequent change.

“But in that field you have to have a regular head on your shoulders!” Mashka’s mother exclaimed and sent Mashka away to do her homework.

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[0] From *The Second of April* by Ilia Zverev, Soviet Pisatel Publishers, 1968

[1] The western reader should look at the year this had been written: at that time computers were at best discussed in the newspapers in Russia.

Leo's father got very excited when he heard about the Olympiad. He started to pace back and forth and rub his bald head that perhaps once grew the same kind of black wire-like hair that his wonderful son had now.

"Listen, Leo," he said at last, "you know, physics is somehow more promising these days. Perhaps there's rocketry physics or something?"

"So what?" Leo said condescendingly. "I, for instance, like math."

Still, Leo's father would be extremely sorry for his son to get involved in a second-rate science, or even a first-rate one, if it were not the main one.

"With your abilities," he cried, "you could ... "

"Enter a school where they teach how to run ministries," Leo prompted gloomily. "It's hard enough to enter this one. They take only one in twenty two."

His father immediately found another subject to worry about: "What if they don't enroll you, Leo? You must go to your headmaster," he said, "and to the Young Communist League, too, and get letters of recommendation from all of them. Make them write that you are one of the best students and a member of the committee ... and about the physics club that you are the monitor of ... "

"Oh, God," Leo said. "And that I bought a light bulb for the physics classroom with my own 30 kopeks. That's also a feature of my character that is a visible sign[1]."

"Don't show your wit here," his father ordered. "I've lived longer and I know better what plays sense in cases like this."

Leo's father was a musician. He played the trumpet and perhaps that's why he thought one could play anything, even sense. He was not too literate because he had joined an orchestra as a prodigy right after his fifth year in elementary school. Of course, now times were different. Prodigies had no privileges. On the contrary, they had to study five times as hard as all the others. Leo put all this into one sentence:

"Daddy, you are out of tune."

But after thinking it over he did decide to get the damn recommendation. It really was highly unlikely that anything of the sort would be required.

Finally, came the morning of the judgement day. That was how Yura chose to call it. For everyone else it was an easy Sunday morning but for 563 students "talented in mathematics", as they were formally called, that morning was most uncomfortable ... .

The boys crowded the wide University staircase decorated with statues of various bearded thinkers. Some of the crowd stood motionless, staring upward and silently moving their lips, perhaps praying or, much rather, solving problems. Others were nervously discussing tricks from the last Olympiad, and of the one before the last.

The girls stood separately. They were bespectacled and very serious. "Abstract", as Yura put it. One with a forelock was surprisingly cute. It was hard to understand what such a beauty needed mathematics for.

The most brave (or, more precisely, the most anxious) had the nerve to come with their parents, and now, shy and suffering, they received fatherly advice and motherly instructions.

"Most important, don't be nervous," a fat red-faced woman in a fur-trimmed coat kept saying to a fat pink-cheeked boy. "I beg you, Noughty!"

What a mathematical name, Noughty. Wonder, what his real name was? Arnold, perhaps? He was pretty nervous, that Arnold 'boychick'. He'd flunk just from fright. Well, actually everybody was rather nervous that morning. Even Yura and Leo, speaking frankly.

In the midst of this excitedly buzzing, breathing, stirring and even steaming crowd, two boys were distinctly out of place, like an iceberg. They were indifferently sitting on a step playing deadman. The older one, in glasses and ski trousers, lazily pronounced after each move: "Aha, oh, well, if you do that, we do this ... "

"That's Guzikov," Yura whispered respectfully. "Second prize at the National Olympiad." He sighed. "Of course, he can do whatever he wants now, even play deadman."

At last, a tall young man carrying a briefcase appeared at the entrance. He made a frightening grimace and shouted:

"Welcome, friends! We are starting."

Everybody began to push one another and loudly tramped their way through the shining marble hall into a very big room. Only Leo lingered at the entrance before an enormous sheet of white drawing paper. It declared:

"STUDENTS! ADDRESS YOUR QUESTIONS TO A. KONYAGIN, ROOM 9".

Leo just had one. He went to Room 9.

A. Konyagin, the question authority, turned out to be the young giant who had just shouted, "Welcome, friends!" He again made a bestial face and said in a very kind voice:

"Please, ask. I'm listening."

Leo thought this young man could be one of the poor "antipeople" Ryasha told stories about. Ryasha was a dreamer of course, and liked to fib but this particular story sounded real. He said there were such "antipeople". Ryasha even remembered their Latin name, very impressive — "homogeneous lupusest". They could never do what they wanted. If such an antiguy, say, wanted to cry, he would laugh instead, and if he wanted to run around, he would go to bed immediately instead. Ryasha swore that it was a quite established scientific phenomenon, well-known in medicine. He might know, after all, since both his parents were doctors.

"Well, what is it?" Konyagin got angry and his face turned accordingly kind. "Speak up!"

Leo asked his question. "Do they require letters of recommendation? What other papers are needed?"

"Papers? That's where your papers are!" Konyagin knocked on his protruding forehead. "Here is your recommendation and reference, and permit. Clear? Then, go ahead!"

In the big room, called auditorium No. 1, stood twenty rows of benches. Very long benches they were, and each had a desk in front of it.

[1][of a Communist morale]. A word from mass-media cliché of that time; (like pledge of allegiance in American schools).

“Two at each desk, not more,” said the question man Konyagin and headed down the aisle. Five scientists, also young and looking very important, went after him, distributing paper.

“The sheets are stamped,” Konyagin said as he walked. “Don’t even think about cheating. No way! We are not so old here. We still remember all the tricks ourselves. Mind that!”

“So it was OK with you?” someone squeaked challenging. Perhaps it was that Noughty one, that pink boychick with the mathematical name.

“But I never cheated in math!” said the question man proudly, at which his assistants burst out laughing for some reason.

Each had his own problem. No ordinary problem about the Collective Farm “Shining Path” that bought two tractors and three vans while the Collective Farm “Dawn” acquired seven tractors, etc. No, these assignments were quite different.

Yura had one about King Arthur and his knights. Knowing that each knight was at war with half of the others, how should King Arthur’s right hand man, Sir Lancelot, arrange them around the table so that no one should sit beside his enemy?

“What have you got?” Yura asked Leo. Of course, he had to know about Leo’s problem first.

Leo had a problem about chess players. Eight chess players took part in a competition and each finished with a different score. The second best had the same score as the four worst combined. What was the score of the game between the fourth and the fifth?

Leo thrust his fingers through his wild hair and began to breathe, moo and blink. This meant he was starting to work.

“Let’s reason!” he persuaded himself aloud. “Let’s think logically and calmly. Each of these guys played with each other and either won, lost or drew. So the first one ... But what am I doing?” Leo interrupted himself noticing that he plunged into his business while his friend might be in trouble. “So, how many knights do we have?” He said it just like that, we.

Mashka, of course, could solve nothing but she could not go away because the boys might think that she had solved her problems before them and feel uneasy. The possibility was pretty hypothetical: of course, the old friends could guess that Mashka’s poor math wouldn’t work here. Still, she was pleased to think that, sitting there, she could somehow inspire these budding Euclids and Lobachevskys.

She just shuffled her clean sheets of paper with purple official stamps and looked around at the people. There was a lot to see for a detached observer. The great Guzikov wrote his figures as if he were playing piano. He thrust his head upwards, raised his eyebrows and even jerked in rhythm to an inner music. Noughty was strangely calm. His pink face shone with satisfaction. Perhaps he had been lucky enough to draw an easy problem.

Occasionally, a boy went to Konyagin and whispered for permission to go to the bathroom.

“Leave your pen here,” said the question man to one. “You don’t need a pen in there, do you?”

As more of the boys asked to go, the assistants looked at each other meaningfully demonstrating that, of course, they knew the secret aim of those visits though the aim might not be secret but quite a natural one. After all, the Olympiad lasted five hours.

Everybody, except Mashka, was suffering, writing or thinking. The cute girl with the forelock — Mashka could swear she would solve nothing and had just come to show off her beauty to the young intellectuals — well, she was also writing and even confidently and merrily.

She could hardly make anything out by looking at Yura and Leo, though naturally she was looking at them most of all. They were whispering, looking into each other’s notes and arguing.

Unfortunately, not only Mashka saw that. Every now and then Konyagin looked at the friends and shook his head making his antiface and antismiles. The boys continued whispering, writing and whispering again. The fools obviously forgot where they were ...

It all ended rather sadly. When Yura and Leo handed in their papers — not among the first, but far from the last — the question man gave them a fierce smile, took out an enormous red marker and slashed on every sheet.

“Mark my word, something terrible is going to happen,” said Mashka.

But the boys were filled with joy of victory. They didn’t want to listen to reason. They jumped, nudged each other, and shouted, because they had solved all their problems.

The joy reached the two families. Leo’s father was extremely happy but having regained self control he claimed that there was nothing to be glad about, it was quite natural, and he had expected nothing less from his son, whom he knew as well as he knew himself. He was much happier about the system where no papers were required, where they just said “go ahead, show what you can do, and that’s your whole file”.

Of course, his father liked this system because he had always had to write in his application forms: “Education: incomplete secondary school”, and some other things on top of that[1].

Fonarev’s father, having heard Yura’s account, silently took his wonderful *Poliot* watch (written just like that, not in the usual Cyrillic; but export make) off his wrist and gave it to his son. A quarter of a century had passed since his last arithmetic class but his horror of the science had hardly diminished. Every other year or so, Fonarev’s father had the same nightmare: his redheaded math teacher, Faina Yakovlevna, a swimming pool with two pipes, and Berezanskaya’s book of arithmetical problems.

“Yes, Yurka,” he said, “a regular guy you are, that’s it! Nothing more to add.”

Mashka didn’t want any lengthy explanations. She just told her parents that she had not gone to the Olympiad, she had merely changed her mind at the last moment, that’s all.

“But what did you do all Sunday?” Her mother was horrified.

“I was busy with my Russian,” replied Mashka.

“She means she was speaking Russian and no other language,” remarked her father sarcastically.

Two more weeks passed and again they went to the University. In the same auditorium No. 1, at the presidium table were three Academicians, the hairy principal of the special mathematical school, a representative from the municipal board of

[1] In the standard questionnaire the line preceding the question on education required to state the ethnic origin which must have been a trial for him judging from his (manifestly Jewish) name and appearance.

education, and other officials. A. Konyagin was no longer in charge. He was somewhere in the seventeenth row with all the other assistants who turned out to be just graduate students helping to run the Olympiad.

It was a ceremony held to mark the results of the competition of mathematicians. The representative of the education department read a speech in which he emphasized achievements and pointed to some isolated shortcomings.

"We are also concerned," he mumbled indifferently, "about the level of education in some schools." At this point he at last looked up and said sternly: "No, comrades, we are not alarmed. But, comrades, neither are we satisfied."

And then, the chief Academician rose and handed awards to the winners. It turned out that the great Guzikov got only the second prize. So did the cutie with the forelock. The first prize went to that pink Noughty, the mother's darling. The boys thought they knew people well but they made a mistake underestimating him.

That psychological miscalculation was not their worst disappointment. The list of winners was apparently about to be exhausted and the friends hadn't been called yet. At last the chairman finished announcing the winners. He folded his sheet in two and then in two again and before leaving said:

"And, well ... a Fonarev and, a mmm ... Makhervax are requested to come to Room 9."

I don't like to describe what happened in Room 9. There weren't any Academy members there, only the principal of the mathematical school and a representative of the education board, the one who was neither alarmed nor satisfied. Konyagin was also there and kindly smiling started to lecture the boys:

"What shall we do with you? Whom should we give the prize? You solved all your problems properly but you were whispering all the time and we don't know which of you did what."

The boys started to explain that they had solved the problems together, they always did everything together, there was no crime in that because history is full of such cases. They recalled Pierre and Marie Curie, or, say, Lomonosov and Lavoisier, though they were not quite sure about the latter.

"Well, stop it," said the representative, "it's a matter of principle. The Olympiad was for individual work and prizes are given to individuals. We have discussed this with the comrades and decided as follows. You work out who deserves more and we'll give him the award. The other will have to pass the entrance exams on the regular basis."

"Here he is, Fonarev," Leo prompted immediately.

"Write Makhervax in," Yura shouted, regretting that Leo was the first to shout the right thing.

"There, there," said the hairy principal. "Go and think. Come back tomorrow morning with your decision."

When Leo's father heard what had happened, he declared that he wouldn't let it go just like that. To him it was a pure crime to prevent the country from having two geniuses instead of one.

"Why 'prevent'?" Leo felt bad but not bad enough to compromise with his conscience. "It's all right, I'll just have to pass the exams like everyone else."

Leo's father, who used to be a prodigy, as you might remember, didn't like this "like everyone else" at all. He said that it was not Leo's business to discuss such serious matters; he would go to Yura's parents and they would settle everything as serious adults should.

So the parents started to decide. First, they spoke about the weather, then about soccer. At last Yura's father decided that the stress was too much and he said with dignity that he knew his son well, was sure about him and that Yura would reach his goal whatever might happen.

Leo's father readily agreed that Yura was a strong personality, no obstacles could stop him, while Leo, of course, was rather unstable, no athlete, and wore glasses. So it would really be better to make it easier for Leo since Yura, as his father so rightly noted, would manage anyway.

Here Fonarev's father cried, "Oh!" because his wife had stamped on his toe under the table.

"No," she said bitterly, "our Yura only looks strong. And his marks are not so stable. Two B's, you see. But yours is doing just fine. Of course it will be easier for him."

"But he has C in German," Leo's father cried. "Do you understand, C! That's much worse than your two B's."

All three looked at each other rather ashamed. Somehow their talk was strange, even uncivilized.

"Katya, look to the kettle, please. I think it's already boiling," Fonarev's father said crossly to Fonarev's mother.

"Yes, really," sighed Leo's father. "It turns out rather unseemly."

"Yes," agreed Fonarev's father, "looks foolish. "Then he suddenly beamed and suggested: "Let's settle it fairly! Heads or tails."

"Heads or tails?" repeated Leo's father with doubt. "Huh ... all right then, tails."

Fonarev took a coin out of his pocket, put it on his big black-rimmed nail, the coin flipped in the air several times and landed on the table.

"Heads!" shouted the lucky one. Leo's father shrugged and sighed.

"Maybe you think I cheated?!" asked Fonarev warily.

"No," Leo's father said sadly. "I didn't think that."

They didn't talk for quite a long time until the kettle, which had not been about to boil, was ready at last.

But there was no reason for them to be so sad.

Everything was wonderfully settled already. Perhaps not too wonderfully but settled nevertheless. Yura and Leo accompanied by Mashka went to Room 9 and took the award paper from Konyagin. The paper had "Fonarev" on it because Leo had managed to shout Yura's name first.

When the boys came out of the building, they tore the paper in halves. First they wanted to tear it in three, because they said Mashka had a right to a piece, but she protested. She said it was a token of their friendship, hardened in battle in which she, although a friend, hadn't taken part. Mashka said she would sew them special safe bags that they could hang around their necks and hide under their shirts on most festive occasions. The boys nodded. All three thought it was an excellent idea. It was quite proper for real knights, even for those they had helped to seat around King Arthur's table.

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Contents

Part 2: Solutions

HINTS

For unknown words and theorems consult *Notational conventions and prerequisites.*

1.1.A.2. Let CD be the bisector of angle $\angle C$ in triangle ABC. Draw segment BE parallel to CD from point B to a point E on the extension of side AC, see Fig. 1. Then $BE = \frac{b}{a}l$ and $\triangle BCE$ is an isosceles triangle.

FIGURE 1. (Hint 1.1.A.2)

FIGURE 2. (Hint 2.2.2)

1.1.A.3. The angle θ to be found satisfies the condition $\tan\frac{\theta}{2} = \frac{\tan\frac{\alpha}{2}}{\sin\varphi}$.

1.1.B.3. Let α be the angle at the vertex. Then $\sin\frac{\alpha}{2} = \frac{1}{\sqrt{2}}\cos\alpha$ and $V = \frac{1}{3}a^3\cos\frac{\alpha}{2}$.

1.1.C.2. The area of the triangle is a half product of its perimeter by the radius of the inscribed circle.

1.1.B.2. If $b \neq 0$, then $x \neq y$, $x \neq 0$, $y \neq 0$ and we can divide the first equation by the second one term-wise. Further, denote: $t = x/y$; we get the equation for t:

$$t^3 - 2t^2 + 2t - 1 = 0. \qquad (*)$$

Clearly, $t = 1$ is a root of $(*)$. This root, however, does not suit us as is easy to check; dividing $(*)$ by $t - 1$ we get a quadratic equation without real roots.

1.1.B.3. $1^3 + 3^3 + \ldots + (2n-1)^3 = S_{2n} - 8S_n$, where $S_n = 1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$.

2.2.1. $x^5 + y^5 = (x+y)\left((x+y)^4 - 5xy(x+y)^2 + 5x^2y^2\right)$.

2.2.2. In the angle inscribe a circle tangent to the angle's legs at a distance of p from its vertex. Prove then that a straight line which cuts off the angle a triangle of perimeter $2p$ is tangent to the circle.

2.2.3. Let x and y be the lengths of sides. Prove that xy is divisible by 3 and 4. To do this, prove first that the remainder after the division of a perfect square by 3 or 4 can only be equal to either 0 or 1.

2.2.4. If $10^6 = (2^{\alpha_1}5^{\beta_1})(2^{\alpha_2}5^{\beta_2})(2^{\alpha_3}5^{\beta_3})$, then $\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 = 6$. For the number of partitions of 6 into the sum of three numbers see the solution of Problem 1.2.C.2.

3.2.1. Let A, B, and C be given points, O_1, O_2, and O_3 the centers of the circles. Connecting all these points as on Fig. 3, we get a hexagon $AO_1BO_2CO_3$ in which $AO_1 = O_1B$, $BO_2 = O_2C$, $CO_3 = O_3A$ (as radii) and $\angle O_1BO_2 = \angle O_2CO_3 = \angle O_3AO_1 = 90°$ (the circles are perpendicular; hence, so are their radii drawn to the intersection points).

FIGURE 3. (Hint 3.2.1)

FIGURE 4. (Hint 4.2.1)

Denote: $\alpha = \angle BCO_2 = \angle CBO_2$, $\beta = \angle ACO_3 = \angle CAO_3$, and $\gamma = \angle ABO_1 = \angle BAO_1$. If the center O_3 lies on the same side of AC as $\triangle ABC$, we will count the angle β as negative and positive otherwise, see Fig. 3; we will similarly treat angles α and γ.

Now, we have

$$\alpha + \gamma + \angle ABC = 90°, \quad \alpha + \beta + \angle ACB = 90°, \quad \beta + \gamma + \angle BAC = 90°. \tag{$*$}$$

These equations together with the condition

$$\angle ABC + \angle ACB + \angle BAC = 180°$$

allow one to easily find and construct

$$\alpha = \angle BAC - 45°, \quad \beta = \angle ABC - 45°, \quad \gamma = \angle ACB - 45°.$$

3.2.3. Prove that all diagonals of the n-gon intersect each other at $K = \binom{n}{4} = \dfrac{n(n-1)(n-2)(n-3)}{24}$ points. Indeed, with each intersection point of two diagonals one can associate a quadrilateral with vertices at the endpoints of the diagonals and this correspondence is one-to-one. Therefore, K is equal to the number of ways to choose 4 vertices of n given ones, as stated above. The diagonals of the n-gon divide it into polygons. It follows that the total number of vertices of these polygons is equal to $4K + n(n-2)$, and the sum of all angles of these polygons is equal to $K \cdot 360° + (n-2) \cdot 180°$. Now, it is easy to deduce the formula required.

4.1.? Some of the points should lie on one side of the plane, some on the other side.

4.2.1. The composition of two symmetries with respect to points X and Y is a translation by the vector $2\overrightarrow{XY}$, see Fig. 4.

4.2.2. If we have already drawn $m-1$ planes, then the m-th plane adds as many new parts as there are parts in it after its intersection with all other $m-1$ planes. Thus we have reduced our problem to a similar (but simpler) one: *what is the greatest number of parts that m straight lines divide a plane into*? The answer to this auxiliary problem is $2 + 2 + 3 + 4 + \ldots + m = \dfrac{m^2 + m + 2}{2}$, cf. the solution of Problem 1.2.B.3.

5.1.1. Use the identity

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

5.1.2. Construct the regular pentagon $ABCDE$ inscribed into the unit circle centered at O and such that $A = (1, 0)$. Then the sum of the vectors OA, OB, OC, OD, OE is equal to $\vec{0}$ and so is the sum of their projections $OA' = OA$, OB', OC', $\ldots$ to OA-axis. But $OB' = \cos\frac{2\pi}{5}$, $OC' = \cos\frac{4\pi}{5}$.

5.1.4. Let $y = \sqrt{a + x}$. Then $x = \sqrt{a - y}$. Squaring these equations and subtracting one from the other we get $(x + y)(x - y + 1) = 0$. The rest is clear.

5.2.1. First, factor the expression $a^{15} - 1$ in two different ways:

$$\begin{aligned} a^{15} - 1 = (a^5)^3 - 1^3 &= (a^5 - 1)(a^{10} + a^5 + 1) \\ &= (a - 1)(a^4 + a^3 + a^2 + a + 1)(a^{10} + a^5 + 1); \\ a^{15} - 1 = (a^3)^5 - 1^5 &= (a^3 - 1)(a^{12} + a^9 + a^6 + a^3 + 1) \\ &= (a - 1)(a^2 + a + 1)(a^{12} + a^9 + a^6 + a^3 + 1). \end{aligned}$$

Hence,

$$\begin{aligned} a^{10} + a^5 + 1 &= (a^2 + a + 1) \cdot \frac{a^{12} + a^9 + a^6 + a^3 + 1}{a^4 + a^3 + a^2 + a + 1} \\ &= (a^2 + a + 1)(a^8 - a^7 + a^5 - a^4 + a^3 - a + 1). \end{aligned}$$

5.2.6. Reduce the problem to a similar one about circles on a plane. In case of n spheres the answer is $\frac{n(n^2 - 3n + 8)}{3}$. (See also Problem 4.2.2.)

6.1.7-8.4. Let r be the radius of the given circle and R the radius of the circle to construct. Let us draw two circles with radii $|R \pm r|$ concentric with the one needed; these circles pass through the center of the given circle and are tangent to the straight lines parallel the given one and passing at distance r from the given line.

6.1.9-10.1. Factor $x^3 + y^3$ and set $z = x^2 + y^2$. We get the equation $3z^2 - b^2z - 4b^4 = 0$. Now use Vièta's theorem.

6.1.9-10.2. Consider

$$\underbrace{1\dots 9}_{\text{9 figures}}\ \underbrace{10\dots 99}_{\text{180 figures}}\ \underbrace{100\dots 999}_{\text{2700 figures}}\ \underbrace{1000\dots 9999}_{\text{36000 figures}}\ \underbrace{10000\dots 43578}_{\text{167895 figures}}\ 435\underline{\underline{7}}9.$$

6.1.9-10.3. If the four points do not lie on one circle, then the circle to be constructed divides them into two nonempty sets, one set inside the circle and one outside it. The number of solutions in the general position is equal to the number of divisions of the four-element set into two nonempty subsets regardless of the order, i.e., in the generic case there are $(2^4 - 2)/2 = 7$ solutions. This is the number that was expected for the answer at the Olympiad. In Solutions we will also consider singular cases.

6.2.7-8.3. At every vertex of the tiled surface four quadrilaterals should meet, every one with a different angle at the vertex. To consider this, translate one quadrilateral, $ABCD$, parallelly by various vectors $k\overrightarrow{AC} + n\overrightarrow{BD}$ for arbitrary $k, n \in \mathbb{Z}$, see Fig. 5.

FIGURE 5. (Hint 6.2.7-8.3)

6.2.9-10.2. Prove by induction that $(3n)! > n^{3n}$.

REMARK. According to an approximate (Stirling's) formula $m! \approx (m/e)^m$, where $e = 2.718281828...$. For our purposes it is only relevant that $e < 3$.

6.2.9-10.3. Prove that the heights of triangle $\triangle O_1O_2O_3$ are the midperpendiculars of the sides of triangle $\triangle ABC$.

6.2.9-10.5. Remainders after a division of $2^x - x^2$ by 7 repeat with a period of 21.

7.1.7-8.2. Divide 524 000 by $7 \cdot 8 \cdot 9 = 504$ with a remainder.

7.1.9-10.5. Consider the intervals $]-\infty, -1]$, $[-1, 0]$, $[0, 1]$, $[1, 2]$, $[2, \infty[$.

7.2.7-8.1. See Hint to Problem 7.2.9-10.1.

7.2.7-8.4. Substitute $x = 10$ and $x = a$. We get $b, c = -9$ or -11; $a + b = \pm 1$, and $a + c = \pm 1$.

7.2.7-8.6. Points H_1, H_2, H_3 lie on the circumscribed circle, see Fig. 6. (Proof of this fact is contained in solution of Problem 22.2.7.2.) Besides, we have (see Fig. 6):

$$AH_2 = AH_3, \quad BH_2 = BH_1, \quad CH_1 = CH_3.$$

Therefore, the vertices A, B, C we are looking for are the intersection points of the midperpendiculars to segments H_1H_2, H_2H_3, H_3H_1 with the circle circumscribed around $\triangle H_1H_2H_3$.

FIGURE 6. (Hint 7.2.7-8.6)

7.2.9-10.1. Consider the squares adjacent to the smallest one.

7.2.9-10.2. Consider the disc of the smallest radius r with all given points inside it or on the boundary. The points on the boundary — i.e., of the circle — divide the disc into arcs of measure $\leq 180°$. Indeed, prove on your own that otherwise we can shrink the disc.

If on the circle there are 2 or 3 such points, then by the hypothesis it is possible to cover them with a unit disc. But our disc is the minimal one, hence, $r \leq 1$.

If there are ≥ 4 such points on the circle, apply the following Lemma.

Lemma. *Let $n \geq 4$ points lie on a circle. Then 3 of these n points divide the circle into arcs, each $\leq 180°$.*

7.2.9-10.3. Let $x(x-a)(x-b)(x-c)+1 = P(x)Q(x)$. The values polynomials $P(x)$ and $Q(x)$ assume at points $0, a, b, c$ are equal to ± 1. Cf. Problem 7.1.7-8.5.

7.2.9-10.6. We can find legs a, b of any right triangle from the relations

$$a^2 + \left(\frac{b}{2}\right)^2 = m_b^2, \quad \left(\frac{a}{2}\right)^2 + b^2 = m_a^2.$$

8.1.7-8.2. Replace each term of the sum with $\frac{1}{2n}$.

8.1.7-8.3. The sum $\overline{ab} + \overline{ba}$ is always divisible by 11. Hence the answer: 29, 38, 47, 56, 65, 74, 83, 92.

8.1.9-10.2. The numbers $N^k - N$, in particular $N^2 - N$, are divisible by 1 000 for any k.

8.1.9-10.3. The sets of the solutions of the first and second equation of the system are, respectively, a pair of straight lines intersecting at the origin, and a circle whose center slides along the x-axis as the parameter a varies.

8.2.7-8.1. Calculate separately the sums of the units, tens, hundreds and thousands of all these numbers.

8.2.7-8.3. Draw a straight line through point C parallel to BP, see Fig. 7. Clearly, $\frac{AP}{AD} = \frac{1}{n}$ implies $\frac{AQ}{AC} = \frac{1}{n+1}$.

FIGURE 7. (Hint 8.2.7-8.3)

8.2.9-10.1. The given equation can be rewritten in the form $(x-5)(y+3) = -18$.

8.2.9-10.2. Show by induction that the assertion of the problem may be reduced to the following statement: *If* $2\sin\alpha = b$, *then* $2\sin(\frac{\pi}{4}a_1 + \frac{\alpha a_1}{2}) = a_1\sqrt{2+b}$.

9.1.7-8.1. Otherwise the sum of the external angles is greater than $360°$.

9.1.7-8.4. Adding together all equations we get

$$3(x_1 + \cdots + x_8) = 0. \tag{$*$}$$

If we add together the 1-st, 4-th and 7-th equation, we get

$$2x_1 + x_2 + x_3 + \cdots + x_8 = 1. \tag{$**$}$$

Now $(*)$ and $(**)$ imply that $x_1 = 1$. The other unknowns can be found in a similar way.

9.1.7-8.5. Let $P(x)$ be the given polynomial. Then $P(-x) = P(x)$.

9.2.9-10.3. Use the fact that set S does not change when segments AB, CD, and EF move any distance along the sides of triangle PQR.

FIGURE 8. (Hint 9.2.9-10.3)

FIGURE 9. (Hint 10.1.7-8.4)

9.2.9-10.4. First, prove that all routes have the same number of stops, say n, and exactly n routes pass through every stop. Deduce from this the equation $n(n-1)+1=57$.

REMARK. This problem arises from projective geometry. Let the stops be points in a plane and the routes be straight lines. Then the first condition means that a straight line passes through any two points and the second one states that any two straight lines have exactly one intersection point (this condition is not met in the "ordinary", i.e., Euclidean geometry but takes place in the projective geometry). It is clear now that the only difference between the "geometry" of this problem and the ordinary projective geometry is that the former has a finite (57) number of points on the plane; every straight line has 8 points. Such a geometry actually exists. Cf. also Problem 9.2.7-8.5.

10.1.7-8.4. Look at Fig. 9. The straight lines coming from the vertices of the pentagon will cross its five sides if M is within region I; three sides if M is within region II; one side if M is within region III.

10.1.7-8.5. These three circles are obtained by reflecting the circumscribed circle of $\triangle ABC$ through the sides of the triangle.

10.1.9-10.2. It suffices to find among 2, 3, 5, 7, 11, 13 a number which has no common divisors with the remaining 15 numbers. (Cf. Problem 10.1.7-8.2.)

10.1.9-10.5. It suffices to consider the first five factors.

REMARK. In general, the following *Euler's identity* holds:

$$\prod_{n=1}^{\infty}(1-x^n)=\sum_{k\in\mathbb{Z}}(-1)^k x^{(3k^2+k)/2}.$$

for more information on combinatorial identities see [F], [K].

10.2.7-8.3. Consider the *convex hull* of all 5 points. If it has more than three vertices, everything is clear. If it is a triangle, two of the points are inside it. Draw a straight line through them.

10.2.7-8.4. If it is possible to cut a convex polygon into parallelograms, every side of the polygon has a parallel counterpart. Hence, such a polygon has an even number of sides.

10.2.7-8.5. There are two numbers among the given 101 numbers whose ratio is a power of 2.

10.2.9-10.3. Connecting point A_9 with all vertices of the square we form four triangles. Consider the two cases:

1) each of these triangles contains exactly one of the points A_5, A_6, A_7, A_8;
2) at least one of the triangles contains two or more points.

10.2.9-10.4. As in the solution of Problem 10.2.7-8.5, each of 100 numbers can be represented in the form $x_i=\frac{a_i}{2^{k_i}}$, where $100<a_i\leq 200$, $k_i\geq 0$. If among the a_i's there are equal ones, then the corresponding numbers x_i form a pair desired.

11.1.9-10.2. Make use of the fact that $\sqrt{10}\approx 3.16>\pi$.

11.2.7-8.1. Suppose $y>x$. Let $x=p_1^{a_1}\cdot\dots\cdot p_k^{a_k}$, $y=p_1^{b_1}\cdot\dots\cdot p_k^{b_k}$. Since $x^y=y^x$, it follows that $ya_i=xb_i$. Therefore, $a_i\leq b_i$, i.e., $y=kx$, where k is an integer. After obvious transformations we obtain the equation $x^{k-1}=k$.

11.2.7-8.2. Through every vertex of the triangle draw the straight line parallel to the opposite side of the triangle. We obtain a triangle similar to the given one with similarity coefficient 2, see Fig. 10.

FIGURE 10. (Hint 11.2.7-8.2)

11.2.7-8.3. The point symmetric to one of the centers of symmetry with respect to another one is also a center of symmetry.

11.2.9-10.2. First, prove that the center of the largest circle coincides with the center of the cube. Then consider the sphere for which our circle is the equator and consider the parts of the sphere cut off by the cube's faces. Show that if the radius of the sphere is greater than $\frac{a\sqrt{6}}{4}$, then any plane passing through the center of the cube intersects at least one of these 6 cut off parts.

REMARK. There are four different circles of radius $\frac{a\sqrt{6}}{4}$ in a cube of edge a (one circle for each diagonal of the cube).

12.1.7-8.1. Consider the prime factorization of 26 460 and prove that the given expression is divisible by $5 \cdot 7^2$ and by $2^2 \cdot 3^3$.

12.1.7-8.3. The greatest power of 2 that divides the right side of the equation is greater than the greatest power of 2 that divides the left side.

12.1.9-10.1. If x, y, z, u are all odd, then the left-hand side is divisible by 4 while the right-hand side is divisible by 2 only. But in all other cases, i.e., when not all of x, y, z, u are odd, the other way round, the right-hand side is divisible by a higher power of 2 than the left-hand side; cf. Hint to Problem 12.1.7-8.3.

12.2.7-8.3. Step 1: prove that the difference between the weights of any two weighs is an even number of grams. (This follows from the fact that the sum of weights of *any* 12 weighs is an even number.)

Step 2: prove that this difference is divisible by any power of 2. (To do this subtract from the weight of each weigh the weight of the lightest weigh, divide the difference obtained by 2, and repeat the first step.)

12.2.7-8.4. Let us take six equal weights and place them at the vertices of the hexagon. Then the intersection point of the medians of any of our triangles coincides with the center of mass of the system of weights.

12.2.9-10.2. A $2 \times 2 \times 2$ cube can be composed of the "bricks" after two centrally symmetric figures shown on Fig. 11have been constructed.

FIGURE 11. (Hint 12.2.9-10.2)

12.2.9-10.4. Let M be the given combination of digits. Consider M as a number. Prove that one of the segments $[\log M + k, \log(M+1)+k]$, $k = 1, 2, \dots$, contains at least one of the points $\log 2, 2\log 2, 3\log 2, \dots$ by *Dirichlet's principle.*

12.2.9-10.5. Construct a square of perimeter twice as long as that of the initial one, with the same center and the sides parallel to the sides of the initial one. Prove that each of the juxtaposed squares covers not less than $\frac{1}{8}$ of the contour of the square constructed.

13.1.7-8.1. If the circle passes through n squares, then it passes through n nodes. By connecting the nodes with segments we get an n-gon with angles of 135° and 90°. Therefore, $n \leq 8$.

13.1.7-8.3. First, prove the following theorem.

Let a line l be drawn through an intersection point, A, of two given circles with centers O' and O''; let B' and B'' be the other intersection points of l with the circles. Let O be the second intersection point of the circles. Then $\angle B'OB'' = \angle O'OO''$ (i.e., the circles intercept on any line passing through A a segment that subtends a fixed angle with vertex at O).

13.2.7-8.2. $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{99}{100} = \frac{1}{100} = \left(\frac{1}{10}\right)^2$.

13.2.7-8.4. Denote by K_n the number of ways to connect $2n$ points on a circle by n non-intersecting chords. Prove that

$$K_n = K_1K_{n-2} + K_2K_{n-3} + \ldots + K_{n-2}K_1 + K_{n-1}.$$

This gives

$$K_2 = 2,\ K_3 = 5,\ K_4 = 14,\ K_5 = 42,\ K_6 = 132,\ \ldots,\ K_{10} = 16\,796.$$

For an arbitrary n the answer is $K_n = \frac{1}{2n+1}C^n_{2n+1}$.

13.2.9-10.3. Let A, B, C, D be the vertices of the quadrilateral, and K, L, M, N the tangent points to the sphere on, respectively, AB, BC, CD, DA, see Fig. 12. First, prove that segments KL and AC lie on one plane and that MN and AC lie on one plane (perhaps, on another one). Prove then that either both KL and MN are parallel to AC, hence, to each other, or they intersect AC at one point.

FIGURE 12. (Hint 13.2.9-10.3)

14.1.7-8.2. Apply several times the following Lemma.

Lemma. *In triangles ABC and $A'B'C'$, let $AB = A'B'$, $AC = A'C'$, and $\angle A > \angle A'$. Then $BC > B'C'$.*

14.1.7-8.5 and **14.1.9-10.4.** If $k \cdot 2^k \le n \le (k+1)2^{k+1} - 1$, then for an n-link chain k breaks will do but $k-1$ will not. The optimal for $k-1$ breaks is:

$$k + 1 + 2 \cdot k + 1 + 4 \cdot k + 1 + \cdots + 1 + k \cdot 2^{k-1} = (k-1) + k \cdot (2^k - 1) = k \cdot 2^k - 1.$$

14.1.9-10.1. This number is greater than $\frac{0.12345}{0.51505}$ and less than $\frac{0.12347}{0.51504}$.

14.1.9-10.2. The sum of the internal angles at side a is $120°$.

14.2.7-8.2. On the sides of $\triangle ABC$ construct three similar triangles ABC_1, AB_1C, A_1BC, with angles of measure $180° - \angle D$, $180° - \angle E$, $180° - \angle F$, respectively; see Fig. 13. Show that each of straight lines AA_1, BB_1, CC_1 passes through point O. To do this check that the circles circumscribed around triangles ABC_1, AB_1C and A_1BC pass through O.

14.2.7-8.3. Prove by induction that this sum is equal to $(1 + 2 + 3 + \ldots + n)^2$.

REMARK. The statement of this problem was used in the solution of Problem 1.2.B.3.

FIGURE 13. (Hint 14.2.7-8.2)

14.2.7-8.6. Observe that $P(\pm i) = 0$. Hence, $P(x)$ is divisible by $x^2 + 1$. So

$$P(x) = (x^2 + 1)(x^2 + x + 1).$$

Now, notice that $x^{12} - 1 \,\vdots\, P(x)$; indeed:

$$x^{12} - 1 = (x^3 - 1)(x^3 + 1)(x^2 + 1)(x^4 - x^2 + 1).$$

Hence,

$$x^{12} - 1 = (x^8 - x^7 - x^6 + 2x^5 - 2x^3 + x^2 + x - 1)P(x).$$

Therefore,

$$x^{1951} - 1 = (x^{12} - 1)(x^{1939} + x^{1927} + x^{1915} + \ldots + x^{19} + x^7) + (x^7 - 1).$$

15.1.7.2 and **15.1.8.3.** Compare the coefficients of an arbitrary 4th degree monomial in a, b, c, x, y, z; for example, $acxy$, in both sides of the equality.

15.1.7.3. Prove that two of the three edges emerging from one vertex of the parallelepiped are equal. Then one of the faces is a rhombus; hence, all other faces are rhombuses, too.

15.1.8.1. The statement follows from the similarity of right triangles AOH_1 and COH_2, see Fig. 14: the equalities $\dfrac{AO}{OH_2} = \dfrac{CO}{OH_1}$ and $\dfrac{AO}{OH_1} = \dfrac{CO}{OH_2}$ imply that $AO = CO$, $OH_1 = OH_2$, therefrom $AH_2 = CH_1$, i.e., all heights of the triangle are of the same length. Now, prove that in this case the orthocenter divides each height in ratio 1 : 2.

FIGURE 14. (Hint 15.1.8.1)

15.1.8.3. Show that after expansion of squares the set of monomials in the left side is the same as the set of monomials in the right side.

15.1.9.1. The absolute value of each term of the progression exceeds the sum of all preceding terms (this is easy to prove, for instance, by induction). Now, assume the contrary to the statement required, i.e.,

$$aq^{k_1} + aq^{k_2} + \ldots + aq^{k_m} = aq^l \tag{$*$}$$

with $k_1 < k_2 < \ldots < k_m$ and $k_i \neq l$ for all i. Dividing both sides of $(*)$ by aq^s, where $s = \min_i(k_i, l)$, we conclude that 1 is divisible by q, which is impossible. Cf. the solution to Problems 28.1.10.4 and 28.1.11.1.

15.1.9.4. Every a_n with $n \geq 3$ is not greater than $81+81+4 = 166$. All that remains is to appropriately organize numbers from 1 to 166.

15.1.10.1. $\cos \arcsin x = \sqrt{1-x^2} = \sin \arccos x$.

15.1.10.3. Let A, B, C be the points where the sphere is tangent to the faces of the angle. Then triangles OAS, OBS, OCS are equal and the heights dropped to SO meet at one point.

15.2.7.2. If the circle inscribed in triangle ABC is tangent to side AC at point K, then $AK - CK = AB - CB$.

15.2.8.2. Prove that $\triangle NEA \sim \triangle NBD$ and that $NAD \sim NBF$. Hence, $ND = \sqrt{NE \cdot NF}$, see Fig. 15.

FIGURE 15. (Hint 15.2.8.2)

FIGURE 16. (Hint 16.1.8.1)

15.2.8.4. If at least two lines intersect then by adding one more line we increase the number of parts at least by 2. Therefore, we get not less than 198 parts.

16.1.8.1. Prove that the circle passing through the tangent points is inscribed in the triangle formed by the centers of the given circles. But the sides of this triangle are perpendicular to the common tangents of every pair of these circles, see Fig. 16.

Draw perpendicular l' to AB through B and draw perpendiculars DD' and CC' to l through points C' and D' on l'. Denote by X the intersection point of straight lines CD' and $C'D$, and by Y the intersection point of AC' and BC. Then line XY bisects segment AB.

16.2.8.4. All even-numbered gears rotate in the same direction and all odd-numbered gears rotate in the opposite direction. Hence the system of gears can only move if it has an *even* number of gears.

REMARK. If we have sufficiently large odd number of gears, it is possible to arrange them on the surface of the Möbius band so that the system will rotate.

16.2.10.1. Let $P_n(x)$ be the left hand side of the equation. Prove by induction on n that

$$P_n(x) = (1-x)(1-\tfrac{x}{2}) \cdot \dots \cdot (1-\tfrac{x}{n}).$$

16.2.10.3. Make use of the equalities

$$x_n - \sqrt{2} = \frac{x_{n-1} + \frac{2}{x_{n-1}}}{2} - \sqrt{2} = \frac{\left(\sqrt{x_{n-1}} - \sqrt{\frac{2}{x_{n-1}}}\right)^2}{2} = \frac{\left(x_{n-1} - \sqrt{2}\right)^2}{2x_{n-1}}.$$

Deduce from them that $x_n - \sqrt{2} < \frac{x_{n-1} - \sqrt{2}}{2}$ and apply this inequality for n from 1 to 30. For n from 31 to 36 apply the inequality $x_n - \sqrt{2} < \frac{\left(x_{n-1} - \sqrt{2}\right)^2}{2}$.

FIGURE 17. (Hint 16.2.9.5)

FIGURE 18. (Hint 16.2.10.5)

16.2.10.5. Prove this by induction on n.

17.1.9.4. Connect the orthocenter of triangle ABC with the bases of its heights, see Fig. 19. The segments obtained divide $\triangle ABC$ into 3 quadrilaterals. Consider the one with point S inside. Then the square also contains one of the quadrilaterals AB_1SC_1, C_1SA_1B, A_1SB_1C and this latter one fulfills the requirement.

FIGURE 19. (Hint 17.1.9.4)

17.1.9.5. In space construct triangles ABC and BCD with sides 8, 10 and 13 (by the hypothesis) and common side $BC = 13$. Fix triangle ABC and rotate the triangle BCD around the side BC. It is easy to see that AD is the longest when D lies in the plane of $\triangle ABC$. At this instance $ABCD$ is a parallelogram, BC and AD are its diagonals, and it is easy to see that $AD = \sqrt{159} < 13$. Therefore, $AD < 13$ for any position of the point D, hence, the answer to the problem is: no.

17.1.10.1. Denote: $y = \frac{z}{x}$. The discriminant of the quadratic equation for x is $D = 4(\sin z)^2 - 4$. For solutions to be real we should have $D \geq 0$, hence, $|\sin z| = 1$. Then the double root of the quadratic is $x = -\sin z$. Finally, the answer:

$$x = (-1)^k, \quad y = -\tfrac{\pi}{2} + \pi k \quad \text{for} \;\; k = 0, \pm 1, \pm 2, \dots.$$

17.2.7.3. The reflection of one of the axes of symmetry through another one is also an axis of symmetry.

17.2.9.1. As the ray l_2 rotates, the intersection point M draws a circle with center on l_1. Prove that the quotients $\dfrac{A_1M}{A_1A_2}$ and $\dfrac{B_1M}{B_1B_2}$ do not vary as l_2 rotates.

17.2.9.5. The construction of each of subsequent 5 segments increases the number of free endpoints by 4.

18.1.7.2. The median is equal to half the hypotenuse, so $BD = AD$. Now, make use of the fact that segments of tangents drawn from one point to the circle are equal.

18.1.8.5. Clearly, it suffices to prove that either $\angle A_1B_1C_1 = \angle A_1D_1C_1$ or $\angle A_1B_1C_1 + \angle A_1D_1C_1 = \pi$, see Fig. Probl.18.1.8.5. To prove this alternative, make use of similar equalities for the other inscribed angles (e.g., $\angle ABB_1$ and $\angle AA_1B_1$) that may appear.

18.1.9.1. Imagine the table in the form of the entry-wise sum of two tables: the i-th row ($i = 1, 2, \dots, k$) of the first table is: $k(i-1)$, ..., $k(i-1)$; that of the corresponding columns of the second table is: 1, 2, 3, ... , k.

18.1.9.5. Let H be the intersection point of AB with the extension of CD. As points A and B are on equal footing, we can assume that E belongs to AH. Prove that the difference $EC - ED$ decreases as E moves away from H in the direction of A.

18.2.7.2. Denote: $f(x) = ax^2 + bx + c$, then $\sqrt[4]{f(x+1)} - \sqrt[4]{f(x)} < 1$ for a great enough x.

18.2.7.3. Prove that the vertices of triangle ABC lie on the sides of triangle $O_1O_2O_3$, and straight lines OA, OB and OC are perpendicular to the respective sides of triangle $O_1O_2O_3$, where O is the center of the circle inscribed in triangle ABC.

18.2.8.1. Cf. Problem 18.27.2.

18.2.10.4. One should take all triples containing an arbitrary point A.

18.2.10.5. Convert $\triangle A_0B_0C_0$ into an equilateral triangle by an affine transformation. Then all other triangles $A_nB_nC_n$ also become equilateral ones. It is easy to prove that each triangle $\triangle A_nB_nC_n$ lies inside $\triangle A'_nB'_nC'_n$ whose vertices are intersection points of the segments A_0A_1, B_0B_1, and C_0C_1 with the sides of $\triangle A_{n-1}B_{n-1}C_{n-1}$.

19.1.7.2. Since the sum of the digits does not change when the number is multiplied by 9, the original number is divisible by 9. Now, take the numbers divisible by 9 and test them by multiplying them by the remaining digits.

19.1.7.5. The points divide the circle into 652 arcs of length 2, and 652 arcs of length 1 positioned alternately.

19.1.8.1. Build a parallelogram $DAEP$, see Fig. 20, and find the set of its vertices P as D and E vary; the set of the centers of these parallelograms is the locus desired.

FIGURE 20. (Hint 19.1.8.1)

19.1.8.3. See Hint to Problem 19.1.7.5.

19.1.9.1. If the mass of the quadrilateral is concentrated in its vertices (in other words, objects of equal mass are placed in the vertices of a weightless quadrilateral), then the intersection point of the described straight lines is the center of mass of the quadrilateral.

19.1.9.2. See the solution of Problem 19.1.8.2.

19.2.7.1. First, let us show that all sides of the hexagon $A_1AB_1BC_1C$ are equal; next, show that it is centrally symmetric. Further, let us show that triangles ABC and $A_1B_2C_3$ are centrally symmetric with respect to the center of symmetry of the hexagon. Now, we get the assertion of the problem.

19.2.7.3. See the solution to Problem 19.2.10.1.

19.2.7.4. Cf. Problem 19.2.8.2.

19.2.8.2. Assume that the sum of the numbers of the i-th row is $S \geq 518$ and prove that the sum of all numbers in the two rows and two columns whose indices are i and $9-i$ is not less than $4S-112 \geq 1960 > 1956$; this is a contradiction.

19.2.8.3. The set of projections of any point K of segment AB to all possible straight lines passing through point O is the circle with OK as its diameter.

19.2.9.4. See the solution of Problem 13.2.9-10.3.

19.2.10.2. See the solution of Problem 19.2.8.5.

19.2.10.5. See Hint to Problem 19.2.7.2.

20.1.8.3. Prove first that this quadrilateral is a rectangle.

20.1.8.5. The solution is similar to that of Problem 20.1.7.2.

20.1.9.3. Through points B and C draw straight lines tll they meet CD and AB at points B' and C', respectively. Then MM' is midline of the triangle DBB' while NN' is midline of the triangle ACC', see Fig. 21. Deduce from this that $BC \parallel MN \parallel AD$.

FIGURE 21. (Hint 20.1.9.3)

FIGURE 22. (Hint 20.2.7.1)

20.1.10.2 (Cf. Problem 20.1.7.3.) In order to come back to the starting point along the broken line, one should make the same number of steps forward and backward in each of three perpendicular directions.

20.1.10.4. See the solution of Problem 20.1.9.4.

20.2.7.1. Let (a_i, b_i) be the coordinates of the vector of the i-th segment of the broken line, l_i its length; let (a, b) be the coordinates of the endpoint of the broken line. The condition implies that $|a| = \sum |a_i|$ and $|b| = \sum |b_i|$. Since $|a_i| + |b_i| > l_i$, we have

$$(|a| + |b|)^2 = (\sum |a_i| + \sum |b_i|)^2 > (\sum l_i)^2 = l^2.$$

20.2.7.3. Fix the shorter side b (let $b \leq a$) and rotate the left vertex of the triangle along the circle with radius a and the center at one of the endpoints of segment b, see Fig. 23.

20.2.8.5. See the solution to Problem 20.2.7.4.

20.2.9.4. Numbers a_i must be approximately equal.

20.2.10.2. On a plane, consider the points with coordinates (x_1, x_2) and (x_2, x_3). Both these points lie on the parabola $y = 1 - x^2$.

To get from one solution to the other one we must move along the (kakih??) arrows of Fig. 25. The system of equations is satisfied if after moving along $2n$ arrows we come to the original point. Find on your own using Fig. 25 why starting with other values of x_1 we get no new solutions. (Point A is called the attracting one; point B the repulsing one.)

20.2.10.3. Drop perpendiculars OK_i and GH_i to all faces of the tetrahedron ($i = 1, 2, 3, 4$). Now consult the solution to Problem 20.2.8.3.

FIGURE 23. (Hint 20.2.7.3)

FIGURE 24. (Hint 21.1.8.3)

FIGURE 25. (Hint 20.2.10.2)

20.2.10.4. Take an arbitrary digit in a number from the first sequence and write a 0 to the right of it to get a number from the second sequence. This action allows us to establish a one-to-one correspondence between the digits of the first sequence and the zeros of the second.

21.1.7.1. The one who moves first can, for example, ascertain that a solution of the system is $(0, 1, -1)$; for this it is necessary that y and z have equal coefficients.

21.1.7.3. First, find how many four-digit numbers are there, such that the sum of their first two and the sum of their last two digits are both equal to $k = 0, 1, \dots, 18$.

21.1.7.4. Points A and B should be opposite vertices of the square.

21.1.7.5. The sum of any two numbers symmetric through the vertical axis is divisible by 1958 (the induction is obvious). In particular, the sum of the two numbers in the penultimate row is divisible by 1958.

21.1.8.3. First, find what is the minimal diameter of the pines that guarantees that two people moving abou the clearing cannot see one another. Then show that three people moving according to the Rule of the problem will not see each other either, see Fig. 24.

21.1.9.1. r_n grows as n, and S_n as n^2.

21.1.10.1. Just as in Problem 21.1.8.5, consider the intersection of the given rectangles and find the conditions under which the area of the octagon q inscribed in this intersection Q is minimal. (Observe that the vertices of q lie on all sides of Q; besides, Q and q can be degenerate octagons).

First, prove that all vertices of q coincide with some vertices of Q. Next, case-by-case checking. (The value $S = 10$ can be achieved.)

Prove that of all the inscribed polygons the triangle shown in Fig. 26 has the smallest area.

FIGURE 26. (Hint 21.1.10.1)

21.2.7.4. Express $(n!)^2$ as $(1\cdot n)(2\cdot(n-1))(3\cdot(n-2))\cdot\dots\cdot(n\cdot 1)$ and observe that almost each factor in parentheses is $>n$ (save the first and the last, which are equal to n). As the number of factors is equal to n, we are done.

21.2.8.5. Zoom the discs of radius $\frac{1}{2}$ untill their radii double. Show that the zoomed discs cover the whole polygon. Hence, $a \geq b$ (the equality is attained, for example, if the diameter of the polygon is less than 0.1).

Example when $a > b$: a regular hexagon with side 1.

21.2.9.2. The rays should be placed so that each straight line drawn through point O along an arbitrary ray divides the plane into two half-planes, each with an equal number of rays (or one with one more ray than the other).

For example, it is possible to arrange the rays so that the angle between any two neighboring rays is equal to $\frac{2\pi}{n}$.

21.2.9.4. Each of the discs covering polygon M contains not more than one center of any of b non-intersecting circles. Construct examples for $a=b$ and $a>b$ on your own.

21.2.10.4. As a increases k times, the volume of the 1-st cylinder increases k times and the volume of the 2-nd cylinder increases k^2 times.

22.1.7.1. The procedure described in the problem is the multiplication in the binary number system.

22.1.8.2. The first digit of the number is 1, 8, 6 or 9; the next three digits are selected at random from the five given in the hypothesis; the fifth digit is 0, 1, or 8. The remaining digits can be found unambiguously.

22.1.9.3. Prove that the common chords of the circle O and of each one of the circles passing through points A and B meet at the same point (it lies on straight line AB, see Fig. 27). Use the theorem about a tangent and a secant drawn from the same point to a circle.

22.1.9.5 and **22.1.10.3.** It is impossible for the longest edge of the tetrahedron to be adjacent to an obtuse angle, otherwise there would have existed a longer edge opposite this obtuse angle.

22.1.10.3. See Problem 22.1.9.5.

22.1.10.4. Prove that the sum of the numbers in any other row is greater by N than the sum of the numbers in the column with the same index.

22.2.7.3. Since

$$999\,999\,999\cdot A = \overline{A\,000\,000\,000} - A = 111...11,$$

we see that

$$A = \overline{A\,000\,000\,000} - 111...11. \qquad (*)$$

Perform subtraction in $(*)$ and find the last nine digits of the number A; substitute them in the number $\overline{A00...0}$; continue the process: subtract and find the next nine digits of the number A; substitute them again in the number $\overline{A00...0}$; etc.

FIGURE 27. (Hint 22.1.9.3)

22.2.8.2. Prove that 4 numbers in a row cannot be of the same sign (if $a_2 > 0$, $a_3 > 0$, then $a_1 - a_2 + a_3 < 0$ and $a_2 - a_3 + a_4 < 0$; the sum of these inequalities is $a_1 + a_4 < 0$).

REMARK. The estimate of the problem is exact: there is a set of numbers satisfying the condition with only 3 negative or 3 positive numbers. For example: $1, 3, 1, -3,\ 1, 3, 1, -3,\ 1, 3, 1, -3$.

22.2.8.4. See Problem 22.2.9.2.

22.2.10.2. The intersection point of segments K_1K_2 and K_3K_4 is the center of mass of the set of weights of suitable masses (find out on your own which ones) each weight occupying a vertex of the quadrilateral.

22.2.10.3. See the solution to Problem 42.10.4.

23.1.7.2. Connect the given intersection point S with points A_1, A_2, A_3, see Fig. 28.

FIGURE 28. (Hint 23.1.7.2)

FIGURE 29. (Hint 24.1.8.1)

23.1.8.4. Draw straight lines OA and OB and consider different positions of segment CD with respect to two vertical angles with vertex O, one of which contains segment AB.

23.1.9.4. For an arbitrary rectangle $ABCD$ and an arbitrary point M on the plane, $MA^2 + MC^2 = MB^2 + MD^2$. Cf. also Problem 20.1.8.1.

23.1.10.4. If the last digit of the given number A is equal to y, then $y = x^2$, where $x = 1, 2$ or 3. (Number A can not end with a 5 or 6 because $(10x + 5)^2$ ends with a 25 and the penultimate digit of $(10x \pm 4)^2 = 100x^2 \pm 80x + 16$ is odd.) We also see that

$$A^2 - x^2 = (A + x)(A - x) = c \cdot 10^{k+1} = c \cdot 2^{k+1} \cdot 5^{k+1},$$

where $c \leq 9$ and $k \geq 4$ is the number of zeroes at the end of $A^2 - x^2$. The factors of the written product can differ from one another by 2, 4 or 6. Therefore, if one of the factors is divisible by 5, the second factor is not divisible by 5; hence, the first factor is divisible by 5^{k+1}, i.e., it is $\geq 5^{k+1}$ and the second factor is $\leq 9 \cdot 2^{k+1}$, but the factors do not differ more than by 6. Hence, contradiction.

23.2.8.1. All the points lie on the circle whose diameter is the longest segment among those whose endpoints belong to the set of given points.

23.2.8.5. Compose a hexagon from four pentagons and tile the plane with these hexagons.

23.2.9.3. See the solution to Problem 22.2.8.5.

The board has 2 extreme and two "middle" rows. From an extreme row it is only possible to get to one of the middle rows in 1 move. So the knight will have to move at all times from an extreme row to a middle one and back again to meet the condition (otherwise it will pass more middle squares than extreme ones). But under such a Rule, the knight will go over black squares only in extreme rows and over white squares in the middle rows (or vice versa).

24.1.7.3 and **24.1.8.3.** For triangles $A_0B_0C_0$ and $A_1B_1C_1$ the statement is clear. Further, apply induction on n and notice that all triangles are acute.

24.1.8.1. Triangle $M_1M_2M_3$, partly drawn on Fig. 29, is similar to triangle ABC with coefficient $\frac{1}{3}$; hence, $S_{M_1M_2M_3} = \left(\frac{1}{3}\right)^2 S_{ABC}$, Q.E.D.

24.1.9.5. Consider the convex hull of all the points and prove that it is an n-gon.

24.1.10.3. See the solution to Problem 24.1.9.4.

24.2.7.2. Consider 2 cases: $AP = AS$ and $AP \neq AS$.

24.2.8.4. Set $4S = a_1 + \cdots + a_{10}$. First, observe that

$$S = x_1 + \cdots + x_5.$$

Let $a_1 \leq \cdots \leq a_{10}$. Then it is obvious that

$$x_1 + x_2 = a_1$$

(the sum of the two greatest numbers) and

$$x_4 + x_5 = a_{10}$$

(the sum of the two smallest ones). Therefore, $x_3 = S - a_1 - a_{10}$, where x_3 is the middle in value among the x_i. Now it is not difficult to find the other unknowns: for example, $x_1 = a_2 - x_3$.

24.2.10.1. First, prove that it is possible to select a non-decreasing subsequence from any infinite sequence of positive integers.

25.1.7.4. **Lemma**. *$a - S(a)$ is divisible by 9.*

25.1.7.5. Prove that there exist numbers $a_1, \ldots, a_k$, where $k \leq n$, and k cards such that (a_1, a_2) is written on the first card, (a_2, a_3) on the second one, etc., (a_{k-1}, a_k) on the $(k-1)$-st card, and (a_k, a_1) on the k-th card. Delete these cards and use similar arguments several times for the remaining cards.

25.1.8.1. The segments of these straight lines divide triangle ABC into four triangles of equal area.

25.1.9.1. See solution to Problem 25.1.10.1.

25.1.9.4. Multiply the second equation by x_1, the third one by x_1x_2 and so on, and use the first equation.

25.1.9.5. It suffices to consider circles of very small radii, for example, 4 000 times shorter than the shortest side of the rectangle.

25.2.7.2. Let O be the center of circle MBC, let O_1 be the center of circle ABC ($O_1 \in BH$), N the midpoint of segment BM, and L the intersection point of ON and BH. Prove that triangle OO_1L is isosceles.

25.2.7.4. Since 1962 is divisible by 3, the given number is the sum of two cubes: $N = (10^{654})^3 + 1^3$.

25.2.8.1. Prove a stronger statement: there are two *non-adjacent* vertices from which no diagonals are drawn.

25.2.8.2. See the solution to Problem 25.2.9.2.

25.2.8.4. Set $4S = a_1 + \cdots + a_{10}$. First, observe that
$$S = x_1 + \cdots + x_5.$$
Let $a_1 \leq \cdots \leq a_{10}$. It is clear that
$$x_1 + x_2 = a_1$$
(as the sum of the two greatest numbers) and
$$x_4 + x_5 = a_{10}$$
(as the sum of the two smallest numbers). Therefore, $x_3 = S - a_1 - a_{10}$, where x_3 is the middle in value among the x_i. Now it is not difficult to find the other unknowns: for example, $x_1 = a_2 - x_3$.

25.2.10.1. The algorithm for constructing point B from the given point C inside the circle is as follows, see Fig. 30:

FIGURE 30. (Hint 25.2.10.1)

FIGURE 31. (Hint 25.2.10.4)

1) drop the perpendicular from C on straight line l until the perpendicular intersects the circle at point C';

2) draw a tangent to the circle through point C' until it intersects l. The intersection point is B.

25.2.10.4. The area of the projection of triangle $\triangle ABC$ (or an analogous triangle with other vertices) is always equal to a half area of the whole projection of the parallelepiped, and the area of the projection of the given triangle to the horizontal plane is equal to the area of the triangle itself times the cosine of the angle between the planes.

26.1.7.2. See the solution to Problem 26.1.8.1.

26.1.7.3. It suffices, for example, to fill in each of 100 columns of height 4 with two cards. Cf. also Problem 26.1.8.5.

26.1.9.1. If the difference of the progression is less than 10^k, there is a term of the progression consisting of not less than $k+1$ digits and beginning with any digit (9 included).

26.1.9.4. See the solution to Problem 26.1.10.2.

26.1.10.5. Let M' be the meeting point of the perpendiculars to straight lines AM, BM and CM. Consider the circle with MM' as a diameter.

26.1.11.4. Each of the digits 1, 2, ... , 7 occurs in each of the places 720 times, and 720 is divisible by 9.

26.2.7.3. This point is the triangle's center of mass.

26.2.7.4. Prove that if $a+b$, $a+c$ and $b+c$ are divisible by 26, then the residues after division of a, b, c by 26 are equal.

26.2.7.5. First, prove that there cannot be a closed cycle in this system (neither segments AB, BC, and CA nor AB, BC, CD and DA). Then prove that there is a point to which only one path leads.

26.2.8.3. Consider the limit case of right triangles.

26.2.9.3. Circumscribe the circle around the decagon; the angles of each of the triangles constructed are multiples of $\frac{2\pi}{10}$. In this way we reduce the problem to a combinatorial problem: *on the circle find the total number of different (in size) arcs such that the sums of any three of them are equal.* (See Problem 26.2.7.1.)

26.2.11.1. If, for example, x is the greatest of these four numbers, then

$$x^x > x^z + x^t > z^z + t^t.$$

26.2.11.3. Prove successively that $P(x)$ is divisible by x, by $x-1$, etc., by $x-25$, therefrom

$$P(x) = x(x-1)(x-2)\dots(x-25)Q(x).$$

Then prove that $Q(x-1) = Q(x)$ for any x; hence, $Q(x) = const.$

27.1.7.5. Prove by induction that, beginning with the third number, the remainders after division of the terms of the sequence by 4 are with period 3 as follows: 2, 3, 3, 2, 3, 3, etc.

27.1.8.4. See the solution to Problem 27.1.9.4.

27.1.8.5. The 1's are obtained from numbers of the form $9k+1$, and 2's from numbers of the form $9k+2$.

27.1.9.1. Prove first that either all three numbers are ≥ 1 or all three are ≤ 1. Then prove that all of them are equal to one another, and $xyz = 1$.

27.1.10.2. The point O — the center of mass of the given points — is fixed during the movement described.

27.1.10.3. See the solution to Problem 27.1.9.4.

27.1.10.5. Each integer $n > 1$ occurs on the cards as many times as there exist irreducible proper fractions $0 < \frac{m}{n} < 1$ with denominator n, in other words, as many times as there are integers that are less than n and relatively prime to n.

27.1.11.3. See the solution to Problem 27.1.9.4.

28.1.8.2. Use the fact that 37 is a divisor of 999 and represent the number $\overline{abc000}$ in the form of $\overline{abc} \cdot 999 + \overline{abc}$.

28.1.9.4. See the solution of Problem 28.1.8.4 where the cities are replaced with the teams and each air line is replaced with a match.

28.1.9.5. First, observe that since the sides of mats are parallel to the walls of the corridor, it follows that the position of any mat is completely determined by the coordinates of the beginning and the endpoints of one of the sides parallel to the walls (counting, say, from the entrance). Let a point be covered by mats whose endpoints have coordinates (a_i, b_i), (a_j, b_j), (a_k, b_k). If a_i is the least of the numbers a_i, a_j, a_k and b_j is the greatest of the numbers b_i, b_j, b_k, then the mat (a_k, b_k) can be removed.

28.1.10.4. Prove that the greatest of all numbers in the pockets is strictly greater than the sum of the remaining "pocketed" numbers (cf. the solution to Problems 28.1.11.1 and 15.1.9.1).

28.1.11.2. The tangents to the circles at the given points A, B, C meet at one point D — the center of the circumscribed circle of $\triangle ABC$.

28.2.9.2. The ball can get reflected at the midpoints of opposite sides only if it moves along a closed trajectory. See also the solutions to Problem 28.2.8.2 and 28.2.10.3.

28.2.11.2. Prove that the last digits of the numbers n and $(n+20)^{n+20}$ coincide (verify this for different last digits of n). The period is

$$(0, 1, 4, 7, 6, 5, 6, 3, 6, 9, 0, 1, 6, 3, 6, 5, 6, 7, 4, 9).$$

29.2.8.2. Show that the first terms of the sequence $a_1, \dots, a_k, \dots$ are:

$$1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, \dots.$$

Each number in this sequence occurs twice except for 1, which occurs 4 times, and numbers of the form 2^n, which occur 3 times each. Now it is simple to find a_{1000}.

29.2.9-11.5. By changing places of the rows and the columns we can always come to a "diagonal" arrangement of the kind shown on Fig. 32. We only have to find the number of partitions of 11 into summands not less than 2.

30.1.8.1. The smallest such pair consists of numbers N and $N-1$, where $N = 899\dots900\dots0$ (13 digits 9 and 14 zeros).

FIGURE 32. (Hint 29.2.9-11.5)

FIGURE 33. (Hint 30.1.8.4)

30.1.8.4. To make the triangle ABM isosceles, point M must belong to the set shown in Fig. 33: the union of (1) the straight line — the midperpendicular to AB, (2) the circles of radius AB with centers at A and B, each circle without points on line AB. Thus, the solution is the intersection of this set with the analogous set constructed for segment BC.

30.1.8.5. Prove that if the trades union members are given x elephants each and the non-union members get y elephants each, then $x \leq 37$, $y \leq 28$.

30.1.10.2. The solution is similar to that of Problem 29.1.9-11.3.

30.1.10.3. Show that the remainder of the left hand side after division by 9 is never equal to 5.

30.1.10.5. $S + 1 = (2 + 1) \cdot (3 + 1) \cdot (5 + 1) \cdot \dots \cdot (p_k + 1)$.

30.2.7.1. Prove that triangle ABC is an equilateral one.

30.2.7.4. Prove first that if the sum of two integers is equal to a number of the form 999...99, then the addition will not cause a transfer into the next order in any order of the number.

30.2.8.3. The center of the circle circumscribed around the figure coinsides with the intersection point of the three midperpendiculars to the sides of the triangle, i.e., it coincides with the midpoint of the hypotenuse. Now, prove that the triangle is isosceles.

30.2.8.4. See the solution to Problem 47.7.5.

30.2.9.2. The longest sequence begins with the numbers $x_1 = M - 1$ and $x_2 = M$.

30.2.10.4. See solution to Problem 30.2.7.5.

30.2.10.5. Prove that there are two numbers which differ in only one digit and which end with different digits. Let, for example, digit 1 be written onto the end of number $11\dots1$ and digit 2 be written onto the end of number $211\dots1$. Then, inevitably, digit 3 is written onto the end of number $33\dots3$, and digit 2 onto the end of number $22\dots2$. Now, it is not difficult to prove that the first digit of any number is written onto its end (first, prove this for the numbers beginning with a 3 and consisting of 2's and 3's).

31.2.7.4. The remainder after division of a number by 99 is equal to the remainder after division of the sum of its "two-digit parts" by 99; for example, the remainder after division of $\overline{444x18y27}$ by 99 is equal to the remainder after division of $4 + 44 + \overline{x1} + \overline{8y} + 27$ by 99.

31.2.7.5. Let the bullet's trajectory be parallel to the ventillator's shaft (as usual, we visualize the shaft in the form of a line segment) and the shortest distance between the bullet and the shaft be smaller than the radius of the blade. Consider the coordinate system (x, t), where x is the bullet's position and t is time. On the coordinate plane draw 4 sets of points corresponding to the intersection of the blades with the trajectory of the bullet. (What is the form of the trajectory?) Show that one can move the trajectory as a solid in such a way that it intersects all four sets.

32.1.7.1 and **32.1.8.3.** The rook should move along tone of the two midlines of the chessboard keeping one knight's (bishop's) move behind the bishop (knight).

32.1.7.2 and **32.1.8.3.** Draw medians MF, EL, and NK in triangle EKM.

32.1.7.4. No number ending with four identical digits is divisible by 16.

FIGURE 34. (Hint 32.1.10.5)

32.1.7.5 and **32.1.8.2.** See the solution to Problem 32.1.9.5.

32.1.10.4. The fractional part of the number $\frac{1969^n}{1968}$ is equal to $\frac{1}{1968}$ for any n.

32.1.10.5. Two circles, one passing through A, B, M and the other one through C, D, M, see Fig. 34, should have equal radii and their centers should lie on the midperpendicular to segments AB and CD. These circles either coincide, or are symmetric through the center of the square, or are displaced with respect to one another by $\overrightarrow{AD}$.

32.2.7.1. $m\left(\frac{1}{a}+\frac{1}{b}+\cdots+\frac{1}{k}\right)=a+b+\cdots+k.$

32.2.8.3. All four straight lines meet at the second intersection point of the circles circumscribed around the pentagons.

REMARK. The problem can be extended to arbitrary regular n-gons.

32.2.8.5 and **32.2.9.2.** ??

32.2.9.4. Put $c = a - b$. Then $a = b + c$; now, use the binomial formula together with the prime factorization of n.

32.2.9.5. ??

32.2.10.1. The optimal strategies for both wizards are as follows: Let the first wizard strike out all odd numbers; then the second one strikes out all numbers greater than 512 (there remain exactly 256 of them); next the first wizard strikes out all numbers not divisible by 4; the second one strikes out all numbers greater than 256, etc.

32.2.10.3. See the solution to Problem 32.2.7.2.

32.2.10.4. Begin with the farthest end of the chessboard. The easiest way is to fill in with zeros the squares that you do not know how to fill (they won't bother you later).

33.1.7.4. Place the weights in pairs (one weight on every pan), placing each lighter weight on the heavier pan.

33.1.7.5. Number the cottages so that a lodger from the first one moves into the second one, from the second one into the third one, and so on. There is then a k such that the lodger from the k-th cottage moves into the first one, and the cycle is closed. Now prove the statement of the problem for the first k cottages and then consider the remaining cottages.

33.1.9.4. Use the fact that the ratio of the length of any side of the triangle to the sine of its opposite angle is equal to the diameter of the circumscribed circle.

33.1.9.5. The cockroach learns in not more than 7 steps in which of the four quadrants the Truth is, and then (s)he begins to move parallel to the sides of this quadrant and makes not more than $D\sqrt{2} < \frac{3D}{2}$ steps before (s)he reaches the Truth.

33.2.8.5 and **33.2.9.2.** Assuming that it is possible to streighten the broken wire, consider the moment when the segment (link) connecting the endpoints of the broken line constitutes an angle of 45° with the legs of the angle.

33.D.7.1. Multiply any number n by a one-digit number a such that $an \geq 10^k$ and $(a-1)n < 10^k$, and then delete the first digit of the product. We get a number which is less than the original n.

34.1.8.2. See the solution to Problem 34.1.10.1.

34.1.10.5. See the solution to Problem 34.1.8.3.

35.1.7.4. See the solution to Problem 35.1.8.5.

35.1.10.4. The set T is the intersection of n half-planes bounded by the sides of the n-gon, see Fig. 35.

FIGURE 35. (Hint 35.1.10.4)

FIGURE 36. (Hint 35.2.9.3)

35.2.7.3. Making use of the fact that one of 12 consecutive integers is divisible by 12 prove that the sum of all the proper divisors of this number is greater than the number.

35.2.9.3. Arrange the figures shown in Fig. 36 a) along the spiral so that they fill in the whole 20×20 square and do not intersect. To this end we need not less than 60 such figures, and in every one of them there is at least one subway station. This means that there are not less than 60 stations.

Fig. 36 b) shows that it is possible to arrange exactly 60 stations by means of the same construction.

36.1.8.5. Consider grasshoppers on an infinite checkered plane with each cell equal to the given square. Paint the nodes four colors as shown on Fig. 37. Let at the initial moment the grasshoppers be in the nodes of one cell.

FIGURE 37. (Hint 36.1.8.5)

FIGURE 38. (Hint 36.2.9.2)

Each grasshopper can only hop from one node to a node of the same color.

36.1.9.1. It suffices to consider the case when the parallelogram is a unit square.

36.1.9.4. Let A be an arbitrary subway station. Then close the subway station which is the farthest one from A.

36.1.10.3. Make use of the fact that $P(x) - P(y)$ is divisible by $x - y$. (See the solution to Problem 36.1.9.3.)

36.2.8.4. Assume that the common side is the side of the pentagon whose two adjacent angles exceed 60°.

36.2.9.1. The digits at the positions numbered 1, 6, 11, 16, ..., 96 can be rearranged at random; only the number of 1's among them will remain constant; it may vary from 0 to 20, i.e., there are 21 possibilities. The same refers to the places 2, 7, ..., 97, and so on.

36.2.9.2. Every black square has not more than 4 adjacent white ones, see Fig. 38. Prove that not less than $k - 1$ white squares are adjacent simultaneously to two black squares and we have counted them twice by now.

36.2.9.4. Any number $n > 1$ will be written $\varphi(n)$ times, where the function φ (*Euler's function*) sends n to the number of positive integers less than n and relatively prime to n. But 1973 is a prime.

37.2.7.4. See the solution to Problem 37.2.8.3. (The number of marbles in one pile corresponds to a number in one column.)

38.2.7.2. Prove first that two of the three angles of 120° adjoin one of the sides of the heptagon (cf. the solution to Problem 31.1.8.3).

Figure 39. (Hint 38.2.7.2)

Next, prove that if the heptagon is an inscribed one and if $\angle ABC = \angle BCD$, then $AB = CD$, see Fig. 39.

38.2.7.3 and **38.2.10.2.** Kolya loses if the number of stones is equal to $2^k - 1$; in all other cases he wins, every time cornering Vitya into a situation when the number of stones in the greatest pile is of the form $2^l - 1$ (for example, if $n = 100$, then Kolya's first move is to divide the pile into 63 and 37 stones). Vitya wins if $n = 31$.

39.2.9.1. By dividing $n!$ by the greatest possible power of 10 we obtain a number divisible by a large power of 2. But 1976 is not even divisible by 16.

39.2.10.1. Find an n such that $10^n + 1 \,\vdots\, c^2$ for an integer $c > 1$. Take $A = \left(\frac{c-1}{c}\right)^2 (10^n + 1)$.

39.2.10.2. Consider a 1975-gonal pyramid and the projection of all the vectors on the pyramid's height.

40.1.10.3. To pull wire cube through, one needs an aperture of the shape of the Greek capital letter Π with the horizontal bar of length 1 and the vertical legs long enough; to pull tetrahedron through, one needs an aperture of the shape of letter T with bars long enough.

FIGURE 40. (Hint 40.2.7.2)

40.2.7.2. See Fig. 40.

40.2.7.4. If one of the nails, A, is connected by wires with two nails, B and C, then the player's move is to connect nails B and C; thus (s)he will win.

41.1.7.1. Transfer the 1 to the right hand side and factorize the right hand side. Then the left hand side, $3 \cdot 2^x$, has to be the product of two factors that differ by 2, that is $3 \cdot 2^k = 2^{x-k} \pm 2$.

Now, show that $k < 2$ and investigate the cases $k = 0$ and $k = 1$.

42.8.5. See the solution to Problem 42.9.5.

43.7.1. Prove step by step that the last (fifth) digit of A is 0 (if it is ≥ 1, then passing from it to the first digit we see that the first digit is≥ 15); the penultimate digit is 1 (if it is ≥ 2, then the first digit is ≥ 12); the third digit is 2 and the second digit is ≥ 4. If the second digit is > 5, then the first digit is > 9. Therefore, the first two digits must be 9 and 5.

43.9.1. Make use of the fact that a_k has either the same number of digits as a_{k-1} or one digit more. Therefore, if our fraction is a periodic one, there is a k such that a_k coincides with the period. Now, considering a_{k+1} we come to a contradiction.

44.7.1. It is equal to the remainder after division of 35 by 14.

44.7.4. If $\left[\sqrt{\sqrt{x}}\right] = n$, then $n^4 \leq x < (n+1)^4$. Now it is easy to prove that $\left[\sqrt{[\sqrt{x}]}\right] = n$.

44.8.1. Use the fact that the sum of the angles at the vertices of the "star" is 180°; then as they are all equal, each angle is equal to 36°, and all angles of the inner pentagon obtained are equal to 108°.

44.10.6. The minimal number of place changes is attained as follows. Divide the round table by its diameter into two parts; and divide all people into two equal groups (if the total number of people is even) or two groups differing by 1 person (if the total number is odd). In each group let, first, one person change places with all other persons in this group. Let then the second person of each group change places with all other persons in the same group (except the first person). Then the third person starts all over again, etc.

45.8.2. Prove that either the union of two of the five rectangles is also a rectangle, or that we have the situation shown in Fig. 41. In the latter case the rectangles having the same letters (A or B) satisfy the required condition.

45.8.3. The number obtained is of the form $3k + 2$ and, therefore, is not a perfect square.

45.8.5. a) Proof by induction.

45.9.2. See the solution to Problem 23.2.7.1.

46.7.4. Cf. Problem 15.1.8.2.

FIGURE 41. (Hint 45.8.2)

46.9.3. Prove the general statement: $S = 1^n + 2^n + \cdots + n^n$ *is divisible by* $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ *for* n *odd.*

Since $GCD(n, n+1) = 1$, the general statement is equivalent to the statement that $2S$ is divisible by n and by $n+1$. To prove the latter statement, sum pairwise the symmetric (trough the middle) summands of S; i.e., consider sums of the form $k^n + (n-k+1)^n$. Each sum is divisible by $n+1$. To prove that $2S \,\vdots\, n$ consider $S - n^n$, and proceed as above.

47.9.2. The number of the cables of one (arbitrary) color must be equal to $\frac{13}{2}$, which is not an integer.

47.9.5. Prove that if the shortest distance between four points takes the greatest possible value, then the points lie in the vertices of a rhombus with side 25/8, two opposite vertices of the rombus coincide with the opposite vertices of the given rectangle and two other vertices are on the longer sides of the rectangle.

0

47.1.2. Evaluate every problem according to the 6-mark system; the value of marks runs 0 to 5. Assign o every participant the following 8-digit number: its two first digits is the total score and each of the other six digits is the number of points for the corresponding problem.

48.7.1. $(x-1)(y-1) = 0$.

48.7.3. Prove that there exists a triangle with the sides b, c, $d-a$ and then construct a parallelogram adjacent to it with sides a and c and an angle equal to the angle of triangle between sides c and $d-a$ (see the solution to Problem 23.2.7.2).

48.7.5. First, get 17 liters of milk into one can in 5 weighings and leave two other cans empty. Then fill the empty cans with 17 liters each in 2 weighings; pour the milk from one can into the other and get 34 liters (without weighing); pour 34 liters into an empty can at the 8-th weighing and lastly pour together 34 l, 34 l and 17 l.

REMARK. It is possible to similarly weigh

$$L = (2^{n_1} + 1)(2^{n_2} + 1)\dots(2^{n_k} + 1)$$

liters of milk in $N = (n_1 + 1) + \cdots + (n_k + 1)$ weighings. In our case $L = 85 = (2^2+1)(2^4+1)$ and $N = 8$.

48.9.3. If the 2×2 square covers two nodes with distance $2\sqrt{2}$ between them, then it obviously covers all 9 nodes.

48.10.3. If $a_n = \frac{M}{2^n}$ has the greatest "complexity" among the numbers with denominator equal to 2^n, then the number $a_{n+1} = N/2^{n+1}$, where $N = 2^{n+1} - M$, has the greatest "complexity" among the numbers with denominator 2^{n+1}.

49.8.4. Consider the length of the whole boundary of the domain overgrown with weeds. Then show that its total length will not increase after weeds begin to grow in new plots.

49.9.3. See the solution of Problem 49.8.5.

49.9.4. The solution is similar to that of Problem 49.7.4.

50.9.1. If $a_1, \dots, a_7$ are the given digits and $b_1, \dots, b_7$ are the last digits of the numbers $n - a_1, \dots, n - a_7$, then some two of these 14 digits coincide, say, $a_i = b_j$. So $a_i + a_j$ ends with the same digit as n. All that remains is to exclude the possibility of coincidence of a_i with a_j for $i \neq j$. This is not difficult.

50.9.5. Prove that the polygon formed by the sides of the pool and the planks farthest away from it (cf. Fig. Probl. 50.9.5 and Fig. 42) is non-convex and contains a triangle which, in turn, contains point $(2, 2)$.

FIGURE 42. (Hint 50.9.5) FIGURE 43. (Hint 50.10.1)

50.10.1. Observe that if $x = \tan\alpha$, $y = \tan\beta$, then $\frac{x-y}{1+xy} = \tan(\alpha-\beta)$. Let us prove that for any 4 numbers (or 3 positive numbers) indicated on the tangent line one can find the angles from $[0,\pi]$ corresponding to theese numbers and such that two of these angles do not differ by more than $\frac{\pi}{4}$. This is indeed the case since any pair of neighboring points among 4 points on the upper semicircle of the unit circle cannot all be farther than $\frac{\pi}{4}$ away from each other, see Fig. 43.

?? no hints

52.8.1. Both summands are equal to zero.

52.8.5. Send any scout to the team in which (s)he has not more friends than in the other team.

52.9.4. Consider the map as a graph. Then examine the nodes at which even number of edges meet and those at which odd number of edges meet.

53.9.3. Prove that any circle drawn of radius 2 is tangent to the circle centered at A and of radius R that satisfies the condition $R(4-R) = \lambda$, where λ is equal to the product of the lengths of the segments that A divides the chord into (λ does not depend on the choice of a chord). Since $\lambda < 4$, this equation has a positive solution.

?? ??

56.8.1. The sum of digits of a number has the same residue after the division by 3 as the number itself. Investigate these residues.

56.8.3. Consider how the number of all (not necessarily neighboring) pairs of chips of distinct colors with the red chip standing to the left of the blue one varies under each operation.

56.8.4. Compare the usual watch (UW) with its mirror reflection (LW).

56.8.5. Consider the 33-th word in the sequence (composed in the same way of the letters of the Russian alphabet)

$$a, \quad aba, \quad abacaba, \quad abacabadabacaba, \quad \dots$$

56.8.6. Prove that $\angle ADB = \angle ACB$. Then $\angle ADB$ is inscribed in the same circle with chord AB as $\angle ACB$. Hence, points A, B, C, D lie on one circle.

56.9.1. Clearly, $\angle A < 90°$ if and only if point C lies on the same side of the perpendicular to AB through A as B does. Similarly, $\angle B < 90°$ if and only if point C lies on the same side of the perpendicular to AB through B as A does. Finally, $\angle C < 90°$ if and only if point C lies outside the circle with diameter AB.

Now, $\angle B \leq \angle A \leq \angle C$ if and only if $AC \leq BC \leq AB$. The latter pair of the inequalities is satisfied if C lies inside the circle centered at B with radius AB and on the same side of the midperpendicular to AB as A does. Similarly, $\angle C \leq \angle A \leq \angle B$ if and only if $AB \leq BC \leq AC$, i.e., if C lies outside the circle centered at B with radius AB and on the same side of the midperpendicular to AB as B does.

Figure 44. (Hint 56.9.1)

56.9.2. Write down several consequtive first terms of the sequence and find the differences of the neighboring terms.

56.9.4. One of the two is possible: either there is a classmate with 28 friends or the most friendly classmate has 27 friends. In the first case the least friendly person has just one friend — the most friendly one. In the second case there necessarily exists an outistic classmate without any friends. In either case let us transfer the most and the least friendly classmates in another class. We get the same situation as earlier but with 26 classmates. Let us treat it in a similar way, etc.

56.9.6. If AM is indeed the bisector of $\angle BMC$, then it should divide the segment of the perpendicular to AM from point B confined between MB and MC in halves. The converse is also true.

56.10.1. The hypothesis implies that the length of a period of $A + B$ is equal to 12, hence the length of the minimal period is a divisor of 12. The length of the minimal period can not be equal to 6 since this would mean that $B = (A + B) - A$ is of period 6. Similarly, the length of the period is not equal to either 3 or 2.

Figure 45. (Hint 56.10.3)

56.10.3. A counterexample to heading (a) is depicted on Fig. 45a).

In heading (b) we have to show that there is no such route but one can place a disc of radius 800 m in the river. Use this fact to complete the proof.

56.10.4. The transformation $\{b\} \mapsto \{b + a\}$ (where $\{\cdot\}$ is the fractional part) is a rotation of the circle through an angle of $\{a\}$, where the full revolution, i.e., an angle of 2π radian is taken for the unit of measure.

The sequence $x_n = \{an + b\}$ is the sequence of points on the circle obtained from b by n-fold repeated rotation by an angle of a. We have: $p_n = 0$ if $x_n \in [0, \frac{1}{2}[$, i.e., if x_n lies on the half-open upper semicircle; $p_n = 1$ if x_n lies on the half-open lower semicircle. Any sequence of 4 zeroes or units can be realized; examples are easy to construct. The set 00010 is not realized: if three points lie on the upper halfcircle, then the next two should necessarily lie on the lower one.

56.10.5. Compute in two ways the number of distinctions between all possible pairs of plants.

56.11.4. The most profitable is to choose the number $k = \left[\sqrt{n+2}\right]$, where n is the maximal number of stones in a box. Therefore, for $n = 460$ the game is determined by the table:

	1	2	3	4	5	stones remained after the 5-th move
n	460	40	10	4	2	1
k	21	6	3	2	2	0

The same method shows that for $n = 461$ there remains 2 stones after 5 moves.

56.11.5. b) Mark all singular points: 0, and the endpoints of the intervals and their images under the action of f. These singular points are divided into foursomes symmetric through 0; the function f cyclicly permutes each foursome. The same is true for small intervals between marked points. Therefore, it turns out that each of the intervals $(-1, 0)$ and $(0, 1)$ will be divided by an even number of points into an even number of smaller intervals which is impossible.

57.7.4. See Problem 57.6.4.

57.8.5. The time is infavorable only if the second hand belongs to the angle vertical to that between the minute and hour hands. If at the beginning of the minute this angle were $\alpha°$, than during this minute the total amount of infavorable time is $\leq \frac{\alpha+6}{360}$ min.

57.8.6. One should mentally draw an axial-symmetric 18×18 square and repeat the opponent's moves symmetrically through the axis.

57.9.3. Seek a solution satisfying $y = -z$.

ANSWERS

For unknown words and theorems consult *Notational conventions and prerequisites.*

1.1.A.1. $\frac{16}{9}$ or $\frac{9}{16}$.

1.1.B.1. The speed of the train is $\dfrac{l}{t_2 - t_1}$ and its length is $\dfrac{lt_1}{t_2 - t_1}$.

1.1.B.3. $V = \frac{1}{6}\sqrt{1+\sqrt{5}}\,a^3 \approx 0.3a^3$.

1.1.C.1. 24, 18, 12, 6 and 3, 9, 27, 81.

1.1.C.3. $\varphi = \arctan 4$.

1.1.D.1. There are two solutions:

$$(a^2+a+1, a^2-a+1, a^2+1) \text{ and } (a^2-a+1, a^2+a+1, a^2+1).$$

1.1.D.3. $\varphi = \arccos \dfrac{1}{\sqrt{17}}$.

1.2.B.1. $x = y = 1$, $z = 0$.

1.2.B.2. If $b = 0$, then $x = y$; if $b \neq 0$, then there is no solution.

1.2.B.3. $n^2(2n^2 - 1)$.

1.2.C.1. a) $5 \cdot 3! = 30$ ways; b) $11 \cdot 2 \cdot 9! = 7\,983\,360$ ways.

2.1.1. $7744 = 88^2$.

2.1.2. A triangle with angle α at the base meets the condition if $\tan\alpha = 2$, or, equivalently, the height must be equal to the base.

2.1.3. $N = -\log_2 \log_2 \sqrt{\dots\sqrt{2}}$ (N-many radicals).

2.1.5. 5, 6, 7, 8.

2.2.4. The number of ways is $28^2 = 784$.

2.2.5. 0 to 14 ways.

3.2.2. 30 ways (3 times the number of greatest diagonals of the dodecahedron).

3.2.3. Into $\dfrac{(n-1)(n-2)(n^2-3n+12)}{24}$ parts.

4.1.? $\frac{1}{2}\binom{4}{2} + \binom{4}{1} = 7$.

4.2.2. The greatest number of parts is $\dfrac{n^3+5n+6}{6}$.

4.2.3. For the construction see Fig. 46:

5.1.1. If $b \neq 0$, then there are no real solutions at all. If $b = 0$, the solutions are $(x, y, z) = (0, a, -a)$ or $(a, 0, -a)$, where $a \in \mathbb{R}$.

5.1.5. The solution becomes obvious after examination of Fig. 47.

5.2.1. $(a^2+a+1)(a^8-a^7+a^5-a^4+a^3-a+1)$.

5.2.4. 5.

FIGURE 46. (Answ. 4.2.3)

FIGURE 47. (Answ. 5.1.5)

5.2.6. 30 parts.

6.1.7-8.1. $3(a-b)(b-c)(c-a)$.

6.1.7-8.2. 35 days. If we replace 5 and 7 with m and n, respectively, then the answer is $\dfrac{2mn}{n-m}$.

6.1.7-8.3. 24 zeros (the number of 5's in the prime factorization of 100!).

6.1.9-10.2. 7.

6.1.9-10.4. For the answer see Fig. 48. It is the union of 8 rays emerging from points A, B, C, D.

FIGURE 48. (Answ. 6.1.9-10.4)

6.2.7-8.4. 10 153 pairs, if the pairs (x, y) and (y, x) are not distinguished or 142^2 if they are.

6.2.9-10.1. The greatest integer n such that $n \cdot \alpha < 180°$.

6.2.9-10.2. $300! > 100^{300}$.

6.2.9-10.5. 2857.

7.1.7-8.2. 523 152 or 523 656.

7.1.7-8.4. The part of the circle with OP as its diameter, where O is the center of the given circle.

7.1.7-8.5. $n(n+1)(n+2)(n+3)+1 = (n(n+3)+1)^2$.

7.1.9-10.5. $x = -2$ or any $x \geq 2$.

7.2.7-8.4. $a = 8$ or $a = 12$.

7.2.9-10.3. The numbers 0, a, b, c can be any four consecutive integers (e.g. take $a = -1$, $b = 1$, $c = 2$).

7.2.9-10.4. $(0,0)$, $(1,0)$, $(0,1)$, $(2,1)$, $(1,2)$, $(2,2)$.

By multiplying (both parts of) the equation by 4 we get $(2x-y-1)^2+3(y-1)^2 = 4$. Hence $|y-1| \leq 1$, therefrom $y = 0$, 1, or 2.

7.2.9-10.5. The circle whose center is the midpoint of the common perpendicular to the given straight lines.

8.1.9-10.1. $a-b$.

8.1.9-10.2. 625 and 376.

8.2.7-8.1. 1 769 580.

8.2.7-8.2. 240.

8.2.9-10.1. 12 solutions.

9.1.7-8.1. 3.

9.1.7-8.4. $x_1=1$, $x_2=2$, $x_3=3$, $x_4=4$, $x_5=-4$, $x_6=-3$, $x_7=-2$, $x_8=-1$.

9.2.7-8.1. 7 or 14. (Cf. the solution of Problem 9.2.9-10.1.)

9.2.7-8.5. 7 routes, see Fig. 49.

FIGURE 49. (Answ. 9.2.7-8.5)

9.2.9-10.1. 10 points.

9.2.9-10.4. 8.

10.1.9-10.5. 0.89001.

11.1.7-8.1. $1=\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$.

11.2.7-8.1. $(x,y)=(2,4)$, and $(4,2)$.

11.2.7-8.2. $\frac{a}{b}$ =??.

11.2.7-8.3. No.

11.2.9-10.1. $x=\left(\frac{p+1}{p}\right)^p$, $y=\left(\frac{p+1}{p}\right)^{p+1}$ for any $p\in\mathbb{N}$.

11.2.9-10.2. The radius of the largest circle is equal to $\frac{a\sqrt{6}}{4}$, where a is the length of the cube's edge. This is the circle inscribed in a section of the cube which has the form of a regular hexagon.

11.2.9-10.4. 4. Consider, for example, the rays emitted from the center of an equilateral tetrahedron to its vertices.

12.1.9-10.1. $x=y=z=u=0$.

12.2.7-8.1. A cyclic permutation is possible: $1\longrightarrow 4\longrightarrow 2\longrightarrow 3\longrightarrow 1$ (red $\longrightarrow$ blue $\longrightarrow$ yellow $\longrightarrow$ green $\longrightarrow$ red).

13.1.7-8.1. The greatest circle with $R=\frac{1}{2}\sqrt{1^2+3^2}=\sqrt{10}/2$ passes through eight black squares, see Fig. 50. Its center is the center of a black square.

13.1.9-10.1. Yes.

Figure 50. (Answsw. 13.1.7-8.1)

Figure 51. (Answ. 13.1.9-10.2)

13.1.9-10.2. Here is an example, see Fig. 51. Are there other examples?

13.2.9-10.1. 1949. See Problem 13.2.7-8.1.

13.2.9-10.4. Yes, it is possible.

14.1.7-8.3. The second number is greater.

14.1.7-8.5 and **14.1.9-10.4.** 3 links; 4 links, respectively.

14.1.9-10.1. 0.239... .

14.1.9-10.2. This is an equilateral triangle with side a.

14.1.9-10.3. The star-shaped polygon is an unfolding of a pyramid if and only if the radius of the external circle is no less than twice the radius of the internal one.

14.2.7-8.4. As a result of a central projection of a triangle one can obtain the figures plotted on Fig. 52and only them. The form of the projection depends on the mutual disposition of the triangle and the line whose projection is the infinite line.

Figure 52. (Answ. 14.2.7-8.4)

14.2.7-8.6. -1.

15.2.7.4. This set is shown on Fig. 53.

15.2.8.4. Either 100 or 198.

15.2.9.1. If n is odd, then $x_1 = x_2 = \cdots = x_n = \pm 1$.

15.2.9.2. The axes of the cylinders must be perpendicular to each other, see Fig. 54.

15.2.9.5. A is not shorter than B which settles the bet.

16.1.7.4. Indeed, $n^2 + 8n + 15 = (n+4)^2 - 1$.

FIGURE 53. (Answ. 15.2.7.4)

FIGURE 54. (Answ. 15.2.9.2)

FIGURE 55. (Answ. 16.2.9.5)

16.1.9.1. $y = k\pi - x$ for $k = 0, \pm 1, \pm 2 \dots$ (a mesh of parallel straight lines forming an angle of 135° with the x-axis and intersecting it at points 0, $\pm\pi$, $\pm 2\pi$, ...).

16.2.7.5. See the solution to Problem 16.2.8.5.

16.2.8.4. Gears can rotate if n is even. For an odd n rotation is impossible.

16.2.8.5. $x_1 = x_3 = \dots = x_{99} = -1$; $x_2 = x_4 = \dots = x_{100} = 1$.

16.2.9.4. Yes.

16.2.9.5. Four-faced pyramids $AC'CB'B$, $AC'D'A'B'$ and $AC'CDD'$ are the desired ones, see Fig. 55.

16.2.10.1. $x = 1, 2, \dots, n$.

16.2.10.5. After $2n$ moves ($n > 1$) the knight can get from square A to any black square whose center lies in the convex octagon shown in Fig. 56: for $n = 1$ there are four inaccessible cells inside the 8-gon. The center of this octagon coincides with that of A, each of its angles equals 135° and every side passes through $2n + 1$ black squares.

17.1.7.1. For the desired partition see Fig. 57. The result is trapezoid $ABCD$.

17.1.7.2. They are congruent.

17.1.7.3. 1014 (first divide by 3, and then by 2) and 1035 (divide by 9 and 5).

17.1.7.4. There are no such integers.

17.1.7.5. 100.

17.1.8.1. For the answer see Fig. 58.

17.1.9.2. 99 999 78 59 60 61 ... 99 100.

17.1.9.5. No.

17.2.7.1. Two letters. Place them in the vertices of the squares in “chess order”.

17.2.7.3. 0, 1 or 7 axes of symmetry.

17.2.7.5. The cars will never meet.

Figure 56. (Answ. 16.2.10.5)

Figure 57. (Answ. 17.1.7.1)

Figure 58. (Answ. 17.1.8.1)

Figure 59. (Answ. 18.1.7.3)

17.2.9.4. No.

17.2.9.5. Yes, it can.

17.2.10.1. 0, 1, 2, 3 or 6 planes.

17.2.10.5. All numbers with an odd number of 2's are to be contained in the first set; and numbers with an even number of 2's in the second set; cf. Problem 30.1.7.3.

18.1.7.2. $\angle BAC = 60^\circ$, $\angle ACB = 30^\circ$.

18.1.7.3. The union of the height BH and arc $\smile AC$ of angle measure 120° with endpoints A and C; see Fig. 59.

18.1.7.5. Only the rectangles $a \times 13a$.

18.1.8.2. This set of points fills a parallelogram; see Fig. 60.

Figure 60. (Answ. 18.1.8.2)

18.1.8.4. The entire plane, a half-plane, or the strip between two parallel straight lines.

18.1.9.1. $(k \cdot 0 + k \cdot 1 + \cdots + k(k-1)) + (1 + 2 + \cdots + k) = \frac{1}{2}k(k^2+1)$.

18.1.9.2. The annulus (with the boundary) bounded by the two concentric circles whose center is the midpoint of the segment connecting the centers of the initial circles and whose radii are equal to a half sum and a half difference of the radii of initial circles, respectively. If the initial radii are equal, the annulus becomes a disc, see Fig. 61.

FIGURE 61. (Answ. 18.1.9.2)

18.1.10.4. Such a section always exists.

18.2.7.1. $x = y = z = 0$.

18.2.7.4. 28 games.

18.2.9.5. 5 games were played; there is a unique distribution of players up to renumbering of players and games.

18.2.10.4. $\binom{1954}{2}$.

19.1.7.2. 18, 45, 90, 99.

19.1.7.4. 13. See the solution to Problem 19.1.8.4.

19.1.7.5. $1956 \cdot \frac{2}{3} = 1304$.

19.1.8.1. The segment connecting the midpoint M of side AC with the midpoint N of segment BK, where K is a point on side AB such that $AK = BC$ (we assume that $BC \leq AB$). On Fig. 62 $AD = EC$, $AK = BC$; $DP \parallel AE$, $CP \parallel MN$, $\angle ECP = const$.

FIGURE 62. (Answ. 19.1.8.1)

FIGURE 63. (Answ. 19.2.8.3)

19.1.8.2. 0; 0.51; 0.52; … ; 0.98; 0.99; 1.00.

19.1.9.4. 8, 4, 2 and 0.

19.2.8.3. The set to be found fills in the two "holes" contained between the circles with diameters AO and BO, see Fig. 63.

19.2.9.2. This is possible.

19.2.10.5. The polygon $B_1B_2 \dots B_n$ is similar to polygon $A_1A_2 \dots A_n$ with coefficient $\tan\frac{\alpha}{2}$, where $\alpha = \frac{n-2}{n}\pi$ is an angle in the n-gon but turned by an angle of 90° relative $A_1A_2 \dots A_n$.

FIGURE 64. (Answ. 20.1.7.1)

FIGURE 65. (Answ. 20.2.7.1)

20.1.7.1. One of the diagonal of the regular pentagon cuts the trapezoid from it, see Fig. 64.

REMARK. Thhe solution is unique if $AD \neq BC$; if we allow to consider a trapezoid as a particular case of a parallelogram, there are other solutions: any rhombus.

20.1.7.5. 40 signs.

20.1.8.3. A square.

20.1.10.1. For any even n.

20.2.7.1. The part between the circle with radius l and the square whose vertices are the intersection points of straight lines OA and OB with this circle (including the circle but excluding the square); see Fig. 65.

20.2.7.2. This is possible; and the simplest variant is as follows: number the contacts of the lamp 1 to 7 clockwise, and the holes of the socket counterclockwise; see Fig. 66.

FIGURE 66. (Answ. 20.2.7.2)

FIGURE 67. (Answ. 20.2.7.3)

20.2.7.3. The third side must be as long as the longer of the two other sides.

20.2.7.4. $60°$, $60°$, $60°$.

20.2.8.1. If $b \geq a\sqrt{2}$, the length of the third side should be equal to $\sqrt{b^2 - a^2}$ and if $a \leq b \leq a\sqrt{2}$ it should be equal to a.

20.2.8.4. $x_1 = x_2 = x_3 = 0$ and $x_1 = x_2 = x_3 = 1$.

20.2.9.3. This is possible (for any even number of contacts).

20.2.9.4. One of the summands is equal to 164, and the other summands to 163.

20.2.10.1. Let us draw the directions required. The axes l_1 and l_2 pass through two neighboring vertices (C and B, respectively) of the quadrilateral; points P and Q are the intersection points of AB with l_1 and CD with l_2, respectively, see Fig. 68. The intersection point of PQ and AD is one of the vertices of the rectangle.

FIGURE 68. (Answ. 20.2.10.1)

If PQ and AD do not intersect, there is no solution.

20.2.10.2. If n is even there are four solutions: x_1 is equal to 0, 1, $\frac{\sqrt{5}-1}{2}$ and $-\frac{\sqrt{5}+1}{2}$, respectively, and it is clear that x_1 *does* determine $x_2, \dots, x_n$ *are* and *how* it does so).

If n is odd, then the solutions are only the last two of the above.

20.2.10.5. It is possible.

21.1.7.3. $1^2 + 2^2 + 3^2 + \cdots + 10^2 + 9^2 + \cdots + 1^2 = 670$.

21.1.10.4. $(1 + 2 + \cdots + 100) \cdot 2$.

21.2.7.3. Yes, this is possible, see Fig. 69.

FIGURE 69. (Answ. 21.2.7.3)

FIGURE 70. (Answ. 21.2.7.5)

21.2.7.5. It is possible to draw such a broken line, see Fig. 70.

21.2.8.5. $a \geq b$.

21.2.9.3. 6 pieces.

21.2.9.4. $a > b$.

21.2.9.5. 100 and the circuit design is either 10 parallel chains, each consisting of 10 successively connected resistors, or a chain of 10 links consisting of 10 parallel resistors each.

21.2.10.1. $x = 3$, $y = 1$.

21.2.10.4. b/a.

21.2.10.5. Let k be the number written on the visible side of the last card and let p, $p+1$, ... , q be the longest sequence of consecutive numbers shown, one of them being equal to k. Guesser can determine the number on the back of "k" in the following cases:

a) $p = 0$;

b) $q = n$ (if Guesser knows the number of cards n; the formulation of the problem is vague here);

c) one of the numbers from $[p, q]$, but not k, was shown twice.

22.1.7.3. It is possible.

22.1.7.5. The answer is shown on Fig. 71. Figure it out.

FIGURE 71. (Answ. 22.1.7.5)

22.1.8.1. It is impossible.

22.1.8.2. $4 \times 5^3 \times 3 = 1\,500$.

22.1.10.4. $N^2 - N$.

22.2.7.2. The intersection point H of the heights, i.e., the orthocenter of the triangle.

22.2.7.3. Multiply by $A = \overline{XX\dots X}$, where the number

$$X = 0...0\,1...1\,2...2\dots 7...7\,8...8\,9$$

with 9 many 0's, 1's, etc, 7's and with only 8 many 8's is repeated arbitrarily many times as a part of A.

22.2.8.3. $\pi - 2\angle A$, $\pi - 2\angle B$, $\pi - 2\angle C$.

23.1.7.1. Any amount of $N \geq 10$ roubles satisfies the condition and only such amounts do satisfy.

23.1.7.3. 26.

23.1.8.1. The given number is divisible by 3 and is not divisible by 9.

23.1.8.4. 0, 2 or 4.

23.1.8.5. There are infinitely many such numbers *even* among the set of perfect squares.

23.1.9.4. The union of two circles of radii $\sqrt{2R^2 - (OA)^2}$ and OA, concentric to the given circle of radius R and centered at O.

23.1.10.1. The convex hull of the given pair of triangles is a prism or an octahedron. The desired locus is the section of the convex hull by the plane lying in the middle between triangles; it is either a triangle equal to the given ones (if their sides are pairwise parallel) or a hexagon.

23.2.7.4. The only date is the 31-st. (It suffices to consider the remainders after division by 7.)

23.2.8.1. $n = 4$.

23.2.9.1. $l < m$, and at least one of the numbers l or m must be even.

24.1.7.4. The center of this circle is the midpoint of a longest segment with the endpoints in two of the given points.

24.1.8.5. Here are examples of the segments:

$$(a+0,\ a+981,\ a+1,\ a+982,\ a+2,\ \dots,\ a+978,\ a+1959,\ a+979,\ a+1960,\ a+980);$$

$$(b+980,\ b+0,\ b+981,\ b+1,\ b+982,\ \dots,\ b+1958,\ b+978,\ b+1959,\ b+979,\ b+1960).$$

Figure 72. (Answ. 24.1.10.2)

24.1.10.2. The axes of all these strips must meet at the same point O, see Fig. 72.

24.2.7.4. If there are only 6 asterisks, it is always possible to find two columns which have no fewer than 4 asterisks together, to delete these columns, and, by deleting some rows, to eliminate the two remaining asterisks. An example for 7 asterisks is shown in Fig. 73.

Figure 73. (Answ. 24.2.7.4)

Figure 74. (Answ. 25.2.10.4)

24.2.7.5. No solutions.

24.2.8.1. No, it is impossible.

24.2.8.3. 39. See the solution to Problem 24.2.7.3.

24.2.9.5. The best way for Nick is the first one; the worst ways are the second and the third ones.

25.1.9.4. Each of the unknowns $x_2, \dots, x_{1961}$ can assume any of the following tree values: 1, $\dfrac{1+\sqrt{5}}{1-\sqrt{5}}$, $\dfrac{1-\sqrt{5}}{1+\sqrt{5}}$.

25.2.7.1. You have to strike the ball so that its path cuts an isosceles triangle off each corner of the billiard table, as shown on Fig. 75.

25.2.10.4. The parallelepiped must be positioned so that the plane passing through points A, B, C, see Fig. 74, is horizontal.

26.1.7.3. It is possible.

26.1.7.5. It is impossible.

26.1.8.1. $0 = (a_1 + a_2 + \cdots + a_n)^2 = a_1^2 + a_2^2 + \cdots + a_n^2 + 2S$. Hence, $S = -\frac{1}{2}(a_1^2 + a_2^2 + \cdots + a_n^2) \leq 0$.

26.1.8.5. It is possible if n is divisible by 4. It is shown on Fig. 76 how to fill in a 4×5 rectangle with cards, and it is possible to construct a $4n \times 5$ rectangle of such small rectangles.

If n is odd, then it is impossible to cover the rectangle with cards; if n is divisible by 2, and not divisible by 4, the rectangle is to be covered with an odd number of cards, and, therefore, the product of all numbers in the table is odd. It is clear now that the product of numbers in each row cannot be even.

26.1.9.5. It is impossible.

Figure 75. (Answ. 25.2.7.1)

Figure 76. (Answ. 26.1.8.5)

Figure 77. (Answ. 26.1.10.3)

26.1.10.3. $m + n - 1 = (m - 1) + (n - 1) + 1$.
See the solution to Problem 26.1.7.5 and Fig. 77, where $m = 4$, $n = 17$.

26.1.10.5. The circumscribed circle of $\triangle ABC$ without points A, B, C.

26.1.11.2. There is no such a brocken line. Cf. the solution to Problem 39.2.10.2.

26.1.11.5. 15 parts.

26.2.7.1. Red beads divide the blue ones into three groups. Clearly, the answer is equal to the number of ways one can represent 7 as the sum of three non-negative integers. There are 8 representations:

$$7 + 0 + 0,\ \ 6 + 1 + 0,\ \ 5 + 2 + 0,\ \ 5 + 1 + 1,\ \ 4 + 3 + 0,\ \ 4 + 2 + 1,\ \ 3 + 3 + 1,\ \ 3 + 2 + 2.$$

26.2.7.4. 76. Take all odd numbers divisible by 13; there are $\left[\frac{1963}{2 \cdot 13}\right] + 1 = [75.5] + 1 = 76$ of them.

26.2.7.5. It is impossible.

26.2.8.2. ?????????????????.

26.2.8.3. If O is the center of the given disc of radius R, the locus desired is the disc (without the boundary) with center O and radius $R/3$.

26.2.8.4. 655 numbers.

26.2.9.2. 17 points of self-intersection. An example of this broken line is shown in Fig. 78.

26.2.10.3. Any 12 or 13 consequtive vectors.

27.1.7.1. $90°$, $45°$ and $45°$.

Figure 78. (Answ. 26.2.9.2)

Figure 79. (Answ. 27.1.7.4)

27.1.7.4. 41, see Fig. 79. Plotted are the segments to be erased.

27.1.8.2. Numbers of the form p and $2p$, where p is prime, and also 8 and 9.

27.1.8.5. The number of 1's is greater by 1 than the number of 2's (thanks to the number 10^6).

27.1.9.1. $x = y = z = 1$.

27.1.10-11.3. $a = 27^{1964}$. See the solution to Problem 27.1.8.3.

27.2.8.1. For n equal to a power of 2.

27.2.8.3. The first player must make his/her moves symmetrically (through the center) to the moves of the second player.

27.2.10.1. It is possible.

28.1.9.2. The straight lines parallel to the sides of the triangle and displaced from them inwards by $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$ of the corresponding heights meet at point O.

28.1.10.5. The number of the triangles is $n = 3932$, the number of the cuts is $k = 5896$.

28.1.11.4. The spatial curve which is the intersection of the sphere with diameter AB and the cone with vertex at the point B and the given circle as its base.

28.2.8.4. The robber must move along the street in the general direction "towards" the cop at speed $2v$ or $\frac{1}{2}v$.

28.2.10.2. No.

29.1.8.1. The segment connecting the midpoint of the base AB of triangle ABC with the midpoint of the height dropped to base AB (the endpoints of the segment do not belong to the locus); see Fig. 80.

29.1.8.2. All the numbers that are not divisible by 20.

29.1.8.5. It is impossible.

29.1.9-11.4. $k = 30p + 29$ for $p = 0, 1, \ldots$.

29.1.9-11.5. Fig. 81 shows an arrangement of 16 kings satisfying the condition.

29.2.8.1. The solution to the problem is shown in Fig. 82, where the radius of circle No. 7 is $\frac{1}{6}$, and that of circle No. 8 is $\frac{1}{3} = \frac{2}{6}$.

29.2.8.2. $a_{1000} = 495$.

29.2.9-11.2. Cf. Problem 29.2.8.2.

29.2.9-11.5. 14.

Figure 80. (Answ. 29.1.8.1)

Figure 81. (Answ. 29.1.9-11.5)

Figure 82. (Answ. 29.2.8.1)

Figure 83. (Answ. 29.2.9-11.5)

30.1.8.1. They exist.

30.1.8.2. Point M is the base of height CM of triangle ABC.

30.1.8.3. The spy's assignment is hopeless, there is none.

30.1.8.5. $2 \cdot 27 \cdot 38 = 2\,072$ elephants.

30.1.9.1. The pedestrians will meet.

30.1.9.2. It is impossible.

30.1.9.4. 13. For example, for the numbers 993 and 1006.

30.1.10.1. It is always possible to divide the square into $k+1$ triangles, but to divide it into k triangles might be impossible (if all the points lie on one half of the square's diagonal), see Fig. 84.

30.2.7.2. The least height is equal to a half of the square's diagonal.

30.2.8.2. $N = 8p$ $(p \geq 3)$ and $12p$ $(p \geq 5)$ for a prime p and also 8, 9, 12, 18.

FIGURE 84. (Answ. 30.1.10.1) FIGURE 85. (Answ. 31.1.7.2)

31.1.7.1. 4 and 1.

31.1.7.2. The only possible answer is shown on Fig. 85.

31.1.7.4. The answer is to connect one of the towns with 49 other towns.

31.1.8.1. $66 - 12 = 54$.

31.1.8.5. No.

31.1.9.1. Yes, there is. For example, any trapezoid with bases 1 and $\sqrt[3]{2}$ satisfies the condition of the problem.

31.1.9.4. Yes, it is possible.

31.1.10.3. It is impossible. The greatest possible number of subsets satisfying the Rule is equal to $2^{10} = 1024 < 1968$.

31.2.7.3. 49.

31.2.7.4. $x = 4$, $y = 6$.

31.2.8.1. It is possible.

31.2.8.4. 990.

31.2.9.1. The set to be found is shown in Fig. 86.

FIGURE 86. (Answ. 31.2.9.1)

31.2.9.2. Not more than three of them.

31.2.9.3. The white wins.

31.2.10.4. It is possible.

32.1.8.5. It is possible. For example:

$$-a, b, b, -a, b, b, \dots, -a, b, b, -a, b, \dots\ ,$$

where a and b are such that $2b - a > 0$ and $13b - 7a < 0$, i.e., $\frac{13}{7} \cdot b < a < 2b$. Take for example $a = 17$, $b = 9$; $a = 25$, $b = 13$, etc.

32.1.9.3. It is impossible.

32.1.10.4. There is. For example, $h = \frac{1969^3}{1968}$.

32.1.10.5. For the answer see Fig. 87.

FIGURE 87. (Answ. 32.1.10.5)

FIGURE 88. (Answ. 32.2.7.3)

32.2.7.3. It is possible. One of the ways to do this is shown in Fig. 88. See also the solution to Problem 32.2.10.2.

32.2.7.5. To find the pearl after 33 cuts it suffices to make the cuts parallel and equidistant to one another. Now show that it is impossible to find the pearl in 32 cuts.

32.2.9.1. The first player wins.

32.2.9.4. It is possible.

32.2.10.4. It is possible.

33.1.7.1. It is impossible.

33.1.7.3. $53°$, $63°$ (by $60°$ less than the given angles) and $64°$.

33.1.8.3. 4 sides.

33.1.8.4. Only for $k = 1$ or $k = 2$.

33.1.9.2. 9 checkers, see Fig. 89.

FIGURE 89. (Answ. 33.1.9.2)

FIGURE 90. (Answ. 33.2.8.3)

33.2.7.3. It is possible.

33.2.8.3. $\frac{10\,000}{4} = 2\,500$ trees: clearly, among the four trees put in a square on Fig. 90 not more than one can be chopped off.

Figure 91. (Answ. 33.2.9.2)

Figure 92. (Answ. 34.1.8.3)

33.2.9.2. $2+2\sqrt{2}$.

A suitable wire piece is, for example, the one that bounds a segment of a circle of width (height?) 1 based on a chord of length $2+2\sqrt{2}$, see Fig. 91.

33.2.9.3. No.

33.2.9.4. 1699 (see the solution to Problem 33.2.7.4).

33.2.9.5. They can.

33.2.10.3. Yes, they can. Cf. Problems 33.2.7.3, 33.2.8.2 and 33.2.9.5.

33.D.7.1. 0.

33.D.7.3. No.

33.D.7.4. It is impossible.

34.1.8.3. The Tbilisi team could even be the last one, see Fig. 92.

34.1.9.3. This is possible for any n except for $n = 2^k$ ($k \geq 2$).

34.1.10.2. One cannot say whether Peter is lying or not.

34.1.10.4. $k = 1$.

34.1.10.5. The first player.

34.2.7.1. It is not difficult to guess (and then to verify) that this number is 111 111 111.

34.2.7.5. The solution is shown in Fig. 93.

Figure 93. (Answ. 34.2.7.5)

Figure 94. (Answ. 34.2.8.2)

34.2.8.2. 501, see Fig. 94.

34.2.8.4. It is possible.

34.2.9.3. It is possible.

34.2.9.4 and **34.2.10.1.** $n = (\sum x_i)^2$.

34.2.10.4. Such a sphere exists.

35.1.8.4. 50.

35.1.9.4. The first sum is greater than the second sum by $m - n$.

35.1.9.4. No.

35.1.10.3. 20 000.

35.1.10.5. Not always.

36.1.8.1. Yes, it is; see Fig. 95.

Figure 95. (Answ. 36.1.8.1)

36.1.8.2. No, it is impossible.

36.1.8.5. No.

36.2.7.1. No, it can not.

36.2.7.3. a) it is possible; b), c) not necessarily.

36.2.9.1. 21^5.

36.2.9.2. $4k + 1$.

36.2.9.3. See the solution of Problem 36.2.8.5.

36.2.9.4. 1972 times.

37.1.10.4. It is impossible.

37.2.7.2. 197. See the solution to Problem 37.2.8.2.

37.2.7.3. 4 or 5 (all diagonals in a square or in a regular pentagon are equal).

Figure 96. (Answ. 37.2.7.3)

37.2.7.4. ?See the solution to Problem 37.2.8.3. (The number of marbles in one pile corresponds to the numbers in one column.)

37.2.9.1. Yes.

37.2.9.4. See the solution to Problem 28.1.9.3.

38.1.10.1. $x = y = z = t = 0$.

38.1.10.4. It is impossible.

38.1.10.5. It is possible.

38.2.8.5. The greatest difference in the scores of "neighboring" teams may be equal to n.

EXAMPLE: the winner, having beaten all, has $2n - 2$ points while the other plays ended in a draw, and each team except the winning one scored $n - 2$ points.

38.2.9.5. No.

38.2.10.4. For any $n > 0$.

39.1.10.2. Yes.

39.1.10.3. The first four digits are 1000.

39.1.10.4. It is possible.

39.1.10.5. It is possible. You can, for example, arrange the domino tiles in the way shown in Fig. 97.

FIGURE 97. (Answ. 39.1.10.5)

FIGURE 98. (Answ. 39.2.7.4)

39.2.7.2. It can.

39.2.7.3. There is. For example, $n = 10\,111\,111\,111$.

39.2.7.4. The desired arrangement is shown in Fig. 98 (every point is connected with the three nearest points).

39.2.9.1. It is impossible.

39.2.10.1. There is. In particular, the *minimal* A with this property is

$$\frac{(10^{11}+1)16}{121} = 13\,223\,140\,496.$$

39.2.10.2. There exists, see Fig. 99.

FIGURE 99. (Answ. 39.2.10.2)

FIGURE 100. (Answ. 41.9.1)

40.1.10.2. Yes, they must.

40.1.10.3. It is possible in both cases.

40.2.7.2. a) It is impossible. b) It is possible.

40.2.7.4. The second.

40.2.7.5. 50.

40.2.9.1. $n = 2$.

40.2.9.2 and **40.2.10.2.** There are such integers (for all cases in question).

FIGURE 101. (Answ. 40.2.10.1)

40.2.10.1. It is possible, see Fig. 101. For example, one can arrange the discs far from one another so that their centers are situated on the parabola $y = x^2$.

41.7.1. $(0, \pm 2)$; $(3, \pm 5)$; $(4, \pm 7)$.

41.7.2. The angles of the triangle shall be equal to one of the following triples:

$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right), \quad \left(\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{6}\right), \quad \left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\right), \quad \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\right).$$

41.9.1. $n - 2$ points, see Fig. 100.

41.9.2. No, there is not.

41.9.5. Yes, it is possible.

42.7.1. a) It is possible. b) It is impossible.

42.7.4. Vitya always wins.

42.8.1. a) yes; b) no. Cf. the solution to Problem 42.7.1.

42.9.1. $D = \frac{10}{11}$.

42.9.2. It is possible.

42.9.3. a) It exists. b) It does not exist.

42.10.3. The greatest value is $\frac{1}{8}$; the function $g(x) = \dfrac{x(1-x)}{2}$ assumes this value at $x = \frac{1}{2}$.

43.7.1. 95 210.

43.9.3. mn.

44.7.1. This remainder is 7.

44.7.4. It is.

44.8.1. One must mark 5 angles of the vertices of the "star" formed by the diagonals, and 2 more angles between the diagonal and the sides of the pentagon which together with this diagonal form a triangle.

44.9.4. It is possible.

44.10.5. $a_1 = \dfrac{32}{\sqrt{15}}$, $a_2 = \dfrac{24}{\sqrt{15}}$, $a_3 = \dfrac{16}{\sqrt{15}}$.

44.10.6. If $n = 2k$, then the minimal number of place changes is $2C_k^2 = k(k-1)$; if $n = 2k+1$, then this number is equal to $C_k^2 + C_{k+1}^2 = k^2$.

45.7.4. 8.

45.8.1. $1 + \sqrt[4]{5}$.

45.8.5. b) The equality holds, for example, for $a_1 = 1$, $a_2 = 2$, ... , $a_n = n$.

45.9.1. All integers of the forms $6k+1$ or $6k+2$, where $k = 0, 1, 2, \dots$.

45.9.4. The number A is a periodic decimal fraction $10/81 = 0.(123456790)$ and, therefore, its expression does not contain a digit 8 (cf. [Tr], Problem 92).

45.9.5. $p = 2 + 4\sin 15°$.

45.10.4. $n = 1$ and 4.

46.7.1. $(4, 1)$; $(4, -3)$; $(-4, 1)$; $(-4, -3)$.

46.7.2. Two vertices of the same color of an equilateral triangle with side l are the desired points.

46.7.4. Cf. Problem 15.1.8.2.

46.7.5. No, there is not.

46.8.3. Yes: take $N = \underbrace{9\dots9}_{1983}5$.

46.8.4. There exists (exactly one).

46.9.2. $\frac{40\sqrt{3}}{7}$, see Fig. 102.

FIGURE 102. (Answ. 46.9.2)

FIGURE 103. (Answ. 47.8.5)

46.10.3. No, he is not.

47.7.1. It is possible.

EXAMPLE: tickets 159 999 and 160 000. In general, any pair of numbers meets the requirement if the first of the numbers of the form $\overline{ab9\,999}$, where $a+b$ is equal to either 6 or 13. Prove on your own that there are no other suitable pairs. (See also solutions to Problems 30.1.8.1 and 30.1.9.4.)

47.7.3. 25 roubles (money borrowed and returned to the neighbor is not to be taken into consideration).

47.8.1. $x = \sqrt{2}$.

47.8.5. The solution is shown on Fig. 103.

47.8.6. It is odd.

47.9.2. It is impossible.

47.9.6. They do.

EXAMPLES: $333...334^2 = 11...155...56$; $666...667^2 = 44...488...89$.

47.10.3. No solutions.

48.7.1. One of the numbers is equal to 1, the second one is arbitrary.

48.7.2. There is exactly one way.

48.7.3. It is possible.

48.7.4. It is possible.

48.8.1. $y = x$ and z arbitrary or $y = z$ and x arbitrary.

48.8.3. Two, see Fig. 104.

FIGURE 104. (Answ. 48.8.3)

FIGURE 105. (Answ. 49.7.3)

48.9.2. It is possible.

48.9.3. 9.

48.10.1. $x = 0;\ 99;\ 49 + \frac{50}{99}$.

48.10.3. The number $\frac{2}{3} + \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{50}$ is of the greatest "complexity" equal to 101.

48.10.4. $44 \cdot 1985 + 1$.

49.7.3. The meeting place is the house of the first dwarf (whose speed is 1 km/h), see Fig. 105.

49.9.2. It is possible.

50.7.5. The first thief can get 50 coins, the second one 26 coins, and all other thieves (beginning with the third and including Ali-Baba) at least one coin each, provided each thief is doing his best at division of coins.

50.8.5. There is not.

50.10.4. It is impossible.

51.7.4. Three colors are enough.

51.7.4. 3 weighings are required in the general case since two weighings may be not enough.

51.10.1. $\min\{a, b\} = \dfrac{a + b - |a - b|}{2} = \dfrac{a + b - \sqrt{(a-b)^2}}{2}$.

52.7.1. The letters should be placed in the first, second, third, and fourth horizontal, for example, in the following order: $ABCD$, $DCBA$, $BADC$, and $CDAB$.

52.7.3. 5 socks.

52.7.5. One number: 1989.

52.8.1. $x = -1$.

52.8.3. The least number of lines in both cases is 3, the third line being the perpendicular desired.

52.8.4. 55.

52.8.6. $\max\limits_{a,b,c}(|a|+|b|+|c|) = 17$, for example, for the polynomial $8x^2-8x+1$ (that satisfies the condition).

52.9.3. Yes, one can.

52.9.4. Depending on how we should interpret the formulation of the problem (A: the outer rim of the square is not counted in the set of streets; B: the outer rim of the square is counted) we have to choose one of the following solutions or solve both cases. A: 60; B: 68.

53.8.2. 2 (e.g., for $m = 5$, $n = 1$).

53.8.3. No, they cannot.

53.8.5. 6.

53.9.4. One can.

53.9.5. $n(10^n - 1)$, for example, for the fraction $(10^n - 1)^{-1} = 0.00...100...1...$.

53.10.1. One can.

53.10.2. $(2, 3, 17)$, $(3, 2, 17)$.

53.10.4. In the vertices of the inscribed square. (This statement is known as *I. Schur's theorem.*)

53.10.5. $\dfrac{\sin\beta}{\sin\alpha}$.

53.11.2. Example. $f(x) = x^{\sqrt{2}}$.

53.11.3. No.

53.11.5. No three of the points lie on one line and all four points do not lie on a circle.

54.8.2. Yes, it is possible.

54.8.3. 4.

54.9.1. $x = 0$ or -1.

54.9.2. Yes.

54.10.1. $f(x) = \begin{cases} \dfrac{1}{\frac{1}{2} - x} & \text{for } x \neq \frac{1}{2}, \\ \frac{1}{2} & \text{for } x = \frac{1}{2} \end{cases}$, in particular, $f(0) = 2$, $f(1) = -2$.

54.10.2. Any even number > 2.

54.10.5. 3.

54.11.1. a) between the 996-th and the 997-th; b) between the 995-th and the 996-th.

54.11.2. Through the midpoint O of segment AB draw line OD perpendicular to AB (D is the intersection point of OD with the projection of the equator; see Fig. 106 a)).

Figure 106. (Answ. 54.11.2)

Draw the chord parallel to AB and passing through D. On OD, mark point C', so that (the length of) the segment OC is equal to a half of the chord. Then point C' will be the projection of the North pole, C.

54.11.4. 1991.

55.8.2. No.

55.8.4. 9 weights, e.g. 3, 3, 4, 5, 6, 7, 8, 12, 12 g.

55.8.6. Yes, see Fig. 107.

55.9.2. There are equally many numbers in each set.

55.9.3. No, this is impossible.

55.9.4. 678 ways.

55.9.6. Yes. Examples for $n = 5$ are plotted on Fig. 108 a).

55.10.2. In the shaded pentagon on Fig. 109.

55.10.3. When $n + m$ is even the black wins, when $n + m$ is odd the white wins.

55.10.4. 11 weights, e.g., 2, 2, 3, 3, 4, 5, 6, 7, 8, 10, 10 g.

55.10.6. No.

Figure 107. (Answ. 55.8.6)

Figure 108. (Answ. 55.9.6)

Figure 109. (Answ. 55.10.2)

55.11.1. a) yes; b) no.

55.11.2. $\angle A = \angle C = 70^\circ$, $\angle B = 120^\circ$, $\angle D = 100^\circ$.

55.11.5. No.

56.8.1. a) no solutions; b) $x = 1963$.

56.8.3. No, this is impossible.

56.8.4. The favorable and unfavorable time are equally distributed.

56.8.5. Yes, there is.

56.9.1. See Fig. 110.

56.9.2. 501500.

56.9.4. Pete has 14 friends.

56.9.5. 58.

56.10.1. 4 or 12. Examples: 1) $A = -0.(000\,001)$, $B = 0.(000\,101\,010\,002)$, $A + B = 0.(0001)$; 2) $A = 0.(000\,001)$, $B = 0.(000\,000\,000\,001)$, $A + B = 0.(000\,001\,000\,002)$.

56.10.4. a) yes, b) no.

FIGURE 110. (Answ. 56.9.1)

FIGURE 111. (Answ. 56.11.2)

56.11.2. It is. Fig. 111 shows one such partition. In the lower half of the greatest square we find one square of side $\frac{1}{2}$ (marked 1), 2 squares of side 2^{-2} (marked 2), etc., 2^{1993} squares with side 2^{-1993} (marked 1993) — all aligned with a side of the great square. A symmetric picture occupies the upper half of the square.

Moreover, there are 2^{1993} squares with side 2^{-1993} situated along the main diagonal.

56.11.3. n.

56.11.4. (a) it is possible, (b) it is impossible.

56.11.5. a) it is possible (the graph of f is plotted on Fig. 112); b) it is impossible.

FIGURE 112. (Answ. 56.11.5)

56.11.6. The least perimeter is equal to $\frac{a}{\sqrt{10}}$.

57.6.1. It can: for example, Ivan Lukich Zuev, Ivan Fomich Pnin, Petr Lukich Pnin, Petr Fomich Zuev.

57.6.4. See Fig. 113.

FIGURE 113. (Answ. 57.6.4)

57.6.7. a) Yes, b) No.

57.6.8. It could (see Fig. 114).

57.7.1. By 30%.

Figure 114. (Answ. 57.6.8)

Figure 115. (Answ. 57.9.1)

57.7.4. See Problem 57.6.4.

57.7.5. It can not.

57.8.2. $143143 = 7 \cdot 143 \cdot 143$. The solution is similar to that of Problem 10.1.

57.8.5. There is more favorable time than infavorable. (Lucky us!)

57.8.6. The first wins.

57.9.1. There exists, see Fig. 115. The solid line denotes the sides, the dotted one the diagonals of the pentagon. (We can interchange the roles of solid and dotted lines.)

57.9.2. If $\frac{k}{l} \leq 1$, Leo wins; otherwise Nick wins.

57.10.5. a) no. b)????

57.10.6. Example: $P(x) = x^4 + x^3 - 0.1x^2 + x + 1$. It suffices to ensure that all the coefficients of $P^2(x)$ and $P^3(x)$ are positive. Indeed, all the other powers of P can be represented as products of squares and cubes of P.

57.11.1. A thrihedral prism with one angle sown off, see Fig. 116.

Figure 116. (Answ. 57.11.1)

Figure 117. (Answ. 58.9.2)

57.11.2. See Probl. 57.10.2.

57.11.3. The cherry will touch the bottom for $r \leq \frac{3}{4}\sqrt[3]{2}$.

58.8.4. It suffices in all three cases.

58.9.2 and **58.10.2.** The locus is the union of the height of $\triangle ABC$ from vertex B with an arc γ of measure 120° lying inside $\triangle ABC$, see Fig. 117.

58.9.3. For $n <= 998 = \frac{1996}{2}$ and for $n > 2 \cdot 1995 + 1$.

58.9.4. No it can not.

58.10.1. a) 4; b) 3.

58.10.2. See Problem 58.9.2.

58.10.4. See Problem 58.9.5.

58.11.2. a) it is possible; b) it is impossible.

58.11.5. $n = 1994$. Examples: $A = 1\underbrace{0\dots0}_{1994\text{times}}$ and $B = 1\underbrace{0\dots0}_{1995\text{times}}$.

SOLUTIONS

For unknown words and theorems consult *Notational conventions and prerequisites.*

1.1.B.2. Choose a vertex A on one of the lines and rotate one of the other lines through $90°$ around point A, see Fig. 118. The point where the rotated line l_2' intersects the third line l_3 is a second vertex of the square.

FIGURE 118. (Sol. 1.1.B.2)

1.1.D.2. Triangles $\triangle ADG$, $\triangle DBF$ and $\triangle ABC$ are similar, therefore, $\frac{R_1}{AD} = \frac{R_2}{DB} = \frac{R}{AB}$, which implies $R = R_1 + R_2$.

FIGURE 119. (Sol. 1.1.D.3)

1.1.D.3. See Fig. 119. The angles that form the progression are $45°$, $60°$, $75°$. The angles $\angle AOB$, $\angle BOC$, $\angle COA$ are twice as large, respectively: $90°$, $120°$, $150°$. Hence, $AB = \sqrt{2}R$, where R is the radius of the base of the cone, $OH = \frac{\sqrt{2}}{2}R$. Since the angle of the unfolding is $120°$ (i.e., $\frac{1}{3}$ of the full angle), the radius of the corresponding circle is equal to $3R$. In $\triangle AHS$ we have: $AS = 3R$, $AH = \frac{\sqrt{2}}{2}R$ and, therefore, $SH = \sqrt{\frac{17}{2}}R$. Hence,

$$\cos\varphi = \frac{\frac{\sqrt{2}}{2}R}{\sqrt{\frac{17}{2}}R} = \frac{1}{\sqrt{17}}.$$

1.2.A.1. Let X, Y, and Z be the points, where the median, the bisector, and the height drawn from vertex A intersect the circle, see Fig. 120. Any circle can be recovered from three points, the circumscribed one as well. Since bisector AY divides arc $\smile BC$ in halves, we have:

a) $OY \perp BC$ and, therefore, OY is parallel to height AH. Drawing a straight line parallel to OY through Z we find vertex A, see Fig. 120.

b) OY passes through the center of BC, as median AX does.

Therefore, we can find the midpoint M of side BC as the intercection point of OY with AX. Finally, vertices B and C lie on the perpendicular to OY through M.

FIGURE 120. (Sol. 1.2.A.1)

1.2.A.2. Consider the sphere whose diameter is a diagonal of the cube. Any point *inside* this sphere serves as the vertex of an obtuse angle the diagonal subtends; any point *on* the sphere is the vertex of a right angle the diagonal subtends. All points of the cube, except its vertices, lie inside the sphere. So it is clear that the points to be found are the vertices of the cube that lie not on the diagonal.

1.2.A.3. Straight line BB_1 is the intersection line of planes ABA_1B_1 and CBC_1B_1. If plane CAC_1A_1 is not parallel to BB_1, then the intersection of plane CAC_1A_1 with the dihedral angle formed by planes ABA_1B_1 and CBC_1B_1 is a planar angle with legs AA_1 and CC_1 and the vertex on line BB_1. Therefore, lines AA_1, BB_1, and CC_1 meet at one point.

If plane CAC_1A_1 is parallel to BB_1, then straight lines AA_1 and CC_1 are also parallel to BB_1.

1.2.B.1. There exists only one solution because by the inequality between the arithmetic mean and geometric mean

$$xy \leq \frac{(x+y)^2}{4} = 1 \quad \text{for } x+y=2.$$

1.2.B.3. The general recipe to solve such problems is as follows. The unknown sum $f(n)$ satisfies the equation

$$f(n) + (2n+1)^3 = f(n+1)$$

or

$$f(n+1) - f(n) = (2n+1)^3 = 8n^3 + 12n^2 + 6n + 1. \tag{$**$}$$

Denote: $P_0 = 1$ and

$$P_k(x) = x(x-1)\dots(x-k+1) \quad \text{for } k > 0.$$

Then

$$P_k(x+1) - P_k(x) = kP_{k-1}(x).$$

Now, let us express the rhs of $(**)$ as a linear combination of the $P_k(x)$. Since

$$\begin{aligned} P_0(x) &= \ 1, \\ P_1(x) &= \ x, \\ P_2(x) &= \ x^2 - x, \\ P_3(x) &= \ x^3 - 3x^2 + x, \\ P_4(x) &= \ x^4 - 6x^3 + 11x^2 - 6x, \end{aligned}$$

it follows that

$$f(x+1) - f(x) = 8P_3(x) + 36P_2(x) + 26P_1(x) + P_0(x).$$

Therefore,

$$f(x) = 2P_4(x) + 12P_3(x) + 13P_2(x) + P_1(x) + cP_0(x).$$

Substituting the above values of $P_i(x)$, we get $f(x) = 2x^4 - x^2 + c$; from the condition $f(0) = 0$ we deduce that $c = 0$. Hence, $f(x) = (2x^2 - 1)x^2$.

1.2.C.1. a) Assume that the bottom face of the cube is painted the first color. Then the upper face can be painted one of 5 colors. We can color the remaining 4 faces in 6 different ways. Indeed, by rotating the cube around its vertical axis we can fix the color of the back face. *Only* 3 *faces are left* and all $3! = 6$ possible colorings of these faces that do not coinside under rotations of the cube. So we have $5 \times 6 = 30$ different ways to color a cube.

b) In this case we argue similarly to (a). Place all dodecahedrons not coinciding under rotations on the plane so that all their bottom faces are of a given color, A. Then the top face can be of any of the remaining 11 colors. If we fix now any of the remaining 10 colors and paint one of the two fixed faces adjacent to the top or the bottom face this color, then the remaining 9 faces can be arbitrarily painted any of the remaining 9 colors. Altogether $11 \cdot 2 \cdot 9!$ ways to paint.

1.2.C.2. If $x + y + z = n$, then x can be one of 1, 2, 3, ... , $n - 2$ and $y + z = n - x$. So there are $n - x - 1$ possibilities for y and z and the number of ways to be found is equal to

$$\sum_{x=1}^{n-2}(n - x - 1) = 1 + 2 + \cdots + (n - 2) = \frac{(n-2)(n-1)}{2}.$$

1.2.C.3. a) We know that $a = k_1 D(a, b)$, $b = k_2 D(a, b)$, and $M(a, b) = k_1 k_2 D(a, b)$ for some k_1, k_2, see *Prerequisites*. The rest is clear.

b) If $D = D(a, b, c)$ and $a = k_1 k_2 D$, $b = k_1 k_3 D$, $c = k_2 k_3 D$, then $D(a, b) = k_2 D$, $D(a, c) = k_2 D$, $D(b, c) = k_3 D$, and $M(a, b, c) = k_1 k_2 k_3 D$, Q.E.D.

2.1.1. Since $\overline{aabb} = 11 \cdot \overline{a0b}$ is a perfect square, $\overline{a0b} = 11 \cdot k^2$ for$10 \leq k^2 \leq 90$. The possible variants are: $k^2 = 16, 25, 36, 64, 81$ and only $11 \cdot 64 = 704$ has a 0 as the second digit.

2.1.4. First, construct the center O of the circle (choose three points on the circle and draw the mid-perpendiculars to the sides of the triangle obtained). Then construct the point P' symmetrical to P with respect to O. If AB is the diameter to be found, i.e., $\angle APB = \alpha$ is the given angle, then $\angle PAP' = \pi - \alpha$.

FIGURE 121. (Sol. 2.1.4)

So point A is an intersection point of the circle and the angle $\angle PAP'$ of measure $\pi - \alpha$. Finally, we find the diameter AB by drawing straight line AO.

2.1.5. How can we guess the answer? If $n(n+1)(n+2)(n+3) = 1680$, then $n < \sqrt[4]{1680} < n + 3$, which gives $3.4 < n < 6.4$. This leaves only three possibilities: $n = 4$, 5, or 6. Now, test them.

ANOTHER SOLUTION. Put $x = n^2 + 3n$. Then $x(x + 2) = 1680 = 40 \cdot 42$. Hence, $x = 40$ and $n = 5$.

2.2.5. Three planes can be arranged in 5 ways:

1) The planes intersect at one point. In this case the balls that are tangent to all three planes form 8 cones, one in each of the parts that the planes divide the space into. If the given ball intersects the cone, then there is one or two balls inscribed into the cone and tangent to the given ball, see Fig. 122.

FIGURE 122. (Sol. 2.2.5)

If the intersection point of the planes lies inside the given ball, then inside of each cone there would be only one inscribed ball tangent to the given ball; the sum total of 8 balls.

If the intersection point of the planes lies outside the given ball, then it is possible to show that the given ball cannot intersect all cones and, therefore, the total number of variants does not exceed 14 (two times for each of the 7 cones).

2) Pairwise intersections of the planes are three parallel lines. In this case the balls tangent to all three planes form 4 cylinders (the inscribed and three escribed ones). In each of the cylinders there are not more than 2 inscribed balls tangent to the given ball; the total number of variants: not more than 8.

3) Two planes are parallel, the third one is not. In this case instead of 4 cylinders (see case 2)) we have only 2; hence, not more than 4 variants.

4) all planes are parallel and 5) the planes intersect along one line. In these cases the given ball cannot be tangent to all three planes.

Now, let us show that for any n from 0 to 14 there exists a position of planes and a ball that realizes the answer "n variants".

Let the planes be the coordinate planes. Then the cones' axes are the diagonals of the cube with vertices at points $(\pm 1, \pm 1, \pm 1)$. The plane passing through the cube's center and perpendicular to one of the cube's diagonals (let us mark this diagonal) intersects 6 cones. Therefore, it is possible to find a ball intersecting all these 6 cones and one of the remaining cones whose axis lies on the marked diagonal.

For such a position there exist 14 balls tangent to the three given planes and the given ball.

Moving the given ball along the diagonal and slightly in a transversal (perpendicular) direction we can arrange the ball so that it is tangent to one of the side cones only. There are 13 variants in this case.

Moving the ball further along the diagonal (now, parallely to it) we diminish the number of variants to 2 (the case when the given ball lies inside of one of the cones) having encountered on the way all intermediary sets of variants 3 to 12.

The cases with 1 or 0 solutions are the ones when we place the given ball in the slot between the cones; in particular, 1 solution occurs when the ball is tangent to exactly one of the cones.

3.1.1. Cubing the first equation and subtracting the third one from the result we get $3(x+y)(y+z)(z+x) = 0$. If $x+y=0$, then $z=a$ and from the second equation we derive $x=y=0$. The cases $z+x=0$ and $y+z=0$ similarly give two solutions.

3.1.2. (A "scientific" solution), see Fig. 123 a). If $MA+MB=a$ then point M lies on the ellipse with focuses A and B. Let us transform the plane so that the ellipse becomes a circle: compress the plane in the direction of segment AB without moving its midperpendicular. Then straight line l turns into straight line l_α and all we have to do now is to find the intersection points of l_α with the circle and perform the inverse transformation: decompress the plane.

ANOTHER SOLUTION (A "school version"), see Fig. 123 b). Construct the circle with radius $AM+MB$ and center A. Let point B' be symmetric to B through the given straight line l. With the help of inversion with center at B' draw the circle through B and B' and tangent to the first circle. Let C be the tangent point of the circles.

FIGURE 123. (Sol. 3.1.2)

The point to be found is the intersection point of straight lines l and AC.

3.1.3. Let AB and CD be the given segments. Construct a parallelepiped with edges AB, AC and $AA' = CD$, see Fig. 124.

FIGURE 124. (Sol. 3.1.3)

Then it is clear that:

a) the height of the parallelepiped is equal to the distance between the skew lines, so it does not depend on the position of the segments;

b) the base parallelogram of the parallelepiped does not change when the segments move;

c) the volume of the tetrahedron is $1/6$ of the volume of the parallelepiped.

3.2.1. To completely solve the problem, one has also to consider the cases when several of the right hand sides of $(*)$, see Hints, are replaced by $270°$ or $-90°$ (the cases of selfintersection).

3.2.2. For each of the ten greatest diagonals of the dodecahedron there exist three suitable planes perpendicular to the diagonal. In order to prove it, select a greatest diagonal of the dodecahedron and start dragging from infinity the plane perpendicular to this diagonal.

This plane will consecutively cut off the dodecahedron:

1) a *point*;

2) an *equilateral triangle*;

3) an *irregular hexagon* which at some moment becomes a

4) *regular* hexagon and then turns again into

5) an *irregular* hexagon until the plane reaches the center of the dodecahedron (the midpoint of the diagonal); at this moment the section becomes again

6) a *regular* hexagon.

As the plane moves further, it cuts the dodecahedron along a regular hexagon just once — the hexagon is the centrally symmetric one to the hexagon from step 4). We get the total of $3 \cdot 10 = 30$ planes.

4.2.3. For the construction see Fig. 125:

FIGURE 125. (Sol. 4.2.3)

FIGURE 126. (Sol. 5.1.3)

i) Denote the height of triangle ABC dropped from vertex C by h; let $\gamma = \angle A - \angle B$. Draw straight line MN parallel to AB at distance h from it.
ii) Find point B' symmetric to B through MN.
iii) Construct arc $\smile AB'$, the locus of vertices of angles of measure $\pi - \gamma$.
iv) Find the intersection point of this arc with MN. This is vertex C.

4.2.4. Strike out all numbers divisible by 5 from the sequence 1, 2, ... , 999. There are $\left[\frac{999}{5}\right] = 199$ such numbers. Then strike out the numbers divisible by 7 from the same sequence 1, 2, ... , 999; there are $\left[\frac{999}{7}\right] = 142$ of them. But among thenumbers stricken out there are $\left[\frac{999}{35}\right] = 28$ numbers which are divisible by both 5 and 7. So 28 numbers are stricken out twice and there are $999-(199+142-28) = 999-313 = 686$ numbers left.

REMARK. If we only need an approximate answer N, this answer can be found from the formula $N \approx 1\,000\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right) = 1\,000 \times \frac{24}{35} = 685.7...$ Try to prove this formula yourselves.

This formula is useful if we wish to find how many numbers simultaneously nondivisible by primes p_1, p_2 , ... , p_k are there. In our case $k = 2$, $p_1 = 5$, $p_2 = 7$.

5.1.1. Thanks to the hint

$$-b^3 = 2b(x^2 + y^2 + z^2 - xy - yz - zx).$$

For $b \neq 0$ we see that

$$x^2 + y^2 + z^2 - xy - yz - zx = -\tfrac{1}{2}b^2.$$

This is equivalent to

$$(x + y + z)^2 - 3(xy + yz + zx) = -\tfrac{1}{2}b^2.$$

But $x + y + z = 2b$, therefrom

$$xy + yz + zx = \tfrac{3}{2}b^2 \quad \text{and} \quad x^2 + y^2 + z^2 = b^2. \tag{$*$}$$

Since $x^2 + y^2 - z^2 = b^2$, then the second of formulas $(*)$ implies that $z = 0$, $x + y = 2b$, $xy = \frac{3}{2}b^2$, i.e., x and y are roots of the equation $t^2 - 2bt + \frac{3}{2}b^2 = 0$. These roots are

$$\left((1 + \tfrac{\sqrt{2}}{2}i)b,\ (1 - \tfrac{\sqrt{2}}{2}i)b\right) \quad \text{and} \quad \left((1 - \tfrac{\sqrt{2}}{2}i)b,\ (1 + \tfrac{\sqrt{2}}{2}i)b\right).$$

The organizers of the Olympiad did not plan to consider complex solutions. This means, they only considered $b = 0$.

For $b = 0$ we have $z^2 = x^2 + y^2$, $z = -x - y$, hence, $x^2 + 2xy + y^2 = x^2 + y^2$, $xy = 0$ and either $x = 0$ or $y = 0$.

5.1.3. a) When points B and C lie *on different* sides of the straight line to be constructed, do as follows, see Fig. 126:

i) Construct the circle with diameter BC.

ii) Construct chord BK of given length h.

iii) Draw a straight line, l, through A perpendicular to BK. This line is the needed one, as $\angle BKC = 90°$ (i.e., $l \parallel KC$) and $BP + PK = h_1 + h_2 = h$.

b) The case when points B and C lie on the same side of the straight line to be constructed may be reduced to case a) by replacing point C with point C' symmetric to C through A.

c) Investigate on your own the cases without solutions.

5.2.2. Both polynomials cannot have only even coefficients (otherwise all coefficients of their product would be divisible by 4). Suppose that each of the polynomials has an odd coefficient. Substitute 0 for every even coefficient and 1 for every odd one. Then the product of the polynomials has a non-zero coefficient. Contradiction.

5.2.3. Suppose that chord CD is already drawn. Draw chord DK parallel to straight line AB, and then straight line KC until it meets AB at point P. It is not difficult to show that triangles ACP and ABX are similar. Hence, $AP = \dfrac{AC \cdot AX}{AB}$ does not depend on X and the segment AP can be constructed, see Fig. 127.

FIGURE 127. (Sol. 5.2.3)

Now, to construct chord CD, note that $\angle KDC$ is the angle between the given straight lines AB and MN. Therefore, arc $\smile KC$, and, consequently, chord KC, are known. It only remains to draw a straight line through P so that the circle would intercept on the line a chord of given length $|KC|$. Take any chord of this length and construct the concentric circle tangent to this chord. The tangent to this circle passing through P gives us point C. After that, point D can be easily found.

5.2.4. Let $n = 6q + r$. Prove on your own that $10^6 \equiv 3^6 \equiv 1 \pmod 7$, hence, $10^n \equiv 10^r \pmod 7$. This implies that each term of the sum has the same remainder after division by 7 as $10^4 \equiv 3^4 \equiv 4 \pmod 7$. Hence, the remainder to be found is equal to $10 \cdot 4 = 40 \equiv 5 \pmod 7$. Hence, the answer.

5.2.5. The sum to be found is equal to the sum of the distances from P to the edges of the base times the tangent of the dihedral angle between the base and a side face. The sum of the lengths of perpendiculars dropped from P to the sides of a regular polygon is equal to twice the ratio of the polygon's area to its side. Hence, it is a constant.

6.1.9-10.1. Let $xy = t$. Then the first equation can be written as

$$(b^2 - 2t)(b^3 - 3bt) = 2b^5. \qquad (*)$$

If $b \neq 0$, then $(*)$ is a quadratic equation for t with roots $t = b^2$ and $t = -\dfrac{b^2}{6}$. Now it is not difficult to find x and y.

If $b = 0$, then $(*)$ is an identity and any pair $x = a$, $y = -a$ is a solution.

6.1.9-10.3. The number of solutions in the general position is what was expected for the answer at the Olympiad. There are, however, particular cases with quite different number of solutions:

— all points lie on one circle: infinitely many solutions;

— all points lie on one line: infinitely many solutions if points lie symmetrically and none otherwise;

— three points of four lie on one line: 6 solutions $(3 + 3)$;

— the points are vertices of a non-isosceles trapezoid: 6 solutions $(4 + 2)$;

— the points are vertices of a parallelogram (but not a rectangle): 5 solutions $(4+1)$.

FIGURE 128. (Sol. 6.1.9-10.3)

CONSTRUCTION. If the points are divided by the circle into groups $3+1$, let us draw another circle, S, through 3 points of one group. The center of S is the center of the circle to be constructed and the radius to be found is the arithmetic mean of the radius of S and the distance from the center of S to the fourth point.

If the points are divided into groups $2+2$, let us construct two concentric circles on which these points lie (the center is the intersection point of midperpendiculars, see Fig. 128b)). The circle to be constructed has the same center and its radius is the arithmetic mean of the radii of the circles. Now, let us consider what may happen.

$3+1$: the generic case: 4 solutions;
the given points are on one circle: 1 solution;
the given points are on one line: 0 solutions;
3 of given points are on one line: 3 solutions;

$2+2$: the generic case: 3 solutions,
the given points are on an isosceles trapezoid: ∞ solutions;
the given points are on a non-isosceles trapezoid: 2 solutions;
the given points are on a parallelogram: 1 solution;
the given points are on one line: 0 or ∞ solutions.

6.1.9-10.5. Since $7! > 1000$, it follows that $a, b, c < 7$. Since $6! = 720$, neither one of a, b, c can equal 6. On the other hand, if each digit of the number is less than 5, then $\overline{abc} \leq 4! + 4! + 4! = 72 < 100$. Hence, there is at least one 5 among these digits. Since $5! + 5! + 5! = 360 < 500$ and $\overline{a55} \neq a! + 240$ for $a = 1, 2, 3, 4$, there is exactly one 5 among the digits. Then $\overline{abc} \leq 5! + 4! + 4! < 200$ and our number begins with a 1. Now, considering all the remaining variants, we find the only answer: $145 = 1! + 4! + 5!$.

6.2.7-8.2. On CD, draw segment CE such that $CE = AD$, see Fig. 129. Triangles $\triangle BCE$ and $\triangle BAD$ are equal in two sides and the angle between them (the angles subtend the same arc $\smile DB$). In $\triangle DBE$, we have $DB = BE$ and $\angle BDE = 60^\circ$. Hence, $\triangle DBE$ is an equilateral triangle. Therefore, $BD = DE$ and $CD = CE + ED = AD + BD$.

6.2.7-8.4. Considering the remainders after division by 7, it is not difficult to show that both x and y must be divisible by 7. Combining pairs of $\left[\frac{1000}{7}\right] = 142$ numbers, which are less than 1000 and divisible by 7, we get $\frac{142^2 - 142}{2} + 142 = 10\,153$ pairs if the pairs (x, y) and (y, x) are not distinguished. The formulation of the problem is, however, ambiguous and, to be on the safe side (who knows the juri's whims), we must consider the other possibility: when pairs (x, y) and (y, x) are considered *distinct*. Do it on your own.

6.2.9-10.1. The curve is the image of a straight line l on the plane after the plane is folded into a cone with vertex O not on the line, see Fig. 130. The points of selfintersection are: $A = A'$, $B = B'$, ... such that $AO = A'O$ and $\angle AOA' = \alpha$; $BO = B'O$ and $\angle BOB' = 2\alpha$, etc. The total number of such pairs is equal to $\left[\frac{180^\circ}{\alpha}\right]$.

FIGURE 129. (Sol. 6.2.7-8.2)

FIGURE 130. (Sol. 6.2.9-10.1)

6.2.9-10.4. By the Cauchy inequality

$$\frac{\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_n}{a_1}}{n} \geq \sqrt[n]{\frac{a_1}{a_2}\cdot\frac{a_2}{a_3}\cdot\cdots\cdot\frac{a_n}{a_1}} = 1, \quad \text{Q.E.D.}$$

7.1.7-8.1. Let m, h, R be the lengths of median CM, height CH, and the circumscribed circle's radius, respectively. Construct right triangle $\triangle CMH$ with the hypotenuse m and a leg h; see Fig. 131.

FIGURE 131. (Sol. 7.1.7-8.1)

FIGURE 132. (Sol. 7.1.9-10.2)

The center O of the circumscribed circle is constructed as the intersection point of the straight line MN parallel to CH with the circle of radius R centered at C (and therefore, two solutions are possible). The points A and B are the intersection points of MH with the circumscribed circle.

Since $OM = h \pm \sqrt{R^2 - m^2 + h^2} < R$, then

$$\text{for } R > h \text{ we have: } \begin{cases} \text{one solution} & \text{if } m^2 \leq 2Rh, \\ \text{two solutions} & \text{if } 2Rh < m^2 \leq R^2 + h^2, \\ \text{one solution} & \text{if } m^2 = R^2 + h^2, \\ \text{no solutions} & \text{if } m^2 > R^2 + h^2. \end{cases} .$$

$$\text{for } R \leq h \text{ we have: } \begin{cases} \text{one solution} & \text{if } m^2 < 2Rh, \\ \text{no solutions} & \text{othersise.} \end{cases}$$

7.1.7-8.3. $\triangle BCP = \triangle ADQ$ in three sides. Indeed, each side of the triangles is equal to either a half of one of quadrilateral's sides or a half of one of its diagonals.

7.1.9-10.2. Connect each vertex A_i of the parallelogram with the centers of the two squares that contain this vertex, see Fig. 132.

Now, prove that all four obtuse triangles thus obtained are equal. This fact implies that all sides of the quadrilateral constructed are equal and the angle at vertex B_i is equal to $\angle A_iB_iA_{i+1} = 90°$. Hence, this quadrilateral is a square.

7.1.9-10.3. First of all observe that $P(x) - P(0) \vdots x$ and $P(x) - P(1) \vdots (x-1)$.

Let $P(n) = 0$. If n is even, then the difference $P(n) - P(0)$ is also even and $P(n)$ is odd, contradiction. If n is odd, then it is the difference $P(n) - P(1)$ that is even, whereas $P(1)$ is odd, contradiction again.

7.1.9-10.4. CONSTRUCTION, see Fig. 133:

FIGURE 133. (Sol. 7.1.9-10.4)

i) Draw the midperpendicular to segment MN. This midperpendicular meets the given line l at point O.

ii) Draw the circle with radius $OM = ON$ and the center O.

iii) The points, where this circle intersects line l, are vertices A and B.

iv) Connect A with N, and B with M. Lines AN and BM meet at the third vertex of $\triangle ABC$.

7.1.9-10.6. Plot the sketches of the graphs of the functions $y = \sin x$ and $y = \frac{x}{100}$ and take into account that $\frac{100}{\pi} \approx 31.8$. It is clear that the number of roots is equal to $31 \cdot 2 + 1 = 63$.

7.2.7-8.2. Most plausible, the answer (for the triangles in the general position) is 3, see Fig. 134 a), where O is the center of the inscribed circle. Certain triangles can be divided into 2 parts or even into 1 part (isosceles ones). Fig. 134 b–e) illustrate the only ways to divide into two parts we know (into two symmetric polygons).

On Fig. 134c) $\alpha + k\beta = (k-1)\pi$, where $k \geq 3$ is an integer; on Fig. 134d) $2\alpha = k\beta$, where $k \geq 1$ is an integer.

FIGURE 134. (Sol. 7.2.7-8.2)

7.2.7-8.3. There are 6 segments, see Fig. 135. Indeed, each step cuts off the triangle ABC a similar triangle with coefficients k or $1-k$, one after the other; the similarity coefficient at the first step is $k<1$. Therefore, at the sixth step we will pass again through point M.

FIGURE 135. (Sol. 7.2.7-8.3)

FIGURE 136. (Sol. 7.2.7-8.6)

REMARK. If M lies on the midline of $\triangle ABC$ parallel to BC, we will already have returned to M at the third step.

7.2.7-8.5. Every prime $p>3$ is of the form $6n\pm 1$; hence, $p^2=12n(3n\pm 1)+1$.

7.2.9-10.2. PROOF of Lemma from Hints. If there are two neighboring arcs the measure of whose union is $\leq 180^\circ$, then the point that separates the arcs can be erased without violating the property mentioned in Lemma.

Let us show that for $n\geq 4$ such an extra point always exists. Indeed, if for all neighboring pairs of arcs the measure of their union is $>180^\circ$, then the sum total of the measures of all pairs is $>n\cdot 180^\circ\geq 720^\circ$ for $n\geq 4$. But, on the other hand, this sum is equal to 720° since each arc was counted twice. Contradiction. Hence, we can diminish the number of points until n becomes equal to 3. Q.E.D.

Now, finish the proof of the statement of the problem on your own.

7.2.9-10.4. By multiplying (both parts of) the equation by 4 we get $(2x-y-1)^2+3(y-1)^2=4$. Hence, $|y-1|\leq 1$, therefrom $y=0$, 1, or 2.

7.2.9-10.5. Let one of the lines be x-axis, L the other line, see Fig. 137.

FIGURE 137. (Sol. 7.2.9-10.5)

Denote the length of the segment by l, the distance between the lines by h. Since the distance from $(2x,0,0)$ to $(0,2y,h)$ is equal to l, it follows that $(2x)^2+(2y)^2+h^2=l^2$, i.e., the locus is the circle:

$$x^2+y^2=\frac{1}{4}(l^2-h^2).$$

8.1.7-8.4. Let $\triangle ABC$ be the triangle considered and let $\triangle CPB=\triangle ACP$. The areas of these triangles are equal, hence, $AP=PB$. Side CP is the common one; hence, the remaining sides are equal, i.e., $AC=CB$, but this contradicts the fact that $\triangle ABC$ is a scalene one.

8.1.7-8.5. Observe that the polygon whose sides connect the tangent point is an isosceles trapezoid. Let R and r be the lengths of the radii of the circles, and α the angle between the larger base of this isosceles trapezoid and the radius of the larger circle drawn to one of the trapezoid's vertices. Then the side of the trapezoid is equal to $(R+r)\cos\alpha$ and the distances from the trapezoid's vertices to the straight line connecting the centers of the circles are $R\cos\alpha$ and $r\cos\alpha$. This implies the assertion of the problem, Q.E.D.

8.1.9-10.4. The locus that point A fills in is segment A_1A_2 shown on Fig. 138. The length of A_1A_2 is equal to $c-a$, where c is the length of the hypotenuse and a is the length of the shorter leg of $\triangle ABC$.

FIGURE 138. (Sol. 8.1.9-10.4)

8.2.7-8.4. The midpoints of the segments AA_1, BB_1, CC_1 lie on the sides of the triangle formed by the midlines of triangle ABC. But no straight line can intersect all sides of a triangle unless it passes through its vertex.

8.2.9-10.2. To prove the statement of Hint make use of the fact that $a_1 = \pm 1$ and the formula

$$\sqrt{2+b} = \sqrt{2\left(1 + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)} = \sqrt{2}\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right) = 2\sin\left(\frac{\alpha}{2} + \frac{\pi}{4}\right).$$

8.2.9-10.3. Let ABC be the equilateral triangle and let the circle be rolling along side BC. Consider triangle $AB'C'$ symmetric to triangle $\triangle ABC$ through vertex A. Then $\angle B'AC'$ and $\angle BAC$ subtend equal arcs. But $\angle BAC$ is equal to a half sum of the angle measure of these arcs and, on the other hand, its measure is 60°.

9.1.7-8.1. Suppose in a convex n-gon there are k acute angles; the sum of these angles is $< 90^\circ \cdot k$ and the sum of the other angles is $< 180^\circ \cdot (n-k)$. Making use of the formula for the sum of the angles of a polygon we can rewrite the inequalities as

$$90^\circ \cdot k + 180^\circ \cdot (n-k) > 180^\circ \cdot (n-2),$$

implying $k < 4$. It is easy to constuct an example with exactly 3 acute angles for any n.

9.1.7-8.2. If we rotate the plane through 60° about point B, we transfer points A and A_1 to points C_1 and C, respectively. Therefore, the midpoint M of segment AA_1 becomes the midpoint N of segment CC_1. It follows that $BM = BN$ and $\angle MBN = 60^\circ$. Hence, triangle BMN is an equilateral one, see Fig. 139.

FIGURE 139. (Sol. 9.1.7-8.2)

9.1.7-8.3. Let $x = 131a + 112 = 132b + 98$ be the desired number. Then $131(a - b) = b - 14$ and since $b < 100$, it follows that $a = b = 14$. Hence, the answer is 1946.

9.1.9-10.1. In plane α, draw a line B_1B_2 parallel to the intersection line l of planes α and β, see Fig. 140. Let us assume that $B_1A \perp l$. The orthogonal projection of line B_1B_2 to β is line $C_1C_2 \in \beta$. With the help of the theorem on three perpendiculars it is not difficult to see that $\triangle AB_1B_2$ (in plane α) is a right one and AB_2 is its hypotenuse; hence, $AB_2 > AB_1$. If now we compare right triangles AB_1C_1 and AB_2C_2, we see that they have equal legs $B_1C_1 = B_2C_2$ but distinct hypotenuses: $AB_2 > AB_1$. Hence, $\angle B_1AC_1 > \angle B_2AC_2$, Q.E.D.

FIGURE 140. (Sol. 9.1.9-10.1)

FIGURE 141. (Sol. 9.1.9-10.2)

9.1.9-10.2. Let $AM = MB$, see Fig. 141; let KL be another segment passing through M. Draw AP parallel to KB with P on KL. Now it is easy to see that $\triangle AKM = \triangle BPM$ and, therefore, $S_{\triangle AKM} = S_{\triangle BPM} < S_{\triangle BLM}$.

9.1.9-10.3. Observe that $n^2 + 3n + 5 = (n - 4)^2 + 11n - 11$. Hence, this sum is divisible by 11 if $n - 4$ is divisible by 11. In this case $(n - 4)^2$ is divisible by 121 but $11(n - 1)$ is not divisible by 121.

9.1.9-10.4. Consider the identities

$$\begin{aligned}(2n)! &= (1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n - 1))(2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n)\\ &= (2n - 1)!! \cdot 2^n \cdot (1 \cdot 2 \cdot 3 \cdot \dots \cdot n) = 2^n \cdot (2n - 1)!! \cdot n!\end{aligned}$$

9.1.9-10.5. The derivative of the function $y = \frac{\tan x}{x}$ is $\dfrac{x - \frac{\sin 2x}{2}}{x^2(\cos x)^2}$. It is positive when $x \in]0, \frac{\pi}{2}[$, since $\sin 2x < 2x$.

9.2.7-8.2. The expression in the left hand side can be written as

$$(x - 2y)(x - y)(x + y)(x + 2y)(x + 3y).$$

Every two factors in this product are, clearly, distinct if $y \neq 0$. But 33 cannot be represented as a product of more than four distinct factors.

9.2.7-8.3. Let $OA = a$, $OB = b$, $\angle AOB = \alpha$. There are two ways to select points M and N, as illustrated on Fig. 142a) and b).

FIGURE 142. (Sol. 9.2.7-8.3)

a) $f(x) = (MN)^2 = (a-x)^2 + (b-x)^2 - 2(a-x)(b-x)\cos\alpha$. Since $f(0) = f(a+b)$, the minimum of the quadratic is attained at point $x_1 = \frac{1}{2}(a+b)$, the midpoint between 0 and $a+b$; the minimum is equal to

$$f(x_1) = \frac{(a-b)^2}{2}(1+\cos\alpha) = (a-b)^2\cos^2\tfrac{\alpha}{2}.$$

b) Set $g(x) = (a-x)^2 + (b+x)^2 - 2(a-x)(b+x)\cos\alpha$. Clearly, $g(0) = g(a-b)$; hence, the minimum is attained at point $x_2 = \frac{1}{2}(a-b)$; it is equal to

$$g(x_2) = \frac{(a+b)^2}{2}(1-\cos\alpha) = (a+b)^2\sin^2\tfrac{\alpha}{2}.$$

By comparing the two minima of cases a) and b) we see that $f(x_1) \le g(x_2)$ for $\tan\frac{\alpha}{2} \ge \frac{a-b}{a+b}$, and $f(x_1) \ge g(x_2)$ otherwise. Thus:

$$x = \begin{cases} \frac{1}{2}(a+b) & \text{if} \quad \tan\frac{\alpha}{2} \ge \dfrac{a-b}{a+b}; \\ \frac{1}{2}(a-b) & \text{if} \quad \tan\frac{\alpha}{2} \le \dfrac{a-b}{a+b}. \end{cases}$$

ANOTHER SOLUTION. In order to avoid trigonometry we can reason as follows. Let us prove that $\triangle OMN$ is an isosceles triangle ($OM = ON$). Let $OM \ne ON$; let M' and N' be some points such that $MM' = NN'$. Let us prove that $M'N' > MN$. Let us construct segment $N'P$ equal and parallel to $M'M$, see Fig. 142c).

Then $M'N' = MP$; to prove that $MP > MN$ we can compare the angles $\angle MNP$ and $\angle MPN$. Since $\triangle NPN'$ is an isosceles one, $\angle PNN' = \angle NPN'$; so we can compare $\angle MNN'$ with $\angle MPN'$. But $\angle MNN' = 180^\circ - \angle MNO$ and $\angle MPN' = 180^\circ - \angle PMO > 180^\circ - \angle NMO$; therefore, $\angle MNN' > \angle MPN'$ (since $\angle MNO = \angle NMO$); hence, $MP > MN$.

9.2.7-8.4. 1°. Denote by a_n the road emerging from A_n. If the car m, i.e., from A_m, is stopped at intersection P_{mn}, then the angle α_n is closer to 90° than angle α_m (the road a_n has greater slope than a_m).

2°. Let $m < n$. The roads a_m and a_n intersect if $\alpha_m < \alpha_n$ and do not intersect if $\alpha_m \ge \alpha_n$.

3°. It follows from 1° that if the road a_n has a greater slope than any of the roads which intersect it, then the car n will pass all intersections.

4°. Let a_m have the greatest slope of all the roads intersecting a_n. If the slope of a_m is greater than the slope of a_n, then car m will not be stopped before point P_{mn} (prove by the rule of contraries).

5°. It follows from 4° and 1° that if the car n passes all the intersections, then every road intersecting a_n has a lesser slope than a_n (cf. with statement 3°). It follows from the table that the road with the greatest slope is a_{14}. Hence the car 14 will not be stopped. Every car on a road intersecting a_{14} will be stopped. These are cars: 1, 2, 3, 4, 6, 7, 8, 10, 12, 13, 18, 19, 22, 27, 28, 29, 30.

Proceeding with this analysis we finally see that only the cars 14, 23, 24 will not be stopped, and the answer does not depend on the distances between the points, see Fig. 143.

FIGURE 143. (Sol. 9.2.7-8.4)

FIGURE 144. (Sol. 9.2.7-8.5)

REMARK. We encounter similar problems in crystallography. Crystals grow in random directions. If the crystals are long and thin then these directions can be determined by the angle between the crystal and the surface of the initial crystal body. The long crystal halts the growth of any other crystal which bumps into it. This corresponds to closing a gate in our problem.

9.2.7-8.5. If N is the total number of all routes in the town and every route has n stops, then $n(n-1)+1=N$. Since by the hypothesis $n=3$, we have $N=7$. An example with 7 routes is plotted on Fig. 144, cf. Problem 9.2.9-10.4.

9.2.9-10.1. Let n be the number of 9-th graders and let them score, say, m points. Then the number of 10-th graders is equal to $10n$ and they scored $4.5m$ points. So the total number of participants in the tournament is $11n$ and they scored $5.5m$ points. There were $\frac{11n(11n-1)}{2}$ games played during the tournament, and every game adds exactly one point. Hence, $5.5m = \frac{11n(11n-1)}{2}$ or $m = n(11n-1)$. Everyone played $11n-1$ games. So the 9-th graders played $n(11n-1)$ games. But this is exactly the number of points they scored, which means that each of them won all his/her games. This is only possible when $n=1$ because no 9-th grader cannot beat another 9-th grader. Thus, $n=1$ and $m=10$.

9.2.9-10.2. Consider the sequence $\{a_k\}_{k\in\mathbb{Z}}$, where a_k is the number formed by the last four digits of the k-th term of the given Fibonacci sequence and $a_k \leq 10^4$. Then the number a_{k-1} can be determined from a_k and a_{k+1} (explain why).

Now, if $a_k = a_{n+k}$ and $a_{k+1} = a_{n+k+1}$, then $a_{k-1} = a_{n+k-1}, a_{k-2} = a_{n+k-2}, \dots, a_1 = a_{n+1}$. But $a_1 = 0$ and so $a_{n+1} = 0$ and the $(n+1)$-th term of the given sequence ends with four zeros.

It remains to show that among 10^8+1 pairs (a_1, a_2), (a_2, a_3), $\dots$, (a_{10^8+1}, a_{10^8+2}) there are two identical ones. Make use of Dirichlet's principle: there is no a_k greater than 10^4 and one can compose only $(10^4)^2 = 10^8$ different pairs from the numbers 0, 1, 2, $\dots$, 10^4.

Extension. Show that the first term of the Fibonacci sequence which end with four zeros is the 7501-th one.

9.2.9-10.3. The answer does not change if we move any of segments AB, CD, EF along PQ, QR, RP, respectively, and also if we zoom all three segments with the same coefficient. Therefore, we may assume that $A = P = E$ and $B = Q = C$, see Fig. 145. Now the answer is obvious: this is the segment parallel to FD and passing through S_0: the sum of areas to be found is equal to $S_{PQR} - S_{RDSF}$.

Figure 145. (Sol. 9.2.9-10.3)

If $\frac{AB}{PQ} = \frac{CD}{QR} = \frac{EF}{RP}$, the points F and D coincide with vertex B and the locus to be found is the whole triangle.

10.1.7-8.1. Subtract 1 from every monomial. Now, all new summands are divisible by $x-1$. Hence, the remainder is equal to 6.

10.1.7-8.2. Only 2, 3, 5, 7 can be prime common divisors of any pair (since the difference between the numbers is ≤ 8) and 7 can only be divisor of the first and the penultimate or of the second and the last numbers.

If 5 of the 9 numbers are odd, then among them there is a number not divisible by either 3, or 5, or 7; it is the desired number.

If 4 of the 9 numbers are odd, then the only case when there is no number not divisible by 3, 5, 7 is the one when the smallest and the greatest numbers are divisible by 3 whereas the numbers in between them are divisible one by 5 and the other by 7. In this case the number divisible by 7 (but not by 2, 3, 5) is relatively prime with the other numbers.

10.1.7-8.3. There is no term with x^{18} because 18 cannot be written in the form of a sum of 5's and 7's with nonnegative coefficients. The coefficient of x^{17} is $20 \cdot \binom{19}{2} = 3420$.

10.1.9-10.1. Denote the given polynomial expression by $P_k(x)$. The constant term of $P_k(x)$ is equal to

$$P_k(0) = ((\dots((0-2)^2-2)^2-\dots)^2-2)^2 = \dots = (4-2)^2 = 4.$$

Denote the coefficients of x and x^2 by A_k and B_k, respectively. Now, use the fact that $P_k(0) = 4$ and $P_k(x) = (P_{k-1}(x)-2)^2$ to prove that $A_k = 4A_{k-1}$ and $B_k = A_{k-1}^2 + 4B_{k-1}$. We have

$$A_k = 4A_{k-1} = \dots = -4^k$$

because $(x-2)^2 = x^2-4x+4$ and $A_1 = -4$. The formula for B_k expressed in terms of A_{k-1}, A_{k-2}, ..., A_1 and $B_1 = 1$ takes the form:

$$B_k = 4^{k-1}\cdot\frac{4^k-1}{4-1} = \frac{4^{2k-1}-4^{k-1}}{3}.$$

10.1.9-10.3. There are $(9-n)^2$ squares of size $n\times n$ on the chessboard. Hence, the total number of squares is $1^2+2^2+\dots+8^2 = 204$.

10.1.9-10.4. Denote the first polynomial by $P(x)$ and the second one by $Q(x)$. Substitute $(-x)$ for x. Then we find that the coefficients at even powers of x do not change. Hence, the coefficients of x^{20} in $P(x)$ and $Q(x)$ are the same as in the polynomials $(1+x^2+x^3)^{1000}$ and $(1-x^2-x^3)^{1000}$, respectively. But, clearly, the first of these new polynomials has a greater coefficient of x^{20} than the other one.

10.1.9-10.6. Consider the given straight line as the axis of a cylinder of radius d. Through point M draw two planes tangent to the cylinder. Every straight line p in these planes is also tangent to the cylinder. It follows that the distance between p and the given straight line equals d. Evidently, no other straight line meets this condition.

10.2.7-8.1. Let us divide all cubes into 10 pairs. Let us place the first pair on the left pan of the balances, while on the right pan we consequently place the second pair, the third pair, etc., until we will have found two pairs of different weight. In the heavier pair one of the cubes might be duralumin one; we recognize the other cube by comparing the cubes of this pair. (If their weights are equal they are both of duralumin; if not, all is clear.) Similarly, we use one weighing to determine the content of the lighter pair. Thus, we know the number of alumin cubes in all pairs that we have already weighed.

Next, let us put two cubes of different weight on the left pan and start comparing with them the remaining pairs of cubes. In this way we determine the number of alumin cubes in each pair.

Altogether we have compared pairs of cubes 9 times and a cube with a cube 2 times. Thus, the total number of weighings is equal to 11.

It remains to consider the case when all pairs are of the same weight (we will have established that this is the case in 9 weighing). Then either all cubes are from alumin, or each pair has an alumin cube and a duralumin one. We can settle the case in one (tenth) weighing.

10.2.7-8.2. Since $2^{10} = 1024$, the number 2^{100} is a bit greater that $1000^{10} = 10^{30}$. Hence, 2^{100} consists of 31 digits.

10.2.7-8.5. Any number from 1 to 200 can be represented in the form $\frac{a}{2^k}$, where $100 < a \le 200$, $k \ge 0$. Among the 101 numbers there are two with equal numbers a. This is the pair to be found.

10.2.9-10.1. The case $k = 1$ is obvious.

Suppose $k \ge 2$. Consider a triangle T with vertices 1, 2, 3. There are $k-1$ triangles adjacent to each vertex of T. All of these $3(k-1)$ triangles are different (otherwise some two of them would share two vertices). Every triangle has a common vertex with T; hence, there are no triangles but triangle T and $3(k-1)$ triangles adjacent to it; altogether $n = 3(k-1)+1$. Hence, the total number of vertices is equal to $\frac{3n}{k} = \frac{3(3k-2)}{k} = 9 - \frac{6}{k}$ implying $k = 1, 2, 3, 6$.

The case $k = 6$ is, however, impossible: in this case there are only 8 points and the number of edges, $3n = 48$, is greater then $\frac{8\cdot 7}{2} = 28$, the number of segments with endpoints in the 8 points.

Thus, $n = 3(2-1)+1 = 4$ for $k = 2$ and $n = 3(3-1)+1 = 7$ for $k = 3$, see Fig. 146a) and b), respectively.

10.2.9-10.2. The parity of the first four numbers in any row of our triangle depends only on the parity of the first four numbers in the previous row. We can see that this parity in our table will repeat with period of 4 rows. It remains to check which of the first four rows is the first to contain even numbers.

FIGURE 146. (Sol. 10.2.9-10.1)

10.2.9-10.4. Afer Hint, there remains to be considered the case when all the a_i are distinct. Clearly, among them there are numbers divisible by 1, 3, 5, ... , 15. It remains to consider the cases when the first of the numbers selected is an even one.

1) $x_1 = 14$. Then either $\frac{126}{2^k}$ is divisible by x_1 or divides 189.

2) $x_1 = 12$ or 6. Then $\frac{144}{2^k}$ either is divisible by x_1 or equal to 9 (in which case we may assume that $x_1 = 9$), or is equal to 18. But then $\frac{108}{2^l}$ either is divisible by 18 or equal to 27 (in which case it divides 135).

3) $x_1 = 10$. Then $\frac{110}{2^k}$ either is divisible by 10, or divides 165.

4) $x_1 = 8$, 4 or 2. Then $\frac{192}{2^k}$ is either divisible by x_1, or is equal to 12, 6, or 3. Therefore, we may assume that $x_1 = \frac{192}{2^k} = 12$, 6, or 3.

FIGURE 147. (Sol. 10.2.9-10.5)

10.2.9-10.5. Draw a plane through edge AB of the tetrahedron parallel to CD and project the edge CD to the plane. It is not difficult to show that the projection $C'D'$ intersects AB at a point E, the midpoint of both segments. Indeed, it suffices to observe that heights CM and DN of triangles $\triangle CAB$ and $\triangle DAB$, respectively, are equal, see Fig. 147.

This means that quadrilateral $AC'BD'$ is a parallelogram, i.e., $AC' = BD'$ and $BC' = AD'$. It is easy to see now that $AC = BD$ and $CB = AD$. The equality $AB = CD$ can be proved in the same way.

11.1.7-8.2. All possible solutions are shown in Fig. 148, where segments of length b are plotted by solid lines.

FIGURE 148. (Sol. 11.1.7-8.2)

If three of the 4 points form an equilateral triangle with side a, then the fourth point is either at distance b from all vertices (case 2)), or at distance a from one vertex and at distance b from two vertices (cases 4), 6)), or at distance a from two vertices and at distance b from one vertex (case 3)).

If no three of the 4 points form an equilateral triangle, then some three of the 4 points always form an isosceles triangle with sides a, a and b. The fourth point can only be at distance b from the vertex of the triangle (the vertex outside the base) and the distance from the other vertices (at the base) are equal to either a, a or to a, b (in the other cases there are equilateral triangles). Prove on your own that in these cases there are unique solutions — those depicted on Figs. 1) and 5), respectively.

The task to determine the exact value of $\frac{a}{b}$ is left to the reader.

11.1.7-8.3. Let us apply induction. Suppose we have managed to paint all domains formed by k straight lines in the required way. Draw a $(k+1)$-st straight line and alter the color of every domain on the left of the line. Then the condition of the problem is still satisfied. Hence, the induction is complete, since for $k=1$ the statement is obviously true.

11.1.9-10.1. Let $m = \dfrac{2^n-2}{n}$. Then $2^n-2 = mn$ and

$$\frac{2^{2^n-1}-2}{2^n-1} = 2\frac{2^{mn}-1}{2^n-1} = 2((2^n)^{m-1} + (2^n)^{m-2} + \cdots + 2^n + 1).$$

11.1.9-10.3. The statement of the problem is equivalent to the following one: the sum of all planar angles at vertices B, C, and D of pyramid $ABCD$ is *greater* than the sum of planar angles at the same vertices of pyramid $A'BCD$, see Fig. 150.

But such an inequality holds for *each* of the vertices B, C, and D *separately*, i.e.,

$$\angle ABC + \angle ABD > \angle A'BC + \angle A'BD \tag{$*$}$$

and similar statement is true for the other two vertices. Inequality $(*)$ follows from the fact that in a trihedral angle the sum of two planar angles is always greater than the third angle:

$$\begin{aligned}\angle ABC + \angle ABD = \angle ABC + \angle ABA'' + \angle A''BD &> \angle A''BC + \angle A''BD \\ = \angle A''BA' + \angle A'BC + \angle A''BD &> \angle A'BC + \angle A'BD,\end{aligned}$$

where A'' is the intersection point of CA' with plane ABD.

FIGURE 149. (Sol. 11.1.9-10.4)

11.1.9-10.4. Denote by t_x the length of the tangent to the given circle drawn from point x, see Fig. 149. Let $ABC\ldots KLA$ be the route under consideration. Then its "length" with signs is

$$(t_A - t_B) + (t_B - t_C) + \cdots + (t_K - t_L) + (t_L - t_A) = 0.$$

FIGURE 150. (Sol. 11.1.9-10.3)

FIGURE 151. (Sol. 11.2.7-8.2)

11.2.7-8.2. To finish the proof, get another triangle by drawing tangents to the circumscribed circle parallel to the sides of the triangles, see Fig. 151. The new triangle is also similar to the initial one and the similarity coefficient is equal to R/r. Moreover, this triangle contains the first of the constructed triangles. Hence, $\frac{R}{r} \geq 2$ and $R \geq 2r$ (the equality is only possible for an equilateral triangle). Q.E.D.

11.2.7-8.4. In quadrilateral $ABCD$ let M and N be the midpoints of sides AB and CD, respectively; let K be the midpoint of diagonal AC, see Fig. 152.

FIGURE 152. (Sol. 11.2.7-8.4)

Segments MK and KN are the midlines of triangles ABC and ACD, respectively; therefore, $MK = \frac{1}{2}BC$, $KN = \frac{1}{2}AD$, $MK \parallel BC$, $KN \parallel AD$. From triangle MNK we have $MN \leq MK+KN = \frac{1}{2}(BC+AD)$, so if $MN = \frac{1}{2}(BC + AD)$, then we immediately get $K \in MN$ and $MN \parallel BC \parallel AD$.

11.2.7-8.5. Suppose $\alpha = \frac{90^\circ}{n}$; let us "rectify" the trajectory of the ray of light by reflecting the angle with respect to its sides $2n$ times.

FIGURE 153. (Sol. 11.2.7-8.5)

We obtain an angle of 180°, see Fig. 153a). It is obvious that the ray of light will stop reflecting in the sides of the angle after the reflection of the straightened ray from the side of the angle of 180°. By making the reflections in reverse order we see that the first ray will finally move parallel to its initial movement but in the opposite direction. In this case the ray makes $2n = \frac{180^\circ}{\alpha}$ reflections.

If $\alpha \neq \frac{90^\circ}{n}$, then the number of reflections is $\left[\frac{180^\circ}{\alpha}\right] + 1$, see Fig. 153b).

11.2.9-10.1. Let $y = kx$ with a rational k. Then the given equation implies that $x = k^{\frac{1}{k-1}}$ and $y = k^{\frac{k}{k-1}}$. Let the fraction $\frac{p}{q}$ be the irreducible form of $\frac{1}{k-1}$. Then we have $x = \left(\frac{p+q}{p}\right)^{\frac{p}{q}}$ and $y = \left(\frac{p+q}{p}\right)^{\frac{p+q}{q}}$. But x and y are rational and p and q are relatively prime, therefore, the numbers p and $p+q$ are perfect q-th powers. If $q \geq 2$ and $p = n^q$, then $n^q < p+q < (n+1)^q$. Hence, $q = 1$.

11.2.9-10.3. Let us count the number of ordered pairs (x, y) for which $|x| + |y| = k$. If $x = l$, $y = k - l$ is a solution and $l \neq 0$, $k - l \neq 0$, then there are three extra solutions: $(-l, k-l)$, $(l, -k+l)$ and $(-l, -k+l)$. If $l = 0$ or $l = k$, then we have only two solutions: $(0, k)$ and $(0, -k)$ or, respectively, $(k, 0)$ and $(-k, 0)$. Therefore, when $k \neq 0$ the total number of solutions equals $2 + 4(k-1) + 2 = 4k$. For $k = 0$ there is only one solution, $(0, 0)$.

Thus the inequality $|x| + |y| < 100$ has $1 + 4(1 + 2 + 3 + \cdots + 99) = 19\,801$ integer solutions.

11.2.9-10.4. Let us prove that it is impossible to find 5 such rays. Assume the contrary and through the common vertex of the rays draw the plane perpendicular to one of the rays. The four remaining rays form obtuse angles with the chosen one. Hence, the rays lie on one side of the plane and form the edges of a convex tetrahedral angle, the sum of its planar angles is $< 360^\circ$. It follows that one of these planar angles is $< 90^\circ$. Contradiction.

11.2.9-10.5. Let us associate with the given trihedral angle a rectangular coordinate system $Oxyz$. The ray of light that moves in the direction of vector (x, y, z) will move after being reflected in plane Oxy in the direction of vector $(x, y, -z)$. Therefore, after it will has been reflected from all faces, it will move in the direction of vector $(-x, -y, -z)$.

12.1.7-8.2. If one of the axes of symmetry is reflected with respect to another, then the reflected line is also an axis of symmetry (prove this). If three axes form a triangle, then by reflecting this triangle through its sides one gets an infinite number of symmetry axes which is impossible.

If two axes are parallel, then we get a similar contradiction reflecting them repeatedly with respect to each other.

Another solution. Use the fact that the axes of symmetry pass through the center of mass of a weightless polygon with equal masses at the vertices.

12.1.7-8.4. Take a point A on the broken line and find a point B on it such that the lengths of both parts of the broken line connecting A with B are equal to $\frac{1}{2}$. Let O be the midpoint of segment AB; let us prove that *the circle with center O and radius $\frac{1}{4}$ contains the whole broken line*, see Fig. 154.

Indeed, let M be a point on the broken line and M' be the point which is symmetric to it through O. The quadrilateral $AMBM'$ is a parallelogram. It follows that $MM' \leq MB + BM' = AM + MB \leq \frac{1}{2}$, which gives $OM \leq \frac{1}{4}$. Hence, point M lies inside the circle.

Figure 154. (Sol. 12.1.7-8.4)

Figure 155. (Sol. 12.1.7-8.5)

12.1.7-8.5. Let M be the midpoint of segment IE that connects the centers of the inscribed and an escribed circles (this segment is a part of the bisector of $\angle A$). To prove that points A, B, M, and C lie on one circle we have to prove that $\angle ABM + \angle ACM = 180°$, see Fig. 155.

Observe that the angles $\angle IBE$ and $\angle ICE$ are right ones (formed by bisectors of supplementary angles). Therefore, $BM = IM = CM$ and $\angle IBM = \angle BIM$, $\angle ICM = \angle CIM$. Hence,

$$\angle ABM = \angle ABI + \angle IBM = \angle ABI + \angle BIM = \angle CBI + \angle BIM$$

and, similarly, $\angle ACM = \angle BCI + \angle CIM$, whereas $\angle ABM + \angle ACM$ is equal to the sum of the angles of $\triangle IBC$, i.e., to 180°.

12.1.9-10.2. A finite silid body cannot have either skew or parallel axes of symmetry (prove this on your own).

Let O be the intersection point of the axes of rotation. If point A belongs to the body, then the surface of a sphere centered at O and of radius OA also belongs to the body. Therefore, the body may only be the union of the following bodies: (1) concentric spheres, (2) a hollow ball (spherical layer), or (3) a sphere. Hence, any plane passing through O is a plane of symmetry.

12.1.9-10.3. Denote by y and y_1 the left and right sides of the equation, respectively. By expressing x in terms of y_1 we get $x = y_1^2 + 2ay_1 + \frac{1}{16}$, that is the expression for x in terms of y_1 is the same as the expression for y in terms of x. This means that $y_1(x)$ is the inverse function of $y(x)$ and the graphs of y and y_1 (both are parabolas) are symmetric with respect to the straight line $y = x$. The roots of the equation $y_1 = y$ correspond to the intersection points of the graphs. But these points lie on the axis of symmetry $y = x$, which gives $y = x = y_1$. This equation gives immediately all real roots (recall that $0 < a < \frac{1}{4}$):

$$x_{1,2} = \frac{1-2a}{2} \pm \sqrt{\left(\frac{1-2a}{2}\right)^2 - \frac{1}{16}}.$$

12.1.9-10.4. Assume the contrary. Then it is possible to choose five different numbers $a_1 < a_2 < a_3 < a_4 < a_5$ from our set. Concider geometric progressions a_1, a_2, a_3, a_4 and a_1, a_2, a_3, a_5. We obtain: $a_1a_4 = a_2a_3 = a_1a_5$. Thus, $a_4 = a_5$, which contradicts the inequality $a_4 < a_5$.

12.1.9-10.5. Consider the trapezoid whose bases are a pair of opposite sides of the hexagon, see Fig. 156. By the hypothesis, the trapezoid's diagonals are equal. Therefore, the midperpendiculars to the considered pair of sides of the hexagon are the bisectors of the angle between the diagonals of the hexagon, i.e., bisectors of the angles of the triangle formed by three diagonals of the hexagon. The bisectors of this triangle meet at a point I. This is the point where the midperpendiculars to the sides of the hexagon meet.

FIGURE 156. (Sol. 12.1.9-10.5)

FIGURE 157. (Sol. 12.2.7-8.2)

12.2.7-8.1. Change the order of points on the circle so that a checker can get from one point to an adjacent one in one move. That is, place 6 after 1, then place 11, 4, 9, 2, 7, 12, 5, 10, 3, 8 and again 1. Now, it is clear that the only way to rearrange checkers is to move them along the circle in one direction or the other.

12.2.7-8.2. Points Z fill in a polygon whose sides are parallel to the sides of the triangles ABC and DEF, see Fig. 157.

If no two sides of these triangles are parallel, then the polygon (it is called the *vector sum of the sets* $\triangle ABC$ and $\triangle DEF$) has 6 sides. They are equal and parallel to the respective sides of triangles ABC and DEF. Hence, the perimeter of the polygon is equal to the sum of the triangles' perimeters.

If the triangles have parallel sides, then the number of sides of the polygon may be less than 6. One of its sides in that case may be parallel to a pair of parallel sides of the triangles and its length be equal to the sum of the lengths of the parallel sides. Therefore, the statement on perimeters holds, nevertheless.

12.2.7-8.2. v russkom est'

12.2.7-8.5. Consider 100 sums a_1, a_1+a_2, $a_1+a_2+a_3$, $\dots$, $a_1+a_2+\dots+a_{100}$. If none of these sums is divisible by 100, then, by Dirichlet's principle, some two of these sums have equal remainders after division by 100. Subtracting the lesser of these sums from the other one we get the sum to be computed.

12.2.9-10.3. Let us prove that of the centrally symmetric polygons inscribed in $\triangle ABC$ with the given center of symmetry O the greatest in area is the intersection of triangle ABC with triangle $A'B'C'$ symmetric to ABC with respect to O, see Fig. 158.

Let point O be at distance $x\cdot h_a$ from BC, at distance $y\cdot h_b$ from AC, and at distance $z\cdot h_c$ from AB. Then $x+y+z=1$ (from comparison of area of $\triangle ABC$ with the sum of areas of $\triangle AOB$, $\triangle BOC$, $\triangle COA$) and the areas of white triangles, see Fig. 158, are equal to $(1-2x)^2S_{\triangle ABC}$, $(1-2y)^2S_{\triangle ABC}$, and $(1-2z)^2S_{\triangle ABC}$, respectively. It remains to find the minimum of the function $(1-2x)^2+(1-2y)^2+(1-2z)^2$ provided $x+y+z=1$.

Now, choose point O so that the area of the intersection $\triangle ABC\cap\triangle A'B'C'$ were maximal. This happens when O coincides with the intersection point of the medians of $\triangle ABC$.

Figure 158. (Sol. 12.2.9-10.3)

Figure 159. (Sol. 12.2.9-10.5)

12.2.9-10.5. Let the initial square $PQRS$ be a unit one. Let us construct the square $P'Q'R'S'$ with respectively parallel sides at the distance $\frac{1}{2}$ from the sides of the initial square. The perimeter of $P'Q'R'S'$ is equal to 8. It suffices to prove that each juxtaposed square covers a part of the contour of $P'Q'R'S'$ of length ≥ 1.

If the square constructed intersects the initial one along a segment of the contour, this is obious. If the intersection is a point, 4 cases are possible:

1) A segment AB of straight line is covered; its endpoints lie on the neighboring sides of the adjacent square, see Fig. 159a)

Let M be the midpoint of AB. Since $\angle BCA=90^\circ$, it follows that $CM=\frac{1}{2}AB$. Let us drop perpendicular CH to AB; clearly, $CH=\frac{1}{2}$. Since $CM\geq CH$, it follows that $AB=2CM\geq 2CH=1$.

2) A segment AB of straight line is covered; its endpoints lie on opposite sides of the juxtaposed square. Then the length of AB is greater than or equal to that of the side of the adjacent square, i.e., ≥ 1.

3) Covered are: a vertex of the constructed square and the two segments that go out of it; the vertex of the adjacent square coincides with a vertex of the initial square, see Fig. 159b). Let us drop perpendiculars PH and PK to $P'A$ and $P'B$, respectively. We have:

$$PH=PK=\frac{1}{2},\quad \angle HPA=90^\circ-\angle APK=\angle KPB;$$

hence, $\triangle PHA=\triangle PKB$ and $AH=KB$. But then $AP'+P'B=HP'+P'K=1$.

4) Covered are: a vertex of the constructed square and the two segments that go out of it; vertex C of the adjacent square lies on a side of the initial square, see Fig. 159c). Let us prove that $AP'+P'B\geq 1$. Let us move the adjacent square by the vector $\overrightarrow{CP}$. This movement diminishes the covered brocken line $AP'B$ by segment AA' but augments by segment BB' resulting in brocken line $A'P'B'$ of length 1, cf. case 3).

It suffices to prove that $AA'>BB'$. Let M be the intersection point of lines SP with BC, and N that of PQ with $P'B$. Clearly, $BB'=MP$, $AA'=CP$ and $\triangle MPC$ is similar to $\triangle BNC$. But $BN<\frac{1}{2}<NC$; hence, $MP<PC$ and $BB'<AA'$, Q.E.D.

13.1.7-8.2 (a)**.** It is easy to divide the weights of 1, 2, 3, …, 9 grams into three groups of equal mass: $(1,5,9)$; $(2,6,7)$; $(3,4,8)$.

(b) Divide the weights from 1 to 549 grams into 61 groups of 9 weights each: $(1,2,\dots,9)$; $(10,11,\dots,18)$, etc., and divide each group into three piles as in (a) .

(c) Divide the remaining 6 weights into 3 groups as follows: $(550,555)$; $(551,554)$; $(552,553)$. The rest is clear.

13.1.7-8.3. One solution is indicated in Hints.
ANOTHER SOLUTION: see Problem 13.1.9-10.5.

13.1.7-8.4. Since the angle subtending the longer side of the triangle is wider than the angle subtending the shorter side, it follows that

$$(A-B)(a-b)\geq 0,\quad (B-C)(b-c)\geq 0,\quad (C-A)(c-a)\geq 0.$$

The sum of these inequalities yields the one to be proved.

13.1.7-8.5. Suppose that for 2 km one must walk. Then there is even more reason for one to walk for 1 km. Hence, 2 km should take twice as much time as 1 km, which is not the case; hence, to overcome 2 and 3 km, one must wait after the 1st km and take a cab.

Let v be the speed of the pedestrian, V the speed of the cab, t the time spent awaiting the cab. Let speed be measured in km per hour, time in hours. From the condition it follows that

$$\min(\frac{1}{v},\ t+\frac{1}{V})=\frac{1}{6},\quad \min(\frac{2}{v},\ t+\frac{2}{V})=\frac{1}{4},\quad \min(\frac{3}{v},\ t+\frac{3}{V})=\frac{7}{24}.$$

This gives $\frac{1}{v}\geq\frac{1}{6}$; therefore, $\frac{2}{v}>\frac{1}{4}$ and $\frac{3}{v}>\frac{7}{24}$. Hence, $t+\frac{2}{V}=\frac{1}{4}$ and $t+\frac{3}{V}=\frac{7}{24}$, which implies $\frac{1}{V}=\frac{1}{24}$ and $t=\frac{1}{6}$. Therefore, the trip from A to a point distant by 6 km takes

$$\min(\frac{6}{v},\ \frac{1}{6}+\frac{6}{V})=\min(1,\ \frac{1}{6}+\frac{6}{24})=\frac{10}{24}h=25\text{ min}.$$

ANOTHER SOLUTION. Suppose that one must walk for 2 km. Then there is even more reason for one to walk for 1 km. But 2 km should take twice the time required for 1 km which is not the case; hence, to cover 2 and 3 km one waits and takes a cab. Now we have a system of two equations to solve.

13.1.9-10.1. From the inequalities

$$\sin B\sin C+\cos B\cos C=\cos(B-C)\leq 1 \text{ and } \cos A\leq 1$$

it follows that

$$\sin B\sin C\leq 1-\cos B\cos C\leq 1-\cos B\cos C\cos A.$$

Therefore,

$$0<\frac{\sin B\sin C}{1-\cos B\cos C\cos A}\leq 1.$$

13.1.9-10.2. In addition to examples from Answers there are lots of other examples.

FIGURE 160. (Sol. 13.1.9-10.2)

Let A be the vertex of the octant, $AC=AD=1$, $AB=3$; let A' be chosen close enough to vertex B (for example $A'B=1$). Then $AB+AC+AD=5$, and $A'B+A'C+A'D=1+2\sqrt{5}>5$.

13.1.9-10.3. First, divide 9 weights of n^2, $(n+1)^2$, …, $(n+8)^2$ grams into three groups:

1st group: n^2, $(n+5)^2$, $(n+7)^2$;

2nd group: $(n+1)^2$, $(n+3)^2$, $(n+8)^2$;

3rd group: $(n+2)^2$, $(n+4)^2$, $(n+6)^2$.

The first two groups have equal masses and the third is 18 g lighter. Next, divide the next 9 weights so that the first and the third groups were of equal mass; and, finally, divide the next 9 weights into three groups so that the first group would be 18 g lighter than the other two.

Gathering all weights from the first three groups together and doing the same with the second and third groups, we obtain a distribution of any 27 consecutive weights into three groups of equal total weight. Repeating this three times we get the desired distribution of 81 weights.

13.1.9-10.4. Expressing the terms under outer radicals as perfect squares we get:

$$|\sqrt{x-1}-2|+|\sqrt{x-1}-3|=1 \Longleftrightarrow 2\le\sqrt{x-1}\le 3.$$

Squaring the latter inequalities we get the answer: $5\le x\le 10$.

13.1.9-10.5. First, let $n=3$. To prove that B_4 coinsides with B_1, it suffices to verify that points B_1, B_3 and B_3 lie on one straight line. This is equivalent to the fact that $\angle B_1+\angle B_2+\angle B_3=\pi$ in $\triangle B_2B_3B_4$, see Fig. 161.

Observe that $\angle B_1=\frac{1}{2}\smile A_3OA_1$, as an inscribed one. On the other hand, O_3O divides arc $\smile A_3O$ in halves; hence, the central angle $\angle O_3O_1O$ is equal to $\frac{1}{2}\smile A_2OA_3$. This implies that

$$\angle O_3O_1O_2=\angle O_3O_1O+\angle OO_1O_2=\frac{1}{2}\smile A_3O+\frac{1}{2}\smile OA_1=\frac{1}{2}\smile A_3OO_1=\angle B_1.$$

Similarly, $\angle B_2=\angle O_1O_2O_3$ and $\angle B_3=\angle O_2O_3O_1$; hence, $\angle B_1+\angle B_2+\angle B_3$ is equal to the sum of angles of $\triangle O_1O_2O_3$, i.e., π.

FIGURE 161. (Sol. 13.1.9-10.5)

To prove that points B_1 and B_4 coincide, it suffices to demonstrate that $\smile A_1OA_2+\smile A_2OA_3+\smile A_3OA_1=2\pi$ (this sum is equal to $2(\angle A_1B_2A_2+\angle A_2B_3A_3+\angle A_3B_1A_1)$). But as soon as we consider $\triangle O_1O_2O_3$ with its vertices at the centers of the circles, we see that the sum of the halves of arcs $\smile A_1OA_2$, $\smile A_2OA_3$ and $\smile A_3OA_1$ is equal to π (the sum of the angles at the vertices $O_1O_2O_3$). Q.E.D.

The general case can be easily reduced to the above, if you notice that for n circles the broken line $B_1B_2\dots B_{n+1}$ is divided by segment B_3B_1 into triangle $B_1B_2B_3$ and broken line $B_1B_3B_4B_5\dots$ with a smaller number of links, both broken lines satisfying the condition of the problem.

13.2.7-8.1. Obviously, among the sides of this polygon there are at most two diagonals beginning at one vertex. This easily implies that the polygon has at most 13 vertices. On the other hand, starting with a regular 13-gon we get a polygon with 13 vertices. Hence, the answer is 13.

The case of a polygon with any odd number of vertices is treated similarly.

EXTENSION. Try to investigate yourself the case of a polygon with an even number of vertices. (See Problem 13.2.9-10.1.)

FIGURE 162. (Sol. 13.2.7-8.3)

13.2.7-8.3. Call the union of two segments of neighboring sides of the square from their common vertex to the points of contact with the circle an "*angle*" of the square, see Fig. 162. Then

1°. It is clear that at least one "angle" of the square lies entirely inside the triangle.

2°. If two "angles" lie inside the triangle, we are done: at least $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$ of the perimeter of the square lies inside the triangle.

3°. Suppose now that only one "angle" of the square lies inside the triangle (the other three lie partly outside it). It is not difficult to show (using the fact that the doubled length of the hypotenuse of a right triangle is longer than the sum of the lengths of the triangle's legs) that in this case more than one-third of each "angle" lies inside the triangle. This implies that more than $\frac{1}{4}+3(\frac{1}{3}-\frac{1}{4})=\frac{1}{2}$ of the perimeter of the square is contained in the triangle.

13.2.9-10.1. If MN is the diagonal of the $2n$-gon that connects the opposite (in order) vertices and our polygon lies, say, on the right of MN, then all diagonals of the $2n$-gon that constitute the small polygon should have at least one endpoint to the right of MN. But to the right there are only $n-1$ vertices; hence, there can be not more than $2(n-1)$ diagonals: two from each vertex. Together with diagonal MN there are $2n-1$ diagonals.

13.2.9-10.2. From the given sequence $a_1, a_2, \dots, a_{101}$ select a decreasing subsequence $b_1, b_2, \dots$ by the following rule: $b_1=a_1$; let b_2 be the first among the a_i which are less than b_1; let b_3 be the first among the a_i that lie after b_2 and are less than b_2, etc.

Take the selected numbers b_i away and repeat the procedure with the remaining numbers to obtain the second, third, etc., subsequences. In this way we split all 101 numbers of the sequence $\{a_i\}_{i=1}^{101}$ into m subsequences. If at least one of the subsequences has more than 10 terms, then this is the desired sequence.

Otherwise, $m\geq 11$ and it is possible to take one number from each subsequence so that they form an increasing subsequence (prove this yourself).

REMARK. This problem is a finite analogue of Problem 48.8.4.

FIGURE 163. (Sol. 13.2.9-10.3)

13.2.9-10.3. Let AC intersect the sphere at points P and Q; let S be the intersection point of KL with AC. Let us prove that AS does not depend on the radius of the circle, the section of the sphere with plane ABC. This would imply that MN intersects AC at the same point S.

Let us draw the segment $L'A$ parallel to LC. Triangle $\triangle AKL'$ is an isosceles one (because it is similar to $\triangle BLK$); hence, $AL' = AK$.

From similarity of triangles $\triangle SAL'$ and $\triangle SCL$ we deduce:

$$\frac{SA}{SC} = \frac{AL'}{CL} = \frac{AK}{CL}.$$

But $AK = \sqrt{AP \cdot AQ}$, $CL = \sqrt{CP \cdot CQ}$; $SC = SA + AC$; therefore, $\frac{SA}{SA + AC} = \sqrt{\frac{AP \cdot AQ}{CP \cdot CQ}}$, i.e., the length of SA is expressed in terms of the lengths of AP, AQ, AC, PC, QC and does not depend on the radius of the circle.

For another solution see the solution to Problem 19.1.10.4.

13.2.9-10.4. On a plane, consider 10 straight lines no two of which are parallel and no three of which intersect at one point. Take the lines for the bus routes and their intersection points for the bus stations. Then one can go from any station to any other station without changing buses if they are on the same straight line and changing buses only once otherwise.

Further, after closing one of the routes it is still possible to get from any station to any other with at most one bus change. But when two routes are closed, the station at their intersection is not connected to the other stations by the remaining routes.

14.1.7-8.1. If $x \geq 1$, then the value of the polynomial is also ≥ 1 since $x^{12} \geq x^9$ and $x^4 \geq x$. The same is true when $x \leq 0$, since $-x^9 \geq 0$ and $-x \geq 0$. Lastly, when $0 < x < 1$ we have $-x^9 + x^4 > 0$, $-x + 1 > 0$, and $x^{12} > 0$. Hence, the polynomial always takes a positive value.

14.1.7-8.2. Let $\angle A \geq \angle A'$. Then $BD > B'D'$ and so $\angle C > \angle C'$. If, moreover, the inequality $\angle B \geq \angle B'$ were true, then thanks to Lemma from Hint it would have implied $\angle D \geq \angle D'$, which is impossible (the sum of four angles is a constant).

14.1.7-8.3. Let $\alpha = 1.00...04$, $\beta = 1.0...02$ (ten zeros after the decimal point in both α and β). Then the first number is $\frac{1+\alpha}{1+\alpha+\alpha^2}$ and the second one is $\frac{1+\beta}{1+\beta+\beta^2}$. The identity

$$\frac{1+x+x^2}{1+x} = 1 + \frac{1}{\frac{1}{x} + \frac{1}{x^2}}$$

shows that

$$\frac{1+\alpha}{1+\alpha+\alpha^2} < \frac{1+\beta}{1+\beta+\beta^2} \Longleftrightarrow \alpha > \beta.$$

14.1.7-8.4. Let us mirror trapezoid $ABCD$ through its midline MN and then move by vector $\overrightarrow{MN}$. As a result, we get trapezoid $A'B'C'D'$, see Fig. 164. Let P' be the image of P under the above transformation. Quadrilateral $PCP'D$ meets the condition of the problem.

FIGURE 164. (Sol. 14.1.7-8.4)

14.1.9-10.5. Introduce the same uniform scale of measurement on each of the straight lines. Call these lines x, y, and z axes and place them horizontally one above the other, their origins on a single vertical. Then introduce the scale $I = 10^x$ onto the x-axis; the scale $R = 10^{-2y}$ onto the y-axis, and the scale $V = 10^{-z}$ onto the z-axis. This means that instead of x on the old scale we write the number $I = 10^x$, etc.

Since the midline of a trapezoid is equal to a half sum of the bases, the line connecting points I and R on the first two axes meets the third axis at point $V = I \cdot R$.

14.2.7-8.1. This number is greater than $(10^{50})^3$, not equal to $(10^{50}+1)^3$, and less than $(10^{50}+2)^3$. Thus, it is not a perfect cube.

14.2.7-8.5. Suppose that student A_1 has solved problems a_1 and a_2. Ask this student to divulge the solution of problem a_1. By the condition, there is only one student, A_2, who has also solved problem a_2. Let A_2 tell the solution of the other problem (s)he has solved. Proceeding like that we will eventually reach with our question a student who has also solved problem a_2.

Think on your own: what shall we do if not all of the students are asked?

14.2.9-10.1. Let S be the vertex of the pyramid; O the center of the inscribed sphere; H and P the points where the sphere touches the base and face SAB of the pyramid, respectively; let AB be one of the sides of the base.

It is not difficult to see that $OH \perp AB$ and $OP \perp AB$. Let plane POH intersect edge AB at point M. Then $MP = MH$ as two tangents to a sphere drawn from one point. Rotate face SAB about edge AB until it coincides with the base, then point P coincides with H.

The second statement follows from the first one and the equality of all tangents drawn to the sphere from a point (vertex S).

14.2.9-10.2. There are $\left[\frac{1951}{a_n}\right]$ multiples of the number a_n in the sequence 1, 2, ..., 1951. Since the least common multiple of any two of the given numbers $a_1, a_2, \dots, a_n$ is greater than 1951, it follows that no number from 1 to 1951 is divisible by any two of the a_i. Therefore, there are exactly $\left[\frac{1951}{a_1}\right] + \left[\frac{1951}{a_2}\right] + \dots + \left[\frac{1951}{a_n}\right]$ numbers from 1 to 1951 which are divisible by at least one of the a_i. But there are not more than 1951 such numbers. Hence,

$$\left(\frac{1951}{a_1} - 1\right) + \dots + \left(\frac{1951}{a_n} - 1\right) < 1951 \quad \text{and} \quad \left[\frac{1951}{a_1}\right] + \dots + \left[\frac{1951}{a_n}\right] \le 1951.$$

This gives

$$\frac{1951}{a_1} + \dots + \frac{1951}{a_n} < 1951 + n < 2 \cdot 1951$$

and

$$\frac{1}{a_1} + \dots + \frac{1}{a_n} < 2, \quad \text{Q.E.D.}$$

14.2.9-10.3 (See also solution to Problem 25.2.9.4.) Let the projection of the tetrahedron — call it $ABCD$ — coincide with the projection of some of its two faces, say ABC and BCD. Let K, L, M, N be the midpoints of AB, AC, CD and BD, respectively. Then it is easy to see that $KLMN$ is a square the area of whose projection is equal to a half area of the projection of the tetrahedron. Therefore, for the area of the tetrahedron's projection to be the greatest possible, the plane of the projection must be parallel to that of the square. The area of this projection is equal to $\frac{a^2}{2}$, where a is the length of the tetrahedron's edge.

If the tetrahedron's projection is a triangle, its area does not exceed $a \cdot \frac{h}{2} < \frac{a^2}{2}$.

14.2.9-10.4. After circle, the "next in simplicity" curve with the required property is the union of two arcs each of measure 120° with the radii of the corresponding circles equal to the side of the triangle (Fig. 165).

Extension. How to construct other curves with the required property (there are infinitely many of them)?

14.2.9-10.5. a) Consider all possible pairs of stops (i, j), $i < j$. On the plane with coordinates (i, j), distinguish points that correspond to such pairs, see Fig. 166.

Let no passenger ride from the i-th to the j-th stop; mark all such points (i, j) on our diagram, see Fig. 166. Now, we have to prove that on the diagram there are four marked points no two of which lie in one row or in one column.

Consider the 7×7 square $P = \{(i, j) : i \le 7, j \ge 8\}$. Since between the 7-th and the 8-th stops there are not more than 25 passengers in the bus, the number of unmarked points in square P is ≤ 25. It follows that P contains at least $49 - 25 = 24$ marked points.

Figure 165. (Sol. 14.2.9-10.4)

Figure 166. (Sol. 14.2.9-10.5)

Let us prove that it is possible to choose from these 24 points the desired 4 ones. Solution of the problem follows then from the next Lemma.

Lemma. *In a 7×7 square 24 cells are marked by a cross. Then it is possible to select 4 rows, r_1, r_2, r_3, r_4, and 4 columns, c_1, c_2, c_3, c_4, so that on the intersection of r_i with c_i there stands a cross for all $i = 1, 2, 3, 4$.*

Proof of Lemma. Let us show that it is possible to delete *one* row and *one* column on whose intersection a cross stands (in what follows such an operation will be referred to as *deleting a cross*) and leave *not less* than 15 crosses on the 6×6 square obtained.

If each row and each column has *not more* than 5 crosses, the statement is clear: we can delete any cross. If a row has 6 crosses, then one of the corresponding 6 columns has no more than 4 crosses, so we may select this column together with the row.

The case of 7 crosses in a row (i.e., in succession) is similarly treated.

Further, we should show that from a 6×6 square with 15 crosses a cross can be deleted so that in the 5×5 square obtained there remains not less than 8 crosses. Finally, delete a row and a column for the third time leaving at least 1 cross. This will successively finish the proof. Do these steps on your own. □

b) Let us divide the proof in steps.

i) Let us enumerate all stops from 1 to 14 and connect with arrows all pairs (i, j) of the first ten stops so that if $1 \leq i < j \leq 10$, then the arrow connecting i and j begins in i and ends in j. An arrow from i to j signifies that there is a passenger who takes the bus at stop i and gets off at stop j. (For the last four stops the bus is empty.)

ii) Let us prove that example from step *i*) meets the condition of the problem, i.e., at every moment there are no more than 25 passengers in the bus. Consider any two successive stops k and $k + 1$. When the bus drives from stop k to stop $k + 1$ there are as many passengers in it as there are arrows beginning in one of the stops 1, 2, …, k and ending in one of the stops $k+1$, $k+2$, …, 10. One can see now that the number of passengers is $k(10 - k) \leq 25$.

iii) Let us now prove the assertion of the problem. To this end, let us consider the pairs (A, B) of stops between which the passengers take the bus. Let us connect all such pairs of points by segments. It is clear that any of these segments has one of the stops 11, 12, 13, 14 as at least one of its endpoints.

iv) Let us call a selection of exactly one endpoint on each segment a "projection" of a segment. Let us now perform a "projection" that results in 4 points: 11, 12, 13, and 14. We will say that *segments do not intersect* if they have no endpoint in common. Now, assuming that there exist 5 *nonintersecting segments* we get a contradiction: these segments cannot be "projected" into 4 points (because a "projection" of 5 nonintersecting segments always results in 5 points). Q.E.D.

15.1.7.1. Triangles $\triangle ALM$ and $\triangle BLN$ are isosceles, see Fig. 167. Therefore,

$$\begin{aligned}\angle MLN &= 180^\circ - \angle BLN - \angle ALM \\ &= 180^\circ - \left(\frac{180^\circ - \angle A}{2} + \frac{180^\circ - \angle B}{2}\right) = \frac{\angle A + \angle B}{2} = \frac{180^\circ - \angle C}{2}.\end{aligned}$$

This is, obviously, an acute angle.

FIGURE 167. (Sol. 15.1.7.1)

FIGURE 168. (Sol. 15.1.8.1)

15.1.7.3. Indeed, let B, C, D be three adjacent to A vertices of the parallelepiped. Then one of the parallelograms is constructed on AB and AC and another on AB and AD. If $AB \neq AC$, then AC must be equal to AD for the parallelograms to be equal. But then the third parallelogram constructed on AC and AD is a rhombus. This means that all these parallelograms are rhombuses.

15.1.8.2. From the condition of the problem it easily follows that the trip takes least time if A and B simultaneously arrive in N. (Cf. Problem 15.1.7.4.) Then the distances they walked are equal, and the same holds for the distances they rode on bicycles. If each of them walks x km, then the length of the remaining trip is equal to $15 - x$ km. Hence, while B walks, A and C together ride bicycles for $(15 - x) + (15 - 2x)$ km. This gives $\frac{x}{6} = \frac{30 - 3x}{15}$; i.e., $x = \frac{60}{11}$ km.

All this means that C should walk $\frac{60}{11}$ km from N until the meeting with A, and A should ride a bicycle $15 - \frac{60}{11} = \frac{105}{11}$ km. To do this, it takes $\frac{10}{11}$ of an hour for C and $\frac{7}{11}$ of an hour for A. Therefore, C must have started from N by $\frac{3}{11}$ of an hour earlier than A started from M.

15.1.9.2. Without loss of generality we may assume that $x \geq y$. Then the conditions imply that the numerator and the denominator are positive. Therefore, we can omit the absolute value sign and multiply (both parts of) the inequality by the denominator. Now, the inequality takes the form $(1 - x)(1 + y) > 0$ which is a direct consequence of the condition. Cf. the solution to Problems 28.1.10.1.

15.1.9.3. It can be seen from Fig. 169 that triangles ABD and BCD contain discs the sum of whose radii is equal to $R = r_1' + r_2'$. But $r_1' < r_1$, $r_2' < r_2$, hence, $R < r_1 + r_2$, Q.E.D.

FIGURE 169. (Sol. 15.1.9.3)

15.1.10.2. Proof by induction. For $n = 2$ the inequality is obvious if $|x| < 1$. If it holds for some n, then

$$(1 - x)^{n+1} + (1 + x)^{n+1} < \big((1 + x)^n + (1 - x)^n\big)\big((1 - x) + (1 + x)\big) < 2^n \cdot 2 = 2^{n+1}.$$

15.1.10.4. If $a > 0$, then it is clear that for a large enough p there are no solutions at all. If $a < 0$, then it is also clear (look at the graph) that for a large enough p the greatest root is negative. Therefore, $a = 0$.

15.1.10.5. The statement follows from the fact that there is only one straight line which is perpendicular to two skew lines and intersects both of them.

15.2.7.3 and **15.2.8.1.** If $\alpha < 1$, then $1 > \alpha > \alpha^2$. The square root of these inequalities yields $1 > \sqrt{\alpha} > \alpha$. Hence, if $\beta = 0.99...9$ (n nines), then $1 > \sqrt{\beta} > 0.99...9$ (n nines).

FIGURE 170. (Sol. 15.2.8.2)

15.2.8.1. See the solution to Problem 15.2.7.3.

15.2.8.3. Any number consisting of digits 1, 2, ..., 7 is congruent to 1 modulo 9. Therefore, if the ratio of two such numbers is an integer $k > 0$, then $k - 1$ is divisible by 9. It follows that $k = 1$ or $k \geq 10$. On the other hand, it is clear that the ratio of any two such numbers is not greater than 7.

15.2.9.1. If n is even, then

$$\begin{aligned} x_1 = x_3 = x_5 = \dots \quad &= a; \\ x_2 = x_4 = x_6 = \dots = x_n \quad &= \frac{1}{a}, \end{aligned}$$

where a is any non-zero number.

15.2.9.4. Draw straight line AR through vertex A, so as to form an angle of 60° with base AC, see Fig. 171.

Connect point P with point S, the intersection point of AR with QC. Prove that $\triangle QPS = \triangle QRS$. It is easy to observe that triangles $\triangle ACS$ and $\triangle RQS$ are equilateral ones; hence, $RQ = QS$ and $AC = CS$. Calculating angle $\angle CPA$, let us establish that $\triangle ACP$ is isosceles and, therefore, $CP = CA = CS$. Since $CP = CS$, it follows that $\triangle CPS$ is also an isosceles one. The angles of $\triangle ACP$ are: 20°, 80° and 80°. Now, calculate the angles of $\triangle SPQ$; we find that $\triangle SPQ$ is also an isosceles triangle (with angles 100°, 40° and 40°). Therefore, $PR = PS$ and triangles $\triangle PRQ$ and $\triangle PSQ$ are equal in three sides.

This gives $\angle PQS = \angle PQR$ but the sum of these angles is 60°. Hence, $\angle PQC = 30°$.

15.2.9.5. Consider the soldier C who stands in the same row as A and in the same column as B; see Fig. 172. Then the heights of the soldiers are as follows: $H_A \geq H_C \geq H_B$.

15.2.10.1. Denote the sum given by $f(x)$. Since $\int\limits_0^{2\pi} \cos kx\, dx = 0$ for any $k \in \mathbb{N}$, we have

$$\int\limits_0^{2\pi} f(x)dx = 0. \tag{$*$}$$

Equation ($*$) and continuity of $f(x)$ imply that if f takes positive values, then it takes negative values as well. All that remains is to show that $f(x)$ is not identically equal to zero. But this is obvious.

15.2.10.3. Let t_1, t_2 be the roots of the equation $3t^2 + at + 2 = 0$. Then $t_1 \cdot t_2 = \frac{2}{3}$. Hence, the roots of the first and the second polynomials are $x_i = \sqrt[n]{t_i}$ and $x_i' = \sqrt[m]{\frac{1}{t_i}}$ $(i = 1, 2)$, respectively. We have $x_1 x_2 = \sqrt[n]{\frac{2}{3}} < 1$ and $x_1' x_2' = \sqrt[m]{\frac{3}{2}} > 1$.

If the first polynomial is divisible by the second one, then all roots of the second polynomial are roots of the first polynomial as well, but this is impossible thanks to the above inequalities.

FIGURE 171. (Sol. 15.2.9.4)

FIGURE 172. (Sol. 15.2.9.5)

16.1.7.1. Let AD be the longer base of the trapezoid, and BC the shorter. Draw the line through point B parallel to CD. It cuts off the trapezoid a triangle ABE, where E lies between A and D (because $AD > BC$); therefore,

$$\angle A + \angle D = \angle A + \angle AEB = 180^\circ - \angle ABE < 180^\circ.$$

Hence, $\angle B + \angle C > 180^\circ$.

16.1.7.2. If the number 11...11 (n-many 1's) is divisible by 33...33 (100-many 3's), then n must be divisible by 3 and by 100. Hence, the smallest possible n is 300.

16.1.7.3. Let AB be the given segment, see Fig. 173. Draw the perpendicular l to AB through A. Mark points C and D on l. The rest is clear from Fig. 173.

FIGURE 173. (Sol. 16.1.7.3)

16.1.8.2. Denote by a the radical in the numerator. It is easy to establish that $a < 2$ (e.g., by induction on the number of radicals). The radical in the denominator is equal to $a^2 - 2$ and, therefore, $\frac{2-a}{2-(a^2-2)} = \frac{1}{2+a} > \frac{1}{4}$ because $a < 2$. Q.E.D.

16.1.9.2. Let $\overrightarrow{A_1A} = \vec{a}$, $\overrightarrow{B_1B} = \vec{b}$, $\overrightarrow{O_1O} = \vec{c}$. Then we have two equalities:

$$\vec{c} = \overrightarrow{O_1A_1} + \vec{a} + \overrightarrow{AO} \qquad \text{and} \qquad \vec{c} = \overrightarrow{O_1B_1} + \vec{b} + \overrightarrow{BO}.$$

Summing them and bearing in mind that $\overrightarrow{AO} + \overrightarrow{BO} = \overrightarrow{O_1A_1} + \overrightarrow{O_1B_1} = \vec{0}$ we get $\vec{c} = \frac{1}{2}(\vec{a} + \vec{b})$. Hence, $|\vec{c}| \le \frac{1}{2}(|\vec{a}| + |\vec{b}|)$.

16.1.9.3. Suppose there are two polynomials f and g such that $x^{200} \cdot y^{200} + 1 = f(x) \cdot g(y)$. Let a and b be the constant terms of f and g, respectively. (This means that $f(0) = a$ and $g(0) = b$.)

The substitutions: $x = 0$ (y arbitrary); $y = 0$ (x arbitrary) and $x = 0$, $y = 0$, give, respectively, $ag(y) = 1$, $bf(x) = 1$, and $ab = 1$. It follows that $f(x)g(y) = 1$ for any x and y which, clearly, is impossible.

REMARK. The polynomial is, actually, factorizable but in a different way. Set $z = (xy)^{40}$. We have $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$.

16.1.9.4. Since $BF \parallel CE$ and $AC \parallel EF$, the quadrilateral $AFEC$ is a parallelogram (even a rhombus), see Fig. 174. Therefore, $\triangle AFE = \triangle ACE = \triangle ABD$. Let us cut polygon $ABB_1CC_1DED_1$ into three triangles: $\triangle ABD$, $\triangle B_1CC_1$, and $\triangle DED_1$. Let us cut quadrilateral AD_1EF into triangles as well: $\triangle AEF$, $\triangle E_1ED_1$, and $\triangle AEE_1$. Clearly, these triangles are pairwise equal: $\triangle ABD = \triangle AEF$, $\triangle B_1CC_1 = \triangle E_1ED_1$, $\triangle DED_1 = \triangle AEE_1$.

16.1.10.2. Construct a cone with vertex A congruent to the given one. Then the cone symmetric to the constructed one with respect to A is the locus to be found.

16.2.7.1. Denote: $N = LCM(m, n)$. If a prime p divides N, it also divides one of the numbers, m or n. Since p divides $m + n$, it divides the other number, too, so $GCD(m, n)$ has the same prime factors as $GCD(N, m + n)$. Verify on your own that their powers are also the same.

FIGURE 174. (Sol. 16.1.9.4)

16.2.7.2. Each diagonal bisects two opposite angles of the quadrilateral. Thus,

(1) each diagonal divides the quadrilateral into two equal triangles.

Furthermore, this quadrilateral is circumscribed, hence

(2) its longest and shortest sides are opposite one another.

From these two statements it follows that the sides of the quadrilateral are of the same length, i.e., the quadrilateral is a rhombus.

16.2.7.4. Let us find the sum of the angles of all triangles obtained after the division of the 1000-gon. The sum in question is equal to the sum of all angles of the 1000-gon plus 360° for every given point inside the 1000-gon, i.e., the sum is equal to $998 \cdot 180^\circ + 500 \cdot 360^\circ$. Therefore, there are $998 + 2 \cdot 500 = 1998$ triangles.

16.2.8.1. Let us divide the quadrilateral by its diagonal into two triangles whose areas are not greater than $\frac{1}{2}ab$ and $\frac{1}{2}cd$, respectively. Hence, $S \leq \frac{1}{2}(ab + cd)$. Drawing the other diagonal we prove that $S \leq \frac{1}{2}(ad + bc)$ and the sum of these inequalities gives the statement desired.

16.2.8.2. Let us prove that if a number $A = \overline{a_1 a_2 ... a_{1953}}$ is divisible by 27, then the number $B = \overline{a_{1953} a_1 ... a_{1952}}$ is also divisible by 27. To this end, it suffices to show that 27 divides $10B + a_{1953}$ with remainder of a_{1953}. This statement follows from the equation

$$10B + a_{1953} = a_{1953} 10^n + A = a_{1953} \cdot B + A + a_{1953}, \text{ where } B = \underbrace{99...9}_{1953-\text{times}}$$

because A and B are divisible by 27 (since $999 = 27 \cdot 37$ and $1953 = 3 \cdot 651$).

The rest is clear: we must move the figures one by one from the end of the number to its beginning until we encounter the initial number.

16.2.8.3. The first group contains not less polygons than the second one because adding the vertex A_1 to any polygon from the second group we get a polygon from the first group. But the cardinalities of these groups cannot be equal for it is not always possible to throw away a vertex from a polygon (e.g., when it is a triangle) from the first group to get a polygon from the second group.

16.2.8.5. Let us subtract from every equation the preceding one. Then from the last equation we find $x_{100} = 1$, from the penultimate one we get $x_{99} = 1 - 2x_{100} = -1$, and so on.

16.2.9.2. Let us introduce a coordinate system with l as the x-axis and let (a_1, b_2), (a_2, b_2), and (a_3, b_3) be the coordinates of the given points A_1, A_2, A_3, respectively. It is not difficult to find that the equations of the straight lines passing through these points are:

$$\begin{aligned} \frac{x - a_1}{a_2 - a_3} + \frac{y - b_1}{b_2 - b_3} &= 0 && (1) \\ \frac{x - a_2}{a_3 - a_1} + \frac{y - b_2}{b_3 - b_1} &= 0 && (2) \\ \frac{x - a_3}{a_1 - a_2} + \frac{y - b_3}{b_1 - b_2} &= 0. && (3) \end{aligned}$$

We multiply the first equation by $(a_2 - a_3)(b_2 - b_3)$, the second by $(a_3 - a_1)(b_3 - b_1)$, and the third by $(a_1 - a_2)(b_1 - b_2)$. Then it is easy to see that the sum of the left hand sides of the equations obtained is identically zero. This means that if the first two equations are satisfied, the third equation is also satisfied; hence, all the straight lines meet at the same point.

16.2.9.3. The function $f(x) = \frac{1}{2}ax^2 + bx + c$ takes values $-\frac{1}{2}ax_1^2$ and $\frac{1}{2}3ax_2^2$ at x_1 and x_2, respectively. These values have opposite signs which means that one of the roots of $f(x)$ lies between x_1 and x_2.

16.2.9.4. Let us tile the big $20\,200 \times 20\,200$ square with rectangular pieces of paper of size 101×200 placing their longer sides horizontally. Draw the diagonal of the big square from the lower left corner to the upper right one. The diagonal crosses some of the rectangles; let us call them *marked* rectangles. Repeatedly reflecting the initial rectangle through its sides so that it consecutively covers the marked rectangles, we get a (rather fancy) broken line (the trace of the big square's diagonal) on the initial big square from one corner of the big square to the opposite one.

17.1.7.2. Yes they are equal. Make countours of both polygons of wire. Denote by A, B, C and, respectively by A', B', C' the vertices of the n-gons for which the equalities we wish to prove take the form:

$$\angle A = \angle A', \quad \angle B = \angle B', \quad \angle C = \angle C'.$$

Cut both contours at these vertices. Each contour splits into three brocken lines. Place the broken lines obtained, $A' \dots B'$, $B' \dots C'$ and $C' \dots A'$, respectively, upon the broken lines $A \dots B$, $B \dots C$ and $C \dots A$. They coinside since the links and the angles between them are respectively equal. Therefore, $AB = A'B'$. Similarly, $BC = B'C'$ and $AC = A'C'$; hence, $\triangle ABC = \triangle A'B'C'$.

Now let us place polygons so that the vertices A, B and C coincides with A', B' and C', respectively. Observe that a brocken line with fixed points can be placed on the plane in two ways, but for one of the ways the inner angles become the outer ones, hence, will not coincide. Thus, the contours of these n-gons completely coincide in each of the three parts, hence, the n-gons are identically equal.

17.1.7.3. Let x be the 4-digit number to be found. According to the first division we have $x = \overline{pqr} \cdot a \geq 1000$ but $\overline{pq0} \cdot a < 1000$. Therefore, $1000 \leq x \leq 1081$. Since $p \cdot a = 9$, $q \cdot a \leq 9$ and $r \cdot a \geq 10$, it follows that either $a = 9$, $p = q = 1$ or $a = p = 3$, $q \leq 3$ (and then from $\overline{3qr} \cdot 3 \geq 1000$ we deduce that $q = 3$, $r \geq 4$).

Thus, the following cases are possible:

$$x = \overline{11r} \cdot 9 \ \ (r \geq 2) \quad \text{and} \quad x = \overline{33r} \cdot 3 \ \ (r \geq 4).$$

The second division yields: $x = \overline{s0t} \cdot b$, $s \cdot b = 10$, $t \cdot b \geq 10$. The following cases are possible:

$$x = \overline{20t} \cdot 5 \ \ (t \geq 2) \quad \text{and} \quad x = \overline{50t} \cdot 2 \ \ (t \geq 5).$$

The rest is a case-by-case checking.

17.1.7.4. Indeed, if $m^2 - n^2$ is not odd, then it is divisible by 4, but 1954 is not divisible by 4.

17.1.7.5. Indeed, $\dfrac{100a + 10b + c}{a + b + c} \leq 100$.

17.1.8.2. If we turn these vectors by $90°$ and multiply by 2, they become equal (as vectors) to the respective sides of the polygon. But the vector sum of the sides is zero, so the initial sum is also zero.

17.1.8.4. Getting rid of the denominators we reduce the system to the form

$$\begin{cases} 6x + 8y + 9z & = & 0, \\ 45x + 48y + 50z & = & 0, \\ 140x + 144y + 147z & = & 0. \end{cases}$$

Subtract the first equation from the second, and the second from the third:

$$\begin{cases} 39x + 40y + 41z & = & 0, \\ 95x + 96y + 97z & = & 0. \end{cases}$$

Subtracting one of these equations from the other we get $x + y + z = 0$. This easily implies $x = y = z = 0$.

Extension. It is obvious that this problem is a particular case (for $N = 3$) of a more general similar system in N unknowns where n takes the values 1, 2, …, N. Try to solve this general system similarly.

17.1.9.1. Let $P(x) = x_0^4 + a_1x_0^3 + a_2x_0^2 + a_3x_0 + a_4$. The condition of the problem means that $P(x_0) = P'(x_0) = 0$. Therefore, x_0 is a repeated root of $P(x)$ of multiplicity ≥ 2. Hence, $P(x) \,\vdots\, (x - x_0)^2$.

17.1.9.2. The greater of two numbers with the same number of digits is the one that has the greater first (from the left) digit which is not equal to the corresponding digit in the other number. So we will begin by deleting out the first (from the left) 84 figures, without touching the nines. We obtain 99 999 505152 . . . 99100. Then delete the next 15 digits: 5, 0, 5, 1,. . . , 5, 6, 5. All that remains is to delete only one figure — it is the 5 in 58. Hence, the answer.

17.1.9.3. The condition implies the following inequalities:

$$\begin{array}{rcl} a_1 - a_2 & \geq & 2(a_2 - a_3), \\ a_2 - a_3 & \geq & 2(a_3 - a_4), \\ \dots\dots\dots & & \dots\dots\dots \\ a_{100} - a_1 & \geq & 2(a_1 - a_2). \end{array}$$

Their sum is $0 \geq 0$; hence, all these inequalities are equalities. It follows that

$$a_1 - a_2 = a_2 - a_3 = \cdots = 0; \text{ hence, } a_1 = a_2 = a_3 = \cdots = a_{100}.$$

17.1.9.4. If the triangle is not an acute one, the solution is even simpler than the one indicated in Hints. If $\angle A$ is an obtuse angle, then the two adjacent angles are acute and, clearly, either C_1SA_1B or A_1SB_1C satisfies the condition.

FIGURE 175. (Sol. 17.1.9.4)

FIGURE 176. (Sol. 17.2.7.5)

17.1.9.5. With our Hint we can similarly prove that there exist no points A, B, C, D with the given distances between them only if $AD^2 + BC^2 > AB^2 + AC^2 + BD^2 + CD^2$.

REMARK. The problem is associated with an interesting question: *whether any metric space containing $n + 1$ distinct points can be immersed into an n-dimensional space*. If $n = 2$ the answer is in affirmative. Our problem shows that for $n = 3$ the answer is negative. We do not know what is the answer for other n.

17.1.10.3. See the solution to Problem 17.1.9.3.

17.2.7.2. The given system has the following property: *the absolute value $|a_k|$ of the coefficient of x_k in the k-th equation is greater than the sum of absolute values of all other coefficients (in the same equation).*

Let us make use of this property; let x_m be one of the numbers $x_1, \dots, x_7$ with the greatest absolute value. Then in the m-th equation $|a_m x_m|$ is greater than the sum of absolute values of all other summands. But the right hand side is equal to 0, hence, $x_m = 0$. Since $|x_m|$ is the greatest, it follows that $x_1 = \cdots = x_7 = 0$.

17.2.7.4. If 1 is written below the divisor $2q$, then there will be a -1 below the divisor q, and vice versa. As all divisors are divided into pairs $(q, 2q)$, the sum in question is equal to 0.

17.2.7.5. Let the car routes be plotted on a transparent map. Let us fold the map along a street. After folding, the routes remain *admissible* (since the angles of the turn remain equal to 120°). Let us fold the map until both routes are mapped into triangles. To achieve this, we have to take an equilateral triangle with side of length 2^k that contains both routes inside of it and proceed with folding as shown on Fig. 176. Now, both cars move around the inner triangle in one direction and with the same speed and, clearly, will never meet.

17.2.8.1. In the 17×17 square, there are 144 pairs of squares symmetric through the center of the square; the sum of the numbers in any of such pair of squares has a value from 2 to 140. As there are fewer than 144 different values of the sums, there exist two pairs of centrally symmetric squares in which these sums are identical. The centers A, B, C and D of these four squares are the vertices of the parallelogram to be constructed, the center of the parallelogram coincides with the center of the 17×17 square.

REMARK. It can happen that A, B, C and D lie on the same straight line (a "degenerate" parallelogram).

17.2.8.2 and **17.2.9.2.** Let O be the origin of the coordinate system; let m_1 be the y-axis, and let $|OA_1| = 1$.

If k_1, k_2, k_3 are the slopes of the other lines, then

$$A_1 = (0,1); \qquad A_2 = \Big(\frac{1}{k_1 - k_3},\ \frac{k_1}{k_1 - k_3}\Big); \qquad A_3 = \Big(\frac{1}{k_1 - k_3},\ \frac{k_2}{k_1 - k_3}\Big);$$

$$A_4 = \Big(\frac{k_1 - k_2}{(k_1 - k_3)^2},\ k_2\frac{k_1 - k_2}{(k_1 - k_3)^2}\Big); \qquad B = \Big(0,\ \frac{(k_1 - k_2)(k_3 - k_2)}{(k_1 - k_3)^2}\Big);$$

$$|OB| = \frac{(k_3 - k_2)(k_2 - k_1)}{(k_3 - k_1)^2} \le \frac{\left(\frac{k_3 - k_1}{2}\right)^2}{(k_3 - k_1)^2} = \frac{1}{4}.$$

17.2.9.3. Write the numbers $x_1, \dots, x_{100}$ and their squares as shown in Fig. 177. Let x, y, z be the greatest numbers among the first, second and the third hundreds, respectively (obviously it is possible to arrange them inside the intervals $(0, 100)$, $(100, 200)$, and $(200, 300)$ as shown in Fig. 177. Then the total area of the squares is less than the area below the broken line $ABCDEF$, which is equal to $100x + 100y + 100z$. Hence, $x + y + z \ge 100$.

FIGURE 177. (Sol. 17.2.9.3)

ANOTHER SOLUTION. (A *fantastic* one.) This solution contains the least possible number of implications. It is as short, as it is unclear what is the idea behind it. Here it is:

Let $x_1 \ge x_2 \ge \dots \ge x_{100} > 0$. If $x_1 \ge 100$, everything is proved. So, let us assume that $x_1 < 100$. Since

$$100 - x_1 > 0, \quad 100 - x_2 > 0, \quad x_1 - x_3 \ge 0, \quad x_2 - x_3 \ge 0,$$

we get:

$$\begin{aligned}
&100(x_1 + x_2 + x_3)\\
&\ge 100(x_1 + x_2 + x_3) - (100 - x_1)(x_1 - x_3) - (100 - x_2)(x_2 - x_3)\\
&= x_1^2 + x_2^2 + x_3(300 - x_1 - x_2) > x_1^2 + x_2^2 + x_3(x_3 + x_4 + \dots + x_{100})\\
&\ge x_1^2 + x_2^2 + \dots + x_{100}^2 > 10\,000
\end{aligned}$$

implying $x_1 + x_2 + x_3 > 100$. Q.E.D.

REMARK. Were the inequalities of the conditions of the problem not strict, the sum of the three greatest numbers could have been equal to 100 in exactly two cases: a) the greatest number is 100, and the rest are 0; and b) nine numbers are $\frac{100}{3}$, and the rest are 0. Prove on your own that there are no other possibilities.

17.2.9.4. It is clear that if the numbers a_i are rearranged in increasing order, then the sequence b_i' constructed for the rearranged sequence a_i is the sequence b_i whose terms are also arranged in increasing order:

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 5,\ 6,\ 7,\ 8,\ 8,\ 9,\ 10,\ 13,\ 13.$$

Hence, $a_1 < a_2 < a_3 < a_4 < a_5 < a_6 = a_7$. This contradicts the condition $b_8 = 6$.

17.2.10.1. The plane of symmetry should contain one edge of the tetrahedron and divide the opposite edge in halves. Therefore, there are not more than 6 planes of symmetry. Let us give examples when there are 1, 2, 3 and 6 planes; see Fig. 178:

— a regular pyramid $ABCD$: 6 planes;
— pyramid $ABCS$ (S lies on the height of pyramid $ABCD$): 3 planes;
— pyramid $ABLK$ (where $CL = DK$): 2 planes;
— pyramid $ABCK$: 1 plane.

Clearly, the case with 0 planes of symmetry is also possible. Let us prove that 4 or 5 planes of symmetry are impossible.

Indeed, each symmetry implies a) equality of two triangles (the pyramid's faces), b) two equalities of the pyramid's edges. Moreover, distinct symmetries imply distinct equalities. Therefore, 4 symmetries imply 8 equalities of edges. Since there are only 4 faces, one of them will be involved in ≥ 2 equalities; hence, this face is an isosceles triangle.

Furthermore, 4 symmetries imply 4 equalities of edges. Clearly, this is only possible if all faces are equal. With the above, all faces are equilateral. So our pyramid is, actually, a regular one, but the regular pyramid has 6 planes of symmetry.

FIGURE 178. (Sol. 17.2.10.1)

FIGURE 179. (Sol. 18.1.7.3)

17.2.10.4. It is easy to see that if $|x| \geq 1$, then the left hand sides of both given equations are positive. Therefore, their half-sum is positive, in particular, is not equal to 0, contradiction, Q.E.D. (To make this clearer consider the graphs of quadratics.)

18.1.7.3. Cf. the solution of Problem 58.9.2.

18.1.7.4. If $n^2+n+1 = 1955k$, then the remainder after division of $n^2+n = n(n+1)$ by 5 is 4. But it is easy to see that this remainder can only equal 0, 1 or 2. This means that such an integer n does not exist.

ANOTHER SOLUTION. If n^2+n+1 is divisible by 5, then $4(n^2+n+1) = (2n+1)^2+3$ is also a multiple of 5. But this is impossible since $a^2 \not\equiv 2 \pmod 5$ for any integers a.

18.1.8.1. The remainder after a division of 2^n by 30 is equal to either 2, or 4, or 8, or 16. Therefore, either $b = 2$, 4, 8 and a is divisible by 3, or $b = 6$, $a = 3k+1$.

18.1.9.2. Indeed, fix one endpoint, A, of some segment on the first circle (with radius r_1 and center O_1), and make the second endpoint move around the second circle (with radius r_2 and center O_2).

FIGURE 180. (Sol. 18.1.9.2)

The midpoint M of segment AB will move along the circle of radius $\frac{r_2}{2}$ and center at the midpoint of segment AO_2. The union of all circles formed as point A moves along the first circle gives the desired locus of all midpoints of segments AB (the set of the centers of the circles from the union is the circle with the center at the midpoint of segment O_1O_2 and radius $\frac{r_1}{2}$).

18.1.9.3. It follows from the second equation that $|x| \leq 1$, $|y| \leq 1$. But then $x^3 \leq 1$, and the first equation yields $y \geq 0$; similarly, $x \geq 0$. If $0 < x < 1$, then $x^3 > x^4, y^3 \geq y^4$ and the equations contradict one another. Therefore, either $x = 0$, $y = 1$, or $x = 1$, $y = 0$.

18.1.9.4. Examine the remainders after division of all numbers a_i by p. There are exactly p of them and there is no zero among them (otherwise the number a_i corresponding to it would not be a prime), therefore, there are two numbers, a_i and a_j, with equal remainders. Then $|a_i - a_j| = k \cdot d$, where d is the difference of the progression and k is an integer smaller than p and divisible by p. Since this is impossible, d is divisible by p.

18.1.10.1. Obviously, each number i occurs in the table n times. The number of i's not on the main diagonal of the table is even, and as n is odd, there must be an odd number of i's on the diagonal, i.e., there must be no fewer than one 1, one 2, etc., one n. But the main diagonal crosses exactly n small squares, therefore, each number i occurs on it exactly once, Q.E.D.

18.1.10.4. Indeed, draw a plane through points A, B, C equidistant from vertex O. Then the triangles OAB, OAC, OBC are isosceles ones; hence, the angles at sides AB, BC, AC are acute ones.

18.2.7.1. Clearly, x^3 is even and, moreover, divisible by 8. Divide the equation by 2, and notice that y^3 is even and, moreover, divisible by 8. Divide the equation by 2 once again; we see that z^3 is divisible by 8. Dividing the equation by 2 provides us with a modification of the original equation with all the unknowns halved. Therefore, by the same arguments, these halved unknowns are also even and you can repeat the process of division by 2 *ad infinitum.*

18.2.7.3. Now, that you have solved the problem required in Hint, use the fact that any side of triangle ABC subtends an obtuse angle with vertex at O.

18.2.7.4. The champion is determined according to the olympic system in $25 - 1 = 24$ games. The second best is a person who played with the champion, and there are 5 of them. The champion among these 5 is similarly determined in $5 - 1 = 4$ more games.

REMARK. In general, *the minimal number of games needed to determine two leaders among chess players is equal to* $n - 2 + \lceil \log n \rceil$, where $\lceil x \rceil$ is an approximation of x to the nearest greater integer, e.g., $\lceil 6.1 \rceil = \lceil 6.9 \rceil = \lceil 7 \rceil = 7$.

18.2.7.5. It is possible to obtain the desired partition for any number n of rectangles beginning with $n = 5$, arranging n rectangles "along a spiral" as it is shown in Fig. 181.

FIGURE 181. (Sol. 18.2.7.5)

18.2.8.1. Denote $f(x) = \sqrt{ax^2 + bx + c}$ and let x approach $+\infty$. It is easy to see that conditions imply $a > 0$ and

$$f(x+1) - f(x) = \frac{(f(x+1))^2 - (f(x))^2}{f(x+1) + f(x)} = \frac{a(2x+1) + b}{f(x+1) + f(x)} \to \sqrt{a} \text{ as } x \to \infty.$$

Since $f(x)$ and $f(x+1)$ are integers for an integer x, the number $d = \sqrt{a}$ is also an integer.

Moreover, since the difference $f(x+1) - f(x)$ is an integer, it must be equal to its limit $f(x+1) - f(x) = d$ for a great enough x. Let this be true beginning with some x_0 and let $y = x_0 + n$; then $f(y) = f(x_0) + nd$, i.e.,

$$ay^2 + by + c = (f(x_0) + nd)^2 = (dy - dx_0 + f(x_0))^2.$$

Hence, the assertion of the problem.

18.2.8.2 and **18.2.9.4.** ?ask Prasolov!

Denote the first two circles by α and β, and the third one by γ; see Fig. 182. Denote the intersection point of γ with the interior tangent to α and β by A. Denote the tangent point of α and β by B. Draw two common tangents to α and β: the inner one AC andf the outer one MN.

Let us draw the circle δ with center A and radius AB and prove that it passes through points M and N of the intersection of γ with the outer tangent to α and β.

Let us prove tha tA is the midpoint of the arc $\smile MAN$. Make the inversion with respect to circle δ. Since α and β are orthogonal to δ, the inversion sends α, β and δ into themselves; it sends circle γ into straight line l tangent to α and β and intersecting AB, i.e., γ goes into l. Similarly, the inversion sends l into γ. Hence, the intersection points of l with γ are fixed, i.e., they lie on δ. Thus, $AM = AB = AN$. Therefore, $\smile AM = \smile AN$ on γ.

FIGURE 182. (Sol. 18.2.8.2)

FIGURE 183. (Sol. 18.2.8.3)

18.2.8.3. a) An example for $n = 9$ is shown on Fig. 183.

b) Let us add up all numbers on the sides of the triangles $\triangle OA_iA_{i+1}$. As each segment OA_i is taken into consideration twice (as a side of two triangles), this sum is equal to

$$(1 + \cdots + n) + 2(1 + \cdots + n) = \frac{3n(n+1)}{2}.$$

This means that the sum of the numbers on the sides of any of n triangles is equal to $\frac{3(n+1)}{2}$. This sum must be an integer but for $n = 10$, and any other even n it is not. As this sum *is* an integer, $n \neq 10$.

18.2.8.4. Observe that if at least one of the numbers A, B or C is equal to 0, then the nonstrict variant of the given inequality holds; for example: for $A = 0$ we get $2BCbc \geq \frac{BCa^2}{2}$, whence $a^2 \leq 4bc \leq (b+c)^2$, that is $a \leq b + c$; and the equality is only attained if $4bc = (b+c)^2$, i.e., $b = c$.

The inequalities $b \leq a + c$ and $c \leq a + b$ are similarly obtained.

If no two of the numbers a, b, c are equal to half the remaining one, then it is possible to construct a triangle from these segments.

18.2.8.5. Let $a > 0$. Then all numbers arising in the problem are non-negative. If $a < \frac{N-1}{N}$, then $[Na] < N-1$ and, therefore, among the numbers $[a], \dots, [Na]$ some are identical. Thus, $a \geq \frac{N-1}{N}$. But then $\frac{1}{a} \geq \frac{N-1}{N}$ because the condition (2) is the same as (1) with a replaced by $\frac{1}{a}$. Therefore, $\frac{N-1}{N} \leq a \leq \frac{N}{N-1}$. It is easy to see that all numbers of this interval will do. The case $a < 0$ is similar.

18.2.9.1. Consider vectors $AB = u$, $BC = v$, $CA = w$. Then a direct count shows that

$$C_1A_1 = w_1 = \frac{nu + v}{n+1}, \quad A_1B_1 = u_1 = \frac{nv + w}{n+1}$$

and, therefore,

$$B_2C_2 = v_2 = \frac{w_1 + nu_1}{n+1} = \frac{nu + nw + (n^2+1)v}{(n+1)^2}.$$

Take into consideration that $u+w=-v$; then $B_2C_2 = \dfrac{n^2-n+1}{(n+1)^2}BC$. A similar formula holds for C_2A_2, A_2B_2.

18.2.9.2. Prove on your own that, for example, the system of segments $[1,\ 2]$; $[2\frac{1}{2},\ 3\frac{1}{2}]$; $[3\frac{3}{4},\ 4\frac{3}{4}]$; ... (the intervals between the segments form the geometric progression $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}, \dots$) symmetrically complemented in the negative part of the real line will do.

18.2.9.3. Divide the left hand side by x^n. The obtained function

$$f(x) = 1 - \frac{a_1}{x} - \frac{a_2}{x^2} - \dots - \frac{a_n}{x^n}$$

is an increasing one for $x>0$; hence, it can not have two positive roots.

18.2.10.1. According to *Bezout's theorem*, $f(x)$ is divisible by $x-\frac{p}{q}$. This means that $qf(x)$ is divisible by $qx-p$. Since the quotient is a polynomial with integer coefficients, the number $qf(k)$ is divisible by $qk-p$. As p and q are relatively prime, $f(k)$ is divisible by $qk-p$.

18.2.10.3. The answer depends on the relative values of the numbers H, h, r and 1. If, for example, $\frac{H}{r+1} > \frac{h}{r}$, it is easy to see that the entire disc will be illuminated. If $H>h$ but $\frac{H}{r+1} < \frac{h}{r}$, then the shadow of the cone is triangular-like; its vertex A is the projection of the vertex of the cone, the sides AB and AC are tangents from A to the base of the cone, see Fig. 184. Now it is not difficult to determine the arc that serves the base of the "triangle".

FIGURE 184. (Sol. 18.2.10.3)

FIGURE 185. (Sol. 19.1.8.1)

Analyze on your own the case of $H<h$.

19.1.7.1. If A, B, C, D lie in the vertices of a convex quadrilateral, then one of them is a vertex of a non-acute angle. If one of the points, say A, lies inside $\triangle BCD$, then one of the angles at vertex A is not less than $120°$.

19.1.7.3. Two segments pass through each point of self-intersection, and exactly one intersection point lies on each segment. Therefore, the number of the segments is twice the number of intersection points.

19.1.8.2. Let a_0 be the approximation and $d=a-a_0$. If $a<0.01$, then $a_0=0$ and $\frac{a_0}{a}=0$. If $a\ge 0.01$, then $\frac{a_0}{a} = 1-\frac{d}{a}$ and $a_0 \ge 0.01$, $d \le 0.01$.

This means that $0 \le \frac{d}{a} < \frac{1}{2}$ and $\frac{1}{2} < 1-\frac{d}{a} \le 1$. Prove on your own that $\frac{d}{a}$ can take any value b such that $0 \le b < \frac{1}{2}$.

19.1.8.4. If both the numerator and the denominator are divisible by k, then $c(al+b)$ and $a(cl+d)$ are divisible by k, and, therefore, their difference $ad-bc$ is also divisible by k.

19.1.8.5. Each number that is not on an edge (because it is equal to the arithmetic mean of its four adjacent numbers) is not greater than the greatest of the four numbers; hence, in particular, it is not the greatest number on the cut-off piece of paper.

19.1.9.4. The locus D is the union of D_1 with D_2, where D_1 is the intersection of the plane π_1 with the cylinders C_1 and C_2 while D_2 is the intersection of the plane π_2 with the cylinders C_1 and C_2. We can represent D_1 as the intersection of D_1' with D_1'', where D_1' is the intersection of the plane π_1 with C_1 while D_1'' is the intersection of π_1 with C_2.

The intersection of the cylinder with the plane is either

1) two parallel straight lines (if the radius of the cylinder is $> h$);

2) one line, the tangent to the cylinder along its element (if the radius of the cylinder is equal to h);

3) the empty set (if the radius of the cylinder is $< h$).

All the lines obtained are parallel to the axis of the cylinder. Since the lines from the sets D_1' and D_1'' are parallel to the intersecting lines AC and BC, respectively, they must also intersect each other. Therefore, D_1' may consist of a foursome of points (the intersection of two pairs of parallel straight lines); be a pair of points (the intersection of a pair of parallel lines with a line); be one point (the intersection of two lines); or no points at all (when one of the sets D_1' or D_1'' is empty). It is not difficult to see that the set D_2 has as many points as D_1, which implies the answer: the locus to be found has 8, 4, 2 or 0 points.

19.1.10.1. Let us construct the inscribed and the circumscribed circles of the square; let us draw tangents to these circles parallel to the sides of $\triangle ABC$, as shown on Fig. 186.

FIGURE 186. (Sol. 19.1.10.1)

FIGURE 187. (Sol. 19.2.7.2)

This construction yields triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ similar to $\triangle ABC$.

Let r_1 and r_2 be the radii of the inscribed circles of triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$, respectively. Clearly, $r_1 < r < r_2$. It is also clear that $r_1 = \frac{1}{2}a$ and $r_2 = \frac{1}{\sqrt{2}}a$. Therefore, $\frac{1}{2}a = r_1 < r$ and $\frac{1}{\sqrt{2}}a = r_2 > r$, i.e., $\sqrt{2}r < a < 2r$.

19.1.10.4. Let $h_1, h_2, \dots, h_n$ be the distances between points $A_1, A_2, \dots, A_n$ and the given plane. Then the product $(*)$ is equal to

$$\frac{h_1}{h_2} \cdot \frac{h_2}{h_3} \cdots \frac{h_{n-1}}{h_n} \cdot \frac{h_n}{h_1} = 1.$$

REMARK. One can similarly solve Problem 13.2.9-10.3: draw a plane through three tangent points and similarly prove that the fourth point also lies in the same plane.

19.1.10.5. If $x_i > 0$ for all i, then

$$|a| + |b| < x_1 + x_2 + x_3 + x_4 = 1.$$

Conversely, let $|a| + |b| < 1$. If $a, b \geq 0$ we may assume that

$$x_2 = x_4 = \frac{1-a-b}{4}, \quad x_1 = x_2 + a, \quad x_3 = x_4 + b, \quad \text{Q.E.D.}$$

Let us show that it suffices to consider non-negative a and b. Indeed, if for example, $a < 0$, then replace a by $-a$ and interchange x_1 with x_2.

19.2.7.2. Denote the bases of the perpendiculars dropped to rays l_6, l_5, l_4, l_3, l_2 by B_2, B_3, B_4, B_5, B_6, respectively, see Fig. 187.

Let $A_2B_1 = x$; then $A_1B_2 = \frac{A_1B_1}{2} = \frac{1+x}{2}$ from the right triangle $A_1B_1B_2$, as $\angle A_1B_1B_2 = 30°$. Similarly, from the right triangles $A_6B_2B_3$, …, $A_2B_6B_1$ we get

$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}(1+x)+1\right)+1\right)+1\right)+1\right)+1\right) = x.$$

This means that $x = 1$ and $A_1B_1 = 1 + x = 2$.

Remark. Use similar arguments for an arbitrary n-gon (see Problem 19.2.10.5) to get $x = 1$.

19.2.7.5. Let us generalize the problem to n leaves and prove by induction.

Obviously, there exists a leaf in the tree which, when torn off, leaves $a_1 \geq 1 - \frac{1}{n}$ of the shadow. There exists a leaf among the $n-1$ leaves left which, when torn off, leaves

$$a_2 \geq a_1 - \frac{1}{n-1}a_1 = a_1\left(1 - \frac{1}{n-1}\right) \geq \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n-1}\right) = 1 - \frac{2}{n}$$

of the shade.

Induction: if $k-1$ leaves are torn off the tree so that the remaining part of the shade is equal to $a_{k-1} \geq 1 - \frac{k-1}{n}$, then there is a leaf among those still hanging which, being torn off, leaves

$$a_k \geq a_{k-1} - \frac{1}{n-(k-1)}a_{k-1} \geq \left(1 - \frac{k-1}{n}\right)\left(1 - \frac{1}{n-(k-1)}\right) = 1 - \frac{k}{n}$$

of the shade. Q.E.D.

19.2.8.1. Distribute the freight in the following way. Keep on putting loaded containers into the first truck until their mass becomes greater than 1.5 tons, then take away the last container and put it beside the truck. After this, put as many loaded containers into the second truck as there stand on the first truck and put one contained beside it; then similarly load the third one, and so on, until 8 trucks become full.

The total mass of the containers in these 8 trucks and the containers beside them is greater than $1.5 \cdot 8 = 12$ tons. Therefore, the mass of the containers that stand by is less than 1.5 tons and the 9-th truck can take them away.

Divide 8 containers among the trucks into two sets of four; the mass of each set of containers is less than $4 \cdot 350 = 1400$ kg, and, therefore, it is possible to put them into two trucks.

19.2.8.5. Assume that this is not so. Let us number our 9 rectangles in an arbitrary way and begin to paint them: first the first one, then the second one and so on. Painting the first one we paint over an area equal to 1, painting the second one we paint over an area greater than $\frac{8}{9}$ (an area less than $\frac{1}{9}$ might be already painted as it is contained in the first rectangle).

In the same way, painting the third rectangle we paint over an area greater than $\frac{7}{9}$, and so on.

As a result the painted area is greater than $1 + \frac{8}{9} + \frac{7}{9} + \cdots + \frac{1}{9} = 5$ square units. Contradiction.

19.2.9.2. Divide the cube with planes parallel to its faces into $13^3 = 2197$ unit cubes. By Dirichlet's principle, there exists a unit cube among them which does not contain any of the 1956 points chosen.

19.2.9.3. The main steps of the solution:

1) the numbers x, y, z are non-negative;

2) there is a zero among them;

3) there is a zero among numbers x_1, y_1, z_1 as well, therefore, two of the numbers x, y, z are equal.

Now, it remains to try out two sets of three numbers: $(0, 0, 1)$ and $(0, 1, 1)$.

19.2.9.5. Let us circumscribe about our triangle a rectangle with sides parallel to the lines of the lattice. None of the vertices of the triangle can occur inside the rectangle: if this were the case, the vertex would be the vertex of the obtuse angle, see Fig. 188 a). The case when all vertices of the triangle lie in vertices of the rectangle is also impossible: if this were the case, our triangle would have been a right one.

Hence, at least one of the vertices of the triangle lies on a side (but not in vertex) of the rectangle. Take this vertex as the origin and the side of the rectangle as the coordinate axis Ox. Let us direct Oy-axis so that the triangle were in the 2-nd and 3-rd quadrants, see Fig. 188 b). Then point with coordinates $(0, 1)$ lies inside of our triangle.

Therefore, at least one vertex of the triangle lies not in the vertex of the rectangle. Denote this vertex by O; let the other two vertices of the triangle be A and B. Let O be the origin of the coordinate systemin which the x-axis goes along the side of the rectangle and the y-axis is directed so that the rectangle be in the upper half plane, see Fig. 188 b).

FIGURE 188. (Sol. 19.2.9.5)

Observe that neither A nor B lies on Ox (otherwise, $\triangle ABC$ is obtuse), hence, their ordinates are ≥ 1. Let y-axis intersect AB at point M. The ordinate of M is not less than the minimum of of ordinates of A and B, i.e., ≥ 1. Therefore, a point with coordinates $(0, 1)$ lies on OM, i.e., inside or on the boundary of the triangle.

ANOTHER SOLUTION. The following lemma is rather widely known under the name of Pick's Lemma. Its proof, however, is not so well known, so we give it for completeness. We borrowed it from [P?]

Lemma. *Let polygon P with the vertices in nodes of a lattice be such that p nodes of the lattice lie inside P and r nodes on the border (on sides and at vertices) of P. Then the area of P can be evaluated by the formula*

$$S(P) = p + \frac{r}{2} - 1 \tag{$*$}$$

PROOF of Lemma. Let us prove first, that formula $(*)$ is *additive*: if a broken line (with vertices at the nodes) cuts polygon P into polygons P_1 and P_2, then $S(P) = S(P_1) + S(P_2)$. Indeed, let p_1 and p_2 be the number of nodes inside P_1 and P_2, respectively; r_1 and r_2 the number of nodes on the borders, and t the number of nodes on the shearing line (endpoints including). Then we have $p_1 + p_2 + (t - 2)$ nodes inside P and $(r_1 - t) + (r_2 - t) + 2 = r_1 + r_2 - 2t + 2$ nodes on the border of P, therefrom

$$\begin{aligned} S(P) =& p_1 + p_2 + t - 2 + \frac{1}{2}(r_1 + r_2 - 2t + 2) - 1 = \\ & p_1 + p_2 + \frac{1}{2}r_1 + \frac{1}{2}r_2 - 2 = S(P_1) + S(P_2). \end{aligned}$$

Now, let us prove that formula $(*)$ yields an expression for the area; we start with simplest polygons and then using additivity of $(*)$ and additivity of area pass to arbitrary polygons.

1°. *Rectangle with sides parallel to the lines of the lattice*. Let the size of the rectangle be $a \times b$. Then $p = (a - 1)(b - 1)$, $r = 2(a + b)$, and $S(P) = (a - 1)(b - 1) + \frac{1}{2} \cdot 2(a + b) - 1 = ab$.

2°. *Right triangle with legs parallel to the intersecting lines of the lattice*. Two such triangles constitute a rectangle; now use additivity.

3°. *Arbitrary triangle* is obtained from a circumscribed rectangle by chopping off right triangles; see Fig. 188 a)–b); now use additivity.

4°. Diagonals divide an *arbitrary polygon* into the union of triangles; now use additivity. □

In our case $p = 0$, $r = 3$, so $S_{\triangle ABC} = \frac{1}{2}$.

REMARKS. 1) We may interpret formula $(*)$ as follows: the area of the n-gon is equal to the number of the lattice's cells the n-gon contains if a cell on the boundary is counted as half a cell and a cell at any of the four vertices for a quarter of a cell.

2) For figures whose boundary consists of several polygons or of selfintersecting polygons formula $(*)$ requires a modification (taking into account the number of pieces and holes).

3) Do *not* conclude that *some two adjacent vertices of the triangle with area* $\frac{1}{2}$ *are situated in two neighboring nodes of the lattice*: for example, the *Fibonacci triangles* with vertices in points $(0, 0)$, (f_n, f_{n-1}), (f_{n+1}, f_{n+2}), where f_i is the i-th Fibonacci number, do not satisfy this condition but have the area of $\frac{1}{2}$. All these triangles are, however, obtuse.

We have $S_{\triangle ABC} = \frac{1}{2}ab \sin C = \frac{1}{2}$, i.e., $ab \sin C = 1$. If $a = b = 1$, i.e., the triangle is a half cell, all is clear. Otherwise, $a \geq 1$, $b \geq \sqrt{2}$ (or $a \geq \sqrt{2}$, $b \geq 1$); hence, $\sin C \leq \frac{1}{\sqrt{2}}$, i.e., $C \leq 45°$ or $C \geq 135°$. But in a triangle all angles can not simultaneously be $\leq 45°$; hence, there is an angle $\geq 135°$.

19.2.10.1. Let a_1, a_2, …, a_n be the given numbers, a_i one of the underlined numbers, and k the least possible number such that the sum $a_i + a_{i+1} + \dots + a_{i+k}$ is positive (if $a_i > 0$, then $k = 0$). Then it is easy to see that all numbers a_{i+1}, a_{i+2}, …, a_{i+k} are underlined.

In fact, for any shorter sum $a_i + a_{i+1} + \dots + a_{i+l} \leq 0$ for $l < k$, otherwise, k is not the minimal one. Then

$$a_{i+l+1} + a_{i+1} + \dots + a_{i+k} = (a_i + a_{i+1} + \dots + a_{i+l}) - (a_l + a_{l+1} + \dots + a_{i+l}) > 0$$

which means that a_{i+l+1} is underlined.

Now, ler us write the sum of all the underlined elements in the natural order and let us start inserting parentheses in it. Open the parenthesis i front of the first number, a_i and close it after the last positive summand for a_i. Then, perform the same for the first of the remaining numbers (summands?), and so on. As a result, all the sum gets divided into the sum of positive summands in parentheses, hence, it is positive itself. (The solution to Problem 19.2.7.3 is similar.)

19.2.10.4. Let S be the apex of pyramid $SABC$. Consider the unfolding $S_1AS_2BS_3C$ of the pyramid obtained by putting its lateral faces to the plane of its base ABC. It follows from the condition that the sums of the three planar angles at any vertex are equal. Because the sum of the planar angles at all the vertices is equal to 4π (the sum of the angles of one face multiplied by 4), then the sum of the planar angles at any vertex is equal to π, see Fig. 189.

FIGURE 189. (Sol. 19.2.10.4)

Hence, the unfolding is a triangle $S_1S_2S_3$ with triangle ABC inscribed in it. Since $\triangle S_1S_2S_3$ is an unfolding, $S_1A = AS_2$, $S_2B = BS_3$, $S_3C = CS_1$. Hence, AB, BC, and AC are the segments connecting the midpoints of the sides of triangle $S_1S_2S_3$. Therefore, all four small triangles are equal (in 3 sides). Q.E.D.

20.1.7.1. Let $ABCD$ be a trapezoid with equal sides ($AB = CD$), AD being the greater base, and obtuse angle $\angle B$. Then $AC > AB$ (AC subtends obtuse angle B in triangle ABC); therefore, the hypothesis implies that $AB = BC$. In the same way, $AC > CD$ (since $CD = AB$) and then $AC = AD$ by the hypothesis. Draw AN parallel to BD and DN parallel to AC. We get a regular pentagon, see Fig. 190, all of whose angles are equal to $108°$.

FIGURE 190. (Sol. 20.1.7.1)

20.1.7.2. Let $x = 0$, then d is divisible by 5. Setting $x = 1$, $x = -1$ or $x = 2$ it is easy to show that $2b$, $a + c$ and $a - c$ are divisible by 5. Hence, the assertion of the problem.

REMARK. Two circumstances are important here: a) the degree of the polynomial is less than 5; b) 5 is a prime. Consider on your own (1) if there exists a 5-th degree polynomial divisible by 5 with not all coefficients divisible by 5 for all x and (2) if there exists a 3-rd degree polynomial divisible by 4 with not all coefficients divisible by 5 for all x.

20.1.7.3. Let a *link* be a segment that the snail covers in-between turns. The number of horizontal links along which the snail moves away from the origin is equal to the number of horizontal links along which it moves towards the origin.

This means that the number of all horizontal links is even. The number of all vertical links is even by the same argument. But the numbers of vertical and horizontal links are equal. This means that the number of all links is divisible by 4.

20.1.8.1. Let R_1, R_2 be the radii of circles O_1 and O_2, respectively. Two positions of rectangles are possible: either neighboring or opposite vertices can lie on the first circle.

FIGURE 191. (Sol. 20.1.8.1)

In the first case the fourth vertex lies on the second circle and the locus to be found is (Fig. 191 a), b)) one of the following:

— the whole second circle without four points that corresponds to degenerate rectangles if $R_1 \geq R_2$;
— the four arcs of the second circle without four points that corresponds to degenerate rectangles if $R_1 < R_2$.

In the second case the fourth vertex should lie on the circle of radius $\sqrt{2R_1^2 - R_2^2}$ and inside the square with side $2R_1$ and the same center; hence, the locus to be found is (Fig. 191 c)) one of the following:
— the four arcs intercepted by the square if $R_1 \geq R_2$;
— the whole circle if $R_1 < R_2$.

20.1.8.4. Let the table have m rows and n columns. Denote the sum of the numbers in the j-th row by S_j, the sum of the numbers in the k-th column by T_k, and the total sum of the numbers in the table by A. Then

$$S_1 + S_2 + \dots S_m = T_1 + T_2 + \dots + T_n = A.$$

But then it follows that

$$S_l = S_lT_1 + S_lT_2 + \dots + S_lT_n = S_lA. \tag{$*$}$$

Hence, if $A \neq 1$, equation $(*)$ implies that each S_l is 0. But then the conditions of the problem imply that each number in the table is also equal to 0.

20.1.9.2. Since $0 \leq x - [x] < 1$, the equality $x^3 - x + \{x\} = 3$ implies $2 < x^3 - x \leq 3$. If $x \geq 2$, then $x^3 - x = x(x^2 - 1) \geq 6$; if $x < -1$, then $x(x^2 - 1) < 0$; if $|x| \leq 1$, then $x^3 - [x] \leq 2$, and, lastly, on the segment $[1, 2[$ the equation takes the form $x = \sqrt[3]{4}$. This is the answer.

Another solution. Let $[x] = t$. Then $x = t + \alpha$, $0 \leq \alpha < 1$, and

$$(t + 1)^3 > x^3 = (t + \alpha)^3 = t + 3 > t^3.$$

Solving these inequalities in integers, we see that $t \leq 1$ if $3 > t^3 - t$ and $t \geq 1$ if $(t + 1)^3 > t + 3$. So $t = 1$ and $x = \sqrt[3]{4}$ is the unique answer.

20.1.9.3. Observe that if $M = N$, thenm $ABCD$ is a parallelogram and $BC \parallel AD$. Therefore, in what follows we assume that $M \neq N$. On line CD, construct point B' and on line AB construct point C' so that $BB' \parallel CC' \parallel MN$, see Fig. 192. In $\triangle DBB'$ segment NN' passes through the midpoint of side BD and is parallel to the base BB', hence, NN' is a midline and $BB' = 2NN'$. Similarly, $CC' = 2MM'$, but $MM' = NN'$, hence, $BB' = CC'$.

Suppose $B' \neq C$. then in the quadrilateral $BB'C'C$ the sides BB' and CC' are equal and parallel, hence, $BB'C'C$ is a parallelogram and $BC' \parallel B'C$. But this means that $ABCD$ is a trapezoid and MN is its midline; but the midline can not intersect the bases AB and CD: a contradiction. Hence, $B' = C$ and $BC = BB' \parallel MN$. We similarly prove that $AD \parallel MN$, which amounts to $BC \parallel AD$.

Figure 192. (Sol. 20.1.9.3)

Figure 193. (Sol. 20.1.10.5)

20.1.9.4 and **20.1.10.4.** First, observe that if the student has at least one 20 kopek coin then the change is exactly 30 kopeks. The case when (s)he has two 10 kopeks coins or two 15 kopek coins is also obvious. What remains is to investigate the case when the student has just one 10 kopek coin and one 15 kopek coin and the rest of his money are coins of denomination of 1, 2, 3 and 5 kopeks. But then it is easy to see that the coins can add up to 10 or 15 kopeks, which immediately proves what the required.

Problem 20.1.10.4 can be solved by the same case-by-case checking (the case when the student has two 20 kopek coins or two 15 kopek coins is again obvious, etc.).

20.1.10.1. If $n = 2k$, then $16^n - 1 = 256^k - 1$ is divisible by $256 - 1 = 17 \cdot 15$, in particular, it is divisible by 17. The number $20^n - 3^n$ is also divisible by 17. Moreover, if $n = 2k$, then both $20^n - 1$ and $16^n - 3^n$ are divisible by 19. Therefore, N is divisible by $17 \cdot 19 = 323$ if $n = 2k$.

If $n = 2k + 1$, then the remainder after division of $16^n - 1$ by 17 is 15, and $20^n - 3^n$ is divisible by 17; therefore, the entire expression is not divisible by 17, nor is it divisible by 323.

20.1.10.5. Select any link AB of the broken line — the boundary of the polygon — and let its length be a. Consider a point M that divides the length of the remaining broken line in halves. If M is a node, then we obtain the triangle required by straightening AM and BM (it is isosceles).

Suppose M is not a node and belongs to a link CD of length b (see Fig. 193). Let the lengths of the broken lines AC and BD be x and y, respectively. If it is possible to construct a triangle from the segments whose lengths are a, $x + b$, y or a, $y + b$, x, then the problem is solved.

But if it is *not* possible to construct a triangle from these segments, then one of the segments is not shorter than the sum of the two others. However, $x + y + b > a$ (a broken line is always longer than a straight segment) and since $x + b > y$ and $y + b > x$ (as dictated by the choice of M), the impossibility of constructing a triangle from the given segments leads to the inequalities

$$x + b \geq a + y, \quad y + b \geq a + x$$

whose sum yields $2b \geq 2a$, that is $b \geq a$.

Now, select the longest link of the broken line; let it be AB. Then $a \geq b$ while $b \geq a$ by proved above. Hence, $a = b$. The inequalities $x + b \geq a + y$ and $y + b \geq a + x$ become $x \geq y$, $y \geq x$, and so $x = y$.

The condition $n > 4$ implies that at least one node lies on either AC or BD (let it lie on AC for definiteness sake). The node divides AC in two parts: $x = u + v$, see Fig. 193.

Let us demonstrate that a triangle can be constructed in this case from the three segments, whose lengths are $a + u$, $b + v$, y. For this purpose it suffices to verify three inequalities:

$$a + u + y > b + v, \quad a + u + b + v > y, \quad b + v + y > a + u.$$

They are easy to prove taking into account that $a = b$ and $y = x = u + v$:

$$\begin{array}{ccccccc} a+u+y & = & b+u+x & = & b+u+u+v & > & b+v, \\ a+u+b+v & = & a+b+x & = & a+b+y & > & y, \\ b+v+y & = & a+v+y & = & a+v+u+v & > & a+u. \end{array}$$

20.2.7.4. Denote the angles of the original triangle by α, β, γ and find the angles of the second and the third triangles; they are $\frac{\alpha+\beta}{2}$, $\frac{\beta+\gamma}{2}$, $\frac{\alpha+\gamma}{2}$ and $\frac{\alpha+2\beta+\gamma}{4}$, $\frac{\alpha+\beta+2\gamma}{4}$, $\frac{2\alpha+\beta+\gamma}{4}$, respectively. Equate the angles of the first triangle to the angles of the third one. From the system obtained it follows that $\alpha = \beta = \gamma$.

20.2.7.5. **Lemma**. *Let a_i be the i-th term of the given Fibonacci sequence. Then $a_{n+2} = 2+a_1+\dots+a_n$.*

PROOF. By the induction: if $n = 0$, then $a_2 = 2$.

The inductive step:

$$a_{n+3} = a_{n+2} + a_{n+1} = a_{n+1} + 2 + a_1 + \dots + a_n = 2 + a_1 + \dots + a_{n+1}, \quad \text{Q.E.D.}$$

If $S = a_{k+1} + a_{k+2} + \dots + a_{k+8}$, then (use Lemma or the second inequality)

$$a_{k+9} = a_{k+7} + a_{k+8} < S < a_{k+10} = 2 + a_1 + a_2 + \dots + a_{k+8}.$$

20.2.8.1. Fix $AC = b$ and assume that $b \geq a$. Then $\angle B \geq \angle A$ and the smallest angle is either $\angle A$ or $\angle C$. The value of $\angle A$ is the greatest possible when AB is tangent to the circle of radius a centered at C. In this case we have $\angle B = 90°$ and $AB = \sqrt{b^2 - a^2}$. We also have $AB \geq BC$ and $\angle C \geq \angle A$ for $b \geq a\sqrt{2}$, i.e., $\angle A$ is the smallest in the triangle.

But if $b \leq a\sqrt{2}$, then $\angle C \leq \angle A$ and $\angle A$ decreases as $\angle C$ increases. So the smallest angle ($\angle C$) reaches its maximum when $AB = CB = a$.

20.2.8.3. Obviously, $\frac{A'O}{A'G}$ is equal to the ratio of the heights in triangles COB and CGB or, which is the same, to the ratio of their areas. The same holds for $\frac{B'O}{B'G}$ and $\frac{C'O}{C'G}$. But the area of each of the triangles GBC, GAC and GAB is equal to $\frac{1}{3}S_{ABC}$, see Fig. 194.

Thus,

$$\frac{A'O}{A'G} + \frac{B'O}{B'G} + \frac{C'O}{C'G} = \frac{S_{OAB} + S_{OBC} + S_{OAC}}{\frac{1}{3}S_{ABC}} = 3\frac{S_{ABC}}{S_{ABC}} = 3. \quad \text{Q.E.D.}$$

FIGURE 194. (Sol. 20.2.8.3)

20.2.8.4. It is easy to see that x_1, x_2, x_3 vanish only simultaneously. If $x_1x_2x_3 \neq 0$, then we can rewrite the system in the following form:

$$\begin{cases} \dfrac{x_1^2+1}{x_1^2} - \dfrac{2}{x_2} & = & 0, \\ \dfrac{x_2^2+1}{x_2^2} - \dfrac{2}{x_3} & = & 0, \\ \dfrac{x_3^2+1}{x_3^2} - \dfrac{2}{x_1} & = & 0. \end{cases}$$

Adding up these equations we get:

$$(1-\frac{1}{x_1})^2 + (1-\frac{1}{x_2})^2 + (1-\frac{1}{x_3})^2 = 0;$$

hence, each term is equal to 0. Therefore, $x_1 = x_2 = x_3 = 1$ is another solution and there are no more solutions.

20.2.9.1. Any parallel translation can be represented as the sum of a horizontal and a vertical translation. Therefore, it suffices to consider a horizontal movement of the second rectangle. With a horizontal movement points A and C move to the right, while B and D move upward at the same speed.

The area of a quadrilateral is equal to a half product of the diagonals by the sine of the angle between them, see *Prerequisites*. During the movement described in the problem the lengths of the diagonals and the angle between them do not vary.

20.2.9.3. Assume the contrary. Then after each rotation exactly one (by the Dirichlet principle) contact will get to its place. Let the i-th contact have the number a_i on the socket and the number b_i on the plug. Then the difference $a_i - b_i$ must attain all values $0, 1, \dots, 19$ modulo 20. Therefore, the sum

$$(a_1 - b_1) + (a_2 - b_2) + \dots + (a_{20} - b_{20})$$

is congruent to $1 + 2 + \dots + 19 = 190$ modulo 20. But $a_1 + \dots + a_{20}$ and $b_1 + \dots + b_{20}$ themselves are equal to 190. Contradiction.

REMARK. If the number of contacts is odd, a similar argument does not lead us to a contradiction; cf. Problem 20.2.7.2.

20.2.9.5. Let r be the radius of the small (the fourth) circle with the center O; see Fig. 195. Let O_1, O_2, O_3 be the centers of the first three equal circles; let their radii be equal to $R-r$ ($O_iO = R$ for all $i = 1, 2, 3$); let $\rho_i = O_iM$ be the distance between an arbitrary point M on the small circle and O_i; let l_i be the length of the tangent to the circle with center O_i drawn from point M; let α_i be the value of the angle $\angle O_iMO$.

Applying the law of cosines to triangle $\triangle O_iMO$ we get:

$$\rho_i^2 = R^2 + r^2 - 2Rr\cos\alpha_i.$$

In the right triangle MA_iO_i we have:

$$l_i^2 = \rho_i^2 - (R-r)^2 = R^2 + r^2 - 2Rr\cos\alpha_i - R^2 - r^2 + 2Rr = 2Rr(1-\cos\alpha_i). \tag{1}$$

Taking into consideration that $\alpha_2 = \alpha_1 + 120°$ and $\alpha_3 = \alpha_1 + 240°$, we find

$$\begin{cases} \cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3 & = & 0, \\ (\cos\alpha_1)^2 + (\cos\alpha_2)^2 + (\cos\alpha_3)^2 & = & \frac{3}{2}. \end{cases} \tag{2}$$

It suffices now to prove that

$$(l_1 + l_2 - l_3)(l_1 - l_2 + l_3)(l_2 + l_3 - l_1) = 0, \tag{3}$$

therefrom it follows that one tangent is equal to the sum of the other two. Let us multiply (3) by $l_1+l_2+l_3 \neq 0$ and prove that

$$P = (l_1 + l_2 + l_3)(l_1 + l_2 - l_3)(l_1 - l_2 + l_3)(l_2 + l_3 - l_1) = 0.$$

The product P can be reduced to

$$P = (l_1^2 + l_2^2 + l_3^2)^2 - 2(l_1^4 + l_2^4 + l_3^4)$$

(see the solution to Problem 19.2.9.5). But by (1) and (2)

$$l_1^2 + l_2^2 + l_3^2 = 2Rr(3 - \sum \cos\alpha_i) = 6Rr;$$

$$l_1^4 + l_2^4 + l_3^4 = 4R^2r^2(3 - 2\sum \cos\alpha_i + \sum(\cos\alpha_i)^2) = 4R^2r^2(3 + \tfrac{3}{2}) = 18R^2r^2.$$

Therefore,

$$P = (6Rr)^2 - 2 \cdot 18R^2r^2 = 0, \quad \text{Q.E.D.}$$

FIGURE 195. (Sol. 20.2.9.5)

FIGURE 196. (Sol. 20.2.10.2)

20.2.10.2. Consider the case $n = 1$. Then $x_1 = 1 - x_1^2$, hence, $x_1 = \frac{-1\pm\sqrt{5}}{2}$. Observe that we also get two solutions for the general case:

$$x_1 = \cdots = x_n = \frac{-1+\sqrt{5}}{2} \text{ and } x_1 = \cdots = x_n = \frac{-1-\sqrt{5}}{2}.$$

Consider the case $n = 2$. Then $x_1 = 1 - x_1^2 = 1 - (1 - x_1)^2$, hence, $x_1^4 - 2x_1^2 + x_1 = 0$. Let su factor the left hand side: $x_1(x_1 - 1)(x_1^2 + 2x_1 - 1) = 0$ implying $x_1 \in \{0, 1, \frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$. We thus get two solutions for the general case:

$$(x_1, x_2) \in \{(0, 1), (1, 0), (\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}), (\frac{-1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2})\}.$$

Observe that the existence of the last two solutions was predicted at the previous step.

Observe also that we have obtained two solutions for an arbitray even $n = 2m$:

$$x_1 = x_3 = \cdots = x_{2m-1} = 1, x_2 = x-4 = \cdots = x_{2m} = 0 \quad \text{and the other way round}$$
$$x_1 = x_3 = \cdots = x_{2m-1} = 0, x_2 = x-4 = \cdots = x_{2m} = 1.$$

Let us prove that these are all the possible solutions for any n, i.e., there are two solutions for n odd and 4 solutions for n even.

Here is the idea. For any solution each of the points $A_i = (x_i, x_{i+1})$ lies on the parabola $y = 1 - x^2$. From the point A_i we can costruct the point A_{i+1} as followds: through A_i draw the horizontal line to its intersection with the line $y = x$, next, through the intersection point draw the vertical line to its intersection with the parabola at point A_{i+1}. Besides, from point A_n we should get in this way the point A_1.

Now, on the parabola take an arbitrary point A_1 and construct in the above described way the sequence of points $A_2, \dots, A_k, \dots$. This sequence gives a solution of the problem if and only if some of the points of the sequence coinsides with A_1.

Let us provbe now that if A_1 does not coincide with any of the points $A(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$, $B(\frac{-1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2})$, $O(0,1)$ or $C(1,0)$, then the sequnce constructed does not contain another point A_1.

Study of the graph shows, see Fig. 196, that if we take A_1 on the right branch of the parabola below C, then the sequnce will never return there; if we take $A_1 \neq B$ on the left branch of the parabola, then the sequnce will either run away from B to infinity or to the square K with diameter OC. Having reached the square the sequence will never exit it; if the point of the sequence got inside the square, but not in A, then the horizontals and verticals constructed form an unwinding spiral that tends to the sides of the sqaure (That is why mathematicians call the curves such as this square attractors.)

Let us document our observation as a proof. Let $x_1, \dots, x_n$ be a solution of the problem. Then

(1) $x_1 \leq 1$. Indeed, $x_1 = 1 - x_n^2 \leq 1$.

(2) $x_1 \geq \frac{-1-\sqrt{5}}{2}$. Indeed, having solved the inequality $x > 1 - x^2$ we see that for $x < \frac{-1-\sqrt{5}}{2}$ it is verified, hence, $x_2 = 1 - x_1^2 \leq x_1 \leq \frac{-1-\sqrt{5}}{2}$; similarly, $x_3 < x_2$, etc. Finally, $x_1 = x_{n+1} < x_n < \dots < x_2 < 1$, a contradiction.

(3) If $x_k \geq 0$ for some k, then all the $x_i \geq 0$. Indeed, since $x_k \leq 1$, it follows that $x_{k+1} = 1 - x_k^2 \in [0,1]$, etc., along the circle.

(4) If $x_1 < 0$, then $x_1 = \frac{-1-\sqrt{5}}{2}$. In fact, by (3), all the x_i are < 0 (otherwise $x_1 \geq 0$). If $x_1 \neq \frac{-1-\sqrt{5}}{2}$, then by (2) $x_1 \in]\frac{-1-\sqrt{5}}{2}, 0[$. For x from this open interval the inequality $x < 1 - x^2$ is satisfied, and, therefore, $x_1 < x_2 < \dots < x_n < x_1$, a contradiction.

(5) If $x_i \in]0,1[$ and $x_i \neq \frac{-1+\sqrt{5}}{2}$, then x_i and x_{i+1} lie on either sides of $\frac{-1+\sqrt{5}}{2}$, as follows from the fact that the function $1 - x^2$ monotonosly decreases on $]0,1[$ and is equal to $\frac{-1+\sqrt{5}}{2}$ at $x = \frac{-1+\sqrt{5}}{2}$.

(6) If $x_i \in]0, \frac{-1+\sqrt{5}}{2}[$, then $x_{i+2} < x_i$, whereas if $x_i \in]\frac{-1+\sqrt{5}}{2}, 1[$, then $x_{i+2} > x_i$. In fact, $x_{i+2} = 1 - (1 - x_i^2)^2$. Knowing the roots of the equation $x = 1 - (1 - x^2)^2$, it is not difficult to verify that on the intervalls indicated the needed inequalities hold.

(7) If $x_1 \geq 0$, then $x_1 \in \{0, \frac{-1+\sqrt{5}}{2}, 1\}$. In fact, suppose that $x_1 \in]0, \frac{-1+\sqrt{5}}{2}[$, then by (5) all the x_i with even indices are $> \frac{-1+\sqrt{5}}{2}$, i.e., are $\neq x_1$ while all the x_i with odd indices belong to $]0, \frac{-1+\sqrt{5}}{2}[$, and by (6) for them the chain of inequalities $x_1 > x_3 > x_5 > \dots$ holds. In other words, there is no x_1 anong them either.

The assumption $x_i \in]\frac{-1+\sqrt{5}}{2}, 1[$ leads to a contradiction in a similar way.

20.2.10.5. Distribute the numbers in the following way: place the two greatest numbers, a_n and a_{n-1}, in two different groups; then place each next number (a_{n-2}, a_{n-3}, etc.) into the group the sum of whose numbers is lesser.

Let S_1 and S_2 be the sum of numbers in the respective group. Observe that after we have placed the i-th number, a_i, then $|S_1 - S_2| \leq a_i$. The proof is carried out by induction (backwards!).

The base of induction: for the number a_n we have $S_1 = a_n$; $S_2 = 0$ and $|S_1 - S_2| = a_n$.

The step of induction: before we place the number a_{i-1}, let $S_1 \leq S_2 \leq S_1 + a_i$. We have

$$(S_1 + a_{i-1}) - S_2 = a_{i-1} + S_1 - S_2 \leq a_{i-1};$$
$$S_2 - (S_1 + a_{i-1}) = -a_{i-1} + (S_2 - S_1) \leq -a_{i-1} + a_i \leq -a_{i-1} + 2a_{i-1} = a_{i-1}.$$

Therefore, placing the last number, 1, we get $|S_1 - S_2| \leq 1$ but $S_1 + S_2$ is even; hence, $S_1 = S_2$.

21.1.7.2. Let O be the center of the circle. Then points P and Q lie on the circle with diameter OM, and the length of the chord PQ is a constant because it subtends angle POQ.

21.1.8.1. Let $\overrightarrow{OS}$ be the sum of given vectors $\overrightarrow{OA_1}$, $\overrightarrow{OB_1}$ and $\overrightarrow{OC_1}$. Let us draw the unit circle with center O and its diameters, A_1Q and B_1P. In order for O to be inside $\triangle ABC$, point C_1 should lie on arc $\smile PQ$. Let us draw the second unit circle shifted with respect to the first one by vector $\overrightarrow{OA_1} + \overrightarrow{OB_1}$. After the shift arc $\smile PQ$ becomes $\smile A_1B_1$ and point C_1 becomes point C_1' lying on $\smile A_1B_1$. But $OC_1' < 1$ and

$$\overrightarrow{OC_1'} = \overrightarrow{OA_1} + \overrightarrow{A_1O_1} + \overrightarrow{O_1C_1'} = \overrightarrow{OA_1} + \overrightarrow{OB_1} + \overrightarrow{OC_1}, \quad \text{Q.E.D.}$$

21.1.8.2. If $p_1 \neq p_2$, subtract one equation from the other one; the root $x = \dfrac{q_1 - q_2}{p_2 - p_1}$ of the equation obtained is non-integer by the hypothesis, hence, it is rational. But it is well-known (prove it yourself if it is news to you) that any rational root of the equation $x^2 + p_i x + q_i = 0$ $(i = 1, 2)$ is an *integer*.

21.1.8.5. By the hypothesis, the given polygon is contained within the intersection of two rectangular domains, the size of the first being 4×5, same of the second $3\sqrt{2} \times 4\sqrt{2}$ (the sides of the second rectangle are slanted at an angle of 45° relative to the respective sides of the first rectangle). The intersection of the rectangular domains is an octagonal domain, all vertices of one rectangle being outside the other (on its sides in the degenerate case) since otherwise it is easy to show that one of the projections would be shorter than the given in the condition.

Denote the distances from the vertices of the second rectangle to the sides of the first one by x, y, z, t. Then the area of the octagon is equal to $S_8 = 24 - (x^2 + y^2 + z^2 + t^2)$, where 24 is the area of the second rectangle.

Let x and y be the distances from the vertices of the second rectangle to the sides of length 5 of the first rectangle. Then $x + y = 3$ and $z + t = 2$.

So $x^2 + y^2 \geq 2 \cdot 1.5^2 = 4.5$ and $z^2 + t^2 \geq 2 \cdot 1^2 = 2$; hence, the estimate for the area S of the given polygon:

$$S \leq S_8 \leq 24 - 6.5 = 17.5.$$

To see that estimates are exact, take $x = y = 1.5$ and $z = t = 1$.

21.1.9.2. The symmetry through each axis should send either (a) each of the given straight lines into themselves or (b) one of the straight lines into itself and transpose the other two.

For a pair of nonparallel lines there exist exactly two axial symmetries that interchange the lines; hence, there are altogether not more than 6 axes of symmetry of type (b). (Investigate yourself the case then two lines are parallel.)

The axes of type (a) are either the lines themselves (not more than 3 of them); or the common perpendicular to all three lines *and intersecting them*.

The sum total of possible axes is equal to 10. But if each of the lines is a type (a) axis of symmetry, the lines are pair-wise perpendicular and can not have a common perpendicular in 3-dimensional space[1]. Hence, not more than 9 axis of symmetry altogether are possible. Example: coordinate axes; the axes of symmetry: the lines themselves and the bisectors of planar angles they form.

21.1.9.3. The left (and, therefore, the right) hand side is greater than $\frac{1}{2}$. Therefore, $x_1 = 1$. Then from the equation

$$1 - \frac{1}{2+a} = \frac{1}{1+z}$$

we find that $z = \frac{1}{1+a}$. Now, if two numbers are equal, their integer parts are also equal; hence $x_2 = 1$. By continuing similarly, we get

$$x_3 = 3, \quad x_4 = 4, \quad \dots, \quad x_n = n.$$

Remark. As a result we have got a beautiful and unexpected identity:

$$\cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \dots \cfrac{1}{(n-1) + \cfrac{1}{n}}}}} + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \dots \cfrac{1}{(n-1) + \cfrac{1}{n}}}}}} = 1.$$

21.1.9.4. The numbers equal to the distance between a marked point and the segment's left endpoint will be referrred to as *marked* as well. Let us prove now for all k which are $\leq n$ that every number x from 1 to 3^k can be represented as the difference of two marked numbers not exceeding 3^k. Perform induction on k. The base for $k = 0$ is obvious. Let for $k - 1$ the statement is true. Call the marked numbers $\leq 3^{k-1}$ *small*. Observe that if a is small, then $a + 2 \cdot 3^{k-1}$ is a marked number (this follows from the fact that the left and the right thirds of the segment are on equal footing). Consider three cases:

1) If $x \leq 3^{k-1}$, then by inductive hypothesis $x = a - b$, where a and b are small numbers.

2) If $x \in [2 \cdot 3^{k-1}, 3^k]$, then $y = x - 2 \cdot 3^{k-1} < 3^{k-1}$, hence, $y = a - b$, where a and b are small numbers, and, therefore, $x = (a + 2 \cdot 3^{k-1}) - b$.

3) If $x \in]\cdot 3^{k-1}, 2 \cdot 3^{k-1}[$, then $z = 2 \cdot 3^{k-1} - x < 3^{k-1}$, hence, $z = a - b$, where a and b are small numbers, and, therefore, $x = 2 \cdot 3^{k-1} - z = (b + 2 \cdot 3^{k-1}) - a$.

Another solution (idea). Write the integers — the distances between the marked points and the segment's left endpoint — using the *ternary* system. The marked points are the numbers which, in the ternary system, either do not contain a 1_3 or end with $10...0_3$ and have no more 1_3's.

---

[1] Note that they can have a common perpendicular in an n-dimensional space for $n > 3$.

Let us prove that any number 1 to 3^n can be represented in the form of the difference of two such numbers. Let $x = \overline{a_1a_2...a_k}$ (we skip subscript 3 hereafter in order not to confuse with indices) be a number in the ternary system. Consider the sum $x + y$, where y is writen under x according to the following rule:

• first, if x ends with zeros write zeroes under them;

• beginning from the *left* write 0 under digit a_i if after addition $(x+y)$ the transfer to the higher order is not needed and 2 otherwise. (The transfer is needed in order to replace 1 by 2 or 2 by 0, with the subsequent transfer in the higher order.)

Therefore, the second summand, y, contains 0's and 2's only, and the sum $x + y$ can only contain one 1 followed by 0's.

21.1.10.2. Let us estimate n. Clearly, $n > 1155^{979}$. But, on the other hand, $n < 1155^{979} + 2$ since $34^2 = 1156$ and

$$\left(\frac{1156}{1155}\right)^{979} = \left(1 + \frac{1}{1155}\right)^{979} < \left(1 + \frac{1}{1155}\right)^{1155} < 4.$$

(Actually, even a stronger estimate holds: $\frac{1156}{1155}^{979} < e = 2.73...$) This implies that

$$(1155^{979} + 2)^2 = 1155^{1958} + 4 \cdot 1155^{979} + 4 > 1155^{1958} + 1156^{976} = 1155^{1958} + 34^{1958}.$$

Now, we must check that

$$1155^{1958} + 34^{1958} \neq (1155^{979} + 1)^2.$$

But this is clear since the numbers on the left and on the right are of different parity (i.e., their residues modulo 2 are different).

21.1.10.4. Let us rotate the n-gon counterclockwise. As it rotates, the changes in the order of numbers occur at the moments when two of the numbers occure at the same distance from the edge of the table (the left number was after the right one, but became infront of). At this moment the segment that connects the numbers — either a side or a diagonal — becomes parallel to the edge of the table.

But the angles between such segments are multiples of $\frac{\pi}{n}$. In fact, the n-gon is inscribed in the circle, divides its circumference into arcs of measure $\frac{2\pi}{n}$ and the angle between the chords is equal to a half sum of the arcs subtended if the chords meet or to a half difference if they do not.

It is easy to see that for each segment there is another segment that shares a vertex and form an angle of $\frac{\pi}{n}$. Therefore, there will be $2\pi/\frac{\pi}{n} = 2n$ permutations altogether. Let us rotate each time therough the angle of $\frac{\pi}{n}$, it is clear that we get the initial collection only afer the full revolution.

Since from our point of view all vertices are on equal footing, each number will visit each place twice. Hence, the answer: $2 \cdot (1 + 2 + \cdots + n) = n(n + 1)$.

21.1.10.5. First, let us prove a more general assertion:

Let n pedestrians move along n pairwise non-parallel foot paths, each at a constant speed. Let it be known that the first pedestrian meets all other pedestrians and the second pedestrian meets all other pedestrians. Then

(a) *each pedestrian meets every other one*;

(b) *all pedestrians are situated on one straight line at all times.*

Proof. Erect a vertical line (the axis of time t) to the plane in which all paths are situated and consider the graphs of the movement (scientifically called *world lines*) of all pedestrians.

These graphs are straight lines; the first and the second world line intersect with the rest of the world lines. But two straight lines, in particular, the first and the second world line, determine a plane in space. An arbitrary i-th world line intersects both the first and the second world line, and this means that it is situated in the same plane. Hence, all world lines are situated in the same plane; see Fig. 197.

Assume now that some two pedestrians will not meet, i.e., their world lines do not intersect. Then these world lines, lying in the same plane, must be parallel. But then their projections to the plane in which the paths are situated are also parallel; this contradicts the hypothesis. Therefore, any two pedestrians will meet and (a) is proved.

At any moment $t = c$ the pedestrians are situated on the plane $t = c$, and, on the other hand, they are on the plane we have found. These planes intersect along a line, implying (b).

Another solution (idea). In a movable coordinate system associated with the first pedestrian, this pedestrian does not move, which means that the other pedestrians pass the point at which the first pedestrian is standing.

Since the 2-nd pedestrian meets the 3-rd one and the 4-th one, it follows that all of the pedestrians walk along one straight line. Therefore, the 3-rd pedestrian will meet the 4-th one.

Figure 197. (Sol. 21.1.10.5)

Remarks. 1) Both solutions are, essentially, equivalent. The scheme of the second solution is obtained from the scheme of the first one by projection the pedestrians' Universe parallel to the world line of the 1-st pedestrian.

2) It could happen that the rendez-vous of the 3-rd and the 4-th pedestrians took place at $t < 0$. Hence, the statement of the problem is false if t runs not over an *open* interval, say, from $-\infty$ to ∞, but if, as for *our* Universe, the Beginning of Time is assumed. Construct a counterexample with world rays instead of world lines.

21.2.7.1. We will make use of *Euler's formula* (and give its proof for convenience and completeness).

Let polygons be countries. By removing the superfluous boundaries, i.e., frontiers, attach the neutral territories to their neighbors (the outer region included).

Denote the number of "nodes" (points of frontiers of ≥ 3 countries) by N; the number of pieces of frontiers between neighboring nodes by F; the number of countries by C.

Euler's formula

$$C - F + N = 2 \qquad (E)$$

holds if the boundary is *connected* (does not split into the union of separate pieces) and does not contain loops without nodes.

Figure 198. (Sol. 21.2.7.1)

Sketch of the proof of (E). Let us delete one-by-one pieces of the boundary without violating its connectedness. As we delete one piece we diminish by 1 either C (by uniting two countries, see Fig. 198 a)) or N (see Fig. 198 b)), but the value of $C - F + N$ does not vary. After the last removal we get $C = 1$, $F = 0$, $N = 1$ (see Fig. 198 c)) which satisfy the formula.

It is not difficult to see that in our case the condition of Euler's formula are valid; hence, so is the formula itself.

If each country neighbors all other countries, then $F \geq \frac{1}{2}C(C-1)$. Since ≥ 3 pieces of boundary meet at each node and there are $2F$ endpoints altogether, $N \leq \frac{2}{3}F$. But under these consitions

$$C - F + N = C - F + \frac{2}{3}F = C - \frac{1}{3}F \leq C - \frac{1}{3} \cdot \frac{1}{2}C(C-1) = \frac{1}{6}(7C - C^2).$$

The right-most function diminishes for $C > 3.5$ and is equal to 2 at $C = 4$. Therefore, $C - F + N < 2$ for $C > 4$ which contradicts Euler's formula.

21.2.7.2. It suffices to prove that after several moves it is possible to change the sign of exactly one of the numbers of the first row. Then change the signs of the numbers of the first row, which are distinct from the corresponding numbers of the second row.

Note, first, that it is possible to change the signs of exactly two numbers as follows: consider 10 arbitrary numbers of the first row (call them "added") and add them to the first number, change the signs of all 11 numbers, and then change the signs of the "added" 10 numbers and of the second number.

Now, "add" 10 more numbers to the first number, change the signs of all 11 numbers, divide the 10 numbers "added" into pairs and change the signs of the numbers in each pair as above.

As a result only the first number in the row changes its sign.

21.2.7.5. The length L of the brocken line is less by 1 than the number of vertices in the rectangle, i.e., $m+n+mn=(m+1)(n+1)-1$. It is impossible to draw a shorter broken line since the distance between any two nodes is not less than 1 and the total number of nodes is equal to $(m+1)(n+1)$.

21.2.8.1. Fig. 199 illustrates the polygon, the parts over which the glueing was performed are shaded. All sides that do not belong to the shaded polygons constitute parts of perimeters of both the initial and the obtained polygons. The sides of the shaded polygons are of two kinds: those that lie on the line along which the polygon was folded contribute to the perimeter of the obtained polygon but not to that of the initial one, whereas the other sides of the shaded polygons do just the opposite: contribute to the perimeter of the initial polygon but not to that of the obtained one.

FIGURE 199. (Sol. 21.2.8.1)

FIGURE 200. (Sol. 21.2.8.3)

Since for any polygon the sum of its sides that lie on a line is less than the sum of the other sides, the perimeter of the initial polygon is always greater than that of the obtained one.

21.2.8.2. It is very difficult to solve the problem if you try to find a suitable formula to express b_1 and b_2 in terms of a_1 and a_2 in the general form.

But the problem is easy to solve if you divide it into two cases:

a) $a_1 < \frac{1}{4}$. Then $(\frac{5}{4} - a_1) > 1$ and if you choose b_1 close enough to 1, then even the first term is greater than 1.

b) $a_1 \geq \frac{1}{4}$. Then $a_2 \leq \frac{3}{4}$, and $3(\frac{5}{4} - a_2) \geq \frac{3}{2}$. Therefore, if $b_2 > \frac{2}{3}$, then even the second term is greater than 1.

21.2.8.3. An auxiliary construction is shown in Fig. 200. The heights PD and QE meet at the point C in the triangle OQP, whence $OC \perp PQ$. Therefore, to solve the problem it is necessary to prove that $MN \parallel PQ$.

Construct the circle with PQ as its diameter; it is obvious that points D and E are on the circle. Therefore, angles $\angle EDP$ and $\angle EQP$, both subtending arc $\smile EP$, are equal. Similarly, by constructing the circle with DE as a diameter, we see that $\angle MND = \angle MED$. As $EM \parallel PD$, the angles EDP and MED are also equal; hence, $\angle MND = \angle EQP$.

Lastly, since $ND \parallel EQ$, it follows that $MN \parallel PQ$. Q.E.D.

The case when angle $\angle AOB$ is an obtuse one is dealt with similarly.

If the angle $\angle AOB$ is a right one, points M and N coincide.

21.2.8.4. The right hand side has the same number of factors as the left hand side but all factors on the right are equal to n whereas on the left there are some factors which are less than n.

21.2.9.3. Note that the rhombus can be deformed into a square, see Fig. 201 a); observe also the auxiliary lines which help to understand the construction: it is possible to move along the horizontal from all small squares which are above straight line AC, along the vertical from the small squares to the right of BD, and along the diagonals from the small squares of square $AOBK$.

FIGURE 201. (Sol. 21.2.9.3)

Now, let only 5 chips be positioned, and let the first chip be situated in the top right 5×5 square (in Fig. 201 b) it is the square $ODLC$). But then it is only possible to move along the diagonals of square $AOKB$. But to do this one needs not less than 8 chips. Contradiction.

It is easy to notice that if 5 chips are arranged not in the top right corner but arbitrarily, then 32 small squares are similarly distinguished, and they correspond to square $AOBK$ (these squares are not shaded in Fig. 201 c)) and a similar argument holds.

21.2.9.5. On each resistor, write a number equal to the length of (the number of resistors in) the *shortest* pass from A to this resistor. If all resistors indexed by i ($1 \le i \le 10$) become open, then so will all passes from A to B of length $\ge i$. But there are no passes of length shorter than 10 (otherwise this pass might be shortcircuted). Therefore, there are not less than 10 resistors indexed by i and not less than $10 \cdot 10 = 100$ resistors altogether.

21.2.10.1. If $y = 1$, then $x = 3$ (the second root of the quadratic, $x = -1$, is negative).

Let $y > 1$, then the numbers x and $x + 2$ are of the same parity; therefore, $x + 1$ is even: $x + 1 = 2k$. We get

$$(2k-1)^{2y} + (2k)^{2y} = (2k+1)^{2y} \tag{$*$}$$

and it is not difficult to see (by the binomial formula the initial formula turns into $2^{2y-3} \cdot k^{2y-1} + \dots - y = 0$) that y is divisible by k (for $y > 1$).

Dividing now equation ($*$) by $(2k)^{2y}$ we get

$$2 > (1 - \frac{1}{2k})^{2y} + 1 = (1 + \frac{1}{2k})^{2y} > 1 + \frac{2y}{2k}.$$

Hence, $y < k$ and, therefore, y cannot be divisible by k. This means that there are no solutions for $y > 1$.

21.2.10.2. If segment AB is contained in the polygon, then straight line AB divides the perimeter of the polygon into two parts, the length of each being greater than 1 by the hypothesis. Therefore, the perimeter is greater than 2.

Consider the case when segment AB is not contained in the polygon. On straight line AB determine points A' and B' of its intersections with the boundary of the polygon; let A' and B' lie on the extensions of segment AB and be nearest to points A and B, respectively.

In a similar way determine points A'' and B'' as the closest to A and B, respectively, intersection points of line AB with the boundary of the polygon, see Fig. 202. The segments $A'A''$ and $B'B''$ divide the whole polygon into three domains: the boundary of one of them is the union of the segment $A'A''$ and the part α of the boundary of the polygon; the boundary of the second domain is the union of segment $B'B''$ and the part β of the boundary of the polygon; the third domain is bounded by segments $A'A''$, $B'B''$, and the broken lines γ and δ, see Fig. 202.

Figure 202. (Sol. 21.2.10.2)

The broken lines $AA''\gamma B'B$ and $AA'\delta B''B$ lie inside the polygon and their lengths are greater than 1 (by the hypothesis). Therefore, $A'A''| + |\gamma| + |B'B''| + |\delta| > 2$. But $|\alpha| > |A'A''|$ and $|\beta| > |B'B''|$, hence, the whole perimeter is $|\alpha| + |\beta| + |\gamma| + |\delta| > 2$.

21.2.10.3. Let us consider 2^{2n} students having all possible combinations of marks in the transcript. From all the students we can arrange $(2n)!$ chains of $2n+1$ student each, so that each next student in a chain has better marks than its predecessor.

In our arrangement the student that has k-many A's and $(2n-k)$-many B's should stand in $k! \cdot (2n-k)!$ chains. The minimal of such products is equal to $(n!)^2$.

Let m of these students study in our school. By the hypothesis each chain has not more than 1 student from our school; therefore, $m \cdot (n!)^2 \leq (2n)!$ or $m \leq \frac{(2n)!}{(n!)^2} = \binom{2n}{n}$.

The estimate is an exact one because there exists $\binom{2n}{n}$ different combinations with n marks A and n marks B.

22.1.7.2. The difference $2^{2^{1959}} - 1 = 4^{2^{1958}} - 1$ is divisible by $4 - 1 = 3$.

22.1.7.3. Mark 9 points in a plane and number them 1 to 9. Draw all segments that connect pairs of these 9 points. With each 3-digit number $\overline{abc}$ that does not end with a 0 associate the segment drawn from the point with the number a to the point with the number c.

Now, it suffices to prove that there exists a path which passes each segment once and begins and ends at the same point. Moving along this path 10 times (for different digits b) we get the sequence to be found.

Prove the existence of this path on your own.

22.1.7.4. Each column either contains a vertical line or intersects with 8 horizontal lines. Therefore, the route of the rook contains either 8 vertical or 8 horizontal lines. To pass from one of parallel lines to another one we need two turns; the total of not less than $2 \cdot (8-1) = 14$ turns. An example is plotted on Fig. 203.

Figure 203. (Sol. 22.1.7.4)

Figure 204. (Sol. 22.1.8.3)

22.1.8.1. Indeed, we can pour $a = k(2-\sqrt{2}) + l\sqrt{2}$ liters, where k and l are integers. If $l - k \neq 0$, then a is an irrational number but if $l = k$, then $a = 2k$ is an even number. In both cases $a \neq 1$.

22.1.8.3. Draw diagonal BD. Since DK is a median in $\triangle ADB$, we have $S_{AKD} = \frac{1}{2}S_{ABD}$; similarly $S_{BMC} = \frac{1}{2}S_{BCD}$; see Fig. 204. Draw the second diagonal AC, then: $S_{BKC} + S_{AMD} = \frac{1}{2}S_{ABCD}$. This means that (the areas of triangles BCP and AOD are counted twice in the sum)

$$S_{ABCD} = S_{AKD} + S_{BMC} + S_{AMD} + S_{BKC} = S_{ABCD} - S_{MPKO} + S_{BCP} + S_{AOD}.$$

It follows that $-S_{MPKO} + S_{BCP} + S_{AOD} = 0$. Q.E.D.

22.1.8.5. By symmetry, see Fig. 205, the point O lies on the straight line O_1O_2 and is equidistant from the tangents a_3 and a_4; hence, the first circle exists. Let r be its radius, T its tangent point with a_3, T_1 and T_2 the tangent points of the circles centered in O_1 and O_2, respectively, with a_3, see Fig. 205a). Let $O_1T_1 = r_1$, $O_2T_2 = r_2$, $OO_1 = p$ and $OO_2 = q$. Since the segments OT_1, OT, OT_2 are all perpendicular to a_3 and, therefore, are parallel. If $r_2 > r_1$, drop height on O_2T_2 in trapezoid $O_1T_1T_2O_2$. It cuts off two similar triangles, hence, $\frac{r - r_1}{p} = \frac{r_2 - r_1}{p+q}$, implying $r = \frac{pr_2 + qr_1}{p+q}$, where $p = OO_1$ and $q = OO_2$.

For $r_2 < r_1$ the formula is similarly proved, while if $r_2 = r_1$ the formula is obvious, since $r_1 = r_2 = r$.

Similarly, by symmetry there exist the circle with center at O tangent to a_5 and a_6. Denote its radius by p_1; let the tangent point with a_5 be P and that of a_5 with the circle centerd at OO_1 by P_1, see Fig. 205b). Then the triangle O_2OP and $O_2O_1P_1$ are similar, hence, $\frac{r_1}{p+q} = \frac{\rho_1}{p+q}$, implying $\rho_1 = \frac{qr_1}{p+q}$.

Similarly, the radius ρ_2 of the circle centered at O tangent to the straight lines a_7 and a_8 is equal to $\rho_2 = \frac{pr_2}{p+q}$.

FIGURE 205. (Sol. 22.1.8.5)

This already implies that $\rho_1 + \rho_2 = r$. Now, recall that O is the intersection point of the inner tangents a_1 and a_2. Let a_2 be tangent to the circle centered in O_1 at Q_1 and that centered in O_2 at Q_2, see Fig. 205c). Since the triangles OO_1Q_1 and OO_2Q_2 are similar, it follows that $\frac{r_1}{p} = \frac{r_2}{q}$, hence, $qr_1 = pr_2$ and $\rho_1 = \rho_2 = \frac{1}{2}r$.

22.1.9.1. Raise $a_1 + a_2 + \cdots + a_{1959} = 1$ to the 1000-th power. We get the sum whose summands are products of one thousand of our numbers in arbitrary order. Among these products choose only the products needed; clearly, their sum is strictly less than 1.

REMARK. Numbers a_i can be interpreted as probabilities of random independent events. Then the sum of the products of 1000 factors is the probability of 1000 events which take place simultaneously. This probability is, therefore, strictly less than 1.

22.1.9.3. Let us prove the assertion of Hint. To this end draw two circles O_1 and O_2 intersecting the given circle O and passing through points A and B. Let XY and KN be the common chords of O_1 and O and of O_2 and O, respectively. Let us denote the intersection point of straight lines XY and KN by M. Connect M with A. Let straight line MA intersect circles O_1 and O_2 at points C and D, respectively; see Fig. 206 b).

Now, let us prove that all three points, C, D, and B coincide. Using the theorem on secant and tangent we get the equality $MY \cdot MX = MN \cdot MK$ for the circle O, while $MD \cdot MA = MY \cdot MX$ holds for O_1 and $MC \cdot MA = MN \cdot MK$ holds for O_2. This implies that $MD \cdot MA = MC \cdot MA$; hence, $MD = MC$ and, therefore, $C = D = B$. Q.E.D.

Thus, point M lies on AB.

Construction. 1) First, find the intersection point M of extensions of all common chords of the circles passing through A and B. To this end, draw any circle intersecting O at X and Y and passing through A and B. Then $M = AB \cap XY$.

FIGURE 206. (Sol. 22.1.9.3)

2) By construction, $XY = a$. Then $MA{\cdot}MB = MX{\cdot}MY$, see Fig. 206 a), or $MA{\cdot}MB = MX{\cdot}(MX{+}a)$. From these formulas we find the length of MX:

$$MX = \frac{1}{2}(-a + \sqrt{a^2 + 4MA \cdot MB}).$$

3) Draw the circle of radius found in step 2) centered at M. Find point X. Finally, draw the circle trough A, B, X. This is it.

22.1.9.4. It is possible to move from node O along one of four links. Any other node on the paper leaves us with only three possibilities to choose the next link of the broken line.

Thus, we have three ways to choose the second, the third, ... , the k-th link of the broken line. This means that there are not more than

$$p_k \leq 4 \cdot 3^{k-1} = 2 \cdot 2 \cdot 3^{k-1} < 2 \cdot 3^k$$

distinct nonselfintersecting broken lines.

22.1.10.1. Let $x \leq y$. Then

$$x^k + y^k = z^k \leq 2y^k \leq y^k + ky^{k-1} \leq (y+1)^k.$$

Hence, $y < z < y + 1$; that is z is not an integer.

22.1.10.5. Assume the contrary. Then choosing k numbers $a_1, a_2, \dots, a_k$, we get $a_k \leq \frac{a_1}{2}$. Choosing k numbers $a_k, a_{k+1}, \dots, a_{2k-1}$ we get $a_{2k-1} \leq \frac{a_k}{2} \leq \frac{a_1}{2^2}$ and, similarly, $a_{n(k-1)+1} \leq \frac{a_1}{2^n}$.

Consider the sum

$$S_1 = a_1 + a_k + a_{2k-1} + \dots + a_{n(k-1)+1} + \dots \leq a_1(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} + \dots) = 2a_1.$$

Therefore,

$$S_2 = a_2 + a_{k+1} + \dots + a_{n(k-1)+2} + \dots \leq 2a_1$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$S_{k-1} = a_{k-1} + a_{2k-2} + \dots + a_{n(k-1)+(k-1)} + \dots \leq 2a_1.$$

Add up these inequalities; we get:

$$S = S_1 + S_2 + \dots + S_{k-1} \leq 2(k-1)a_1 = \frac{2(k-1)}{2k} < 1.$$

Contradiction, since by the hypothesis

$$S = S_1 + \dots + S_{k-1} = a_1 + a_2 + \dots + a_n + \dots = 1.$$

22.2.7.1. Let us arrange the summands of the left and the right sums into groups of two summands in each, from different ends. Then the difference of these sums is equal to the sum of the following expressions:

$$(a_k b_k + a_{n-k+1} b_{n-k+1} - a_k b_{n-k+1} - a_{n-k+1} b_k) = (a_k - a_{n-k+1})(b_k - b_{n-k+1}); \qquad (*)$$

both factors in the right hand side of $(*)$ are positive as $a_k > a_{n-k+1}$ and $b_k > b_{n-k+1}$ by the hypothesis. This means that the left sum is greater than the right one.

22.2.7.2. Let point A_1 be symmetric to H through side BC. Then $\angle HA_1C = \angle A_1HC$. Besides, $\angle A_1HC = \angle CBA$, since the respective sides of these angles are perpendicular. Therefore, $\angle AA_1C = \angle ABC$, i.e., point A_1 lies on the circumscribed circle. Similarly, points symmetric to point H through sides AB and AC lie on the circumscribed circle. (See also Problem 7.2.7-8.6.)

22.2.7.4. Let $a_1, \dots, a_6$ be the digits of a 6-digit number. Rearrange the digits so that

$$a_1 \geq a_4 \geq a_2 \geq a_5 \geq a_3 \geq a_6.$$

Then

$$0 \leq (a_1 + a_2 + a_3) - (a_4 + a_5 + a_6) \leq (a_1 + a_2 + a_3) - (a_2 + a_3 + a_6) = a_1 - a_6 \leq 9.$$

22.2.7.5 and **22.2.8.1.** Clearly, the reformulation of Problem 22.2.8.1 goes as follows: each knight is replaced by x_i; the knights of the same sign are friends; those of opposite signs are enemies.

Half of the summands are equal to 1, and half are equal to -1; hence, $n = 2k$. But $x_ix_{i+1} = -1$ if and only if the factors are of different signs. This means that k is the number of the alternations of signs in the sequence $x_1, x_2, \dots, x_n, x_1$, and since the sequence begins and ends with the same number, $k = 2l$. Thus, $n = 2k = 4l$.

22.2.8.2. Let us prove that among the first four numbers there are both positive and negative ones. Suppose, $a_2 > 0$ and $a_3 > 0$; then $a_1 - a_2 + a_3 < 0$, $a_2 - a_3 + a_4 < 0$. Having added these four inequalities, we get $a_1 + a_4 < 0$, i.e., either a_1 or a_4 is negative. similarly, if $a_2 < 0$ and $a_3 < 0$; then $a_1 + a_4 > 0$.

The same statement is true for the second and the third foursome of numbers.

22.2.8.3. Thanks to the symmetry, triangles $\triangle ABC$, $\triangle A_1BC_1$, $\triangle A_2B_2C$ are equal, see Fig. 207.

FIGURE 207. (Sol. 22.2.8.3)

FIGURE 208. (Sol. 22.2.8.5)

Therefore, the four angles of pentagon $A'B_2CBC_1$ are equal to $\pi - \angle B$, $\pi - \angle C$, $\pi - \angle B$, $\pi - \angle C$. Since the sum of all five angles is equal to 3π,

$$\begin{aligned}\angle A' &= 3\pi - (\pi - \angle B) - (\pi - \angle C) - (\pi - \angle B) - (\pi - \angle C)\\ &= 2\angle B + 2\angle C - \pi = \pi - 2\angle A.\end{aligned}$$

22.2.8.5. Denote the cells of the upper and lower rows by e (from edge rows), those of the other rows by m (from middle rows). Since it is impossible to go from edge to edge, the only admissible moves are those of types $e - m$, $m - e$ and $m - m$. Since there is equally many edge and middle cells, they should alternate, with one exception: if the beginning and the end of the route is in edge rows, then a chain

$$e - m - e - \cdots - m - e - m - m - e - \cdots - m - e$$

is possible. The same applies to the columns. This means that all moves, except two ones, should pass along the segments plotted on Fig. 208. The figure formed by these segments consists of four quadrilaterals and can not be connected by two segments in a continuous pass.

EXTENSION. Is it possible to move the knight in the same way over a 4×3 chessboard? And what about a 4×5 chessboard?

22.2.9.1. First consider the numbers $x_1, x_2, \dots, x_{50}$. Let the difference between their sum and $\frac{1}{2}$ be greater than $\frac{1}{100}$. Let it be, for example, less than $\frac{1}{2}$; but then $x_{51} + \dots + x_{100}$ is greater than $\frac{1}{2}$.

Let us begin to "move" the numbers of our set to the right replacing first x_{50} by x_{51}, then x_{51} by x_{52}; then x_{49} by x_{50}, and so on. Since in doing so we finally get the set $\{x_{51}, \dots, x_{100}\}$, this sum will eventually become greater than $\frac{1}{2}$.

It is easy to see that exactly at this moment the difference between the sum of the numbers of the set and $\frac{1}{2}$ becomes less than $\frac{1}{100}$.

22.2.9.2 and **22.2.8.4.** The angle between some two segments is not less than $\frac{\pi}{n}$. Consider one of two vertical angles which is at least $\frac{\pi}{n}$ and such that the sum of the lengths of its two legs, a and b (the parts of the given segments of length 1), is not less than 1.

It is not difficult to verify that the length of the third side of the triangle with sides a, b and the angle $\alpha \geq \frac{\pi}{n}$ between them is not less than the length of the third side of the triangle with sides $\frac{1}{2}$, $\frac{1}{2}$ and the angle $\frac{\pi}{n}$ between them.

22.2.9.3. Suppose that two vertices, A and B, have the property indicated.

Then $\angle CAB + \angle DAB > 180°$, and $\angle CBA + \angle DBA > 180°$; but the sum of all six angles of the triangles CAB and DAB is equal to $180° + 180°$. Contradiction.

22.2.9.4. The cube of any number is of the form either $9k$ or $9k \pm 1$. Hence, the numbers of the form $9k + 4$ and $9k + 5$ cannot be represented as the sum of three cubes.

Another solution: estimate how many numbers among the first n numbers can be represented in the form of the sum of three cubes. Using the fact that among the first thousand there are only ten cubes, prove that among the first thousand there are no more than 200 numbers which can be represented in the form of the sum of three cubes. In general, only approximately $\frac{n^3}{6}$ numbers among the first n^3 numbers can be represented as the sum of three cubes.

22.2.9.5. Let us enumerate the squares of the chessboard (except the central one) in the order of the knight's movements, see Fig. 209.

Figure 209. (Sol. 22.2.9.5)

Figure 210. (Sol. 22.2.10.4)

The white knights stand on squares 1 and 3, the black ones on squares 5 and 7. On Fig. 209these squares are situated along the circle, a white small circle corresponds to a white knight, a black small circle corresponds to a black knight. Each move of the knight on the chessboard corresponds to its shift to the neighboring position on the circle and the other way round, hence, it suffices to prove the statement of the problem for the circle.

Observe that on the circle the knights can not jump over each other, therefore, there order (clockwise) does not vary. But then teh wits and the blacks can only change places if the whole picture is rotated, as a solid body, through $180°$. Since to move to the opposite point of the circle each knight needs ≥ 4 moves,the total of moves required is ≥ 16.

Remark. This estimate is an exact one: compose an example how to make it for exactly 16 moves.

22.2.10.4. Denote: $d_1 = c_1 - z$, $\dots$, $d_n = c_n - z$. Clearly, $\frac{1}{d_1} + \dots + \frac{1}{d_n} = 0$. Let us translate the whole picture by the vector zO; then z becomes O, the points $c_1, \dots, c_n$ become $d_1, \dots, d_n$, respectively.

Clearly, the polygons $M = c_1 \dots c_n$ and $M' = d_1 \dots d_n$ are equal and it suffices to prove that O lies inside M', see Fig. 210.

Suppose that O lies outside M'. Then M' lies entirely an angle of measure α with vertex O and such that $\alpha < \pi$.

As is known, if f is an arbitrary complex number and $f = \frac{1}{d}$, then the rays Od and Of are symmetric with respect to the x-axis. Therefore, the points $\frac{1}{d_1}, \dots, \frac{1}{d_n}$ lie in a certain angle β symmetric to α with respect to the x-axis and $\beta = \alpha < \pi$. The radius-vectors corresponding to these points also lie inside β; hence, their sum is not equal to $\vec{0}$, see Lemma below. Contradiction.

Lemma. *Consider several nonzero vectors with the common initial point. Let this point be the vertex of an angle on the plane and let the vectors lie inside the angle. The sum of the vectors is not equal to* $\vec{0}$.

Proof. Consider the bisector of the angle as a directed real axis. Each of the vectors constitutes with the bisector an acute angle, hence their projections on the axis are all positive. Since the sum of projections is nonzero, the projection of the sum to which it is equal is nonzero; hence the sum itself is nonzero.

22.2.10.5. Let us put the discs so that the boundaries of the sectors were identical. There are $2n$ different ways to put the 1-st sector upon the 1-st one, 2-nd one, etc. Let at the first identification the colors of s_1 sectors coinside, at the second identification the colors of s_2 sectors coinside, etc.

Let su compute the sum $s_1 + \dots + s_{2n}$ by a different method. Each of $2n$ sectors of the smaller disc coinsides in color with the sectors of the larger disc n times and n times the colors do not coinside; hence, the contribution of each sector in the sum total is equal to n. Therefore, $s_1 + \dots + s_{2n} = n \cdot 2n$. All the summands can not be smaller than n, since otherwise the sum $< n \cdot 2n$. Hence, at some position not less than n sectors will coinside and the sum of their arc lengths willconstitute not less than a half circumference.

23.1.7.1. Let S be a sum that meets the conditions. If $S \geq 10$, then $S = 1 + 1 + \cdots + 1$ (S-many 1's) and also $S = 10 + 1 + \cdots + 1$ (($S - 10$)-many 1's). Clearly, it is impossible to represent an even sum of less than 10 roubles by an odd number of bills of 1, 3 and 5 roubles.

23.1.7.2. Observe (see Fig. 211) that $\frac{1}{2}\smile SA_i = \angle SA_iA_j$ for any i and $j \neq i$; hence, a halfsum of all 6 arcs is equal to the sum of angles in triangle $\triangle A_1A_2A_3$. Therefore, the sum of distinct arcs is $\smile A_1S + \smile A_2S + \smile A_3S = 180^\circ$.

Let us prove that $\triangle A_1A_2O_3 = \triangle O_1O_2A_3$. Indeed, they have two equal sides (the radii of the circles); it remains to prove that $\angle A_1O_3A_2 = \angle O_1A_3A_2$. But

$$\angle A_1O_3A_2 = \smile A_1A_2 = \smile A_1S + \smile A_2S = 180^\circ - \smile A_3S,$$
$$\angle O_1A_3A_2 = 180^\circ - 2\angle O_3A_1A_2 = 180^\circ - 2 \cdot \tfrac{1}{2}\smile A_3S.$$

Therefore, $A_1A_2 = O_1O_2$. We similarly prove that $A_1A_3 = O_1O_3$, $A_2A_3 = O_2O_3$. Hence, $\triangle A_1A_2A_3 = \triangle O_1O_2$.

FIGURE 211. (Sol. 23.1.7.2)

FIGURE 212. (Sol. 23.1.7.4)

23.1.7.3. Let a_k be the number of k-th year students, x_k the number of problems each of them suggested. Then

$$a_1 + a_2 + a_3 + a_4 + a_5 = 30, \quad a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 40.$$

Without loss of generality we may assume (after a renumeration of years) that $x_1 < x_2 < x_3 < x_4 < x_5$ and $x_1 \geq 1$; $x_2 \geq 2$; $x_3 \geq 3$; $x_4 \geq 4$; $x_5 \geq 5$.

Then $40 > (a_1 + a_2 + a_3 + a_4 + a_5)x_1 = 30x_1$, therefrom $x_1 < 4/3$; hence, $x_1 = 1$.

Thus, freshmen suggested one problem each. It remains to find the number of freshmen, a_1. Since

$$\begin{aligned}40 = a_1 \cdot 1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 \\ &= 30 + a_2(x_2 - 1) + a_3(x_3 - 1) + a_4(x_4 - 1) + a_5(x_5 - 1) \\ &\geq 30 + a_2 + 2a_3 + 3a_4 + 4a_5 \geq 30 + 1 + 2 + 3 + 4 = 40\end{aligned}$$

we get: $a_2 = a_3 = a_4 = a_5 = 1$ and $x_2 = 2$; $x_3 = 3$; $x_4 = 4$; $x_5 = 5$. Hence, from each year except the first one, only one student suggested one problem and the 4 non-freshmen suggested $2+3+4+5 = 14$ problems. The remaining $a_1 = 30 - 4 = 26$ freshmen suggested $x_1 = 40 - 14 = 26$ problems.

23.1.7.4. Points A_1, N, B_2 lie on the same straight line ($\angle A_1NB_2 = \angle A_1NM + \angle B_2NB = \pi$, since the inscribed angles A_1NM and B_2NM subtend diameters A_1M and B_2M, see Fig. 212). Since $\angle A_1B_1M = \frac{\pi}{2}$ (subtends diameter A_1M) and $\angle B_2A_2M = \frac{\pi}{2}$ (subtends diameter B_2M), then A_1B_1 and B_2A_2 are heights of the triangle A_1MB_2, and MN is also a height of this triangle. But the heights of a triangle meet at one point, H.

23.1.7.5. If d_1 is a divisor of n, then $d_2 = \frac{n}{d_1}$ is also a divisor of n, and, therefore, the lesser of them is not greater than $\sqrt{n}$. In this way all divisors can be arranged in pairs (except $\sqrt{n}$ when n is the perfect square), which means that there are not more than $\sqrt{n} + \sqrt{n} = 2\sqrt{n}$ of divisors altogether.

23.1.8.2. Let there be n players. The first $n-3$ players played $\frac{(n-3)(n-4)}{2}$ games with one another and scored the same number of points. Then they played $3(n-3)$ games with the last three players.

But the last three players played three games with one another and scored 3 points; by the hypothesis they scored the same number of points in the games with the first $n-3$ players. Therefore, the first $n-3$ players scored only $3(n-3)-3$ points after $3(n-3)$ games. Now, by the hypothesis, $\frac{(n-3)(n-4)}{2} = 3(n-3)-3$, whence either $n=4$ or $n=9$.

The case $n=4$ is to be excluded since in this case the last three players scored 3 points from the first player which means that (s)he is in the last place but not the first.

The case $n=9$ remains. An example of the tournament table is shown in Fig. 213.

FIGURE 213. (Sol. 23.1.8.2)

FIGURE 214. (Sol. 23.1.8.3)

23.1.8.3. Draw a straight line through vertex B parallel to diagonal AC; see Fig. 214. Let P be the point of its intersection with the extension of side DC and let R be the midpoint of segment DP. It is easy to see that straight line AR is the desired one: $S_{ARD} = S_{ARP}$, $S_{ABC} = S_{APC}$; hence, $S_{ABCR} = S_{APR} = S_{ARD}$ and $S_{ABCR} = \frac{1}{2} S_{ABCD}$.

23.1.8.5. If $m^2 = p + n^{2k}$, then $p = (m-n^k)(m+n^k)$. Therefore, as p is a prime, then $m - n^k = 1$ and $p = m + n^k = 2n^k + 1$. But the number $2n^k+1$ is not a prime for an infinite set of numbers of the form n^k and, since $n^k = m-1$ is uniquely determined from m, the same applies to infinitely many m's.

23.1.9.1. Let $\frac{a}{b}$ be a proper fraction. Find the least possible integer q_1 for which $q_1 \cdot \frac{a}{b} \geq 1$. Let $\frac{a_1}{b_1} = q_1 \cdot \frac{a}{b} - 1$ and find the least possible integer q_2 for which $q_2 \cdot \frac{a_1}{b_1} \geq 1$.

Let $\frac{a_2}{b_2} = q_2 \cdot \frac{a_1}{b_1} - 1$ and in the same way find an integer q_3, etc. Note that

$$\frac{a}{b} > \frac{a_1}{b_1} > \frac{a_2}{b_2} > \dots;$$

the denominators of these fractions are not greater than b. Therefore, this process stops after a finite number of steps, say, there is an n such that $q_n \cdot \frac{a_{n-1}}{b_{n-1}} = 1$. It is easy to see that

$$\frac{a}{b} = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \dots + \frac{1}{q_1 q_2 \dots q_n}.$$

REMARK. Such a representation is not always unique. For example,

$$\frac{13}{50} = \frac{1}{4} + \frac{1}{100} = \frac{1}{5} + \frac{1}{17} + \frac{1}{50 \cdot 17}.$$

23.1.9.3. Let an arbitrary straight line l passing through point O meet the sides of the polygon M at points A and B and let, say, $OA > OB$. Let then A' and B' be the intersection points of the polygon M with another straight line l' rotated through an angle of α about l. If α is small, then, obviously, $OA' > OB'$ and, therefore,

$$S_{OAB} = OA \cdot OB \cdot \sin\alpha > OA' \cdot OB' \cdot \sin\alpha = S_{OA'B'}.$$

This contradicts the fact that both l and l' divide the area of M in halves. This means that $OA = OB$ holds at all times. Q.E.D.

23.1.10.2. Let n be odd. Suppose that $\frac{a^n+b^n}{a+b}$ is an integer. Then

$$\frac{a^n+b^n}{a+b} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1},$$

and it is easy to show that any two consecutive terms $a^{n-k}b^k$ and $-a^{n-k-1}b^{k-1}$ have the same remainder after division by n. Indeed, their difference $a^{n-k-1}b^k(a+b)$ is a multiple of $a+b$, Q.E.D.

For even n **the statement to prove is wrong**, e.g. for $n=2$: $\frac{1^2+1^2}{1+1}=1$, for $n=10$: $\frac{5^{10}+5^{10}}{5+5}=5^9$.

23.1.10.5. Suppose the absolute value of one of the numbers a_i is strictly greater than that of the others; let a_1 be this number. Divide the given equality by a_1^n; the left hand side is equal to 1 plus the sum of the terms each one of which approaches zero as $n \to \infty$.

Therefore, for an n which is great enough the sum is non-zero and this means that among the numbers a_i there is a number, say a_2, equal to $-a_1$. Then $a_1^n + a_2^n = 0$ for all odd n, and we can forget about a_1 and a_2.

Repeat the same argument for the remaining numbers. (Cf. Problem 31.2.10.3.)

23.2.7.1. If points A, B, C, D form a convex quadrilateral, the inequalities $MA + MC \geq AC$ and $MB + MD \geq BD$ hold for any point M. Adding up the inequalities we see that the point to be found is the intersection point of diagonals AC and BD, see Fig. 215 a).

FIGURE 215. (Sol. 23.2.7.1)

If one of the points, say D, lies inside triangle ABC (or on its side), then for any point M we have

$$AM + BM + CM + DM \geq AD + BD + CD.$$

Indeed, for any position of point M point D lies inside one of the triangles $\triangle ABM$, $\triangle BCM$, $\triangle CAM$. Let, for instance, D lie inside $\triangle ABM$. Then $AM + BM \geq AD + DB$. Besides, $CM + DM \geq CD$. Therefore, the point to be found is D; see Fig. 215 b).

Let M be inside (or on the boundary) of $\triangle BDA'$. Then

$$AM + MC \geq AQ + QC \geq AD + DC; \quad DM + MB \geq DB.$$

Finally, if A, B, C, D lie on one line (suppose that C and D lie between A and B) then the point to be found is any point M of segment CD, Fig. 215 c).

23.2.7.2. Let $a \geq b \geq c \geq d$ be the lengths of the sides of the quadrilateral. Then it is possible to construct a triangle with sides of lengths $a-d$, b, c.

Indeed, $a-d+b \geq c$, $a-d+c \geq b$ and $b+c \geq a-d$. The first two inequalities are obvious and the third means that the sum of lengths of the three sides of the quadrilateral is greater than the length of the fourth side. Starting with this triangle, it is easy to build a trapezoid by adjoining to the triangle the parallelogram with sides c and d, see Fig. 216.

FIGURE 216. (Sol. 23.2.7.2)

23.2.7.3. The convex hull of a pentagon is a triangle, quadrilateral or pentagon formed by diagonals and/or sides of the initial pentagon. If the convex hull is formed by diagonals only, it is is a pentagon (identical with the initial pentagon, actually). It is clear that the diagonals of any pentagon form a pentagon, not a triangle or quadrilateral. But then the diagonals of the convex hull (i.e., the sides of the initial pentagon) intersect in contradiction with the condition.

Hence, at least one side of the convex hull is a side of the initial pentagon; it is the side we are looking for.

23.2.8.2. Consider the three cases:

a) If both m and n are odd, every move of the piece changes the parity of the number of the column in which it is situated. Therefore, it takes an even number of moves to return into the original column and a certain "*even more*" even number of moves to return to the original square.

b) If only one of the numbers m and n is odd, the piece changes the color of the square on which it is situated with each move.

c) If both m and n are even, let 2^k be the greatest power of 2 by which both m and n are divisible. Then consider a geometrically "similar" piece whose moves are 2^k times shorter and apply the arguments of a) or b).

23.2.8.4. It is clear that during the time covered by k observers the snail had crawed at most k meters (one should subtract from the total sum the path observed twice, trice, etc.). We can, however, consider only part of the whole set of observers, provided the snail is watched each moment of the 6 minutes considered.

Let a_1 be the first observer, a_2 the last of all observers who began to watch the snail before observer a_1 finished watching it, a_3 the last of the observers who began to watch the snail before observer a_2 finished watching it, etc. The observation intervals of *odd* observers $a_1, a_3, a_5, \dots$ do not intersect and the observation intervals of *even* observers $a_2, a_4, a_6, \dots$ do not intersect either (otherwise, one of the observers is not correctly chosen).

FIGURE 217. (Sol. 23.2.8.4)

But each of these intervals lasts exactly 1 minute; the whole period of observation lasts 6 minutes, the total number of odd intervals is *not greater* than 6, while the total number of even intervals is *not greater* than 5, thanks to the initial shift.

If there exactly 6 odd intervals, they precisely cover the whole observation time and the snail had crawled ≤ 6 m. If the total number of odd intervals is < 6, the total number of both odd and even intervals is *not greater* than 10. Hence, the snail had crawled ≤ 10 m.

Hence, the number of the observers is not greater than 10, which means that the snail crawled not more than 10 m. (It can crawl exactly 10 m, for example, if it only crawls while being watched by exactly one observer and stands still otherwise; see Fig. 217.)

23.2.8.5. Since $2\alpha+\beta+2\cdot 90^\circ = 540^\circ$, it follows that $2\alpha+\beta = 360^\circ$; hence, it is possible to compose a hexagon from four pentagons; for an example see Fig. 218a). Fig. 218b) shows how to cover the plane with such hexagons.

FIGURE 218. (Sol. 23.2.8.5)

23.2.9.2. Let K, L, M and N be the third vertices of equilateral triangles constructed on sides BC, AB, AF, and FE, respectively; let B_1, A_1, and F_1 be the midpoints of segments KL, LM and MN, respectively, see Fig. 219. Let us prove that vector $\overrightarrow{A_1B_1}$ can be obtained from vector $\overrightarrow{F_1A_1}$ by rotation through an angle of 60° (and similarly for the other sides). This would, clearly, imply that the hexagon is regular.

Let, further, $\mathbf{a} = \overrightarrow{BC} = \overrightarrow{FE}$, $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{AF}$; let R be the rotation by 60° that sends $\overrightarrow{BC}$ to $\overrightarrow{BK}$. Then $\overrightarrow{AM} = -R^2\mathbf{c}$ and $\overrightarrow{FN} = -R^2\mathbf{a}$. Therefore, $2\overrightarrow{A_1B_1} = R^2\mathbf{c} + R\mathbf{a} + \mathbf{b}$ and $2\overrightarrow{F_1A_1} = R^2\mathbf{a} + R\mathbf{b} - \mathbf{c}$, i.e., $\overrightarrow{F_1A_1} = R(\overrightarrow{A_1B_1})$.

23.2.9.4. If the rectangle is circumscribed around the triangle, then a vertex of the triangle coincides with a vertex (denote it A) of the rectangle. Denote the opposite vertex of the rectangle by N, and remaining the vertices of the triangle by B and C, see Fig. 220.

The endpoint N of the diagonal AN circumvents a part of the semicircle constructed outside triangle ABC with side BC as a diameter. What is this part?

Erect perpendiculars to sides AC and AB at points C and B. Let us determine the intersection points of the perpendiculars with this semicircle. These intersection points are the endpoints of the arc to be found. The center of the rectangle — the midpoint of segment AN — circumvents a similar part of the arc of the semicircle for which segment $B'C'$ connecting the midpoints of sides AC and BC of triangle ABC is a diameter.

FIGURE 219. (Sol. 23.2.9.2)

FIGURE 220. (Sol. 23.2.9.4)

Now, it is easy to solve the problem: the locus of centers of rectangles is the curvilinear triangle formed by the arcs of the semicircles whose diameters are segments connecting the midpoints of the sides of triangle ABC, see Fig. 220.

23.2.9.5. Consider figure F — the locus of points whose distances to the segment of length 10 are not greater than 1, see Fig. 221.

FIGURE 221. (Sol. 23.2.9.5)

Divide the whole 100×100 square into 50 vertical strips of width 2 each, and in each of them arrange 8 nonintersecting figures identical to F. Then by the hypothesis, each of the 400 figures contains at least one center of a circle; therefore, there are no fewer than 400 circles. Q.E.D.

23.2.10.1. Assume the contrary. First of all, observe that there are not more than 7 digits 1 among $a_1, \dots, a_k$. Adding the rest of the numbers a_i one by one we can make the sum less than A but greater than $A - 8$ (one number may still be added as long as the sum is less than A).

Therefore, if among the a_i there are 8 digits 1 we can make the sum equal to A. Note that there are no more than 7 digits 2 among $a_1, \dots, a_k$. Use a similar argument adding either one even number or two odd numbers at one step (to keep the sum even). At some moment we obtain a number less than A but greater than $A - 18$ and congruent to A modulo 2. Add the needed number of 2's to it; we get A.

We similarly show that among the a_i there are not very many 3's, 4's, ... , 9's. But then their sum is surely $< 5040 = 2520 \cdot 2$: indeed, 2520 is the first number divisible by 1, 2, ... , 9. Contradiction.

23.2.10.2. Let b be the last digit. Replace it with a 0 (by subtracting a b) and write it in front of the number (add $b \cdot 10^{6n}$). The increment of the number is equal to $b(10^{6n} - 1)$. The number in parentheses is $\underbrace{9\dots9}_{6n\text{ nines}}$. Clearly this number is divisible by 999 999 and 999 999 is divisible by 7. Thsu, the initial number was divisible by 7, we added to it a number divisible by 7, so the result is divisible by 7. This result is 10 times greater than we need, it is true, but since 10 and 7 are relatively prime, the number with the last 0 deleted (and this is the number we need) is divisible by 7.

23.2.10.3. Assume that some person X has m acquaintances $A_1, A_2, \dots, A_m$. By the hypothesis all these acquaintances are not acquainted with one another. This means that for every two people (A_i, A_j) there is one more person Y (not X) who is their common acquaintance, and who is not acquainted with X; different people Y correspond to distinct pairs (A_i, A_j).

So there are at least $C_m^2 = m(m-1)/2$ people who are not acquainted with X. But every person Y who is not acquainted with X has exactly two acquaintances among $A_1, A_2, \dots, A_m$, and distinct pairs (A_i, A_j) correspond to different people Y.

This means that the number of people who are not acquainted with X is not greater than C_m^2. Hence, exactly C_m^2 people are not acquainted with X.

Therefore, $n = 1 + m + C_m^2$, where 1 corresponds to X himself/herself, m is the number of his/her acquaintances, and C_m^2 is the number of people who are not acquainted with X.

Therefore, $m = m(X)$ is a constant, it is the same for every X. Try to find out on your own for what m and n the situation described in the problem is possible.

23.2.10.5. Mark those of numbers $1, 2, \ldots, 2n$ whose corresponding segments are directed to the right or upwards (C_{2n}^n ways). It is possible to independently choose n numbers whose corresponding segments are directed to the right or downwards (also C_{2n}^n ways). These choices uniquely determine the broken line, and the total number of ways is equal to $(C_{2n}^n)^2$.

24.1.7.1. Divide all rows into pairs and number the pairs from 1 to $\frac{n}{2}$. Write the numbers

$$2(k-1)n+1, \quad 2(k-1)n+2, \quad \ldots, \quad 2(k-1)n$$

in the first row of the k-th pair and the numbers

$$2kn, \quad 2kn-1, \quad \ldots, \quad 2(k-1)n+1$$

in the second row of this pair. Then the sum of two numbers of an arbitrary column from the k-th pair of rows is equal to $4kn-2n+1$, i.e., it is constant for this pair of rows. This means that the sum of all numbers in an arbitrary column does not depend on the number of the column and is equal to $\dfrac{n(n^2+1)}{2}$, as is not difficult to calculate.

24.1.7.2. Clearly, all numbers beginning with $\overline{a_1b_1c_1}$ are divisible by 9 and (if they are not equal to 0) their middle digit is 9. Therefore, it suffices to test ten numbers: 99, 198, 297, ... , 990, which is easy.

24.1.7.5. Let us use a stronger statement:

Lemma. *For any two squares of different colors there exists a way of the rook passing each square once and with endpoints in the given squares.*

Let us traverse the way indicated in Lemma from the first selected square to a square neighboring the other selected square and with the last move put the rook on the other selected square. This solves the problem.

PROOF of Lemma. Let us divide the chessboard into 2×2 squares and mark a chain of squares connecting the squares selected. It is not difficult for the rook to traverse all squares of the marked chain with the given squares as the endpoints of the pass. Now, it only remains to incorporate the remaining squares into the itinerary; do it yourselves. (Cf. solution of Problem 56.10.2.)

24.1.8.1. Let A_1 and B_1 be the midpoints of sides AB and BC, respectively. Then A_1B_1 is the midline in $\triangle ABC$ and $A_1B_1 \parallel AC$, $A_1B_1 = \frac{1}{2}AC$. But M_1 and M_2 are the points on the segments OA_1 and OB_1, respectively, that divide the segments in ratio $2:1$ counting from O; therefore, $M_1M_2 = \frac{2}{3}A_1B_1$ and $M_1M_2 \parallel A_1B_1$. Thus,

$$M_1M_2 = \frac{2}{3}\cdot\frac{1}{2}AC = \frac{1}{3}AC \text{ and } M_1M_2 \parallel A_1B_1.$$

Similar statements can be similarly proved for the other sides of triangle $M_1M_2M_3$.

FIGURE 222. (Sol. 24.1.8.1)

FIGURE 223. (Sol. 24.1.8.4)

24.1.8.2. One question will do if the second player takes a sequence of fast growth, e.g.,

$$a_1 = 1, \;\; a_2 = 100, \;\; a_3 = 10000, \;\; \ldots, \;\; a_n = 10^{2^n-2}.$$

24.1.8.4. Essentially, we have to construct a closed path of vertical and horizontal segments through the centers of the board's cells. One of these paths for an even n is shown on Fig. 223. If the number of cells is odd, then a pass can not be closed, since it should contain equally many black and white cells.

24.1.9.3. For n even see the solution of Problem 24.1.7.1 for an even n.

If n is odd the situation is more difficult to treat: a column has an odd number of places and it cannot be filled up completely by pairs. So you may do the following: fill the first row with integers from the middle of the segment $[1, n^2]$, i.e., the integers from $\frac{n^2-n}{2}+1$ to $\frac{n^2+n}{2}$; the second row with integers $1, \dots, n$; and the third row with integers $n^2-n+1, \dots, n^2$ arranging the integers so that the sum of the three integers in each column were the same (see the solution of Problem 24.1.8.5). Then you can fill the remaining $n-3$ places in each column with pairs of integers in the same way as in Problem 24.1.7.1, since $n-3$ is even.

FIGURE 224. (Sol. 24.1.9.4)

24.1.9.4. Since no passenger has coins of denomination smaller than 10 kopeks, every passenger ought to get the change. This means that $4k$ coins have to be left with the passengers. But the fare of all $4k$ passengers is equal to $20k$ kopeks, and not less than k coins are needed to pay the fare. Hence, not less than k coins were placed into the cash-box; the total number of coins is, therefore, not less than $5k$.

An example of the procedure with $5k$ coins is as follows. Divide all passengers into k groups, 4 in each group. Let 5 coins in each group be distributed in the following way: the first passenger has 15 kopeks, the second one has $10+10$ kopeks, the third one has 15 kopeks, the fourth one has 20 kopeks.

How should the passengers of each group exchange coins? See Fig. 224.

24.1.10.1. Use induction. Let u_{5k} be divisible by 5, and let the remainder after division of u_{5k+1} by 5 be $r \geq 0$. Then the remainders after division of u_{5k+2}, u_{5k+3}, u_{5k+4}, u_{5k+5} by 5 are r, $2r$, $3r$, and $5r \equiv 0 \pmod 5$, respectively.

24.1.10.2. ANSWER: all axes of these strips must meet at the same point O, see Fig. 225.

The intersection domain of maximal area is a polygon such that

FIGURE 225. (Sol. 24.1.10.2)

FIGURE 226. (Sol. 24.2.7.1)

a) its sides are pairwise parallel (this follows from the fact that otherwise a shift of the strip whose edge has no parallel match would increase the area of the intersection);

b) parallel sides are equal (for the same reason).

This implies that the polygon is centrally symmetric and its center of symmetry is the intersection point of the strips that form its sides, Q.E.D.

24.1.10.4. Draw tangents OA and OB to S. Let C be the center of disc S and D the intersection point of segments OC and AB. The desired locus of centers is the punctured disc (without O) constructed on OD as its diameter in the plane perpendicular to the one of S.

To prove this, observe that any circle, along which the sphere is tangent to the cone, passes through A and B, i.e., the plane of this circle rotates around straight line AB, and the axis of the cone is a perpendicular to this plane.

24.1.10.5. Assume the contrary. Since the difference of arguments of the numbers z_2 and z_1 is less than 120°, it follows tat z_2 lies strictly inside an angle of 240° (120° on both sides of the vector Oz_1). Similar arguments apply to $z_3, \dots, z_n$. Actually all these vectrs lie inside a still smaller angle, namely of measure 2α with z_1 as the bisector, where α is the largest of the differences of arguments of the z_i and z_1. Since $\alpha < 120^\circ$, the complement is of measure $360^\circ - 2\alpha > 120^\circ$.

Let z_i and z_j be teh two numbers closest to the legs of the angle. The angle β between them is also smaller than 120°. Observe, however, that the angle β can include the complement $360^\circ - 2\alpha$, hence, it lies completely inside the angle 2α. Due to the choice of z_i and z_j, all the other numbers lie inside the angle β, i.e., all the numbers lie inside an angle of 120°. But then by Lemma from the solution of Problem 22.2.10.4 the sum of htese numbers does not vanish. Contradiction.

24.2.7.1. Use induction on the number of diagonals drawn. Let one of the polygons be completely painted on the outside after drawing k diagonals. If the $(k+1)$-st diagonal AB does not intersect any of the small polygons, then after drawing this $(k+1)$-th diagonal we get the polygon to be found, see Fig. 226 a).

If the $(k+1)$-st diagonal, AB, *does* intersect a small (painted) polygon, then it is easy to see that one of the two polygons, into which the diagonal divides it, is the one to be found, see Fig. 226 b).

24.2.7.3. Let m be a number the sum $S(m)$ of whose digits is divisible by 11. Let us show that among the next 39 numbers there is a number with the same property.

a) If the last digit of m is not 0 and the penultimate one is not 9, then $S(m+9) = S(m)$, and $m+9$ is the desired number.

b) If m ends with a zero and the penultimate digit is not greater than 7, then the number $m+29$ will do, as $S(m+29) = 11 + S(m)$. For example, $S(470) = 11$, $S(499) = 22$.

c) If the penultimate digit of m is 8 or 9, then not more than 20 can be added to m in order to get a number 4n4 which ends with two zeros. Let $S(n) = k$. Then the sums of the digits of the numbers n, $n+1$, $\dots$, $n+9$, $n+19$ are, respectively equal to k, $k+1$, $\dots$, $k+9$, $k+10$. Among 11 consequtive numbers at least one is divisible by 11 but all these numbers do not exceed $m+29$.

The example $m = 999\,980$ shows that this estimate cannot be improved.

24.2.7.5. Since $abcd - a = a(bcd - 1) = 1961$, the number a is odd. It follows from the other equations that b, c, d are also odd. But then the number $abcd$ is odd and, therefore, the difference $abcd - a$ is even.

24.2.8.1. Assume that it is possible to draw such a broken line; it may be considered an open one. The broken line enters each of 5 domains and leaves them, and for a given domain there is an exit corresponding to each entrance, except the case when the broken line enters a domain for the last time and never leaves it.

FIGURE 227. (Sol. 24.2.8.1)

Now look at Figure. Since the two upper domains of Fig. 227 is bounded by five (an odd number) segments, one endpoint of the broken line must be in one of these domains, and the second endpoint in the other domain.

But the lower middle domain is also bounded by five segments. Therefore, some endpoint of the broken line must also be contained in this domain.

But a line, even a broken line, does not have a third endpoint.

24.2.8.2. Drop perpendicular OK from point O (the intersection point of the diagonals of the rectangle) to the common outer tangent to circles 1 and 3. Draw radii r_1 and r_3 to the tangent points in order to get the right-angled trapezoid in which segment OK connects the midpoints of the lateral sides. This means that $OK = \frac{1}{2}(r_1 + r_3)$. Other perpendiculars dropped from point O to common tangents to circles $1, 3$ and $2, 4$ have the same length (because $r_1 + r_3 = r_2 + r_4$), see Fig. 228.

FIGURE 228. (Sol. 24.2.8.2)

FIGURE 229. (Sol. 24.2.9.1)

This means that O is the center of the circle inscribed in the quadrilateral obtained.

24.2.8.5. Let some set of four numbers coincide with the initial one. The solution goes as follows. We should

a) *prove that* $abcd = 1$ (otherwise the product of the numbers obtained does not equal the product of the initial ones);

b) *prove that* $ac = 1$ *and, therefore,* $bd = 1$ (this follows from the fact that for all sets of four numbers, beginning with the second one, this is true);

c) *analyze what happens with the set* a, b, $\frac{1}{a}$, $\frac{1}{b}$ *under our transformations.*

24.2.9.1. To solve the problem, identify our plane with the *complex line*[1] $\mathbb{C}$. Let a, b be the centers of circles O_1 and O_2, and let $c = \overrightarrow{O_1A}$ and $d = \overrightarrow{O_2B}$ at the initial moment $t = 0$. (Here a, b, c, d are considered as complex numbers.) Let ω be the angle velocity of each of the points A and B.

Then at any arbitrary moment t we have:

$$A(t) = a + ce^{i\omega t}; \quad B(t) = b + de^{i\omega t}; \quad \overrightarrow{AB}(t) = (b - a) + (d - c)e^{i\omega t}.$$

Hence, $\overrightarrow{AC} = \overrightarrow{AB} \cdot e^{i\frac{\pi}{3}}$; therefore, $C = A + e^{i\frac{\pi}{3}} \cdot \overrightarrow{AB}$ and, finally,

$$\begin{aligned} C(t) &= a + ce^{i\omega t} + \left((b - a) + (d - c)e^{i\omega t}\right) e^{i\frac{\pi}{3}} \\ &= \left(a + (b - a)e^{i\frac{\pi}{3}}\right) + \left(c + (d - c)e^{i\frac{\pi}{3}}\right) e^{i\omega t} = \alpha + \beta e^{i\omega t}. \end{aligned}$$

Hence, point C moves with the same angular velocity ω via the circle with the center at $\alpha = a + (b - d)e^{i\frac{\pi}{3}}$ and the radius $R = |c + (d - c)e^{i\frac{\pi}{3}}|$.

24.2.9.2. The number of tables obtainable along the indicated procedure is a finite one. Let us consider a table with the greatest sum of *all* numbers. If the sum of numbers in a row or a column of this table is negative, change the signs of these numbers thus increasing the sum total of *all* numbers. Q.E.D.

24.2.9.3. Prove first that there exists a point with exactly one segment beginning in it. To prove this, let us move away from an arbitrary vertex A. We will never return to the vertex we have passed (otherwise this would mean that there is a closed route, which contradicts the hypothesis). Therefore, sooner or later, we come to a vertex from which there is no route out. And this is the vertex from which only one route emerges, the route we came there.

Discard this vertex together with the segment leading to it, and appeal to the induction on the number of points.

24.2.9.4. Denote: $d = GCD(b, p - a)$. Then $b = kd$ and $p - a = ld$, where k and l are relatively prime. But then

$$ak + bl = a\frac{b}{d} + b\frac{p - a}{d} = \frac{b}{d}p = kp, \quad \text{Q.E.D.}$$

[1] Sometimes called the *plane* of the complex argument. Do not confuse it with $\mathbb{C}^2$.

24.2.9.5. In the first way Nick can get $n+1$ nut. For this it suffices to divide the pile into 2 and $2n-1$ nuts. It is impossible, however, to gain more since Pete may divide the piles into almost equal parts and it is impossible to pick more than $n+1$ nut from piles of a, a, b and $b+1$ nuts.

The second way is that of a loser for Nick, since Pete can divide the smaller pile in halves and the larger into 1 and $b-1$ nuts.

The third way is also to be abandoned by Nick, since to pay the whole nut for the dubious privilege of choice between the first (good) and the second (bad) ways is a bad bargain.

24.2.10.1. Having done with Hint, the rest is simple: choose a non-decreasing subsequence $a_1' \le a_2' \le \dots$ from the sequence $a_1, a_2, \dots$; choose a non-decreasing subsequence $b_1'' \le b_2'' \le \dots$ from the subsequence $b_1', b_2', \dots$ corresponding (with the same indices) to the previous subsequence $a_1' \le a_2' \le \dots$. Next, choose a non-decreasing subsequence $c_1''' \le c_2''' \le \dots$ of the corresponding subsequence $c_1'' \le c_2'' \le \dots$. Then

$$a_1''' \le a_2''' \le \dots; \quad b_1''' \le b_2''' \le \dots; \quad c_1''' \le c_2''' \le \dots.$$

24.2.10.2. In order to place into the rectangle a disc not intersecting the squares, the center of the disc must lie in a 19×24 rectangle and the distance between the center of the disc and each square must be greater than $\frac{1}{2}$, i.e., the disc's center must not lie inside the figure shown in Fig. 230.

FIGURE 230. (Sol. 24.2.10.2)

FIGURE 231. (Sol. 24.2.10.4)

But the sum of the areas of 120 such figures is equal to $120(3+\frac{\pi}{4}) < 19 \times 24$, and so one can find room for the center of a disc of diameter 1 (cf. the solution to Problem 23.2.9.5).

24.2.10.4. Draw ray BM so that $\angle CBM = \angle ABP$ and $\angle PBM = 60°$, see Fig. 231.

On BM, mark segment BP' equal to BP and connect P' with C and P. Then triangle PBP' is an equilateral one, whence $PP' = PB = 3$, and $\triangle BCP' = \triangle ABP$, implying $P'C = PA = 2$. Therefore, $PC \le PP' + P'C = 5$. The equality is attained if $PP'C$ is a straight line.

24.2.10.5. It is easy to prove by induction that the set

$$a_1 a_{2^p+1}, \quad a_2 a_{2^p+2}, \quad \dots, \quad a_{2^k} a_{2^p+2^k}$$

will be obtained in 2^p steps. In particular, we will get the set

$$a_1^2, \quad a_2^2, \quad a_3^2, \quad \dots, \quad a_{2^k}^2$$

consisting of 1's only in 2^k steps.

25.1.7.1. Let us confine yourself to the case when points M and N lie in one half-plane (on one side of straight line AB). Then M is the center of the circle inscribed in triangle ABN, and $|AN - BN|$ is the absolute value of the difference in length of the two parts of segment AB. Indeed, since $NL = NP$, $AL = AK$, $BP = BK$, we have $|AN - BN| = |AL - BP| = |AK - BK|$, see Fig. 232. Therefore, $|AN - BN| = const$, Q.E.D.

25.1.7.2. Construct a hexagon of six equilateral triangles, see Fig. 233. As the initial triangle rolls (is being reflected) inside of the hexagon, the vertices A, B, C assume positions marked by the same letters. Let us pave the whole plane with such hexagons using parallel translations. Clearly, as the triangle rolls over from one hexagon to another, the vertices A, B, C assume positions marked by the corresponding letters. Therefore, if after several reflections the triangle returns back to its initial place each of its vertices will return home, to its initial position.

FIGURE 232. (Sol. 25.1.7.1)

FIGURE 233. (Sol. 25.1.7.2)

FIGURE 234. (Sol. 25.1.7.3)

25.1.7.3. Let $a \le b \le c \le d$. Since a, b, c and d are the lengths of the quadrilateral's sides, we have $(a+b)+c > d$ and $(a+b)+d > c$. Moreover, since the initial quadrilateral is not a rhombus, we have $a < d$, and $(a+b) < c+d$. Therefore, we can construct triangle $\triangle AED$ with sides $a+b$, c, d and parellelogram $BEDC$ with sides b, c, as shown on Fig. 234. Then $ABCD$ will be a selfintersecting quadrilateral to be found.

25.1.7.4. PROOFof Lemma from Hints. Represent the difference as a sum in which each summand is divisible by 9:

$$\begin{aligned}&a - S(a)\overline{a_1 \dots a_k} - (a_1 + \dots + a_k) =\\ &a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \dots a_k \cdot 1 - a_1 - a_2 - \dots - a_k =\\ &a_1 \cdot \underbrace{99\dots9}_{k-1} + a_2 \cdot \underbrace{99\dots9}_{k-2} + \dots + a_2 \cdot 9. \quad \square\end{aligned}$$

Let us now solve the problem. In the expression $a = 2a - a = (2a - S(2a)) - (a - S(a))$ each term in parentheses is divisible by 9, hence, so is the whole expression.

25.1.8.1. Clearly, $AX = BY = CZ$. Under the rotation of $\triangle ABC$ about O through the angle of $60°$ the figure (see Fig. 235) will turn into itself.

FIGURE 235. (Sol. 25.1.8.1)

Therefore: (1) $\triangle KLM$ is an equilateral one; (2) $\triangle AKB = \triangle BLC = \triangle CMA$.

Denote the areas of $\triangle AKB$ and $\triangle ABC$ by S' and S, respectively. Then

$$S = S_{\triangle AKB} + S_{\triangle BLC} + S_{\triangle CMA} + S_{\triangle KLM} = 3S' + \frac{1}{4}S$$

wherefrom $S' = \frac{1}{4}S$. Since $\triangle AKB$ and $\triangle ABC$ have a common base, the ratio of their heights is equal to the ration of their areas, i.e., K is 4 times closer to AB than to C. This means that K lies on the line l drawn parallel to AB four times closer than C.

We have: $\angle AKB = 180^\circ - \angle AKL = 180^\circ - 60^\circ = 120^\circ = \angle AOB$. Two equal angles subtend the same segment AB, hence, points A, O, K and B lie on one circle.

Now we are able to construct K (and, further, X, Y and Z) as the intersection of line l with the circle circumscribed about $\triangle AOB$; two intersection points give rise to two symmetric sets of points X, Y and Z.

25.1.8.3. If $d \neq -1$, transform the ratio to express n:

$$n = m^2 - \frac{(m-1)^2}{d+1}.$$

Now, take $m = d + 2$, and $n = m^2 - (d + 1)$. If $d = -1$, then you may take $m = 1$ and an arbitrary n.

25.1.9.2. If n is odd, then Boo does not change the color of its square by its move, and, therefore, it cannot get from a black square to a white one, or vice versa. If n is even, the problem is always solvable. Look at Fig. 236 to see that Boo can get to the square adjacent to its original one in $n + 1$ moves.

FIGURE 236. (Sol. 25.1.9.2)

FIGURE 237. (Sol. 25.1.10.1)

25.1.10.1. Denote the angle between the rays by α. Draw straight line MP parallel to BN through point M and mark segment MP equal to AM, see Fig. 237.

The triangle AMP is an isosceles one, and $\angle MAP = \frac{\pi - \alpha}{2}$. This means that $\angle MAP$ does not depend on the position of M on ray AM, and this means that point P always lies on the straight line l drawn through A at an angle $\beta = \frac{\pi - \alpha}{2}$ with respect to ray AM. Since $MNBP$ is a parallelogram, $MN = BP$. But BP can not be shorter than the distance from B to l.

Let us construct an example when MN and BP are equal to this distance: let us drop the perpendicular BP to l and construct PM in parallel to the ray from B.

See also the solution to Problem 9.2.7-8.3.

25.1.10.2. Suppose that from the square $ABCD$ we got an octagon $APBQCRDS$, see Fig. 238. Let us extend BQ to its intersection with AP at point T and PB to its intersection with CQ at point U.

If we rotate the figure about the center of the square by 90° counterclockwise, then AP becomes BQ; hence, $AP \perp BQ$ and $\angle ATQ = 90^\circ$.

Suppose we managed to split the octagon into parallelogramms. Then there will necessarily exist a parallelogramm with vertex B and a side adjacent to BP or BQ. (Let, for definiteness, this be BP.) Let us construct all the sides of the parallelogramms with exactly one endpoint on BP or BQ. Let us pass from P to B, slightly stepping inside of the boundary of the octagon. Each next side intersected is slanted with respect to PB as or more than the preceding one. In particular, if MB is the last side, then $\angle MBU \leq \angle APB$. This implies that $\angle MBQ \leq \angle ATB = 90^\circ$.

Let us now move similarly from B to Q. Let $Q'N$ be the last intersected side not exiting Q. Analogously, $\angle NQ'Q \leq \angle MBQ \leq 90^\circ$.

Figure 238. (Sol. 25.1.10.2)

Let us consider the parallelogramm with side QQ'. In it, $\angle Q' \leq 90°$ and $\angle Q' \leq \angle CQB < 90°$ by hypothesis.

The sum of angles adjacent to side QQ' is less than 180°: contradiction.

25.1.10.3. If n is any positive integer, and $a_m \leq n < a_{m+1}$, then $n - a_m < a_{m-1}$. Assuming by the induction that the statement holds for any $k < a_{m-1}$, we find that it also holds for $n = a_m + (n - a_m)$.

25.2.7.1. Let the ball was consequently reflected from all the sides and returned to the initial point M, see Fig. 239. Consider the triangles cut off each angle. The neighboring triangles have two equal angles (thanks to the law of reflection and the fact that all the angles of the $2n$-gon are equal); hence, the triangles are similar; therefore, all the triangles are similar. Denote the corresponding sides of the i-th triangle by a_i and b_i, see Fig. 239. Then $b_i = ka_i$, where k does not depend on i. But a half of all the sides of the $2n$-gon is composed of the a_i and the other half of the b_i; hence

$$a_1 + \cdots + a_n = b_1 + \cdots + b_n = k(a_1 + \cdots + a_n),$$

so $k = 1$. All the triangles are, therefore, isosceles ones.

Figure 239. (Sol. 25.2.7.1)

Figure 240. (Sol. 25.2.7.3)

So the angle α at which the ball is to be shot is uniquely determined: $\alpha = \dfrac{180° - \gamma}{2}$, where $-\gamma$ is the angle of the $2n$-gon. (In actual fact, $\alpha = \dfrac{360°}{4n}$.)

Conversely, if one shoots the ball to the right or to the left at such an angle α, then the second angle will also be equal to α; hence, all the triangles will be isosceles. Let l be the length of the $2n$-gon's side. We consequently compute: $a_2 = b_2 = l - a_1$, $a_3 = b_3 = l - a_3 = a_1$, ... , $a_{2i} = b_{2i} = l - a_1$, $a_{2i+1} = b_{2i+1} = l - a_1$, ... , i.e., the ball will return to the initial point M.

Further, the length of the base of the i-th triangle is equal to $(a_i + b_i)\cos\alpha$, i.e., the total length of the path is equal to $P\cos\alpha$, where P is the perimeter of the $2n$-gon; so the path does not depend on M.

Remark. The original formulation was wrong. Shoot the ball from the center of the polygon so that having missed two sides it would hit the third. Then the ball will return to the starting point having rolled along a longer (than for the consequtive circomvention) self-intersecting pass.

25.2.7.2. The statement of Hint follows from the fact that sides of $\triangle OO_1L$ are perpendicular to the respective sides of $\triangle ABC$. Indeed, $ON \perp AB$, since ON is the radius that divides the chord in halves; $O_1L \perp AC$, since O_1L lies on a height; $OO_1 \perp BC$ since OO_1 is the axis of symmetry of a pair of circles.

Further, the symmetry of isosceles triangles $\triangle OO_1L$ and $\triangle OKB$ implies that $BL = O_1K$ and since $O_1M \perp AB$ (as the radius of the circle ABC that divides chord $\smile AB$ in halves), $O_1M \parallel LN$. Since $BN = MN$, it follows that LN is the midline of $\triangle O_1MB$; this implies that $O_1L = LB$. Thus, $BL = LO_1 = O_1K$; this means that $BK = \frac{3}{2}BO_1 = \frac{3}{2}R$.

25.2.7.3. a) The area of this figure is equal to 3 but 1961×1963 is not divisible by 3.

b) Divide the 1963×1965 rectangle into three rectangles: 1958×1965, 5×1956, and 5×9. The length of one side of the first rectangle is divisible by 3, that of the second one is divisible by 2; this means that it is possible to divide the first rectangle into 2×3 rectangles, and each of the 2×3 rectangles can be represented as the union of two L-shaped figures. Divide the second rectangle into 5×6 rectangles, and subdivide each of them into 10 L-shaped figures as on Fig. 240 a). Divide the third rectangle into 15 L-shaped figures as on Fig. 240 b).

25.2.7.5. Choose two points, A and B, that have the greatest distance between them, and construct two unit discs with their centers at these points. For an arbitrary point C take the one of the points A and B which is nearer to C, see Fig. 241. Then the distance between point C and the nearest to it of the given points is less than 1 (otherwise we come to a contradiction with the condition of the problem).

FIGURE 241. (Sol. 25.2.7.5)

Thus, all 25 points lie inside the two discs constructed. Hence, by Dirichlet's principle, the 13 of the given points lie inside one of the discs.

25.2.8.1. This is easy to prove by the induction on the number of vertices drawn. The base of induction: for the quadrilateral the statement is obvious.

The step of induction: let the n-gon ($n > 4$) be divided by a diagonal into two polygons with a lesser number of sides. By the inductive hypothesis each of these polygons has a free vertex, the one that does not belong to the division line. (If one of the polygons is a triangle this is also true, though this does not follow from the inductive hypothesis.) It follows that the original n-gon also has two non-neighboring free vertices.

25.2.8.3. Denote by (AB) the number of sides between vertices A and B as we move from A to B clockwise. If under rotation A turns into A' and B into B', then $(AB) = (A'B')$. We see that than

$$(AA') = \begin{cases} (AB) + (BA') = (BA') + (A'B') = (BB') & \text{if } A' \text{ does not lie between } A \text{ and } B \\ (AB) - (BA') = (A'B') - (AB') = (BB') & \text{otherwise.} \end{cases}$$

Thus, $(AA') = (BB')$, i.e., under the rotation through an angle of φ all the vertices move by the same number of sides, $m < n$.

Let n be a prime. Then by the lemma on residues (see below) there exists a k, such that $km \equiv 1 \pmod{n}$. This means that under the rotation through an angle of $k\varphi$ each vertex (and each side) will turn into the next (clockwise) one. All the neighboring angles and sides are, therefore, equal and the n-gon is a regular one. The contradiction shows that n is not a prime.

Lemma (On residues). *If p is a prime and m is not a multiple to p than among the residues after the division of $0, m, 2m, \dots, (p-1)m$ by p we encounter all the integers 0 to $p-1$.*

PROOF. Suppose $km \equiv lm \pmod{p}$, where $0 \le k \le l < p$. Then $(l-k)m \,\vdots\, p$, but m is not divisible by p, hence, $l - k \,\vdots\, p$. Since $l - k < p$, it follows that $l = k$. Thus, p numbers give p different possible residues; therefore, we encounter all the residues 0 to $p-1$, Q.E.D.

FIGURE 242. (Sol. 25.2.8.5)

25.2.8.5. Drop perpendiculars from centers O_1 and O_2 to chords AM and BM, and extend them to their intersection at point R, see Fig. 242.

Let us prove that $RP \perp MP$. Let us extend MO_1 and MO_2 to diameters MC and MD, respectively. Since $\angle CPM = \angle DPM = 90°$, point P lies on CD and $MP \perp CD$. Observe that $O_1R \parallel MO_2$ (both are perpendicular to AM) and $O_2R \parallel MO_1$. Since O_1R and O_2R are midlines of $\triangle MCD$, point R also lies on CD; hence, $MP \perp PR$. Then $MR = RH$, and since $AR = MR$ and $MR = RB$ (due to the way we determined point R), it follows that R is the center of the circumscribed circle of $MAHB$.

25.2.9.1. Let $a_1, \dots, a_7, a_8$ be the total numbers of problems solved by the student in some consecutive 8 days. By the hypothesis $a_1 + a_2 + \dots + a_7 = 25$ and $a_2 + a_3 + \dots + a_8 = 25$, whence $a_1 = a_8$. Hence, the total number of problems solved by the student in a day is to be repeated every 7 days, and the student must make a plan to solve $a_1, a_2, \dots, a_7$ problems every next day, respectively, for one week only. If the student solved all 25 problems on Monday, it would take him/her S_1 hours for all problems during the year. In the same way, the numbers $S_2, S_3, \dots, S_7$ are defined. Solving on Mondays not 25 but a_1 problems (s)he will need $\frac{a_1}{25}S_1$ hours for all Mondays of the year; similarly, it will take him/her $\frac{a_2}{25}S_2$ hours for all Tuesdays, and so on. The total time that the student will need to solve all problems is $S = \frac{1}{25}(a_1S_1 + \dots + a_7S_7)$. Choosing the minimal number S_k from the numbers $S_1, S_2, \dots, S_7$ take $a_k = 25$ and $a_i = 0$ for $i \neq k$. Then S is the least possible.

25.2.9.2. The sum in question is $\pm(a_1 - a_2) \pm (a_2 - a_3) \pm \dots \pm (a_{25} - a_1)$. Remove the parentheses. Then 25 summands have the plus sign and 25 summands the minus sign. This means that the sum is not greater than

$$2b_{25} + 2b_{24} + \dots + 2b_{14} + b_{13} - b_{13} - 2b_{12} - \dots - 2b_1,$$

where the b_i are our numbers a_i placed in increasing order. To get the greatest possible sum arrange the numbers in the following order: $b_{25}, b_1, b_{24}, b_2, \dots, b_{14}, b_{12}, b_{13}$.

25.2.9.3. It is easy to see from Fig. 243 that the area of the polygon will increase by more than $12 + \pi > 15$.

25.2.9.5. The sequences are divided into 3 groups:

1) "short" ones, of length less than n,
2) "full" ones, of length equal to n,
3) "long" ones, of length greater than n.

Observe that it is impossible for our set to contain only sequences of types 1) and 2). Indeed, each short sequence can be made into a full one in all possible ways, without affecting the conditions on the set or increasing the number of the sequences, and this contradicts the fact that there exist exactly 2^n full sequences.

If in our collection of segments there are no short sequences, everything is proved. Let the collection contain all three types of sequences. Let us shorten, as much as possible, all long sequences by erasing the last digits. Now, take the longest sequence; let it be of the form $\overline{a0}$. There should be found the sequence of the form $\overline{a1}$ as well, since otherwise the zero could have been erased.

Consider also the shortest sequence $\overline{d}$, and replace sequences $\overline{a0}$, $\overline{a1}$ and $\overline{d}$ by $\overline{a}$, $\overline{d0}$ and $\overline{d1}$, respectively. This operation does not increase the sum of the lengths of all sequences and conforms with the Rule (*no sequence is the beginning of another sequence*). We can repeat this operation until there will be no either long sequences or short sequences. But we have already considered such cases.

This means that the sum of the lengths of the original sequences in the set is not less than $n \times 2^n$.

FIGURE 243. (Sol. 25.2.9.3)

FIGURE 244. (Sol. 25.2.10.1)

25.2.10.1. Indeed, see Hint, let $OC = a$, $OB = b$. It follows from the construction that $ab = r^2$, where r is the radius of the circle.

Let AA'' be the chord passing through C so that point A'' is symmetric to A' through line l. We have to prove that points A', A and B lie on one line.

Let $y = k(x - a)$ be the equation of line AA''. Then abscissas of points A and A'' are found from the equation

$$x^2 + k^2(x - a)^2 = r^2 = ab. \tag{1}$$

Let $y = m(x - b)$ be the equation of a line through point B. Its intersection points with the circle are found from the equation

$$x^2 + m^2(x - b)^2 = r^2 = ab. \tag{2}$$

For $m^2 = \dfrac{ak^2}{b + (b - a)k^2}$ the coefficients of equations (1) and (2) are proportional; hence, their solutions coincide, i.e., line $y = x(m - b)$, where $m^2 = \dfrac{ak^2}{b + (b - a)k^2}$, passes through points A, A'. Q.E.D.

25.2.10.5. Prove it by induction. For $k = 2$ the statement is obvious. If for $k = n$ the participants $a_1, \dots, a_n$ are arranged in increasing order, put the $(n + 1)$-st participant in this line before the first player who did not beat him.

26.1.7.1. Let us connect point B with points A' and C'. Then the hypothesis immediately implies that $A'B = AB$ and $BC' = BC$. Besides, $\angle A'BC = \angle ABC'$ which follows from the equality $\angle ABM = \angle CBN$, see Fig. 245. Hence, $\triangle A'BC = \triangle ABC'$ implying $A'C = AC'$, Q.E.D.

FIGURE 245. (Sol. 26.1.7.1)

26.1.7.5. A straight line intersecting 50 squares must intersect not less than 49 lines dividing the squares, and we have only 48 such lines on our paper (19 vertical lines and 29 horizontal lines).

26.1.8.2. The segments connecting the midpoints of the sides of quadrilateral $ABCD$ form a parallelogram P, and it is easy to see that $S_P = \frac{1}{2}S$.

On the other hand, these segments are also the segments that connect the midpoints of the sides of the four triangles with the common vertex M into which the quadrilateral $EFGH$ is divided; see Fig. Probl. 26.1.8.2. Hence, $S_{EFGH} = 4S_P = 2S$.

26.1.8.3. We may confine ourselves to finding only positive solutions, since if (x_0, y_0, z_0) is a solution, then by changing the sign of any two numbers of this triple we also get a solution. Multiply the given equation by $2xyz$ and use the inequality $a^2 + b^2 \geq 2ab$:

$$\begin{aligned}&6xyz = 2x^2y^2 + 2x^2z^2 + 2y^2z^2\\ &= (x^2y^2 + x^2z^2) + (x^2y^2 + y^2z^2) + (x^2z^2 + y^2z^2)\\ &\geq 2x^2yz + 2y^2xz + 2z^2xy = 2xyz(x + y + z),\end{aligned}$$

whence $x + y + z \leq 3$. Since x, y, z are positive integers, $x = y = z = 1$ is the only positive solution.

26.1.8.4. Translate all straight lines parallel with themselves so that all of them meet at one point, call it O, see Fig. 246.

Figure 246. (Sol. 26.1.8.4)

The angles between the lines are preserved under the translation and we see that the lines divide the angle of 360° into 14 parts. If every part is $\geq 26°$, their sum is $\geq 14 \times 26° = 364° > 360°$. Contradiction.

26.1.9.3. Set $x = a + b$, $y = b + c$, $z = a + c$.

Then the left hand side of the given inequality is:

$$\begin{aligned}S &= \frac{1}{2}\left(\tfrac{x-y+z}{y} + \tfrac{x+y-z}{z} + \tfrac{-x+y+z}{x}\right)\\ &= \frac{1}{2}\left(\tfrac{x}{y} + \tfrac{y}{x} + \tfrac{x}{z} + \tfrac{z}{x} + \tfrac{y}{z} + \tfrac{z}{y} - 3\right)\end{aligned}$$

and since

$$\frac{x}{y} + \frac{y}{x} \geq 2, \quad \frac{x}{z} + \frac{z}{x} \geq 2, \quad \frac{y}{z} + \frac{z}{y} \geq 2,$$

it follows that $S \geq \frac{6-3}{2} = 3/2$.

26.1.9.5. Let α be the angle between a side of the inscribed rectangle and a side of the circumscribed rectangle; let 4 and 7 be lengths of the sides of the smaller rectangle (Fig. 247).

Then the lengths of the different sides of the larger rectangle are equal to $4\sin\alpha + 7\cos\alpha$ and $7\sin\alpha + 4\cos\alpha$, and their ratio, as you can prove yourselves, is always greater than $\frac{4}{7} > \frac{9}{16}$.

26.1.10.2. Consider the convex hull of all the six points. It is an n-gon with $n \leq 6$. The sum of its n angles is equal to $180°(n-2)$, the least angle does not exeed

$$\frac{180°(n-2)}{n} = 180° - \frac{360°}{n} \leq 180° - -\frac{360°}{6} = 120°.$$

Connect the vertex of this angle, call it A, with all other points. We thus divide the angle into 4 angles the least being $\leq \frac{\angle A}{4} = \frac{120°}{4} = 30°$. Now, construct a triangle with this angle.

FIGURE 247. (Sol. 26.1.9.5)

26.1.10.4. If all numbers a, b, c are positive or n is even, there is nothing to prove. Otherwise one of them, say, a, is positive, and two others are negative: $b < 0$, $c < 0$. But then $a > |b| + |c|$, and, therefore, $a^n > |b|^n + |c|^n$. This is equivalent to the inequality to be proved.

26.1.10.5. Let M' be the meeting point of the perpendiculars to segments AM, BM and CM. Consider the circle with MM' as a diameter. Since $\angle MAM'$ is a right one, the point A lies on this circle. Similarly, the points B and C also lie on this circle; hence, this circle is the circumscribed circle of $\triangle ABC$.

Conversely, if M lies on the circumscribed circle of $\triangle ABC$ and does not coinside with A, B or C and M' is the diametrically opposite point to M, then $M'A \perp AM$, $M'B \perp BM$ and $M'C \perp CM$. Thus, the locus to be found is the circumscribed circle of $\triangle ABC$ without points A, B, C.

26.1.11.1. Let $x = \tan\alpha$, $y = \tan\beta$, $z = \tan\gamma$. Then $0 < \alpha < \frac{\pi}{2}$, $0 < \beta < \frac{\pi}{2}$, $0 < \gamma < \frac{\pi}{2}$; as we know from trigonometry $\tan(\alpha+\beta) = \dfrac{x+y}{1-xy}$ and, therefore,

$$\tan(\alpha+\beta+\gamma) = \tan\left((\alpha+\beta)+\gamma\right) = \frac{\dfrac{x+y}{1-xy} + z}{1 - z\dfrac{x+y}{1-xy}} = \frac{x+y+z-xyz}{1-xy-yz-xz}. \qquad (*)$$

Three cases are possible:

a) $\alpha+\beta+\gamma < \pi/2$, then $\tan(\alpha+\beta) > 0$ and $\tan(\alpha+\beta+\gamma) > 0$. Therefore,

$$\tan(\alpha+\beta) < \tan\left(\frac{\pi}{2} - \gamma\right) = \cot(\gamma) = \frac{1}{z}.$$

Hence, $\dfrac{x+y}{1-xy} < \dfrac{1}{z}$, wherefrom $xz + yz < 1 - xy$, i.e., the denominator of the fraction $(*)$, and, therefore, its numerator, are positive.

b) $\alpha+\beta+\gamma > \frac{\pi}{2}$. Then the fraction is negative. Arguments similar to those of case a) show that its denominator is also negative. This means that the numerator is positive.

c) $\alpha+\beta+\gamma = \frac{\pi}{2}$, i.e., $\gamma = \frac{\pi}{2} - (\alpha+\beta)$ and $z = \tan(\gamma) = \dfrac{1}{\tan(\alpha+\beta)} = \dfrac{x+y}{1-xy}$. We have to prove that $x+y+z > xyz$. Let us substitute $z = \dfrac{x+y}{1-xy}$ into this inequality, transport everything to the left hand side and simplify. We get an equivalent inequality:

$$\frac{1+x^2+y^2+x^2y^2}{x+y} > 0$$

which is always true. Q.E.D.

26.1.11.2. If such a broken line exists, then by orienting its links counterclockwise we see that the sum of 25 vectors is equal to $\vec{0}$. Project the given vectors to a straight line perpendicular to l: the sum of the vectors-projections is not equal to $\vec{0}$ because their lengths are equal and their number (25) is odd. Contradiction.

26.1.11.5. The tetrahedron is divided into 11 tetrahedrons (1 at the vertex, 3 in the middle layer, and 7 at the bottom) and 4 octahedrons (1 in the middle layer and 3 in the bottom layer); the total of 15 parts.

26.2.7.3. 1) Let us prove that for any line l passing through the center of mass M and intersecting the sides AB and AC the condition of the problem is satisfied. From point P, the midpoint of BC, drop perpendicular to l. Then from trapezoid BB_1C_1C, see Fig. 248, we have $PH = \frac{1}{2}(h_1 + h_2)$. Let the angle between the median AP and l be equal to α. Then $PH = PM \cdot \sin\alpha$, $h = AA_1 = AM \cdot \sin\alpha$, and, therefore, $\dfrac{PH}{h} = \dfrac{PM}{AM} = \frac{1}{2}$. This implies that $PH = \frac{1}{2}h$ and, therefore, $h_1 + h_2 = h$, Q.E.D.

FIGURE 248. (Sol. 26.2.7.3)

2) Let us prove that for any line l' not passing through the center of mass M the condition of the problem is violated. For this let us draw a line l parallel to l' and passing through M. For l the statement is true by heading 1) so it is false for l', Q.E.D.

26.2.8.1. Compose a table (called *Young tableau*) as shown in Fig. 249, with a_i squares in the i-th row. Then b_j is the number of squares in the j-th column. Hence, the statement to be proved.

FIGURE 249. (Sol. 26.2.8.1)

FIGURE 250. (Sol. 26.2.9.2)

26.2.8.2. Assume the contrary. Then if the way from one square to another takes k steps, the difference between the numbers in these squares is not greater than $4k$. But $64 - 1 = 63$, and the number of steps between the squares that contain 1 and 64 is not greater than 14. As $4 \times 14 < 63$, we come to a contradiction.

The solution to Problem 26.2.9.4 is similar.

26.2.8.4. Let a and b be two numbers choosen, $a < b$. If $b - a = 1$, then $b + a$ is divisible by 1. If $b - a = 2$, then either both a and b are even or both odd, hence, $b + a$ is divisible by 2. Therefore, $b - a \geq 3$. It is clear then that among three consequtive numbers not more than one can be chosen.

Divide all the numbers into 655 groups:

$$\{1, 2, 3\}, \quad \{4, 5, 6\}, \quad \dots, \quad \{1960, 196, 1962\}, \quad \{1963\}.$$

From each group we can select not more than one number, hence, the total number of of the numbers chosen is ≤ 655. It is, however, possible to select 655 numbers:

$$1, \quad 4, \quad 7, \quad \dots, \quad 1960, \quad 1963-$$

all the numbers of the form $3k + 1$. In any pair the difference is divisible by 3 whereas the residue after division of the sum by 3 is equal to 2; hence, the sum is not divisible by the difference.

26.2.8.5. The gentlemen — call them G_6, G_3, G_2 — pass the 100 meter-long alley in 6, 3 and 2 minutes, respectively. Hence, every 12 minutes their position reiterates: every 12 minutes each of them appears in the original point of the path and moves in the same direction along it, as at the beginning of the stroll.

Let us take the stop-watch with a hand and the full circle of 12 minutes long (we measure the length in units of time, so these minutes are parts of an hour, not a degree). Let us divide the dial into 12 sectors, each

1 min long. Let us start the stopwatch so that the hand occurs on the boundary of sectors at the moment when G_2 turns (on Fig. 251 these positions are marked with solid lines).

FIGURE 251. (Sol. 26.2.8.5)

Mark with a dotted line the positions of the hand when G_3 makes a turn. Shade the 4 sectors whose beginning or insides contain the dotted line. Now, one shaded sector alternates with a pair of unshaded ones on the dial.

Observe that the hand passes the midpoints of these pairs in 3 minutes while G_2 makes a turn in 2 minutes; hence, at the moment that occurs at one of the midsectors G_2 does not turn. But then G_2 turns at the beginning and at the end of this pair.

G_3 does not turn at the middle of this pair either (he only turns when in a shaded sector).Therefore, G_2 and G_3 move either in the same direction or in the opposite directions along the whole pair.

If they move in the same direction, select this pair; if they move in the opposite directions, select the pair that comes 6 minutes after this one (inside it, G_3 continues in the same direction, while G_2 moves in the opposite one).

Thus, inside the pair selected, G_2 and G_3 move in the same direction. Let us turn the dial so that the midline of this pair would point upwards and number the minutes clockwise starting from this line as on Fig. 251.

It is not difficulat to see that if during the 12-th and 1-st minutes both G_2 and G_3 go, say “thither”, then during the 3-rd and 10-th minutes they go “hither”.

Now, if G_6 does not turn during the first 3 minutes, then he moves along with the other gentlemen either during the 1-st or 3-rd minute. If he does turn inside the first 3 minutes, then he moves along with the other gentlemen either during the 10-th or 12-th minute.

26.2.9.1. Consider the vectors $\overrightarrow{XA} = \vec{a}$, $\overrightarrow{XB} = \vec{b}$, $\overrightarrow{XC} = \vec{c}$. Then

$$\overrightarrow{XM} = \frac{\vec{b}+\vec{c}}{2}, \quad \overrightarrow{XN} = \frac{\vec{a}+\vec{c}}{2}, \quad \overrightarrow{XQ} = \frac{\vec{a}+\vec{b}}{2}$$

and the areas of the triangles in question are equal, respectively, to halves of the lengths of the vector products

$$\frac{\vec{a}+\vec{c}}{2} \times \vec{b}, \quad \frac{\vec{a}+\vec{b}}{2} \times \vec{c}, \quad \text{and} \quad \frac{\vec{b}+\vec{c}}{2} \times \vec{a}.$$

But

$$\frac{\vec{a}\times\vec{b}+\vec{c}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}}{2} = \vec{0}$$

because $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ for any $\vec{u}$, $\vec{v}$. Since the result of all three vector products are perpendicular to the plane of triangle ABC, the sum of any two of them is equal to the third one with the opposite sign. Q.E.D.

ANOTHER SOLUTION (a sketch). Consider the projection of all triangles on a plane. Under the projection the triangle's median passes over into a median and the ratio of the areas does not change. Therefore, it suffices to prove the statement for an equilateral triangle $\triangle ABC$ (any triangle can be projected into an equilateral triangle); then it is equivalent to the following:

in an equilateral triangle, the distance from a point to one of the heights is equal to the sum of its distances to two other heights.

The easiest way to prove this is to consider first a point on one of the heights (the assertion is obvious for it) and then to watch how the distances vary as the point moves in parallel to the other height.

26.2.9.2. Since there is one horizontal and one vertical link beginning at each vertex, our broken line has 7 horizontal and 7 vertical levels. Number the horizontal levels consecutively and find the greatest number of self-intersection points that can lie on each level.

No link can intersect the *first* (upper-most) level; two links with the beginning points on the first level can intersect the *second* level; four links with the beginning points on the first two levels can intersect the *third* level; five links can intersect the *fourth* (middle) level — all except the two links with the beginning points on the fourth level.

The three lower levels are similarly treated. We get at most $2+4+5+4+2 = 17$ self-intersection points. Such a number of self intersections is possible, see Fig. 250.

26.2.9.4. Whatever squares are occupied by 1 and 81, any square can be reached from another (and not via one but via several paths) in not more than 16 steps passing from one square to a neighboring one. So at some step the difference between numbers becomes greater than 5. (See also the solution to Problem 26.2.8.2.)

26.2.10.1. If n is odd, then $z^n = x^n + y^n$ is divisible by $x+y$. But then z is divisible by $x+y$ which means that $z \geq x+y$. Hence, $z^n \leq (x+y)^n > x^n + y^n$.

26.2.10.2. If $i_1, i_2, \dots, i_n$ is an arbitrary numeration of horizontal straight lines (there are $n!$ such numerations), and $j_1, j_2, \dots, j_n$ is an arbitrary numeration of vertical straight lines (there are $n!$ of such numerations also), then the broken line that traverses along lines indexed by $i_1, j_1, i_2, j_2, \dots, i_n, j_n$ in the indicated order is uniquely constructed. The total number of such lines is equal to $(n!)^2$. On each of the broken lines thus obtained we can choose the beginning point in n ways and the direction in 2 ways. Therefore, each line is counted $2n$ times and the total number of them is equal to $\frac{(n!)^2}{2n}$.

26.2.10.3. If we turn the 25-gon by $\frac{360^\circ}{25}$ the whole set of vectors turns into itself. Their sum does not change under the rotation, hence, it is equal to $\vec{0}$.

Let the sum $\vec{S}$ of several selected vectors be of the greatest possible length. Then the sum of other (ignored) vectors is the vector $-\vec{S}$ of the same length. Let us draw through the origin the line perpendicular to $\vec{S}$. We get two half-planes.

Every vector from the set that belongs to the half-plane, where $\vec{S}$ lies, should enter the sum (otherwise, adding it we increase the length of $\vec{S}$). Similarly, every vector from the opposite half-plane should enter the sum of vectors ignored. This means that selected are the vectors lying in the same half-plane with $\vec{S}$ and only them. But any half plane contains 12 to 13 consequtive vectors from the collection.

26.2.10.4. Let A', B' be the midpoints of sides AB and BC, respectively, see Fig. 252. Clearly, $S_{A'BB'} = \frac{1}{4}S_{ABC}$ and a similar formula is used for other triangles. Therefore, the sum of the areas of triangles ABC, BCD, CDE, DEA, EAB is 4 times the sum of the areas of the small triangles complementing $A'B'C'D'E'$ to $ABCDE$.

FIGURE 252. (Sol. 26.2.10.4)

FIGURE 253. (Sol. 26.2.11.5)

On the other hand, these five triangles cover pentagon $ABCDE$ but do not cover it twice; therefore, the sum of their areas is less than $2S_{ABCDE}$. The inequality to be proved follows.

26.2.10.5. Let us prove by induction that $a_n = 4a_{n-1} - a_{n-2}$. Check the formula for $n = 3$, 4. Then the inductive step:

$$\frac{a_{n-1}^2 + 2}{a_{n-2}} = 4a_{n-1} - a_{n-2}.$$

Then the formula

$$a_{n-1}^2 + 2 = 4a_{n-1}a_{n-2} - a_{n-2}^2$$

implies

$$a_{n-2}^2 - 4a_{n-1}a_{n-2} + a_{n-1}^2 + 2 = 0. \tag{$*$}$$

Making use of the inductive hypothesis and equality $(*)$ we have:

$$\begin{aligned} a_{n+1} &= \frac{a_n^2 + 2}{a_{n-1}} = \frac{a_n^2 - a_{n-1}^2 + 4a_{n-1}a_{n-2} - a_{n-2}^2}{a_{n-1}} \\ &= \frac{-a_{n-1}^2 + a_n^2 + a_{n-2}a_n}{a_{n-1}} = \frac{-a_{n-1}^2 + a_n 4a_{n-1}}{a_{n-1}} = 4a_n - a_{n-1} \end{aligned}$$

which proves the statement.

26.2.11.2. Let us write all the fractins one under the other so that the decimal points are aligned. Each column has 11 digits, hence, at least 2 of them coincide. For every column write down the pair of fractions with coinciding (in this column) digits.

The number of columns is infinite while the number of pairs is finite; hence, some pair will occure infinitely many times. This means that the fractions of this pair coincide in infinitely many places. The difference of these fractions has either 0 or 9 at these places. There are infinitely many such places and only two figures for these places, hence, one of the figures occures infinitely many times.

26.2.11.5. Assume the contrary. Number these arcs and complement them to circles retaining the same numeration. Denote diametrically opposite intersection points of circles 1 and 2 by A and A_1; those of circles 2 and 3 by B and B_1; those of circles 1 and 3 by C and C_1.

These intersection points are on at least one of the arcs; let A_1 and C_1 lie on arc 1 and B_1 on arc 2, then B and C lie on arc 3 and A lies on arc 2. Denote the planar angles of the trihedral angles by α, β, and γ, as shown in Fig. 253, where O is the center of the sphere.

Since neither A nor C lies on arc 1, we see that $360° - \beta > 300°$. Similarly, neither B nor A_1 lies on arc 2; hence, $180° + \alpha > 300°$, and, lastly, for arc 3 we have: $360° - \gamma > 300°$; therefore, $\beta < 60°$, $\alpha > 120°$, $\gamma < 60°$, which means that $\alpha > \beta + \gamma$. This contradicts the theorem on the planar angles of a trihedral angle.

Extension. The estimate of 300° for three arcs on the sphere is an exact one.

27.1.7.1. The answer follows from the chain of inequalities: $c \geq h_a \geq a \geq h_c \geq c$, where a, c are sides BC and AB, and h_a, h_c are the lengths of the heights dropped onto them.

27.1.7.2. Let O be the center of the circle, K the intersection point of the tangent at C with tangent BM, see Fig. 254.

FIGURE 254. (Sol. 27.1.7.2)

FIGURE 255. (Sol. 27.1.7.4)

Then $KC = KB$ (as two segments of tangents from one point), and if $\angle CAB = \alpha$, then $\angle ACO = \alpha$. Since $\angle OCK = 90°$, it follows that $\angle KCM = 90° - \alpha$. But from triangle AMB we have $\angle KMC = 90° - \alpha$; hence, $KC = KM$. Therefore, $KB = KM = \frac{1}{2}BM$.

27.1.7.3. The number with the sum of its digits equal to 5 is of the form $9k + 5$, and, therefore, after division by 3 gives a remainder of 2. But the remainder after division of any perfect square by 3 is equal to either 0 or 1.

27.1.7.4. There are $9 \times 9 = 81$ inner points; 4 links come out of each. If we erase ≤ 40 links, they "take away" ≤ 80 points and, therefore, there exists a point with 4 non-erased links, which is not allowed.

Fig. 255shows how to erase 41 links.

27.1.8.2. If $n = p \cdot q \cdot r$, where p and q are nonequal primes and $r > 1$, then a factorization of $(n-1)!$ contains factors p, q, pr and qr and, therefore, is divisible by $(pqr)^2$. It remains to consider numbers n of the form pq and p^k, where p, q are prime.

If $n = pq$, $2 < p < q$, then $(n-1)!$ has divisors p, $2p$, q, $2q$.

If $n = p^k = p \cdot p^{k-1}$, then $(n-1)!$ has divisors p, $2p, \dots, (2^{k-1}-1) \cdot p$ and, for $p^{k-1} - 1 \geq 2k$, is divisible by p^{2k}.

Finally, if $p^{k-1} - 1 < 2k$, then it is not difficult to find all solutions of this inequality: $k = 2$, $p < 5$ and $p = 2$, $k \leq 4$. Let us check; we see that the case $n = 2^4 = 16$ does not fit; the other cases are listed in the answer.

27.1.8.3. In the equation given $\sqrt{x + \sqrt{x + \dots + \sqrt{x}}} = z$, the left-most square root is an integer. Consequtevely squaring and sending x to the right hand side we see that all other square roots encountered are also integers. In particular, if $y > 1$, then so is $\sqrt{x + \sqrt{x}}$. But if $x \neq 0$, the number of the form $x + \sqrt{x}$ is never a perfect square since $(\sqrt{x})^2 \leq x + \sqrt{x} \leq (\sqrt{x} + 1)^2$.

Therefore, $y = 1$. Hence, $x = n^2$, $z = n$, where n is any positive integer. If $x = 0$, then $z = 0$ and y is an arbitrary integer.

27.1.9.1. Let $x \geq 1$. Then $z = x^y \geq 1$, $y = z^x \geq 1$. Therefore, $z = x^y \geq x$ and similarly, $x \geq y \geq z$. Therefore, $x = y = z$. From any of these equations we deduce that $x^x = x$ implying $x = 1$.

The case $x < 1$ can be similarly considered.

27.1.9.2. If $m(m+1) = a^k$, then each of these numbers is also a k-th power, because m and $m+1$ are relatively prime. But this is impossible.

27.1.9.3. Since $27^3 - k^3 \vdots 27 - k$, it follows that $a = 27^3$ is a solution. Let now certain a also be a solution. Since both $27^3 - k^3$ and $a - 27^3$ are divisible by $27 - k$, then so is their difference: $a - 27^3 \vdots 27 - k$. But $27 - k$ can be any integer, i.e., $a - 27^3$ is divisible by any number which is only possible when $a - 27^3 = 0$.

27.1.9.4. a) Draw straight lines parallel to sides FE, AB and CD through vertices A, C and E, respectively; see Fig. 256, where $a_1 - a_4 = a_5 - a_2 = a_3 - a_6$.

FIGURE 256. (Sol. 27.1.9.4)

FIGURE 257. (Sol. 27.1.9.5)

Then inside the hexagon a triangle is formed; the sides of the triangle are equal to the differences mentioned in the condition of the problem. It is easy to see that it is an equilateral triangle.

b) This, converse, statement is proved by constructing the same Figure but in reverse order: starting from within, with the inner triangle.

27.1.9.5. If two heights are drawn in a triangle, their bases and the third vertex of the triangle are the vertices of a triangle similar to the initial one because $BM : BN = BC\cos\angle B : BA\cos\angle B = BC : BA$, see Fig. 257.

Therefore, each of the four triangles into which quadrilateral $MNPQ$ is divided by its diagonals is similar to its corresponding triangle among the four triangles into which quadrilateral $ABCD$ is divided by its diagonals. This means that the quadrilaterals are also similar. Q.E.D.

27.1.10-11.1. Replace the two last digits of N with zeros. Obviously, we get a perfect square, n^2. Then $\sqrt{N} \geq n+1$, hence, $(n+1)^2 - n^2 \leq 99$. Thus, $n \leq 49$.

On the other hand, n obviously ends with a 0. Thus, the greatest suitable n is 40 and $N = 1681 = 41^2$.

27.1.10-11.5. Not less than two tetrahedrons are adjacent to each of the two opposite faces of the cube, and all 4 of these tetrahedrons are distinct. It is easy to see that the volume of each of them is not greater than $\frac{1}{6}$ of the cube's volume. Therefore, these 4 tetrahedrons do not fill the whole volume of the cube, and a fifth tetrahedron is needed. An example of such five tetrahedrons is shown on Fig. 258.

FIGURE 258. (Sol. 27.1.10-11.5)

FIGURE 259. (Sol. 27.2.7.1)

27.2.7.1. Denote the radii of circles T_1, T_2, T_3 by r_1, r_2, r_3; let O_1, O_2, O_3 be their centers, see Fig. 259. Consider triangles $\triangle O_1RO_3$ and $\triangle O_2SO_3$. We have $O_1R = O_3O_2 = r_1$, $O_3O_1 = O_2S = r_2$ and $\angle O_3O_1R = \angle O_3O_2S$ (this follows from similarity of isosceles triangles $\triangle O_3R_1B$ and $\triangle SO_2B$). Therefore, $\triangle O_1RO_3 = \triangle O_2SO_3$; hence, $O_3R = O_3S$. Let us drop perpendicular O_3H on chord PQ. We obtain $RH = SH$ and $PH = QH$. This means that $PR = PH - RH = QH - SH = QS$.

27.2.7.2. If everyone is acquainted with each other, the problem is solved. Let now A and B be not acquainted. Then $2n-2$ persons are left, and more than n people are acquainted with both A and B. This means that more than $(n+n)-(2n-2)=2$ people are common acquaintances of both A and B. Now, offer opposite seats to A and B and the seats between them to two of their common acquaintances.

27.2.7.3. Select any point A from the given ones. Through the remaining points draw the rays with the origin at A. The rays divide the plane into 101 angles. All the angles, except, perhaps the largest one, are $< 180°$ (there can not be two angles of measure $\geq 180°$). Let us connect with segments the pairs of given points lying on the legs of each angle, except, perhaps the largest one, see Fig. 260.

FIGURE 260. (Sol. 27.2.7.3)

FIGURE 261. (Sol. 27.2.7.4)

We get 100 nonoverlaping triangles inside the square. Let us prove that they do not cover the whole square. Indeed, on the side of the square there lies not more than 8 of the given points, otherwise there are 3 points lying on one side.

Among given points, select one, B, that does not lie on the square's side and one of the angles, e.g., $\angle CAB$, with leg AB, see Fig. 260. Then $\triangle ABC$ does not cover the whole intersection of $\angle CAB$ with the square.

Thus, the total area of the 100 triangles constructed is < 1 and, therefore, the area of the smallest triangle is < 0.01, Q.E.D.

27.2.7.4. By the hypothesis, $BMON$ and $DPOQ$ are rhombuses; therefore, AB, BC, CD and DA are parallel to ON, OM, OQ and OP, respectively. Therefore, $\angle A+\angle\varphi = 2\pi-\alpha-\beta = \angle C+\angle\psi$, see Fig. 261.

But the angles φ and ψ are measured by arcs they subtend; $\angle A$ and $\angle C$ are measured by halves of such arcs. It follows that $2\angle A+\angle\varphi = 2\angle C+\angle\psi = 2\pi$, and we find that

$$\angle A = \angle C = \alpha+\beta = \angle B+\angle D.$$

Therefore, we immediately deduce that

$$\angle B+\angle D = \tfrac{1}{3}(\angle A+\angle B+\angle C+\angle D) = 120°.$$

27.2.7.5. Suppose (x, y) is a solution. If $d = GCD(x, y) > 1$, then $(\frac{x}{d}, \frac{y}{d})$ is also a solution. Select a solution with relatively prime x and y. Suppose $x > 1$; then there exists a prime p such that $x \,\vdots\, p$. But then $y^2 = x(ay-x) \,\vdots\, p$ and $y \,\vdots\, p$: contradiction.

Hence, $x = 1$ and, similarly, $y = 1$; wherefrom $a = 2$.

27.2.8.1. If $n = 2^k$, the method of refill is clear. Let us prove that for other values of n the refill is impossible. Let n be an arbitrary number, and let all water be in one glass after m refills. Obviously, there are two glasses with an equal amount of water before the last pouring. Thus, if the total amount of water is equal to 1, the distribution of water in non-empty glasses at the $(m-1)$-st step is: $(\frac{1}{2}, \frac{1}{2})$.

By the inductive hypothesis the distribution of water among the glasses at the $(m-k)$-th step is of the form

$$\left(\frac{x}{2^a}, \frac{y}{2^b}, \ldots, \frac{z}{2^c}\right) \quad \text{where} \quad x, y, z, a, b, c \in \mathbb{N}.$$

What was the water distribution during the previous step?

As the numeration of glasses is arbitrary, we can assume that at this step water is poured from the second glass into the first one. There are two possibilities: either the second glass becomes empty or some water is left in the second glass. The corresponding distributions of water at the previous step are:

$$\left(\frac{x}{2^{a+1}}, \frac{x}{2^{a+1}}, \frac{y}{2^b}, \ldots, \frac{z}{2^c}\right) \text{ and } \left(\frac{x}{2^{a+1}}, \frac{y}{2^b} + \frac{x}{2^{a+1}}, \ldots, \frac{z}{2^c}\right).$$

The sum of two fractions whose denominators are powers of 2 is also a fraction of the same form. We see that in all cases all denominators are powers of 2. In particular, such were the denominators before the first refill, but the glasses had equal amount of water, hence, all fractions were equal to $\frac{x}{2^a}$. Since $n\frac{x}{2^a} = 1$, then $nx = 2^a$ and, therefore, $2^a \,\vdots\, n$, i.e., n is also a power of 2.

27.2.8.2. We assume that C lies between A and B, see Fig. 262. Let us prove that $\angle O_1O_3O = \pi - \angle OCA$. Indeed, since circles O_1 and O_3 are symmetric through the line that connects their centers, it follows that $\angle O_1O_3O = \angle AO_3O_1$ and, therefore, $\angle O_1O_3O = \frac{1}{2}\smile ACO$. The angle $\angle ACO$ subtends the complement of the same arc to the circle; hence, $\angle ACO = \frac{1}{2}(2\pi - \smile ACO) = \pi - \angle O_1O_3O$.

FIGURE 262. (Sol. 27.2.8.2)

We similarly show that $\angle O_1O_2O = \pi - \angle OCB$. This implies that $\angle O_1O_3O + \angle O_1O_2O = \pi$; therefore, quadrilateral $O_1O_3O_2O$ can be inscribed in a circle.

27.2.8.3. The first player must make his/her moves symmetrically (through the center) to the moves of the second player until the moment (s)he can move into a corner square.

27.2.8.4. Let the side of the heptagon be equal to 1. Then $A_2H_1 = 1 - A_1H_1$. Connect O with all the vertices and drop all perpendiculars from point O to the sides, so that the heptagon becomes divided into 14 right triangles, see Fig. 263.

Expressing from the two triangles the square of the hypothenuse OA_2 via Pythagoras' theorem we get:

$$OH_2^2 + A_2H_2^2 = OH_1^2 + (1 - A_1H_1)^2.$$

Write such equalities for each hypothenuse OA_i, add them up term-wise and strike out the equal summands OH_i^2 and $A_iH_i^2$ on the left and right. We get:

$$0 = 7 - 2(A_1H_1 + A_2H_2 + \cdots + A_7H_7).$$

This implies that $A_1H_1 + A_2H_2 + \cdots + A_7H_7 = \frac{7}{2}$ is exactly half of the perimeter.

27.2.8.5. Divide a horizontal side of the square into 50 equal segments. Drawing vertical straight lines through their endpoints we obtain 50 vertical strips of width $\frac{1}{50}$ each and with areas also equal to $\frac{1}{50}$. Obviously, no fewer than three of the given 101 points are contained in at least one of the strips. Therefore, the triangle that has them as its vertices is entirely contained in the strip and its area is not greater than a half area of this strip, i.e., it is not greater than $\frac{1}{100}$.

FIGURE 263. (Sol. 27.2.8.4)

FIGURE 264. (Sol. 27.2.9.3)

27.2.9.3. The statement of the problem is equivalent to the following statement which is often used:

There exists a one-to-one correspondence between the set of non-negative integers and the set of pairs of non-negative integers.

This correspondence is given (in one direction) by the function:

$$f(x, y) = \frac{(x+y)^2 + 3x + y}{2}.$$

To make sure that this is indeed a one-to-one correspondence arrange all possible pairs of numbers in a table and number them as shown in Fig. 264.

Then it is easy to see (verify it yourself) that the number of the pair (x, y) is precisely $f(x, y)$. Q.E.D.

FIGURE 265. (Sol. 27.2.9.4)

27.2.9.4, **10.3** and **11.3.** Let $AB = c$, $AC = b$ and let L be the base of the bisector AL, see Fig. 265. By the known theorem about segments into which the bisector divides the side, $BL = \frac{c}{2}$, $CL = \frac{b}{2}$.

Let O be the center of an inscribed circle, D and E the midpoints of sides AB and AC, respectively. Then triangles BOD and BOL are equal; therefore, $OD = OL$, and, similarly, $OE = OL$. This means that $OD = OE$. Let us circumscribe a circle around triangle ADE. If P is the intersection point of this circle with AL, then $PD = PE$, as chords of equal arcs.

Suppose $P \neq O$, then AL is the midperpendicular to DE; hence, $AD = AE$ and $AB = AC$. This contradicts the conditions of Problems 27.2.9.4 and 27.2.10.3.

In Problem 27.2.10.3 we get $BC = AB = AC$; hence,

$$\angle DAC + \angle DOE = 60^\circ + 120^\circ = 180^\circ.$$

This means that the points A, D, E and O lie on one circle; implying $P = O$.

Thus, $P = O$, i.e., the circle circumscribed around triangle ADE passes through point O.

But triangle ABC is homothetic to ADE with coefficient 2 and A is the center of homothety. This means that the radius O_1A of the circumscribed circle coincides with the diameter of the circle circumscribed around triangle ADE. Therefore, $\angle O_1OA$ is a right angle, as the angle intercepting a diameter. This completes the solution of Problems 27.2.9.4 and 27.2.10.3.

Now, observe that the line tangent to the small circle and passing through point O is parallel to chord DE ($DO = OE$), which means that the tangent is also parallel to BC. Since BO is a bisector of triangle ABL, it follows that $LO : OA = LB : BA = 1 : 2$. Therefore, the tangent divides median AM of triangle ABC in the ratio $1 : 2$ as well, i.e., the tangent passes through the center of mass of the triangle. This completes the solution of Problem 27.2.11.3.

27.2.9.5. Introduce the coordinate system whose axes are directed along the sides of the squares of the paper and with the origin in a node. Let x_i and y_i be oriented projections of segments of the broken line to the axes. Then

$$x_1 + x_2 + \cdots + x_n = 0; \quad y_1 + y_2 + \cdots + y_n = 0;$$

and $x_i^2 + y_i^2 = c$ for $i = 1, \ldots, n$. Consider separately 3 cases:

a) c is divisible by 4. Then both numbers in each pair (x, y) are even. Divide them by an appropriate power of 2, and we come to one of the cases b) or c) below.

b) Let c be of the form $4k + 2$. Then x_i and y_i are both odd. Therefore, we have n odd terms with zero sum; this means that n is even.

c) Lastly, let c be odd, then x_i and y_i are of opposite parities. Let there be exactly m odd numbers x_i, then there are exactly $n - m$ odd numbers among the y_i. It follows from the fact that the sums of x_i and y_i are all equal to zero, that m and $n - m$ are both even, and, therefore, n is also even.

27.2.10.1. Use induction. The base $n = 1$ is obvious. By the inductive hypothesis the uniform mixtures of the $n - 1$ liquids are already made in $n - 1$ beakers, one beaker is empty, and one more is "put aside" with the same liquid that was in it from the very beginning.

Now, using the scale on the beakers, let us pour an equal amount of the mixture from each of $n - 1$ beakers into the empty one so as to get an equal amount of the mixture in n beakers. Next, let us divide the content of the additional beaker (with a pure liquid) in equal quantities between the beakers with the mixture. We get the desired uniform mixture.

27.2.10.5. Let us assume that the numbers on the cards are denominators of some fractions. Let us recover their numerators so that all fractions were irreducible, distinct, and of minimal (positive) value. For example, if there were 4 cards with number 10 on them, then our procedure will turn them into cards with numbers $\frac{1}{10}$, $\frac{3}{10}$, $\frac{7}{10}$, and $\frac{9}{10}$.

Let us prove that *as a result the cards will contain all rational numbers between 0 (excluded) and 1 (included)*. Indeed, let us consider the cards whose denominators are divisors of n. There are precisely n of them.

Let us reduce the numbers depicted on the cards to a common denominator, n. Let us prove by induction on n that we get

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}.$$

For $n = 1$ the statement is obvious. Let $n > 1$. Assume the contrary. Then among the numbers obtained there is no fraction $\frac{l}{n} < 1$ but there is instead a fraction $\frac{m}{n} > 1$. Suppose $GCD(l, n) = k > 1$, i.e., $l = kl_1$, $n = kn_1$. Then for $n_1 < n$ the fraction $\frac{l_1}{n_1}$ should be lacking among the n_1 fractions contradictiong the inductive hypothesis.

Thus, l and n are relatively prime. But then while selecting the numerators for the denominator n we have chosen m and not chosen $l < m$; this contradicts the minimality of the numerators.

Now we see that a card with an n exists, at least, due to the fact that a card with fraction $\frac{1}{n}$ occurs. (Actually, n occurs $\varphi(n)$ times, where $\varphi(n)$ is the number of positive integres smaller than n and relatively prime with n.)

27.2.11.1. The sum of the lengths of the projections of all vectors to the coordinate axis is ≥ 4 (since the sum of the lengths of two projections of the same vector is not less than its length). Consider the vectors for which the sum of the lengths of their projections to one of four coordinate half-axes (rays emitted from O) is the greatest one. The projection of the sum of selected vectors (to the corresponding half-axis) is equal to the sum of projections. If it is greater than 1, these vectors are the desired ones since the length of the sum is > 1.

If the length of the projection is equal to 1, then (1) the sum of the lengths of projections to each half-axis is equal to 1; (2) each vector is parallel to one of the axes. By (1) and (2) there exists a vector perpendicular to the sum of the selected ones. If we add it to the selected ones, we increase the length of the sum.

27.2.11.4. Make a construction as shown in Fig. 266.

FIGURE 266. (Sol. 27.2.11.4) FIGURE 267. (Sol. 27.2.11.5)

It is easy to see that if O is the center of the pie and K and L are the midpoints of sides AB and BC, respectively, the circle constructed on BO as a diameter passes through K and L.

Let P be the point at which the cuts (or their extensions) made from points K and L meet. If P lies inside circle $OKBL$, then $KP \leq BO = 1$ and $LP \leq 1$; so, not the extensions but the cuts themselves meet at point P. Thus, a piece $KPLB$ is cut off the pie.

Let now P lie outside the circle. Then $\angle KPL < \angle KOL$; hence, $\angle OKP < \angle OLP$, i.e., $\alpha < \beta$. Notice that since points $K, L, M, \dots$ lie along the circle, the inequalities $\alpha < \beta$, $\beta < \gamma, \dots, \delta < \alpha$ cannot hold simultaneously. And this means that at least one piece is cut off.

REMARK. The solution shows that if each cut is *shorter* than 1, then for $n > 3$ it is always possible to cut the pie in such a way that not a single piece is cut off.

27.2.11.5. First, arrange the knights in an arbitrary way. Let some two enemies — knights A and B — be neighbors, B sitting on the right of A. Now, among the friends of knight A find a knight C whose right neighbor D is a friend of B (such a knight exists, as A has n friends and the number of enemies of B without A is only equal to $n-2$); see Fig. 267.

Now change the order of the part of the table $B \dots E \dots F \dots C$ — from B to C — backward:

$$C \dots F \dots E \dots B.$$

It is easy to see that this decreases the number of the neighbors who are enemies. By repeating the procedure, we arrive at the arrangement desired.

28.1.8.1. If line b is drawn and N is the midpoint of segment AB that the circle, call it S, intercepts on line b, then $AN = NB$ and $ON \perp b$, see Fig. 268. Therefore, $\angle ONM = 90°$, and $\angle ONM$ is inscribed in the circle, call it S', constructed on segment OM as on diameter.

FIGURE 268. (Sol. 28.1.8.1)

Now we have the following way to construct b: Construct the circle S' with diameter OM. If it does not intersect straight line a inside circle S, there is no solutions.

Let S' has common points with a inside S. It can be one or two points. By drawing lines through M and either these points we get one or two solutions.

28.1.8.3. Let O be the intersection point of straight lines AB and CD. Mark segments $OX = AB$ and $OY = CD$ on these straight lines, and then construct parallelogram $OXYZ$. Point M, where lines OZ and a meet, is the desired one, see Fig. 269.

FIGURE 269. (Sol. 28.1.8.3)

FIGURE 270. (Sol. 28.1.9.3)

Indeed, the areas of triangles ZOY and ZOX are equal. Denote by h_M and h_Z the distance from line OX to M and Z, respectively. Then

$$\frac{S_{\triangle MOX}}{S_{\triangle ZOX}} = \frac{\frac{1}{2}h_M \cdot OX}{\frac{1}{2}h_Z \cdot OX} = \frac{h_M}{h_Z} = \frac{OM}{OZ}.$$

Similarly, $\frac{S_{\triangle MOY}}{S_{\triangle ZOY}} = \frac{OM}{OZ}$, hence, $S_{\triangle MOX} = S_{\triangle MOY}$. But $S_{\triangle MAB} = \frac{1}{2}h_M \cdot AB = \frac{1}{2}h_Z \cdot OX = S_{\triangle MOX}$ and, analogously, $S_{\triangle MCD} = S_{\triangle MOY}$. Therefore, $S_{\triangle MAB} = S_{\triangle MCD}$, as required.

Arguing now backwards we see that $S_{\triangle MAB} = S_{\triangle MCD}$ implies $S_{\triangle ZOX} = S_{\triangle ZOY}$, where Z is an arbitrary point on the line OM. Choose Z so that $XZ \parallel OY$. Then the equality of the areas implies $XZ = OY$ and $XOYZ$ is a parallelogram.

Observe, however, that line OM must not lie inside angle $\angle AOD$; it can lie in the angle smezhnyj to $\angle AOD$. Constucting OY in the opposite direction we get another parallelogram and another point M. In the general case — when none of the lines OM is parallel to a — the problem has two solutions.

28.1.8.4. (The solution to Problem 28.1.9.4 is similar.) Assume the contrary. Let a_i be the number of games played by the i-th team. Obviously, $0 \le i \le 29$. If all numbers a_i differ from each other, then one of them is equal to 0, some other one to 1, ... , the last one to 29. The team that played 29 games played at least one game with every other team, including the one which played 0 games. Contradiction.

28.1.9.1. We have $N = \overline{abcdef} \,\vdots\, 37$. The figures c and f are different, therefore, $N \neq N' = \overline{abfdec}$. Since $a > 0$, the number N' is also a 6-digit one. Since

$$\begin{aligned} N' - N &= \overline{ab0de0} + 1000 \cdot c + f - (\overline{ab0de0} + 1000 \cdot f + c) \\ &= 999 \cdot c - 999 \cdot f = 37 \cdot 27(c - f) \,\vdots\, 37, \end{aligned}$$

hence, $N' = (N' - N) + N \,\vdots\, 37$.

REMARK. The statement of the problem is also true for k-digit numbers for $k > 4$.

28.1.9.3. Due to a property of bisectors, see *Prerequisites*,

$$\frac{AM}{MB} = \frac{AC}{BC} > \frac{AC}{AB} = \frac{CK}{BK}.$$

It follows that M is farther from AC than K. Therefore, straight lines MK and AC meet at point P to the right of C along AC, see Fig. 270.

Hence, $\beta > \varphi$, since $\angle MCA$, an outer angle of triangle MCP, is greater than its inner angle $\angle CMP$.

Similarly, from $\triangle AKP$ we deduce that $\psi > \alpha$.

In $\triangle AMK$, the side subtending angle ψ is AM, and the side subtending angle α is MK. Now, $AM > MK$ since $\psi > \alpha$. By the same reason it follows from $\varphi < \beta$, true in $\triangle CMR$, that $MK > KC$, and we are done.

28.1.10.1. Select points K on BC and L on AB so that O_1 were inscribed in $\triangle KBL$ and $KL \parallel AC$. Select points M on AB and N on AC so that O_2 were inscribed in $\triangle AMN$ and $MN \parallel BC$, see Fig. 271.

Since triangles $\triangle BKL$ and $\triangle MNA$ intersect (both of them contain the tangent point of the circles), M lies between B and L. These triangles are similar to the initial one with coefficients $k_1 < 1$ and $k_2 < 1$. But then $BL = k_1 AB$, $MA = k_2 AB$ and since $BL + MA > AB$, we have $k_1 + k_2 > 1$.

Figure 271. (Sol. 28.1.10.1)

Figure 272. (Sol. 28.1.10.3)

Denote by r_1, r_2 and r the radii of the circles O_1, O_2 and the circle inscribed into $\triangle ABC$, respectively. Clearly, $r_1 = k_1 r$, $r_2 = k_2 r$ and $r_1 + r_2 > (k_1 + k_2)r > r$. Q.E.D.

Remark. Problem 15.1.9.3 can be solved in a similar way.

28.1.10.3. Considering segment AB fixed, move the angle around it. Then vertex C of this angle draws arc $\smile AB$ of a circle in which AB is a chord, see Fig. 272.

The perpendicular to the midpoint X of the chord intersects the bisector of angle $\angle C$ at the midpoint Y of the arc α that complements $\smile ACB$ to the full circle. Hence, the length of segment XY does not vary while C moves.

28.1.10.5. First, calculate the sum of all angles of all triangles. The sum of the angles of the triangles at all 1965 points is equal to $1965 \cdot 2\pi$, the sum of the angles at all vertices of the square is 2π. So the sum of all the angles of all triangles is equal to $1965 \cdot 2\pi + 2\pi = 1966 \cdot 2\pi$. On the other hand, this sum is πn; therefore, $n = 3932$.

To calculate the number of cuts, find the number of sides of the triangles inside the square. It is equal to $k = \frac{1}{2}(3n - 4) = 5896$ because each such side is counted twice, cf. Problem 16.2.7.4.

28.1.11.1. Prove that if $|x| \geq 2$, then

$$|x^n| > |x^{n-1}| + |x^{n-2}| + \cdots + 1.$$

Therefore, $f(x)$ has the same sign as x^n for $x \notin]-2; 2[$ and, consequently, $f(x) \neq 0$ for $|x| \geq 2$.

28.1.11.3. Clearly, the maximum of $x = -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}$ is equal to $\frac{1}{2} + \sqrt{\frac{1}{4} + 1} = \frac{1}{2}(1 + \sqrt{5})$. Now, let y be a root of an equation $x^2 + px + q = 0$ satisfying the conditions of the problem, and $|a| \leq 1$. Then $z = ay$ is a root of the equation $z^2 + paz + qa^2 = 0$ also satisfying the conditions. Therefore, the set of values to be determined is the set of numbers of the form $a\frac{1}{2}(1 + \sqrt{5})$, where $|a| \leq 1$, that is the segment $[-\frac{1}{2}(1 + \sqrt{5}), \frac{1}{2}(1 + \sqrt{5})]$.

28.1.11.5. Denote by a_{ij} the number of cards with number j lying between the two cards with number i. Clearly, $a_{ij} \leq 2$ for any i and j; moreover, $a_{ii} = 0$.

Observe further that $a_{ij} + a_{ji} = 2$ for $i \neq j$. In fact, if both cards with number i lie between the two cards with number j, then the cards with number j do not lie between the cards with number i; if only one card with number j lies between the cards with number i, then only one card with number i lies between the cards with number j.

Hence, the sum of all the a_{ij} is even. But, on the other hand, $a_{i0} + a_{i1} + \cdots + a_{i9}$ is the number of all the cards between the cards with number i. If we could manage to arrange the cards as required the sum total of all the a_{ij} would have been equal to $1 + 2 + \cdots + 9 = 45$, an odd number. Therefore, the required arrangement does not exist.

28.2.8.1. Let $a_p = a_q$ and assume that $a_{p+1} - a_{q-1} = b \neq 0$. For instance, let $b > 0$. Then one can easily prove by the induction that $a_{p+2} - a_{q-2} > 2b$ and $a_{p+3} - a_{q-3} > 4b$. Generally, the difference $a_{p+k} - a_{q-k}$ increases more than twice when k increases by 1. On the other hand, for $k = q - p$ these numbers are equal. Thus, if $a_p = a_q$, then $a_{p+1} = a_{q-1}$, whence the assertion of the problem.

Another solution: Observe that the condition of the problem is satisfied by a geometric progression whose denominator q is a root of the equation $\frac{1}{2}(q - q^{-1}) = 1$.

Thus, q is either $2 + \sqrt{3}$ or $2 - \sqrt{3}$. One can also easily see that for arbitrary t and s the sequence

$$a_n = t(2 + \sqrt{3})^n + s(2 - \sqrt{3})^n \qquad (*)$$

fulfills the conditions.

Since arbitrary numbers a_0 and a_1 can be represented in the form $(*)$ for certain s and t and since a_0 and a_1 completely determine the sequence described in the formulation of the problem, it is clear that $(*)$ gives the general term of the sequence required.

Knowing the general form of a term of the sequence, it is not difficult to prove the assertion of the problem.

28.2.8.2. If the ball would have roamed about the billiard table indefinitely, it would have, sooner or later, moved along the path already covered (the billiard table has only a finite number of segments that can serve as paths for the ball). But trying to point out the first moment when it started the same path for the second time we immediately face a contradiction since the ball was to follow the old path before. So the ball will drop into one of the pockets, but into which one?

Note that the ball can only reach the points (with integer coordinates) whose coordinates are of the same parity. Hence, it gets either into $(0, 0)$, wherefrom it came, or into $(26, 0)$. But to get into $(0, 0)$ the ball should pass along the way already taken; this requires a reflection. Hence, the only possible pocket for the ball to wind up into is the upper left one, cf. Problem 16.2.9.4.

28.2.8.3. Make all 1965 possible rotations. Obviously, the total number of coincidences of red sectors is equal to $200 \times 200 = 40\,000$. Assume now that in all but 59 positions not less than 21 red sectors coincide. Then the total number of coincidences is not less than $59 \cdot 0 + (1965 - 59) \cdot 21 = 40\,026 > 40\,000$.

28.2.8.4. Obviously, if the robber goes the same direction as the cop, or stays, (s)he, the robber, will immediately be "blown". Hence, the robber must go the other direction. In order to avoid being seen by the other cops the robber should "meet", i.e., occure on the same perpendicular to the sides of the street; the other cop right in the middle of the houses (if the robber is shifted, however slightly, from the center at some moment, then the shift will increase at each the moment of "meeting" with the next cop, etc., until the cop sees the robber). But the distance from cop to cop is three houses, hence, the "meeting" occurs either at the neighboring house or at the one after it. Therefore, the speed of the robber is either $\frac{1}{2}v$ or $2v$.

Let us verify that if the robber moves with speed $\frac{1}{2}v$ (s)he will never be seen. Indeed, point A_1, see Fig. 273, will always be on the segment connecting the robber and a cop, point A_2 shields the robber against the next cop, etc.

If the robber moves with speed $2v$ arguments are the same but points A_1, A_2 are on the opposite sides of houses.

Figure 273. (Sol. 28.2.8.4)

This means that cops and robbers may exchange roles (just as it happens in real life sometimes). Hence, if $w = av$ is a solution of the problem, then $w = \frac{v}{a}$ is also a solution. Therefore, one may assume that $|w| < v$.

28.2.9.5. Let an equilateral triangle $\triangle XYZ$ is circumscribed about $\triangle ABC$ so that $A \in YZ$, $B \in XZ$, $C \in XY$. Draw a circle with side AB of the given triangle ABC as a chord, so that the arc outside the triangle is equal to $240°$ and the one inside is equal to $120°$. Let C_1 be the midpoint of the internal arc. Let $\triangle XYZ$ be the circumscribed triangle. Obviously, point Z lies on the external arc we have constructed, and $\angle C_1ZA = \angle C_1ZB$. Therefore, if O is the center of mass of $\triangle XYZ$, then straight line OZ contains C_1.

Next, construct similarly two circles with BC and AC as chords and denote the midpoints of the obtained arcs by A_1 and B_1, respectively. Now, we can easily see that $\angle XOZ = 120°$. It follows (not immediately!) that the circle circumscribed around $\triangle A_1B_1C_1$ is the locus to be found.

28.2.10.1. The problem has many solutions. Here is one of them.

First weighing. On one pan we put one coin from each of 10 sacks; on the other pan, 10 coins from the last sack.

Second weighing. On the first pan we put one coin from the first sack, 2 coins from the second sack, 3 from the third sack, ... , 10 coins from the tenth sack. On the other pan we put 55 coins from the last sack (the same numbering of sacks as in the first weighing).

Extension. Think on your own how by comparing the differences of weights obtained in these weighings one can determine which of the eleven sacks contains the counterfeit coins.

28.2.10.2. The contents of each layer is uniquely determined from two preceding layers. Therefore, if the process is infinite it is periodic.

Observe also that from two neighboring layers we can recover the one that precedes them (according the same rule!). But this means that the process is periodic *in both directions* and, therefore, we can start counting the period from the first layer.

Let l be the length of the period. Since the $(l+1)$-st layer coincides with the 1-st and the $(l+2)$-nd with the 2-nd, the Rule's requirements for the $(l+1)$-st layer are satisfied without taking into account the cubes of the l-th layer. But this is only possible if the cubes of the l-th and the $(l+1)$-st layers are identical. But then the Rule's requirements are satisfied for the l-th layer without taking into account the cubes of the $(l+1)$-st layer; hence, the process could have been stopped on the l-th layer.

28.2.10.3. Let after several reflections the ball fall in one of the corner pockets. Meanwhile it went x times along the table and y times across it. Since the way along is equal to that across, $p \cdot x = 2q \cdot y$, implying that x is even. Till the middle of its way the ball went $\frac{x}{2}$ times (an integer!) along and $\frac{y}{2}$ times (*not* an integer, since otherwise the midpoint of the way would lie in the angle pocket) across. But an integer number of ways along and a half-integer across is precisely the midpoint of a side of length $2q$, the place of a middle pocket. Q.E.D.

28.2.10.4. Call the numbers obtained after their indices are added *new*. Their sum is equal to $2(1+2+\cdots+2n) = 2n(2n+1)$. This sum is divisible by $2n$. If we replace each new number with the remainder of its division by $2n$, then the sum of these remainders will also be divisible by $2n$. Now, suppose that all the $2n$ remainders are different. Then each of the remainders — $1, 2, 3, \ldots, 2n$ — should be encountered once. The sum of these remainders is, however, equal to $n(2n-1)$, hence is not divisible by $2n$. Contradiction shows that there are equal remainders, cf. Problem 20.2.9.3.

28.2.10.5. Let us call the biggest box the *box of rank* n. Two boxes next in size will be called *boxes of rank* $n-1$, and so on. The smallest boxes are *boxes of rank* 1. This is where the coins are.

The difference between the number of heads and tails in any box will be called the *defect* of this box. Denote the defect of the biggest box by d.

Let us prove that there is always a box which when turned upside down reduces the total defect at least twice. Then the problem is solved since $|d| \leq 2^n$ and d is always even.

Assume the contrary and let $d > 0$. When all coins of the box with defect d' are turned upside down, the total defect d becomes $2d'$. Then by assumption we have $|d - 2d'| > \frac{1}{2}d$; hence, either $d' < \frac{1}{4}d$ if $d - 2d' > \frac{1}{2}d$ or $d' > \frac{3}{4}d$ if $2d' - d > \frac{1}{2}d$.

Clearly, $\frac{3}{4}d > 1 = |d'|$ for an even d; hence, the defect of any box of rank 1 is $d' < \frac{3}{4}d$. Due to the alternative above, $d' < \frac{1}{4}d$.

This means that the defect of any box of rank 2 is $< \frac{1}{2}d$. Hence, it is $< \frac{1}{4}d$, etc.; hence, the defect of any box of rank 3 is $< \frac{1}{4}d$, and so on.

Finally, we find that the defect of any box of rank n is less than $\frac{1}{4}d$, although it is just equal to d; contradiction.

Remark: to obtain all tails in the “up” position one needs 2^{n-1} flippings over.

28.2.11.1. It is clear that $p < 19$. Besides, if p is odd, then $p^p + 1$ is divisible by 2 and if p has an odd divisor, for example, $p = 5k$, then $p^p + 1$ is equal to $(p^k)^5 + 1^5$ and is divisible by $p^k + 1$; therefore, it is not a prime.

Thus, we must check $p = 1, 2, 4, 8, 16$. But $16^{16} = 2^{64} = 2^{60} \cdot 16$ and, since $2^{10} > 1000$, then $2^{60} > 10^{18}$; hence, it is clear that 16^{16} has more than 19 digits.

If $p = 8$, then $8^8 + 1 = 2^{24} + 1 = (2^8 + 1)(2^{16} - 2^8 + 1)$ and it is not prime. We have to test $p = 1, 2, 4$ only. A direct verification shows that 2, 5, 257 are primes.

28.2.11.3. Take point M in which segment AB intersects with plane P. Set $AM = a$, $BM = b$. A sphere passing through points A and B cuts a disc in plane P, see Fig. 274.

FIGURE 274. (Sol. 28.2.11.3)

Point M divides the passing through it diameter of this disc into two parts; let their lengths be x and y. Then by the theorem on intersecting chords, $ab = xy$.

Thus, the product xy is a constant for all spheres passing through points A and B.

Find x and y so that $x + y$ is the least possible. As $x + y \geq 2\sqrt{xy} = 2\sqrt{ab}$, so $\min(x + y) = 2\sqrt{ab}$ (attained at $x = y = \sqrt{ab}$) and M is the center of the smallest disc to be found; its radius is equal to $\sqrt{ab}$. Hence, the construction:

1) find $M = P \cap AB$;
2) draw the perpendicular l to plane P through point M;
3) draw the plane π through the midpoint of segment AB perpendicularly to AB;
4) the center O of the sphere to be constructed is the intersection point of l and π.

28.2.11.4. If we cannot draw a single diagonal in a polygon (this never happens in reality, but suppose we cannot do this for the argument's sake), then there is nothing to prove.

Proof: induction on the number of sides of P. Let at least one diagonal, AB, can be drawn entirely within the polygon. It divides the given polygon into two polygons with a smaller number of sides.

By the inductive hypothesis the points of the diagonals form linearly a connected set in each of them, i.e., the assertion of the problem is satisfied. Hence, all diagonals inside the polygon can be divided into three classes: those on the left of AB, those on the right of AB, and those which cross AB.

Each class forms a linearly connected set and the proof will have been completed when we demonstrate that it is possible to get to AB both from the right-hand class and from the left-hand class. Such a possibility is a consequence of the following statement.

Proposition: *In any n-gon with $n > 3$, a diagonal entirely within the n-gon can be drawn from one of two adjacent vertices.*

Observe that we cannot assume that a diagonal can come out of any vertex.

28.2.11.5. Consider all rows and all columns and choose the row (or column) with the minimal sum of numbers, let it be equal to l. If $l \geq \frac{1}{2}M$, all is proved. If not, the M-tuple chosen (a row or a column; let it be a row for definiteness sake) contains not less than $k = M - l$ zeros. Then the sum of numbers in the k columns corresponding to zeros of the row is not less than k (by the hypothesis) and that in the remaining l columns is not less than l (this follows from our choice of the row).

Thus, the sum total is not less than $k^2 + l^2 = k^2 + (M - k)^2 \geq \frac{M^2}{2}$.

Find on your own an example of the table for which this inequality turns into an equality.

29.1.8.5. It is impossible because the same digits — collections of dots on a half-tile — of the chain are arranged in pairs of neighboring digits, and there is only an odd number of each digit left after removal of the 6's.

29.1.9-11.1. Add up the equations and after simplification we get

$$(x-1)(y-1)+(z-1)(t-1)=2.$$

As both summands of the left side are non-negative, there are only two cases:

a) both are equal to 1; then, obviously, $x=y=z=t=2$;

b) one of them is equal to 2, the other one to 0. In this case

$$x=3, \quad y=2, \quad z=5, \quad t=1$$

is a solution and there are 7 more solutions you get by permuting the unknowns. Thus, the total number of solutions to the problem is equal to 9.

29.1.9-11.2. Obviously, if we pass from k to $k+1$, the denominator increases $k+1$ times.

And what about the numerator? Its summands are multiplied by 19 and 66, respectively. Therefore, it is multiplied by some complicated fraction which is certainly less than 66 but does not differ much from 66 (for large k the first summand is much less than the second one and, therefore, is not very important).

It is easy to show that if k is not very small (for example, if $k>5$), then the numerator increases by more than 65 times. One directly shows that $A_1<A_2<\dots<A_5$; this is easy.

Thus, A_k increases for $k<65$ and decreases otherwise; hence, A_{65} is the greatest.

FIGURE 275. (Sol. 29.1.9-11.3)

29.1.9-11.3. Let us consider the projection $\mathcal{B}$ of the given pentagon $\mathcal{A}$ to the circle along the radii, see Fig. 275. Clearly, $\mathcal{B}$ is an inscribed pentagon. Assume that the circle is a unit one and the length of the side of a regular inscribed pentagon is equal to a. (Obviously, $a>1$.)

The radii connecting the center of the circle with the vertices of $\mathcal{B}$ split $\mathcal{B}$ into 5 isosceles triangles with side 1. Take one of these triangles (let is be $\triangle AOB$) whose angle at the vertex O is $\le 72^\circ$. Let S denote the sum of five angles at the vertex O. Then

$$\begin{cases} S=360^\circ & \text{if } O \text{ lies inside } \angle A,\\ S<360^\circ & \text{if } O \text{ lies outside } \angle A. \end{cases}$$

The *diameter of the triangle* (see Prerequisites) is equal to either AB (if $\angle AOB>60^\circ$) or OA (if $\angle AOB\le 60^\circ$). In both cases the diameter of the triangle is $\le a$. Hence, the side CD of the initial pentagon $\mathcal{A}$ which lies inside $\triangle AOB$ does not exceed a, Q.E.D.

29.1.9-11.4. Let us prove that the remainders after a division of numbers n^n by p appear periodically. Clearly, it suffices to consider prime p's. We have:

$$(n+kp)^{n+kp}\equiv n^{n+kp}\equiv n^n\cdot n^{kp} \pmod p$$

and to complete the proof it suffices to get a k such that $n^k\equiv 1 \pmod p$ for any n nondivisible by p. Take $k=p-1$. Indeed, by *Fermat's small theorem* $n^p-n \,\vdots\, p$ for any n. We will not carry out the proof; for our needs it suffices to check the cases $p=2,3,5$.

29.1.9-11.5. Let us prove that it is impossible to arrange 17 kings. Assume that there are 17 kings on the checker board. Obviously, it is impossible to place a king on an edge square; therefore, the kings fill the whole central 6×6 square of the board excluding one small square. But then in the central 4×4 square there are no fewer than 7 kings. To make it possible for the kings to get jumped they must all be adjacent to a single vacant square in the central 6×6 square. But there are only 4 squares neighboring the vacant one.

29.2.8.2. PROOF will be carried out by induction. The base of the induction is obvious. Let the sequence start with $1, 1, \dots, k-1, k-1$. Its partial sum corresponding to this beginning is equal to

$$\begin{aligned} S &= 1 + 1 + \dots + (k-1) + (k-1) \\ &= 2(1 + 2 + \dots + (k-1)) + (1 + 1 + 2 + 4 + \dots) = k(k-1) + 2^p, \end{aligned}$$

where $k \leq 2^p < 2k$. Therefore, $k^2 \leq S < k^2 + k$ and the equality $k^2 = S$ can only be attained for $k = 2^p$. This implies that $\left[\sqrt{S}\right] = k$, $\left[\sqrt{S+k}\right] = k$ while the next term, $\left[\sqrt{S+2k}\right]$, is equal to k only for $k = 2^p$.

29.2.8.2. See Problem 29.2.9-11.3.

29.2.8.3. See solution to Problem 29.2.9-11.3

29.2.8.4. By the hypothesis the greatest number of train changes from the given subway line is three, and the greatest number of changes from the second lines is 2. Therefore, the number of lines is not greater than $1 + 3 + 2 \times 3 = 10$. Fig. 276 (in which white dots are stations, thick dots are the transfer stations) shows how to change trains on 10 lines to satisfy the condition.

FIGURE 276. (Sol. 29.2.8.4)

29.2.8.5. Consider a number $N = 1\,000\,100\dots$ (the number of zeros at the end of N will be given later).

Find how the first 4 digits of the number $k!$ change for $k = N, N+1, \dots$. If $k! = \overline{abcde\dots}$, then $(k+1)! = k! \times 1\,000\,100\cdots = \overline{abcx}$, where x is equal to either d or $d+1$, depending on the transfer of the sum $a + e$ into the next order under multiplication. At each step we augment k by 1 and see how the first 4 digits change.

As the fifth digit of the product mentioned increases by a, this transfer takes place at least once every 10 steps and the number $\overline{abcd}$ increases by 1. Let now $N = 1\,000\,100\,000$. During the first $100\,000$ steps the number $\overline{abcd}$ increases by not more than 1 after each multiplication but also by not less than 1 every 10 multiplications.

Thus, the first four digits of the number $k!$ will consequently take one of $10\,000$ values among which there sure will be the number 1966 as well.

29.2.9-11.3 and **29.2.8.3.** The complete scheme of the investigation is given of Fig. 277.

In the trial we test a group of balls. Fix such a group for the first trial. consider now a first pair and suppose that exactly these two balls are radioactive ones. (whereas the other balls are not). Give the pair a "+" if the group is radioactive and a "-" otherwise. Let us endow each pair with a "+" or a "-". Let us perform the first trial. If the group is a radioactive one the set of answers shrinks to the set of pairs with a "+", if the group is not radioactive, then the set of answers shrinks to the set of pairs with a "-".

Suppose our luck is tough and a considerable set is to be investigated, with more than a half of pairs left. It has $l_{n-1} \geq \binom{k}{2}/2 > 2^{n-1}$ pairs. Let us repeat the procedure, this time with the pairs left. There remain

Figure 277. (Sol. 29.2.9-11.3)

$n-2$ trials and $l_{n-2} > 2^{n-2}$ pairs, etc. Finally we have 0 trials and more than 1 pair left that satisfies all the trials. The answer is not found.

Let us illustrate with two particular cases how this scheme works.

For the first trial, take balls 1 to 5 (group (1–5) on Fig. 277.). If "+", take the group (6–13) of 8 balls for the second trial. If "+" again, then two radioactive groups are found and each of them has a radioactive ball each. in the group (1–5) such a ball can be found in $s(5) = 3$ trials; in the group (6–13) such a ball can be found in $s(8) = 3$ trials; taking into account two trials already performed, we get 8 trials altogether.

If the second trial resulted in "-", check the group (14–17), if "-" again, check the group (18, 19). Let we draw "-" again. Then both the radioactive balls are in the group (1–5). They are identified in not more than $d(5) \leq 5-1 = 4$ trials; with the 4 trials already performed this constitutes 8 trials.

Here is the general principle as to how to solve similar problems.

1°. *If one of 2^n (or less) balls is radioactive, then it is possible to find it in n tests*: the first step is to test half of the balls and thus to find which half contains the radioactive ball, the second step is to take the radioactive half of this half, and so on.

2°. *If there are more than 2^n balls, then it is impossible to find one radioactive ball in n steps.*

Indeed, assume the contrary. Make n tests marking the presence of radioactivity by "+" sign and its absence by "−" sign. By the hypothesis, knowing the sequence of signs we can find which of the balls is radioactive. But there exist only 2^n distinct sequences. If we mark a radioactive ball for each of them we will come to any absurd conclusion that the balls which do not correspond to any sequence cannot be radioactive.

3°. *If two of m balls are radioactive, then there are $C_m^2 = \dfrac{m(m-1)}{2}$ variants depending on which pair of balls is radioactive. If $\dfrac{m(m-1)}{2} > 2^n$, it is impossible to find the radioactive pair in n tests* (for the same reasons as in 2°).

4°. *If during the first step we test k of m balls, then the result "−" corresponds to C_{m-k}^2 variants* (both radioactive balls are among the $m-k$ remaining balls), *and the result "+" corresponds to the remaining $C_m^2 - C_{m-k}^2$ variants.* If we have only l tests left, then neither C_{m-k}^2 nor $C_m^2 - C_{m-k}^2$ can be greater than 2^l. Naturally, the best strategy is to have the numbers C_{m-k}^2 and $C_m^2 - C_{m-k}^2$ approximately equal.

Now, let us pass to the particular cases:

29.2.8.3. Let us number the balls and take those numbered 1 to 5 for the first test.

1a) Assume that we get a "+".

Second test: Take the balls numbered 6 to 13.

2a) If we get a "+" once again, then one of the radioactive balls is in the first set of five, the second one is in the second set of eight, and it is possible to find each of them in 3 tests by 1°.

2b) If the result of the second test is a "−", then go to 3-rd test.

Third test: Take the balls numbered 14 to 17

3a) "+" case. In this case you can find 1 of 5 balls in 3 tests, and 1 of 4 balls in 2 tests.

3b) "−" case. Solve this case on your own.

1b) Let the result of the first test be "−". Then we have to find 2 of 14 balls in 7 tests. Take the balls numbered 6 to 9 for the next test.

Finish the solution on your own in a way similar to 1a).

29.2.9-11.3. Assume that during the first test we took 2 balls and got a "−". Then we have to find 2 of 9 balls in 5 tests. But since $C_9^2 = 36 > 2^5$, this is impossible by 4°. Similarly, 1 ball will not do.

Let us now take 4 balls for the first test. If we got a "+", then the number of the variants left is equal to $C_{11}^2 - C_7^2 = 34 > 2^5$ which is also too much. Similarly, 5 or more balls will not do.

Thus, it is necessary to take 3 balls for the first test.

Now, suppose that we got "$-$". So we have to find 2 of 8 balls in 5 tests. Similarly, you have to take more than one but less than 3 balls for the second test.

So, take 2 balls for the second test. Assume that we again drew a "$-$".

We have to find 2 of 6 balls in 4 tests. But if we take 1 ball, then in the "$-$" case we have $C_5^2 = 10 > 2^3$ variants left; if we get 2 balls for the third test, then in the "+" case there are $C_6^2 - C_4^2 = 9 > 2^3$ variants left.

Thus, to find 2 of 11 balls in 6 tests is only possible if one is lucky but impossible in general. To do it in 7 tests is not difficult; one can even select 2 of 14 balls in 7 tests, see the solution of Problem 29.2.8.3.

29.2.9-11.4. The set of weights 26, 25, 24, 22, 19 g and 11 g satisfy the condition. Let now some 7 weights be chosen. Note that it is forbidden to take the weights 26, 25, 24, 23 g simultaneously. Therefore, the sum of the masses of any four weights of these seven is less than 98 g. But of 7 weights we may compose $\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 98$ different sets of 1, 2, 3 or 4 weights. Therefore, the weights of some two sets coincide.

30.1.8.2. We have $R_1 + R_2 = \dfrac{AC + BC}{2\sin\angle BMC} \geq \frac{1}{2}(AC + BC)$, whence $\angle BMC = 90°$.

30.1.8.3. Let an asterisk stand for an arbitrary set of 8 characters, for example, 8 dots. Consider 4 words whose first 8 characters denoted by $*$ are identical:

$$(* \cdot \cdot), \quad (* \cdot -), \quad (* - \cdot), \quad (* - -).$$

By the hypothesis these words must get into different groups but there are only two groups.

Prove on your own that:

a) it is impossible to divide a set of words of length 10 into fewer than 11 groups so that the words of one group differ in at least 3 places;

b) there exists a corresponding division into 16 groups.

Extension. Perhaps you will be able to improve the result, i.e., find a division into less than 16 groups. (We can not.)

30.1.9.1. The pedestrians will definitely meet on the longest circle as its length is greater than the sum of the lengths of all the other circles put together.

30.1.9.2. It is impossible, since the total area of the shaded parts in Fig. 278 is equal to $\frac{1}{2}$ (i.e., is not $\frac{5}{9}$).

FIGURE 278. (Sol. 30.1.9.2)

30.1.9.5. Let the first 12 digits of the 120 numbers obtained form 12-digit numbers $A_1, \dots, A_{120}$, and let the other 108 digits of each number form the number B. Then prove that $(A_1 + \dots + A_{120}) \cdot 10^{108} + 120B$ is divisible by 120, i.e., the first summand is divisible by 120. But $A_1 + \dots + A_{120}$ is divisible by 3 (as the remainders after a division of each A_i by 3 are equal), and 10^{108} is divisible by 40.

30.1.10.4. 1) Each of the planes damages raisins whose centers lie in the layer of thickness ≤ 0.2 mm (for our purposes the thickness of the layer is of no importance, what matters is that it is finite, not as vanishing). In space, consider all the planes of the form $x = k$, where k is an integer. No two of the planes can be completely contained inside one layer (the distance between them is greater than the thickness of the layer). There are infinitely many such planes, so there is one plane that is not completely contained in any layer. Fix this plane and consider its intersections with the layers. Each layer cuts off this plane a strip of width $\frac{0.2}{\sin\alpha}$, where α is the angle between the plane and the layer.

2) Take an integer n greater than the width of any strip; on the plane, consider the straight lines $y = 0$, $y = n$, and $y = 2n$. No two of these lines lie in one strip (the distance between them exceeds the strip's width). There are infinitely many such lines; hence, htere is one that does not completely lie in one strip.

3) In much the same way we show that on the line there si a point with integer coordinate lying outside all strips. By the choice of the plane, the line and the point all the coordinates of the point are integer, it does not belong to any layer, hence, the raisin with the center in this point is undamaged.

Another solution is to consider a cube with edge l great enough; the total number of raisins in the cube is proportional to l^3 but the number of raisins damaged by k planes is only proportional to kl^2.

30.2.7.2. The union of 4 identical discs of radius $r < \frac{\sqrt{2}}{2}$ cannot cover 4 vertices and the center of a unit square.

30.2.7.3 and **30.2.8.1.** Let us show how to find a multiple of 2^q whose decimal expression has no zeros. Take any multiple a of 2^q whose decimal expression has not less than q digits. If among the digits there are no zeros, we are done.

Suppose there is a 0 and the right-most 0 occupies k-th place (counting from the right). Observe that the last digit of 2^q is not a 0. Let us write on the right of (the decimal expression of) 2^q as many 0's as is needed to move the last digit to the k-th place and add this number to a. Clearly, the sum is divisible by 2^q; the digits to the right of the k-th one ar ethe same as those of the initial number whereas the k-th digit became nonzero, i.e., the tail of zeros became shorter. Again find the right-most zero, etc. and repeat the procedure unteal we get a number B with the last q nonzero digits. Consider the number N formed by these last q digits. We have: $B = C \cdot 10^q + N$, B and $C \cdot 10^q + N$ are divisible by 2^q, hence, so is N.

Another solution: prove the following statement S:
S: *if a number consisting of k digits is divisible by 2^k, then you may write a* 1 *or a* 2 *in front of this number so that the number becomes divisible by 2^{k+1}.*

Therefore, there even exists a number divisible by 2^n and consisting of 1's and 2's. (This problem was suggested at the 1972 National Olympiad, see also Problem 53.11.4.)

Remark. Statement S is actually a *lemma* needed to solve a more general problem: *if a number m is not divisible by* 10*, then there exists an N divisible by m and without zeros in its decimal expression.* Prove this on your own.

30.2.7.5. Choose two spotlights which are the "northernmost" (if several spotlights are situated on the same parallel, choose any two of them). It is easy to see that if we direct them southeast and southwest we illuminate the whole half-plane to the south of them, see Fig. 279.

Figure 279. (Sol. 30.2.7.5)

Now it is easy to direct the two remaining spotlights so that the whole plane is illuminated.

Remark. The main idea of this solution is to divide the set of spotlights in halves, the "northern" and the "southern" ones, and to reduce the problem on a plane to a similar problem on a straight line, i.e., to illuminate the whole straight line with two spotlights, each being able to illuminate a half-line, which is trivial.

One can similarly reduce spatial Problem 30.2.10.4 to a planar problem.

30.2.8.2. **Lemma**. *Let $N = bp$, where p is a prime. Then $d(b) \geq \frac{1}{2}d(N)$.*

Proof. Let us divide all the divisors of N into two groups: C, consisting of those that are divisible by$p$ and D, the other ones. Let B be the set of all divisors of b. Clearly, $C = B$: if q divides N but is not divisible by p, then q divides b. The other way round, if q divides N and is divisible by p, then q/p divides b, i.e., under division by p all the elements of D turn into B; hence, B has no less elements than D. But either D or C has not less than half of all the divisors of N; hence, $d(b) \geq \frac{1}{2}d(N)$. Lemma is proved.

Let $\frac{N}{d(N)} = p$. Then $N = d(N)p$ and, having denoted $d(N)$ by b we get thanks to Lemma

$$d(b)\frac{1}{2}b. \qquad (*)$$

This is a very strong condition: all the divisors of b, except b itself, do not exceed $\frac{1}{2}b$; hence, the condition means that all the numbers smaller than $\frac{1}{2}b$, except, perhaps, one, divide b. Let us find all the b satisfying this condition.

If b is odd, $b = 2k+1$, then, by condition $(*)$, b is divisible by 1, 2, … , k. But

$$2k+1 \vdots k \iff 1 \vdots k \iff k = 1 \iff b = 3.$$

If b is even, $b = 2k$, then b is divisible by all the numbers 1, 2, … , $k-1$, except, perhaps, one of them. It is easy to verify that for $k < 3$ the numbers 2 and 4 satisfy condition $(*)$. Let $k \geq 3$. Then b is divisible by $k-1$ or by $k-2$ and we have

$$2k \vdots (k-2) \iff (2k - 2(k-2)) \vdots (k-2) \iff 4 \vdots (k-2) \iff k-2 \in \{1,2,4\} \iff b \in \{6,8,12\}.$$

Similarly,

$$2k \vdots (k-1) \iff b \in \{4,6\}.$$

It is easy to verify that all the numbers found, i.e., $b \in \{12,8,6,4,3,2\}$ satisfy condition $(*)$.

Now seek N in the form $N = pb$. It is easy to verify that for $p = 2$ of the 6 candidates only 8 and 12 fit; for $p = 3$ only 9, 18 and 24 fit. If $p > 3$, then p and b are relatively prime and, therefore, $d(bp) = 2d(b)$: by lemma the cardinalities of C and D are equal!. But $d(bp) = d(N) = b$ implying $\frac{b}{d(b)} = 2$; as is esy to verify, only $b = 8$ and $b = 12$ will do; we have already found them. Hence, the answer:

$$\{8, 9, 12, 18, 24, 8p, 12p \mid p \text{ is a prime }, p \geq 5\}.$$

30.2.8.5. To *not* satisfy the assertion of the problem the children must see 8 shows in one theater and at least 2 shows in each of the other 6 theaters — 20 in all. But they can see only 2 shows during one visit (6 of them can see one show and the seventh can see another show), i.e., they will see only 16 shows during the day.

30.2.9.1. Let us write one number under the other one (as we do to add them). Clearly, several last columns (perhaps, none) are filled with zeros; next stands the column with sum of its digits equal to 10 and the sum of the digits of the other columns is equal to 9.

Since the sum total of the digits is even, there is an even number of columns with the sum of their elements equal to 9. Hence, there is an odd number (at least one) of zero columns.

Let us show that the sum 10 is represented by partition $5+5$. Indeed, let it be, for example, the partition $7+3$ instead. Then in columns with sum 9 the upper number will contain fewer 7's than the lower number and, therefore, the lower number has more 2's than the upper one. But this is impossible since both numbers have equal amounts of 2's.

30.2.9.2. Let us prove that *the sequence that begins with $x_1 = M-1$ and $x_2 = M$ is the longest.* (It is not difficult to verify that its length is equal to $L_M = M + 1 + \left[\frac{1}{2}(M+1)\right]$.)

Proof: by induction. The base of the induction: for the cases $M = 2$ and 3 the statement is easy to prove.

The inductive step: it is not difficult to verify that in any case $x_5 \leq M-2$. If in a sequence $x_4 \leq M-2$, then its length L is not greater than $3 + L_{M-2} = L_M$. If, contrarywise, $x_4 > M-2$, then there is the only possible variant: $M-1$, M, 1, $M-1$, … , and its length is equal to L_M.

30.2.9.4. Note that it is possible to get rid of zeros at the end of the number N (if $N = A \cdot 10^l$, then $\overleftarrow{N} = \overleftarrow{A}$ is divisible by k and, therefore, $A = \overleftarrow{\overleftarrow{A}}$ is also divisible by k.) Let $N = \overline{a_1a_2...a_{n-1}a_n}$ be divisible by k and $a_n \geq 1$.

Subtract the number $\overleftarrow{N}$ from the number $N \cdot 10^{n+2}$; we get

$$M = \overline{a_1a_2...a_{n-1}(a_n-1)99(9-a_n)(9-a_{n-1})...(9-a_2)(10-a_1)}.$$

Now, add M to $\overleftarrow{M}$. We get the number

$$Y = 10\underbrace{999...999}_{n-1}989\underbrace{00...000}_{n-1}$$

divisible by k. If we do the same operation with the numbers $\overleftarrow{N}$ and $N \cdot 10^{n+3}$, we get the number

$$X = 10\underbrace{999...999}_{n-1}9989\underbrace{00...000}_{n-1}$$

also divisible by k. But then $\overleftarrow{X - 10Y} = 99$ is divisible by k. Q.E.D.

30.2.9.5. Number the kings 1 to n. Obviously, it suffices to prove that one can get the order $1, 2, \dots, n$ from any original arrangement of the portraits.

To do so, let us first move the first portrait to the place of the second one (clearly, this is possible to do, as one is allowed to exchange the places of the first portrait and any other one) and then move the portraits 1 and 2 in a tow (mind the Rule) until the moment they preceded the third portrait, etc.

30.2.10.1. Obviously, it suffices to show that it is impossible for any number to return to its place in less than n moves. You can easily find examples showing that the $(n+1)$-st row written according to the same rule coincides with the first row. (In fact, the $(n+1)$-st row *always*, i.e., for any m and $n-k$, coincides with the first one.)

Let us follow the movement of some number l. When moved into the next row, it is moved to the right by one of the positive or negative numbers $n-k$, $n-m-k$, or $-m$. (If the displacement number is negative, l actually moves to the left.) Let during several moves l occurred x times in the first group (in one of the places from 1 to k), y times in the second group, and z times in the third group. This means that it moved to the right by $N = x(n-k) + y(n-m-k) + z(-m)$ places.

Let l finally return to its original place without being in the same place twice. Then $N = 0$ and

$$0 \le x \le k, \quad 0 \le y \le m-k, \quad 0 \le z \le n-m. \qquad (*)$$

Let us express z from the equation $N = 0$, i.e., from $x(n-k) + y(n-m-k) - zm = 0$:

$$z = \frac{1}{m}(x(n-k) + y(n-m-k)) = \frac{(n-k)(x+y)}{m} - y.$$

Since m and $n-k$ are relatively prime, $x+y$ is divisible by m. But the condition $0 \le x \le k$, $0 \le y \le m-k$ implies that either $x = y = 0$ or $x = k$, $y = m-k$. In the first case, $z = 0$, this is the first row of the table, i.e., 0 moves are made; in the second case $z = n-k-y = n-m$, which means that n moves are made.

30.2.10.3. By the hypothesis the numbers 1, 2, 3, 10, 11, 12 cannot be neighbors. Therefore, they alternate in the circle. But then it is impossible to place number 4 and satisfy the condition.

30.2.10.5. Let us write $A \perp B$ if A and B differ in all decimal places; let A_i be the i-th digit of A and $f(A)$ the digit ascribed to A. The rule of ascription defines a function f of integer argument that satisfies

$$\text{if } A \perp B, \text{ then } f(A) \ne f(B). \qquad (*)$$

Let us give examples of such functions:

1) Fix i and set $f(A) = A_i$ (to the number its i-th digit is ascribed);

2) Fix i and a permutation s of the integers 1, 2, 3; set $f(A) = s(A_i)$.

In these examples the digit ascribed is determined by a fixed decimal place and does not depend on the other decimal places. Let us show that only such functions satisfy $(*)$. First, find two numbers that differ in one decimal place and with different digits ascribed. To this end, select any integers P and Q, so that $f(P) \ne f(Q)$; e.g., for $n = 3$ take 222, 212, 211 and 111. Somewhere in the chain the value of f changes under the passage from a neighbor to the neighbor; these neighboring integers are A and B to be found, e.g., 211 and 111. Let them differ in the i-th place (in our example $i = 1$).

Let us assume that the i-th place can be occupied by a, b, c while the digits ascribed are α, β and γ; let $A_i = a$, $B_i = b$; let further $f(A) = \alpha$, $f(B) = \beta$. (In the example $a = 1$, $b = 2$, $c = 3$.)

Lemma. *If $D_i = a$, b or c, then $f(D)$ is equal to α, β or γ, respectively.*

Proof. For A and B Lemma is true. Denote by Ω the set of all the numbers that differ form A (hence, from B) in all the digits except, perhaps, the i-th one. It is possible to divide Ω into three subses Ω^a, Ω^b and Ω^c depending on the i-th digit. (In the example $\Omega^a = \{122, 123, 132, 133\}$, $\Omega^b = \{222, 223, 232, 233\}$ and $\Omega^c = \{322, 323, 332, 333\}$.)

If $D \in \Omega^c$, then $D \perp A$ and $D \perp B$; hence, $f(D) \ne \alpha$ and $f(D) \ne \beta$, i.e., $f(D) \ne \gamma$. So Lemma holds for Ω^c.

An important observation: *for any two numbers P and Q there exists a number R such that $R \perp P$ and $R \perp Q$*. In fact, it suffices to take $R_k \ne P_k$ and $R_k \ne Q_k$ for all k (we have three digits, don't we).

Let now $D \in \Omega^a$, then $D \perp B$; hence, $f(D) \ne \beta$. Select E so that $E \perp D$ and $E \perp B$ (In our example if $D = 122$, then $E = 333$.) Clearly, $E \in \Omega^c$; hence, $f(E) = \gamma$ and $f(D) \ne \gamma$. So $f(D) = \alpha$.

Similarlly, if $D \in \Omega^b$, then $f(D) = \beta$. Lemma holds for the whole set Ω.

Let now D be an arbitrary number (e.g., $D = 123$). Select E so that $E \perp D$ and $E \perp A$ (say, $E = 232$). Clearlym $E \in \Omega$. Construct a new number, E', having changed only the i-th digit of E but so that $E'_i \neq D_i$ (in our example $E' = 332$). We again have $E' \perp D$ and $E' \in \Omega$. Consider three pairs:

$$(E_i, f(E)), \quad (E'_i, f(E')), \quad (D'_i, f(D)).$$

The first two pairs are among the "good" pairs (a, α), (b, β), (c, γ) (since $E, E' \in \Omega$). Since D_i differs from E_i and E'_i and $f(D)$ from$f(E)$ and $f(E')$, it follows that $(D_i, f(D))$ is also among the "good" pairs, i.e., Lemma holds for D.

Let us return to the problem. If $(1, 1)$ is among the "good" pairs, then take the numberwith the i-th digit 1 and the other digets are 9's. The number ascribed to it is 1.

If the units occure in "good" pairs, say $(a, 1)$ and $(1, \beta)$, then take the number with 1 on some (but not the i-th!) place and the other digits being equal to a; the number ascribed to it is 1. (In our example, if $\alpha = 1$, take 122; if $\beta = 1$, take 212; if $\gamma = 1$, take 313.)

31.1.7.1. We exclude all numbers $\frac{1}{2}q^2$ to $q-1$ for various $q \geq 2$, i.e., 2 to 3, 5 to 8, 8 to 15, 13 to 24, etc.

31.1.7.3. There are 10 primes between 50 and 100; namely, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97. Assume that neither of them is relatively prime with some three numbers A, B, C less than 10^6. Hence, each of these primes divides one of the numbers A, B or C. But then one of the numbers A, B or C has to be divisible by at least 4 of these numbers, and, therefore, by their product. But the product of any four of these primes is greater than $50^4 = 6\,250\,000 > 1\,000\,000$.

31.1.7.4. Prove on your own that if there are less than 49 airlines, then there are two towns such that it is impossible to get from one to the other by any number of plane changes.

31.1.8.1. Prove that each participant of the tournament has won only one game. Observe that if nobody but the players from the $(k-1)$-st list are in the k-th list of a fixed player, then there is nobody else on this player's $(k+1)$-st, $(k+2)$-nd, ..., 12-th lists. This means (by Dirichlet's principle) that there is a new player on each new list. As the number of lists a player has is equal to the number of players, this means that exactly one person gets into a list each time. Therefore, the second lists of all players consist of exactly two persons, i.e., every participant won exactly one game. And this means that exactly 12 games did not end in a draw.

Since the total number of games was $\frac{11 \cdot 12}{2} = 66$ and 12 of them did not end in a draw, we are done.

Here is an example satisfying the conditions: the players won in a cycle: 1-st won 2-nd, 2-nd won 3-rd, etc., 11-th won 12-th and 12-th won 1-st; the other games ending in a draw.

31.1.8.2. Only one number must be left after the fourth striking out. Obviously, it is necessary to leave the number 44.

But on the other hand, since the sum of the given numbers is divisible by 11, it is easy to see that 44 has to be stricken out during the first step. Contradiction.

31.1.8.3. If the measure of the angle $\angle ABC$ is equal to $a°$, then it subtends the arc $2a°$; hence, $\smile ABC = 360° - 2a°$. Therefore, $\smile A_1A_2A_3 = 360° - 240° = 120°$; similarly, $\smile A_3A_4A_5 = 120°$ and $\smile A_5A_6A_7 = 140°$. But $360° < \smile A_1A_4A_7 = \smile A_1A_2A_3 + \smile A_3A_4A_5 + \smile A_5A_6A_7 = 380°$ — contradiction.

31.1.8.4. Let us rewrite the given expression as follows:

$$S = x_1 + (-x_2 + x_3) + (-x_4 + x_5) + \cdots + (-x_{98} + x_{99}) - x_{100}.$$

Since $x_{k+1} - x_k \leq x_k$ and $-x_{100} \leq 0$, it follows that

$$S \leq x_1 + x_2 + x_4 + \cdots + x_{96} + x_{98}. \qquad (*)$$

Since $x_2 \leq 2x_1 \leq 2$, $x_3 \leq 2x_2 \leq 4$, etc., $x_{k+1} \leq 2x_k \leq 2^k$, it follows that

$$S \leq 1 + 2 + 2^3 + \cdots + 2^{95} + 2^{97}. \qquad (**)$$

Setting $x_k = 2^{k-1}$ for $k = 1, 2, \ldots, 99$ and $x_{100} = 0$ we see that the inequalities $(*)$ and $(**)$ turn into equalities; hence, we have found the numbers sought.

31.1.8.5. Choose a coordinate system so that the coordinate axes are not parallel to any of the segments. Then the endpoints of one of the segments with the greatest abscissa (or ordinate) cannot touch the second segment.

31.1.9.1. Let us prove Answer. Assume that the areas of all four triangles are rational. The height of the trapezoid of area 1 with bases as in Answer is equal to $\frac{2}{1+\sqrt[3]{2}}$. The heights of the triangles adjacent to the trapezoid 's bases are equal to a and $\frac{b}{\sqrt[3]{2}}$, respectively, where a and b are rational. Thus,

$$a + \frac{b}{\sqrt[3]{2}} = \frac{2}{1+\sqrt[3]{2}}.$$

Hence, $\sqrt[3]{2}$ is a solution of the quadratic equation

$$ax^2 + (b + a - 2)x + b = 0,$$

which is impossible.

31.1.9.3. In the corridor the covered and uncovered parts alternate. Call the set of rugs that cover one covered part of the corridor a *group*. If there are k groups, then there are not more than $k+1$ uncovered parts of the corridor. Suppose there is an uncovered part. Then the combined length of the rugs is 10 times that of the covered part. Hence there is a point covered with > 10 rugs, i.e., with ≥ 11 rugs. All these rugs belong to one group. Then the other groups contain not more than 9 rugs altogether; hence ther are not more than 10 groups. Therefore, there are not more than 11 uncovered parts of the corridor.

Here is an example with exactly 11 uncovered parts of the corridor: Put one upon the other 11 equal rugs 90.5 m long, then alternate uncovered parts of 40 cm long and rugs of 50 cm long, 9 rugs altogether.

31.1.9.4. Let us take, for example, all the numbers the sum of whose digits is divisible by 10. Any digit A of such a number is uniquely determined by the other digits. Indeed, if x is the remainder after the division of the sum of all digits except A one by 10, then

$$A = \begin{cases} 0 & \text{if } x = 0 \\ 10 - x & \text{otherwise.} \end{cases} \tag{$*$}$$

Therefore, if two numbers coinside after striking out, the sums of their digits are equal; hence, so are the digits stricken out, i.e., the numbers themselves were equal. Conversely, the rule $(*)$ allows one to complete a 5-digit number to a 6-digit number with the sum of digits divisible by 10, as was required.

31.1.9.5. Observe, that p is odd (otherwise q is even and not prime); hence, $p = 2m+1$, where $m \in \mathbb{N}$. Then $p+q = 2p+2 = 4m+4$, $p^2 - 1 = 4m^2 + 4m = m(4m+4)$, i.e.,

$$p^2 - 1 \,\vdots\, p + q. \tag{$*$}$$

Now, observe, that $p^q + q^p = (p^p + q^p) + (p^q - p^p)$; let us prove that each parenthesis is divisible by $p+q$.

Indeed, $(p^p + q^p) \,\vdots\, (p+q)$ because p is odd and $p^q - p^p = p^{p+2} - p^p = p^p(p^2-1) \,\vdots\, (p+q)$ due to $(*)$.

31.1.10.1. Let us number the airplanes as they land. We may assume that the tanks of all flying airplanes constitute one reservoir and at each moment the only kerosine used is that from the nonempty tank with the least number. It is clear now that the airplane with the empty tank can not help the other planes and therefore has to land.

In reality this means that the airplane has to land as soon as she pumps the kerosine left in the tanks of the other planes. As a result the flagman plane will cover

$$(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{100})1000 \text{ km.}$$

31.1.10.2. Call a pile an a-pile, if the residue after division of the number of candies in the pile by 5 is equal to a. Call 2-piles and 3-piles "good" ones, the rest will be "bad" ones. Observe that (a) under a division of a bad pile into two piles, at least one pile is a good one. (For example, 3-pile can be divided into a 1-pile and a 2-pile or a 0-pile and a 3-pile or into two 4-piles and the first of the piles in all the divisions is a good one. For a 2-piles verify property (a) on your own.)

Observe also that (b) a good pile can be always divided into two bad ones: 0-pile into 2- and 3-piles; 1-pile into two 3-piles; 4-pile into two 2-piles.

Now we can produce the winning strategy of the first player: select a good pile and eat up the other pile; divide the good pile into two bad piles. Properties (a) and (b) show that the strategy is always applicable. In particular, by the first move one can, e.g., eat up 33 candies and divide the remaining 35 candies into piles of 17 and 18 candies. Then by property (a) the second player willnever get a pile of 1 candy, hence, (s)he will lose.

31.1.10.3. Let us prove that there are not more than 1024 classes. Let us select all numbers one can obtain by striking out digits of the number 9876543210. There are exactly $1023 = 2^{10} - 1$ such numbers, since any digit can be independently either striken out or not but we can not simultaneously strike out all the numbers. Let us add 99 to the numbers selected.

Now, let us take any number. Using admissible transformations we can interchange any two of its neighboring digits (see Hint). If the number has two identical digits, let us herd them by transpositions to each other and strike out (strike out the pairs of zeroes first). In this way we finally get numbers with all digits different or exactly two digits, equal to each other.

In the first case let us reorder digits in decreasing order, in the second case add 99 infront of the number and strike out the last two digits. In either case we get one of selected numbers. This means that any number lies in one class with one of selected numbers; hence, there are $\leq 1024 \leq 1968$ classes.

31.1.10.4. Let us prove by induction on n that $|f(_n(x) \geq |f_{n-1}(x)|$ for $|x| \geq 1$. The base of induction, for $n = 1$ is obvious. Let all be proved for n. Then

$$\begin{aligned}&|f_{n+1}(x)| \geq |(1+q_n)xf_n(x) - q_nf_{n-1}(x)| \geq \\ &(1+q_n)|x| \cdot |f_n(x)| - q_n|f_{n-1}(x)| \geq \\ &|(1+q_n)|f_n(x)| - q_n|f_n(x)| = |f_n(x)|.\end{aligned}$$

Since $f_1(x)$ has no roots for $|x| \geq 1$, it follows that $f_1(x) > 0$ and, therefore,

$$|f_n(x)| \geq |f_{n-1} \geq \dots \geq |f_1(x)| > 0.$$

31.1.10.5. The center of the parallelogram should be the midpoint of segment A_1A_3 connecting straight lines l_1 and l_3. It is easy to see that the set of midpoints of such segments is a plane P parallel to and equidistant from lines l_1 and l_3, and that for any point $x \in P$ there exists a single segment A_1A_3 whose midpoint is x.

Construct similarly plane Q corresponding to straight lines l_2 and l_4. For the center of the parallelogram we can take any point on the straight line m of intersection of planes P and Q.

Thus, the problem has infinitely many solutions. To construct a parallelogram with given center O you must reflect straight line l_3 through O; the straight line obtained lies in the same plane as l_1 and the intersection point of these lines is A_1. The other vertices are similarly constructed.

31.2.7.1. The first move of the player who opens the game should be to draw a diagonal through the center of the 1968-gon and then, during the other moves, to maintain symmetry with respect to this diagonal.

31.2.7.2. If the initial 3 points lied on one line, then all the points obtained would lie on the same line. If the point were not on one line, let us draw a circle through them. Then each time we will make a symmetry through the diameter of this circle; hence, all the new points will lie on the circle.

Now it suffices to take any line not intersecting the initial line or the initial circle.

31.2.7.3. The color-blind painter can ensure 49 changes of color if (s)he will alternate colors each time and paint the left-most unpainted part. Then under the passage from the previous (?) part to the today's one the color will change at least once irrespective the number and colors of the parts painted in between the time by the other painter.

It is impossible to ensure more than 49 changes of color: if the other painter will color the left-most unpainted part the color of the neighboring part, then of the total 99 boundaries between parts 50 boundaries will not give a change of color.

31.2.7.4. Divide the number $8925\dots25$ by 10^{198}; the quotient is approximately (but with a very good accuracy) equal to[1] $89\frac{25}{99}$. Therefore, the product is approximately equal to $n + \frac{k}{99}$, also with a very good accuracy. The condition of the problem implies that $k = 1$, whence

$$25 \cdot \overline{444x18y27} \equiv 1 \pmod{99}.$$

Show on your own that the accuracy is high enough to state that the last congruence (both sides of which are integers) holds.

To complete the solution use Hint.

31.2.7.5. The bullet's trajectory on the coordinate plane (x, t), see Hint, is a straight line and the i-th blade lies on the plane if the segment

$$x = a_i, \qquad t \in \bigcup_{n\in\mathbb{Z}} \left[t_i + \frac{n}{50},\ t_i + \frac{n+\frac{1}{2}}{50}\right] \quad \text{for } i = 1, 2, 3, 4$$

(this is one of the vertical segments shown in Fig. 280) intersects this blade.

So we have to prove that whatever times t_1, t_2, t_3, t_4 are (they correspond to the angles through which the blades are rotated relative to one another), there exists a straight line intersecting 4 of the drawn segments.

As is clear from Fig. 280 a) through the first two blades one can shoot at any point of a segment of length three times the blade's diameter and lying in the plane of the 3-rd blade. Hence, the 3-rd blade is completely under fire and through the first three blades one can shoot at any point of a segment of length two times the blade's diameter and lying in the plane of the fourth blade. Hence, one can hit the fourth blade, too.

The blades are equal semicircles attached to the shaft at the midpoint of the diameter. The planes of the semicircles are perpendicular to the shaft, the distance between the two neighboring attachment points

[1] This is not $89 \cdot \frac{25}{99}$, but a mixed fraction.

Figure 280. (Sol. 31.2.7.5)

are equal, but the diameters that bound the blades can be turned at arbitrary angles with respect to each other, see Fig. 280b).

31.2.8.1. Take all integers one after the other except the last one in each group. Begin with 1 and excluding all already taken, and let the last number be of the form $m^7 - a_1 - a_2 - \cdots - a_k$.

31.2.8.2. Let the flea make 2 jumps along the broken line ABC as shown in Fig. 281. Find point C' symmetric to C through OB. It is easy to see that $\angle ABC' = 180^\circ - 2\alpha$. Hence, we see that if you reflect the way of the flea first through OB, then through OC', etc., then the points A, B, C', D', E', and so on, will get on the same circle and separate equal arcs. Hence, the statement desired easily comes.

Figure 281. (Sol. 31.2.8.2)

Figure 282. (Sol. 31.2.8.3)

31.2.8.3. The movement of a point of the pie after several operations is shown on Fig. 282. You can see that at some moment this point takes the place symmetric to the initial one. Hence, the result desired.

31.2.8.4. The maximal difference is $M \leq 1001$, since all the numbers of the form $abcabc$ (let us call them *principal*) are colored blue and the difference between two principal numbers is equal 1001. Actually, $M < 1001$, since for any values of a, b, c there is a blue number z between $x = abcabc$ and the next principal number y (and the case $a = b = c = 9$ is impossible because then there is no principal number succeding x):

a	b	c	z
any	< 9	< 9	$abca(b+1)(c+1)$
any	9	< 9	$a9(c+1)a8c$
< 9	any	9	$ab9(a+1)b8$
9	< 9	9	$9(b+1)08b0$

By the condition for divisibility by 11 all blue numbers are divisible by 11, hence, $M \,\vdots\, 11$ and, therefore, $M \leq 990$. Such a difference is possible: let us show that ther are no blue numbers between blue numbers $x = 908919$ and $y = 909909 = x + 990$. Indeed, assume the contrary: there is a blues z between x and y. Then $x < z < y$ implies $z = 90abcd$ and $a \geq 8$. If $a = 9$, then $b + d = 9 + 9 + c$, implying $b = d = 9$, $c = 0$, i.e., $z = y$. If $a = 8$, then $z > x$ implying $b = 9$, $c > 1$, and, since $9 + 8 + c = 0 + b + d$, then $d = 8 + c > 9$: contradiction.

31.2.8.5. Let at the initial moment Major Pronin can swim to the island, where the spy is, in n days. Then Major Pronin must swim to this island and then to follow the spy's route. Each 13-th day the distance (in the days of travel) between Major Pronin and the spy along this route will diminish by 1 day, therefore, after n-many 13-th days Major Pronin will catch the spy. (We-ell, it will certainly take time ...)

31.2.9.2. If A and B are points on the circle, then AB will denote the clockwise arc from A to B. The arc O_kO_1 will be called a (nondivisible) *piece* if it has no points O_1 in its interior. Observe that if the left endpoints coincide, then so do the right ones and vice versa.

Let a be the angle measure of any of the arcs O_kO_{k+1}. Then the arcs O_kO_1 and $O_{k+r}O_{1+r}$ are of equal length, since the second is obtained from the first one by turning through an angle of ra.

Let point O_1 separates the pieces O_mO_1 and O_1O_n, the point O_{1968} separates O_pO_{1968} and $O_{1968}O_q$. Let us prove that the length of any piece is equal to that of O_mO_1, O_1O_n or O_pO_q. Let $s_0 = O_kO_l$ be a piece; $s_i = O_{k-i}O_{l-i}$. The lengths of all these arcs are equal to that of s_0. Let r be the greatest number such that s_i for $0 \le i \le r$ are all pieces. Then s_{r+1} either does not exist or is not a piece. Three cases are possible:

1) $k - r - 1 = 0$; then for the piece s_r we have $s_r = O_1O_j = O_1O_n$;

2) similarly, $l - r - 1 = 0$; then $s_r = O_mO_1$;

3) point O_t lies inside of s_{r+1}. Then if $t < 1968$, it follows that O_{t+1} lies inside of s_r which is impossible and, therefore, $t = 1968$. Hence, O_{1968} is the only point on s_{r+1}. Therefore, O_{1968} divides s_{r+1} into 2 pieces that coincide with O_pO_{1968} and $O_{1968}O_q$, implying $s_0 = s_r = O_pO_q$.

Remark. The problem is associated with the ergodic theory. A branch of this theory called "shifting" investigates transformations (shiftings) which rearrange a finite number of segments (for the details see the book by Kornfeld I. P., Sinai Ya. G., Fomin S. V. *Ergodic theory*. Birkhäuser, 1987).

Any power T^n of a shifting T of the given r segments is also a shifting, but of a greater number of shorter segments. How many different lengths are there among all of these shorter segments? It has been discovered that their number does not depend on the exponent n but on the original number r of the shifted segments only: it is not greater than $3(r - 1)$.

This statement was obtained by M. Boshernitzan (*A condition for minimal interval exchange maps to be uniquely ergodic*, Rice University, 1983) as a byproduct of the proof of the following very important and interesting mathematical result: *almost all shiftings are ergodic*.

The construction considered in problem 31.2.9.2 corresponds to the shifting of two segments (when you make the first cut with the compass you divide the circle into 2 "segments"); here $n = 1968$, $r = 2$ and the number of different lengths is not greater than $3(r-1) = 3(2-1) = 3$. The above generalization of the statement of Problem 31.2.9.2 has important applications in modern mathematics.

31.2.9.3. To win, the white must move the king along the path

$$a1 - b1 - c1 - \dots$$

until the black king leaves the eighth horizontal (if he does not leave it the white wins with the eighth move), see Fig. 283.

Figure 283. (Sol. 31.2.9.3)

As soon as the black king leaves the eighth horizontal row, the white king must immediately take a position with an even number of horizontal rows and vertical columns between the kings (for example: 1. Ka1–b1, Kh8–h7. 2. Kb1–c2) and then preserve the parity of the distance and move closer to the black king whenever possible. It is easy to see that the kings will eventually assume one of the positions shown in Fig. a), b), c) after a black move. After that the black must move either upwards or to the right and the white king will "take the black king in tow" and win.

31.2.9.4. The following is another solution as compared with the one hinted at in Hints. Consider the following three cases:

1) $n = p$ is a prime;
2) $n = p^k$ is a degree ($k > 1$) of a prime;
3) n is neither of the above.

Let us subdivide these cases depending on whether

a) $a - b$ is not divisible by p or b) $a - b$ is divisible by p.

Thus, we have:

1°a) Then $\dfrac{a^p - b^p}{a - b}$ is divisible by p.

1°b) Then $\dfrac{a^p - b^p}{a - b} = a^{p-1} + a^{p-2}b + \dots + b^{p-1}$ is the sum of summands congruent to each other modulo p (since the difference between two neighboring summands is equal to $a^{p-k-1}b^k(a - b)$ and is divisible by p). Since there are exactly p terms, their sum is congruent to $0 \pmod p$.

2°a) See 1° a).

2°b) Then

$$\frac{a^{p^k} - b^{p^k}}{a - b} = \frac{a^{p^k} - b^{p^k}}{a^{p^{k-1}} - b^{p^{k-1}}} \cdot \frac{a^{p^{k-1}} - b^{p^{k-1}}}{a^{p^{k-2}} - b^{p^{k-2}}} \cdot \dots \cdot \frac{a^{p^2} - b^{p^2}}{a^p - b^p} \cdot \frac{a^p - b^p}{a - b}.$$

Here each factor is divisible by p, therefore, their product is divisible by p^k.

3° Let p^c be one of the factors in the prime factorization of n. Then $\frac{a^n - b^n}{a^{\frac{n}{p^c}} - b^{\frac{n}{p^c}}} \vdots p^c$ (according to 2°) and, of course, $\dfrac{a^n - b^n}{a - b}$ is always divisible by p^c. Hence, $\dfrac{a^n - b^n}{a - b}$ is divisible by n. (Cf. Problem 23.1.10.2).

31.2.9.5. Let us divide the sets of digits into groups separated by zeros to eliminate transfer of a 1 from one group to another; let us operate separately with each group but simultaneously with all of them:

$$\begin{array}{r} + \begin{array}{r} \dots\ \text{(2nd group)0(1st group)} \\ \dots\ \text{(2nd group)0(1st group)} \end{array} \\ \hline \dots\dots\dots\dots\dots\dots\dots \end{array}$$

In 5 moves a group of ≤ 9 digits can be reduced to 1 digit:

$$1)\ \begin{array}{r} + \begin{array}{r} *****\\ **** \end{array} \\ \hline ***** \end{array} \qquad 2)\ \begin{array}{r} + \begin{array}{r} ***\\ ** \end{array} \\ \hline *** \end{array} \qquad 3)\ \begin{array}{r} + \begin{array}{r} **\\ * \end{array} \\ \hline ** \end{array} \qquad 4)\ * + * \leq 18 \qquad 5)\ 1 + * \leq 10.$$

The left-most asterisks in the sums 1)–3) may denote a 10, but this is immaterial because we will delete a 0 anyway.

Among $n \geq 9$ digits we can always select a group of digits whose sum is a multiple of 9; see Problem 12.2.7-8.5. In this way all digits are divided into groups of not more than 9 digits in each so that the sum of digits in all groups but, perhaps, one group is a multiple of 9.

In 5 moves each group is made in a digit and all these digits, except, perhaps, that from the last group are 9's. Now, the problem is solved in 2 moves:

$$99\dots9 + x = \overline{10\dots0(x-1)}, \qquad 1 + (x - 1) = x.$$

Thus, 7 moves suffices.

Remarks. 1) Previously published solutions made it faster than in 15 moves required but slower than our 7. Actually, 4 moves suffice. A scheme of solution is as follows.

1°. Let us divide the number into groups so that the sum of digits in the first group are congruent to 1 modulo 9, that in every other group except the last were divisible by 9; cf. step 3° below. The first group should have 2 to 10 digits; the other groups not more than 9 digits each.

2°. In 4 moves the first group can be reduced to the number 10, 100 or 1000, the other groups can be reduced to one digit (and all these digits, except the last group, are equal to 9). Let us *not* insert zeros between the groups at the 4-th move; so the 1 from the first group is transferred to the second group, etc. As a result, in all decimal places, except the highest, stand zeros.

3°. We must separately consider the numbers that are impossible to divide into groups as indicated in step 1°, i.e., the numbers from which it is impossible to separate a group of digits with the sum congruent to 1 modulo 9. All such numbers are expressed with the help of a certain amount of digits 3, 6, 9 and certain combinations (there are several tens of variants of such combinations) of other digits. In two moves they can be reduced to several 9's and ≤ 2 other digits; two more moves reduce them to one digit.

The estimate "4 moves" is an exact one. For example, it is impossible to reduce the number 111 111 111 to one digit in 3 moves.

2) The minimal irreducible number is 22 223.

31.2.10.1. In the general case, the greatest disc (with radius R) is tangent to three sides of polygon M; the extensions of these sides form triangle ABC all of whose heights h_i are not shorter than the segment to be rotated, i.e., $h_i \geq 1$ $(i = 1, 2, 3)$, see Fig. 284.

Then $S_{ABC} \geq \frac{p}{3}$, where p is a semiperimeter, because

$$S_{ABC} = \frac{a_1 h_1}{2} = \frac{a_2 h_2}{2} = \frac{a_3 h_3}{2} = \frac{1}{3}\sum_{i=1}^{3} \frac{a_i h_i}{2} \geq \frac{1}{3}\sum \frac{a_i}{2} = \frac{p}{3}.$$

On the other hand, $S_{ABC} = Rp$. Therefore, $R \geq \frac{1}{3}$ and $R = \frac{1}{3}$ for an equilateral triangle with height 1. Q.E.D.

Investigate on your own the case when the greatest possible circle is tangent to only two parallel sides of M.

FIGURE 284. (Sol. 31.2.10.1)

FIGURE 285. (Sol. 31.2.10.4)

31.2.10.2. Observe that either all s_i or all t_j occur in Table B. (Indeed, if there are some s_i and t_j that do not enter the table, then the (i, j)-th slot cannot be occupied by the least one of these numbers.) But if all t_j occur in the table they can be enumerated so that $t_1 \leq 91$, $t_2 \leq 92, \dots, t_{10} \leq 100$ (by the hypothesis).

Therefore, the maximum is equal to $91 + 92 + 93 \cdots + 100$ and it can be achieved if $s_1 = 1$, $s_2 = 11$, $s_3 = 21, \dots$, $s_9 = 81$ while $t_1 = 91$, $t_2 = 92, \dots$, $t_{10} = 100$ (clearly, s_{10} can be easily calculated).

31.2.10.3. (Cf. Problem 23.1.10.5.) First, find the solution of the system

$$\begin{cases} x_1 + x_2 + x_3 &= 0, \\ x_1^3 + x_2^3 + x_3^3 &= 1. \end{cases} \tag{$*$}$$

Clearly, $(1, -2, 1)$ is a solution of the first equation; substituting it into the left hand side of the second equation yields -6. Therefore, divide the numbers 1, -2, 1 by $\sqrt[3]{-6}$. We get the solution of system $(*)$:

$$x_1 = x_3 = -\frac{1}{\sqrt[3]{6}}, \quad x_2 = \frac{2}{\sqrt[3]{6}}.$$

Using this result solve the following system

$$\begin{cases} x_1 + x_2 + \dots + x_9 &= 0, \\ x_1^3 + x_2^3 + \dots + x_9^3 &= 0, \\ x_1^5 + x_2^5 + \dots + x_9^5 &= 1. \end{cases}$$

Let the first and the last triples of unknowns coincide with the solution (x_1, x_2, x_3) found above, the second triple of unknowns be obtained from this solution by multiplying it by $-\sqrt[3]{2}$. Then the first two equations are satisfied automatically, and the left hand side of the last one is equal to a number, a. Dividing all 9 unknowns by $\sqrt[5]{a}$ (how to see that $a \neq 0$?); we get the solution of the system.

Further, proceed in a similar way: increase the number of equations by 1 and simultaneously increas the number of unknowns 3 times; all 27 unknowns are divided into 3 groups of 9; the first and third groups coincide with the above solution, and the middle group is obtained by multiplying that solution by $-\sqrt[5]{2}$. Then all equations but the last turn into identities and the left side of the last one is a number, a. Divide all x_i by $\sqrt[7]{a}$; we get the solution of the system.

Continue this process; we get the system of 11 equations in 3^{10} unknowns which has a solution. (Try to figure out how much time it will take to write the complete answer.)

31.2.10.4. Let us chop three angles off triangle ABC with the straight lines parallel to its sides; then let us chop 6 angles off the obtained hexagon in the same way; see Fig. 285; then chop 12 angles off the obtained 12-gon, etc., 19 times.

At the k-th move the vertices of the $3 \cdot 2^{k-1}$-gon chop $3 \cdot 2^{k-1}$ triangles ("of rank k") off the $3 \cdot 2^{k-1}$-gon and we obtain a $3 \cdot 2^k$-gon ($k = 1, \dots, 19$). Any straight line intersects not more than two triangles of every rank and, perhaps, the $3 \cdot 2^{19}$-gon, i.e., all in all it will intersect not more than $2 \cdot 19 + 1 = 39 < 40$ polygons whereas the total number of polygons is equal to

$$1 + 3 + 3 \cdot 2 + 3 \cdot 2^2 + \dots + 3 \cdot 2^{18} = 1 + 3(2^{19} - 1) > 2^{20} = (2^{10})^2 > (10^3)^2. \text{ Q.E.D.}$$

31.2.10.5. Assume the contrary. It is easy to see that either all or none of 8 vertices are marked. Thus, there are 100 or 92 marked points besides the vertices. But for every point not in a vertex it is necessary to add 5 more points if the point is in the center of a face, 11 more points if the point is in the center of an edge, and 23 more points in any other case, so the total number of points is divisible by 6. But neither 100 nor 92 is divisible by 6.

32.1.7.3. Club "Colo-Colo" scored 8 points; hence, that there exist at least 4 teams besides "Colo-Colo", and the total number of teams is not less than 5. But if there were exactly 5 teams, then "Colo-Colo" would have beaten every team, "Dynamo" in particular. Even if "Dynamo" beat the other three teams, still it would have scored only $3 \cdot 2 = 6$ points. This means that there were $N \geq 6$ teams.

But if there are x more teams than we know, i.e., $N = 4 + x$, then as every team got on the average one point after one game, the mean number of points for one team is equal to $N - 1 = 3 + x$. For the four already known winners, the mean number of points is equal to $\frac{8+7+4+4}{4} = 5\frac{3}{4}$, whence $3 + x \leq 5\frac{3}{4}$ and $x \leq 2$ (x is an integer). This means that $N \leq 6$. So $N = 6$.

Prove on your own that there are two more teams that scored 4 and 3 points. For one of various possible examples of the tournament table see Fig. 286.

FIGURE 286. (Sol. 32.1.7.3)

FIGURE 287. (Sol. 32.1.8.4)

32.1.8.4. If BM and AH are a median and a height of the same length, then by dropping perpendicular MK from point M on side BC we get: $MK = \frac{1}{2}AH = \frac{1}{2}BM$. Now from triangle MKB, we deduce: $\angle MBK = 30°$, see Fig. 287 a).

By reflecting symmetrically the vertex B through point M we get a point B' and see that $\angle AB'B = 30°$. The locus of points B' is two arcs of $300°$ each tied by chord AB. Now, the locus of points C is easily found: it is a shift of the locus of points B' by vector $\overrightarrow{AB}$ ($= \overrightarrow{B'C}$), see Fig. 287 b).

32.1.9.1. Denote by x_a the number obtained by subtracting a from every digit of x. If $x - a \neq 0$, then it is obvious that $x_a < x - a \leq (x - a)^2$. By the hypothesis this means that $x = a$, i.e., x is any integer from 1 to 9.

32.1.9.2. Proof by induction on the number of countries. Clearly, one can paint a map of one country. Suppose one can paint any map of $n - 1$ countries. Take a map with n countries and delete from it any country neighboring the outer border. Let us paint the remaining map 3 colours (by inductive hypothesis). Since the country deleted has not more than 2 neighboring countries, there is a color different from that of these countries. Let us apply it to the deleted country.

32.1.9.3. Let $a_1, \dots, a_{50}$ be such numbers. Then $a_1 + a_2 + \dots + a_{34} > 0$ and $a_{31} + a_{32} + a_{33} + a_{34} > 0$. We similarly prove that $a_{35} + a_{36} + a_{37} + a_{38} > 0$, etc. Summing up these inequalities we find that $a_{31} + a_{32} + \dots + a_{50} > 0$, which contradicts the second condition of the problem.

32.1.9.5. Let the wall be a polygon $ABCD \dots E$. Observe that each of the angles $\angle ABC$, $\angle BCD$, and so on, contains inside it the angle of a 37-gon circumscribed around one of the towns, and that is why it is not less than the angle of a regular 37-gon, i.e., $\pi - \frac{2}{37}\pi$. Since the sum of the exterior angles of any polygon is equal to 2π and each exterior angle is not greater than $\frac{2}{37}\pi$, the number of the angles is not less than 37. Q.E.D.

32.1.10.1. The line should be on the plane perpendicular to both hoops and form an angle of $45°$ with the plane of each hoop. The distance to be found is then equal to a half radius of the hoops. (Verify!)

Let us prove that this distance is the least possible one. Indeed, let the particles cross the plane of the first hoop at distance x from its center. The distance from the point where the particles cross the plane to the first hoop is equal to $r - x$, where r is the radius of the hoops and that to the second hoop is $\geq x$ because the second hoop passes through the center of the first one. But $\min(x, r - x)$ cannot be greater than $\frac{1}{2}r$.

32.1.10.2. Let us prove that the sequence $\{a_i\}_{i \in \mathbb{Z}_+}$ is periodic only if $a_1 + a_2 + \dots + a_{100} = 0$. Indeed, if $a_1 + \dots + a_{100} = -A < 0$ and $N \cdot A > a_1$ for some positive integer N, then

$$0 \leq a_1 + \dots + a_{100N} + a_{100N+1} = (a_1 + .. + a_{100N}) + a_1 = -NA + a_1 < 0.$$

Contradiction. Thus, $a_1 + \dots + a_{100} = 0$.

Then $-a_{100} = a_1 + \dots + a_{99} \geq 0$; hence, $|a_{100}| = -a_{100}$. But $B = a_1 + \dots + a_{98} \leq 0$, therefore, $|a_{100}| = a_{99} + B \leq a_{99} = |a_{99}|$. Q.E.D.

32.1.10.3. It suffices to show that it is always possible to shorten the sequence of five cards.

Indeed, out of the four cards on the right only the suits of the first one and the fourth one can coincide, otherwise one of the cards is immediately removed. Let their suits coincide, i.e., let them be positioned in the order $\dots abca$. Then it suffices to put a card of suit c onto the right end to make it possible to remove two cards (c and a).

Consider the case when the suits of the 4 last (right) cards are different and only the first of five suits coincides with one of the last four suits. Then there are two possibilities: $\dots dabcd$ and $\dots cabcd$. In the first case we add cards of suits cbc onto the right and in the second case we add suits bd onto the right.

32.1.10.4. Let us prove that $h = \frac{1969^3}{1968}$ has the needed property. Observe that $\left[\frac{1969^m}{1968}\right] = \frac{1969^m - 1}{1968}$ for any $m > 0$. Indeed, $\frac{1969^m - 1}{1968}$ is an integer since $1969^m - 1 = (1969 - 1)(1969^{m-1} + 1969^{m-2} + \dots + 1)$ is divisible by 1968 and is less than $\frac{1969^m}{1968}$ by $\frac{1}{1968}$ only. This implies that

$$[h \cdot 1968^n] = \frac{1969^{n+3} - 1}{1968}. \tag{$*$}$$

Then $p = \frac{[h \cdot 1968^n]}{[h \cdot 1968^{n-1}]} = \frac{1969^{n+3} - 1}{1969^{n+2} - 1}$. Let us transform p:

$$p = \frac{1969^{n+3} - 1969 + 1969 - 1}{1969^{n+2} - 1} = 1969 + \frac{1968}{1969^{n+2} - 1}.$$

If $n \in \mathbb{N}$, then the last summand is, clearly, < 1; hence, p is not integer and $[h \cdot 1968^n] \not\vdots\, [h \cdot 1968^{n-1}]$, Q.E.D.

32.2.7.2. Observe that almost always $a + 2b < \overline{ab} = 10a + b$, namely, always except for the cases $a = 0$ and $\overline{ab} = 19$. Therefore, any number, except 19 and one-digit numbers, can be diminished by the operation indicated.

Now, let us deal with the exceptional cases:

$$19 \to \overline{(0 + 2 \cdot 1)9} = 29 \to 2 + 2 \cdot 9 = 20 \to 2 \to \text{(see 02)} \to 1$$

and one-digit numbers:

01 $\to 1$;
02 $\to 0 + 2 \cdot 2 = 04 \to 0 + 2 \cdot 4 = 08 \to 0 + 2 \cdot 8 \to 16 \to$ (see 08) $\to 1$;
03 $\to 6 \to 12 \to 5 \to$ (see 05) $\to 10 \to 1$;
04 $\to$ (see 02) $\to 1$;

$$
\begin{array}{ll}
05 & \to 10 \to 1; \\
06 & \to 12 \to (\text{see } 03) \to 1; \\
07 & \to 14 \to 9 \to 18 \to 17 \to 15 \to 11 \to 2 \to (\text{see } 02) \to 1; \\
08 & \to 16 \to 013 \to \overline{(0+2\cdot 1)3} = 23 \to 023 \to \overline{(0+2\cdot 2)3} \\
 & = 43 \to 4 + 2\cdot 3 = 10 \to 1; \\
09 & \to (\text{see } 07) \to 1.
\end{array}
$$

32.2.7.4. In the first move the first player must strike out 9 numbers: 47 to 55. Then (s)he mentally divides the remaining numbers into pairs: $(1, 56)$, $(2, 57)$, ..., $(46, 101)$ in which each second number is by 55 greater than the first one. Further, after the move of the second player, the first must strike out all the pair-less numbers and as many whole pairs as the second player (does not matter which pairs). After any move of the first player only whole pairs will be left; therefore, in the end there will only remain one whole pair with difference 55.

32.2.7.5. To show that it is impossible to find the pearl in 32 cuts notice that no matter how the k cuts are made, it is possible to inscribe not more than $k+1$ circles into the $k+1$ parts, with the sum of the radii of all circles equal to 10 cm.

FIGURE 288. (Sol. 32.2.7.5)

Indeed, let us prove this by induction. For $k = 0$ this is obvious. Let it be true for k cuts, and then let us make one more cut. If the cut does not touch the inscribed circles, then they satisfy the condition; if one of the circles is cut, then let us replace it by two other circles as shown in Fig. 288. The sum of the radii of the circles does not change, and the assertion is proved.

Thus, after 32 cuts it is possible to construct 32 circles with the sum of their radii equal to 10 cm. Then the radius of at least one of them is greater than 3 mm, and if the pearl lies inside this circle, it can never be found.

32.2.8.2. Let a *c-distance* between clean squares be the least number of moves needed for the knight to get from one clean square to the other one stepping only on clean squares. It is clear that at any moment we can daub with glue the square most c-distant from $a1$. Therefore, no matter how the partners play, the player, whose turn is to make a move, can always make it if at least one square is not covered with glue. As there are 63 squares on the chessboard without the square $a1$, the player who begins will always win.

32.2.8.4. Let us use induction and assume that for any $l < 1969$ the number of sequences ending with l is less than l. Consider all sequences ending with 1969 and erase their last terms. Then all these sequences end with one of the numbers $1, 2, 3, \dots, \left[\sqrt{1969}\,\right]$. For each one of the numbers the number of sequences ending with this number is less than $\sqrt{1969}$. Hence, their total number is less than $\sqrt{1969}\cdot\left[\sqrt{1969}\,\right] < 1969$.

32.2.8.5. (See also Problem 32.2.10.5.) Let us begin with the right end. The first cube is a black one. Indeed, if it were a white one, then, by the hypothesis, in any one-cube set there would be not less than one white cube, i.e., all cubes would be white.

Similarly, we can show that the first 4 cubes from the right are black (otherwise among any four cubes there would be one white cube, and the number of white cubes would be not less than 25). The fifth cube is white, otherwise among any five cubes there would be ≤ 1 white ones, i.e., the number of white cubes would not be greater than 20.

Proceeding with the argument, it is easy to prove that there are exactly $[\frac{23}{100}k]$ white cubes among the first k cubes from the right. It follows that the color of every cube is uniquely determined.

32.2.9.1. The winning strategy of the first player: his/her first move is to strike out 27, then unite the numbers k and $25-k$ ($1 \le k \le 12$) in pairs, and organize the numbers 5, 10, 15, 20, 25, 26 in a separate group. If the second player strikes out a number k not divisible by 5, then the first one strikes out $25-k$ (for $k \neq 26$).

If the second player strikes out some other number from the last formed group, then the first one strikes out 26; if the second player strikes out 26, then the first one strikes out 25.

In the end either two numbers $(k, 25-k)$ or two numbers divisible by 5 will be left. The first player wins, having a possibility to make the last move in both cases.

32.2.9.2. **Lemma**. *Let three circles of radius r pass through the common point M and let A, B, C be three other points of their pair-wise intersections. Then*

a) If A and B lie on the same circle, segment AB is parallel and equal to the segment connecting the centers of the other two circles;

b) A, B, C lie on a circle of radius r that does not coincide with any of the given circles.

FIGURE 289. (Sol. 32.2.9.2)

PROOF. Denote by A_1, B_1, C_1 the centers of the given circles on which points A, B, C, respectively, do not lie; let K be the intersection of B_1C_1 with AM; L that of BM with A_1C_1, see Fig. 289. Since AB_1MC_1 is a rhombus (with sides r), we see that K is the midpoint of B_1C_1 and AM; similarly, L is that of BM and A_1C_1. Then KL is the midline of triangles $A_1B_1C_1$ and ABM; hence, $A_1B_1 = 2 \cdot KL = AB$ and $A_1B_1 \parallel KL \parallel AB$.

We similarly prove that $A_1C_1 = AC$ $B_1C_1 = BC$ and, therefore, triangles $\triangle A_1B_1C_1 \equiv \triangle ABC$. But the radius of the circumscribed circle of $\triangle A_1B_1C_1$ is equal to r (since $MA_1 = MB_1 = MC_1 = r$); hence, same is the radius of $\triangle ABC$. □

In what follows instead of "by heading (b) of Lemma applied to circles c_1, c_2, c_3 with common point M" we will briefly write "by (b) for $(M; c_1, c_2, c_3)$" and write similarly for heading (a).

To construct the fourth vertex of the parallelogram with three given vertices A, B, C it suffices to construct 5 circles with the help of the coin (of radius r). First, let us draw two circles c_1 and c_2 through A and B and a circle c_3 through B and C. Denote the center of c_3 by O_3 and the intersection points of c_3 with c_1 and c_2 by E and F, respectively. Let us draw through C and E a circle c_4 distinct from c_3; let us draw through C and F circle c_5 also distinct from c_3; the circles drawn intersect at D. Let us prove that D is the point required.

By (b) for $(C; c_3, c_4, c_5)$ the points F, E and D lie on the circle c_6 of radius r and c_6 intersects c_3. By (b) for $(B; c_1, c_2, c_3)$ the points A, E and F also lie on a circle of radius r not identical to c_3; hence A, E, F and D lie on one circle c_6. Denote its center by O_6.

By (a) for $(E; c_1, c_3, c_6)$ segment AB is parallel and equal to O_6O_3; by (a) for $(F; c_3, c_4, c_6)$ segment CD is parallel and equal to O_6O_3. Hence, segments AB and CD are parallel and equal and, therefore, $ABCD$ is a parallelogram.

32.2.9.3. Given a sequence x of length n, call sequence $\bar{x}$ obtained from x by replacing 0's by 1's and 1's by 0's the *inverse* of x. Denote by $x \cdot y$ the product of sequences x and y place-wise.

Proof of the problem follows from the next three lemmas (prove them on your own).

Lemma 1°. *For any sequence x there is exactly one of two sequences: x or $\bar{x}$ among our 2^{n-1} sequences.*

Indeed, otherwise for an arbitrary sequence y we get $x \cdot \bar{x} \cdot y = 0$, which contradicts the assumption. In particular, the sequence $0 \ldots 0$ does not belong to the given system.

Lemma 2°. *If x and y are from our system, then so is $x \cdot y$.*

Proof is similar to the proof of Lemma 1°.

Lemma 3°. *Let $x_1, x_2, \dots, x_{2^{n-1}}$ be all given sequences. Then their product, i.e., the sequence $X = x_1 \cdot \dots \cdot x_{2^{n-1}}$, consists of one* 1 *and* $n-1$ *zeros.*

Indeed, $X \neq 0$ due to Lemmas 1° and 2°. If X contains at least two 1's, then every sequence x_i contains a 1 in these two orders. And then, putting 0's and 1's arbitrarily in the remaining $n-2$ places, we get only 2^{n-2} distinct sequences, not 2^{n-1} as the condition requires.

Thus, all sequences x_i have a 1 in exactly the same place as X does.

32.2.9.4. It is possible, all right.

There are 200 000 supporters of the President. Let the smallest voting groups have 5 people; then it is necessary for the President's victory to have 3 of his supporters in each group. Thus, 200 000 supporters will yield, if correctly distributed, 66 666 voters "of the first order" out of the total number of 4 million.

Let us divide the 4 million again, into groups of 4 people. It is necessary to have 3 people in each one of them for a majority, then Miraflores will get 22 222 voters of "the second order" out of the total number of 1 million. Then dividing people into groups of either 5 or 4 (the order is of no importance), Miraflores will get:

7 407 votes out of 200 000;	2 469 votes out of 50 000;
823 votes out of 10 000;	274 votes out of 2 000;
91 votes out of 500;	30 votes out of 100;
10 votes out of 25;	3 votes out of 5,

and in this way he will win.

Extension. How could Miraflores win if the number of his supporters were $3^8 \cdot 5^2 = 164\,025$?

32.2.9.5. Circumscribe the circle around the 1000-gon; the vertices of the 1000-gon cut the circle into 1000 equal arcs that measure the lengths of the respective chords. Among the triangle the 1000-gon is divided into, consider a triangle containing the center of the circle. The length of one of the triangle's sides is $\geq \left[\frac{1000}{3}\right] + 1 = 334$.

The longer of the other two sides of an adjoining triangle is not shorter than $\frac{334}{2}$; this side adjoins a triangle with the longer of the other two sides not shorter than $\frac{334}{2^2}$; etc.; the eighth triangle has the longer of the other two sides not shorter than $\frac{334}{2^8} > 1$. It is easy to see that all 8 indicated diagonals have different lengths.

32.2.10.1. The optimal strategy of the first wizard: to strike out the numbers so as the minimal difference between the remaining ones was the greatest possible. If before the move of the first wizard the difference were equal to a, then by renumbering the numbers in the increasing order 1 to 2^{n+1} and crossing out the numbers with even numbers, the first will make the difference $\geq 2a$. Since at first the difference between the neighboring numbers were equal to 1, then after 5 moves he will make in a play the difference will become $\geq 2^5$, i.e., the first will win not less than 32 roubles.

The optimal strategy of the second wizard: to strike out the numbers so as the difference between the greatest and the smallest numbers were the least possible. If before the move of the second wizard the difference were equal to b, then (s)he can siminish it at least twice by renumbering the numbers in the increasing order 1 to $2m+1$ and selecting the m-th number. The absolute value of the difference between it and one of the extremal (first or last) numbers is $\leq \frac{1}{2}b$. If the number selected is closer to the first number, one should strike out m largest numbers, otherwise m smallest numbers. At the beginning the difference between the extremal numbers were equal to 1024; after 5 moves of the second wizard it becomes $\leq \frac{1024}{2^5}$, i.e., the second one will lose not more than 32 roubles.

Thus, if both wizards play optimally, the second wizard will pay the first one exactly 32 roubles.

Extension. Invent an extension of the strategy for the first wizard for the case when the second wizard plays arbitrarily to prove that *the first wizard can always achieve a difference not less than* 32. Invent an extension of the strategy for the second wizard for the case when the first wizard plays arbitrarily to prove that *the second wizard can always achieve a difference not greater than* 32.

32.2.10.2. It is shown on Fig. 290 how to fold the triangle so as to obtain a closed segment passed twice.

Similarly, one easily cooks a hexagon out of a segment. (Make a segment of the hexagon and then unfold the segment in the hexagon in the reverse order.)

FIGURE 290. (Sol. 32.2.10.2)

32.2.10.5. Take k white cubes and $1969 - k$ black ones and arrange them in a row, positioning white ones in places (counting from right to left) with the numbers $\frac{1969}{k}$, $2 \cdot \frac{1969}{k}$, $3 \cdot \frac{1969}{k}, \dots$ (non-integers must be rounded off upwards).

Show on your own that such an arrangement satisfies the condition. Assuming $k = 0, 1, 2, \dots, 1969$, we get 1970 arrangements. Now, show that there exist no other arrangements satisfying the condition; cf. Problem 32.2.8.5.

33.1.7.1. It is impossible since the white checker jumping the black one always covers an even number of verticals, see Fig. 291.

FIGURE 291. (Sol. 33.1.7.1)

FIGURE 292. (Sol. 33.1.7.3)

Therefore, if it jumps one checker, it will always be positioned at a distance of an even number of verticals from the second checker and will never jump it.

33.1.7.2. There are 50 odd numbers among 99 numbers. Each is written twice: from both sides. Altogether there are 100 odd numbers written on 99 cards; hence, by Dirichlet's principle, there is a card with two odd numbers. Then the sum of the numbers on it is even and so is the total product.

33.1.7.3. Reflect point O symmetrically through the sides of triangle $\triangle ABC$ to obtain points O_1, O_2, O_3, see Fig. 292. Let $OA = a$, $OB = b$, $OC = c$. By symmetry, $\angle OAB = \angle O_1AB$; similarly, $\angle OAC = \angle O_1AC$; therefore, in the isosceles triangle AO_1O_3 we have

$$\angle O_1AO_3 = \angle O_1AO + \angle OAO_3 = 2\angle OAB + 2\angle OAC = 120^\circ.$$

Similarly, the angles at the vertices B and C of isosceles triangles BO_1O_2 and CO_2O_3 are also equal to 120°. Therefore, $O_1O_2 = b\sqrt{3}$, $O_2O_3 = c\sqrt{3}$, $O_1O_3 = a\sqrt{3}$. This means that $\triangle O_1O_2O_3$ is similar to the triangle with sides a, b, c and thus has the same angles. But $\angle O_3O_1O_2 = \angle AO_1B - 60^\circ = \gamma - 60^\circ$; similarly, $\angle O_1O_2O_3 = \alpha - 60^\circ$, $\angle O_2O_3O_1 = \beta - 60^\circ$.

33.1.7.4. Let us prove a stronger statement: *If it is possible to divide* $2n$ *weighs into pairs so that in each pair the weighs do not differ more than by* w*, then it is possible to divide them between two pans,* n *weighs on each, so that the weights of the pans will not differ more than by* w.

Let us arbitrarily divide the weighs from the first pair between different pans. The weights on pans differ not more than by w. Further, let us put the weighs from each next pair onto different pans: the lighter weigh on the heavier pan. Suppose the weights on the pans were: a on the light one, b on the heavy one; the light weigh weights p, the heavy one q. Then $w > b - a \geq 0$, $w > q - p \geq 0$. The weights on the pans become $a + q$ and $b + p$. But then

$$(a + q) - (b + p) = (a - b) + (q - p) < 0 + w = w$$

and

$$(b + p) - (a - q) = (b - a) + (p - q) < w + 0 = w,$$

i.e., the difference of the weights on the pans is again less than w. By repeating the operation with the other pairs, we will distribute all the weighs as required.

33.1.8.2. Let L and M be the tangent points of straight lines AE and DE with the circle, see Fig. 293.

Comparing the segments of the tangents, we see that $BK+AL = KC+DM = 1$. Therefore, $EL-BK = AE - 1$ is an integer and $EM - KC$ is also an integer. Since $EL = EM$, it follows that $BK = KC$ is an integer. Since $BK + KC = 1$, we deduce that $BK = KC = \frac{1}{2}$.

33.1.8.3. Consider 4 points A, B, C, D such that a rectangle with the sides passing through these 4 points, contains all remaining points; see Fig. 294. Then the rectangle is red.

FIGURE 293. (Sol. 33.1.8.2) FIGURE 294. (Sol. 33.1.8.3)

33.1.8.4. Let the masses of the weights on the left pan be equal to $a_1, \dots, a_k$, and on the right one to $b_1, \dots, b_k$. Set $\sum a_i = S$, $\sum b_i = T$. By the hypothesis $S > T$ but $S - a_i + b_i \leq T - b_i + a_i$ for any $i = 1, 2, \dots, k$ and, therefore, $0 < S - T < 2(a_i - b_i)$.

Adding up these k inequalities, we get:

$$0 < k(S - T) \leq 2\left(\sum a_i - \sum b_i\right) = 2(S - T);$$

hence, $k \leq 2$.

It is easy to find examples for $k = 1$ and $k = 2$.

33.1.8.5. Let A be the player who lost the largest number of games (if there are several such losers, take any of them). By condition A did not lose all the games, so there exists a player B who lost to A. If A lost in m games and B in n ones, $m \geq n$. Among the other players, A lost m players and B lost to $n - 1$ players (in the n-th game (s)he lost to A). Since $m > n - 1$, among the remaining ones there exists a player C to whom A lost and B did not. Then A lost to C, C lost to B and B lost to A, as was required.

33.1.9.1. Suppose there is a "leftist" among the kings, i.e., the one who chooses left directions more often than right ones. Then it is not difficult to see that any king who lives to the right of the leftist covers a longer way during the year. Hence, the king who does not live on the very edge does not cover the longest way.

33.1.9.2. Let us mark half of black fields as shown below:

	*		*		*		*	
	*		*		*		*	
	*		*		*		*	
	*		*		*		*	

Suppose the wight chip first stood on an unmarked field. Then after each jump over a black chip it occures again on an unmarked field and, therefore, it can only jump chips standing on marked fields. The jumped chips can not stand on the border of the table either, therefore, we have only 9 fields for them to occupy.

We similarly prove that the chip that first occupied an unmarked field can jump not more than 9 chips. In the example on Fig. 295we show how to jump 9 chips.

33.1.9.3. Let us prove that all digits of the number are zeros except, perhaps, the first 50 digits. Indeed, if $x = \overline{ab...p}$ and $y = \overline{b...pq}$ are divisible by 2^{50}, then so is $y - 10x = q - 10^{50}a$. But this is only possible for $q = 0$.

33.1.9.4. Let us draw a circle of the given radius and in it, draw a chord, AK, intercepting an arc of $90°$. On this chord mark the segment AL equal to the bisector, on the contractible (?explain!!) arc of $90°$ mark a point B so that $\angle BLA = 45°$ and draw line BL. Denote by C the other intersection point of this line with the circle.

FIGURE 295. (Sol. 33.1.9.2)

FIGURE 296. (Sol. 33.1.10.2)

Let us verify that $\angle B = \angle C + 90°$ and AL is the bisector of $\angle A$. Indeed, let $\angle C = \alpha$; then $\smile AB = 2\alpha$; hence, $\smile BK = 90° - 2\alpha$ (by construction) and, therefore, $\angle BAK = 45° - \alpha$. From triangles $\triangle ABL$ and $\triangle ACL$ we obtain: $\angle ABC = 180° - 45° - (45° - \alpha) = 90° + \alpha$ and $\angle CAL = 180° - 135° - \alpha = 45° - \alpha$, Q.E.D.

33.1.10.1. The mass total of all weights does not exceed $17 \cdot 70 = 1330$ g, so we can not compose from them more 1330 collections of different mass. Let us show that at least 100 of these collections are impossible to get.

Indeed, let S be the total mass of the 17 heaviest weights. Clearly, if a collection has not more than 17 weights its mass does not exceed S. There are not more than 20 collections of mass $> S$: 19 collections of 18 weights and one collection of all 19 weights.

There are exactly $1330 - S$ integer masses between S (included) and 1330 (excluded). From these we should delete 20 exceptional collections, thus, at least $1330 - S - 20$ masses can not be obtained. But $S \leq 17 \cdot 70 = 1190$, so $N \geq 120$, even more than required.

33.1.10.2. Let MK and PL be these chords, see Fig. 296. Then $PK \cdot PM = (PM')^2 = (MP')^2 = ML \cdot PM$, i.e., $PK = ML$ and, therefore, $PL = MK$.

33.1.10.3. Let us prove that after each move the initial and the obtained numbers are congruent modulo 9. Indeed, let A be the initial k-digit number, b its first digit, C the number obtained from A by striking out its first digit, D the number obtained finally. Then $A = b \cdot 10^{k+1} + C$, $D = C + b$ and we see that $A - D = b(10^{k+1} - 1) = b \cdot 99\dots9 \,\vdots\, 9$. Hence, the remainders after division of A and D by 9 coincide.

The initial number is not divisible by 9, so neither will the 10-digit number obtained from it. However, if all digits of the initial number were distinct, their sum $(0 + 1 + \dots + 9 = 45)$ whould be divisible by 9 togethere with the initial number. Contradiction.

33.1.10.4. On the plane, draw a line so that 100 fixed points would lie on one side of it and another 100 on the other side. Let us start rotating the line counterclockwise about an arbitrary point on the line until one of fixed points occurs on the line. Let us give it number 1. Then let us rotate the line about point 1 until another of fixedr points sits on the line, let its number be 101.

Let us discard these points and repeat the procedure, but call next pairs 2 and 102, etc. In this way for half a turn we enumerate all fixed points. Indeed, if two of the points become missed by the procedure then clearly, they initially lay on one side of the line: otherwise it would have been impossible to rotate the line by 180°. But on each side there were equal amounts of points, contradiction.

All straight lines connecting the chosenpairs of points have distinct directions and, therefore, intersect.

33.1.10.5. 1°. The case when each row and each column host *exactly* one cross is obvious.

2°. Let us successively delete a row and a column without violating the Rule for the remaining table. Let us show that this is always possible except in case 1°.

Indeed, in $m \times l$ table denote by c_i the number of rows that become "cross-free" after the i-th column is deleted and consider the sum $c_1 + c_2 + \dots + c_m$. If there are no zeroes among the c_i (i.e., no column can be deleted), then $c_1 + c_2 + \dots + c_m \geq m$. But, on the other hand, $c_1 + c_2 + \dots + c_m \leq l$, since this is the total number of rows that contain only one cross each.

Therefore, it is always possible to delete a column if there are more columns than rows; if $m = l$, then we still can delete one column if there is at least one row with more than one cross. Q.E.D.

33.2.7.1. There is a test of divisibility of a number by $N(k) = 9...9$ (k many 9's) similar to the test of divisibility by 9: *a number is divisible by $N(k)$ if and only if the sum of its k-digit segments (cut from right to left) is divisible by $N(k)$*. For example, 24 354 is divisible by 99 because $2 + 43 + 54$ is divisible by 99. Prove the validity of this test on your own.

But it is easy to see that if a number has ≤ 8 non-zero digits, then the sum of its k-digit segments cannot be divisible by $N(9) = 999\,999\,999$.

33.2.7.2 and **33.2.8.1.** Consider two diametrically opposite points A and B distinct from the marked points. It is clear that the sum of the distances from points A and B to any of the marked points is greater than 2, see Fig. 297. Therefore, the sum of the distances from one of points A and B to all the marked points is greater than 100.

FIGURE 297. (Sol. 33.2.7.2)

FIGURE 298. (Sol. 33.2.7.3)

33.2.7.3. Father should control pass AB for Kolya not to run through the intersection A or B, see Fig. 298b).

To this end, Father has to divide segment AB into three equal parts by points P and Q as indicated and move so as to be at point P when Kolya is out of the rectangle $P_2Q_2Q_1P_1$ and out of segment P_3A; on segment AP exactly 3 times closer to A than Kolya when Kolya is on any of AP_i; on segment BQ exactly 3 times closer to B than Kolya when Kolya is on any of BQ_i; and, finally, on segment PQ when Kolya is on P_1Q_1 or P_2Q_2, namely, 3 times closer to P than Kolya to P_i, $i = 1, 2, 3$.

Observe that since Father's segments are 3 times shorter than Kolya's, then occupying, with Mother's help, a position on AB corresponding to Kolya's position once, Father only has to move 3 time slower than Kolya to occur at points A, P, Q and B when Kolya is at A, P_i, Q_i and B, respectively.

Now, Mother has to persue Kolya along the roads as on Fig. 298a), where she will easily catch him.

33.2.7.4. The sum of a square's angles is equal to 2π. It is easy to see that every cut increases the total sum of the angles of all parts by 2π. Thus, after k cuts we have $k+1$ polygons, with the sum of their angles equal to $2\pi(k+1)$. The sum of the angles of 73-many 30-gons is equal to $73 \cdot 28\pi$.

The area of each of the rest of the polygons is not smaller than that of any of the triangles, and as the number of them is equal to $(k+1) - 73$, the sum of their angles is $\geq (k-72)\pi$. Thus, $73 \cdot 28\pi + (k-72)\pi \leq 2\pi(k+1)$; hence, $k \geq 73 \cdot 27 - 1 = 1970$.

It is easy to verify that the estimate is an exact one. Cutting a triangle off the angle of the n-gon we make the latter into an $(n+1)$-gon. In this way we can make a 30-gon from a square after 26 cuts and from a triangle after 27 cuts. Let us make the first 30-gon from a square, the other 72 triangles from the triangles obtained as byproduckts of manufacturing the first 30-gon; we need $26 + 72 \cdot 27 = 1970$ cuts altogether.

33.2.7.5. If we arrange the courtiers along the circle in the order that they spy on one another and if we assume that their number is an even n, then we see that a courtier No. $\frac{n}{2}$ spies on the one who spies on courtier No. 1, although this should be courtier No. n. Fig. 299 shows this for $n = 8$.

33.2.8.2. The guards will catch the monkey if they act according to the following plan.

First, they are to occupy the vertices A and B, see Fig. 300, while the monkey is in the lower part of the figure. Now one of the guards has to go down along AC, and the second one is to control the segment AB so that the monkey cannot pass A or B (clearly this is possible). The rest is simple.

33.2.8.4. If we cut the 80-digit number into 38 parts: 34 of 2-digit numbers and 4 of 3-digit ones then their sum is $\leq 34 \cdot 99 + 4 \cdot 999 < 34 \cdot 100 + 4 \cdot 1000 = 7400$. Let us count the number of such cuttings. If we enumerate the parts left to right 1 to 38, then to each cutting a 4-tuple of 3-digit parts corresponds. Conversely, from every 4 numbers we uniquely recover a cutting: we start cutting from the left — 2 digits

Figure 299. (Sol. 33.2.7.5)

Figure 300. (Sol. 33.2.8.2)

at a time for a nonselected number or 3 digits at a time for a selected one. The total number of such tuples is equal to

$$\binom{38}{4} = \frac{38!}{34! \cdot 4!} = \frac{38 \cdot 37 \cdot 36 \cdot 35}{24} = 73815.$$

There are more cuttings than different sums, so there are two cuttings with the same sum.

33.2.9.1. Let the *center of the component* be the center of the circle whose quarter the component constitutes. Let us divide the plane into equal squares with side 20 cm so that the center of the first component would coincide with the center of one of the squares, the beginning point with the center of this square. Then moving from a component to a component we without difficulty establish that the centers of the other components would coincide with the centers of the squares and the end points with the midpoints of the sides. Let us paint the squares as on Fig. 301.

Figure 301. (Sol. 33.2.9.1)

Let us move the train from beginning to the end. Let along the first component the train moves clockwise about the center. Then to exit the first square it has to start moving counterclockwise. Accordingly, to exit the second square it has to change direction again, etc. As a result, in all white squares the train will move clockwise and in all black squares counterclockwise (this will happen even if the train gets into some square again!).

But if the endpoint would coincide with the beginning one, as in the formulation of the problem, the train would have returned to the starting point moving clockwise in a black square, which is impossible.

33.2.9.3. Observe that the order in which we change the signs is not important (check it, for example, with one row or one column). Therefore, we may say that we first change the sign in k rows, and then in l columns. But after that the number of minuses is equal to $100(k+l) - kl$. Equating this to 1970 we get the equation for integers not greater than 100:

$$kl - 100(k+l) + 1970 = 0.$$

It is easy to deduce the answer because the equation can be rewritten as

$$(100-k)(100-l) = 8030 = 2 \cdot 5 \cdot 11 \cdot 73.$$

33.2.9.5. One of the spiders must control the edge AB, and the other one the edge CC', so that the fly cannot pass any of the corners A, B, C, C'; see Fig. 302 a). Then there are no closed ways on the remaining part of the cube (Fig. 302 b)), and so the third spider will catch the fly.

FIGURE 302. (Sol. 33.2.9.5)

FIGURE 303. (Sol. 33.2.10.5)

33.2.10.1. The area of the given sphere is equal to 400π. Therefore, the area of the 19-hedron into which the sphere is inscribed is greater than 400π, and the area of one of its faces is not greater than $\frac{400\pi}{19}$.

Let A be the tangent point of the face to the sphere, and B a point of the face such that $AB > \sqrt{21}$ (such a point exists, otherwise the whole face would be situated inside a circle of radius $\sqrt{21}$ and have an area less than $21\pi < \frac{400\pi}{19}$).

If O is the center of the sphere, then $BO > \sqrt{121} = 11$. Hence, *the length of the segment drawn from B through O to the intersection with the polyhedron is greater than* $11 + 10 = 21$. (Actually, its length is greater than 22, as you can prove on your own.)

33.2.10.2. Let $10\,101\,010\,101 = A$, $K = A \cdot B$. If $K \leq 10^{12}$, then $B \leq 99$, and the statement is obvious.

Let $K > 10^{12}$. Then K consists of $(m+1)$ digits, where $m \geq 12$. Let us transform K into $K' < K$: for this let us separate from it the first significant figure, d, translate it to the right by 12 digits and add to the remaining part: for example,

$$96, 343, 434, 343, 380 \mapsto 63, 434, 343, 433, 380 + 900$$

(we add the first 9 to the third from the left digit). We can express this as follows:

$$K' = K - (10^m - 10^{m-12})d.$$

the expression in parenthesis, $10^1 - 10^{1-12}$, is divisible by $\underbrace{99\dots9}_{12 \text{ nines}}$, hence, $10^1 \,\vdots\, A$ and, therefore, $K \,\vdots\, A$. We may assume proved (induction on K) that K' has at least 6 digits.

Observe that by striking out the first digit, we diminish the number of significant figures by one; adding to the number a 1-digit number (the tail of zeroes does not count), the number of significant figures can not increase by more than 1 (the first 0 on the right may turn into a 1). Therefore the total number of significant figures in K' is not greater than in K; hence, K has not less than 6 significant digits.

33.2.10.3. One of the spiders controlls three edges with a common vertex (the endpoints of these edges included). As a result, the fly can not crawl along any of the closed passes and the other spider will easily catch it moving with however small speed.

33.2.10.4. Among the numbers smaller than 2^n there are numbers of the form $m = k \cdot 2^l$, where $k > 1$ is odd and $l < n$; let t_1 be the total number of them. We have $2^n \not\,\vdots\, m$ for such an m but the remainder after the division of 2^n by m is divisible by 2^l and, therefore, this remainder is $\geq 2^l$. Such numbers to the total sum of remainders S with $S_1 \geq t_1 2^l$.

Let us estimate t_1 from below. As is known, among the numbers $< n$ there is exactly $[\frac{n}{s}]$ divisible by s.

Recall, that $[x]$ here denotes the integer part of x, i.e., satisfies the inequality

$$x - 1 < [x] \leq z. \qquad (*)$$

The numbers m of the form indicated are multiples of 2^l but not of 2^{l+1}; moreover, we have to discard $m = 2^l$; hence, $t_1 = \left[\frac{n}{2^l}\right] - \left[\frac{n}{2^{l+1}}\right] - 1$. Due to $(*)$ we have

$$t_1 > \left[\frac{n}{2^l}\right] - \left[\frac{n}{2^{l+1}}\right] - 2 = t_1 = \left[\frac{n}{2^{l+1}}\right] - 2; \quad S_1 > 2^l\left(\frac{n}{2^{l+1}} - 2\right) = \frac{n}{2} - 2^{l+1}.$$

In particular, taking the summands for $l = 0, 1, 2, 3, 4$ only we get

$$S \geq (\tfrac{n}{2} - 2^1) + (\tfrac{n}{2} - 2^2) + (\tfrac{n}{2} - 2^3) + (\tfrac{n}{2} - 2^4) + (\tfrac{n}{2} - 2^5) = (\tfrac{5n}{2} - 62)$$
$$> 2n + \tfrac{n}{2} - 62 > 2n + \tfrac{1970}{2} - 62 > 2n.$$

33.2.10.5. Clearly, 100 shillings will suffice (you may simply outline the squares along the diagonal of the table).

Let us prove that 99 shillings are not enough. Assume that the magic table is filled with some numbers $x_1, x_2, \dots, x_{100}$ as shown in Fig. 303.

Then whatever square is outlined we can learn only one of the numbers x_i after one question (the one in the last column or in the first row of the outlined square). Therefore, we learn only 99 numbers after 99 questions, and the sum of the numbers along the diagonals remains unknown.

33.D.7.2. The exact answer was unknown at the time of the Olympiad. The solutions with greater number of rugs was given higher score.

It suffices to take all the carpets of perimeter 18. Indeed, if one of the rectangles lies inside of the other, then its perimeter is smaller; hence, the set of all the carpets of the same perimeter serves as a perimeter of a covering consisting of nonintersecting carpets.

It is possible to place $(14 - K) \cdot (14 - L)$ carpets of size $K \times L$ or $L \times K$ (it suffices to look where can the bottom left cell be); hence, the total number of carpets of perimeter $2p \leq 26$ is equal to $13 \cdot (15 - p) + 2 \cdot (16 - p) + \dots + (15 - p) \cdot 13$. It is easy to see that the maximal amount — 680 carpets — is obtained for $p = 9$.

33.D.7.3. To position the domino tiles according to the condition of the problem one must always put an even number beside an odd one. But on the domino tiles there are 32 even numbers and only 24 odd numbers.

33.D.7.4. It is impossible since $1 + 2 + 3 \cdots + 33$ is odd, and the sum of the numbers in every group must be even.

33.D.7.5. It is easy to position at least three switches "up" (first do this with an adjacent pair, then with a pair of diagonal ones).

If the door into the cave is not opened, then the fourth switch is "down", and Ali-Baba must put his fingers inside the holes along a diagonal; if he feels that one of the switches is "down", he must flip it "up" and then he can enter the cave.

If both of the switches are "up", one of them should be turned "down". Now it is clear that two adjacent switches are "up" and two of them are "down".

Next, Ali-Baba puts his fingers in the holes along a side of the square: if both switches are in the same position, he flips both of them and opens the door. If they are in different positions, he flips both of them and then puts his fingers in the holes along a diagonal and flips both switches for the last time.

33.D.7.6. The convex hull of the points at which the cameras are situated is a 1000-gon (otherwise there exists a camera from which one cannot see all of the other cameras). But the sum of all angles of a 1000-gon is equal to $180° \cdot 999 > 179° \cdot 1000$, i.e., not every camera "sees" all other cameras.

34.1.8.1. Let A and B be a pair of the most distant from each other vertices of the 1000-gon (if there are several such pairs, take any of them). Draw the line AB and replace the segment AB by a brocken line passing inside the town. The line l obtained divides the plane into 2 parts I and II, see Fig. 304a).

Let us show that the guards from vertices C and D from different parts do not see each other. Let K be the intersection point of CD and line AB. Suppose K is outside of segment AB, say, on the same side as B. Let us construct $\triangle ACD$, see Fig. 304b). From the choice of A and B we have $AB \geq AC$, but $AK > AB \geq AC$ hence, in $\triangle AKC$, we have $\angle AKC < \angle C$. Similarly, $\angle AKD < \angle D$ and $180° = \angle AKC + \angle AKD < \angle C + \angle D$. Contradiction. Thus, K lies on the brocken line AB, i.e., inside the town and the guards C and D do not see each other.

Points A and B lie on the boundary of the parts, the other 998 vertices lie inside of which k_1 in part I and k_2 in Part II. Let, say, $k_2 \geq k_1$.

If $k_2 > k_1 > 0$, take any guard from part I. Then $k_1 \leq 498$ and the guard sees not more than 499 guards (including A and B and excluding self), as required.

If $k_2 = k_1$ or $k_1 = 0$, select a vertex, E, most distant from line AB (if there are several such points, take any of them). From point E outside the town draw the ray perpendicularly AB and inside the town a brocken line to intersection with l. Let E were of part I. Then the line m drawn divides part i into two: I' and I'', see Fig. 304b). Let us prove that the guards from I' and I'' do notsee each other.

Let F be from I' and G from I'' and M the intersection of segment FG with the boundary of the parts. Point M does not lie on the ray m outside of the town, since M is not further from AB than the endpoints of FG. By the proved before, M does not lie on AB outside of the town either. Hence, M is inside of the town (or on its boundary) and the guards E and G do not see each other. Observe that the proof can be carried on if either $F = A$ or $G = B$.

Now, we have tree parts with 997 vertices inside of them and 3 vertices — A, B, E — on the boundary. If two of the parts are not empty, then in one of them we have $0 < k < 499$ guards and any of them will do. If there are two empty parts, the guard standing on their boundary sees only his/her two neighbors.

34.1.8.2. See the solution to Problem 34.1.10.1.

34.1.8.4. Draw 4 segments connecting every point with 4 neighboring points (3 or 2 segments instead of 4 if the point is situated on a side or in a vertex). It is easy to see that the number of segments drawn from red points is equal to the number of segments drawn from blue points. Now, erase the segments connecting red points with blue ones.

34.1.8.5. **Lemma**. $k(2^n) + k(5^n) = n + 1$.

Proof. Let $k(2^n) = p$ and $k(5^n) = q$. Then $10^{p-1} < 2^n < 10^p$ and $10^{q-1} < 5^n < 10^q$. Multiply these inequalities term-wise. We get: $10^{p-+q-2} < 2^n \cdot 5^n = 10^n = 10^{p+q}$ or, equivalently, $p + q - 2 < n < p + q$, implying $n = p + q - 1$ which is equivalent to the required statement. □

Since $n = 1090701$ is odd, $k(2^n) + k(5^n) = n + 1$ is even; hence, either both $k(2^n)$ and $k(5^n)$ are even or both are odd and $k(2^n) - k(5^n)$ is even.

34.1.9.1. The sum of all numbers on the circle is doubled after each operation. The sum of all original numbers is 1. Therefore, after 100 operations this sum is equal to 2^{100}; hence, one of 25 numbers is not less than

$$\frac{2^{100}}{25} > \frac{10^{30}}{100} = 10^{28} > 10^{20}.$$

34.1.9.2. The sides of polygon M are segments that connect midpoints of the sides of the triangles formed by the adjacent sides and short diagonals of polygon P, see Fig. 305. Therefore, the perimeter of M is equal to a half sum of the lengths of all short diagonals and is greater than the semi-perimeter of the star formed by parts of the short diagonals.

Figure 304. (Sol. 34.1.8.1)

Figure 305. (Sol. 34.1.9.2)

The perimeter of the star is greater than the perimeter of polygon P; hence, the perimeter of polygon M is greater than $2 \cdot \frac{1}{2} = 1$.

34.1.9.3. If n is odd, then the straight lines should go along the sides of a regular n-gon.

If $n = 2^k m$, where $m \geq 3$ is an odd number, then all straight lines are divided into 2^k groups of m straight lines in each, and all straight lines of one group are situated along the sides of a regular m-gon. All these 2^k regular m-gons have a common center at O and are slightly turned with respect to one another so that no two straight lines are parallel.

After a rotation through an angle of $\frac{2}{m}\pi$ about O the system of straight lines turns into itself.

If $n = 2^k$ it is impossible to arrange n straight lines in the required way. Indeed, assume the contrary. It is easy to show that no straight line must pass through O, the center of rotation. Next, determine the angle of a rotation about O when the system of straight lines turns into itself and no parallel straight line exists. The angle α of such a rotation is necessarily equal to $\frac{2}{p}\pi$ for an integer p. Hence, 2^k is divisible by p and, therefore, $p = 2^s$.

Let us rotate the system 2^{s-1} times about O through an angle α each time. The system of straight lines turns into itself. But the total angle of this rotation is 180°; hence, some lines turn into parallel lines. Contradiction.

34.1.9.4. Assume the contrary. Permute the digits of 2^k so as to get 2^n, where $n > k$. Then $2^n - 2^k$ is divisible by 9. Hence, so is $2^{n-k} - 1$. But $2^{n-k} = \frac{2^n}{2^k} \leq 9$, because by permuting digits in a number we cannot make the new, permuted, number *more* than 9 times greater than the old one. Contradiction.

34.1.9.5. Clearly, $2^n\sqrt{2} = \left[2^n\sqrt{2}\right] + a$, where $0 < a < 1$. Now, consider two cases.

a) If $a < \frac{1}{2}$, then $\left[2^{n+1}\sqrt{2}\right]$ is an even number.

b) If $a > \frac{1}{2}$, then $2^{n+1}\sqrt{2} = 2\left[2^n\sqrt{2}\right] + 2a$, where $\left[2^{n+1}\sqrt{2}\right]$ is odd and $\{2a\} < a$. Multiplying $2^n\sqrt{2}$ by successive powers of 2 , we decrease the fractional part of the result every time until it gets less than $\frac{1}{2}$. Now, see heading a).

Another solution. Let us prove that in the sequence there are infinitely many even numbers. Indeed, if this sequence has only odd numbers from some place on, then each of their binary expression ends with a 1. This contradicts the irrationality of $\sqrt{2}$. (Why?)

34.1.10.1. Let the segment A_1A_2 intersect the sphere at points a_1', a_2'', etc., the segment $A_{n-1}A_n$ at points a_{n-1}', a_n'' and the last segment A_nA_1 at points a_n', a_1'', see Fig. 306. We have to prove that equations

$$A_1a' = A_1a_1'', \quad A_2a' = A_2a_2'', \quad \dots, \quad A_{n-1}a_{n-1}' = A_{n-1}a_{n-1}''$$

imply $A_na_n' = A_na_n''$.

Triangles OA_1a_1 and OA_1a_1' are equal in three sides, hence, $\angle Oa_1a_2'' = \angle Oa_1''a_n'$; therefore, in the isosceles triangles $\triangle Oa_1a_2'$ and $\triangle Oa_1''a_n'$ the angles at the bases are equal. Since their lateral sides are equal to the radius of the sphere, these triangles are equal; hence, $a_n'a_1'' = a_1'a_2''$.

Similarly, $a_1'a_2'' = a_2'a_3''$, … , $a_{n-2}'a_{n-1}'' = a_{n-1}'a_n''$ implying $a_n'a_1'' = a_{n-1}'a_n''$.

Figure 306. (Sol. 34.1.10.1)

Figure 307. (Sol. 34.1.10.2)

Apply now the theorem (prove it yourself) that *if a ray beginning at a point X outside a sphere intersects the sphere at two points Y and Z (say, $XY < XZ$), then the product $XY \cdot XZ$ is a constant that only depends on X and does not depend on Y and Z.*

Therefore,

$$A_na_n' \cdot A_na_1'' = A_na_n'' \cdot A_na_{n-1}';$$

therefrom we get

$$A_na_n' \cdot (A_na_n' + a_n'a_1'') = A_na_n'' \cdot (A_na_n'' + a_n''a_{n-1}'). \qquad (*)$$

Assume that, say, $A_na_n' < A_na_n''$. Then the left hand side of equation $(*)$ is smaller than the right hand side. Contradiction.

In the same way we can prove the impossibility of the opposite inequality.

34.1.10.2. The simplest example is shown in Fig. 307.

The principal idea behind the construction of such examples is to take odd numbers of L-shaped tiles. Then if one of the L-shaped tiles is lost, it is possible to divide the others into pairs and to place every pair in a rectangle of size 2×3.

34.1.10.3. Let $y_k = \frac{k}{2} - \frac{1}{4}$; obviously $3y_n - y_{n-1} = n$.

Clearly, $|x_k - y_k| = \frac{1}{3}|x_{k-1} - y_{k-1}|$ and, therefore, $x_k \approx y_k$ for large k with a very good accuracy. In particular, $x_{1971} \approx y_{1971} = 985.250\,000$.

34.1.10.4. **Lemma**. *Let inside $\triangle ABC$ a point M be marked such that the triangles MAB, MAC and MBC are isosceles. Then*

a) M is the center of the circumscribed circle of $\triangle ABC$;

b) $\triangle ABC$ is an acute triangle.

Proof. Among the angles with vertex M there are two non-acute ones (otherwise the measure of the third angle is $> 180°$), say, $\angle AMC$ and $\angle BMC$. But in isosceles triangles $\triangle AMC$ and $\triangle BMC$ only the angle at the vertex can be non-acute, so $AM = MC$ and $CM = MB$; i.e., M is the center of the circumscribed circle. For an obtuse triangle, however, this center lies outside, for a right one on the hypothenuse. Hence, $\triangle ABC$ is an acute one. □

Let us prove by induction on n that $k = 1$.

Base: $n = 3$. Taking any marked point inside the given triangle we get under the conditions of Lemma, hence it is the center of the circumscribed circle that is marked; therefore, all the marked points coinside and $k = 1$.

The inductive step. Let $n \geq 4$ and everything is proved for lesser n. In the n-gon there is an angle $\geq 90°$. Assume the contrary; then the sum of the angles S satisfies $S < n \cdot 90°$. On the other hand, $S = (n-2) \cdot 180°$ and the inequality $(n-2) \cdot 180° < n \cdot 90°$ implies $n < 4$: contradiction.

Let us take a vertex with angle $\geq 90°$ and cut from the n-gon the triangle formed by it and the two neighboring vertices. The triangle is not acute one, hence, by heading (b) of Lemma it has no marked points and all the k points lie in the complementary $(n-1)$-gon. This $(n-1)$-gon also satisfies the conditions of the problem, Q.E.D.

34.1.10.5. The strategy of the first player: as (s)he may take 1, 2, 3, 4 or 5 matches, (s)he leaves the second player a number of matches divisible by 6 whatever the move of the second player. Hence, after finitely many moves the first player leaves exactly 6 matches to the second one. Then after a move of the second player the first one takes the rest of the matches and finishes the game.

34.2.7.2. Rotate square $ABCD$ about its center through an angle of $90°$. Then the rotated square coincides with the original one and the straight lines AH_1, BH_2, CH_3 and DH_4 turn into the straight lines AO, BO, CO, DO, respectively; the latter lines intersect as desired.

34.2.7.3. The number of bacteria k minutes after Time Zero will be $(n-k)2^k$; hence, in n minutes there will be no bacteria left.

34.2.7.4. Let three successive nodes on a horizontal line l be painted colors a, b, c, respectively. (If there is no such node, then line l is the desired one.)

It is easy to see that the three nodes above them are to be painted c, d, a, respectively, and the next three are to be painted a, b, c once again, etc., see Fig. 308.

Figure 308. (Sol. 34.2.7.4)

Now, it is clear that any of the lines I, II, III satisfies the condition of the problem.

34.2.7.5. First, we arrange the searchlights 1, 2, 3 and 4 to satisfy the requirements and then add to them other searchlights (5, 6, 7).

34.2.8.1. Forget for a while that our number consists of 29 digits; and let us find the greatest number of digits that it can have.

Let $k(A)$ be the number of digits of X equal to A. It follows from the hypothesis that if digits B and C are symmetric with respect to the middle of the decimal expression of number X, then $k(B) = C$ and $k(C) = B$. But then these digits are to be situated in symmetric places everywhere; hence, $k(B) = k(C)$ and $B = C$.

Hence, $k(A) = A$ always; the digit situated in the 15-th (middle) place occurs an odd number of times, and all other digits occur an even number of times. But then the number of digits of the number X is not greater than

$$k(2) + k(4) + k(6) + k(8) + k(9) = 2 + 4 + 6 + 8 + 9 = 29.$$

By the hypothesis it is equal to 29. This means that in X there are exactly 2 twos, 4 fours, 6 sixes, 8 eights, and 9 nines. The sum of its digits is equal to $2^2 + 4^2 + 6^2 + 8^2 + 9^2 = 201$.

34.2.8.2. Let us prove that the line can cut the 1000-gon into ≤ 501 part. Let us mark on the line the intersection points with the contour of the 1000-gon (if the whole side lies on the line, we mark the endpoints only and ignore the side in the further arguments). Assign to every point index 1 or 2 — the number of non-excluded sides that contain it. (Clearly, the points of index 2 are the vertices of the 1000-gon.)

Every side has ≤ 1 point; hence, the sum of indices ≤ 1000 (in particular, there are ≤ 1000 points). These points divide the line into two rays and several segments; each segment passes either inside or outside (or along the boundary) of the 1000-gon. Let k be the number of inner segments; then they have $2k$ endpoints. Certain endpoints could have been counted twice, but if a point separates two inner segments, then its index is equal to 2.

Hence, $2k \leq$ the sum of indices ≤ 1000 and, therefore, $k \leq 500$. Let us perform a straight cut along each segment, in turns. A new part (one) can only appear after a cut along an inner segment, so there will appear ≤ 501 part.

On Fig. 309 an example of cutting a 1000-gon into 501 triangle is plotted.

FIGURE 309. (Sol. 34.2.8.2)

FIGURE 310. (Sol. 34.2.9.1)

34.2.8.3 and **34.2.10.5.** Denote the sum of the digits of a natural number X by $S(X)$. Let us prove two lemmas first.

Lemma 1. $S(A + B) \leq S(A) + S(B)$.

PROOF: by induction on $A + B$. For $A + B < 10$ the inequality turns into equality. Observe that if $X = 10 \cdot K + k$, where k is a one-digit integer, then $S(X) = S(K) + k$.

Let a and b be the last digits of A and B, respectively; then $A = 10 \cdot M + a$ and $B = 10 \cdot N + b$, where M and N are integers ≥ 0 and $S(A) + S(B) = S(M) + S(N) + a + b$.

Consider two cases:

a) $a + b = c < 10$. Then $M + N < A + B$, hence,

$$S(A + B) = S(10(M + N) + c) = S(M + N) + c \leq S(M) + S(N) + a + b = S(A) + S(B).$$

b) $a + b > 10$, then $a + b = 10 + c$, where $c < 10$. Since $M + N + 1 < A + B$, it follows that

$$S(A + B) = S(10(M + N + 1) + c) = S(M + N + 1) + c \leq$$
$$S(M) + S(N) + 1 + (a + b - 10) = S(A) + S(B) - 9 < S(A) + S(B).$$

Lemma 1 is proved. □

Lemma 2. $S(AB) \le S(A) \cdot S(B)$.

PROOF. If A is a 1-digit number, then $S(A) = A$ and the inequality $S(AB) \le A \cdot S(B)$ follows easily by replacing the products AB and $A \cdot S(B)$ with the sums of A summands.

Let $A = a_n 10^n + a_{n-1}10^{n-1} + \dots + a_0$, where $a_n, \dots a_0$ are all digits of A. Then

$$\begin{aligned} S(AB) &= S(Ba_n 10^n + Ba_{n-1}10^{n-1} + \dots + Ba_0) \\ &\le S(Ba_n 10^n) + S(Ba_{n-1}10^{n-1}) + \dots + S(Ba_0) \\ &\le S(B)S(a_n 10^n) + S(B)S(a_{n-1}10^{n-1}) + \dots + S(B)S(a_0) \\ &= S(B)(a_n + a_{n-1} + \dots + a_0) = S(A)S(B). \end{aligned}$$

Lemma 2 is proved. □

The estimates required follow from Lemmas. Indeed,

Problem 8.3: $S(K) = S(1000K) = S(125 \cdot 8K) \le S(125)S(8K) = 8S(8K)$,
Problem 10.5: $S(N) = S(10^5 N) = S(2^5 \cdot 5^5 N) \le S(32)S(5^5 N) = 5S(5^5 N)$.

These estimates are unimprovable, as $S(125) = 8S(1000)$ and $S(32) = 5S(10^5)$.

34.2.8.4. Let us perform the opposite operations. Take any n. If n ends with 1, 2, 3, 5, 6, 7, 8 or 9, then multiply n by 4, 5, 8, 2, 4, 2, 5, or 6, respectively. We get a number $M < 10n$ that ends with a 0 or 4, and since we are allowed to get rid of the last digit, we get $M_1 < M$.

Prove on your own that the obtained number M_1 is indeed less than n.

As a result of several similar operations we obtain a one-digit number, either 0 or 4. With this number we begin the direct procedure.

34.2.9.1. Choose the vertex A and its left and right neighboring vertices B, C and B', C', see Fig. 310.

The sum of the angles of every triangle $C'AB$, CAB', $B'BC$, $B'BC'$, is equal to 180°:

$$\angle 4 + (k+1)\angle 1 + 2\angle 2 = 180^\circ; \qquad \angle 5 + (k+1)\angle 2 + 2\angle 3 = 180^\circ;$$
$$\angle 3 + (k+1)\angle 1 + 2\angle 5 = 180^\circ; \angle 2 + (k+1)\angle 5 + 2\angle 4 = 180^\circ.$$

It follows:

$$\angle 4 - \angle 3 = 2(\angle 5 - \angle 2) = 4(\angle 4 - \angle 3), \quad \angle 5 - \angle 2 = 2(\angle 4 - \angle 3) = 4(\angle 5 - \angle 2),$$

whence $\angle 3 = \angle 4$ and $\angle 2 = \angle 5$.

Similarly, we can show that for any triangle constructed from two neighboring sides and a short diagonal of the polygon the angles at the base of the short diagonal are equal; hence, so are the lateral sides. This means that the polygon is a regular one.

34.2.9.3. Let us assume that the side of the square is equal to 1. Take four distinct primes p, q, r, s, each greater than 2^{101}, and divide the side of the square into segments of length $\frac{a_1}{p}$, $\frac{a_2}{p}, \dots, \frac{a_{100}}{p}$, where $a_i \ge 2a_{i-1}$ and $a_1 + \dots + a_{100} = p$. Similarly, replace p with q, r, s, respectively, and divide the other three sides.

Let us prove that from the obtained segments one cannot compose another rectangle. Let this be not so. Then the length of the shorter side of the rectangle is a rational number $\frac{n}{p \cdot q \cdot r \cdot s} < 1$. It is easy to show that this number can be represented in no more than one way in the form

$$\frac{a}{p} + \frac{b}{q} + \frac{c}{r} + \frac{d}{s},$$

where all numerators are nonnegative. Each of the numerators, in its turn, can be represented in no more than one way as the sum of the lengths of the segments (since each the lengths of each segment a_i is longer than the sum of the lengths of all the segments that follow it). Therefore, it is impossible to compose from our segments *two* sides of the rectangle. Contradiction, Q.E.D.

34.2.9.4 and **34.2.10.1.** As $\sum\limits_{i<j} x_i x_j = 0$ and $\sum\limits_i x_i^2 = n$ by the hypothesis, then

$$n = \sum_i x_i^2 + 2\sum_{i<j} x_i x_j = (\sum_i x_i)^2.$$

EXTENSION. *What is n equal to*? It is possible to prove that n is divisible by 4 (see Problem 22.2.7.5) and that $n \ne 16$. We do not know whether there exists $n \ne 4$ that will do.

34.2.9.5. Through the center of the n-gon draw a straight line neither parallel to any of the polygon's sides nor passing through any of its vertices. Take this straight line for the coordinate x-axis; at every vertex of the polygon write a number x_i equal to the distance from the vertex to the x-axis, with the corresponding sign. The numbers x_i are distinct; none of them is equal to 0, and the sum of the numbers written at the vertices of any regular k-gon is equal to 0 as the second coordinates of vectors whose sum is equal to 0, cf. Lemma from Problem ???.

34.2.10.2. We can simultaneously interchange the numbers $\{a_i\}_{i=1}^n$ with $\{b_i\}_{i=1}^n$ in order to make $\{b_i\}_{i=1}^n$ a decreasing sequence; this operation permutes the sums $a_i + b_j$ but does not, clearly, affect their maximum. Therefore, we can assume from the beginning that $b_1 \geq b_2 \geq \dots \geq b_n$; let us prove that the greatest sum $b_k + a_{i_k}$ is of the smallest value when the numbers $\{a_{i_k}\}_{k=1}^n$ are arranged in the increasing order.

If for some permutation of the numbers $\{a_{i_k}\}_{k=1}^n$ this maximum is attained at the sum $b_s + a_{i_s}$ and $a_{i_s} \geq a_{i_l}$ for some $l > s$, then *transpose* a_{i_s} and a_{i_l}. Now, both sums $b_s + a_{i_l}$ and $b_l + a_{i_s}$ are not greater than $b_s + a_{i_s}$, i.e., the maximum of the sums $b_k + a_{i_k}$ does not increase under the transposition.

Choosing this new maximum and repeating the same argument we obtain the required arrangement of the numbers $\{a_i\}_{i=1}^n$.

34.2.10.3. First, Gambler has to double the number A_1 and to consider the answer $2A_1 = \overline{\alpha_1 ... \alpha_k ... \alpha_n}$, where α_k is the middle digit, as the sum $x_1 + y_1$, where $x_1 = \overline{\alpha_1 ... \alpha_k 0 ... 0}$ and $y_1 = \overline{\alpha_{k+1} ... \alpha_n}$. Then (s)he has to name the number $B_1 = \dfrac{x_1 - y_1}{2}$.

Next, $A_1 + B_1 = x_1$ and $A_1 - B_1 = y_1$. Thus, both obtained numbers (x_1, y_1) have approximately half the number of digits as compared with (A_1, B_1) not counting zeros at the end of the number. In the same way Gambler gets the numbers (x_2, y_2), (x_3, y_3), etc., till the moment when (s)he gets a one-digit number.

One can turn a one-digit number into 1 or 10 after at most 5 moves (do it yourself). And so Gambler needs not more than $(\log 1000 + 3) + 5 \leq 18$ moves.

34.2.10.4. Let us connect all straight lines by pairwise perpendicular line segments and take a sphere of radius r containing all these segments inside it; let S be its center. If the angle between two straight lines is of measure α, and the projection of point A lying on one of the lines on the other line is point A', then

$$SA' \leq r + (SA + r)\cos\alpha.$$

This implies that if we take a sphere of radius $R \geq \dfrac{1+c}{1-c} r$, where c is the maximal value of the cosine of the angles between lines, then the projection of a point that lies on a straight line inside the sphere is a point inside the sphere.

35.1.7.1. Assume that not all numbers are equal; say, $a_1 > a_2$. Let us make use of the fact that *if the powers are equal and the bases are different, then the greater exponent corresponds to the lesser base*. Assume that $a_1 > a_2$. As $a_1^{a_2} = a_2^{a_3}$, it follows that $a_2 < a_3$; hence, $a_1 > a_2 \Rightarrow a_2 < a_3$. Analogously we derive:

$$a_1 > a_2 \Rightarrow a_2 < a_3 \Rightarrow a_3 > a_4 \Rightarrow \dots \Rightarrow a_{16} < a_{17} \Rightarrow a_{17} > a_1 \Rightarrow a_1 < a_2.$$

Contradiction.

Remark. If instead of 17 integers a_i there were an even number of them, then the assertion of the problem would have been false. Give a counter-example.

35.1.7.2. Of any three delegates, there are two who can always speak with one another. Dismiss them and go ahead with the procedure for the remaining delegates. After 498 such operations there are only 4 delegates left, call them A, B, C, D. If any two of them can speak with each other, let us didtribute them in pairs at random. If not, there exist two, say, A and B, which can not speak. This means that C can speak with both A and B: otherwise the triple A, B, C could not communicate; similarly, D can speak with A and B. So, let A share room with C and let B share room with D.

35.1.7.3. As 13 is an odd number, some two neighboring vertices are of the same color. Let us number the vertices clockwise 1 to 13. Let the chosen vertices of the same color have numbers 2 and 3. Then some three of the vertices 1, 2, 3, 4 and 9 are the vertices of an isosceles triangle and have the same color.

35.1.8.1. Induction on n. The base for $n = 1$ is obvious. Suppose the statement holds for $k = n - 1$. In an $n \times n$ table T select a column with one asterisk; mark the column and the row with the asterisk. Observe that the intersection of any nonmarked row with the marked column is an empty cell. For an $(n-1) \times (n-1)$ table T', obtained from T by striking out the marked column and row, the conditions of the problem hold, hence, holds the statement. Suppose now, several columns are striken out from T; let us strike them out of T' as well. Two cases are possible:

a) Not everything is stricken out of T'. Then by hypothesis there is a row in T' with one non-stricken asterisk. Take the same row in T. Only one new element can be added to it — the intersection with the marked column, which is not an asterisk. Thus, we have found a row desired; it is not marked.

b) Everything is stricken out of T'. Then, in T, all columns are stricken out, bar the marked one and it suffices to take the marked row.

35.1.8.2. Clearly, X_1, X_2, X_3, ... , X_{n-1} all lie on the same circle (AA' is a chord of the circle), as the angles between the corresponding straight lines are equal, see Fig. 311.

FIGURE 311. (Sol. 35.1.8.2)

FIGURE 312. (Sol. 35.1.9.1)

35.1.8.3. Let us divide the coins into two groups of 500 coins each and compare their weights (*the first weighing*). Two cases are possible.

A) One of the pans (say, the right one) is heavier. This means that there are counterfeit coins (one or two), and all of them are in one pan (were they in different pans, the balance would be in equilibrium).

Divide now the heavier group into two groups of 250 coins each and compare them (*the second weighing*).

If one of the pans is heavier, then the counterfeit coins are among these 500, and they are heavier; in this case a third weighing is not necessary.

If the pans are in equilibrium, then either the counterfeit coins are among the 500 left (and they are lighter) or they are divided: one of them is in every group of 250. To learn which is the case divide one of the groups of 250 coins into two parts and compare them (*the third weighing*).

If the balance is in equilibrium, we have the first case, if it is not, we have the second case.

B) The first weighing shows that the pans are in equilibrium. This means that we have either 0 or 2 counterfeit coins (one in every pan). Once again divide one of the groups into two of 250 coins each (*the second weighing*).

If the pans are in equilibrium, each pan has an even number of counterfeit coins. But as there is not more than one in every pan, this means that we have none at all. If one of the pans is heavier, then we have two counterfeit coins and exactly one of them is in one of the pans. Dividing the heavier group into two parts and weighing them for the third time we learn whether the counterfeit coin is heavier or lighter than a genuine one.

REMARKS. 1) It is essential for the solution that the total number of coins be divisible by 8; otherwise three weighings will not suffice.

2) If we wanted to know how many counterfeit coins are there, then we would have needed at least 8 weighings.

3) It is easy to see that we could make the three weighings described above without paying attention to which pan is heavier: first, we could put into each pan 500 coins, then half that many, and, lastly, half of this half. We dealt with the heavier group for clarity only.

35.1.8.4. It is clear that 50 numbers $(2, 2, \ldots, 2)$ satisfy the condition of the problem. On the other hand, 49 numbers (or fewer) can be divided into 7 groups of 7 numbers in each group; the sum of the numbers in every group would not be greater than 14, and the total sum $S \leq 7 \cdot 14 = 98$. Hence, the answer.

35.1.8.5. The triangles $\triangle ADC$ and $\triangle BEC$ are similar because two corresponding angles are equal. If a is the coefficient of similarity, then $AC = a \cdot BC$ and $\frac{1}{2}BC = DC = a \cdot EC = a \cdot \frac{1}{2}AC$, whence $a = 1$ and $AC = BC$. Then, in triangle BEC, the side EC opposite the angle of $30°$, is equal to a half of its base BC. Hence, $\angle BEC = 90°$ and $\angle C = 60°$.

35.1.9.1. Let us prove a stronger assertion: *five points K, E, H, M, C lie on the same circle*, see Fig. 312.

As $KCHE$ and $CMHE$ are isosceles trapezoids, each of them is inscribed in a certain circle. The points C, E and H, which are situated on some third circle, are also situated on the first two. This means that all three circles coincide.

35.1.9.2. **Lemma**: *The identity $a_{11} + a_{km} = a_{k1} + a_{1m}$ holds for any k and m.*

PROOF. If $k = 1$ or $m = 1$, the identity is clear. Otherwise let us place the rooks so as to satisfy the requirements and that two of them would occupy squares $(1, 1)$ and (k, m). Such an arrangement exists: e.g., place the rooks at squares $(1, 1)$, $(2, 2)$, etc., (n, n). If $k = m$, then the needed squares are occupied, if not let us place the (k, k)-th rook on the (k, m)-th place and (m, m)-th rook on the (m, k)-th place.

Let us fix the arrangement obtained; let us finally get from it one more arrangement sending the $(1, 1)$-th rook on the $(1, k)$-th place and (k, m)-th rook on the $(1, m)$-th place. Since the sum of the numbers under all the rooks did not vary, neither did that under the rooks that changed positions in the final move, just the statement of the lemma.

Lemma implies that that $a_{km} = a_{k1} + (a_{1m} - a_{11})$ for any k and m. Now, it suffices to take $x_i = a_{i1}$ and $y_i = a_{1i} - a_{11}$ for all $i = 1, 2, \ldots, n$.

35.1.9.3. Plot each tree by a dot on a map. Arrange the heights of the trees in decreasing order and in the same order connect the trunks of the trees — these dots — by segments. The length of the obtained broken line (which may be self-intersecting) is not greater than the sum of the differences of the heights of neighboring trees, and so not greater than the height of the highest tree, see Fig. 313.

FIGURE 313. (Sol. 35.1.9.3)

Hence, it suffices to fence in the obtained broken line from two sides, and the length of the fence does not exceed $100 + 100 = 200$ m.

35.1.9.4. Let us prove that by *discarding from every sum the last summands* (i.e., the numbers m and n) *we get equal sums S_1 and S_2.*

Consider all integer points in the $m \times n$ rectangle, $1 \le x \le m-1$, $1 \le y \le n-1$. There are $(m-1)(n-1)$ such points, exactly half of them lying above the diagonal passing through $(0, 0)$, and half of them below it (no integer point lies on the diagonal since m and n are relatively prime). But the number of integer points with abscissa k is equal to $\left[k \cdot \frac{m}{n}\right]$. This means that $S_1 = S_2 = \frac{(m-1)(n-1)}{2}$.

35.1.9.5. Let us number the avenues 1 to 10 from top to bottom (on the map). Let the first cop be on the first avenue and the second one on the second avenue, and let us assume that the robber is in the strip between these avenues. Then the first cop will find the robber passing not more than 100 blocks along his/her avenue.

If the cop does not find the robber, then the robber is not between these avenues but below them, and the cops can move down to the next pair of avenues. Patrolling now the second and the third avenues the cops ensure that the robber will not get to the first avenue.

Acting as before and moving down every time (if necessary) the cops will find the robber.

35.1.10.1. All inhabitants are divided into two nonintersecting sets (club), any two members of the same set are friends and any two members of distinct sets are enemies. Then "turning a new leaf" places every inhabitant from one set into another set. So it suffices for all citizens from the second set to "turn a new leaf".

35.1.10.2. **Lemma**. *If p is relatively prime with 10, then among the numbers $3, 33, \dots, \underbrace{33...3}_{p \text{ times}}$ there is a number divisible by p.*

Proof. Assume the contrary. Then we have p numbers with $p-1$ nonzero residues after division by p. By Dirichlet's principle, there are two numbers with equal residues. Their difference is of the form $\underbrace{3\dots3}_{r \text{ times}}0\dots0$ and is divisible by p and, since p is relatively prime with 10, we see that $\underbrace{3\dots3}_{r \text{ times}} \vdots\, p$. Q.E.D.

Assume that there is at most one composite number. Consequently, all numbers beginning with some a are primes. What digits must be added to the number a?

To add even digits or a digit 5 is not allowed, and to add a 1 or a 7 is allowed not more than once, since adding these digits we increase by 1 the remainder after division by 3. This means that, beginning with some place, we can only add a digit 3.

Let p be a prime in this series and 3 its last digit. Then p is not divisible by 2 nor is it divisible by 5 but by Lemma among the numbers $3, 33, \dots, \underbrace{33...3}_{p \text{ times}}$ there is a number divisible by p.

35.1.10.3. Obviously, all of the tetrahedron's faces are equal; let S be the area of a face. Assume that one of the faces is the base and project the other faces on it. They will cover it; hence,

$$S = S\cos\alpha + S\cos\beta + S\cos\gamma \iff \cos\alpha + \cos\beta + \cos\gamma = 1,$$

where α, β, γ are the dihedral angles at the base. By adding 4 such equations for each face we see that the doubled sum of cosines of all dihedral angles is equal to 4, hence, the sum itself is equal to 2.

35.2.7.1. Prove first that point O is the midpoint of the diagonals. If a, b, c, d are the lengths of four segments OA, OB, OC, OD and $c \geq a$, $d \geq b$, then marking the segment OM of length a on the segment OC, and the segment ON of length b on the segment OD, we get parallelogram $ABMN$, divided into 4 triangles of the same perimeter. Contradiction.

Hence, $ABCD$ is a parallelogram and comparing the perimeters of two adjacent triangles we find that, moreover, it is a rhombus.

35.2.7.2. Consider the triangle formed by lines b, c, and d. Let straight line a through a vertex of this triangle be parallel to median PM. Consider two cases.

1) The midpoint M does not lie on b. Denote by Q the intersection point of C with D; let T_1 be the triangle with sides PM, S and d; let T_2 be the triangle with sides a, c and d. The respective sides of T_1 and T_2 are parallel, hence the triangles are similar and homothetic with center at the common vertex Q. In triangle T_1 draw median m from vertex M. In triangle T_2, median m_1 is parallel and homothetic to m. On the other hand, m is the midline in the triangle with sides b, S, d; hence, $m \parallel b$ and, therefore, $m_1 \parallel b$.

2) The midpoint M lies on b. Let us draw the straight line a' parallel to PM through the intersection point of b with d. In the triangle with sides b, c and a' we have: $a' \parallel PM$ and M is the midpoint of the other side. Hence, PM is the midline and P is the midpoint of the third side. Let us consider one more midline, PR, in this triangle. Clearly, $PR \parallel b$. Let T_1 be the triangle with sides d, c and a'; let T_2 be the triangle with sides a, c and d. Since $a \parallel a'$, triangles T_1 and T_2 are homothetic with center at P and PM is identical with a median in triangle T_2.

35.2.7.4. Assume that the knight travels long enough *to take a certain road not less than* 6 *times.* Then he took it at least three times in the same direction; therefore, he took one of the two roads to which he may turn after this one not less than twice in the same direction. But then his next way coincides with the previous one (before he took this road) and passes through the same castle he left first.

35.2.7.5. Let $MB = a$, $BK = b$, $KC = c$, $CA = d$, $AN = e$. Since $S_{BMK} > S_{MKC}$ and $S_{BMK} > S_{AMK}$, it follows that $b > c$ and $a > e$. Assume that $\frac{a+b}{c+d+e} < \frac{1}{3}$, i.e.,

$$3a + 3b < c + d + e < a + b + d.$$

Then $2a + 2b < d$; hence, $a + e + b + c < 2a + 2b < d$ and $AB + BC < AC$. This contradicts the triangle's inequality.

35.2.8.2.

$$\begin{aligned} d = d(af - be) = adf - bed = (adf - bcf) + (bcf - bed) \\ = f(ad - bc) + b(cf - ed) \geq f \cdot 1 + b \cdot 1 = f + b, \quad \text{Q.E.D.} \end{aligned}$$

35.2.8.3. Use the induction on the number n of road intersections. The base of the induction: $n = 1$. In this case all roads are circular and meet at one point; the solution is obvious.

Inductive hypothesis: let the statement hold for n. Consider the $(n+1)$-st intersection C. It is connected with two intersections A and B considered in the inductive hypothesis. One can get from A to B along the net of streets and then back to A through some intermediate intersections. Contract the obtained ring into one point and introduce one-way traffic in the obtained town as follows: one can get from C to A, from A to B (the one-way traffic for n intersections) and from B to C.

The traffic in the original town is uniquely recovered after we "zoom" the point back into the original ring $A \to B \to A$.

35.2.8.4. Consider all fractions, both reducible and irreducible, whose numerators and denominators are not greater than 100; there are 100^2 of them. If $GCD(a, b) = n$, then the fraction $\frac{a}{b}$ is divisible by n, and the irreducible fraction $\frac{a_1}{b_1}$ obtained after simplification is such that $a_1 \leq \dfrac{100}{n}$ and $b_1 \leq \dfrac{100}{n}$, i.e., it is accounted in the number $I\left(\dfrac{100}{n}\right)$ of irreducible fractions.

Hence, *the sum to be found is equal to the number of all fractions considered at the beginning of the proof*, 10 000.

35.2.9.1. Let angle $\angle C$ in pentagon $ABCDE$ be non-obtuse and let the lengths of all sides of the pentagon be equal to 1. Draw diagonals AC and CE which divide the pentagon into 3 triangles, two of them — $\triangle ABC$ and $\triangle CDE$ — isosceles, with vertex angles less than $120°$ and base angles greater than $30°$, see Fig. 314.

FIGURE 314. (Sol. 35.2.9.1)

Consequently, the third triangle ACE is acute, the angle at its vertex, $\angle C$, is less than $90° - 2\cdot 30° = 30°$, and the lengths of the sides of triangle ACE (these sides are bases of triangles ABC and CDE) are less than $\sqrt{3}$. The other angles of the triangle measure $60°$ to $90°$. By the law of sines

$$AE = AC \cdot \frac{\sin\angle ACE}{\sin\angle AEC} \leq \frac{\sin 30°}{\sin 60°} \cdot \sqrt{3} = 1.$$

Contradiction. Q.E.D.

35.2.9.4. Set $R = \sqrt{2}$. Observe that if a number is of the form $a + bR$ for rational a and b, then such a representation is unique. Indeed, let $a + bR = c + dR$ for rational a, b, c, d. Then if $b = d$, then $a = c$; otherwise $R = \frac{a-c}{d-b}$ is rational, which is false (see *Prerequisites*).

For every $x = a + bR$ let $C(x) = a - bR$ be its *conjugate*. It is subject to straightforward verification that conjugation satisfies:

$$C(x + y) = C(x) + C(y); \quad C(xy) = C(x)C(y); \quad C(x^n) = C(x)^n \text{ for } n \in \mathbb{N}.$$

Now, apply C to the equation of the problem. We get

$$(a - bR)^{2n} + (c - dR)^{2n} = 5 - 4R.$$

The left hand side is positive while the right hand side is negative: contradiction.

35.2.9.5. a) Consider straight line a and the nearest to a point A at which the two other lines, b and c, meet.

No other straight line intersects the triangle constructed by the straight lines a, b, c: otherwise there would have existed another point which would be closer to a than A. This means that a triangle is adjacent to each of the straight lines, and as each of the triangles is considered three times when the total number is counted, there are not less than $3000 \cdot \frac{1}{3} = 1000$ of them.

b) *There are not more than two straight lines such that all intersection points of the other lines are situated on one side of each of them*, see Fig. 315.

FIGURE 315. (Sol. 35.2.9.5)

Indeed, if there would have been more than two such straight lines, we could have considered three of them. These three lines divide the plane into 7 parts, and all intersection points of the other straight lines must be located only in one of these 7 parts, namely, in the triangle because it is only the triangle that contains all three intersection points of the given lines. This, however, is impossible because the fourth line intersects the extension of one of the sides of this triangle. Therefore, $3000 - 2 = 2998$ straight lines have one more triangle in addition to those from the solution of a), i.e., there are $2998 + 3000$ triangles. But every one of them is counted three times, whence $N \geq 2 \cdot \frac{3000}{3} - \frac{2}{3}$. Since N is integer, $N \geq 2000$.

ANOTHER SOLUTION of b): We give an *exact* lower estimate of the number of the triangles: *For n lines in the general position, the number Δ_n of the triangles is $\geq n - 2$.*

PROOF. Assume the contrary and let $\Delta_n \leq n - 3$.

Fix two arbitrary lines, l_1 and l_2. Start moving all $n - 2$ remaining lines $l_3, l_4, \dots, l_n$ at once in parallel with themselves at some constant rates (v_i is the speed of the i-th line) in such a manner that $v_1 = v_2 = 0$ and not all of the speeds $v_3, v_4, \dots, v_n$ are equal to 0. This means that we consider a one-parameter family of straight lines, $l_i(t)$, whose equations in a Cartesian coordinate system are $a_i x + b_i y + (c_i + v_i t) = 0$, where each original line $l_i = l_i(0)$ is given by equation $a_i x + b_i y + c_i = 0$. Select the speeds v_i so that *the dimensions of all triangles remain the same* at all times. Why can we find such speeds?

Watch the movement of three lines, l_i, l_j, l_k, that compose a fixed triangle $\triangle$. It is transformed into a similar triangle $\triangle(t)$ as the lines move. For $\triangle$ to be independent of t, the rates v_i, v_j, v_k should be constrained by a *linear* (explain on your own: why) relation

$$\lambda_i v_i + \lambda_j v_j + \lambda_k v_k = 0 \qquad (*)$$

with uniquely determined coefficients λ_i, λ_j, λ_k.

Since by the hypothesis the total of all triangles, Δ_n, does not exceed $n - 3$, we obtain a system of $\leq n - 3$ homogeneous equations for $n - 2$ unknowns $v_3, \dots, v_n$ by imposing all linear relations $(*)$ between the rates of $l_3, \dots, l_n$. This system has a non-zero solution $(v_3^*, \dots, v_n^*)$, since the number of the equations is strictly less than the number of the unknowns and defines the required collection of the speeds of the lines.

The moment when some three or more lines cross will be called a "catastrophe". (The moment of a catastrophe always exists: it is clear if at least one of the lines $l_3, \dots, l_n$ is moving towards the intersection point O of l_1 and l_2; but if all of them move away from O, we replace all rates with their opposites.) Then there exist however small triangles at moments "just before" the catastrophe. Contradiction; so $\Delta_n \geq n - 2$. Q.E.D.

If you got that far you might be tempted to make the final touch:

EXTENSION. Give an example with $\Delta_n = n - 2$.

35.2.10.1. The lengths of the sides and heights of the projected triangle $\triangle A_1 B_1 C_1$ are not greater than the lengths of the sides and heights, respectively, of triangle $\triangle ABC$, which is projected onto plane P. Now consider on your own the two cases:

a) two angles of triangle ABC are not less than the corresponding angles of triangle $\triangle A_1 B_1 C_1$;

b) two angles of triangle $\triangle ABC$ are not greater than their corresponding angles.

Consider the cases separately; they will lead you to the desired result.

35.2.10.2. Prove that for a set $a_1 \geq a_2 \geq a_3 \cdots \geq a_n \geq 0$ we have

$$a_1x_1 + \cdots + a_nx_n \geq a_1y_1 + \cdots + a_ny_n.$$

Indeed, take non-negative numbers $b_1, \dots, b_n$ such that

$$a_n = b_1, \quad a_{n-1} = b_1 + b_2, \; \dots, \; a_2 = \sum_{i\leq n-1} b_i, \quad a_1 = \sum_{i\leq n} b_i.$$

We have

$$\begin{aligned} a_1x_1 + \cdots + a_nx_n &= (\sum_{i\leq n} b_i)x_1 + (\sum_{i\leq n-1} b_i)x_2 + \cdots + b_1x_n \\ &= b_1(\sum_{i\leq n} x_i) + b_2(\sum_{i\leq n-1} x_i) + \cdots + b_nx_1 \geq b_1(\sum_{i\leq n} y_i) + \cdots + b_ny_1 \\ &= (\sum_{i\leq n} b_i)y_1 + \cdots + b_1y_n = a_1y_1 + \cdots + a_ny_n. \end{aligned}$$

Use the above inequality k times:

1) Let $a_i = x_i^{k-1}$, $1 \leq i \leq n$. Then

$$x_1^k + x_2^k + \cdots + x_n^k \geq x_1^{k-1}y_1 + \cdots + x_n^{k-1}y_n.$$

2) Let $a_i = x_i^{k-2}y_i$, $1 \leq i \leq n$. Then

$$x_1^{k-1}y_1 + \cdots + x_n^{k-1}y_n \geq x_1^{k-2}y_1^2 + \cdots + x_n^{k-2}y_n^2.$$

3) Let $a_i = x_i^{k-3}y_i^2$, $1 \leq i \leq n$. Then

$$x_1^{k-2}y_1^2 + \cdots + x_n^{k-2}y_n^2 \geq x_1^{k-3}y_1^3 + \cdots + x_n^{k-3}y_n^3.$$

....

k) Let $a_i = y_i^{k-1}$, $1 \leq i \leq n$. Then

$$x_1y_1^{k-1} + \cdots + x_ny_n^{k-1} \geq y_1^k + \cdots + y_n^k.$$

Uniting all k inequalities into one long chain we come to the assertion of the problem.

35.2.10.3. If $x_i \geq \cdots \geq x_{200}$ are the numbers of the first set and $y_1 \geq \cdots \geq y_{200}$ are the numbers of the second set, and in his/her move player B takes the cards $x_1, \dots, x_{100}$ and $y_1, \dots, y_{100}$, then (s)he will get as the difference $S(B) - S(A)$ the expression:

$$(x_1 - x_{101}) + \cdots + (x_{100} - x_{200}) + (y_1 - y_{101}) + \cdots + (y_{100} - y_{200}) \geq 20000.$$

But if player A in his/her first move puts the cards with numbers 1, 2, ... , 100, 201, ... , 300 into the first set and the other cards into the second one, then A can return to the same position after every move of player B; and $S(B) - S(A) = 20\,000$.

35.2.10.3. 5.2.10.4. Induction on n: For $n = 3$ it is easy to write such numbers: they are $\frac{0}{3}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{3}$ (verify).

Let the statement be true for $n-1$. A new set of n numbers is obtained by adding some numbers to the previous set. If the neighboring numbers of the new set are neighboring numbers in the old one as well, then everything is proved for them. If the fraction $\frac{k}{p}$ neighbors the fractions $\frac{a}{b}$ and $\frac{c}{d}$ of the old set, then let us prove that

$$A = kb - ap = 1 \text{ and } B = cp - kd = 1 \quad (\tfrac{a}{b} < \tfrac{k}{p} < \tfrac{c}{d}).$$

Assume the contrary: let the greatest of the numbers A and B be greater than 1.

$$b + d < bB + dA = p(bc - ad) = p; \quad b + d < p.$$

This means that at least $\frac{a+c}{b+d} \neq \frac{k}{p}$. But $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ and, therefore, $\frac{k}{p}$ does not neighbor the fractions $\frac{a}{b}$ and $\frac{c}{d}$. Contradiction.

REMARK. It is possible to solve the problem with the help of number lattices in the plane: for such a proof see [Cox].

35.2.10.5. Consider the numbers modulo 10 (i.e., consider the last digits of the numbers). Consider the "zero" set consisting of all zeroes and count how many sets we can get by means of our operations. There are $(8-3+1)^2 = 6^2$ squares of size 4×4 and $(8-4+1)^2$ squares of size 3×3; all in all there are $6^2+5^2 = 61$ squares. Consequently, there are not more than 10^{61} sets. But there exist 10^{64} possible different sets. This means that there exists a set which cannot be obtained from the zero set using our operations.

Take this set for an original one.

36.1.8.2. If the number is a perfect square, then it ends with an even number of zeros, and we may strike them out. The number left is of the form $2A$, where A is a number consisting of 600-many 3's and some number of zeros, and ending with a 3. Thus, A is an odd number, which means that $2A$ is not a perfect square.

36.1.8.3. Among the five points considered there are two points (denote them by A and B) such that the other three points, C_1, C_2, C_3, lie on one side of line AB.

Clearly, the angles $\angle AC_1B$, $\angle AC_2B$ and $\angle AC_3B$ are different: if not, say, $\angle AC_1B = \angle AC_2B$, then points A, B, C_1 and C_2 lie on one circle. Suppose $\angle AC_1B < \angle AC_2B < \angle AC_3B$. Then point C_1 lies outside the circle passing through points A, B and C_2, while point C_3 is inside the circle.

36.1.8.4. Multiplying the given equation by xyp we get

$$px + py = xy \quad \text{or} \quad (x-p)(y-p) = p^2.$$

For a prime p it follows that either

1) $x - p = 1$; $y - p = p^2$, or
2) $x - p = p$; $y - p = p$, or
3) $x - p = p^2$; $y - p = 1$.

If p is not a prime, then p^2 can be factorized in other ways, too.

36.1.9.1. Consider points K, L, M, and N on sides AB, BC, CD and DA, respectively. Suppose that diagonal KM is not parallel to side AD. Fix points K, M, and N and let us move point L along BC in one direction. Then the distance from L to KM and, therefore, the area of $\triangle KLM$ will monotonously vary. The same applies to the area of quadrilateral $KLMN$. Hence, the area of $KLMN$ is equal to a half area of the parallelogramm for at most one position of L. Such a position does exist: if $LN \parallel AB$, then $S_{KLN} = \frac{1}{2}S_{ABLN}$. Therefore, $S_{KLMN} = \frac{1}{2}S_{ABCD}$.

36.1.9.2. First, divide all polygons into triangles so that no triangle's vertex is an inner point of a side of another triangle. All inner triangles will be adjacent to three other triangles (i.e., to an odd number of triangles).

FIGURE 316. (Sol. 36.1.9.2)

Next, divide the triangles with one side lying on a side of the square as shown on Fig. 316.

36.1.9.3. Observe that if $a \neq b$ are integers, then $a^k - b^k$ is divisible by $a - b$; hence, $P(a) - P(b)$ is divisible by $a - b$.

Now, let $P(a_1) = P(a_2) = P(a_3) = 2$, $P(b) = 3$. Then $P(b) - P(a_i) = 1$ is divisible by $b - a_i$ $(i = 1, 2, 3)$, whence $|a_i - b| = 1$. These equalities are impossible to satisfy for three distinct a_i.

36.1.9.4. Let A be an arbitrary subway station. Then close a subway station B which is farthest from A, i.e., with the greatest number of stations between A and B. If there are several such stations, close any of them. Observe that the shortest rout from a station C to A does not pass through B: otherwise C would have been further from A than B.

Hence, from any station one can reach A, and from A any other station.

36.1.9.5. There will be printed 99 numbers altogether, the initial and the final squares, inclusive. Let us write the numbers consequtively as they appear and trace a pair of numbers on the opposite faces, say, a and $7-a$.

Between two neighboring appearences of a a $7-a$ will necessarily appear. Indeed, if we roll the cube standing on a in one direction, say, to the right, twice, then the cube will print an $7-a$.

If we roll differently, for example, first, to the right, then upwards, then $7-a$ remains on the right face, a remains on the left face and none of them will be printed. Still, $7-a$ will be printed as soon as the cube rolls to the right.

Thus, for each pair $(a,\ 7-a)$ there are printed as many a's as there are $7-a$'s or the number of prints differs by 1. Therefore, all the prints, except for at most 3 prints, can be split into pairs with sum 7, not less than 48 such pairs. Hence, the sum total takes values between $342 = 48\cdot 7+1+2+3$ and $\leq 351 = 48\cdot 7+4+5+6$.

The extremal values can be attained. To get 351, let 4, 5 and 6 are on its bottom, right and front faces (in any order) of the cube; roll the cube once to the right, then upwards as far as possible, then to the right as far as possible.

To get 342, do as above but replace 4, 5 and 6 with 1, 2 and 3.

36.1.10.1. Use the fact that $a\geq 2,\ b\geq 2 \implies ab\geq a+b$. Then $f(k)\leq k+1$ for any k. If k is an even number greater than 7, then

$$f(k) = 2+p_2+p_3+\dots+p_n+1 \leq 3+p_2\dots p_n = 3+\frac{k}{2} < k.$$

Therefore, the sequence considered takes values in $[1, k+1]$ only. This means that it takes some of the values twice and, therefore, is a periodic one.

36.1.10.3. Let $P(a)=1$, $P(b)=2$, $P(c)=3$, $P(d)=5$. By Lemma from Problem 36.1.9.3 we have: $P(x)-P(y) \,\vdots\, (x-y)$ for integer x and y. Then $1 = P(b)-P(a) \,\vdots\, (b-a)$ implying that $b-a=\pm 1$, i.e., $a=b\pm 1$.

Similarly, $c=b\pm 1$. Thus, two cases are possible: A) $a=b-1$, $c=b+1$ and B) $a=b+1$, $c=b-1$.

Further on, $3 = P(d)-P(b) \,\vdots\, (d-b)$; hence, $d-b=\pm 1$ or ± 3. The case $d-b=\pm 1$ is impossible, because $P(b+1)$, as well as $P(b-1)$, are not equal to 5; so $d=b\pm 3$. Of these 2 possibilities in each case not more than 1 can be realized. For example, in case A) $d\neq b-3$, because otherwise $P(d)-P(c) = 2 \not\vdots\, d-c=-4$.

36.1.10.4. Let there exist a polyhedron all of whose faces are polygons with different numbers of sides. Select a face with the greatest number of sides, n. Each other face can have from 3 to $n-2$ different numbers of sides; hence, among n neighboring faces of the selected face, there are two faces with an equal number of sides. Contradiction.

36.1.10.5. Let us solve the problem in several steps.

a) It is clear that for any position of the switches it is possible to change the condition (on or off) of any fixed lamp by changing the position of some switch, as the change of the condition of different lamps corresponds to the change of the position of different switches.

b) Let us turn the switches in a position such that all the lamps are switched off. This position of every switch will be called “off”, and the opposite one will be called “on”.

Now if k switches are turned “on”, then $r\leq k$ lamps are switched on. And if $r<k$, then we can change the position of r switches so that all lamps are switched off, see heading a), although not all switches are turned “off”. Contradiction implies that $r=k$.

c) Enumerate the switches 1 to n and enumerate the lamps so that if we turn “on” the i-th switch (keeping other switches “off”), then the i-th lamp is lighted.

Let us prove that if the switches $i_1, i_2, \dots, i_k$ are turned “on”, then the lamps with the same numbers are lighted. It follows from b) that k lamps are lighted. Let one of these lamps have the number j different from $i_1, \dots, i_k$. Changing positions of switches $k-1$ times one can switch the other $k-1$ lamps off thanks to a). As a result one of the switches $i_1, \dots, i_k$ is turned “on”, but the lamp lighted is the j-th one. This goes against the choice of j, hence the assertion.

36.2.7.1. Let $\overline{a_1a_2a_3a_4} - \overline{a_4a_3a_2a_1} = 1008$. Let us start subtracting from the end. Since $a_1 > a_4$, the subtraction requires borrowing from tens: $(10+a_4)-a_1 = 8$, $(a_3-1)-a_2 = 0$; hence, $a_3 = a_2+1$.

But then to get the next digit we must borrow again: $(10+a_2)-a_3 = 9$ and not a 0. Contradiction.

36.2.7.2. Let O be the intersection point of the heights AK, BL, CM of triangle ABC. The height divides triangle ABC into 3 quadrilaterals $AMOL$, $BMOK$, $CLOK$ each of which is covered by a corresponding disc (even by the disc of the smallest radius).

Let us show, e.g., that $\triangle ALO$ is covered by a disc centered at A. Let T be any point on LO. Then $AT \leq AO$ (the inequality of two slanted lines) and $AO < AK$; hence, line segment AT lies inside the disc.

36.2.7.3 and **36.2.8.2.** First, let us verify that a regular coloration of the whole piece exists. This is not difficult: it suffices to paint diagonals one color, see Fig. 317 a).

FIGURE 317. (Sol. 36.2.7.3)

a) Let now one square be not painted. This means that there are 100 squares of each of some 99 colors and 99 squares of the 100-th color. Clearly, this is the color the unpainted square should be colored.

b), c) For counterexamples see Fig. 317 b), c).

36.2.8.1. Let the minimal distance from an arbitrary point of the blot to its boundary take the greatest value (say, r) at a point A, and let the maximal distance be equal to the minimal value (say, R) at point B. Then the disk of radius r with center A can be contained inside the blot, and the disk of radius R with center B contains the blot.

FIGURE 318. (Sol. 36.2.8.1)

But by the hypothesis $r = R$, which means that points A and B coincide, and the blot is of the form of a disk of radius r.

36.2.8.2. See Problem 36.2.7.3.

36.2.8.3 and **36.2.9.5.** If the speed of the cop is $\frac{1}{2}v$, where v is the speed of the robber, it suffices for the cop to run first directly towards the robber and, when the robber runs along a side of the square, to run directly to that side.

If the cop's speed is smaller then $\frac{1}{2}v$, (s)he must use the fact that if (s)he runs inside a square with the same center but three times as small as the original square, (s)he can stay abreast with the robber, i.e., on the same ray coming out of the center of the square, or even outrun the robber. So the cop should get out beyond the boundary of this square and then rush back to the boundary at an appropriate moment.

36.2.8.4. The sum of the angles of a convex pentagon is equal to $540°$. If the pentagon would have contained 3 angles of measure $< 60°$ each, and the remaining two angles are $< 180°$ each, then the sum total of all the angles would have been $< 3 \cdot 60° + 2 \cdot 180° = 540°$. Contradiction. Therefore, among 3 angles that are $\geq 60°$ there are two neighboring ones.

a) Observe that in an equilateral convex pentagon two non-neighboring angles can not be simultaneously $< 60°$. Indeed, assume the contrary: let in a pentagon $ABCDE$ with side a we have $\angle B < 60°$ and $\angle E < 60°$. Then $\angle CAB = \frac{180° - \angle B}{2} \geq 60°$; similarly, $\angle DAE \geq 60°$.

Since $\angle B$ is the least in $\triangle ABC$, then CA is the shortest side, i.e., $CA < AB$. Similarly, $AD < CD$. Then CD is the longest side in $\triangle ACD$, and, therefore, $\angle CAD$ is the largest angle, i.e., $\angle CAD > 60°$. But then $\angle A = \angle BAC + \angle CAD + \angle DAE > 180°$ which is in contradiction with the convexity of $ABCDE$.

b) Let $\angle C$ be the least in $ABCDE$. Let us prove that the triangle can be put to side AE. Let M be the midpoint of side AE, let MK, where K lies on a side of the pentagon, be the midperpendicular to AE. It suffices to prove that $AK \geq a$: then there exists point K' on MK such that $AK' = a$ and $\triangle AK'E$ is the triangle to be found.

Two cases are possible:

1) K lies on a side neighboring with AE, say, on AB. But if K does not coinside with B, then $\angle B < 60°$, $\angle B < 60°$, $\angle B < 60°$ and we get a contradiction with heading a). If K coincides with B, then $AK = a$ and $\triangle ABE$ is the one to be found.

2) Point K lies on a side not neighboring with AE, say, on BC; point K may coincide with C, see Fig. 319.

FIGURE 319. (Sol. 36.2.8.4) FIGURE 320. (Sol. 36.2.9.2)

Since C lies to the right of the midperpendicular, $AC \geq EC$. Triangles $\triangle ABC$ and $\triangle EDC$ have two common sides and $AC \geq EC$; hence, $\angle B \geq \angle D$. We also have

$$\angle C > \angle BCA + \angle ECD = (90° - \frac{1}{2}\angle B) + (90° - \frac{1}{2}\angle D) \geq 2(90° - \frac{1}{2}\angle B) = 180° - \angle B.$$

From $\angle B \geq \angle C > 180° - \angle B$ we deduce that $\angle B > 90°$.

Since $\angle B$ is the largest angle in $\triangle ABK$, it follows that AK is the longest side, i.e., $AK > AB = a$.

36.2.9.1 (See Solution to Problem 36.2.8.5.) Let us divide the digits into groups of digits with distance 5 between them: 1st, 6th, 11th, 16th, ..., 96th (the 1st group), 2nd, 7th, 12th, 17th, ..., 97th (the 2nd group), etc. Each permutation intermixes the digits inside each group only; therefore, the number of 1's in each group does not change. This means that if two numbers have different number of 1's in the same groups (1st to 5th), the numbers are not similar.

Each group can have 0 to 20 digits 1; there are 5 groups, so there are 21^5 possibilities altogether. For every possibility fix a number with the required number of 1's all of which stand to the left of the 2's. Let us prove that the remaining numbers are similar to the fixed ones.

Select 11 digits in a row, and perform a "double permutaion": replace first the 5's in the left 10 digits, next in the right 10 digits. Observe that the 1st, 6th and 11th digits have been cyclicly permuted while the other digits remained fixed. For any triple of neighbors from one group with the help of not more than two such double permutations we can assure that all 2's in the triple are to the right of the 1's.

After sufficiently many such operations we herd all the 2's in each group to the right, thus getting a fixed number similar to the initial one.

37.2.7.4. Take several times a ball from each pile untill the smallest piles have just one ball each. Now, double the number of balls in each pile that consists of one ball and take a ball from each pile. As a result, each pile consisting of one ball retains its size, while the other piles diminish by 1 ball. Repeat this procedure untill each pile becomes a one-ball one. Now, collect all of the piles. (Cf. Solution of Problem 37.2.8.3.)

37.2.8.2. Denote the points with the greatest distance between them by A and B. Connect point A by segments with all other points but B. The midpoints of the obtained $n-2$ segments do not coincide (otherwise the second endpoints of the segments would coincide too) and are situated inside a disc of radius $\frac{AB}{2}$ with center in point A. Similar arguments for point B add $n-2$ midpoints situated in a disc of radius $\frac{AB}{2}$ with center in point B. The two discs constructed have exactly one common point — the midpoint of segment AB.

Thus, we have implicitly constructed $(n-2)+(n-2)+1 = 2n-3$ midpoints of segments. An example of $2n-3$ midpoints: all points lie on the same straight line with a constant step between them. Consequently, the smallest number of midpoints of all possible segments is $2n-3$.

37.2.8.3. First, consider one column. By several subtractions of a 1 from all numbers of this column we make the smallest number in it into 1.

Now, let us double all numbers in all rows containing 1's and then subtract 1 from all numbers of our column. In the same way, in the column, we decrease by 1 all numbers greater than 1 while the 1's remains intact.

Proceeding in this way we soon make all numbers of the column into 1, and then we can make a 0 in each of the column's squares.

Repeat the similar procedure with the rest of the columns. (Cf. solution to Problem 37.2.7.4.)

37.2.8.4. In pentagon $ABCDE$, draw diagonals AC and AD and construct discs on them as diameters with centers in O_1 and O_2. These discs cover triangles ABC and AED, as angles ABC and AED are obtuse; see Fig. 325.

If one of the angles ACD or ADC is obtuse, then one of the discs with center O_1 or O_2 covers triangle ACD. If angles ACD and ADC are non-obtuse, then the disc with center O_1 covers triangle AHC, and the disc with center O_2 covmakwithADC

37.2.9.3. Consider a weitght. We will call this weight an *essential* one if all the weights without the one considered cannot be divided into K groups of equal mass. Otherwise we call the weight considered an *inessential* one.

Assume that the assertion of the problem is false, i.e., there are fewer than K essential weights. Then let us prove that *for any n the mass of any inessential weight is divisible by K^n.* Since this number can be made however great, this means that the mass of any inessential weight is 0. Contradiction.

Let us prove the italicized statement by induction.

It clearly holds for $n = 1$: the mass of all weights is divisible by K, and it is possible to divide the weights into K groups of equal mass without any inessential weight (by definition). Therefore, the mass of this inessential weight is divisible by K.

Assume now that the mass of any inessential weight is divisible by K^n and prove that it is divisible by K^{n+1}. By the hypothesis, the number of essential weights is less than K, which means that there exists a group all of whose weights are inessential (otherwise there would be at least as many inessential weights as there are groups). The mass of the whole group of inessential weights is divisible by K^n, the total number of groups is K, and the masses of any two such groups are equal.

Hence, the mass of all weights together is divisible by K^{n+1}, and this is true both with an inessential weight and without it. Consequently, the mass of an inessential weight is divisible by K^{n+1}, Q.E.D.

37.2.9.5. Introduce a coordinate system with the origin in one of the vertices of the graph paper and the coordinate axes directed along the sides of the paper. The lines $x = n/2$ and $y = m/2$, where m and n are integers, draw a grid on the coordinate plane; let us paint it as a chessboard. If the sides of the rectangular strip are parallel to the coordinate axes and the length of one of the sides is equal to 1, then the sum of the areas of its white parts is equal to that of its black parts.

Indeed, under the translation by the vector $\overrightarrow{AB}$, see Fig. 326, the white parts of the strip become black and the other way round.

FIGURE 326. (Sol. 37.2.9.5)

It remains to prove that if the sum of the areas of the white parts is equal to that of the black parts, then one of the sides of the paper sheet is of integer length. Suppose that both sides are not of integral length. The lines $x = m$ and $y = n$ cut off the paper rectangles one of the sides of each of which is equal to 1 and one rectangle both sides of which are shorter than 1. It is easy to verify that in the latter rectangle the sum of the areas of the white parts can not be equal to that of the black parts.

37.2.10.2. Since $1974^n > 10^{3n}$, the number k of digits in the number 1974^n is not less than $3n$. Assume that $1974^n + 2^n$ has more digits than 1974^n. Then $1974^n + 2^n \geq 10^k$. It is easy to see that

$$987^n < 2^{k-n} \cdot 5^k \leq 987^n + 1$$

as 10^k is divisible by 2^n. Therefore,

$$2^{k-n} \cdot 5^k = 987^n + 1, \qquad k - n \geq 2n.$$

If $n \geq 2$, then $2^{k-n} \cdot 5^k$ is divisible by 8; but $987^n + 1$ is not divisible by 8 (the remainder is equal to 2, 4 or 6). Therefore, the equality is false.

37.2.10.3. Draw a plane P through the center of the planet and two arbitrary asteroids; let P be the equatorial plane, so that the axis of the planet's revolution is perpendicular to P. The axis intersects the planet's surface in poles A and B.

It is clear that the astronomers sitting at points A and B can see together only $37 - 2 = 35$ asteroids, and, therefore, one of them can see less than 18 asteroids.

37.2.10.4. Consider a scientist with the greatest number, say n, of acquaintances. By the hypothesis, each of his/her acquaintances has a different number of acquaintances, greater than 0 but not greater than n. There exist exactly n such possibilities: one acquaintance, two acquaintances, $\dots$, $n-1$ acquaintances and our scientist; and all the possibilities are realized. Therefore, there exists a scientist who has only one acquaintance.

38.1.10.1. One solution is obvious: $x = y = z = t = 0$.

Let us prove that there are no other solutions. Move the right hand side of the equation to the left and separate perfect squares:

$$\frac{x^2}{4} + \left(\frac{x}{2} - y\right)^2 + \left(\frac{x}{2} - z\right)^2 + \left(\frac{x}{2} - t\right)^2 = 0.$$

Hence, the answer.

38.1.10.2. Let O be the center of the disc. Draw an arbitrary straight line l intersecting the disc and let A' and O' be the reflections of A and O, respectively, through l. In triangle OAO', see Fig. 327, we have $OO' \geq |OA - O'A|$, and since $O'A = OA'$, it follows that $|OA' - OA|$ is not greater than 2 cm, i.e., twice the distance between O and l.

FIGURE 327. (Sol. 38.1.10.2)

FIGURE 328. (Sol. 38.1.10.4)

Therefore, after 24 reflections point A becomes less than $24 \times 2 = 48$ cm closer to O and cannot get into the disc. It is possible to drive A into the disc in 25 reflections. Indeed, draw two lines perpendicularly to OA and at the distance of 0.99 cm from O. Starting with A, consider successive reflections through the nearest of these lines (to the point to be reflected).

38.1.10.3. Two of the numbers a, b, c , say a and b, are of the same parity. Since b^c is of the same parity as b, it follows that $p = b^c + a$ is even. But p is a prime, which means that $p = 2$; whence $a = b = 1$. Therefore, $q = 1^b + c = c^a + 1 = r$, Q.E.D.

38.1.10.4. Connect the centers of the 28 squares that form the frame of the chessboard; we get a closed brocken line Γ, see Fig. 328.

Each of the 13 straight lines intersects Γ in not more than two points; hence, the total number of the points of intersection with Γ is not greater than 26 and these 13 lines cannot separate even these 28 centers, to say nothing of all centers.

38.1.10.5. Let the light source be situated at point O. Construct a regular tetrahedron $ABCD$ with center O. Consider then 4 circular infinite cones containing inside them pyramids $OBCD$, $OACD$, $OABC$, $OABD$ with a common vertex O. These cones partly intersect so that *any ray from O lies inside one of the cones*, see Fig. 329.

Now, place a ball into each cone so tightly that the balls do not touch each other (the radii of the balls must differ greatly from one another, for example they might be equal to 10, 10^4, 10^7 and 10^{10}). Clearly, any ray OX from O intersects at least one of the balls.

FIGURE 329. (Sol. 38.1.10.5)

38.2.7.2. If $\angle ABC = a^\circ$ is an inscribed angle, it subtends an arc of measure $2a^\circ$. Hence, $\smile ABC = 360^\circ - 2a^\circ$. In particular, if $\angle ABC = 120^\circ$, then $\smile ABC = 120^\circ$.

If in an inscribed heptagon, three angles, 120° each, have no common leg, then their legs subtend the total of 360° of arcs and, therefore, the 7th angle subtends an arc of 0°, which is impossible.

Let then BC be the common leg of two angles of 120° each; A, B, C, D consequtive vertices of the heptagon. Since $\angle B = \angle C = 120^\circ$, then $\smile ABC = \smile BCD = 120^\circ$ and $\smile AB + \smile BC = \smile BC + \smile CD$. This implies that $\smile AB = \smile CD$; hence, $AB = CD$, Q.E.D.

38.2.7.4 and **38.2.8.3.** a) Replace all even digits of the sequence by 0 and all odd ones by 1. We get a sequence in which $\overline{11110}$ is periodically repeated. The sets $\overline{1010}$ and $\overline{1001}$ corresponding to the sets $\overline{1234}$ and $\overline{3269}$ do not occur in the sequence.

b) There is only a finite number of four-digit numbers including numbers beginning with zeros. Since the whole sequence is infinite, there are two equal four-digit numbers: $\overline{1975...A...A...}$ in it.

Let us prove that one more number $\overline{1975}$ stands between these two equal numbers A. Note that there is a unique extension of this sequence to the left, the conditions being preserved (i.e., knowing 4 consecutive digits we can find not only the digit following them, but also the digit preceding them). Therefore, knowing only one part of the sequence $(\overline{A...A})$ we can restore the infinite sequence to the left and right. If $\overline{1975}$ did not occur between A and A, then $\overline{1975}$ would not occur in the entire sequence, which contradicts the hypothesis. This means that $\overline{1975}$ must occur once again.

c) Note that 8 precedes $\overline{1975}$ in the extension of the sequence to the left. Thus, we have found a set $\overline{8197}$ in the infinite (both ways) sequence; this means that $\overline{1975}$ occurs in the original sequence also.

38.2.8.1. a) Prove by induction that the right number is greater for all $n \geq 3$, although for $n = 2$ the opposite is true.

b) Denote the left number by A_n and the right number by B_{n-1}. Let us prove by induction that $A_{n+1} > 2B_n$. Indeed, assuming that $A_n > 2B_{n-1}$, we have:

$$A_{n+1} = 3^{A_n} > 3^{2B_{n-1}} = 9^{B_{n-1}} = (2.25)^{B_{n-1}} 4^{B_{n-1}} > 2 \cdot 4^{B_{n-1}} = 2B_n, \qquad \text{Q.E.D.}$$

ANOTHER SOLUTION of b): use the result of a) and the fact that $2^{2^{\cdot^{\cdot^{2}}}}$ (with $(n+1)$-many 2's) is greater than $4^{4^{\cdot^{\cdot^{4}}}}$ (with $(n-1)$-many 4's), which one also proves by induction using the inequality $2^{2^a} > 4^b$ for $a > b \geq 1$.

38.2.8.4. In Ourland, towns A and B are not connected by a railroad (otherwise one could get from A to B without changing trains); hence, in the Behindland railroad connects A' with B' . Now, let C and D be any pair of towns in Ourland. Town C cannot be connected with both A and B (otherwise one could get from A to B with only one change), so in the Behindland a railroad connects C' with either A' or B'.

The same is true for D. Therefore, Ecila can get from C' to D' (through A' and B') changing trains not more than twice.

38.2.8.5. Let us prove that the difference in the scores of "neighboring" teams cannot be greater. Let the teams of the places k and $k+1$ have the greatest difference in scores. The teams $1, \dots, k$ played with each other in $\frac{k(k-1)}{2}$ games and gained $k(k-1)$ scores.

Besides, they played $k(n-k)$ games with the teams $k+1, \dots, n$ and gained not more than $2k(n-k)$ scores. And so the teams $1, \dots, k$ gained not more than $k(k-1)+2k(n-k) = (2n-k-1)k$ scores. Hence team No. k could not have gained more than $2n-k-1$ scores since its position in the first group is the last one. The teams $k+1, \dots, n$ played $\frac{(n-k)(n-k-1)}{2}$ games with one another and gained $(n-k)(n-k-1)$ scores. Therefore, team No. $k+1$ gained at least $n-k-1$ scores. This means that the difference between the scores of team No. k and team No. $k+1$ is not greater than n.

38.2.9.4. Let X and Y be two towns such that if the road AB cannot be used, then the shortest road l from X to Y is not less than 1500 km long. It is easy to see that on the road l lies a town, M, distant from either X or Y not less than by 500 km. Hence, the shortest way from X to M before road AB was closed, was a road whose part AB was (it was the road $XABM$), and the shortest way from Y to M was the one that contained as a part either $YABM$ or $YBAM$, see Fig. 330 a).

FIGURE 330. (Sol. 38.2.9.4)

In the first case, when road AB is closed, there are still ways XA and YA, and the road XAY is not longer than 1000 km; but this contradicts our assumption on the shortest way from X to Y.

In the second case the roads XA, BM, YB and AM are still in use, $XA + BM < 500$ km and $YB + AM < 500$ km. Hence, $XAMBY < 1000$ km.

REMARK. It is impossible to find a road between ANY two towns shorter than 1500 km, as it is clear from the following example: let there be only 4 towns A, B, C and D in Mantissa, and $AB = 1$ km, $AC = CD = DB = 498$ km, see Fig. 330 b). Then the shortest way from A to B, if road AB is closed, is almost 1500 km.

38.2.9.5. Assume that a convex polygon M is cut into n non-convex quadrilaterals the sum of their angles being $S = 360° \cdot n$. Call a vertex of a quadrilateral with an angle greater than 180° a *non-convex vertex*. A non-convex vertex X cannot lie on the boundary bd M of polygon M, and no two quadrilaterals can have a common non-convex vertex (otherwise they would intersect), see Fig. 331 a) and b).

FIGURE 331. (Sol. 38.2.9.5)

Hence, the number of non-convex vertices is equal to the number of quadrilaterals, and all non-convex vertices are situated inside M. Find the sum of the angles with non-convex vertices (of quadrilaterals) in points $X_1, \dots, X_n$, see Fig. 331c). It is equal to $360^\circ \cdot n$. But then S is not less than this sum plus the sum of the angles of M. Contradiction.

EXTENSION. It is possible to divide a triangle into infinitely many non-convex quadrilaterals and to divide a sphere into finitely many non-convex quadrilaterals (see Fig. 331d) and e)).

38.2.10.4. For any n, let us enumerate the spotlights 1 to n. Notice that the number of pairs (i, j) is equal to $\frac{n(n-1)}{2}$ and, above the circus ring, arrange n discs of the same radius as the ring. Inscribe similarly a regular $\frac{1}{2}n(n-1)$-gon into each disc and label the sides of the polygon by pairs of numbers $(1, 2)$, $(2, 3), \dots, (n-1, n)$. Let us cut off each disc $n-1$ segment formed by a side of an inscribed polygon and an arc of the circle; namely, we will cut off the k-th disc the segments whose label contains k; see Fig. 332.

Clearly, the obtained system of spotlights satisfies the condition.

39.1.10.1. Let X and Y be the greatest and the smallest among $x_1, \dots, x_4, x_5$. Equations imply that $X^2 \le 2X$ and $Y^2 \ge 2Y$. As $X, Y > 0$, we get: $2 \le Y \le X \le 2$. Therefore, the system has a single solution:

$$x_1 = x_2 = \dots = x_5 = 2.$$

FIGURE 332. (Sol. 38.2.10.4)

FIGURE 333. (Sol. 39.1.10.2)

39.1.10.2. To satisfy the condition, $\angle ABC$ must be almost equal to 90°, side AB (on which height CH is dropped) must be a very short one, and side BC (to which the median AM is drawn) must be a very long one. In the limit position, point H coincides with point B, and the area of the triangle $M'H'K'$ is equal to $\frac{1}{2}$; see Fig. 333. More exactly, let $AB = x$, $CH = \frac{2}{x}$, $BH = x^2$. Then $S_{\triangle ABC} = 1$, $S_{\triangle BCH} = x, S_{\triangle AMC} = \frac{1}{2}$, $S_{\triangle ABK} < \frac{x^2}{2}$; hence,

$$S_{\triangle M'H'K'} > 1 - x - \frac{1}{2} - \frac{x^2}{2} = \frac{1}{2} - x - \frac{1}{2}x^2 \longrightarrow \frac{1}{2} \text{ as } x \longrightarrow 0.$$

Therefore, in a position close to the limit one the area of triangle $\triangle M'H'K'$ can become greater than $0.499 \cdot S_{ABC}$.

39.1.10.3. The first four digits are 1000. Indeed,

$$1^1 + 2^2 + \cdots + 999^{999} < 999 + 999^2 + 999^3 + \cdots + 999^{999} \quad = \quad \frac{999 \cdot (999^{999} - 1)}{999 - 1} < \frac{999^{1000}}{998} < \frac{999^3}{998} \cdot 1000^{997} < 1000^{999},$$

because, clearly, $\dfrac{999^3}{998} < 1000^2$.

So, if S is the sum, then

$$1000^{1000} < S < 1000^{1000} + 1000^{999} = 100100\ldots 0.$$

Hence, $S = \overline{1000x_1x_2\ldots x_n}$.

39.1.10.4. Since the angle between the two diagonals of the cube is acute, it is possible to turn the searchlight situated in the center of the cube so that the searchlight illuminates exactly two vertices of the cube. Arrange seven more searchlights in the center of the cube and turn them so that the whole space is illuminated (there is only one way to do this).

As the cube has 8 vertices and there are 8 searchlights, one of which illuminates 2 vertices, there is one searchlight that by Dirichlet's principle does not illuminate any vertex. This is the searchlight to be determined.

FIGURE 334. (Sol. 39.2.7.2)

39.2.7.1. Call the balls A, B, C and D. At the first weighing put balls A and B on the left pan, ball C on the right pan. The needle might show 99 to 106 g.

Let it show 99 g. This means that the weights of balls A and B are equal to 101 and 102 g and the weight of ball C is equal to 104 g. In this way the weights of balls C and D are already known and it only remains to figure out which of the balls A or B is lighter. For this, one weighing is, certainly, sufficient.

Similar is the situation in the other cases. Namely:

N	left pan A, B	right pan C	the needle points at
1	101, 102	104	99
2	101, 102	103	100
3	101, 103	104	100
4	101, 103	102	102
5	101, 104	103	102
6	101, 104	102	103
7	102, 103	104	101
8	102, 103	101	104
9	102, 104	103	103
10	102, 104	101	105
11	103, 104	102	105
12	103, 104	101	106

As we see, to a given value of the weight indicated by the needle not more than 2 variants correspond.

Let us additionally consider, for example, the case when the needle points at 103 g. The 6-th and the 9-th variants are the corresponding cases. At the second weighing, let us place balls A and D on the left pan and nothing on the right pan. A straightforward case-by-case checking shows that the needle may show

203, 204, 205 or 207 g, namely:

variant	weight of ball D	weights of balls A, D on the left pan	the needle points at
6	103	101, 103	204
6	103	104, 103	207
9	101	102, 101	203
9	101	104, 101	205

Either way the weight of each of the balls is uniquely determined.

Consider the other variants on your own.

39.2.7.2. Let point C be the common vertex of equal obtuse triangles ABC and CDE (the obtuse angle is not equal to $108°$; see Fig. 334) and then arrange these triangles so that the lengths of the diagonals BE and AD are equal to each other and to $AC = CE$; the length of side AE is not equal to 1.

39.2.7.5. The two greatest numbers of the table are marked both red and blue: A and B $(B \leq A)$. If they are situated in different rows and columns, then in the row and in the column with the third greatest number of the table C there is not more than one element greater than C; this means that C is also marked both red and blue. If A and B are situated in the same column (or row), then delete this column (row) and find the greatest number C (not greater than A). Number C must also be marked both red and blue.

39.2.8.2. If the length of each segment is equal to 1, i.e., if $ABCD$ is divided into 1×1 squares, then the total number of these segments (equal to their combined length) is equal to $2n(n-1)$, where n is the length of $ABCD$'s side. This number is divisible by 4, because $n(n-1)$ is even.

Let not all of the segments (or none at all) be of length 1. Divide each small square into 1×1 squares inserting new segments of length 1. We have thus inserted into each square several segments whose total number is divisible by 4, and the initial segments became divided into segments of length 1. The total number of length 1 segments (among the initial segments and the additional segments) is also divisible by 4. Therefore, the total number of length 1 segments into which the initial segments are divided (equal to the sum of the lengths of all the initial segments) is divisible by 4.

39.2.9.2. Consider the spot with the greatest radius and around it draw a concentric (with it) circle of a slightly greater radius that does not intersect any of the other spots. Reflect the circle symmetrically through the center of the Sun.

The reflected circle can not be covered by one spot: its radius is greater; neither can it be covered by several spots: they do not intersect and do not touch each other. Any uncovered point of the circle and the point diametrically opposite it constitute a pair whose existence we have to prove.

39.2.9.3. Let $S(A)$ be the sum of the digits of the integer A. Observe that $S(A) \leq 9k$, where k is the number of digits of A.

Assume that $S(2^n) < S(2^{n+1})$ for any $n > 1000$.

It is easy to verify that if $n = 6k$, then $S(2^n) = 9l+1$ and the remainders after division of the consequtive powers of 2 by 9 are 2 (for $n+1$), 4, 8, 7, 5 and then repeat cyclicly: 1, 2, 4, 8, 7, 5, etc. So we have $S(2^{n+1}) = 9x + 2$ and, therefore, $S(2^{n+2}) \geq 9x + 4$ (considering the remainder of 2^{n+2}); $S(2^{n+3}) \geq 9x + 8$.

Now, $S(2^{n+4}) \geq 9x+9$, but the remainder is equal to 7, hence, $S(2^{n+4}) \geq 9x+16$. Similarly, $S(2^{n+5}) \geq 9x + 23$ and $S(2^{n+6}) \geq 9x + 28$. So, $S(2^{n+6}) \geq S(2^n) + 27$. It follows that

$$S(2^{n+6m}) \geq S(2^n) + 27m \quad \text{for any} \quad m \geq 0. \qquad (*)$$

Let us prove that the inequality $(*)$ does not hold for large m. Indeed, let 2^n have r digits. As $2^9 = 512 < 1000$ and $2^{18} < 10^6$, the number 2^{n+18} has not more than $r+6$ digits. Analogously, 2^{n+18s} has not more than $r + 6s$ digits, so

$$S(2^{n+18s}) \leq 9(r + 6s).$$

Let $s = \frac{m}{3}$; then $S(2^{n+18s}) = S(2^{n+6m}) \leq 9r + 18m$. If $(*)$ holds, we have

$$9r + 18m \geq S(2^{n+6m}) \geq S(2^n) + 27m \quad \text{for any } m.$$

So, $9m \leq 9r - S(2^n)$. But this is false for large enough m. Contradiction with the assumption.

39.2.9.4. First, let us prove that it is possible to interchange any two neighboring digits of N. Denote some such digits by a and b, and write bb on the left of ab and aa on its right:

$$\dots bb\,ab\,aa\dots.$$

Striking out two successive pairs ba we get the desired transposition $ab \leftrightarrow ba$.

Hence, we can simultaneously strike out two *identical* digits even if they are not neighbors. Striking out all we can, we get a number with not more than 9 digits, so it is less than 10^9; cf. Problem 31.1.10.3.

REMARK. In higher (non-elementary) mathematics this is the problem on the structure of a *group* in which the square of any element is equal to 1 (one can either delete or insert a factor $1 = a^2$ in a word).

39.2.9.5. It is easy to arrange similar pentagons with their vertices in the nodes of the graph ("huts") so that they cover all nodes, see Fig. 335.

FIGURE 335. (Sol. 39.2.9.5)

FIGURE 336. (Sol. 39.2.10.2)

It is easy to see that every "hut" can be covered with a coin so that the coins do not touch one another. The center of a coin must be situated on the axis of symmetry of the "hut", and the distance between the hut's roof top and the coin's center must be equal to d, where 1.2 cm $< d < 1.264$ cm. Verify this on your own.

39.2.10.1. Let A be an n-digit number. Then $\overline{AA} = A(10^n + 1)$. Let us show that if $\overline{AA}$ is a perfect square, then $10^n + 1$ is divisible by the square of a prime p. Assume the contrary; let $10^n + 1 = p_1p_2 \dots p_n$ be the prime decomposition and all the p_i's are different. We have: $\overline{AA} \vdots p_i \Longrightarrow \overline{AA} \vdots p_i^2 \Longrightarrow A \vdots p_i$ for each i. Hence, $\overline{AA} \vdots p_1p_2 \dots p_n$ and $A \geq 10^n + 1$. Contradiction, since A is an n-digit number.

Conversely, let $10^n + 1 = bc^2$ for some n and integer b, c, $c > 1$. Choose an integer d such that $d < c$ and

$$\frac{d^2}{c^2} \geq 0.1. \qquad (*)$$

It is clear that c is odd and $10^n + 1$ is not divisible by 3 or 5; hence, $c \geq 7$ and we can take $d = c - 1$. Set $A = \frac{d^2}{c^2}(10^n + 1) = d^2b$. The number A is an integer, an n-digit one thanks to $(*)$. Therefore, $\overline{AA} = \frac{d^2}{c^2}(10^n + 1)^2 = (bcd)^2$.

It remains to find a suitable n. Observe that $10^n + 1 \vdots 11$ for n odd:

$$\frac{10^{2k+1} + 1}{10 + 1} = 10^{2k} - 10^{2k-1} + \dots + 10^2 - 10 + 1 = 909\dots091 \quad (k \text{ nines}).$$

The difference between the sum of the digits on the even positions and that on the odd ones is equal to $9k-1$; for $k = 5$ the difference is divisible by 11; hence, $10^{11} \vdots 11^2$. Take $c = 11$, $d = 4$. We get $A = 13223140496$.

REMARK. To see that this A is the minimal one, consider the prime factorization of $10^n + 1$ for $n < 11$.

39.2.10.2. For example, take an arbitrary pyramid with a convex 1975-gon at the base. Let AH be the hight of the pyramid. Under the projection on AH all the vectors of the base go into $\vec{0}$. Each lateral edge goes into either $\overrightarrow{AH}$ or $-\overrightarrow{AH}$; say, k edges go into $\overrightarrow{AH}$ and $1975 - k$ into $-\overrightarrow{AH}$. The sum of the projections is $(2k - 1975)\overrightarrow{AH} \neq \vec{0}$, because k is an integer. The initial sum can not be equal to $\vec{0}$ either, because its projection is nonzero.

39.2.10.3. Let us prove the general statement: *if the k greatest numbers in every row are marked red and the r greatest numbers in every column are marked blue, then in the whole table there are at least kr numbers marked both blue and red.*

Let us prove this by induction on $k + r$. A basis for induction is evident; let the statement be true for the sum of marked numbers less than $k + r - 1$.

Denote a number marked in a row by a "$-$" and a number marked in a column by a "$|$"; if a number is marked both in a row and in a column, then denote it by a "$+$". Consider the greatest number A of the table not marked with a $+$. This means that A is not marked with either a $-$ or a $|$. Let it lack $-$.

In A's row, there are k different numbers greater than A. Hence, all these numbers are marked with a $+$ (as they are greater than A), i.e., there are k-many $+$'s in the row. Strike this row out and consider the obtained smaller table.

Now, use the inductive hypothesis for k red and $r-1$ blue numbers: we have new deletions, but all $-$'s are the same as in the original (larger) table (as the rows are intact) and if there are some $|$'s in the small table, then they remain in the large table (a number not marked by a $|$ in the small table can be marked in the large one).

By induction, there are $k(r-1)$-many $+$'s in the small table, and all of them are transferred into the large table; besides, one must add k more $+$'s, those that were in the deleted row.

Thus, the whole initial (large) table has $k(r-1)+k = kr$ signs $+$'s, Q.E.D.

39.2.10.4. Choose a subset of fixed points consisting of points with the shortest distances between them and their neighbors (all these distances are equal).

Let our plane be a graphed one; consider the following point A of the set of fixed points: it is the farthest right point of the set; if there are several such points take the upper one. Connect point A with all its neighbors B_1, B_2, $B_3, \dots$. Then each angle $\angle B_1AB_2$, $\angle B_2AB_3, \dots$ is $\geq 60°$. Consequently, point A has not more than three nearest neighbors.

39.2.10.5. If there is a straight line, l, painted not less than 3 colors, then the plane passing through l and a point of the 4th color will be painted 4 colors. Suppose there is no such a plane.

Connect 5 fixed points each with each by 10 straight lines. Each of these lines is painted exactly two colors: the colors of the fixed points belonging to it. Consider a plane not parallel to any of the straight lines drawn. Let the plane be painted not more than 3 colors. Consider the intersection point of this 3-color plane with the line passing through 2 points of the other 2 corors; we get a point of 4th color on the plane.

Thus, there is always a plane painted 4 colors. On the plane, fix 4 points of different colors and repeat the above arguments to show that on the plane there always exists a line painted 3 colors.

40.1.10.1. Observe that if $x_n = 2^k m$, then x_{n+k} is odd. If $x_n = 2^k m + 1$, where m is odd, then we similarly find that x_{n+k} is even. This means that in the sequence there are infinitely many odd numbers as well as infinitely many even numbers.

FIGURE 337. (Sol. 40.1.10.2)

40.1.10.2. Consider the convex hull of all vertices of all squares (both cardboard and plastic). It is a convex polygon. Choose one of its vertices. It is a common vertex of some cardboard and plastic squares; denote these squares by C and P, see Fig. 337. If C and P do not entirely coincide, then a vertex of one of them must be situated *inside* the other.

Assume that one of the vertices of C is situated inside P. But a vertex of another plastic square, P', coincides with this vertex (point A on Fig. 337). But then the squares P and P' intersect. Contradiction.

Consequently, C and P coincide. Getting rid of them, repeat the arguments for the obtained system of squares.

40.1.10.4. Let B and R be the sets of blue and red points, respectively. If B had contained a finite number of points with coordinates divisible by b and R had contained a finite number of points with coordinates divisible by r, then the set of natural numbers would have had only finitely many numbers divisible by br.

40.2.7.1. It is clear that if hunter A shoots hunter B, then the hunter B shoots exactly hunter A and not the other one (otherwise the rabbit would have been at point A, i.e., not inside the polygon). This means that n is an even number. Besides, there is the same number of hunters on the right and left of straight line AB (since the plane is divided in halves by line AB and every hunter shoots from one halfplane to the other), see Fig. 338.

Thus, line AB is uniquely defined by choosing one of the two vertices A or B. Consider another straight line, CD (C does not coincide with A or B). Then O is the intersection point of segments AB and CD, i.e., it is uniquely defined.

FIGURE 338. (Sol. 40.2.7.1)

FIGURE 339. (Sol. 40.2.7.2)

40.2.7.2. See Fig. 339.

a) It is impossible. Otherwise there would have been an odd number of white cubes in vertical columns, but there are 14 of them.

b) It is possible. Arrange 12 black cubes along six edges AB, BC, CC', $C'D'$, $D'A'$, $A'A$ of the big cube $ABCDA'B'C'D'$ and place one more black cube in its center. Then there will be one or three black cubes in each column.

40.2.7.3. The initial equality can be written as

$$\left(\frac{a}{c}\right)^{15} + \left(\frac{b}{c}\right)^{15} = c.$$

Set $\frac{a}{c} = n$ and $\frac{b}{c} = k$. Then $c = n^{15} + k^{15}$, therefrom a and b are uniquely defined:

$$a = n(n^{15} + k^{15}), \qquad b = k(n^{15} + k^{15}).$$

40.2.7.4. If the first player connects A_1 with A_2, then the second player must connect some other nails, say, A_3 and A_4. So the first player has to connect A_5 with A_6, and so on. After the $\frac{1976}{4} = 494$th move of the second player there remains only one free nail. So the first player loses.

40.2.7.5. It is possible to obtain any convex 100-gon as an intersection of 50 triangles alternating every second vertex of the 100-gon, see Fig. 340 a).

For $n < 50$ it is impossible to do the same with a 100-gon one side of which is very long and the other sides very short; see Fig. 340 b).

40.2.9.1. Assume that $n \geq 3$. Consider 3 segments a, b, c and draw three straight lines connecting their midpoints pair-wise. By the hypothesis, every segment is perpendicular to the two straight lines intersecting it and, therefore, it is perpendicular to the plane passing through all three straight lines. This means that these three segments are parallel to one another. Contradiction.

40.2.9.2 and **40.2.10.2.** If n numbers a_1, $a_2, \dots,$ a_{n-1}, a_n with the property required are already constructed, then construct $n+1$ new numbers as follows:

$$\begin{aligned} b_0 &= LCM(a_1, a_2, \dots, a_n),\\ b_1 &= a_1 + b_0,\\ b_2 &= a_2 + b_0,\\ &\dots\dots\dots\\ b_n &= a_n + b_0. \end{aligned}$$

Figure 340. (Sol. 40.2.7.5)

40.2.9.3. Let us prove by induction the following general statement: *if for any group of k teams there is a team that had beaten all other teams (in the group), then the total number of participating teams is not less than $n = 2^{k+1} - 1$.*

The base of induction: $k = 0$. In this case there is at least 1 team, since our statement sounds "there is a team". (The case $k = 1$, $n = 3$ can also be taken as an obvious starting point.)

Let now the statement be true for some k. Let us prove it for $k + 1$. Consider all teams that won a game with the winner. Since for any k of them there exists a team that won games with them and with the winner, then these teams satisfy the hypothesis and, therefore, there are not less than $2^{k+1} - 1$ of them. But only less than half of all teams could have won games with the winner, so there are not less than $2(2^{k+1} - 1) + 1 = 2^{k+2} - 1$ of them, Q.E.D.

In particular cases $k = 2, 3$ we have the estimate: not less than 7 or 15 teams, respectively. This estimate is an exact one: see Fig. 341, where the arrows indicate who was beaten by the first team and the rest of the results is obtained by rotating the figures as solids.

Figure 341. (Sol. 40.2.9.3)

Figure 342. (Sol. 40.2.10.1)

40.2.9.4. Assume that the polyhedron has more than 8 vertices and consider any 9 of them. The first coordinates of at least five of them are either all odd or all even. The second coordinates of at least three of these 5 points are also either all odd or all even. And lastly, the third coordinate of some two of these three points also are either all odd or all even.

This means that we found two vertices of the polyhedron whose first, second and third coordinates are of the same parity. But then the midpoint of the segment connecting these two vertices has integer coordinates. However, since the polyhedron is convex, the midpoint is inside the polyhedron. Contradiction.

40.2.9.5. Since the remainder after a division of x_n by $x_n - 1$ is equal to 1, the remainders after division of x_{n+k} by $x_n - 1$ will appear periodically. Therefore, we should look for a number divisible by $x_n - 1$ among

$x_1, x_2, \dots, x_{n-1}$. But the only possibility for this is $x_n - 1 = x_{n-1}$ because all other numbers are smaller. Thus, $P(x_{n-1}) = x_{n-1} + 1$ for infinitely many points x_n. But if the values of two polynomials are equal at infinitely many points, then the polynomials themselves are equal, i.e., $P(x) = x - 1$.

40.2.10.4. Let the sequence y_n, $n \in \mathbb{N}$, be periodic with period T, beginning with some n_0. Then $x_{n+T} - x_n$ is even for all $n > n_0$, and, on the other hand, it is equal to $\left(\frac{3}{2}\right)^{n-n_0}(x_{n_0+T} - x_{n_0})$. The last number is odd for a large n; see the solution of Problem 40.1.10.1. Contradiction.

41.7.2. Fix an arbitrary vertex of the triangle and begin reflecting the triangle through the sides at this vertex. To make the triangle coincide with its initial position after several reflections the angle α at the fixed vertex should be equal to $\alpha = \frac{2\pi}{n}$.

If the triangle is isosceles (the equal sides intersect in the fixed vertex) then the condition is sufficient.

If the triangle is scalane, then the angle α at this vertex must be equal to $\frac{\pi}{n}$ (the triangle is to be reflected an even number of times).

Let $\alpha \le \beta \le \gamma$ be the angles of the triangle. Then $\gamma \ge \frac{\pi}{3}$, and there are 4 possibilities to consider:

a) $\gamma = \frac{2\pi}{6} = \frac{\pi}{3}$; b) $\gamma = \frac{2\pi}{5}$; c) $\gamma = \frac{\pi}{2}$; d) $\gamma = \frac{2\pi}{3}$.

In case a) it is clear that $\alpha = \beta = \gamma = \frac{\pi}{3}$.

In cases b) and d) the triangle must be isosceles; hence, in case d) we have $\beta = \gamma = \frac{3\pi}{10}$, and in case b) we have $\beta = \gamma = \frac{\pi}{6}$.

Case d) is the only suitable one, since $\frac{3\pi}{10}$ is not of the form $\frac{2\pi}{n}$. The vertices of acute angles cannot serve as the vertex between equal sides of an isosceles triangle in case c); therefore, the acute angles must be equal to $\frac{\pi}{m}$ and $\frac{\pi}{k}$, where $\frac{\pi}{m} + \frac{\pi}{k} = \frac{\pi}{2}$. Hence, either $m = k = 4$ or $m = 3$, $k = 6$ or $m = 6$, $k = 3$.

It is not difficult to see that these triangles meet the condition of the problem: these are an equilateral triangle lattice and a square lattice on the plane.

41.7.3 and **41.8.3.** If the lengths of both sides of the rectangle are even, then divide it into 2×2 squares. It is possible to arrange domino tiles on each square in the desired way, see Fig. 343.

FIGURE 343. (Sol. 41.7.3)

If only one of these lengths is even, then $n = 2k + 1$; divide the side whose length is odd into $k - 1$ segments of length 2 each, and one segment of length 3; divide the side whose length is even into n segments of length 2 each.

Now, the rectangle is divided into 2×2 squares and one 2×3 rectangle; it is easy to cover them with domino tiles in the desired way, see Fig. 343.

41.8.5. (Cf. Problem 33.1.9.3.) Consider any 11 successive digits of A. They form a number, T. Denote the first and the last digits of T by a and b, respectively, and let the first 10 digits of T form a number X, and the last 10 digits a number Y.

So, $T = 10X + b = 10^{10} \cdot a + Y$, whence $b = a \cdot 10^{10} + Y - 10 \cdot X$. But the right side of the last equation is divisible by 2^{10}, and $b \le 9$; hence, $b = 0$. Moving along A from left to right we find that its last 990 digits are zeros. Consequently, $A = B \cdot 10^{990}$, where by the hypothesis, B is a 10-digit number divisible by 2^{10}. Therefore, A is divisible by $2^{10+990} = 2^{1\ 000}$.

41.9.1. Connect an arbitrary vertex of the n-gon with the other vertices; we obtain $n-2$ triangles. This means that the minimal number of points is not less than $n - 2$.

Let us prove that it is possible to arrange these $n - 2$ points in the desired way. Select an arbitrary side AB and consider all triangles with side AB (there are $n - 2$ such triangles).

Very close to the vertex opposite side AB draw a point inside each of the $n - 2$ triangles obtained, so that such a point were contained in every triangle formed by a short diagonal and two sides of the polygon. Now, prove that *there is at least one such point in any triangle $\triangle XYZ$, where X, Y, Z* are vertices of the

n-gon. Indeed, one of the angles of triangle $\triangle XYZ$ contains some angle $\angle AX_kB$, where $X_k = X$ or Y or Z, and, therefore, contains a point situated close to X_k.

FIGURE 344. (Sol. 41.9.1)

FIGURE 345. (Sol. 41.9.5)

41.9.2. Assume the contrary and consider two vectors of our set whose x-coordinates are the greatest; if there are more than two such vectors select from them the two with the greatest y-coordinates. Clearly, the sum of these vectors can not be equal to the sum of the other two vectors.

41.9.4. Choose an arbitrary straight line l as a coordinate axis so that l were non-perpendicular to any of the given lines. Let A lie on l and have a sufficiently great coordinate, R. The distance from point A to any of the given points is not greater than the distance from it to any of the given straight lines (if the former are approximately equal to R, then the distance from A to an arbitrary straight line is approximately equal to $R \sin \varphi$, where φ is the angle between this straight line and l).

41.9.5. Fig. 345 shows a graph of news spread. The graph contains 22 rhombuses and 100 vertices; every vertex is endowed with a number equal to the number of days necessary for the rumors to reach this vertex from vertex 0. The 5-th of March corresponds to vertex 64: the gossips need 8 more days (till March 13) for the rumors to reach them.

41.10.1. Note that 12% can be replaced with $(50 - a)\%$, where a is an arbitrary small positive number.

Indeed, with a central symmetry of the sphere we find that the sphere is not wholly red: it has a completely white spot. The spot centrally symmetric to it on the sphere is also white. Choosing 4 vertices of the square in each one of these white spots (the squares are centrally symmetric with respect to the center of the sphere), we obtain the 8 desired vertices of the parallelepiped.

ANOTHER SOLUTION: after three reflections of the sphere with respect to the three pair-wise perpendicular planes passing through the center of the sphere the obtained sphere has not more than $8 \cdot 12\% = 96\%$ of its area painted red; therefore, there is a white point which, together with its images under the reflections, are the desired white corners of the parallelepiped.

EXTENSION. Is it true that there is an inscribed *cube* with white vertices?

41.10.2. The cop should run about the town and at every opportunity call on the center of the town, see Fig. 346. Consider the sequence of points where the robber stands when the cop stands at the center of the town. All these points lie on the outside perimeter. Observe that the robber can not sneak to the center and return on the perimeter (perhaps, at another point) in between two consecutive moments the cop was at the center. Therefore, the robber must also run around the town.

However, the cop will nevertheless catch the robber (verify on your own that if they run in the opposite directions they can not pass by each other, i.e., it cannot happen that the cop will not see the robber).

FIGURE 346. (Sol. 41.10.2) FIGURE 347. (Sol. 41.10.5)

41.10.4. The minimal n for which $2^n = \overline{...n}$ is equal to 36. Minimal or not, at least for $n = 36$ the condition is fulfilled; indeed, $2^{36} = \overline{...736}$.

Further, let us prove by induction that *If* $2^n = \overline{...an}$, *where* a *is a digit and and* n *a group of several* (k) *digits, then* $2^{\overline{an}} = \overline{...an}$.

Indeed, $2^n \equiv n + 10^k a \pmod{10^{k+1}}$ and

$$2^{\overline{an}} = 2^{n+10^k a} = 2^{n+10^k a} + 2^n - 2^n = 2^n + 2^{k+1} \cdot 2^{n-(k+1)}(2^{a\cdot 10^k} - 1).$$

Now, let us make use of the following lemma, which is proved further on.

Lemma. $2^{10^k} - 1$ *is divisible by* 5^{k+1} *for* $k \geq 2$.

Therefore, $2^{a\cdot 10^k} - 1$ being divisible by $2^{10^k} - 1$ (since $a^n - b^n$ is divisible by $a - b$) is divisible by 5^{k+1}. Hence,

$$2^{\overline{an}} \equiv 2^n \pmod{10^{k+1}} \equiv a \cdot 10^k + n \pmod{10^{k+1}} = \overline{an}. \text{ Q.E.D.}$$

Note that it is impossible to have all zeros to the left of the number n in the decimal notation of 2^n; thus a can be considered non-zero without loss of generality.

PROOF OF LEMMA. Use induction on k.

For $k = 2$ we see that $2^{100} - 1$ is divisible by 5^3, since $2^{100} = (5 \cdot 3 + 1)^{25}$, and using the binomial formula we get the desired conclusion.

If $2^{10^k} - 1 = A \cdot 5^{k+1}$, then $2^{10^{k+1}} - 1 = (A \cdot 5^{k+1} + 1)^{10} - 1$, and simplifying we get the expression divisible by 5^{k+2}. Q.E.D.

41.10.5. Consider four vectors (see Fig. 347) $\mathbf{v}_1 = (a, b)$, $\mathbf{v}_2 = (c, d)$, $\mathbf{v}_3 = (e, f)$, $\mathbf{v}_4 = (g, h)$. The six numbers are equal to the pairwise inner products of these vectors:

$$ac + bd = (\mathbf{v}_1, \mathbf{v}_2), \quad ae + bf = (\mathbf{v}_1, \mathbf{v}_3), \ \dots, \quad eg + fh = (\mathbf{v}_3, \mathbf{v}_4).$$

But one of the angles between the vectors is not greater than $\frac{\pi}{2}$; therefore, one of the six inner products is non-negative.

42.7.1. a) For example, divide the plane into 5 similar angles α_1, α_2, α_3, α_4, α_5 ($\alpha_i = 72^\circ$) and inscribe a disc into each of the angles: $\alpha_1 \cup \alpha_2$, $\alpha_2 \cup \alpha_3$, $\alpha_3 \cup \alpha_4$, $\alpha_4 \cup \alpha_5$, $\alpha_5 \cup \alpha_1$, see Fig. 348.

FIGURE 348. (Sol. 42.7.1)

b) Consider the angles with vertices at point O, with the discs inscribed into them. Since each angle is less than 180°, their sum is less than $4 \times 180^\circ = 720^\circ$. But if an arbitrary ray from point O intersects at least two of these four discs, this sum is not less than $2 \times 360^\circ = 720^\circ$. Contradiction.

42.7.2. Let there be n weights, and let p_k be the mass of k-th weight. Let

$$p_1 \leq \frac{1}{2}, \quad p_2 \leq \frac{1}{4}, \quad \dots, \quad p_n \leq \frac{1}{2^n};$$

then the total mass of all weights is not greater than $1 - \frac{1}{2^n}$, which contradicts the hypothesis.

42.7.3. In this problem for brevity $S(P)$ denotes the area of the lamina bounded by polygon P.

The square inscribed in the given disc has an area greater than the area of any rectangle inscribed in the same disc (prove this yourself).

This means that

$$\tfrac{\pi}{2} S(P) \leq S(K) \qquad (*)$$

for rectangle P inscribed in disc K. Let a square K' of area $S(K')$ be divided into the rectangles P_1, P_2, $P_3, \dots, P_n$.

Then taking into account that the area of the disc circumscribed around K' is equal to $\frac{\pi}{2} S(K')$ and using inequality $(*)$ we obtain:

$$\frac{\pi S(K')}{2} \leq \frac{\pi(S(P_1) + \cdots + S(P_n))}{2} \leq S(K_1) + \cdots + S(K_n)$$

where K_i is the disc circumscribed around the rectangle P_i. Q.E.D.

REMARK. The inequality becomes an equality if and only if all the rectangles are squares.

42.7.4. Vitya must choose the center of some square and mark the points symmetric with respect to this center to point Kolya marked. It is easy to see that Vitya always has a move (the convex polygon remains convex after his move). In 6 moves, Kolya will be able to mark only the nodes lying in 6 shaded triangular domains constructed by each side of a hexagon and the extensions of its neighboring sides, see Fig. 349. Thus, Kolya has only a finite number of possibilities. Consequently, Vitya wins in a finite number of moves. (Cf. the solution to Problem 31.2.7.1.)

42.8.3. Draw a chord AE perpendicular to DA. We see that $\smile AE = \smile CB = 180^\circ - \smile DA$. It follows from the right triangle ADE, see Fig. 350, that $OH = \dfrac{AE}{2}$. Q.E.D.

42.8.5, **9.5** and **10.5.** Arrange all scientists in one line and number them from left to right. Then ask every one, beginning with the first, about his/her right-hand neighbor. Let N be the first number for which the N-th scientist said of his/her right neighbor — the $(N+1)$-st scientist — that the latter is an alchemist; until this moment all scientists said that their neighbors were chemists. Then ask the pair $(N, N+1)$ to get out of the line and ask the $(N-1)$-st scientist about his/her new right-hand neighbor, etc.

FIGURE 349. (Sol. 42.7.4) FIGURE 350. (Sol. 42.8.3)

At last, having asked several pairs to get out of the line, we come to the end of the line. Everyone standing in the line said about his/her right-hand neighbor that the latter was a chemist, and one person of every pair said about the other one that the latter was an alchemist. Therefore:

1°. There is at least one alchemist in every pair; hence,

2°. There are more chemists than alchemists left in the line; in particular, there is at least one chemist in it.

3°. If the i-th person in the line is a chemist, then everybody else is also a chemist. In particular, the last person in the line is always a chemist and if there is more than one person in the line, then the last two are chemists.

Until this moment we used $k-1$ questions. Therefore, if there is more than one person in the line, it suffices to ask the last one $k-2$ questions about all others in the line. The total number of questions is equal to $(k-1)+(k-2)=2k-3$. If the obtained "chain" consists of one person (and (s)he is a chemist), then there is exactly one alchemist in every pair (otherwise, there would have been more alchemists than chemists). Now, it suffices to ask one question about one of the scientists of every pair, and the number of questions is not greater than $k-1+\frac{1}{2}k=\frac{3}{2}k-1$.

REMARK. P. M. Blecher (a winner of the 26-th and 28-th Moscow Math Olympiads) proved that the exact answer to the problem is $\left[\frac{3}{2}(k-1)\right]$, see his paper *On a Logical Problem*, Discrete Math. 43, 1983, 107–110. Here is a slightly edited version of the paper, cf. also P. M. Blecher and L. Czirmaz, *About truth-tellers, liars and inscrutable people* (in Hungarian), Kozepiskolai Matematikai Lapok 62(4), 1981, 145–150.

About truth-tellers, liars and inscrutable people

Pavel M. Blecher

In this note I will consider the following logical problem.

PROBLEM. *There is a group of N persons, some of which are reliable and the rest are unreliable. It is known that the reliable persons constitute a majority. A reliable person tells only the truth to all questions asked while an unreliable one sometimes answers the truth and sometimes lies. A mathematician (not belonging to the group) wants to find out "who is who" in the group. For this (s)he may ask any person about any other one in the group if the latter is a reliable person or not. What is the least number of questions by which (s)he can find out for sure who is who in the group?*

Let $Q(N)$ be the least number of questions. The first (to my knowledge) upper bound, $Q(N)\le 2N-3$ was obtained by Konyagin, the author of the problem. A little later I proved the estimate $Q(N)\le\left[\frac{3}{2}(N-1)\right]$. After that, another proof of this estimate was found by Shlosman.

As concerns the lower bound, Ruzsa proved that $Q(N)\ge\left[\frac{1}{4}(5N-3)\right]$ and Galvin improved his result to the following: If $N\ge 5$, then

$$Q(N)\ge\begin{cases}\frac{4}{3}N-1 & \text{if } N\equiv 0 \pmod 6\\ \left[\frac{4}{3}N\right] & \text{otherwise.}\end{cases}$$

The purpose of this note is to prove the following result.

Theorem. $Q(N)=\left[\frac{3}{2}(N-1)\right]$ *for* $N\ge 3$.

Let at first N be odd, $N=2k+1$. We must prove that $Q(N)=3k$. For this we first prove that $Q(N)\le 3k$ and next that $Q(N)\ge 3k$. As a matter of fact, at the first stage we give an algorithm which solves the problem for $3k$ questions and at the second stage we prove that the problem cannot be solved for lesser number of questions.

The upper bound, $Q(N)\le 3k$.

ALGORITHM. Let us enumerate the persons and ask the 2-nd, 3-rd, etc.persons whether or not the 1-st person is reliable. We will stop as soon as one of the following events will occur:

Event (A). k persons said that the first person is reliable. Then the first person is reliable indeed, and all those who said "no" are unreliable. We ask now the first person about all other persons. The direct computation shows that we use $3k$ questions in this procedure.

Event (B). The number of those who said "no" exceeds the number of those who said "yes". In this case if m persons said "no", then $(m-1)$ person said "yes", $m = 1, 2 \ldots$.

Moreover, it is easy to verify that in the group of the first $2m$ persons the number of unreliable persons is not less than the number of reliable ones, so in the remaining group of $2k+1-2m = 2(k-m)+1$ persons the reliable persons form a majority. So using $3(k-m)$ questions we can sort them out.

Next we choose a reliable person among them and ask about the 1-st person and about those among the first $2m$ persons whose answers about the 1-st one were truthful (others are, clearly, unreliable). In such a way we sort out all persons by $2m-1+3(k-m)+1+m = 3k$ questions, Q.E.D. The upper bound is proved.

The above-mentioned Shlosman's proof of the upper bound is based on another algorithm. It seems to be more complicated but in some sense it is more economic: *If the group contains not more than M $(< \frac{N}{2})$ unreliable persons, they are detected not later than after $N+M$ questions.*

The lower bound, $Q(N) \geq 3k$.

Assume that $Q(N) \leq 3k-1$. Now, we will give a strategy of answers and show that there always exists at least two partitions of the group of N persons into reliable and unreliable persons which agree with all $Q(N)$ answers. Let us divide the interrogation into two stages.

Stage 1°. The first $k-1$ questions. All answers $a_1, \ldots, a_{k-1}$ are "no". Let $(s_1, s_1'), \ldots, (s_{k-1}, s_{k-1}')$ be the sequence of the pairs of persons in the game (we ask the s_i-th person about the s_i'-th person during the i-th question). Let G be a (nondirected) graph whose vertices are

$$V = \{s: \text{ there exists } i = 1, \ldots, k-1 \text{ such that } s = s_i \text{ or }, s = s_i'\}, \text{ i.e., } V = \cup_{i=1}^{k-1}\{s_i, s_i'\},$$

and the edges are $(s_1, s_1'), \ldots, (s_{k-1}, s_{k-1}')$. Let $G_1, \ldots, G_r$ be the connected components of the graph G and $V_1, \ldots, V_r$ be the sets of vertices of the subgraphs $G_1, \ldots, G_r$. Let W be the complement of V to the whole set of N persons.

Finally, let $l_1, \ldots, l_r$ be the number of edges of the subgraphs $G_1, \ldots, G_r$, respectively. Then

$$l_1 + \cdots + l_r = k-1. \tag{$*$}$$

Stage 2°. The last $2k$ questions. Define a set V°, $V^\circ \subset V$, depending on i as follows. First of all, $V^\circ = V^\circ[i]$ contains at most one vertex from any V_m, $m = 1, \ldots, r$. Let

$$V[i] = \{s: \text{ there exist } j, k \leq j \leq i, \text{ such that } s = s_j'\} = \cup_{j=k}^{i}\{s_j'\}$$

and

$$V_m[i] = V[i] \cap V_m, \quad m = 1, \ldots, r.$$

If $V_m[i] \neq V_m$, then, by definition, V° contains no point of V_m.

If $V_m[i] = V_m$, then we take the minimal number n such that $V_m[n] = V_m$ (clearly, n depends on m) and put $v_m^\circ = s_n'$. In such a case v_m° belongs to $V^\circ[i]$. Thus, the set $V^\circ = V^\circ[i]$ is defined.

The answers $a_k, \ldots, a_{3k-1}$ are defined by the following rules (as before, we ask the s_i about the s_i', $i = k, \ldots, 3k-1$):

$$a_1 = \begin{cases} \text{"yes"} & \text{if } s_i' \in W, \\ \text{"no"} & \text{if } s_i' \in V \setminus V^\circ, \\ \text{"yes"} & \text{if } s_i' \in V^\circ. \end{cases}$$

Now, let us present a configuration S of persons which agrees with the answers $a_1, \ldots, a_{3k-1}$. To do this, we consider the (finite) set $V^\circ = V^\circ[3k-1]$ and an auxiliary set V'°, $V'^\circ \subset V$, which is constructed in such a way that each connected component V_m contains exactly one point either from V° or from V'°. More precisely, if V_m has a point $v^\circ \in V^\circ$ it contains no point of V'°; if V_m has no points of V°, then the set $V_m \setminus V[3k-1]$ is non-empty and we choose an arbitrary point $v_m'^\circ \in V_m \setminus V[3k-1]$ as a representative of V_m in V'°. Thus, the sets V°, V'° are defined, and we put in the configuration S:

$$s \text{ is } \begin{cases} \text{"reliable"} & \text{if } s \in W \cup V^\circ \cup V'^\circ \\ \text{"unreliable"} & \text{if } s \in V \setminus (V^\circ \cup V'^\circ). \end{cases}.$$

We state that

(i) S agrees with the answers $a_1, \ldots, a_{3k-1}$;

(ii) the number of unreliable persons in S does not exceed $k-1$;

(iii) Let $s \notin V[3k-1]$ (the last set contains at most $2k$ points so such an s does exist); moreover, if V'° is non-empty let $s \in V'^\circ$; then the change of the type of s gives a configuration S' which agrees as well as S with the answers $a_1, \ldots, a_{3k-1}$.

Thus, the sequence of answers $a_1, \ldots, a_{3k-1}$ does not distinguish S and S' in contradiction with our assumption.

Now, let us verify the statements (i) – (iii).

(i) The following takes place:

(a) the answers $a_1, \ldots, a_{3k-1}$ are truthful;

(b) $s \in W$ did not give the answers $a_1, \ldots, a_{k-1}$;

(c) $s = v_m^\circ = V^\circ \cap V_m$ (or $s = v_m'^\circ = V'^\circ \cap V_m$) tells in the answers $a_1, \ldots, a_{k-1}$ only about $s' \in V_m, s' \neq s$, and so his/her answers "no" are truthful.

Thus, all reliable persons $s \in W \cup V^\circ \cup V'^\circ$ answer only the truth, as stated.

(ii) Let v_m be the number of points of the set V_m, $m = 1, \dots, r$. Then $v_m \le l_m + 1$ (the graph G_m is connected). By construction, the number of unreliable persons in V_m is $v_m - 1$, so their total number is

$$\sum_{m=1}^{r} (v_m - 1) \le \sum_{m=1}^{r} l_m = k - 1$$

(see $(*)$ above), as stated.

(iii) Let $s \notin V[3k-1]$. By construction, $V^\circ \subset V[3k-1]$, so $s \notin V^\circ$. Assume that $s \in W$. Then s is reliable in S and unreliable in S'. Moreover, nobody asked about him/her.

Therefore, S', as well as S, agrees with the answers $a_1, \dots, a_{3k-1}$.

Now, consider $s \in V \setminus V^\circ$. Let $s \in V_m$. Then $s \notin V[3k-1]$ implies that V_m does not contain any element of V°, so V_m consists only of unreliable persons, except maybe s itself (if $s = v_m'^\circ \in V'^\circ$). So the answers $a_1, \dots, a_{3k-1}$ of the person s are truthful; hence, all his/her answers are truthful. Therefore, we can change his/her type and remain in agreement with the answers $a_1, \dots, a_{3k-1}$. (Here we have made use of the fact that the answers about s are only from unreliable persons from V_m.) This completes the proof.

The case $N = 2k+2$ is considered in the same manner.

In conclusion let us mention a generalization of the problem.

Assume that it is additionally known that the number of unreliable persons does not exceed M $\left(0 < M \le \left[\frac{1}{2}(N-1)\right]\right)$. *What is the least number of questions* $Q(N, M)$ *in this case*?

Repeating the proof of the lower bound (with $M-1$ questions at the first stage) one can prove that $Q(N, M) \ge N + M - 1$. Moreover, repeating the proof of the upper bound (with M positive answers in event A one can prove that $Q(N, M) \le N + M - 1$. Thus, $Q(N, M) = N + M - 1$.

42.9.1. Let D be the greatest value of d. Then considering the set of 55 stones each of mass $\frac{20}{11}$ we can see that $D \ge \frac{10}{11}$. Let us prove that $D = \frac{10}{11}$.

Assume that $D > \frac{10}{11}$, i.e., there exists a collection of stones with a total mass of 100 kg such that the mass M of any of its subcollection satisfies $|M - 10| > \frac{10}{11}$. Consider then a subcollection with $M > 10$ whose mass is less than 10 kg after any stone is thrown away (clearly, such a subset exists). Let $x_1, \dots, x_k$ be masses (in kg) of the stones from this subcollection, $M = x_1 + \dots + x_k$. By hypothesis, $M > 10 + \frac{10}{11}$ but $M - x_i < 10 - \frac{10}{11}$ for any i. This means that the mass x_i, $i = 1, \dots, k$, of every stone is greater than $\frac{20}{11}$.

By hypothesis $x_1 \le 2, \dots, x_k \le 2$; hence, $k > 5$. But now it is clear that $M' = x_1 + x_2 + x_3 + x_4 + x_5$ satisfies $5 \cdot \frac{20}{11} < M' < 10$, i.e., the mass of the first five stones differs from 10 kg by less than $10 - \frac{100}{11} = \frac{10}{11}$. Contradiction.

42.9.2. For example, consider a straight line a and draw two skew lines b and c perpendicular to a and intersecting a. Consider then the set of planes parallel to one another and to a and draw straight lines through the intersection points of lines b and c with each of these planes. Lastly, perform all possible rotations of the space around the axis a. The images of the straight lines constructed have the desired property.

Remark. Another (non-elementary) solution is to consider the whole space as the union of cofocal one-sheeted hyperboloids, each of which is "woven" from pairs of intersecting straight lines (the so-called line structure, or a movable rod model, of a one-sheeted hyperboloid). For the corresponding construction see the book [CH].

42.9.3. a) Construct the sequence in parts.

The 0-*th part* consists of one term: $a_0 = 1$, *the* n-*th part* consists of $\frac{b_n}{2} + 1$ terms:

$$\frac{b_n^2}{2}, \quad \frac{b_n^2}{2} + b_n, \quad \frac{b_n^2}{2} + 2b_n, \ \dots, \ b_n^2, \text{ where } b_n = 2^{3^{n-1}}.$$

The sum of the terms of the n-th part is equal to

$$\frac{3}{4} \cdot b_n^2 \left(\frac{1}{2} b_n + 1\right) = \frac{3}{8} b_n^3 + \frac{3}{4} b_n^2 \le b_n^3 - b_n = b_{n+1} - b_n.$$

Consequently, the sum of the terms of the preceding parts is not greater than $b_n - b_0$, i.e., is less than b_n; hence, no term of the n-th part is equal to the sum of some other terms.

Consider now the N-th term of the sequence. The number of terms of the $(n-1)$-st part is $\frac{1}{2} b_{n-1} + 1$ which means that the index of any term of the n-th part is greater than $2^{3^{n-2}}$ and the value of the index is not greater than

$$a_N \le b_n^2 = 2^{2 \cdot 3^{n-1}} = 64(2^{3^{n-2}-1})^6 < 64 \cdot N^6;$$

hence, $a_N < 64 \cdot N^6$ for any N; therefore, $a_N < N^7$ for $N > 64$. For $N \le 64$ the inequality $a_N < N^7$ is directly verified.

b) Prove on your own that even the number a_5 cannot be chosen so as to satisfy the condition of the problem.

42.10.2. Cut the segment $[0,1]$ into 10 equal parts. Then move all odd parts to the right by $\frac{1}{10}$ and all the even parts to the left by $\frac{1}{10}$. The set of marked intervals passes then into the set of intervals not intersecting with the marked ones (this follows from the hypothesis), and the total length of the marked and obtained intervals is not greater than 1. Consequently, the sum of the lengths of the marked intervals is not greater than $\frac{1}{2}$. Q.E.D.

42.10.3. Suppose there is a function f such that $f(0) = f(1) = 0$, $f''(x) < 1$ for all $x \in [0,1]$, and $f(a) > \frac{1}{8}$ for some $a \in \,]0,1[$. Set

$$h(x) = f(x) - \frac{f(a)}{g(a)} g(x).$$

As $g''(x) = -1$ and $f(a) > \frac{1}{8} > g(a) > 0$, then $h(0) = h(1) = 0$ and $h''(x) = f''(x) + \frac{f(a)}{g(a)} > 0$. Besides, $h(a) = 0$. Since $h''(x) > 0$, it follows that $h'(x)$ is monotonously increasing. This means that $h(x)$ does not change the sign on one of the segments $[0,a]$ or $[a,1]$. But then

$$\text{either } h(0) = -\int_0^a h'(x)dx \neq 0 \text{ or } h(1) = \int_a^1 h'(x)dx \neq 0;$$

and either case leads us to a contradiction.

42.10.4. Choose the greatest disc; let r be its radius. Consider the concentric disc of radius $3r$. Take away all discs which are inside this second disc; the remaining discs do not intersect the initial one. Choose the disc with the greatest radius among the remaining discs and repeat the procedure. Do this until you obtain several enlarged discs, the area of their union being greater than 1. The initial discs corresponding to them (with radii that are three times shorter) do not intersect, and their total area is not greater than $\frac{1}{9}$.

43.7.2. Let us consider a table with the sum of numbers at vertices of any rectangle (with sides directed along the paper's sides) is not equal to each other. Let us prove that such a table cannot have two 1' or two -1's.

Let us consider 4 small squares — the vertices of a rectangle — with not all numbers equal to each other. Let them, for example be $\boxed{\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}}$. Let us consider two more numbers (squares) placed as follows: $\boxed{\begin{array}{ccc} x & 1 & -1 \\ y & 1 & 1 \end{array}}$. Clearly, the only possibility is $x = y = 1$ since otherwise this is a rectangle with a zero sum. Therefore, both rows and both of the initially considered columns contain only 1's (except one -1). Taking instead of the second row any other one, we again obtain $\boxed{\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}}$ and, therefore, this other row also consists of 1's only.

Therefore, there is only one -1 in the whole table.

43.7.3. Fix an inner polygon and orient its sides counter-clockwise. To each side of the fixed inner polygon there corresponds the vertex of the 100-gon, namaly the vertex towards which the arrow of the side of the fixed polygon is directed, see Fig. 351.

FIGURE 351. (Sol. 43.7.3)

FIGURE 352. (Sol. 43.8.3)

Owing to convexity of the polygons, distinct vertices correspond to distinct sides. Therefore, the number of sides of the inner polygon is not greater than that of 100-gon. (Cf. Problem 13.2.9-10.1.)

43.7.4. First, the cop should check one of the corridors and then return to the center. Then (s)he — the cop — has to go to the middle of the second corridor and return back. During this time the robber cannot sneak unnoticed from the third corridor to the first one because for this (s)he — the robber — had to run $2r$ units which is more than a half length of the corridor. Hence, the robber is either in the third corridor or at the end of the second one (not closer to the center than by $2r$).

Now, the cop investigates the third corridor ($3r$ units inside the corridor and $3r$ units backwards if the cop does not find the robber). The robber will not be able to sneak unnoticed once again, since (s)he will have to run $2r$ units to the crossing point and r units more from it. Now, the cop can investigate without hurry the second corridor.

43.7.5. Let us circumscribe the circle around the 20-gon and rotate the 20-gon around the center through an angle equal to the arc of the circle whose endpoints coincide with the endpoints of the black diagonal. Thus, one black point will pass into another black point (and into another endpoint of the arc). This means that a certain white point passes into another white point. These two white points are connected by a diagonal of the same length as the black diagonal.

ANOTHER SOLUTION. Let k be the number of diagonals with 2 black points; l that with one black and one white points and m the number of diagonals with 2 white points. Let us count all black endpoints of diagonals; there are $2k + l$ of them.

Since each of the 10 black verteces is an endpoint of 2 diagonals, $2k + l = 2 \cdot 10$. Similarly, $2m + l = 20$. Hence, $k = m$, Q.E.D.

43.8.1. Let $0 < a < b$. Consider an arbitrary $k > 1$. Then $a < b < bk$ (the left-hand side is smaller than 1, the right-hand side is greater than 1), whence $ak + a < ak + bk$. Dividing by $k(k + 1)$ we get $\frac{a}{k} < \frac{a+b}{k+1}$. Therefore, in our case

$$a_1 \leq \frac{1}{2}(a_1 + a_2) \leq \frac{1}{3}(a_1 + a_2 + a_3) \leq \dots \leq \frac{1}{6}(a_1 + \dots + a_6) \leq \frac{1}{10}(a_1 + \dots + a_{10}).$$

43.8.3. It suffices to prove that $\beta = 2\alpha$, see Fig. 352, whence it follows that the triangle BCD is isosceles.

To prove this, connect O with A and D. The angles $\angle AOD$ and $\angle ACD$ subtend on the same arc; therefore, $\angle AOD = \beta$. But the angle $\angle AOD$ is a central one in the circle centered at O, and angle $\angle ABD$ is inscribed in the circle, therefore, $\angle ABD = \dfrac{\angle AOD}{2}$, i.e., $\beta = 2\alpha$.

43.9.2. Let us prove it by induction. Let the number of the states of all lamps on the desk be equal to 2^k, when there are n switches on the panel (for $n = 1$ there are exactly $2 = 2^1$ states: the switch on or the switch off).

Altogether we have 2^k states with the new switch on and we have 2^k states with the new switch off. Let us show that these two sets of states either do not intersect (and then we have 2^{k+1} states) or coincide.

By the hypothesis, as we switch several switches, a tuple of several lamps change their states and this tuple does not depend on the initial position of the switches. Therefore, if two sets of states intersect there exists a set of switches (the new switch included) such that if we apply them the state of the tuple does not vary. But then it does not vary for any initial state of the switches and, therefore, both sets of the states coincide.

43.9.3. Mark the upper right-hand vertex of every square. No two marked vertices of distinct squares coincide (by the hypothesis). This means that the number of squares on the paper is not greater than the number of nodes of the paper not lying on its lower left-hand part and whose number is exactly mn. (Prove on your own that mn squares can be uniquely arranged in the desired way.)

43.9.4. Let the cop first act as in the solution of Problem 43.7.4 and then run to and fro along the 2-nd and 3-rd corridors in order for the robber was unable to escape to the 1-st corridor. The limit distance x the cop can run inside of a corridor is determined from the condition that while (s)he runs along $AOBO$, see Fig. 353a), (the distance is equal to $3x$) the robber should not have time to run a distance of $r + x + r$. Therefore, $3x = 2(x + 2r)$, i.e., $x = 4r$. Hence, if $r > \frac{1}{5}l$, then the cop will corner the robber (if $r = \frac{1}{5}l$ the cop will also corner the robber but after an infinite lapse of time).

Now, let $r > \frac{1}{7}l$. Then the cop must run as in the preceding paragraph until x becomes greater than $2l - 10r$ (this is possible because $2l - 10r < 4r$). After that, the cop must run along the route $AOCO$, see Fig. 353 b). During this time the robber can escape into the upper corridor but will be unable to advance further than by $y = l - 2r - \frac{1}{2}x < 3r$ from the center. In order to catch the robber the cop should run the distance of $2y - 2r$ and, if the robber is not there, return to the center.

FIGURE 353. (Sol. 43.9.4)

During this time the robber will be able to sneak again but not further than by $y_1 = 2y - 3r < y$ away from the center, in other words, the robber becomes closer to the cop than at the previous stage, see Fig. 353 c). Therefore, the robber will eventually get caught.

43.10.4. Suppose there exists a table that contradicts the statement of the problem. It can not have more than 75% of zeros, since otherwise there would have existed a 2×2 square consisting of zeros only. Therefore, no less than $\frac{1}{8}$ of the total number of the numbers are 1's and there are exactly as many -1's (because the sum is equal to 0). Therefore, there exists a row that contains 11 units and a column that contains as many -1's. Let us form a 10×10 square from the rows that contain the chosen -1's and the columns that contain the chosen 1's (a chosen row and a chosen column excluded):

$$\begin{array}{cccccc}
-1 & 1 & & & & \\
-1 & 1 & & & & \\
\vdots & & & & & \\
-1 & & & & & \\
-1 & & & & -1 & -1 \\
 & 1 & 1 & \dots & 1 & 1
\end{array}$$

The square obtained cannot have two 1's in a column nor two -1's in a row. But then this 10×10 square has not more than 10 units and the number of -1's in it is also not more than 10. Therefore it has not less than 80 zeros — more than 75% — and there is a square consisting of 4 zeros. Contradiction.

43.10.5. Let α_1, $\alpha_2, \dots$ be the mentioned arcs. By the hypothesis $\alpha_1 + \alpha_2 + \dots < \pi$. The great circle whose points are equidistant from a point S on the sphere is called the *circle polar to* S, and the point S a *pole*.

Consider an arbitrary arc α_i; choose a point S on it and the circle O_S polar to S. Obviously, the circle polar to an arbitrary point on the circle O_S passes through S. Let now S move along the arc α_i from end to end; the polar circle O_S will pass the domain shaded in Fig. 354.

The area of this domain is $2\frac{\alpha_i}{2\pi}$, the $\frac{\alpha_i}{\pi}$-th part of the area of the whole sphere. The circles which are polar to any point of this domain (and only they) intersect the arc α_i.

Construct analogous domains for all arcs $\alpha_1, \alpha_2 \dots$. These domains do not cover the whole sphere, since $\sum\limits_i \frac{\alpha_i}{\pi} < 1$. Consequently, there exists a point not covered by these domains. The plane of the circle polar to this point is the desired one.

44.7.3. Put the second paper disc on top of the first one so that the dragons coincide; in particular, the eye of the dragon painted on the second paper coincides with the eye of the first dragon, i.e., it is situated in the center O, see Fig. 355. Observe that this does not necessarily mean that the discs coincide.

Cut the second paper disc along the arc AnB; see Fig. 355, and put the cut part on top of the part of the first disc corresponding to it. We get the result desired.

44.7.5. Choose an arbitrary pair of weights and weigh it. Clearly, from the result one immediately deduces the mass of each of the weights: for example, if the total is 2004 g, the weighs are 1000 g and 1004 g. If a 1000 g weigh is among the weighs, we will determine it at the 2-nd weighing.

FIGURE 354. (Sol. 43.10.5) FIGURE 355. (Sol. 44.7.3)

44.8.5. Let $A = LCM(a_1, \dots, a_{10})$. Then $A = k_1 a_1 = \dots = k_{10} a_{10}$, where $k_1, \dots, k_{10}$ are integers, and the condition implies that $k_1 > k_2 > \dots > k_9 > k_{10}$. This means that the lowest value of k_1 is 10, otherwise $k_{10} = 0$, i.e., $A = k_1 a_1 \geq 10 a_1$.

44.9.1. Let a be the original $(2l + 1)$-digit number and a_k the number obtained from a by deleting its k-th digit. For any integer x let $f(x)$ be the difference between the total number of 7's in even decimal places and the total number of 7's in odd decimal places of x. Then $|f(a_{k+1}) - f(a_k)| \leq 1$ for $k = 1, \dots, 2l$ because the corresponding $2l$-digit numbers only differ in one place. Hence, either some of the $f(a_k)$ is equal to 0 (our goal) or all of them are of the same sign.

Let us show that $-f(a_1)$ and $f(a_{2l+1})$ differ not more than by 1. Indeed, let us strike out the first digit in the initial number and then move to the first place the last digit of the number obtained. Denote the result by b. This movement interchanges the parity of the digits' places (makes the odd places even and vice versa). Therefore, $f(b) = -f(a_1)$. But the numbers b and a_{2l+1} only differ in the first digits. Therefore, $-f(a_1)$ and $f(a_{2l+1})$ differ by not more than 1 and the numbers $f(a_1)$ and $f(a_{2l+1})$ cannot be of the same sign. Q.E.D.

44.9.2. Clearly, if a and b are integers, $0 \leq a \leq k$ and $1 \leq b \leq k - 1$, then at least one of the inequalities $|a + b| \leq k - 1$ or $|a - b| \leq k - 1$ holds. Therefore, there are sets of $+$ and $-$ signs such that the inequalities

$$\begin{gathered} |a_n \pm a_{n-1}| \leq n - 1, \\ |a_n \pm a_{n-1} \pm a_{n-2}| \leq n - 2, \\ \dots\dots\dots\dots\dots\dots\dots \\ |a_n \pm a_{n-1} \pm \dots \pm a_1| \leq 1 \end{gathered}$$

hold simultaneously. But the numbers $a_n \pm a_{n-1} \pm \dots \pm a_1$ are even, since $a_1 + \dots + a_n$ is even. Consequently, the sum is equal to 0.

44.9.3. First, let us prove the following statement.

If M is a convex polygon, S is its area, P its perimeter, and r the radius of the greatest disc that can be placed inside the polygon, then $\frac{r}{2} \leq \frac{S}{P} < r$.

PROOF. The first inequality is obtained by calculating the sum of areas of the triangles into which M is divided by (extentions of) all radii drawn to its vertices, see Fig. 356 a). To prove the second inequality, construct (inwards) rectangles of hight $\frac{S}{P}$ on the sides of M as on the bases, see Fig. 356 b); the rectangles can partly intersect one another (as illustrated), can partly lie outside M, and their total area is equal to S. This means that there is a point inside M which is not covered by these rectangles.

Denote the radii of the greatest discs which can be placed inside polygons X and Y by $r(X)$ and $r(Y)$, respectively. Then

$$\frac{S(X)}{P(X)} < r(X) \leq r(Y) \leq 2\frac{S(Y)}{P(Y)}, \quad \text{Q.E.D.}$$

44.9.4. First, consider two sets of positive integers: the set A consists of all integers which have 0's only in all even decimal places counting from right to left, and the set B containing integers which have 0's only in the odd decimal places.

Clearly, every integer is uniquely represented in the form of $n = a + b$, where $a \in A$, $b \in B$.

The desired partition of the set of natural numbers into congruent subsets is the following one:

$$\mathbb{Z} = \cup_{b \in B} A_b, \text{ where } A_b = \{x + b : x \in A\}.$$

FIGURE 356. (Sol. 44.9.3)

44.9.5. The following general statement holds:

In a regular n-gon ($n \geq 4$), let $k \geq [\frac{1}{2}(\sqrt{8n+1}+1)]+1$ vertices be marked. Then there exists a trapezoid with its vertices in the marked points. Our case: $n = 1981$, $k = 64$.

Indeed, the points $A_1, \dots, A_{k-1}, A_k$ define $k(k-1)$ distinct arcs on the circle with endpoints in these points. Since $k > [\frac{1}{2}(\sqrt{8n+1}+1)]+1$, i.e., k is greater than the greater root of the equation $x(x-1) = 2n$, it follows that $k(k-1) \geq 2n+1$.

Assuming the arc length between two neighboring vertices of the polygon for the unit of length, we see that the lengths of all arcs considered are integers 1 to $n-1$. There are $2n+1$ arcs altogether and, by Dirichlet's principle, either

a) some three lengths a, b, c are encountered 3 times each, and at least one of them is not equal to $\frac{1}{3}n$ or $\frac{2}{3}n$, or

b) one of the lengths is encountered 4 times.

In case b) two of the arcs of equal length have no common endpoints; in case a) if any two of the three arcs of equal length have a common endpoint, then the measure of such an arc is equal to either $\frac{2}{3}\pi$ or $\frac{4}{3}\pi$.

Thus, for any one of the two possible cases there are two arcs of the same length, out of the total number of $k(k-1)$ arcs, which do not have common endpoints. The endpoints of these arcs are the vertices of the trapezoid to be found.

44.10.1. The condition implies

$$f(x+k) = \frac{1+f(x)}{1-f(x)}.$$

Then

$$f(x+2k) = -\frac{1}{f(x)}; \qquad f(x+4k) = -\frac{1}{f(x+2k)} = f(x).$$

This means that $f(x)$ is a periodic function with a period of $4k$.

44.10.2. Consider the following polynomials (the i-th differences of the polynomial P for $i = 1, \dots, n$):

$$\begin{aligned}
P_1(x) &= P(x) - P(x-1),\\
P_2(x) &= P_1(x) - P_1(x-1),\\
&\dots\dots\dots\dots\\
P_i(x) &= P_{i-1}(x) - P_{i-1}(x-1),\\
&\dots\dots\dots\dots\\
P_n(x) &= P_{n-1}(x) - P_{n-1}(x-1).
\end{aligned}$$

Clearly, $\deg P_i(x) = n - i$ and the hypothesis implies that $P_i(k)$ is divisible by p for any integer k. But $P_n(x) = n!$ (investigate why); therefore, $n!$ is divisible by p.

44.10.3. Use the following trigonometric inequality, a corollary of the formula for the sine of a difference:

$$|\sin(a-b)| \leq |\sin a| + |\sin b|. \tag{$*$}$$

Assume that on the contrary $\sin(k^2) \to 0$ as $k \to \infty$. Choose $d < \frac{|\sin 2|}{8}$ and N such that $|\sin(n^2)| < d$ for any $n > N$. Using inequality $(*)$ twice we get:

$$|\sin(2n+1)| = |\sin((n+1)^2 - n^2)| \le |\sin(n+1)^2| + |\sin(n^2)| < 2d,$$
$$|\sin 2| = |\sin((2n+3) - (2n+1))| \le |\sin(2n+3)| + |\sin(2n+1)| < 2d + 2d = 4d;$$

hence, contradiction:

$$|\sin 2| < 4d < 4\frac{|\sin 2|}{8} = \frac{|\sin 2|}{2}.$$

REMARK. This even makes clear that the inequality $|\sin n^2| < d$ is impossible for *three* consequtive n's.

44.10.4. Project all links of the broken line on two sides of the square, one horizontal and one vertical. Then the sum of the projections' lengths on one of these sides (let it be the horizontal one) is greater than 100; therefore, some point of this side is covered by at least 101 projections. Consequently, some vertical line intersects the broken line at least 101 times.

44.10.5. Let r be the radius of the inscribed circle and h_i the heights of the triangle. Then $\frac{1}{r} = \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3}$. To prove this, express the area S of the triangle in terms of its sides and heights and compare with the same area expressed in terms of the sides and the radius of the inscribed circle.

By the hypothesis

$$h_1 + h_2 + h_3 = 13, \quad \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{3}{4} \tag{$*$}$$

with integer h_i's. Since $h_1 \le h_2 \le h_3$, then $h_1 \le 4$ and $h_3 \le 6$. After considering various possibilities we solve $(*)$; namely, $h_1 = 3$, $h_2 = 4$, $h_3 = 6$. Hence, the ratio of the sides is $4 : 3 : 2$; denote them by $4x$, $3x$, $2x$. From *Heron's formula* we find $x = \frac{8}{\sqrt{15}}$. Hence, the answer: $a_1 = \frac{32}{\sqrt{15}}$, $a_2 = \frac{24}{\sqrt{15}}$, $a_3 = \frac{16}{\sqrt{15}}$.

45.7.1. Find successively: $y_1 = \frac{1}{x}$, then $y_2 = x + 1$, hence, $y_3 = \frac{1}{x+1}$ and, therefore, $y_4 = y_1 - y_3$; consequently, $y_5 = \frac{1}{y_4}$, and, lastly, $y_6 = y_5 - x = x^2$.

Now find the product xy, where $x > y > 0$. Find $x + y$ and $x - y$ and then find their squares (12 more operations); then find $z_1 = (x+y)^2 - (x-y)^2$, next $\frac{1}{z_1} = \frac{1}{4xy}$, then

$$\frac{1}{4xy} + \frac{1}{4xy} = \frac{2}{4xy} = \frac{1}{2xy}, \qquad \frac{1}{2xy} + \frac{1}{2xy} = \frac{1}{xy} = z_2$$

and, finally, (the 20-th or the 19-th operation) $\frac{1}{z_2} = xy$.

45.7.2. Through the center of the square draw straight lines parallel to its sides. The lines cut the square into 4 equal squares. Some two of the five points considered lie in one of these 4 squares and the distance between them is not greater than the length of the diagonal of the square.

45.7.3. Any computation of a number a is a chain of computations of numbers

$$a_0 = 1, \quad a_1, \quad a_2, \ \dots, \ a_{n-1}, \quad a_n = a,$$

where either $a_k = 3a_{k-1}$ or $a_k = a_{k-1} + 4$.

Clearly, the calculator outputs only *odd numbers* since if a_k is odd, then so are $3a_k$ and $a_k + 4$. We do not know whether or not ALL numbers can occure in the output. Anyway, it is impossible to get 1982. Let us see if it is possible to get 1981.

The price of any computation is equal to

$$P(a_0, a_1, \dots, a_n) = 2m + 5(n - m),$$

where m is the number of additions in the chain of a_i's.

Let $P(a)$ be the lowest price of the computations of the number a. Then

$$P(1) = 0, \quad P(3) = 5, \quad P(5) = 2, \quad P(7) = 7, \quad P(9) = 4, \quad P(11) = 9, \quad P(13) = 6,$$
$$P(15) = 7, \quad P(17) = 8, \quad P(19) = 9, \quad P(21) = 10.$$

Observe that either $P(a) = 2 + P(a-4)$ or $P(a) = 5 + P\left(\frac{1}{3}a\right)$; note also that if $a > 21$ and a is divisible by 3, then $P(a) = 5 + P\left(\frac{1}{3}a\right)$ (this is easy to prove by the rule of contraries). Therefore,

$$\begin{aligned} P(1981) &= 2 + P(1977) = 7 + P(659) = 9 + P(655) = 11 + P(651) \\ &= 16 + P(217) = 18 + P(213) = 23 + P(71) = 25 + P(67) \\ &= 27 + P(63) = 32 + P(21) = 42. \end{aligned}$$

45.7.4. Clearly, 8 points $A_1, A_2, \dots, A_8$ with the distances $A_1A_2 = 1$, $A_2A_3 = 2, \dots, A_7A_8 = 64$ between them satisfy the condition.

Let us prove that it is impossible to arrange a lesser number of points on the plane. For each $k = 0, 1, \dots, 6$ choose a pair of points the distance between which is 2^k and connect them by segments. It follows from the triangle inequality that the 7 segments obtained (or parts of them) do not form a closed polygon. Therefore, the number of points must be at least by 1 greater than the number of these segments.

45.8.1. Let $a = \sqrt[4]{5}$. Since $a^4 = 5$, we see that the square of the expression to be simplified is:

$$\begin{aligned} \frac{4}{4 - 3a + 2a^2 - a^3} &= \frac{4(4 + 2a^2 + 3a + a^3)}{-10 - 5a^2 + 7a^2 + 16} = \frac{4(4 + 2a^2 + 3a + a^3)}{6 + 2a^2} \\ &= \frac{2(4 + 2a^2 + 3a + a^3)}{3 + a^2} = \frac{2(4 + 2a^2 + 3a + a^3)(3 - a^2)}{9 - a^4} \\ &= 1 + 2a + a^2 = (1 + a)^2. \end{aligned}$$

45.8.4. By the hypothesis $EDCD'$ and $DCBC'$ are parallelograms, see Fig. 357.

Therefore,

$$EC' = EB - C'B = EB - DC = EB - ED' = D'B.$$

Since $EC \parallel AB$, it follows that $\triangle EC'B' \sim \triangle AC'B$ and $\triangle ED'C \sim \triangle AD'B$, whence

$$\frac{EB'}{AB} = \frac{EC'}{C'B} = \frac{D'B}{ED'} = \frac{AB}{EC} = \frac{AB}{EB' + B'C} = \frac{AB}{EB' + AB} = \left(1 + \frac{EB'}{AB}\right)^{-1}.$$

FIGURE 357. (Sol. 45.8.4)

Let $x = \dfrac{EB'}{AB}$; then

$$x = \frac{1}{1+x};$$

the positive root of this equation is equal to $x = \frac{1}{2}(\sqrt{5}-1)$. This means that

$$\frac{EC}{AB} = 1 + \frac{EB'}{AB} = 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2}.$$

The other ratios are equal by symmetry.

NOTE. For those who know some higher geometry: our pentagon is the image of a regular pentagon under an affine transformation.

45.8.5. a) Proof by induction. For $n = 1$ we have

$$a_1^7 + a_1^5 \geq 2a_1^6 \Longleftrightarrow (a_1 - 1)^2 \geq 0.$$

Assuming that the inequality holds for $n = k$, let us prove it for $n = k+1$. Let $a_1 < \cdots < a_{k+1}$ be arbitrary integers. By the inductive hypothesis

$$(a_1^7 + \cdots + a_k^7) + (a_1^5 + \cdots + a_k^5) \geq 2(a_1^3 + \cdots + a_k^3)^2.$$

But

$$\begin{aligned} &2a_{k+1}^6 + 4a_{k+1}^3(a_1^3 + \cdots + a_k^3) \\ &\quad \leq 2a_{k+1}^6 + 4a_{k+1}^3(1^3 + 2^3 + \cdots + a_1^3 + \cdots + a_k^3 + \cdots + (a_{k+1}-1)^3) \\ &\quad = 2a_{k+1}^6 + 4a_{k+1}^3 \cdot \tfrac{1}{4}(a_{k+1}-1)^2 a_{k+1}^2 = a_{k+1}^7 + a_{k+1}^5 \end{aligned}$$

(here we use the fact that all a_i are distinct). Adding up all these inequalities we have:

$$(a_1^7 + \cdots + a_k^7 + a_{k+1}^7) + (a_1^5 + \cdots + a_k^5 + a_{k+1}^5) \geq 2(a_1^3 + \cdots + a_k^3 + a_{k+1}^3)^2, \text{ Q.E.D.}$$

45.9.1. Clearly, n is not divisible by 3. If $n \equiv 1 \pmod 3$, then $2^n + 1$ must be divisible by 3, which means that n must be odd, i.e., be of the form $6k+1$. If $n \equiv 2 \pmod 3$, then $2^{n+1}+1$ must be divisible by 3 and, therefore, n must be even, i.e., be of the form $6k+2$.

45.9.3. It is easy to see that the distances between the point $a = (\sqrt{2}, \frac{1}{3})$ and the points with integral coordinates are all different. (To prove this use the irrationality of $\sqrt{2}$ and the fact that $\frac{1}{3}$ is not an integer.)

Arrange the distances between a and the marked points in the increasing order: $R_1 < R_2 < \cdots < R_{1982} < R_{1983}$. Then the disc with the center in a and of any radius R such that $R_{1982} < R < R_{1983}$ contains exactly 1982 marked points.

45.9.4. The statement follows from the fact that for $|x| < 1$ we have

$$\begin{aligned} x + 2x^2 + 3x^3 + \cdots + nx^n &= \sum_{k=1}^n kx^k \\ &= \sum_{k=1}^n x^k + \sum_{k=2}^n x^k + \cdots + \sum_{k=n-1}^n x^k + \sum_{k=n}^n x^k = x\frac{1-x^n}{1-x} + x^2\frac{1-x^{n-1}}{1-x} + \cdots + x^n\frac{1-x}{1-x} \\ &= (1-x)^{-1} \cdot \Big(\sum_{k=1}^n x^k - nx^{n+1}\Big) = \frac{x}{(1-x)^2} \cdot (1-x^n) - \frac{nx^{n+1}}{1-x}. \end{aligned}$$

Since $\lim\limits_{n\to\infty}(1-x^n)=1$ and $\lim\limits_{n\to\infty} nx^{n+1}=0$ for any $|x|<1$, we get

$$x+2x^2+\dots=\sum_{k=1}^{\infty} kx^k=\frac{x}{(1-x)^2}.$$

By substituting $x=1/10$ we get

$$\sum_{k=1}^{\infty}\frac{n}{10^n}=\frac{0.1}{(1-0.1)^2}=\frac{10}{81}=0.1234567890123\dots \text{ Q.E.D.}$$

45.9.5. The sum of the lengths of the diagonals of quadrilateral $ABCD$ is greater than the sum of the lengths of its opposite sides and, therefore, the length of neither of the opposite sides cannot be equal to 1. So we may assume that $AB=BC=1$, $\angle ABC\le 60^\circ$ (as $AC\le 1$), and point D lies inside sector ABC of the unit disc with center B, see Fig. 358.

Now, it is not difficult to show that the perimeter attains its maximum if D lies on the disc, at equal distance from points A and C, and $\angle ABC=60^\circ$.

FIGURE 358. (Sol. 45.9.5)

45.10.1. Let all angles with vertex S and subtending the edges of tetrahedron $ABCD$ be equal to α. Then points B, C, and D lie on the intersection of the cone whose axis is SA with the sphere of radius equal to the edge and centered at A. This intersection is of the form of one or two circles that lie in the planes perpendicular to AS. Of three points B, C, and D, two lie on one of these circles and, therefore, are equidistant from point S.

Since instead of point A we can take any other vertex, we have proved that of any three vertices of the tetrahedron two are equidistant from point S. This is possible in two cases: either there are two pairs of vertices equidistant from S or there are three equidistant vertices.

In the first case point S should lie on the line that connects the midpoints of opposite edges. In order for the angles subtending these edges to be equal, point S should be equidistant from the edges, i.e., S is the center of the tetrahedron.

In the second case S lies on the height of the tetrahedron (or its extension). Consider this case on your own.

45.10.2. a) Clearly follows from b).

b) Denote the left-hand side of the given inequality by $f(a,b,c)$. Then

$$f(a,b,c)=(a+b+c)(abc-(b+c-a)(a+c-b)(b+a-c)).$$

Among numbers $b+c-a$, $a+c-b$ and $b+a-c$ not more than one is negative. (Indeed, if, say, $a+b-c<0$, $b+c-a<0$, then $2b<0$.) If exactly one of the numbers is negative, then their product is non-positive, and, therefore, $f(a,b,c)\ge 0$. If all of them are non-negative, then the inequality

$$a^2b^2c^2\ge\left(a^2-(b-c)^2\right)\left(b^2-(a-c)^2\right)\left(c^2-(a-b)^2\right)$$
$$=(b+c-a)^2(a+c-b)^2(b+a-c)^2$$

implies that

$$abc-(b+c-a)(a+c-b)(b+a-c)\ge 0,$$

whence $f(a,b,c)\ge 0$ for any $a,b,c\ge 0$.

45.10.3. a) Let $g(x, y) = xy + x + y + 1$. The polynomial $x + a$ for any $a \neq 1$ is calculated by the following "program" $P(x + a)$:

$$x; \quad \frac{2-a}{a-1}; \quad a-2; \quad g(x, \frac{2-a}{a-1}); \quad g(g(x, \frac{2-a}{a-1}), a-2).$$

The "program" $P(x + 1)$ to calculate $x + 1$ is as follows: x; 0; $g(x, 0) = x + 1$.

Since $f_n(x) = xf_{n-1}(x) + 1 = g(f_{n-1} - 1, x - 1) + 1)$, the "program" $P(f_n)$ for computing $f_n(x) = 1 + x + \dots + x^n$ for any n is constructed by the induction via *Horner's scheme*, namely:

$$P(f_{n-1}); \quad P(f_{n-1} - 1); \quad P(x-1); g(f_{n-1} - 1, x - 1); \quad P(g(f_{n-1} - 1, x - 1) + 1).$$

b) Let $g_1, g_2, \dots, g_n$ be an arbitrary program.

Lemma. *If $g_n \not\equiv$ const, then $g_n(-1) = -1$.*

PROOF. Let us prove Lemma by the induction on n. For $n = 1$ this is obvious: $g_1(x) = x$. Further, $g_n = g_i g_j + g_i + g_j$ for $i, j < n$, where, say, $g_i \not\equiv$ const. Then $g_i(-1) = -1$ (the inductive hypothesis), and $g_n(-1) = -g_j(-1) - 1 + g_j(-1) = -1$, Q.E.D.

But $g_n = f_n$ by the hypothesis, yet $f_n(-1) = 0$ or $f_n(-1) = 1$ and $g_n(-1) = -1$. This contradiction proves that the desired program does not exist.

REMARK. *A polynomial is computable by means of the calculator with the operation $g(x, y) = xy+x+y$ if and only if it is of the form $F_n(x) = A(x + 1)^n - 1$.* The necessity can be proved by the induction with the help of the identity:

$$g(A_1(x+1)^{n_1} - 1, A_2(x+1)^{n_2} - 1) = A_1 A_2 (x+1)^{n_1+n_2} - 1$$

and the sufficiency can also be proved by the induction with the help of the identity:

$$A(x+1)^{n+1} - 1 = g(A(x+1)^n - 1, x).$$

45.10.4. Let there be k digits after the decimal point in the decimal expression of $1/n$; then $10^k/n$ is an integer; therefore, $n = 2^a 5^b$ and $n + 1 = 2^c 5^d$, where a, b, c, d are non-negative integers. Subtracting, we obtain $1 = 2^c 5^d - 2^a 5^b$, whence $\min(a, c) = \min(b, d) = 0$. Let, for example, $c = b = 0$. We get an equation $2^a = 5^d - 1$ for integers a and d. Prove on your own that its unique solution is $a = 2$, $d = 1$.

45.10.5. Assume that the center O of the larger hexagon, Q, is not inside the smaller hexagon, q. Let d be the (length of the) radius of the circle inscribed in Q (it is also qual to the diameter of the circle inscribed in q, i.e., the distance between its opposite sides). Drop perpendicular OM from O on the farthest from it side of q; it is easy to see that $|OM| > d$; see Fig. 359.

FIGURE 359. (Sol. 45.10.5)

Therefore, the point H of the intersection of OM and the circle inscribed in Q is closer to O than point M. Draw the tangent to the circle through H to its intersections with Q at points A and B; it is obvious that segment AB is parallel to a side of q and is longer than this side. But this immediately leads us to a contradiction: it is easy to show that the length of the segment of the tangent AB to the circle inscribed in a regular polygon with side a is not less than $\frac{a}{2}$.

EXTENSION. The assertion of the problem holds for an arbitrary regular polygon with an even number of sides. Construct a counterexample for a polygon with an odd number of sides.

46.7.1. Let us represent the right-hand side of the equation in the form $(y+1)^2+12$; subtracting $(y+1)^2$ from LHS and RHSwe get $(x-y-1)(x+y+1)=12$. Since each expression in the parenthesis is even, we have 4 possibilities: one of the expressions is equal to ± 2, the other one to ± 6, whence the answer.

46.7.3. Denote the given number by $X=\overline{4ab...c}=4\cdot 10^n+A$, where $A=\overline{ab...c}$. After the first digit of X is placed to its end we get the number $Y=\overline{ab...c4}=10A+4$. By the hypothesis, $4\cdot 10^n+A=4(10A+4)$, whence $39A=4\cdot(9...96)$ with $(n-1)$-many 9's, or $13A=4\cdot(3...32)$ with $(n-1)$-many 3's. Dividing $13A$ by 13 we find the lowest value of A: it is $A=4\cdot\frac{33\,332}{13}=10\,256$. Therefore, the lowest value of X is $410\,256$.

46.7.5. Denote the lengths of segments between vertices and the tangent points (each segments is represented twice) by x_1, x_2, x_3, x_4, x_5. Then

$$x_1+x_2+x_3+x_4+x_5=\frac{1}{2}(13+11+9+4+3)=20.$$

The sum of lengths of two non-neighboring sides of the pentagon is equal to the sum of four of the five numbers x_i and, therefore, is <20. Hence, the sides of lengths 13, 11 and 9 should be pairwise neighboring ones, which is impossible.

46.8.1. The left-hand side of the inequality is $\frac{x^5+y^5}{x-y}$. Since $x,y>\sqrt{2}$, then $x^5+y^5>2(x^3+y^3)$ and

$$\frac{x^5+y^5}{x+y}>2\frac{x^3+y^3}{x+y}=2(x^2-xy+y^2)>x^2+y^2.$$

46.8.2. Observe that $\overrightarrow{AA_1}=\overrightarrow{AC}+\overrightarrow{CA_1}$, $\overrightarrow{BB_1}=\overrightarrow{BA}+\overrightarrow{AB_1}$, $\overrightarrow{CC_1}=\overrightarrow{CB}+\overrightarrow{BC_1}$. (Thus the sides of triangle ABC are oriented counter-clockwise.) Summing all these equalities we get:

$$\overrightarrow{AA_1}+\overrightarrow{BB_1}+\overrightarrow{CC_1}=(\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BA})+(\overrightarrow{AB_1}+\overrightarrow{CA_1}+\overrightarrow{BC_1}).$$

The sum of the vectors in the first parentheses is $\vec{0}$, and each vector in the second parentheses is obtained by rotating the corresponding vector from the first parentheses through an angle of 60° clockwise; hence, the second sum is also equal to $\vec{0}$.

46.8.3. Indeed,

$$N^2=(10^{1984}-5)^2=10^{2\cdot 1984}-10\cdot 10^{1984}+25=\underbrace{9\dots 9}_{1983}\underbrace{0\dots 0}_{1983}25.$$

46.8.4. Let us find this arrangement. Draw an axis of symmetry through the 1983-th vertex. Consider this axis as vertical and let the 1983-th vertex lie on the upper semicircle. The axis of symmetry divides the circle into left and right halves. Denote the numbers at the vertices of the left semicircle by a_1, $a_2,\dots,a_{990}$, a_{991} (beginning with the 1983-th vertex counter-clockwise) and the numbers at the respective symmetric vertices by letters b_i with the same subscripts.

Let $a_1>b_1$, $a_2>b_2,\dots,a_{991}>b_{991}$. Draw now an axis of symmetry through the vertex b_{991} (it passes between the vertices 1983 and a_1). Since the arrangement is a "good" one with respect to this axis, and $1983>a_1$, it follows that

$$b_1>a_2,\quad b_2>a_3,\ \dots,\ b_{989}>a_{990},\quad b_{990}>a_{991}.$$

Uniting these inequalities with the preceding ones into one chain, we get:

$$1983>a_1>b_1>a_2>b_2>\dots>a_{990}>b_{990}>a_{991}.$$

Hence, the arrangement is entirely defined by

$$a_1=1982,\quad b_1=1981,\ \dots,\ b_{990}=2,\quad a_{991}=1.$$

A simple check shows that this arrangement is "good" with respect to any axis of symmetry.

46.8.5. Let H_{ij} be the base of the perpendicular dropped from H to line A_iA_j, see Fig. 360. Consider right triangles $\triangle HA_3H_{34}$ and $\triangle HA_3H_{23}$. We have:

$$\frac{h_{23}}{h_{34}}=\frac{A_3H\cdot\sin\angle A_2A_3H}{A_3H\cdot\sin\angle A_4A_3H}=\frac{\sin\angle A_2A_3H}{\sin\angle A_4A_3H}.$$

Since both the numerator and the denominator do not depend on point A_3 on the circle,

$$\frac{\sin\angle A_2A_3H}{\sin\angle A_4A_3H}=\frac{\sin\angle A_2A_1H}{\sin\angle A_4A_1H}=\frac{h_{12}}{h_{14}},$$

therefrom $h_{23}\cdot h_{14}=h_{12}\cdot h_{34}$, Q.E.D.

FIGURE 360. (Sol. 46.8.5) FIGURE 361. (Sol. 46.9.2)

46.9.1. Let $f(x) = x^{2n} - x^{2n-1} + x^{2n-2} - \dots - x^3 + x^2 - x + 1$. Show yourselves that it suffices to prove that $f(x) > \frac{1}{2}$ for any $x > 0$. But $f(x) = \frac{1+x^{2n+1}}{1+x}$, and, obviously, $f(x) \geq 1$ for $x \geq 1$. If $0 < x < 1$, then $1 + x < 2$ and once again $f(x) > \frac{1}{1+x} > \frac{1}{2}$.

46.9.2. Denote the centers of the circles of radius 3, 4, 5 by O_1, O_2, and O_3, respectively; let AB be the segment to be determined, a part of the common tangent to circles O_1 and O_2, see Fig. 361. For $H \in O_1C$, draw O_3H perpendicularly to O_1O_2. Set $x = CH$. From $\triangle O_1O_3H$ and $\triangle O_2O_3H$ we find:

$$9^2 - (4+x)^2 = 8^2 - (3-x)^2, \quad \text{or, equivalently,} \quad x = \frac{5}{7}.$$

Now, knowing that chord AB in the circle of radius 5 is at a distance of $\frac{5}{7}$ from center O_3, we get:

$$AB = 2\sqrt{5^2 - \left(\frac{5}{7}\right)^2} = \frac{40\sqrt{3}}{7}.$$

46.9.4. Consider an arbitrary town A. Assume that it is possible to get from it only to $n < 20$ towns. Let X be the set of n towns to which one can get from A, and Y the set of the other $20 - n$ towns. Then there are no towns from X from which one can get to any town from Y, since otherwise it would be possible to get from A to some town from Y. Since the total number of all possible direct airlines is $C_{20}^2 = 190$, we have $n(20-n)$ airlines missing.

But $n(20-n) \geq 19$ for any integer $n \in \,]0, 20[$; hence, 20 towns are connected by less than 172 airlines, which contradicts our hypothesis.

FIGURE 362. (Sol. 46.10.1)

46.10.1. Prove that $AC = BC$. Reflect triangle AA_1C with respect to the bisector of $\angle ACB$, see Fig. 362. Then point A_1 passes into point B_1, point C is fixed, and point A passes into point A_2 which lies on straight line BC.

Without loss of generality we may assume that point A_2 lies on segment BC. Prove that (1) point A_2 being thus reflected coincides with vertex B; then (2) $AC = BC$. Indeed, assume the contrary; from the symmetry we have $B_1B = AA_1 = B_1A_2$ and $\angle A_1AC = \angle B_1A_2C$. From the isosceles triangle $\triangle B_1BA_2$ we deduce that $\angle B_1BC = \angle BA_2B_1$. But $\angle BA_2B_1 + \angle B_1A_2C = \pi$; hence, $\angle B_1BC + \angle A_1AC = \pi$. This contradicts the inequality

$$\angle B_1BC + \angle A_1AC < \angle CAB + \angle ABC < \pi.$$

46.10.3. It suffices to prove that if $4^{m-n} - 1$ is divisible by 3^{k+1}, then $m - n$ is divisible by 3^k. It follows from the fact that the minimal a such that $4^a - 1$ is divisible by 3^{k+1} is equal to 3^k.

Let

$$t(x) = \cos 5x + a\cos 4x + b\cos 3x + c\cos 2x + d\cos x + e.$$

Then

$$\begin{aligned}
&t(0) + t(\tfrac{2}{5}\cdot\pi) + \dots + t(\tfrac{8}{5}\cdot\pi)\\
&\quad= 5(1+e) + (a+b+c+d)\Big(\cos 0 + \cos(\tfrac{2}{5}\cdot\pi) + \dots + \cos(\tfrac{8}{5}\cdot\pi)\Big)\\
&\quad= 5(1+e);\\
&\qquad t(\tfrac{1}{5}\cdot\pi) + t(\tfrac{3}{5}\cdot\pi) + \dots + t(\tfrac{9}{5}\cdot\pi)\\
&\quad= 5(-1+e) + (a+c)\Big(\cos 0 + \cos(\tfrac{2}{5}\cdot\pi) + \dots + \cos(\tfrac{8}{5}\cdot\pi)\Big)\\
&\quad+(b+d)\Big(\cos(\tfrac{1}{5}\cdot\pi) + \cos(\tfrac{3}{5}\cdot\pi) + \dots + \cos(\tfrac{9}{5}\cdot\pi)\Big) = 5(-1+e);
\end{aligned}$$

hence, the unknown value of $(*)$ is 10. We have made use of the equalities

$$\cos 0 + \cos(2\tfrac{\pi}{5}) + \dots + \cos(8\tfrac{\pi}{5}) = \cos(\tfrac{\pi}{5}) + \cos(3\tfrac{\pi}{5}) + \dots + \cos(9\tfrac{\pi}{5}) = 0,$$

which follow from the fact that the sum of the vectors from the center of a regular pentagon to its vertices is equal to $\vec{0}$.

46.10.4. a) If the segments do not form any triangle, then their second endpoints are not connected with one another. Let A be the endpoint of the greatest number, say n, of segments. Each of the other $7 - n$ points is an endpoint of not more than n segments, therefore, the total number of segments is not greater than $n + (7-n)n = n(8-n) \le 16$.

Contradiction.

b) As was proved in a), there exists a triangle ABC formed by the segments. *How many triangles may have a common side with it*?

If n_A, n_B and n_C are the numbers of the segments whose endpoints are A, B, C (the sides of triangle ABC are not taken into consideration), then there are at least $n_A+n_B+n_C-5$ such triangles. If $n_A+n_B+n_C \ge 8$, then 4 triangles are formed by the segments (the triangle ABC included).

Let now $n_A + n_B + n_C \le 7$. Without loss of generality we may assume that $n_A + n_B \le 4$. Let us delete points A and B and all segments with endpoints in these points (the total number of such segments is $n_A + n_B + 3$). We have 6 points and at least $17 - 7 = 10$ segments. Let us show that they form 3 triangles.

By a) there is at least one triangle. Now it remains to show that 5 segments between 4 points form 2 triangles. But this is obvious.

46.10.5. Let $B_1, \dots, B_l$ be the knights of some town M. Assume that the goblets pass the entire circle. Then every one of the knights B_i holds each golden goblet once, which means that they all hold golden goblets kl times. Since $1 < k < 13$, we deduce that $kl \ne 13$.

If $kl > 13$, then at some moment the knights from M have two golden goblets simultaneously, and everything is proved.

If $kl < 13$, then at some moment the knights from M have no golden goblets. As the number of golden goblets is equal to the number of towns, this same moment some knights from another town, N, have two golden goblets, Q.E.D.

FIGURE 363. (Sol. 47.7.4)

47.7.4. First, let the sides of the parallelogram be parallel to two sides of the triangle.

Clearly, the greatest of such parallelograms lies as shown in Fig. 363 a). If S is the area of AKL, and S_1 and S_2 are the areas of BKC and DCL, respectively; then $S_1 = a^2 S$ and $S_2 = (1-a)^2 S$, where $a = \frac{KC}{KL}$ and $1 - a = \frac{CL}{KL}$ are similarity coefficients, and

$$S_1 + S_2 = \left(\frac{1}{2} + 2\left(\frac{a-1}{2}\right)^2\right) S \geq \frac{S}{2};$$

the equality is attained at $a = \frac{1}{2}$.

Let one of the sides of the parallelogram be not parallel to any side of the triangle. Draw a straight line through one of the vertices of the triangle as shown in Fig. 363 b). It cuts the triangle and the parallelogram into two smaller ones, with one of the sides of the parallelogram parallel to the side of each of the little triangles. Repeating this operation once or twice, we reduce the problem to the case already considered.

47.7.5. Suppose that the king encounteres neither of the troubles mentioned. Then each rook must have had changed both its initial vertical and its initial horizontal during the game, i.e., it must have made not less than 2 moves. Hence the total number of moves of all rooks must be not less than 20. This number has to coincide with the number of the king's moves, i.e., it must be equal to 19. Contradiction. (Cf. Problem 30.2.8.4.)

47.8.1. It follows from the given equation that $4 > x^2$ and

$$x^3 = (4 - x^2)\sqrt{4 - x^2} = (4 - x^2)^{3/2}. \qquad (*)$$

We see that $x^3 > 0$, hence $x > 0$. Square both sides of $(*)$ and take the cube root. We get: $4 - x^2 = x^2$, whence $x^2 = 2$, i.e., $x = \pm\sqrt{2}$. We see that only $x = \sqrt{2}$ will do since $x > 0$.

47.8.2. Arrange the 6 points (computers) in the vertices of a regular hexagon. Paint every other side of the hexagon colors 1 and 2, and its diagonals colors 3, 4 and 5 (2 parallel short diagonals and a long one perpendicular to them are painted each color, i.e., correspond to 3 cables each).

47.8.3. Let $A_1, \dots, A_7$ be the vertices of the given heptagon, O its center, and B_1 an arbitrary point not coinciding with O.

Rotate vector $\overrightarrow{OB}$ through angles of $\frac{2\pi}{7}$, $2 \cdot \frac{2\pi}{7}, \dots, 6 \cdot \frac{2\pi}{7}$; we get 7 points $B_1, B_2, \dots, B_7$ — the images of B — in the vertices of a regular heptagon. It is easy to see that the sum of the lengths of $\overrightarrow{A_1B_1}, \dots, \overrightarrow{A_1B_7}$ is equal to the sum S of lengths of $\overrightarrow{B_1A_1}, \dots, \overrightarrow{B_1A_7}$, see Fig. 364. But

$$\frac{\overrightarrow{A_1B_1} + \dots + \overrightarrow{A_1B_7}}{7} = \overrightarrow{A_1O},$$

whence the triangle inequality yields

$$7|\overrightarrow{A_1O}| \leq |\overrightarrow{A_1B_1}| + \dots + |\overrightarrow{A_1B_7}| = S.$$

FIGURE 364. (Sol. 47.8.3)

FIGURE 365. (Sol. 47.8.5)

47.8.4. Denote the given numbers by $x_1, \dots, x_5$. Since $\left(x_1 - \frac{1}{5}\right)^2 + \dots + \left(x_5 - \frac{1}{5}\right)^2 \geq 0$, then

$$x_1^2 + \dots + x_5^2 \geq \frac{2}{5}(x_1 + \dots + x_5) - 5 \cdot \frac{1}{25} = \frac{1}{5}.$$

Since

$$\begin{aligned} &1 = (x_1 + \dots + x_5)^2 = x_1^2 + \dots + x_5^2 + \\ &2\big((x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1) \\ &\qquad\qquad + (x_1x_3 + x_3x_5 + x_5x_2 + x_2x_4 + x_4x_1)\big), \end{aligned}$$

it follows that by denoting the two sums in parentheses by S_1 and S_2 we get $2(S_1 + S_2) \leq 1 - \frac{1}{5} = \frac{4}{5}$; hence, $S_1 + S_2 \leq \frac{2}{5}$. Consequently, the lesser of the sums S_1, S_2 is not greater than $\frac{1}{5}$.

Actually, a more general statement holds: *it is possible to arrange on a circle n non-negative numbers whose sum is equal to 1 so that the sum of all n pair-wise products of neighboring numbers is not greater than $\frac{1}{n}$.*

Indeed, consider the sums S_1, $S_2, \dots$ of pairwise products of neighboring numbers in all possible arrangements on the circle. Every x_ix_j and x_kx_l with $i \neq j$ and $k \neq l$ occur an equal number of times in all these sums. Since $x_1^2 + \dots + x_n^2 \geq \frac{1}{n}$ and $(x_1 + \dots + x_n)^2 = 1$, the arithmetic mean of all sums S_1, $S_2, \dots$ is not greater than $\frac{1}{n}$. This means that one of the sums is not greater than $\frac{1}{n}$.

47.8.6. Introduce a new digit, $*$, representing number 10 and let

$$A = \underbrace{* \cdots *}_{64} = \underbrace{1 \dots 1}_{64}0.$$

As is easy to see, A is divisible by 101. For every 64-digit number $a = \overline{a_1 \dots a_{63}a_{64}}$ satisfying the condition of the problem consider the number $b = \overline{(* - a_1)(* - a_2) \dots (* - a_{64})}$ "symmetric" to it. It also satisfies the condition, since it is divisible by 101 and its every digit $x - a_i$ is nonzero.

Can two numbers, a and b, symmetric to each other coincide? If $a = b$, then $a_i = * - a_i$; hence, $a_i = 5$ for all i; thus, this number is $5 \dots 5$ with 64-many 5's.

The other numbers are divided into pairs (a, b) of symmetric numbers not equal to each other.

47.9.1. Denote the planar angles at the pyramid's vertex by α, β, γ. Expressing the area of each face in terms of the lengths of edges and the sines of the angles between them, we find that $\sin\alpha = \sin\beta = \sin\gamma$. It follows that at least two of the three angles α, β, γ are equal. Then two of the three lateral faces are equal triangles and, therefore, their bases are also equal.

47.9.3. In the equilateral triangle of area 1 the side length is equal to $a = \frac{2}{\sqrt[4]{3}}$ and the length of the height is equal to $h = \sqrt[4]{3}$. It is easy to see that it is impossible to cut such a triangle out of a strip of a width less than h. Prove now that any other triangle of area 1 can be cut out of such a strip, see Fig. 366.

Assume the contrary; then any of its heights is lower than h and, therefore, any of its sides is shorter than a. But the area of a triangle with all three sides shorter than a is less than $a^2 \sin\frac{\alpha}{2}$, where α is any of its angles; and as it is possible to choose the angle α to be $\leq 60°$, the area of the triangle is less than 1.

Contradiction.

47.9.4. Let a, b, c, d be 4 consecutive numbers on the circle, and $a \geq b$; then, clearly,

$$ab + bc + cd \leq a(b + c) + (b + c)d.$$

FIGURE 366. (Sol. 47.9.3)

Therefore, by replacing b and c by $b+c$ we do not violate the conditions of the problem and replace the sum S_n by a greater one, S_{n-1}.

Repeating this procedure several times we reduce the problem to four numbers on the circle; if these numbers are x, y, z, t, then

$$S_4 = xy + yz + zt + tx = (x+z)(1-(x+z)) \le \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

47.10.1. The inequality to be proved follows from the chain of inequalities:

$$\sin 1 < \sin\tfrac{\pi}{3} = \frac{\sqrt{3}}{2} < \frac{7}{8} < \log_3\sqrt{7}$$

(the last inequality follows from the fact that $3^7 < 7^4$).

47.10.2. EXTENSION.

47.10.3. Rewrite the equation as follows:

$$19(x^3 - 100) = 84(1 + y^2).$$

The right-hand side is divisible by 7; therefore, $x^3 - 100$ is divisible by 7; hence, so is $x^3 - 100 + 98 = x^3 - 2$. But a perfect cube is not congruent to 2 modulo 7.

47.10.4. Let d and d_s be the greatest common divisors of all numbers n_i and of the numbers $n_1, \dots, n_s$, respectively. Since $d_1 \ge d_2 \ge \dots$, we see that $d_k = d_{k+1} = \dots = d$ starting with some $k \ge 2$.

Next, consider the numbers $n_1, \dots, n_k$ and all amounts that may be paid with these coins. These amounts, arranged in an increasing order, form from some point on an arithmetic progression with the difference $d_k = d$; the same arithmetic progression is also formed by the amounts obtained from the initial infinite set of coins $\{n_i\}_{i=1}^{\infty}$. Therefore, adding to the coins $n_1, \dots, n_k$ the necessary number of coins $n_{k+1}, \dots, n_N$ which are not in the said arithmetic progression we obtain the desired finite set of coins $n_1, n_2, \dots, n_N$ with which one can pay all possible amounts constructed from the initial infinite set of coins.

47.10.5. Let A and B be two vertices of the triangles situated inside the square. Clearly, not less than 5 triangles are adjacent to each of the vertices, and their total number is equal to 10, only 2 of which can be counted twice (those adjacent to side AB, if such a side exists). And so the number of triangles is *not less than* 8.

Analyze on your own the case when not more than one vertex is inside the square, and the rest are situated on the sides of the square or on the sides of another triangle.

REMARK. M. Gardner admitted in his book [??????????] that he could not prove that the number of desired triangles is ≥ 8.

47.10.6. Let ABC be the triangle-shaped section of the cube with edge 2, H the tangent point of triangle ABC with the sphere inscribed in the cube.

Let the sphere be tangent to the faces of the cube, to which segments AB, AC and BC belong, at points C', B', A', respectively; see Fig. 367 a). It is easy to see that

$$\triangle AC'B = \triangle AHB, \qquad \triangle BA'C = \triangle BHC, \qquad \triangle CB'A = \triangle CHA$$

(the triangles are equal because the tangents from one point to the sphere are equal). Consequently, the area of triangle $\triangle ABC$ is equal to the sum of the areas of triangles $\triangle AC'B$, $\triangle BA'C$, and $\triangle CB'A$.

Express the area of each one of the triangles $\triangle AC'B$, $\triangle BA'C$, and $\triangle CB'A$ in terms of the directed distances x, y, z between the points A, B, C and the midpoint of the edge of the cube on which the point is situated, see Fig. 367 b). These three areas are $\frac{1-xy}{2}$, $\frac{1-yz}{2}$ and $\frac{1-zx}{2}$.

Now we have to prove that if $|x| < 1$, $|y| < 1$ and $|z| < 1$, then

$$\frac{1-xy}{2} + \frac{1-yz}{2} + \frac{1-zx}{2} < 2 \text{ or, equivalently, } xy + yz + zx > -1.$$

If all numbers x, y, z are of the same sign, then the last inequality is evident; there remains the case when two of them (say, x and y) are of the same sign, and the third is of the opposite sign. Then $(x-y)(y-z) > 0$; hence, $xy - xz - yz > -z^2 > -1$ as $|z| < 1$. Q.E.D.

FIGURE 367. (Sol. 47.10.6) FIGURE 368. (Sol. 48.7.4)

48.7.4. The rabbit must use the following strategy. First, it must choose an arbitrary vertex A of the square and run towards it along the diagonal with the greatest possible speed until the moment when it is less than $\frac{\sqrt{2}-1.4}{2}$ away from A (for example, a distance of 0.005; we assume that the side of the square is equal to 1).

Then without changing its speed it must turn through an angle of 90° and move perpendicularly to the diagonal toward the side of the square which contains only one wolf (if the wolf is in a vertex at this moment, then the rabbit must turn by 90° to the right or left (does not matter), see Fig. 368.

It is easy to see that at the moment when the rabbit crosses a side of the square, no wolf will be able to be in the same point on this side.

REMARK. If the wolf's speed is $\sqrt{2}$ times greater than that of the rabbit, the wolves will catch the rabbit: at each moment they are on the endpoints of the "cross" whose center is the rabbit and whose bars are parallel to the diagonals of the square.

48.8.1. $(x-y+z)^2-z^2-(x^2-y^2) = 0 \Longrightarrow (x-y)(x-y+2z)-(x-y)(x+y) = 0 \Longrightarrow (x-y)(2y-2z) = 0.$

48.8.2. There are exactly 993 numbers a_k which are not less than 993. Therefore, the index of at least one of them is not less than 993, and for it $ka_k \geq 993^2$.

48.8.3. It is impossible for a 2×2 square to cover fewer than two nodes. Indeed, even *a disc inscribed into such a square covers not less than two nodes.*

FIGURE 369. (Sol. 48.8.3)

To prove this, consider the square in which the center of the disc lies, divide the square into 4 triangles by its diagonals and consider the triangle containing the center. Then the disc covers the hypotenuse of the chosen triangle, see Fig. 369 b). Q.E.D. (cf. Problem 48.9.3.)

48.8.4. a) Let there be no shortest knight in the line. Consequently, for each knight there is another who is shorter, and then the desired chain is easily constructed by successively choosing the knights who are shorter and shorter.

b) If there is a knight A_1 who is the shortest one, let him stand out of the line and choose the knight A_2 who is the shortest of all knights that stand to the right of A_1. If there is none, we are in situation (a).

Now, remove A_1 and A_2, then choose the knight A_3 who is the shortest among those that stand to the right of A_2, etc. As a result we obtain a chain A_1, A_2, $A_3, \dots$ of knights ordered with respect to their heights, so let *them* remain while the *other* knights stand out.

48.8.5. Let $\angle A \geq 60^\circ$ be the greatest angle in triangle ABC, and $AD > 1$ the bisector of angle $\angle A$. Draw all possible straight lines through point D intersecting the sides of angle $\angle A$ and choose a triangle of the smallest area, see Fig. 370.

This triangle is isosceles (compare the areas of the shaded triangles in Fig. 370), and its area is greater than $\frac{1}{\sqrt{3}}$.

FIGURE 370. (Sol. 48.8.5)

FIGURE 371. (Sol. 48.9.2)

48.9.1. Setting $x = x_1^2$, $y = y_1^2$, $z = z_1^2$ and squaring both sides of the equation, we come to Problem 48.8.1.

48.9.2. Place the first 50 airports in the vertices of a regular 50-gon, and all the other airports in its center. Then all airplanes from the center fly to the vertices and all airplanes from the vertices fly to the diametrically opposite vertices, see Fig. 371.

Thus, all planes gather in all 50 vertices of the 50-gon. Taking into consideration that the distances between pairs of airports must be distinct, slightly deform our 50-gon to satisfy the hypothesis.

48.9.3. In Problem 48.8.3 we proved that the square covers ≥ 2 nodes. There remains a case when the square covers a "house": 6 nodes in two rows (three in each one) and one node in the center of the third row, see Fig. 372 c).

Prove that the projection of the "house" on one of two perpendicular directions covers points whose distance from one another is ≤ 2. (Instead we may prove that there is a unique way to squeeze an isosceles triangle with base 2 and side $2\sqrt{5}$ into the 2×2 square.)

48.9.4. Prove by the rule of contraries. Choosing any pair of people (A, B), select from the other 10 people some $C_1, C_2, \dots, C_k$ such that each of them is a friend of one person in the pair.

By the hypothesis, $k \geq 6$, as $10 - k \geq 5$ persons in the considered group satisfy the condition of the problem for $k \leq 5$. Let us calculate the number of triples (A, B, C_i) in two ways.

The total number of pairs (A, B) is $11 \times 12/2 = 66$, and at least six persons C_i correspond to each pair; therefore, $N \geq 6 \times 66 = 396$.

On the other hand, we can fix C_i and count pairs (A, B) for which C_i is a friend of exactly one person in the pair. If C_i has n friends, then the number of desired pairs (A, B) is equal to $n(11 - n) \leq 30$. It is possible to choose C_i in 12 ways, hence, $N \leq 30 \cdot 12 = 360$. Thus, $396 \leq N \leq 360$. Contradiction.

48.9.5. Induction: For $n = 3$ this is true; let $2^n = 7x^2 + y^2$, where x and y are odd.

Consider two pairs of numbers $\left(\frac{x-y}{2}, \frac{7x+y}{2}\right)$, and $\left(\frac{x+y}{2}, \frac{7x-y}{2}\right)$. For each pair the square of the first number times 7 plus the square of the second number is equal to 2^{n+1}.

Now observe that the numbers of one pair are of the same parity, and numbers from different pairs are of opposite parity; therefore, the numbers of one of the pairs are odd.

SCIENTIFIC SOLUTION. This solution provides with a deep connection between the above proof and the structure of the set (polynomial ring) $S = \mathbb{Z}[\sqrt{-7}]$. By definition this set consists of all complex numbers z of the form $a + b\sqrt{-7}$ with integer a and b. Clearly, if $z, z' \in S$, then $z + z' \in S$ and $zz' \in S$.

FIGURE 372. (Sol. 48.9.3)

Let the *norm* of z be $N(z) = a^2 + 7b^2 = z\bar{z}$. As is easy to verify, $N(zz') = N(z)N(z')$.

Now, for each $n \geq 3$ we must find a pair of odd numbers $x, y \in \mathbb{Z}$ such that $x = y + x\sqrt{-7}$ and $N(z) = 2^n$. It is clear that

$$8 = (1 + \sqrt{-7})(1 - \sqrt{-7}); \quad N(3 + \sqrt{-7}) = 16; \quad N(1 + 3\sqrt{-7}) = 64.$$

It is not obvious, however, that if $2^n = N(z)$, then $2^{n+1} = N(w)$ for some $w = A + B\sqrt{-7}$ with odd integers A and B. We will try to get such w by multiplying z by $1 \pm \sqrt{-7}$ as follows:

$$(y + x\sqrt{-7})(1 + \sqrt{-7}) = (y - 7x) + (x + y)(\sqrt{-7}),$$
$$(y + x\sqrt{-7})(1 - \sqrt{-7}) = (y + 7x) + (x - y)(\sqrt{-7}).$$

Note that only one of the numbers $x \pm y$ is divisible by 4. Indeed, if $x = 2k + 1$ and $y = 2l + 1$ for $k, l \in \mathbb{Z}$, then $x + y = 2(k + l + 1)$, $x - y = 2(k - l)$ and exactly one of the numbers, $k + l + 1$ or $k - l$, is even. Now, suppose that $x + y$ is divisible by 4. Then take odd numbers

$$A = \frac{y + 7x}{2} = \frac{y - x}{2} + 4x \text{ and } B = \frac{x - y}{2}.$$

We have $N(A + B\sqrt{-7}) = 2^{n+1}$. Similarly, we show that both $A' = \frac{y - 7x}{2}$ and $B' = \frac{x - y}{2}$ are odd if $x - y$ is divisible by 4. The inductive step is also possible in this case because $N(A' + B'\sqrt{-7}) = 2^{n+1}$. Q.E.D.

48.10.4. By the hypothesis any two sets have one common element. Let us prove that *all sets have only one common element*. Assume the contrary.

Consider set No. 1. There is an element A, which belongs to at least 45 other sets (No. 2, 3, 4, ..., 46): otherwise the total number of sets would not be greater than $44 \times 45 + 1 = 1981$. By inductive hypothesis, there is a set which does not contain A. It has one common element with the sets 1, 2, ..., 46, and, therefore, contains not 45 but 46 elements. Contradiction.

48.10.5. Call the tetrahedron T. Draw three pairs of parallel planes through three pairs of skew edges of T, and get a parallelepiped, P; the diagonals of the faces of P are the edges of T. The volume of P is three times greater than the volume of T and the distances between the parallel faces of P are equal to h_1, h_2, h_3, see Fig. 373.

Prove on your own that the volume of P is not less than $h_1h_2h_3$. (Cf. Problem 3.1.3.)

49.7.1. Fold the quadrilateral twice along its diagonals. If the triangles coincide both times, the quadrilateral is a rhombus.

FIGURE 373. (Sol. 48.10.5) FIGURE 374. (Sol. 49.7.3)

49.7.2. Assume that the system has a solution. Square both sides of each inequality, transfer all right hand sides to the left and factorize the obtained differences of squares. We get:

$$(x-y+z)(x+y-z)<0, \quad (y-z+x)(y+z-x)<0, \quad (z-x+y)(z+x-y)<0.$$

Multiplying these inequalities we get a contradiction:

$$((x-y+z)(x+y-z)(y+z-x))^2<0.$$

REMARK (for higher graders). It is interesting to interpret a geometric meaning of these inequalities in terms of distances between points on the real or complex line.

49.7.3. Let A be the meeting place to be found. Number the houses 1, 2, 3 in accordance with the speeds of the dwarfs. Let a be the distance between house No. 1 and house No. 2 and b be that between house No. 1 and house No. 3; let x, y and z be the distances between A and houses 1, 2, 3, respectively. Then by the triangle's inequality

$$\frac{a}{2} \le \frac{x}{2}+\frac{y}{2}, \quad \frac{b}{3} \le \frac{x}{3}+\frac{z}{3},$$

whence

$$\frac{a}{2}+\frac{b}{3} \le \frac{x}{2}+\frac{y}{2}+\frac{x}{3}+\frac{z}{3} \le x+\frac{y}{2}+\frac{z}{3}.$$

The equality only holds for $x=0$.

49.7.4. Consider sets consisting of all possible tuples of the initial integers: of one, of two, of three, etc. It follows from *Prerequisites* that there are $2^{1986}-1$ such sets.

Multiply the numbers of each set and write the obtained product in the form of a product of the greatest square and several prime factors, for example $M=2^{16}\cdot 3^{15}\cdot 5^{13}\cdot 7^{19}$ takes the form $(2^8\cdot 3^7\cdot 5^6\cdot 17^4)^2\cdot 3\cdot 5\cdot 17$ and $N=2^{16}\cdot 13^{10}$ takes the form $(2^8\cdot 13^5)^2$.

Compare each tuple with the set of primes obtained by isolating the greatest square from the product (in this example M is compared with the set $\{3,5,17\}$ and N is compared with the empty set). It is easy to see that the number of distinct tuples composed of 1985 prime divisors (of one, two, three and so on, the empty set included) is equal to 2^{1985}, which is less than $2^{1986}-1$, the number oftuples of initial numbers. Therefore, one set $p_1,\dots,p_k$ of prime divisors corresponds to certain two tuples of initial numbers, $A=a^2p_1p_2\dots p_k$ and $B=b^2p_1\dots p_k$. Hence, AB is a perfect square.

On the other hand, AB is equal to the product of the numbers of set A and the numbers of set B. Delete the common part of the sets A and B from both sets (the product of the numbers deleted is a perfect square); we see that the product of the remaining numbers is a perfect square, Q.E.D.

49.7.5. There are $3^3=27$ different three-digit numbers written with the digits 1, 2, 3. Except for the first two digits in the sequence of pressed buttons, each of the other digits is the last digit of some 3-digit number. This means that the sequence has to have at least $27+2=29$ digits. To open the lock, 29 digits suffice: for example, the sequence

11 123 222 133 313 121 223 113 233 211

will do the trick.

49.8.1. If $ABCD$ is a rhombus (see the solution to Problem 49.7.1), then fold the paper so that vertex C coincides with vertex B. If vertex D coincides with A, then $ABCD$ is a square.

49.8.2. We must solve the equation $x^2 - y^2 = n$ in positive integers. Since $x^2 - y^2 = (x-y)(x+y)$ and $x - y$ and $x + y$ are of the same parity, n is either divisible by 4 or is of the form $4k \pm 1$.

The converse is also true: *all numbers n of the form $4k$ and $4k \pm 1$ except 1 and 4 are of the form $x^2 - y^2$.*

Indeed, if $(x-y)(x+y) = 4k$, take $x = k+1$, $y = k-1$ ($n = 4$ is an exception as $y = 0$ is not positive in this case); if $(x-y)(x+y) = 4k \pm 1 = 2l + 1$, then take $x = l + 1$, $y = l$ ($n = 1$ is an exception as in this case $y = 0$). This means that only 1, 4 and all numbers of the form $4k + 2$ cannot be written in the desired form.

49.8.3. Let $b_n = \frac{a_n - \sqrt{2}}{a_n + \sqrt{2}}$. Prove that $b_{n+1} = b_n^2$.

Indeed,

$$b_{n+1} = \frac{a_{n+1} - \sqrt{2}}{a_{n+1} + \sqrt{2}} = \frac{\frac{a_n}{2} + \frac{1}{a_n} - \sqrt{2}}{\frac{a_n}{2} + \frac{1}{a_n} + \sqrt{2}} = \frac{a_n^2 - 2a_n\sqrt{2} + 2}{a_n^2 + 2a_n\sqrt{2} + 2} = \left(\frac{a_n - \sqrt{2}}{a_n + \sqrt{2}}\right)^2,$$

whence

$$b_{10} = b_9^2 = b_8^4 = \cdots = b_1^{2^9},$$

i.e.,

$$\frac{a_{10} - \sqrt{2}}{a_{10} + \sqrt{2}} = \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right)^{2^9}.$$

It follows from the inequality between the arithmetic mean and the geometric mean that

$$a_{10} = \frac{1}{2}\left(a_9 + \frac{2}{a_9}\right) > \sqrt{a_9 \cdot \frac{2}{a_9}} = \sqrt{2}$$

(the inequality is a strict one, because $a_9 \neq \frac{2}{a_9}$), hence,

$$\frac{a_{10} - \sqrt{2}}{a_{10} + \sqrt{2}} > \frac{a_{10} - \sqrt{2}}{2\sqrt{2}}.$$

Therefore,

$$a_{10} - \sqrt{2} < 2\sqrt{2}\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right)^{512} = \frac{2\sqrt{2}}{(\sqrt{2} + 1)^{1024}}.$$

Observe that

$$(\sqrt{2} + 1)^8 = (3 + 2\sqrt{2})^4 = (17 + 12\sqrt{2})^2 > (24\sqrt{2})^2 = 1152 > 10^3;$$

therefore,

$$(\sqrt{2} + 1)^{1024} > (10^3)^{128} = 10^{384} > 2\sqrt{2}10^{383}$$

and

$$a_{10} - \sqrt{2} = \frac{2\sqrt{2}}{(\sqrt{2} + 1)^{1024}} < 10^{-383}.$$

Thus, we have proved an even stronger inequality than the one required in the problem.

Remarks. 1) The main idea of the proof of this problem is the same as that of Problem 16.2.10.3: the number of digits after the decimal point is approximately doubled after each iteration; therefore, the accuracy of calculations improves quicker than exponentially.

2) It is impossible, however, to conclude formally from the inequality $a_{10} - \sqrt{2} < 10^{-370}$ that the first 370 digits of the number a_{10} and those of $\sqrt{2}$ coincide: for this, it would suffice to show (and the easiest way to do this is to use a computer) that the digit in the 371-th place after the decimal point in the decimal notation of $\sqrt{2}$ is not 9 (if it is 9, then we cannot even say that at least one of the digits coincides; a counter-example is $1 - 0.999\,999\,999\,99 = 10^{-10}$). How do we show this?

Following the algorithm, we get $\sqrt{2}$ with good accuracy but not precisely. The way out is actually a very simple one: it suffices to perform one more, 11-th, iteration, and see that the 371-th digit of a_{11} is not 9.

49.8.4. Thanks to Hint, if the weeds overgrow the whole field, the total length of the boundary of this domain would be equal to 40, although at the first moment (hence, all time afterwards) it could not have been greater than $4 \times 9 = 36$. Contradiction.

49.8.5 and **49.9.3.** The solution of these problems is almost identical to that of Problem 49.7.2. Perform the following operations:

(1) square the inequalities (the signs of the inequalities do not change);
(2) transfer all right-hand sides to the left;
(3) factor each obtained difference of squares;
(4) multiply all obtained inequalities; the product is positive. On the other hand, it is

$$-((x-y+z-t)(x+y-z+t)(-x+y+z-t)(x-y+z+t))^2.$$

Contradiction.

49.9.1 and **49.10.1.** To determine whether $ABCD$ is a rectangle verify whether $AB = CD$, $BC = AD$ and $AC = BD$; the total number of operations is 9 (three operations for each equality: two measurements and one comparison).

The rectangle $ABCD$ is a square if $AB = BC$; to find this we need one more operation: comparison of the lengths of AB and BC.

Prove that *it is impossible to solve the problem in fewer operations*, i.e., it is necessary to perform all the operations listed above.

Indeed, if we do not know that some of two opposite sides of the quadrilateral are equal, we cannot even say that $ABCD$ is a parallelogram, as it may be an equilateral trapezoid (its diagonals are also equal). Therefore, it is necessary to verify if $AB = CD$ and $BC = AD$. If we do not know that $AC = BD$, then, perhaps, $ABCD$ is a parallelogram. This means that it is necessary to verify whether the diagonals are equal. To make sure that it is a square we must verify the equality of two adjacent sides.

49.9.2. If the homothety coefficient is $k < 1$ and the ant runs the first convolution of the spiral in, say, t hours, then for the remaining convolutions it will need kt, k^2t, $k^3t, \dots$ hours, respectively; therefore, it will need as much as $t(1+k+k^2+\dots) = \frac{t}{1-k}$ hours for the whole way.

FIGURE 375. (Sol. 49.9.5)

49.9.4. Write the product of an arbitrary pair of numbers (a, b) of the given set in the form of the product of a perfect square by the rest of its prime divisors. For example, if $a = 2^{13} \cdot 3^4 \cdot 19^3$ and $b = 5^6 \cdot 7^7 \cdot 19$, then $ab = K^2 \cdot 7 \cdot 2 \cdot 7$, where $K = 2^6 \cdot 3^2 \cdot 5^3 \cdot 7^3 \cdot 19^2$.

Assign this set of prime divisors to the pair (a, b). There are $C_{48}^2 = 48 \cdot \frac{47}{2}$ of all possible distinct pairs (a, b) in a set of 48 numbers, and the total number of subsets of the set of ten prime divisors (of none, one, two, etc. divisors) is 2^{10}. Since $C_{48}^2 > 2^{10}$, there are two distinct pairs (a, b) and (c, d) that correspond to the same set $(p_1, p_2, \dots, p_k)$ of prime divisors $(0 \le k \le 10)$.

Consequently, $abcd$ is a perfect square. If the pairs (a, b) and (c, d) do not have a common element, the numbers a, b, c, d are the desired ones. If there is a common term, say, $b = d$, then ac is a perfect square.

Let us discard for a while the pair (a, c). Then we come to a set of 46 numbers, whose product has not more than 10 different prime divisors. Performing all previous operations and taking into account that $C_{46}^2 > 2^{10}$, we conclude that there are two distinct pairs (x, y) and (z, t) for which $xyzt$ is a perfect square.

If these pairs do not have a common term, then x, y, z, t is the 4-tuple desired; if they have a common term, say $x = t$, then yz is a perfect square. In this case the 4-tuple desired is a, c, y, z.

49.9.5. On a checkered plane with the side of each square equal to 1, choose the northern-most horizontal straight line $y = k$ that intersects the fixed circle of radius 100 with center at O (i.e., the straight line $y = k+1$ does not intersect the circle), see Fig. 375.

If all vertices of squares on this straight line are outside the circle, then it can be easily found that the distance between the vertex nearest to the circle and the straight line is less than $\frac{1}{14}$; therefore, the line $y = k$ intersects the drawn disc of radius $\frac{1}{14}$ with the center in the vertex. Let, therefore, some of vertices on the straight line $y = k$ be situated inside the circle.

Among them, choose a vertex, B, closest to the circle. Let A be the vertex nearest to it, but situated on the straight line $y = k$ outside the circle; $AB = 1$. Assume that the circle does not intersect the drawn disks of radius $\frac{1}{14}$ each with centers A and B.

Then $OA > 100 + \frac{1}{14}$, $99 < OB < 100 - \frac{1}{14}$, whence

$$OA - OB > \tfrac{1}{7}, \qquad OA^2 - OB^2 = (OA - OB)(OA + OB) > 199 \cdot \tfrac{1}{7}.$$

If O' is the projection of O on the line $y = k$ and $O'B = x$, then $O'A = x + 1$ and $(x+1)^2 - x^2 = OA^2 - OB^2 > \frac{199}{7}$, whence $O'B = x > \frac{96}{7}$.

Hence,

$$OO'^2 = OB^2 - O'B^2 < (100 - \tfrac{1}{14})^2 - \left(\tfrac{96}{7}\right)^2 < 99^2,$$

therefrom $OO' < 99$. Therefore, the distance between O and the line $y = k+1$ is equal to $OO' + 1 < 99 + 1 = 100$, i.e., our circle of radius 100 intersects the line $y = k+1$, too. Contradiction.

49.10.2. Let $AB = a$, $AC = b$, $AD = l$, $\angle C = x$, $\angle B = y$, let d be the diameter of the circumscribed circle of triangle ABC, see Fig. 376.

FIGURE 376. (Sol. 49.10.2)

By the law of sines the length of a chord is equal to the product of the diameter times the sine of half the arc that the chord intercepts. As

$$\smile AB = 2x, \quad \smile AC = 2y, \quad \smile AD = \pi - x + y,$$

so

$$a = d \sin x, \quad b = d \sin y, \quad l = d \sin \frac{\pi - x + y}{2} = d \cos \frac{y - x}{2}.$$

The given inequality implies $d \cos \frac{x-y}{2} > \frac{d \sin x + d \sin y}{2}$ and, therefore, the statement of the problem is correct.

49.10.3. Obviously, $x = \sqrt{2}$ is a solution. The function $y = x^{x^4}$ monotonously increases as $x > 1$; therefore, there are no other solutions for $x > 1$. There are also no solutions when $0 < x < 1$ since $y(x) < 1$ for these x. Thus, $x = \sqrt{2}$ is the only solution.

49.10.4. Let us square the given inequalities and add them. Since $|\vec{x}|^2 = (\vec{x}, \vec{x}) = \vec{x}^2$, we get

$$3(\vec{a}^2 + \vec{b}^2 + \vec{c}^2) < 2(\vec{a}^2 + \vec{b}^2 + \vec{c}^2) - 2\big((\vec{a}, \vec{b}) + (\vec{b}, \vec{c}) + (\vec{a}, \vec{c})\big),$$

whence $(\vec{a} + \vec{b} + \vec{c})^2 < 0$. Contradiction.

49.10.5. Answer: $\min\limits_{\alpha,\beta}\max\limits_{x} y(x) = \frac{\sqrt{3}}{2}$ and it is reached for $\alpha = 0$, $\beta = -\frac{1}{6}$.

Solution. For any α and β we have

$$\max_x y(x) \geq \max\Big(y(\tfrac{\pi}{6}),\ y(\tfrac{5\pi}{6})\Big) = \max\left(\left|\frac{\sqrt{3}}{2}+\frac{\alpha}{2}\right|,\ -\left|\frac{\sqrt{3}}{2}+\frac{\alpha}{2}\right|\right)$$
$$\geq \frac{\left|\frac{\sqrt{3}}{2}+\frac{\alpha}{2}\right| - \left|\frac{\sqrt{3}}{2}+\frac{\alpha}{2}\right|}{2} \geq \frac{\left|\frac{\sqrt{3}}{2}+\frac{\alpha}{2} - \left(-\frac{\sqrt{3}}{2}+\frac{\alpha}{2}\right)\right|}{2} = \frac{\sqrt{3}}{2};$$

therefore,

$$\min_{\alpha,\beta}\max_x y(x) \geq \frac{\sqrt{3}}{2}.$$

Assume now that $f(x) = \cos x - \frac{1}{6}\cos 3x$ and find the extremal points of $f(x)$. From the equation $f'(x) = 0$, i.e., $-\sin x + \frac{1}{2}\sin 3x = 0$ or, which is the same, $-2\sin x + (3\sin x - 4\sin^3 x) = 0$, we derive that either $\sin x = 0$ (hence, $x = k\pi$) or $\sin^2 x = \frac{1}{4}$ (hence, $x = \pm\frac{1}{6}\pi + k\pi$).

For $y(x) = |f(x)|$ we have

$$y(k\pi) = \frac{5}{6} < \frac{1}{2}\sqrt{3}, \quad y\left(\pm\frac{1}{6}\pi + k\pi\right) = \frac{1}{2}\sqrt{3}.$$

Therefore, $\max\limits_x y(x) \leq \frac{\sqrt{3}}{2}$ and $\min\limits_{\alpha,\beta}\max\limits_x y(x) \leq \frac{1}{2}\sqrt{3}$.

50.7.1. Suppose that the math club held meetings during any three consecutive days in March. Since it did not meet on weekends, it surely met both on Friday and on Monday. During three days, Tuesday, Wednesday and Thursday, the club met at least once by hypothesis. In March there are 4 weeks and 3 extra days and, therefore, the club met at least 8 times on Mondays and Fridays and no fewer than 4 times in the middle of the week: no fewer than 12 times altogether. Contradiction.

50.7.2. There are exactly 25 primes between 1 and 100. Adding 1 to this set we get 26 relatively prime numbers. By hypothesis, 27 numbers, each less than 100, are selected and the total number of different prime divisors of these numbers does not exceed 25. If there is no number 1 among them, then at least 2 of these 26 numbers are not relatively prime; if there is a 1 among them, it is possible to select two of the 27 numbers (thanks to the Dirichlet principle) with the same prime divisor and then these two numbers are not relatively prime.

50.7.3. First the hunter should stand at the center O of the meadow, see Fig. 377, then the distance from the hunter to any of the sides of the meadow is $OH_i = \frac{1}{3} \cdot \frac{100\sqrt{3}}{2} < 30$ m.

FIGURE 377. (Sol. 50.7.3)

Consequently if the wolf is not yet hit, it stands in one of the quadrilaterals OH_1AH_3, OH_1BH_2 or OH_2CH_3, say, in the latter one, and not at the points H_2 or H_3. If the hunter starts to move from O to point C along the bisector OC, the distance from the hunter to sides BC and AC is less than 30 m, and the wolf must intersect at least one of the segments $O'H_2'$ to get out of quadrilateral OH_2CH_3; then the hunter could hit it.

FIGURE 378. (Sol. 50.7.4)

50.7.4. Suppose that $AD \neq BC$, see Fig. 378a) and b).

Let B' be the image of B under the reflection through line CD. If O is the base of the perpendicular MO to segment AB at its midpoint M (point O lies on line CD), then O lies on segment AB' (Fig. 378 a)). Denote by D' the point on straight line CD symmetric to D with respect to O (Fig. 378 b)).

It is clear from Fig. 378 a) that the perimeter of $\triangle ACB'$ is less than that of $\triangle ADB'$ (or less than that of $\triangle AD'B'$ from Fig. 378 b)), as the first triangle lies inside the second one. Therefore,

$$AC + CB = AC + CB' < AD + DB' = AD + DB$$

$$AC + CB = AC + CB' < AD' + D'B' = AD + DB,$$

contradicting the hypothesis. Therefore, $ABCD$ is an isosceles trapezoid.

Remark. Those who know what an ellipse is may immediately derive a solution: the given equality implies that points C and D lie on the ellipse with focuses A and B so that straight line CD is parallel to the ellipse's axis AB. The symmetricity of the ellipse with respect to its axis l implies that points C and D are symmetric with respect to l (Fig. 378 c)) and, therefore, segments AD and BC are symmetric with respect to l, i.e., $AD = BC$.

50.7.5. If the first thief will divide the coins into two piles, each consisting of more than $20 \cdot 49 = 980$ coins, than one of these piles will be divided into not more than 20 parts (since there are 41 parts altogether) and the largest pile will consist of more than 49 coins.

Let us show that it is impossible to get more than 50 coins. Suppose the first thief divided coins into piles of x and $1987-x$ and let $x = 50a+r$, $1987-x = 50b+s$ be the formulas for division with a remainder $(0 \le r, s < 50)$. Then $1987 = 50(a+b)+(r+s)$, therefrom $r+s = 37$ or $r+s = 87$ and $a+b = 39$ or 38, respectively. Therefore, the other thieves can divide the coins into $a+b$ piles of 50 coins each and two piles of r and s coins (40 or 41 piles altogether).

Let us pass to the second thief. To ensure his share of 26 coins it suffices for him to arrange so that the first thief gets not more than 986 coins: then there will remain no less than $1001 = 25 \cdot 40 + 1$ coins and the largest pile will contain not less than 26 coins. In order to prevent this, the first thief should divide the coins into piles of $x \ge 1973$ and $y \le 14$ coins. But in this case if the second thief divides the greater pile in halves (or almost in halves if it contains an odd number of coins) the first thief gets not more than $\left[\frac{1987}{2}\right] + 1 = 994$ coins and the other thieves will get $993 > 14 + 25 \cdot 39$ coins; hence, one of the piles will consist of more than 25 coins.

It is easy to show that the second thief will not get more than 26 coins: in order to ensure this the first thief has to earmark one coin; and each of the other thieves has to separate 26 coins from the greatest pile.

As far as the other thieves are concerned, if all thieves, except the k-th one, will separate 1 coin from the greatest pile then all piles but two will consist of 1 coin. Hence, the k-th thief $(k \ge 3)$ will get only one coin.

50.8.1. The proof follows from the chain of inequalities:

$$\begin{aligned}
&a > b > 0,\ ay > bx \implies ay(a-b) > bx(a-b) \implies a^2y + b^2x > abx + aby \\
&\implies abx + aby + a^2y + b^2x > 2abx + 2aby \\
&\implies (bx+ay)(a+b) > (x+y)2ab \implies \frac{bx+ay}{2ab} > \frac{x+y}{a+b}.
\end{aligned}$$

50.8.2. a) For $n = 1$ and $n = 2$ the boy can cut out of the square two 1×2 strips and five 1×3 strips, respectively; see Fig. 50.8.2 a1) and a2). It is impossible to cut out more strips because of the lack of area (in the first case the additional area is equal to 0, in the second case it is equal to 1).

b) Using the same standard method the boy can cut eight 1×4 rectangles out of the 6×6 square; see Fig. 379 b1). Here area considerations are of no help: it might seem that the 4 shaded squares from Fig. b1) could be "rearranged" somehow into one 1×4 rectangle.

However, it is impossible to cut out more than eight 1×4 strips for the following reason. Let us shade every square of the 6×6 square one of the colors 1, 2, 3, 4, as shown in Fig. 379 b2) (the "diagonals" are colored the same color which is repeated with period 4). Then in an arbitrary 1×4 rectangle there exists a square of color 4 and there are exactly 8 such squares. Therefore, there can be no ninth strip.

c) The number of strips cannot exceed $4n-4$ for $n > 3$ from the area consideration: $(2n)^2 = (4n-4)(n+1)+4$. It turns out in reality that the boy can cut $4n-4$ strips out of a $2n \times 2n$ square. This is shown in Fig. 379 c2): the $2n \times 2n$ square is divided into 4 rectangles of size $(n+1) \times (n-1)$ and one 2×2 square in the middle. From each rectangle we can cut $n-1$ strips of length $n+1$, altogether $4(n-1) = 4n-4$ such strips.

50.8.3. Take an arbitrary team of $n-1$ students, where n is the total number of students. The team can compete only with one remaining student (whom else?). Therefore, the remaining student corresponds to each team of $n-1$ students; the pairs $(n-1, 1)$ are considered and for them the statement is proved.

Any team of $n-2$ students cannot compete with a team consisting of exactly 1 student, since each student enters a pair $(n-1, 1)$ once and cannot enter any other pair. Therefore, two students (from the remaining students of the class) may compete with the chosen $n-2$ students. At least one of these two enters each of the remaining teams of $n-2$ students. Therefore, to each team of $n-2$ students the two remaining students correspond. Hence, for the pairs $(n-2, 2)$ the statement is also proved.

Similarly, any team of $n-3$ students may only compete with the three remaining students, since it must not compete with one or two because they enter either to pair $(n-1, 1)$ or a pair $(n-2, 2)$.

FIGURE 379. (Sol. 50.8.2)

FIGURE 380. (Sol. 50.8.3)

Further, considering teams of $n-4$, $n-5$, $n-6$, ... , persons, we get the desired statement.

ANOTHER SOLUTION. Suppose that some team A_{n-k} of $n-k$ students competes not with the remaining k students but with some of its subteam B_s of $s<k$ persons. Denote by C_{k-s} the set of remaining $k-s$ students, see Fig. 380. Adding C_{k-s} to the team A_{n-k} we get a team A_{n-s}. It cannot compete with B_s, since B_s already competes with A_{n-k} and, therefore, A_{n-s} must compete with a subteam of B_s, a team B_l, where $l<s$. Denote by C_{s-l} the students from B_s that do not enter B_l. Then the new team A_{n-l} consisting of A_{n-s} and C_{s-l} cannot compete with B_l but only with its subteam, B_r, where $r<l$.

Continuing these arguments we see that this process never stops, i.e., the number of teams must be infinite. But this number is less than 2^n. Contradiction.

REMARK. There is one more solution connected with the calculation of pairs of the corresponding subsets. We leave it to the reader to find such a solution.

50.8.4. Let K be the midpoint of diagonal AC and N the midpoint of diagonal EC. Let us prove that $\triangle BKM = \triangle MND$, see Fig. 381.

Indeed, $BK = MN$, since $BK = AC/2$ as a median in the right triangle $\triangle ABC$, and $MN = AC/2$ since MN is a midline in $\triangle ACE$. For the same reason DN is a median in the right triangle $\triangle CDE$, i.e., $DN = CE/2$ and KM is a midline in $\triangle ACE$, i.e., $KM = CE/2$, and, therefore, $KM = ND$.

Finally, let us prove that $\angle BKM = \angle MND$. Let us represent each of these angles as the sum of two angles into which they are divided by straight lines AC and EC: namely, $\angle BKM = \angle BKA + \angle AKM$, $\angle MND = \angle MNE + \angle END$. But $\angle BKA = 2\alpha$ being the outer angle of $\triangle BKC$, see Fig. 381, and $\angle END = 2\alpha$ by the same reason. The angles $\angle AKM$ and $\angle MNE$ equal to $\angle ACE$, as $KM \parallel CE$ and $MN \parallel AC$. The equality of the indicated angles is proved.

FIGURE 381. (Sol. 50.8.4)

FIGURE 382. (Sol. 50.8.5)

Therefore, we have proved the equality of the triangles with two equal sides and an included angle implying $BM = MD$. Q.E.D.

50.8.5. Let us prove that of all foursomes n, $n + 50$, $n + 1987$, $n + 50 + 1987$ (n is arbitrary) exactly two numbers are chosen and two are not.

Indeed, if n is chosen, then $n + 1987$ is not (by the second condition) and, therefore, $(n + 1987) + 50$ is chosen (by the first condition). But of the numbers $n + 50$, $(n + 50) + 1987$ at least one is not chosen and since $n + 50 + 1987$ is chosen, $n + 50$ is not. We similarly consider the case when n is not chosen: then $n + 50$ and $n + 1987$ are chosen and $n + 50 + 1987$ is not.

Let us paint the numbers n and $n + 50 + 1987$ white, the numbers $n + 50$ and $n + 1987$ black. Numbers of the same color are simultaneously either chosen or not. It is convenient to depict these numbers as the intersections of a plane grid colored alternately white and black, see Fig. 382, where neighboring numbers on the same horizontal differ by 50, and on the same vertical, by 1987.

Let us find out what is the color of $M = n + 50 \times 1987$. Moving to the right from n along a horizontal by steps of length 50, we get a black number M at the 1987-th step, and starting from n along a vertical by steps of length 1987, we get a white number M at the 50-th step. This contradiction shows that there is no such a subset of positive integers.

50.9.2. a) If suffices to draw a line l from the remaining vertex B parallel to AD and find the point C of intersection of l with straight line MK, where $K = BE \cap AD$, $M = BA \cap DE$ (Fig. 383 a), then C is the desired vertex.

All that remains is to draw the line l (BC) parallel to AD from point B chosen outside the given straight line AD. We suggest the following construction procedure:

Draw a straight line l' parallel to AD using a two-sided ruler (l' can be drawn so that the given point B is not between these lines). Connect B by straight lines with A and D and let $A'D'$ be a segment intercepted by these straight lines on l' (Fig. 383 b)). If $P = A'D \cap AD'$, then line BP intersects the segments $A'D'$ and AD at their midpoints K' and K, respectively (prove this yourself). Then straight lines AK' and KD' meet at B'; and BB' is the desired straight line l: as Fig. 383 b) shows, $BB' \parallel AD$.

b) In step a) we have actually proved the following:

Lemma 1. *On a plane, given parallel straight lines l_1 and l_2 and a point B not between them, we can use a one-sided ruler*

1) *to draw a straight line l through B parallel to l_1 and l_2 and*

2) *to divide an arbitrary segment on either of the lines l_1, l_2 in halves.*

Two parallel straight lines can always be drawn using a two-sided ruler and, therefore, the following statement holds:

Lemma 2. *Using a two-sided ruler a straight line parallel to l may be drawn through a point not on l and any of its segments may be divided into halves.*

The following statement also holds:

Lemma 3. *On a plane, given an arbitrary point A and a straight line l, it is possible to use a two-sided ruler in order to drop the perpendicular from A to l (if $A \in l$ is is possible to erect the perpendicular to l at A).*

The construction is illustrated in Fig. 383 c). Choose arbitrary points X and Z on l and place the ruler on the plane in two ways so that X and Z are on different sides of the ruler. Then the intersection of these two positions of the ruler gives a rhombus $XYZT$ whose diagonal YT is perpendicular to l. All that remains is to draw the straight line l' through A parallel to YT. This is the desired perpendicular to l.

FIGURE 383. (Sol. 50.9.2)

Now, suppose that three neighboring vertices A, B, E of pentagon $ABCDE$ survive. Draw straight line BD through B parallel to AE (Fig. 383 d)). Further, divide AB in halves: M is the midpoint of AB. Let us erect perpendicular MD to AB at M; it meets BD at D, an erased vertex of $ABCDE$. Further, draw $BC \parallel AD$ and $CK \perp AE$, where $AK = KE$; then C is the intersection point of BC and CK, the second erased vertex. The pentagon $ABCDE$ is completely recovered.

If the non-erased vertices are A, B, D, draw $BC \parallel AD$, divide BD in halves (P is the midpoint of BD) and draw the perpendicular through the midpoint of BD; straight lines BC and PC meet at an erased vertex C. Finally, recover the remaining erased vertex E.

50.9.3. There are many such examples. The simplest one: take the first 50 primes 2, 3, 5, ..., p_{50} and form 50 products P_1, $P_2, \dots, P_{50}$, each of the products containing exactly 49 factors. Then no P_k is divisible by any P_l for $l \neq k$ and P_kP_l is divisible by any P_m for distinct k, l.

50.9.4. Making use of the inequality $a + b \geq 2\sqrt{ab}$ we get

$$\begin{aligned}\left(\frac{a_1^2}{b_1} + \dots + \frac{a_n^2}{b_n^2}\right)(b_1 + \dots + b_n) &= a_1^2\left(1 + \frac{b_2}{b_1} + \dots + \frac{b_n}{b_1}\right)\\ &+a_2^2\left(1 + \frac{b_1}{b_2} + \frac{b_3}{b_2} + \dots + \frac{b_n}{b_2}\right) + \dots + a_n^2\left(1 + \frac{b_1}{b_n} + \frac{b_2}{b_n} + \dots + \frac{b_{n-1}}{b_n}\right)\\ &= \sum a_i^2 + \left(\frac{a_1^2 b_2}{b_1} + \frac{a_2^2 b_1}{b_2}\right) + \left(\frac{a_1^2 b_3}{b_1} + \frac{a_3^2 b_1}{b_3}\right) + \dots + \left(\frac{a_k^2 b_l}{b_k} + \frac{a_l^2 b_k}{b_l}\right)\\ &+ \dots + \left(\frac{a_{n-1}^2 b_n}{b_{n-1}} + \frac{a_n^2 b_{n-1}}{b_n}\right) > \sum a_i^2 + 2\sum a_i a_j = (a_1 + \dots + a_n)^2, \quad \text{Q.E.D.}\end{aligned}$$

50.9.5. Each plank separates from the angle a piece such that

a) the remaining part is a convex figure with obtuse angles;

b) the area of the separated part does not exceed $\frac{1}{4}$ — the maximal area of the right triangle with hypotenuse 1.

The convexity of the remaining part yields that together with point $(2, 2)$ there should be separated a right triangle with hypotenuse passing through this point and with legs along the banks. The area of this triangle is equal to 8; hence, we need not less than 32 planks.

50.10.2. Let the length of the side of the equilateral $\triangle ABC$ be equal to 1, see Fig. 384.

Let vertex C be on the plane π and A' and B' be the projections of vertices A and B onto π; then $AA' = AC \cdot \sin\alpha = \sin\alpha$, $BB' = BC \cdot \sin\beta = \sin\beta$. Considering $\triangle ADA'$ and $\triangle BDB'$ we get

$$AD = \frac{AA'}{\sin\gamma} = \frac{\sin\alpha}{\sin\gamma}, \qquad BD = AD + 1 = \frac{BB'}{\sin\gamma} = \frac{\sin\beta}{\sin\gamma}.$$

Thus,

$$\frac{\sin\alpha}{\sin\gamma} + 1 = \frac{\sin\beta}{\sin\gamma},$$

implying $\sin\beta = \sin\alpha + \sin\gamma$.

FIGURE 384. (Sol. 50.10.2)

FIGURE 385. (Sol. 50.10.4)

50.10.4. Let us paint the numbers n, $n-50$ and $n+1987$ the colors A, B, C, respectively, and denote by the set-theoretic membership sign $\in$ the fact that a number is painted a certain color. Since $n-50+1987 \notin B$ and $n-50+1987 \notin C$, it follows that $n-50+1987 \in A$. Since $n-100 = (n-50)-50 \notin B$ and $(n-50)-50$ is of a color different from that of $(n-50)+1987$, i.e., not of color A, we see that $n-100 \in C$.

Thus, the colors of the numbers n, $n-50$, $n-2\cdot 50$, $n-3\cdot 50, \dots$, repeat with period 3, i.e., n and $n-3\cdot 50 \in A$; $n-50$ and $(n-50)-3\cdot 50 \in B$, etc. It is convenient to represent the numbers as the intersections of a plane grid with the steps of 50 along the horizontal and with the steps of 1987 along the vertical. The colors repeat themselves with period 3 along any horizontal and any vertical, see Fig. 385.

Moving to the left from n along the horizontal by $1987-50$ steps we get $M = n-(1987-50)\cdot 50 \notin A$, since $1987-50$ is not divisible by 3. But, as was found above, the numbers M and $M+(1987-50)$, $M+(1987-50)\cdot 2, \dots, M+(1987-50)\cdot 50$ are of the same color. However, $M+(1987-50)\cdot 50 = n \in A$, though we have just established that n and M are of different colors. Contradiction.

50.10.5. The king can summon his subjects, for example, as follows. Let him divide the whole kingdom into 4 square blocks of side 1 km each and numbere them counterclock-wise 1 to 4. The summons run in several stages.

First, the messenger should reach any inhabitant of the block the messenger now stands in (the distance of $\le \sqrt{2}$); next (s)he should go over the other blocks clockwise citizen (if at least one block contains no citizen at all messenger must go to this block nevertheless, to reach the point of the small square (a house) nearest to him). The solid lines on Fig. 386 show a possible path of the messenger. After (s)he transmits the information (s)he takes a break. The end of the first stage.

The length of the path covered by the messenger does not exceed $\sqrt{2}+3\sqrt{5}$ and the time needed for the first stage does not exceed $t_1 \le \frac{1}{3}(\sqrt{2}+3\sqrt{5})$ hours.

Second, each of the blocks 1, 2, 3, 4 is divided into 4 squares with side $\frac{1}{2}$ km and simultaneously every inhabitant of each block who got the message assumes the role of the messenger and repeats the whole procedure of the first stage in his block. The whole stage takes $t_2 \le \frac{1}{2}\left(\frac{1}{3}\sqrt{2}+\sqrt{5}\right)$ hours.

The same takes place in the 3-rd, 4-th, etc. stages, see Fig. 386. The number of stages depends on the number of citizens of the kingdom. The time required for each stage is, respectively, equal to

$$t_3 \le \frac{1}{2^2}\left(\frac{1}{3}\sqrt{2}+\sqrt{5}\right), \quad t_4 \le \frac{1}{2^3}\left(\frac{1}{3}\sqrt{2}+\sqrt{5}\right), \quad \text{etc.}$$

FIGURE 386. (Sol. 50.10.5)

Therefore, the total time T needed to pass the message to all citizens does not exceed

$$T \leq \left(1+\frac{1}{2}+\frac{1}{2^2}+\dots\right)\left(\frac{1}{3}\sqrt{2}+\sqrt{5}\right) < 2\cdot(0.48+2.24) = 5.44 \text{ h}.$$

After 5.44 hours all citizens (and the messenger) go to the court. Clearly, each of them can go to the court immediately after they fulfill their mission as messengers. It takes not more than $\frac{2\sqrt{2}}{3} < 1$ hours for each citizen (the longest path for each citizen at this stage is along the diagonal of the kingdom).

Thus, to get to the ball it takes $5.44+1 = 6.44$ hours; therefore, everybody will get to the ball by 7 p.m. in no hurry.

51.7.1. Represent p^4-1 as $p^4-1=(p-1)(p+1)(p^2+1)$. Since p is a prime, it is of the form $p=3k\pm1$, and then one of the factors, $p-1$ or $p+1$, is divisible by 3. Further, the numbers $p-1$ and $p+1$ are two consecutive even numbers. So one of them is divisible by 4; thus, the product $(p-1)(p+1)(p^2+1)$ is divisible by $3\cdot 8\cdot 2 = 48$. Finally, the prime $p>7$ is either of the form $p=5n\pm1$ or $p=5n\pm2$ and so either p^2-1 or p^2+1 is divisible by 5. Finally, p^4-1 is divisible by $48\cdot 5=240$. Q.E.D.

FIGURE 387. (Sol. 51.7.2)

51.7.2. Let us label the vertices of the cube: $ABCDA'B'C'D'$, so that points M and P are the midpoints of edges AB and $B'C'$, respectively. Let us find on the surface of the cube points equidistant from M and P along the surface, i.e., such that the shortest broken line lying on the surface and connecting such a point with M is of the same length as that connecting the point with P.

It is not difficult to observe that all inner points of face $ABCD$ are closer to M than to P and those of face $A'B'C'D'$ are closer to P than to M. Therefore, it suffices to consider faces $BCC'B'$ and $CDD'C'$ (we can skip the consideration of the remaining two lateral faces thanks to the symmetry through the line that connects the midpoints of BB' and DD', see Fig. 387 a).

There are two routes from points of face $BCC'B'$ to point M shown on the unfolding of the cube, see Fig. 387 b), as the routes to points M_1 and M_2. It is clear that the shortest way from $\triangle BB'C'$ is the way to M_1, whereas the shortest way from $\triangle BCC'$ is the way to M_2. Therefore, the points equidistant from

P and M lie on two segments — the intersection of the midperpendicular to M_1P with $\triangle BB'C'$ and the intersection of the midperpendicular to M_2P with $\triangle BCC'$.

Upon investigation of face $CDD'C'$ it turns out that there are three routes from it to point M and two routes to point P, while the points equidistant from P and M lie on segments of three midperpendiculars (to P_1M_2, P_2M_2 and P_2M_3); see Fig. 387 c).

Figure 388. (Sol. 51.7.3)

51.7.3. It is easy to construct with calipers the bisector of a given angle at vertex O by the following method. Plot equal segments OA and OA' on the legs of angle O; then, plot equal segments AB and $A'B'$ on the same legs. Draw straight lines AB' and $A'B$ with the help of a ruler. The lines meet at C. The bisector OC is the desired bisector of $\angle O$ (Fig. 388 a).

Suppose now that we want to draw line l' parallel to l through A. For this purpose: 1) choose an arbitrary point O on l, 2) connect it to A, 3) plot segment $OA' = OA$ on l, 4) construct the bisector of $\angle AOA'$, 5) connect A and A'; denote the intersection of the bisector with AA' by C, and 6) plot segment CO' equal to OC on the bisector. Then $AO \parallel OA'$, since $AOA'O$ is a rhombus; see Fig. 388 b).

Figure 389. (Sol. 51.7.4)

51.7.4. Take an arbitrary telephone A_1 and start moving along one of its outgoing wires till we get to the telephone A_2 (we do not consider wireless phones). Either A_2 does not have another outgoing wire or it has just one. Move along this wire to the telephone A_3 and then in just the same way to A_4 (if A_3 has another outgoing wire), etc. until we have either a closed chain of wires — a "cycle" — as on Fig. 389 a); note that the connection of wires shown in Fig. 389 b) is impossible since 3 wires go to the telephone A_k or we do not get to the telephone A_n which has only one outgoing wire (along which we reached it). In the latter case, having started to move from A_1 along its other outgoing wire (if any) we successively reach telephones A_2, A_3', …, until we reach A_m' which has only one outgoing wire (note again that the case of Fig. 389 b) is impossible). So we have an open chain of wires — a "circuit".

Thus, all wired telephones can be divided into "cycles" and "circuits". Any "cycle" with odd number of wires can be painted 3 colors (two colors are obviously not enough) and all "cycles" with even number of wires and the "circuits" can be properly painted 2 colors, see Fig. 389 d).

51.8.1. The sum of all numbers in a line increases by 7 after each operation (the sum of the inserted numbers is seven). So, after 100 operations it will be equal to $1 + 9 + 8 + 8 + 7 \cdot 100 = 726$.

FIGURE 390. (Sol. 51.8.2)

51.8.2. Assume that we can already draw a straight line in parallel to the given line through the point (see the solution of Problem 51.7.3). Take an arbitrary point C outside AB, draw line AC and plot segment $CD = AC$ on it from C. Connect D with B by a straight line and draw line $l \parallel BD$ through C intersecting AB at X (Fig. 390). Then X is the midpoint of AB (according to *Thales' theorem*).

51.8.3. Suppose the equality $3x^4 + 5y^4 + 7z^4 - 11t^4 = 0$ is satisfied for some positive integers x, y, z, t. Having divided the equality by the maximal possible power of 2, we can assume that not all unknown numbers x, y, z, t are even. Then either all of them or only two of them are odd (since the equation's coefficients are odd). The 4-th power of any even number is divisible by 16 and that of an odd number is divisible by 16 with remainder 1. Indeed, $(2k)^4 = 16k^4$, $(2k+1)^4 = 16(k^4 + 2k^3) + 8k(3k+1) + 1$ and if k is even, then $8k$ is divisible by 16 but if it is odd, then $8(3k+1)$ is divisible by 16 and the statement is proved.

What remains is to consider the remainders after division of $S = 3x^4 + 5y^4 + 7z^4 - 11z^4$ depending on which unknowns are odd. The following numbers can be the remainders:

$$3 + 5 + 7 - 11 = 4, \quad 3 + 5 = 8, \quad 3 + 7 = 10, \quad 3 - 11 + 16 = 8,$$
$$5 + 7 = 12, \quad 5 - 11 + 16 = 10, \quad 7 - 11 + 16 = 12.$$

Thus, S is not divisible by 16 and, therefore, it is nonzero for any positive integers x, y, z, t.

51.8.4. Number the coins 1 to 4. Place coins 1 and 2 on the pan of the balance for the first weighing, coins 1 and 3 for the second weighing, and coins 2, 3, and 4 for the third weighing. If the balance reads 20 g or 18 g after one of the weighings — either the first or second one — the weights of all coins (and thus if they are counterfeit or not) can be determined unambiguously (we leave that for the reader.)

Let now the balance read 19 g after the first and the second weighing. Then the weights of coin 2 and coin 3 are equal but yet unknown. The balance may read any weight from 27 to 30 g in the third weighing. All weights of the coins can be determined unambiguously in each case, namely:

27 g mean that Nos. 2, 3, and 4 weigh 9 g each and No. 1 weighs 10 g;

28 g mean that Nos. 2 and 3 weigh 9 g each, No. 4 weighs 10 g and No. 1 weighs 10 g;

29 g mean that Nos. 2 and 3 weigh 10 g each, No. 4 weighs 9 g and No. 1 weighs 9 g;

30 g mean that Nos. 2, 3 and 4 weigh 10 g each, and No. 1 weighs 9 g.

It is easy to see that to determine all weights unambiguously we need a weighing for each pair of coins A, B such that the pan of the balance contains exactly one of the coins and so two weighings may be not enough.

51.9.1. If S_1, S_2, S_3, and S_4 are the areas of the triangles (counted either clockwise or counterclockwise), then $S_1 \cdot S_3 = S_2 \cdot S_4$. Indeed, each of the products is equal to the product of the lengths of the four segments into which the diagonals divided each other times the sine of the angle (does not matter which of the two we take: their sines are equal) between the diagonals. Since the areas of the triangles are integers, the product $S_1 \cdot S_2 \cdot S_3 \cdot S_4$ is the square of a positive integer. But the square cannot end with 8.

51.9.2. Note that if $p \geq 5$ is a prime number, then $p^2 - 1 = (p-1)(p+1)$ is divisible by $3 \times 8 = 24$ (see the solution of Problem 51.7.1). So $(p_1^2 - 1) + \cdots + (p_{24}^2 - 1)$; thus, the given sum is divisible by 24, too. Q.E.D.

51.9.3. Assume that the given perpendicular lines are the coordinate axes Ox and Oy. Plot equal segments OA and OB on the axes by means of calipers. Let the lengths of the segments be equal to 1. We have $AB = \sqrt{2}$. Then plot segment $OC = \sqrt{2}$ on Oy; we see that $AC = \sqrt{3}$. Plot segment $OD = \sqrt{3}$ on Oy and segment $OE = 1$ on Ox. Then $\triangle AED$ is an isosceles triangle with side 2 (Fig. 391).

FIGURE 391. (Sol. 51.9.3)

FIGURE 392. (Sol. 51.9.4)

51.9.4. (See also Problem 27.2.9.3.) Place all natural numbers in the squares of the infinite square table as shown in Fig. 392. Prove by induction that the square with coordinates (x, y) contains the number $f(x, y)$.

Indeed, in the inductive step we must either pass from square (x, y) to square $(x - 1, y + 1)$ or from square $(1, y)$ to square $(y + 1, 1)$. It is easy to verify that in each case we have:

$$f(x-1, y+1) = f(x, y) + 1, \qquad f(y+1, 1) = f(1, y) + 1,$$

i.e., the next number of the natural series, Q.E.D.

FIGURE 393. (Sol. 51.9.5)

51.9.5. Three colors are not enough: Fig. 393 a) shows a graph which satisfies the conditions of the problem and which can only be painted at least 4 colors. Let us prove that in the general case four colors suffice. We will prove this by induction on the number of vertices (telephones).

First, consider the case when in the graph of telephones and wires there is a vertex A that has 1 or 2 outgoing wires (we will say that A *is a vertex of degree* 1 *or* 2, respectively). Delete A and the wires that go out of it. The remaining wires can be painted 4 colors by the inductive hypothesis. Then the discarded wires can be easily painted in the required way: all is obvious for the vertex of degree 1 while for the vertex of degree 2 (Fig. 393 b)), we paint the wire AX color 3 that differs from colors 1 and 2 and we paint the wire AY color c which differs from a, b and 3.

Assume now that exactly 3 wires go out of each vertex (we call such vertices *vertices of degree* 3). Discard A again and its three outgoing wires AB, AC and AD. Paint the remaining wires 4 colors. Now, the vertices B, C and D have two outgoing wires each. Consider three pairs of colors these three pairs of wires are painted with. The following cases may occur:

a) There are at least 3 different colors — 1, 2 and 3 — among the obtained 6 colors. Then it is possible to paint the wires AB, AC and AD in the required way (prove this on your own.)

b) Much more complex is the case when all three pairs of wires going out of B, C, D are painted the same two colors, for example, colors 1 and 2. Then the wires AB, AC and AD cannot be painted in the required way. For this case, we give below a method to repaint the entire graph (without the wires AB, AC, AD), which leads us to the previous case a).

Consider all wires painted only colors 1 and 3 in the graph with the discarded wires AB, AC, and AD. They form several closed "cycles" and "circuits" (see Problem 51.7.4) — *connected components* — of colors 1 and 3. And it is clear that one "cycle" or one "circuit" cannot have all three vertices B, C, D at once (but can have two vertices) since exactly one wire of color 1 (and no wires of color 3) goes out of each of these vertices.

Now, delete one of the points B, C or D which is not part of any connected component of colors 1, 3 with any of the remaining two vertices. In this component, repaint all wires of color 1 into color 3 and all wires of color 3 into color 1. We get the required coloring of the whole graph and pass to case a).

51.10.2. The graph of the function $y = 2^x$ has exactly one asymptote, $y = 0$ as $x \to -\infty$ (in order to prove this note that $|2^x - (ax + b)| \to 0$ as $x \to +\infty$ leads to contradiction for $a^2 + b^2 \neq 0$). If the graph had an axis of symmetry, then it would have also had a second asymptote symmetric to the first one relative to the axis of symmetry. Contradiction.

51.10.3. Let $ABCD$ be the lower base and $A'B'C'D'$ the upper base of the parallelepiped. Draw segment KH on the face of $AA'B'B$ parallel to AA' and BB' (K lies on $A'B'$). Then draw the plane passing through K and perpendicular to KH. This plane intersects the plane of the upper face $A'B'C'D'$ along a straight line l. Finally, draw the plane passing through l and KH. It is this plane that will cut a rectangle out of the parallelepiped. Indeed, the obtained section is a parallelogram with mutually perpendicular sides ($KH \perp l$), i.e., a rectangle.

51.10.4. A perpendicular to l can be constructed using the standard length, calipers (cf. Problem 51.7.3), and a ruler as follows (Fig. 394). On l, plot segments $OA = OB = 1$ to either side of point O (we assume that the length of the standard length is equal to 1). Then on the same side of l on two arbitrary rays with vertex O plot segments $OC = 1$ and $OD = 1$. After that draw straight lines AD and BC meeting at M, and lines AC and BD meeting at K. The line MK is the desired perpendicular to l.

Indeed, draw a circle of radius 1 centered at O. Then the points A, B, C, D lie on this circle and so $\angle ACB = \angle ADB = 90°$. Hence, AC and BD are hights in $\triangle AMB$ and K is the intersection point of the hights. Therefore, MK is a part of the third hight MH in $\triangle AMB$ and so $MK \perp l$. Q.E.D.

FIGURE 394. (Sol. 51.10.4)

FIGURE 395. (Sol. 52.7.2)

51.10.5. Consider two successive pairs of numbers produced by the *operation*. If the first pair is (a, b), $a \geq b$, then $a = bp + r$ and the second pair must be one of the pairs (p, r) or (r, p) (we assume that the first number in a pair is not smaller than the second). If it is the pair (p, r), that is $p \geq r$, we have:

$$r < b \implies b \geq r + 1, \quad a = bp + r \geq (r + 1)p + r = pr + p + r.$$

But if it is (r, p), then

$$r < b, \ \ p \leq r \implies b \geq p + 1, \ \ b \geq r + 1, \ \ a = bp + r \geq pr + p + r.$$

Therefore, if we know the numbers p, r in a pair, we can give a lower bound for the numbers (a, b) in the previous pair: $b \geq \min(r, p) + 1$, $a \geq pr + p + r$.

Now, let us start moving backwards from the last pair to the first one. Instead of the numbers, we write their lower bounds using the above inequalities:

$$(\geq 1, 0) \longrightarrow (\geq 1, \geq 1) \longrightarrow (\geq 3, \geq 2) \longrightarrow (\geq 11, \geq 3)$$
$$\longrightarrow (\geq 47, \geq 4) \longrightarrow (\geq 239, \geq 5) \longrightarrow (\geq 1439, \geq 6) \longrightarrow (\geq 9079, \geq 7).$$

We see that if ≥ 7 divisions have been done to produce the pair $(\geq 1, 0)$ the greatest of the original numbers is not smaller than 9079. So for the numbers not greater than 1988 the number of divisions does not exceed 6. Q.E.D.

52.7.2. *First move*: choose an arbitrary point A on l and draw a circle of radius AB with center at A (here B is the given point). The circle intersects l at points C and D, see Fig. 395.

Second move: make the span of the compass equal to BC, and draw a circle of this radius with center at D. Let the circle intersect the first circle at E.

Third move: connect E and B with the help of a ruler. We get the desired straight line l', $l' \parallel l$.

The line l' cannot be drawn in less than 3 moves, since we have to find the point $E \in l'$ different from B; this requires 2 moves (intersections of two lines); and one move is needed to draw l' itself.

52.7.3. If the sizes are A and B, and the colors are a and b, then all 8 socks are as follows:

$$(A, a),\ (A, a),\ (B, a),\ (B, a),\ (A, b),\ (A, b),\ (B, b),\ (B, b).$$

Socks can be divided into four pairs: $(A, a) - (B, b)$, $(A, a) - (B, b)$, $(A, b) - (B, a)$ and $(A, b) - (B, a)$. If we choose 5 socks than there will necessarily be one of these pairs among them. If you choose only 4 socks, they may be (A, a), (A, a) and (B, a), (B, a). There will be no two socks of different sizes and colors among them.

52.7.4. The tourist can cover 2.55 km downstream and 0.45 km upstream in one cycle of rowing and rest. This cycle takes 0.75 hrs.

If (s)he leaves the tourist lounge downstream, (s)he needs more than 5 cycles to come back after one cycle going forward. That will take 3 hours and so (s)he will be unable to return to the lounge in time. Consequently, the tourist should go upstream.

Let the tourist complete n cycles before turning back; then (s)he needs $\frac{0.45n}{4.4}$ hrs to return and the following inequality should be satisfied in order for the tourist not to be late:

$$0.75n + \frac{0.45n}{4.4} \leq 2.75 \Longrightarrow n \leq 3.$$

Therefore, the longest distance (s)he can go away from the lounge is to the place where (s)he will appear just before the rest during the third cycle; this distance is equal to:

$$0.9 + 0.8 = 1.7 \text{ km}.$$

52.7.5. Clearly, the product of digits of an integer x greater than 9 is less than this integer: if

$$x = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_0,$$

then

$$a_n a_{n-1} a_{n-2} \cdot \cdots \cdot a_0 < a_n \cdot 10^n \leq x.$$

Thus,

$$0 < 44x - 86\,868 < x \iff 1974 < x \leq 2020.$$

Since 86 868 is not divisible by 44, there are no zeros among the digits of x and so $1974 < x < 1999$. Then at least one of the digits of x is equal to 9. But $86\,868 \vdots 9$ and therefore, $x \vdots 9$. But 86 868 is not divisible by 8 and so x is odd.

Only $x = 1989$ can satisfy all above-listed conditions and one can verify we find that $x = 1989$ satisfies the condition of the problem.

52.8.2. The square at the intersection point of the horizontal, where the grasshopper stands, and the vertical, where the flea rests, is either red or white.

If it is red, the grasshopper should jump on it and if the square is white the flea jumps there. So the two insects find themselves either on the same horizontal or on the same vertical. Any of them must have two adjacent squares of different colors, the red square being for the grasshopper and the white for the flea. Thus, they will be side by side after 3 jumps. Q.E.D.

52.8.3. Fig. 396 a) shows the construction for heading a): take two points on a straight line, and construct two circles centered at these points and passing through the given point A. The second intersection point A' of the circles lies on the perpendicular desired.

FIGURE 396. (Sol. 52.8.3)

b) We construct a circle centered outside line l passing through A. Find M, the second intersection point of this circle with l; then draw the diameter through M and find the point A' on the perpendicular desired, see Fig. 396 b).

The impossibility to draw only two lines can be proved simply by trying all possibilities, see Problem 52.7.2.

REMARK. This problem seems to be good enough to be included in a textbook on elementary geometry.

52.8.4. In the sequences

$$aaaa...., \quad ababab...,$$

where a and b are arbitrary distinct numbers, any number from the set X can be found by the hypothesis. So X contains all numbers of the form $\overline{aa}$ and either the number $\overline{ab}$ or $\overline{ba}$ for any digits a, b. Therefore, X contains ≥ 55 two-digit numbers.

On the other hand, the set X_0 that contains all numbers $\overline{ab}$ for $a \leq b$ has the required property. Indeed, in any sequence $c_1, c_2, c_3, \dots$ there are c_n and c_{n+1} such that $c_n \leq c_{n+1}$ and then $\overline{c_n c_{n+1}} \in X$. Q.E.D.

52.8.6. Let us prove that $|a| + |b| + |c| \leq 17$. Set

$$f(x) = ax^2 + bx + c, \quad f(0) = y_1, \quad f\left(\tfrac{1}{2}\right) = y_2, \quad f(1) = y_3,$$

where $|y_i| \leq 1$ for $i = 1, 2, 3$. We have:

$$\begin{aligned} c &= y_1, \\ \tfrac{1}{4}a + \tfrac{1}{2}b + c &= y_2, \\ a + b + c &= y_3, \end{aligned}$$

whence $a = 2y_1 - 4y_2 + 2y_3$, $b = -3y_1 + 4y_2 - y_3$, $c = y_1$.

Thus,

$$\begin{array}{rcccl} |a| & \leq & 2|y_1| + 4|y_2| + 2|y_3| & \leq & 8, \\ |b| & \leq & 3|y_1| + 4|y_2| + |y_3| & \leq & 8, \\ |c| & = & |y_1| & \leq & 1 \end{array}$$

and $|a| + |b| + |c| \leq 17$. Q.E.D.

52.9.1. If two blue lines are not parallel, then each red line is perpendicular to the plane of the blue lines and so the red lines are parallel to each other.

52.9.2. First, observe that $CKML$ is a parallelogram, so $ML = KC$. Since $\triangle MPL \sim \triangle BPK$, then (see Fig. 397)

$$\frac{PL}{BP} = \frac{ML}{BK} = \frac{KC}{BK}.$$

Consider now trapezoid $BMLC$ and a segment QK in it. According to the *Thales theorem*,

$$\frac{KC}{BK} = \frac{QL}{MQ}.$$

Finally, $\triangle MBL$ and $\triangle QPL$ with a common angle $\angle L$. All above equations imply that

$$\frac{BL}{BP} = \frac{QL}{MQ}.$$

Moreover, $\triangle MBL$ and $\triangle QPL$ have a common angle, $\angle L$. Therefore, $\triangle MBL \sim \triangle QPL$; hence, $QP \parallel MB$, i.e., $QP \parallel AB$. Q.E.D.

FIGURE 397. (Sol. 52.9.2)

FIGURE 398. (Sol. 52.9.4)

52.9.3. Let us find a, b, p, q from the system

$$\begin{cases} a + b = A, \\ aq + bp = B, \\ aq^2 + bp^2 = C, \\ aq^3 + bp^3 = D. \end{cases}$$

First, let us exclude b. For this let us subtract from 2-nd, 3-rd and 4-th equations the preceding one multiplied by p. We get:

$$\begin{cases} a(q-p) = B - pA, \\ aq(q-p) = C - pB, \\ aq^2(q-p) = D - pC. \end{cases}$$

Observe that the product of the left hand sides of the first and the last of the new equations is equal to the square of the left hand sides of the middle equation. Hence,

$$(B - pA)(D - pC) = (C - pB)^2.$$

It is not difficult now to find, first, p and, next, the other variables:

$$q = \frac{C - pB}{B - pA}, \quad a = \frac{B - pA}{q - p}, \quad b = A - a.$$

The two solutions are obtained from each other by the change $p \longleftrightarrow q$, $a \longleftrightarrow b$; therefore, the 5-th term of the sequence is uniquely determined.

52.9.4. On Fig. 398 the hollow nodes denote the intersections from which an odd number of roads go out (not counting deadends in case A). It is clear that we have to pass one of them twice. Therefore, the route, taking which we go along solid segments twice, is the shortest one. Find this route on your own.

52.9.5. We will start with two comments as regards an arbitrary n (remember that $x_k > 0$ for all k).

COMMENTS. 1) Suppose $x_1 + \dots + x_n = S$. Given S, we can express $S_k = x_1 + \dots + x_k$ from the given equations for all $k = 1, \dots, n$:

$$\begin{cases} S_1 & = & x_1 = \frac{1}{S}, \\ S_2 & = & x_1 + x_2 = \frac{1}{S - S_1}, \\ S_3 & = & x_1 + x_2 + x_3 = \frac{1}{S - S_2}, \\ & & \dots\dots\dots\dots\dots\dots \\ S_n & = & \frac{1}{S - S_{n-1}}, \\ S_n & = S. \end{cases}$$

It follows that these constraints can be satisfied for one value of S only.

Indeed, let them be true for some S. As S increases, all $S_1, \dots, S_n$ decrease and vice versa, the last equality being violated. It is easy to construct points $S_1, S_2, S_3, \dots$ for the chosen S:

$$S_{k+1} = \frac{1}{S - S_k}, \quad S_0 = 0, \quad S_1 = \frac{1}{S}, \quad \dots$$

by drawing a "ladder" — a broken line whose vertices are alternately on the hyperbola $y = \frac{1}{S-x}$ and on the straight line $y = x$ while its links are parallel to the axes, see Fig. 399 a).

Note that as S changes continuously, the hyperbola and all vertices of the broken line shift continuously. Hence, it is easy to see that there exists a solution for some S (it corresponds to the case when $S_n = S$).

FIGURE 399. (Sol. 52.9.5)

2) It follows immediately from the given equations that $x_1 = x_n$, $x_2 = x_{n-1}$, $x_3 = x_{n-2}, \dots$.

Indeed, it is clear from the first and the last equations that $x_1 = x_n = \frac{1}{S}$, from the second and the penultimate equations that

$$x_1 + x_2 = \frac{1}{S - x_1} = \frac{1}{S - x_n} = x_{n-1} + x_n \Longrightarrow x_2 = x_{n-1},$$

etc. This symmetry is also seen from Fig. 399 a) since the hyperbola is symmetric through the line $y = x + S$.

Comment 2) allows one to write quickly solutions for small n.

$n = 2$. The answer is found at once: $2x_1^2 = 1$, $x_1 = x_2 = \frac{\sqrt{2}}{2}$.

$n = 3$. Then $S = 2x_1 + x_2 = x_1 + S_1$. We find from equations $x_1(x_1 + S_1) = 1$, $S_1^2 = 1$ that $S_1 = 1$, $x_1^2 + x_1 - 1$, whence (considering $x_1 > 0$)

$$x_1 = \frac{\sqrt{5} - 1}{2}, \quad x_2 = 1 - x_1 = \frac{3 - \sqrt{5}}{2}, \quad x_3 = x_1.$$

$n = 4$. Then $S = 2x_1 + 2x_2 = 2S_1$. We find from equations $2x_1S_1 = 1$, $S_1(2S_1 - x_1) = 1$ that $2S_1^2 = 1 + x_1S_1 = \frac{3}{2}$, i.e., $S_1 = \frac{\sqrt{3}}{3}$, whence

$$x_1 = \frac{\sqrt{3}}{3}, \quad x_2 = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{6}, \quad x_3 = x_2, \quad x_4 = x_1.$$

The answers may give an idea to an expert in geometry that the cases when $n = 2, 3, 4$ are related somehow to the angles $45^\circ, 36^\circ$, and 30°.

Indeed, let us show that for any n there exists a solution connected with the angle $\alpha = \frac{180}{n+2}$ and Comment 1) ensures that this solution is unique.

Let us construct an isosceles triangle $\triangle A_0OA_n$ with sides $A_0O = OA_n = 1$ and the angles $\alpha = \frac{180}{n+2}$ at the base, see Fig. 399 b). Mark points $A_1, A_2, \dots, A_{n-1}$ on the base A_0A_n so that segments A_0A_1, $A_1A_2, \dots, A_{n-1}A_n$ could be seen from O at equal angles α.

Let us prove that the lengths of these segments $x_1, x_2, \dots, x_n$ satisfy our system. It is also convenient to draw a ray OP so that $OP \parallel A_0A_n$ ($\angle A_0OP = \alpha$).

Observe that $\triangle A_0OA_k \sim \triangle A_nOA_{k-1}$, since

$$\angle A_0 = \angle A_n = \alpha, \quad \angle A_0OA_k = k\alpha = \angle POA_{k-1} = \angle OA_{k-1}A_k$$

The sides $A_0O = 1$ and $A_0A_k = x_1 + \dots + x_k$ in $\triangle A_0OA_k$ correspond to the sides $A_{k-1}A_n = x_k + \dots + x_n$ and $OA_n = 1$ in $\triangle O_nAA_{k-1}$ wherefrom we get the required equation:

$$(x_1 + \dots + x_k)(x_k + \dots + x_n) = 1.$$

Thus, the existence of a solution is proved.

It is easy to get explicit formula for the solution using the law of sines: Since

$$OA_0 = 1, \quad \angle A_0OA_k = k\alpha, \quad \angle A_0A_kO = 180^\circ - \angle POA_k = 180^\circ - (k+1)\alpha,$$

we get from $\triangle OA_{k-1}A_k$ and $\triangle OA_0A_k$:

$$\frac{x_k}{\sin\alpha} = \frac{OA_k}{\sin k\alpha}, \quad \frac{OA_k}{\sin\alpha} = \frac{1}{\sin(k+1)\alpha}.$$

Hence,

$$x_k = \frac{(\sin\alpha)^2}{\sin k\alpha \cdot \sin(k+1)\alpha}, \quad \text{for } k = 1, \dots, n, \quad \alpha = \frac{\pi}{n+2}.$$

In particular, for $n = 10$ we have $\alpha = \frac{\pi}{12}$ and the numbers $\sin k\alpha$ for $k = 1, \dots, 6$ are equal, respectively, to

$$\sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}, \quad \frac{1}{2}, \quad \frac{\sqrt{2}}{2}, \quad \frac{\sqrt{3}}{2}, \quad \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}, \quad 1.$$

Thus, the answer is:

$$x_1 = x_{10} = \frac{\sqrt{6} - \sqrt{2}}{2}, \qquad x_2 = x_9 = \frac{2\sqrt{2} - \sqrt{6}}{2}, \qquad x_3 = x_8 = \frac{\sqrt{6}}{3} - \frac{\sqrt{2}}{2},$$
$$x_4 = x_7 = \frac{9\sqrt{2} - 5\sqrt{6}}{6}, \qquad x_5 = x_6 = \frac{3\sqrt{6} - 5\sqrt{2}}{4}.$$

52.10.1. The graphs of the functions $y = \log(x-2)$ and $y = -x^2 + 2x + 3$ intersect exactly at one point, since for $x > 2$ the first function increases and the second one decreases. The unique solution is easy to find: $x = 3$.

52.10.2. There exists, for example, the function $y = x^3$. Indeed, if the line is a generic one, $y = kx + b$ is its equation, and since the cubic $x^3 - kx - b$ has a real root, then for some x_0 we have $x_0^3 = kx_0 + b$ and the cubic $y = x^3$ crosses the line $y = kx + b$ at point (x_0, x_0^3).

The case of the line $x = x_0$ is even simpler.

52.10.3. An example of the required arrangement is shown on Fig. 400.

FIGURE 400. (Sol. 52.10.3)

52.10.4. Let $A = \{x_1, x_2, \dots, x_n\}$ be the set of $n \geq 5$ positive integers. We will construct a sequence $A_1 \supset A_2 \supset A_3 \supset \dots$ according to the following rule.

At the first step we break $A = A_0$ into subsets consisting of even and odd numbers and denote any of them containing $\geq n/2$ numbers by A_1; all numbers of A_1 are congruent modulo 2. Let A_k be constructed and contain n_k elements, all its elements congruent to some r_k modulo 2^k and all elements of its complement in A_{k-1} not congruent to r_k modulo this number. Then all numbers in A_k have the remainder equal either to r_k or to $r_k + 2^k$ modulo 2^{k+1} as follows:

$$2^k 2m + r_k = 2^{k+1}m + r_k, \quad 2^k(2m+1) + r_k = 2^{k+1}m + 2^k + r_k,$$

i.e., A_k is split into two subsets and we take any of the subsets that contains $\geq \frac{1}{2}n_k$ elements as A_{k+1}.

The non-increasing sequence of positive integers $f = (n_1, n_2, n_3, \dots)$ thus obtained stabilizes at a number s (i.e., $n_s = n_{s+1} = n_{s+2} = \dots$) and so all numbers belonging to A_s have the same remainders after division by 2^{s+1}, 2^{s+2}, 2^{s+3}, etc.

However, if 2^t is greater than any element of A_s, then the remainders after division by 2^t coincide with the numbers themselves; so all numbers belonging to A_s are equal to each other. This means that $n_s = 1$.

From conditions $n_1 \geq 5$, $n_s = 1$ and $n_{k+1} \geq \frac{1}{2}n_k$, we deduce that in the sequence $n_1, \dots, n_s$ there is a term n_t equal to either 3 or 4. But then the elements of A_t congruent modulo 2^t are contained in the progression $\{r_t + 2^t d\}$ that does not contain other elements of A. Q.E.D.

52.10.5. Let a be the length of the shorter side of the rectangle, see Fig. 401. Then $\frac{1}{a}$ is the length of the longer side of the rectangle. Let x be the total length of the drawn segments. Then the sum of $2x$ and the perimeter of the square is greater than the perimeter of the rectangle: $2x + 4 > 2(a + \frac{1}{a})$, i.e., $x > a + \frac{1}{a} - 2$. But by the hypothesis $a^2 + \left(\frac{1}{a}\right)^2 = 100^2$ and so

$$(a + \frac{1}{a})^2 = 10^4 + 2 \Longrightarrow a + \frac{1}{a} > 100.$$

Thus, $x > 98$.

Now, we will describe a method for recutting the square into the desired rectangle whereby the total length of the cuts remains less or equal to 102.

Consider the unit square $ABCD$ and the rectangle $AKLM$ arranged as on Fig. 401. Let AK be equal to $100a$ and AM to $\frac{1}{100a}$ (prove on your own that $\frac{1}{100} < a < \frac{1}{99}$). It is easy to change $AKLM$ into the desired rectangle by making 99 cuts, each $\frac{1}{100a}$ long, i.e., the total length of the cuts is $\frac{99}{100a}$.

What remains is to recut the square $ABCD$ into the rectangle $AKLM$. For this purpose, we make cuts DP and MN, replace $\triangle BPK$ with $\triangle MND$ (these triangles are equal) and shift triangle $\triangle DCP$ to assume the position of $\triangle MLK$. The length of these cuts is shorter than

$$MN + DP < PK + DP = DK = \sqrt{1 + (100a)^2} < \sqrt{1 + \left(\frac{100}{99}\right)^2} < 2$$

and the total length of all cuts does not exceed

$$\frac{99}{100a} + 2 < \frac{99}{100} \cdot 100 + 2 = 101.$$

So the desired total length is 100 with an accuracy to 2.0.

52.10.6. Let points K, L, M, P, R, and T be selected on the edges of a tetrahedron $ABCD$, see Fig. 402.

Let planes KMP, KLR and MLT be tangent to the sphere inscribed into $ABCD$ and let plane PRT be not tangent to the sphere. And for definiteness sake, let plane PRT cross the sphere. Through P and R draw plane PRT_1 tangent to the sphere. Consider the octahedron $KLMPRT_1T$ that has 6 faces which are triangles and two faces which are quadrilaterals.

Let us blacken the four faces of the octahedron, which do not belong to the faces of the tetrahedron (i.e., KMP, KLR, MLT and PRT_1) and let the other faces remain white.

On each face take the tangent point to the sphere and connect this point to all vertices in this face.

Thus, each face becomes divided into triangles. And for each black triangle there is a white triangle equal to it: these triangles have a common side — an edge of the polyhedron — and pairwise equal other sides (as tangents drawn from one point).

But there is only a pair of equal white triangles belonging to edge T_1T while the other edges correspond to black triangles. Let the angle opposite TT_1 in the pair of white equal triangles be equal to α. The sum of the angles of all black triangles around the tangent points is equal to $4 \cdot 2\pi$. The sum of the corresponding

Figure 401. (Sol. 52.10.5) Figure 402. (Sol. 52.10.6)

angles of white triangles is equal to $4 \cdot 2\pi + 2\alpha$ (a white triangle corresponds to a black triangle plus one extra pair of white triangles).

But, on the other hand, this sum is also equal to $4 \cdot 2\pi$ for white triangles whence $\alpha = 0$ and the points T_1 and T coincide. Q.E.D.

Remark. The proof of the statement that *a sphere cannot be inscribed into a black-and-white polyhedron whose black faces do not have common edges with, and exceed the number of, white faces* is similar.

53.8.1. Since $a_1 + a_2 < 2a_3$, $a_4 + a_5 < 2a_6$, $a_7 + a_8 < 2a_9$, we have:

$$a_1 + \cdots + a_9 < (2a_3 + 2a_6 + 2a_9) + (a_3 + a_6 + a_9) = 3(a_3 + a_6 + a_9). \quad \text{Q.E.D.}$$

53.8.2. For $m > 1$ the numbers m and $m + 2n^2 + 3$ have opposite parity and so $m(m + 2n^2 + 3)$ has at least two prime divisors: 2 and an odd number. But if $m = 1$, then $(n + 9)(2n^2 + 4) = 2(n + 9)(n^2 + 2)$ has at least 2 distinct prime divisors for the same reason.

53.8.3. Consider the grade represented by the least number of winners, which is ≤ 2 (otherwise there would have been at least $3 \cdot 4 = 12 > 11$ winners in all grades). It is this grade that will not be represented among any five students sitting in succession.

53.8.4. First, observe that $\angle ADB$ and $\angle BDC$ intercept (on the greater circle) equal chords $AB = BC$. Therefore, $\angle ADB = \angle BDC = \alpha$; see Fig. 403.

Figure 403. (Sol. 53.8.4)

The angles ABD and ACD intercept a common arc $\smile AD$; hence, $\angle ABD = \angle ACD = \beta$. Connecting points B and C, we see that $\angle OBE = \angle OCE = \beta$. (These angles intercept a common arc of the smaller circle.) Consequently, $\triangle ABD = \triangle EBD$ in their common side BD and their two adjacent angles α and β. Hence, $AD = DE$. Q.E.D.

53.8.5. We represent n switchings on the display board as a table $n \times 64$ filled with 0's and 1's so that each column of the table (of height n) represents a button, each row represents a switching and a 1 represents a push on the button for a given switching.

To find out from the given set of lightings which button switched a specified bulb, it is necessary and sufficient that all columns were distinct.

But the number of columns of height n is equal to 2^n (each square of a column contains either 0 or 1); and, therefore, we should have $2^n \geq 64$, whence $n \geq 6$. The smallest n is equal to 6, and all 6 pushes can be obtained by writing 64 different columns of height 6 and then by separating the resulting 6 rows composed of 0's and 1's.

53.9.1. Take a boy, A, and three of his brothers, and another boy, B, and three of his brothers. If all boys were different, we would have 8 persons but we have 7 of them. Hence, A and B have a common brother and so A and B are also brothers. Q.E.D.

53.9.2. Consider pairs of numbers $(1, 52)$, $(2, 51), \ldots, (26, 27)$ the sum of numbers for each pair being equal to 53. If among our 53 distinct numbers there are no two numbers whose sum is equal to 53, then each pair $(k, 53-k)$ has at most one such number.

Then the sum of all 53 numbers is not less than

$$(1 + \cdots + 26) + (53 + \cdots + 79) = 2133 > 1990$$

a contradiction which proves that our assumption was wrong. Q.E.D.

53.9.4. Divide the coins into 4 groups of two coins each, A, B, C, and D. Next, perform the following three weighings (comparisons):

$$(A + B) \vee (C + D), \quad (A + C) \vee (B + D), \quad (A + D) \vee (B + C)$$

where the sign $\vee$ stands for $<$, or $>$, or $=$.

If there is at least one equality, the two counterfeit coins weigh as much as two genuine ones. Consider, therefore, the case when there is no equality in all three weighings. The same pan outweighs in at least two weighings.

If the heavy counterfeit coinis *much* heavier than other coins, then the pan with it will *always* go down regardless of where the light coin is. Therefore in this case we get either three $>$'s (if the heavy coin is in group A, or two $<$'s and one $>$ (if it belongs to B, C or D).

A similar conclusion can be drawn if the light coin is much lighter: we get either three $<$'s or two $>$'s and one $<$.

Therefore, the number of signes $>$ and $<$ completely and uniquely determines the solution.

53.9.5. **Lemma 1**. *The period of an irreducible fraction $\frac{p}{q}$ is equal to the smallest positive integer n for which $10^{n+k} - 10^k$ is a multiple of q for some k.*

PROOF follows from the representation of the fraction in the form of a decimal fraction of period n:

$$\frac{S(10^n - 1) + r}{10^{n+k} - 10^k} = \frac{S}{10^k} + \frac{r}{10^k \cdot 10^{n-1}} \quad (r < 10^n - 1). \quad \text{Q.E.D.}$$

Lemma 2. $10^t - 1$ *is divisible by* $(10^n - 1)^2$ *if and only if t is divisible by* $n(10^n - 1)$.

PROOF. Set $t = ln + m$ for $0 \leq m < n$. Since $10^n \equiv 1 \pmod{10^n - 1}$, it follows that $1 \equiv 10^{ln+m} \equiv 10^m \pmod{10^n - 1}$. But since $10^m < 10^n - 1$, we see that the above congruence is actually an equality: $10^m = 1$. i.e., $m = 0$. It remains to prove that $l \,\vdots\, 10^n - 1$ which would be equivalent to the fact that $10^{ln} - 1 \,\vdots\, (10^n - 1)^2$.

Further, the number

$$\frac{10^{ln} - 1}{10^n - 1} = 10^{(l-1)n} + 10^{(l-2)n} + \cdots + 10^n + 1 \equiv 1 + \cdots + 1 (l \text{ times}) \equiv l \pmod{10^n - 1}$$

is divisible by $10^n - 1$ if and only if $l \,\vdots\, 10^n - 1$.

Corollary. *The period of the square of the fraction $\frac{1}{10^n - 1}$ is equal to $n(10^n - 1)$.*

PROOF. With the help of Lemma 1 we find the smallest integer t such that for some k the number $10^k(10^t - 1)$ is divisible by $q = (10^n - 1)^2$. According to Lemma 2, t should be divisible by $n(10^n - 1)$ and so the minimum of t is equal to $10^n - 1$. Q.E.D.

And finally,

Lemma 3. *The square of any periodic fraction p/q of period n is of period $\leq n(10^n - 1)$.*

PROOF. By Lemma 1 we have $10^k(10^n - 1) \,\vdots\, q$; therefore, $10^{2k}(10^n - 1)^2 \,\vdots\, q^2$.

But $10^{2k}(10^{n(10^n-1)} - 1) \,\vdots\, 10^{2k}(10^n - 1)^2$ and so $10^{2k}(10^{n(10^n-1)} - 1) \,\vdots\, q^2$; by Lemma 1 the period of the fraction $\frac{p^2}{q^2}$ does not exceed $n(10^n - 1)$. Q.E.D.

The problem is completely solved.

53.10.1. With the help of horizontal segment MN cut a rectangle $MBCN$ with sides $BM = x$, $BC = 1$ off a square $ABCD$ of side 1. With the help of vertical segment KL ($K \in MN$) divide the remaining rectangle $AMND$ into rectangles $AMKL$ and $LKND$. It is easy to deduce from the similarity of the rectangles and from the fact that $0 < x < 1$ that $AM = KL = 1 - x$, $MK = x(1 - x)$ and $KN = 1 - x(1 - x)$.

Since $1 - x < 1 - x(1 - x)$, we derive from similarity of $MBCN$ and $LKND$:

$$\frac{x}{1} = \frac{1 - x}{1 - x(1 - x)} \Longleftrightarrow 1 - x = x(1 - x(1 - x)).$$

The latter equality is satisfied for some $x \in (0, 1)$ since for $x = 0$ its left side is greater than the right side while for $x = 1$ it is the other way round.

53.10.2. At least one of the numbers p, q, r is even and, therefore, is equal to 2. Since $r \neq 2$, let us assume for definiteness sake that $q = 2$. Then $p^2 + 2^p = r$.

One of the numbers p or r is divisible by 3 (otherwise the left hand side is congruent to $(\pm 1)^2 + (-1)^p \equiv 0$ modulo 3, since p is odd, whereas the right hand side is not congruent to 0 modulo 3. But $r \neq 3$, so $p = 3$, $r = 17$.

53.10.3. Set $x = -\frac{1}{2}\pi \pm \frac{1}{3}\pi \pm \frac{1}{9}\pi \pm \frac{1}{27}\pi$, where the signs are selected to be coinciding with the signs of a, b, c and the sign of 0 is supposed to be "+".

As is easy to verify,

$$\pm \cos x > \frac{1}{2}, \quad \pm \cos 3x > \frac{1}{2}, \quad \pm \cos 9x > \frac{1}{2}$$

in accordance with the signs of a, b, c, implying the inequality desired.

53.10.4. Let us prove that *any arrangement of points A, B, C, D in the disc can be step-by-step transformed into arrangement of points in the vertices of the square so that the product of distances increases.*

Indeed, first let us apply similarity transformation to place tree points, say, A, B, C, on the circle. If D belongs to the interior of triangle ABC, let us reflect D symmetrically through one of the sides of ABC keeping D inside the disc. (This is possible, since otherwise

$$\angle ADB + \angle ACB < \pi, \quad \angle BDC + \angle BAC < \pi, \quad \angle CDA + \angle CBA < \pi$$

and having added these inequalities term-wise we get $2\pi + \pi < 3\pi$; contradiction.)

Further on, replace D and its opposite vertex B with the diametrically opposite midpoints of the two arcs $\smile AC$, call them D' and B'. Then

$$B'D' \geq BD, \quad AB' \cdot CB' = \frac{2S_{AB'C}}{\sin \angle B} \geq \frac{2S_{ABC}}{\sin \angle B} = AB \cdot CB.$$

Similarly, $AD' \cdot CD' \geq AD \cdot CD$, i.e., the product of distances does not decrease.

In exactly the same way, let us replace points A and C by the midpoints of the two arcs $\smile B'D'$, call them A' and C'. These two points together with points B' and D' are the vertices of a square.

53.10.5. Consider the plane of triangle ADC. On the plane, take point B' on the same side of line AC as D lies, so that $\triangle AB'C = \triangle ABC$. Then points A, D, B', C belong to one circle (since the segment AC is seen from D and B' at an angle of β).

FIGURE 404. (Sol. 53.10.5)

Let $a = AB = AB'$, $b = AD$, $d = CD$, $c = CB = CB'$. By the law of cosines

$$BD^2 = a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 - 2cd\cos\alpha$$

implying

$$\cos\alpha = \frac{a^2+b^2-c^2-d^2}{2(ab-cd)} \quad \left(\tfrac{a}{c} \neq \tfrac{d}{b} \text{ since } AB \neq CD\right).$$

Denote: $\angle B'AD = \angle B'CD = \gamma$. We similarly get

$$\cos\gamma = \frac{a^2+b^2-c^2-d^2}{2(ab-cd)} = \cos\alpha$$

implying $\gamma = \alpha$ and $BD = B'D$.

Thus, all edges of tetrahedron $ABCD$ coincide with the corresponding edges of "tetrahedron" $AB'CD$; hence, B' coincide with B.

Finally, by the law of sines we get

$$\frac{AC}{\sin\beta} = 2R = \frac{B'D}{\sin\alpha} \Longrightarrow \frac{AC}{B'D} = \frac{\sin\beta}{\sin\alpha}.$$

53.11.1. Setting $\varphi = \arcsin x$, $\psi = \arcsin y$ ($|x| \le 1$, $|y| \le 1$) we have

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin\varphi\cos\psi + \cos\varphi\sin\psi = \sin(\varphi+\psi) \le 1;$$

the equality being attained, for example, at $x = 1$, $y = 0$ and, generally, at any x, y such that $\arcsin x + \arcsin y = \frac{1}{2}\pi$.

53.11.2. Let us prove that $f(x) \neq x$ and $f(x) \neq x^2$ for any $x \in (0,1)$. Indeed, if $f(x) = x$, then $x^2 = f(f(x)) = x \neq x^2$. If $f(x) = x^2$, then $f(x^2) \neq x^2$ by the above but, on the other hand, $x^2 = f(f(x)) = f(x^2)$. This and the continuity of $f(x)$ imply that $f(x) - x$ and $f(x) - x^2$ are sign-constant on the interval $(0,1)$. But the inequalities $f(x) > x$ and $f(x) < x^2$ are impossible since otherwise we have

$$\text{either } x^2 = f(f(x)) > f(x) > x > x^2 \text{ or } x^2 = f(f(x)) < (f(x))^2 < (x^2)^2 < x^2.$$

Thus, $x^2 < f(x) < x$ for all $x \in (0,1)$.

53.11.3. From the bisector's properties we get:

$$\frac{AB}{BC} = \frac{AE}{EC} = \frac{3}{1}, \quad \frac{AD}{DE} = \frac{AB}{BE} = \frac{2}{1}.$$

Hence, if $BC = a$, then $AB = 3a$, $BE = \frac{3a}{2}$.

Let us express the equality of the areas:

$$\frac{a\cdot 3a}{2}\sin 2\alpha = \frac{a\cdot\frac{3a}{2}}{2}\sin\alpha + \frac{\frac{3a}{2}\cdot 3a}{2}\sin\alpha$$

implying $\cos\alpha = 1$, which is impossible.

53.11.4. First, let the odd number n be of the form $n = m\cdot 5^p$, where m is not a multiple of 5. Then, let us find a number N consisting of odd digits and a multiple of 5^p. Take $N_1 = 5^p = \overline{\ldots 25}$ and add $5^p\cdot 10$ to it:

$$N_2 = N_1 + 5^p\cdot 10 = \overline{\ldots 25} + \overline{\ldots 250} = \overline{\ldots 75}.$$

The number N_2 has already two odd last digits; consider the first from the *right* even digit α and let it stand on the k-th place.

Add N_2 and $10^{k-1}\cdot N_2$:

$$N_3 = N_2 + N_2\cdot 10^{k-1} = \overline{\ldots\alpha\ldots 75} + N_2\cdot 10^{k-1} = \overline{\ldots r\ldots 75}$$

and the digit r is odd. The number N_3 obtained has $\ge k$ odd figures from the right; let its first from the right even digit β stand on the l-th place ($l > k$). Add numbers N_3 and $N_3\cdot 10^{l-1}$:

$$N_4 = N_3(10^{l-1}+1) = \overline{\ldots\beta\ldots r\ldots 75} + N_3\cdot 10^{l-1} = \overline{\ldots q\ldots r\ldots 75},$$

where q is odd. The number N_4 has $\ge l$ odd digits from the right.

Similarly, we get a number which has p odd digits from the right, where p is the exponent of 5^p. These last p digits constitute the number N, a multiple of 5^p, since $10^p \,\vdots\, 5^p$.

Now, to finish the proof, consider the numbers obtained by repeated juxtaposition: N, $\overline{NN}$, $\overline{NNN}$, ... There are two of these numbers which give the same residues after division by m (explain, why?) and, therefore, their difference is a multiple of m. Delete the digits 0 at the end of the decimal expression of this difference; we get a number of the form $\overline{N\ldots N}$. This number is a multiple of m (since m and 10 are relatively prime) and of 5^p. But all digits of this number are odd by construction, Q.E.D.

53.11.5. If A, B, C, D are the projections of M onto the faces of tetrahedron, then no three of them are in one plane with M (otherwise three planes of the tetrahedron's faces do not intersect at one point).

Moreover, points A, B, C, D, M do not belong to one sphere (otherwise the four planes of the tetrahedron's faces pass through one point M', the diametrically opposite to M on the sphere).

Thus, if A, B, C, D do not belong to one plane, take a point M that does not lie on either of the planes ABC, ACD, ABD and BCD, or on the sphere circumscribed around tetrahedron $ABCD$ and draw four planes perpendicular to MA, MB, MC and MD (these planes form a tetrahedron).

If A, B, C, D lie on one plane, a similar construction is possible if these points do not lie on one circle (otherwise these points and M belong to a sphere) and no three of them lie on one line.

54.8.1. It is easy to observe that for $a = b$ the left hand side of the desired inequality vanishes and, therefore, it is natural to conjecture that it factors as $(a - b)(\dots)$. (This follows from *Bezout's theorem* but what we need now is not a proof but a hint as to how to proceed.) Similarly, the expression vanishes for $a = c$ and $b = c$, hence, it should be divisible by $(a - b)(b - c)(a - c)$. It is not difficult to verify that this is indeed the case:

$$a^2(b - c) + b^2(c - a) + c^2(a - b) = (a - b)(b - c)(a - c).$$

Since $a > b$, $b > c$, $a > c$, the expression above is a positive one.

FIGURE 405. (Sol. 54.8.2)

54.8.2. First, note that if $AB = r$, then $C = A'$ (hereafter in this problem we will denote by X' the point symmetric to X through B) is constructed as shown on Fig. 405 a). Now, let us consecutively find points D', E', A'.

a) If $AB < 2r$, the construction can be performed as on Fig. 405 b), where we construct first points D, E; next, points D', E', A'.

b) If $AB \geq 2r$, we construct from A the net of vertices of equilateral triangles with side r until two of them, D and E, see Fig. 405 c), occur inside the circle centered at B and of radius r. Next, construct points D' and E' and the symmetric net of vertices among which we single out vertex A'.

54.8.3. Whatever the schedule, the sentries earn 2.5 full days off for 1 full working day, therefore, the mean number of sentries having their days off should be ≥ 2.5. Hence, the total number of sentries should be ≥ 3.5. On the other hand, it is clear that 4 sentries suffice: let each work 24 hours around the clock and take three full days off.

54.8.4. Let x_k be the mass of the weight with the label "k g". It suffices to verify two relations:

$$x_1 + x_2 + x_3 = x_6 \text{ and } x_1 + x_6 < x_3 + x_5$$

each of which is needed to ensure the correctness of the labels.

Indeed, the first relation and the estimates

$$x_1 + x_2 + x_3 \geq 1 + 2 + 3 = 6 \geq x_6$$

imply that

$$x_6 = 6, \quad \{x_1, x_2, x_3\} = \{1, 2, 3\}.$$

Then the second relation and the estimates

$$x_1 \geq 1, \quad x_3 \leq 3, \quad x_5 \leq 5$$

imply

$$7 \leq x_1 + x_6 < x_3 + x_5 \leq 8.$$

We deduce: $x_1 + x_6 = 7$, $x_3 + x_5 = 8$, implying

$$x_1 = 1, \quad x_3 = 3, \quad x_5 = 5; \quad ; x_2 = 2, \quad x_4 = 4.$$

Hence, $x_k = k$ for all $k = 1, \ldots, 6$.

54.8.5. Let A_i ($i = 1, \ldots, n$) be the towns of the first country. Then for each i the set M_i of towns of the second country to which one can fly from A_i is nonempty by the hypothesis.

Suppose that for any two such sets M_i one is contained in another one.

Then the smaller of every pair of sets is contained in every one of the other sets. Therefore, to each of towns B from this small set M_{i_0} there is a flight from each of the towns A_i but no flight from B to any of them; this contradicts the hypothesis.

Therefore, our assumption is wrong, i.e., there are two sets, M_i and M_j and towns $B_i \in M_i$ and $B_j \in M_j$ such that $B_i \notin M_j$ and $B_j \notin M_i$. Then the desired foursome of towns and the flights are: from A_i to B_i, from B_i to A_j, from A_j to B_j, from B_j to A_i.

54.9.1. Since $x = 1$ is not a root of the equation, the latter is equivalent to

$$\frac{x^3 - 1}{x - 1} \cdot \frac{x^{11} - 1}{x - 1} = \left(\frac{x^7 - 1}{x - 1}\right)^2$$

or, which is the same,

$$x^3(x^4 - 1)^2 = 0.$$

FIGURE 406. (Sol. 54.9.2)

54.9.2. Let us associate with two deals of cards an array on which every card with a pair of numbers (m, n) is depicted as the entry situated at the intersection of the m-th row and the n-th column.

The possible answers are illustrated by Fig. 406 a) and b) respectively, e.g. in the 2nd row of Fig. 406 b) there are 4 crosses which signify that during the first deal one should make 2 piles $m = 2$ cards each.

How to draw these pictures? You can verify that the number of crosses in the arrays is equal to the total number of pairs, the number of crosses in each row and each column is a multiple of the corresponding row's or column's number and, finally, the crosses are placed symmetrically with respect to the main diagonal.

54.9.3. In $\triangle A_3A_5A_{11}$, straight lines A_3A_8, A_5A_1 and $A_{11}A_4$ are the bisectors, since $\smile A_5A_8 = \smile A_8A_{11}$, $\smile A_{11}A_1 = \smile A_1A_3$, $\smile A_3A_4 = \smile A_4A_5$; see Fig. 407.

FIGURE 407. (Sol. 54.9.3)

FIGURE 408. (Sol. 54.9.4)

Therefore, diagonal A_1A_5 passes through point O, the intersection point of the diagonals A_3A_8 and $A_{11}A_4$. Similarly, lines A_2A_6, $A_{11}A_4$ and A_8A_3 are the bisectors of triangle $\triangle A_2A_4A_8$ and, therefore, diagonal A_2A_6 also passes through O.

54.9.4. Draw the straight line through the midpoints of two parallel chords, another line through the midpoints of the other two parallel chords and let the origin be the intersection point of the lines we drew. To recover the coordinate axes, it now suffices to draw the bisector of the first quadrant; to draw it, construct, first, a circle centered at the origin and intersecting the given graph. Then draw the straight line through the origin and the midpoint of the chord intercepted by the circle on the graph, see Fig. 408.

To justify the construction it suffices to check that the midpoints of any section of the hyperbola $y = \frac{1}{x}$ by the lines parallel to a fixed one, $y = -ax$, belong to the line $y = ax$.

Indeed, if the section belongs to the line $y = -ax + b$, then the abscissas of its endpoints, x_1 and x_2, satisfy the equation $-ax + b = \frac{1}{x}$ or, which is the same, $ax^2 - bx + 1 = 0$, whereas the coordinates of the section's midpoint are

$$x_0 = \frac{x_1 + x_2}{2}, \quad y_0 = \frac{\frac{1}{x_1} + \frac{1}{x_2}}{2} = \frac{x_1 + x_2}{2x_1x_2} = ax_0.$$

(Recall that by Viéta's theorem $x_1x_2 = \frac{1}{a}$.)

54.9.5. In the given table replace all numbers by ± 1 in accordance with the signs of the initial numbers. The table still satisfies the hypothesis.

An $m \times n$ table all whose entries are ± 1 and every number is the product of its neighbors will be referred to as *suitable*. Let us prove that any suitable $m \times 15$ table, where $m = 1, 3, 7, 15$, is filled by 1's only, i.e., is *trivial*.

Indeed, it is subject to a straightforward verification that the 1×15 table is trivial: it is completely determined by one of the terminal numbers, one of which must be 1. More exactly, it is indeed transtparent that the infinite row $1 \times \infty$ can only be of the form

$$\dots 1111 \dots \quad \text{or} \quad \dots -1\ -1\ 1\ -1\ -1\ 1\ -1\ -1\ 1 \dots$$

We can only cut the second infinite sequence so that at each edge two -1's stand; hence the length should be equal to $2 + 3k \neq 15$.

Now, suppose that there is a suitable nontrivial 15×15 table. If it is symmetric with respect to the middle row, every number in this row coincides with the product of its horizontal neighbors and, by the above, is equal to 1. Then there is a suitable nontrivial (otherwise the whole table would have been trivial) 7×15 table over it.

If the 15×15 table is not symmetric with respect to the middle row, let us multiply each number of its upper 7×15 subtable by the number symmetric with respect to the middle row. We get a suitable 7×15 (and again nontrivial) table (otherwise the whole table would have been symmetric).

Similarly, beginning with a nontrivial suitable 7×15 table we derive the existence of a nontrivial suitable 3×15 table, and, finally, the existence of a 1×15 table. Contradiction.

54.10.1. Substituting $1 - x$ instead of x we get one more identity:

$$f(1-x) + \left(\frac{3}{2} - x\right) f(x) = 1$$

from which we derive $f(1-x)$; substitute it into the given formula:

$$f(x) + \left(x + \frac{1}{2}\right)\left(x - \frac{3}{2}\right) f(x) + \left(x + \frac{1}{2}\right) = 1.$$

For $x \neq \frac{1}{2}$ this yields the desired function

$$f(x) = \frac{\frac{1}{2} - x}{x^2 - x + \frac{1}{4}} = \frac{1}{\frac{1}{2} - x};$$

and the value at $x = \frac{1}{2}$ is found by substitution of $x = \frac{1}{2}$ into the initial identity.

54.10.2. The doubled number of the tangency points for the indicated arrangement of n balls is $3n$; hence, n is even and > 2.

It is possible to arrange 4 balls so that their centers are the vertices of a regular tetrahedron.

For other even n it is possible to arrange each of the two groups of $\frac{n}{2}$ balls in each in the vertices of a regular $\frac{n}{2}$-gon and then place one string of balls above the other so that each ball is tangent to exactly one ball from the other string.

54.10.3. Let D, E, F, G be the points where the circle is tangent to the angles's sides, K, L be the points where the circle is tangent to the triangle's sides, see Fig. 409.

Denote: $a = BK + CL$, $\varphi = \angle ABC$. We see that the sum of the circles' radii is equal to

$$BK \cdot \tan\frac{\pi - \angle ABC}{2} + CL \cdot \tan\frac{\pi - \angle ACB}{2} = a \tan\frac{\pi - \varphi}{2},$$

where $a = \dfrac{AB + AC - BC}{2}$, since

$$AB + AC - a = AK + AL = AF + AG = FG = DE = BC + BD + CE = BC + a.$$

Therefore, the sum of the radii does not change if from the beginning we will assume that $FG \parallel BC$ and the radii are equal to half the distance between these lines, in which case the inequality to be proved is satisfied.

Figure 409. (Sol. 54.10.3)

Figure 410. (Sol. 54.10.4)

54.10.4. Let us make believe that we have cut the initial cube into cubes with side 2 (parallel to the sides of the initial one). Divide the 1000 small $1 \times 1 \times 1$ cubes into 8 sets as follows: with each of the 8 small cubes from the vertices of the imaginary cube associate the set of small cubes each of which is situated relative to the $2 \times 2 \times 2$ cube in the same way as the corresponding small cube from the vertex; see Fig. 410.

As a result, all black cubes constitute 4 sets: M_A, M_C, M_F, M_H.

Let us prove that in each of these sets an equal number of black cubes is removed (hence their total number is a multiple of 4). To this end, it suffices to verify that from any of two sets, for example, M_A and M_B there are removed exactly 25 small cubes.

If a, b and c cubes are removed from sets M_A, M_B and M_C, respectively, then $a = 25 - b = c$. Indeed, the sets M_A and M_B fill in exactly 25 rows parallel to edge AB in each of which precisely one small cube is missing.

54.10.5. In the solution of Problem 54.9.2, let us divide 54 cards into two sets, as on Fig. 54.9.2: 30 cards corresponding to the crosses on the main diagonal and above it and 24 cards corresponding to the crosses under the main diagonal.

If the third deal corresponds to these 2 piles, the new inscriptions on the cards will preserve the differences between the pairs previously distinct and make the pairs of the form (m, n) and (n, m), where $n \leq m \leq 9$, distinguishable, turning them into triples $(m, n, 30)$ and $(n, m, 24)$, respectively.

Let us prove that 2 deals do not suffice. Indeed, with any two deals we can associate a $k \times k$ table, where k is the number of cards in the greatest of all piles during, say, the first deal. Then in the k-th row there are k crosses, one of which is situated on the diagonal. Therefore, the k-th column is also nonempty and, therefore, it also contains k crosses which should be situated symmetrically with respect to the main diagonal to the crosses of the k-th row.

The symmetric crosses correspond to the cards from the same sets of the form (m, n) and (n, m).

54.11.1. Observe that the number containing the first n 9's in the decimal expression is $2 \cdot 10^n - 1$, and the remaining part of the expression constitutes the number $10^{1992-n} - 9$.

a) The sum $S(n) = 2 \cdot 10^n + 10^{1992-n} - 10$ satisfies:

$$\begin{array}{ll} S(996) \leq 2 \cdot 10^{996} + 10^{996} = 3 \cdot 10^{996}, & \\ S(n) > 10^n \geq 10 \cdot 10^{996} & \text{for } n \geq 997, \\ S(n) \geq 2 \cdot 10 + 10^{997} - 10 > 10 \cdot 10^{996} & \text{for } n \leq 995. \end{array}$$

b) The expression

$$2 \cdot 10^{1992} + 9 - 18 \cdot 10^n - 10^{1992-n}$$

attains its maximum when the value $R(n) = 18 \cdot 10^n + 10^{1992-n}$ attains its minimum. Now, observe that

$$\begin{array}{ll} R(995) = 18 \cdot 10^{995} + 10^{997} = 118 \cdot 10^{995}, & \\ R(n) > 18 \cdot 10^n \geq 180 \cdot 10^{995} & \text{for } n \geq 996, \\ R(n) > 10^{1992-n} \geq 1000 \cdot 10^{995} & \text{for } n \leq 994. \end{array}$$

54.11.2. Indeed, the orthogonal projection of the Earth in the direction of ray AB is as expressed on Fig. 411, where DE is the projection of the equator.

FIGURE 411. (Sol. 54.11.2)

From equal right (since $\angle COD = \angle C'OD' = 90°$) triangles $\triangle OCC' = \triangle ODD'$ we get the equality $OC' = OD'$ used in the construction of Fig. 411 a).

54.11.3. Consider a regular 18-gon $A_1 A_2 \dots A_{18}$ with the vertices selected from the vertices of the initial 54-gon. Then the diagonals A_1A_8, A_2A_9, A_4A_{12}, A_6A_{16} meet at one point.

To prove this consider triangles $\triangle A_2A_6A_{12}$ and $\triangle A_4A_8A_{16}$ with the bisectors A_2A_9, A_6A_{16}, $A_{12}A_4$ and A_4A_{12}, A_8A_1, $A_{16}A_6$, respectively; see Fig. 412 and the solution of a similar Problem 54.9.3.

54.11.4. If $k = 1991$ then for each item only < 10 MP wish to diminish the expenditure. Overall there will be $< 200 \cdot 10 = 2000$ MPs wishing to diminish the expenditure on at least one item. Hence, there is a Member approving all expenditures suggested and, therefore, the budget is deficiency-free (by the condition no Member can approve a budget with deficiency).

If $k = 1990$, it could happen that the first 10 MPs would suggest to give nothing for the first item and provide $\frac{S}{199}$ for each of the remaining items. Then for each of the 200 items of the agenda the expenditure $\frac{1}{199}S$ would be approved with the total $200 \cdot \frac{1}{199}S > S$.

FIGURE 412. (Sol. 54.11.3)

FIGURE 413. (Sol. 54.11.5)

54.11.5. Suppose that at some moment all cells went out. Let us prove then that at the beginning the following two conditions were satisfied:

1) in each row and in each column there was at least one unlighted cell;

2) a rook can go around all unlighted cells (perhaps, jumping over lighted cells but not pausing at them).

To prove this note that at the last moment both conditions were satisfied.

Select the *first* moment when both conditions are satisfied. If it happened after the cell A went out then, first, the row and the column containing A had the unlighted cells B and C before, see Fig. 413, and, second, any unlighted cell connected (by a rook's path) with A had been previously connected with either B or C, too; these cells were connected with each other via the fourth vertex of a 2×2 square. Hence, before A went out both conditions were satisfied contradicting the choice of A.

Now, let us prove by the induction on $k = m+n$ that the number p of unlighted cells was $\geq k-1$. Then the inequality $p < k-1$ implies precisely that the whole screen will not blacken out.

Otherwise, for $m \geq n$ and $k > 2$ there would be a row with not more than 1 unlighted cell, D, (or else $p \geq 2m \geq k$) connected with any unlighted cell via a cell E from the same column. Therefore, having thrown out the found row we get the screen of size $(m-1) \times n$ satisfying both conditions and, therefore, the inequality

$$p - 1 \geq (m-1) + n - 1 = k - 2 \Longrightarrow p \geq k - 1.$$

Finally, one should verify that for $k = 2$ the inequality $p \geq 1 = k - 1$ is satisfied which completes the proof.

55.8.1. Since $c + d < a + b$ and $-(c + d) < a + b$, we have $|c + d| < a + b = |a + b|$.

55.8.2. If the answer would have been yes, then on the black cells that constitute 8 diagonals in one direction as well as 7 diagonals in the other direction, there would have stood simultaneously an even and an odd number of chips — the sum of an even and simultaneously of an odd number of odd numbers.

55.8.3. During the first and the second days each participant P solved as many problems as all other participants put together plus P personally during the second day, i.e., as many as all together during the second day. Therefore, everybody solved the same amount of problems during the 2 days.

55.8.4. Let us prove that fewer than 9 weights will not do. Let for convenience sake the total weight be $60 = 3 \cdot 4 \cdot 5$ g. Then the mass of any weight should not exceed 12 g (clear from decomposition in 5 piles) and their number is not less than 8 (follows from decomposition into 4 piles). If there are precisely 8 weights, then from decomposition into 5 piles we derive that there are 2 weights 12 g each, 2 weights 3 g each (from decomposition into 4 piles) and 3 more weights of mass not divisible by 3 or, more generally, nonintegral (follows from decomposition into 3 piles). But then by a direct case-by-case checking it is easy to demonstrate that it is impossible to separate the weights into 5 piles.

55.8.5. On Fig. 414, where BD is the bisector of right angle $\angle ABC$, we have $2(l + a) = b + c \leq 2a + c$, implying $2l \leq c$.

55.9.1. Each participant won both when white and when black as many games as all other put together when black plus the participant personally when black, i.e., as many as all together when black.

55.9.2. Dividing all numbers n into pairs (n_1, n_2) so that $n_1+n_2 = 10\,000$, notice that $n_1^9+n_2^9 = 10\,000 \cdot k$, since $(n_1^9+n_2^9) \vdots (n_1+n_2)$. Therefore, for the numbers m_1 and m_2 formed by the last four digits of the numbers

Figure 414. (Sol. 55.8.5)

Figure 415. (Sol. 55.9.3)

n_1^9 and n_2^9 we have $m_1+m_2=10\,000$. This implies that the inequality $n_1>m_1$ is equivalent to the inequality $n_2<m_2$.

55.9.3. Let $ABCD$ be the square and, say, AEF the first piece; let E belong to AB. Then the next cut according to the Rule leaves point E intact. Hence, after n cuts certain four-tuples of points from each side of the square — denote them K, L, M and N — remain on the pie. It follows that the quadrilateral $KLMN$ is preserved uncut.Prove yourselves that the raisin belongs to this quadrilateral $KLMN$.

55.9.4. This can be seen from Fig. 416, where in each cell stands the number of ways that lead to the cell from the lower left corner. To fill in the table one should write 0 in the marked cells (no way leads to any of them), 1 in each cell of the left column and the bottom row and the sum of numbers of neighboring cells to the left and below in each other cell starting from the lower left corner upwards and to the right.

Figure 416. (Sol. 55.9.4)

Figure 417. (Sol. 55.9.5)

55.9.5. It suffices to prove that line AB, see Fig. 417, divides in halves the segment FG passing through point B and parallel to segment DE. Observe that $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5$ and $\beta_1=\beta_2=\beta_3=\beta_4$. This implies that $\triangle BCG=\triangle BCD$ and, finally, $BG=BD=BF$.

55.9.6. The difference between the neighboring numbers on the k-th figure is equal to either k or $k-11$ and, therefore, the pairs of neighboring numbers never repeat.

Each of such placings allows one to get 2 placings for $n=10$ in the same way: the number 1 is always written on top and the remaining 20 places are divided into the left and the right halves. The first placing is obtained by filling the right half downwards from the top; the second placing is obtained by filling alternately the left and the right halves from the bottom upwards. The numbers are taken from the initial placing clockwise. The remaining places are filled in with numbers 12 to 21 in order for the sum of the numbers on one horizontal to be equal to 23.

For example, from the first placing — Fig. 418 — we get 2 placings depicted on Fig. 419.

FIGURE 418. (Sol. 55.9.6)

FIGURE 419. (Sol. 55.9.6)

Let us verify that no two numbers will be neighbors twice. Let $k < l$. Let us consider all cases that might occur:

1) $k = 1$. One of the neighbors is one of the former (at the initial placing) neighbors, the one with number l, the other one with number $23 - l$. Hence, all neighbors are distinct.

2) $2 \le k < l \le 11$. In this case all neighbors of the initial placing are retained at the first placing and, therefore, at the second placing there are no such neighbors.

3) $12 \le k < l \le 21$. This case is reduced to the preceding one by the change $k \mapsto 23 - k$, $l \mapsto 23 - l$.

4) $2 \le k \le 11$, $12 \le l \le 21$, $k + l \ne 23$. The first placings have no such neighbors. In the second placing, if k and l are neighbors, then k and $23 - l$ are neighbors in the initial placing. Hence, the neighbors do not repeat.

5) $2 \le k \le 11$, $l = 23 - k$. On Figures these are two bottom numbers in each placing. Each time one of the numbers is the former neighbor of number 1 in the initial placing; hence, all numbers are distinct.

55.10.1. If α, β, γ, δ are the angles of the quadrilateral, then $\gamma + \beta = 360^\circ - (\alpha + \delta)$ and

$$\begin{aligned} 0 = \cos\alpha + \cos\beta + \cos\gamma + \cos\delta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos\frac{\gamma+\delta}{2}\cos\frac{\gamma-\delta}{2} \\ &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - 2\cos\frac{\alpha+\beta}{2}\cos\frac{\gamma-\delta}{2} \\ &= -4\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma-\beta-\delta}{2}\sin\frac{\alpha+\delta-\beta-\gamma}{2}. \end{aligned}$$

This implies either $\alpha + \beta = 180^\circ$, or $\alpha + \gamma = \beta + \delta = 180^\circ$, or $\alpha + \delta = \beta + \gamma = 180^\circ$.

FIGURE 420. (Sol. 55.10.5)

FIGURE 421. (Sol. 55.10.6)

The convex hull of the uncut 5 points of the initial pentagon is a pentagon that lies inside the uncut part and contains the shaded part.

For an n-gonal pie the answer is similar: we must take the figure obtained by cutting off n triangles with the help of small diagonals.

55.10.3. Let us paint the table two colors as a chessboard. Clearly, each move of any of the chips changes the color of its field. Therefore, if initially the chips occupied fields of distinct colors the black can not win, whereas the white can: the white chip can occupy a field on the same diagonal as the black chip and gradually reduce the distance between the chips.

If initially the chips occupied the fields of the same color then, the other way round, the white can not win while the black can.

55.10.4. Let the total weight of weights be equal to 60 g. Then the mass of any weight should not exceed 10 g (from decomposition into 6 piles) whereas their number can not be smaller than 10 (from decomposition into 5 piles). If there are precisely 10 weights then among them there are: two weights 10 g each; 2 weights 2 g each; 4 weights of an odd or noninteger mass each (as is clear from a consideration of a decomposition into 4 piles); and 2 weights of total mass 12 g. But then it is impossible to split these weights into 6 piles. (See also Remark to solution of Problem 55.8.4.)

55.10.5. Let A and B be the most distant from each other points of the polygon. Then the polygon is completely contained in a band of width AB and the midpoint O of segment AB is the center of symmetry. Let C and D be the symmetric points of the boundary of the polygon that belong to the midperpendicular to AB. Then $ABCD$ is a rhombus placed inside a polygon situated on one side of line KL and symmetric to it line MN, see Fig. 420.

Thus, parallelogram $KLMN$ contains the polygon; hence, the area $AB \cdot CD$ of $KLMN$ is twice the area of rhombus $ACBD$.

55.10.6. If each face had the same amount of edges of both colors, then the whole polyhedron would have had the same property and would have had an even number of edges. On Fig. 421 there is depicted a polyhedron, each of its faces has an even number of edges. It is impossible to paint the edges of the polyhedron as desired, since the total number of its edges is odd.

55.11.1. a) Write 1 in every cell of the upper and lower rows; write -1 in the central cell and 0 in the remaining cells.

b) If the answer is "yes", paint the table two colors as a chessboard. We see that the sum of the numbers in black cells is simultaneously equal to 1992 and to 1991 — to the number of black diagonals in one direction and in the other one.

55.11.2. Let us draw a circle through points A, B, C. Then point D can occur only inside the circle, because $\angle B + \angle D > 180°$.

FIGURE 422. (Sol. 55.11.2)

Extending diagonal BD of the quadrilateral to its intersection with the circle at point E, we see that BE is a diameter of the circle: on Fig. 422 $\smile BC = 2\angle BAC = 60°$, $\smile CE = 2\angle DBC = 120°$ which implies that $\smile BCE = 60° + 120° = 180°$.

The equalities $\alpha + \gamma = 50°$ (since $\angle ADB$ is an outer one in $\triangle ADE$) and $\beta + 40° + \gamma = 90°$ (since $\angle BCA = \angle AEB = \gamma$ as inscribed angles intercepting arc $\smile AB$ and $\angle BCE = 90°$ as an angle intercepting a diameter) implying $\alpha = \beta$. Thus, $\smile GE = \smile EF$ and, therefore, points G, F are symmetric through diameter BE. Hence, the answer.

55.11.3. Let us modify Aladdin's way by discarding all instantaneous transitions to diametrically opposite points and preserving all movements to the east and to the west; let us assume that Aladdin's speed is a constant. The modified way passes at least along half of the equator and the graph of the pass with directions being taken into account is a broken line whose maximal and minimal values differ by not less than half the length of the equator and are attained just at the endpoints of the time interval to be found.

55.11.4. Consider unit vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, $\overrightarrow{e_3}$, $\overrightarrow{e_4}$ perpendicular to the four faces of tetrahedron and directed inwards. Let $\vec{a}$ be the vector perpendicular to the given triangle and of length equal to the value of the area of the triangle in square units. Then $P_i = |\vec{a} \cdot \vec{e_i}|$ for $i = 1, 2, 3, 4$. Scalaring $S_1\overrightarrow{e_1} + S_2\overrightarrow{e_2} + S_3\overrightarrow{e_3} + S_4\overrightarrow{e_4} = 0$ by $\vec{a}$ we get

$$P_1S_1 = |a \cdot S_1\overrightarrow{e_1}| = |-(\vec{a} \cdot S_2\overrightarrow{e_2} + \vec{a} \cdot S_3\overrightarrow{e_3} + \vec{a} \cdot S_4\overrightarrow{e_4})| \le P_2S_2 + P_3S_3 + P_4S_4.$$

REMARK. The sum of the vectors perpendicular to the faces of any polyhedron of lengths equal to the values of the corresponding areas of faces in square units is always equal to 0. Indeed, by filling the polyhedron with a gas make use of the *Pascal's law* that states that *the pressure of gas at every point of a closed volume is the same*. This means that the sum of forces acting on all faces from inside is equal to 0.

55.11.5. For a polyhedron each of whose faces is a quadrilateral the requirement of the problem coincides with the requirement of Problem 55.10.6 which is impossible to satisfy for the polyhedron depicted on Fig. 55.10.6.

55.11.6. a) The calculator produces the series of pairs:

$$\log_{25} 75 \text{ and } \log_{65} 260; \quad \log_{25} 3 \text{ and } \log_{65} 4; \quad \log_4 65 \text{ and } \log_3 25;$$
$$\log_4 \frac{65}{4} \text{ and } \log_3 \frac{25}{3}; \quad \log_4 \frac{65}{16} \text{ and } \log_3 \frac{25}{9};$$

and gives the answer (the left number is greater than the right one since $\frac{65}{16} > 4$ and $\frac{25}{9} < 3$).

b) Suppose that it takes forever for the calculator to compare $x_1 = \log_a b < \log_c d = y_1$. Take positive integers m_0, m_1 for which $z_1 < \frac{m_0}{m_1} < y_1$. Then applying the first rule several times the calculator passes to the pair $x_1 < y_1$ between which stands the number $\frac{m_2}{m_1}$ with $0 < m_2 < m_1$.

After applying the second rule once the calculator passes to the pair $x_2 = \frac{1}{\{y_1\}} < y_2 = \frac{1}{\{x_1\}}$ between which the number $\frac{m_1}{m_2}$ is situated. Now, the first rule is applied again, etc.

As a result we get a sequence of positive integers $m_1 > m_2 > \dots$ which can not be infinite. Contradiction.

56.8.1. a) Since x, $s(x)$, $s(s(x))$ and $s(s(s(x)))$ have the same residues after the division by 3, then $x + s(x) + s(s(x))$ is always, unlike 1993, divisible by 3 and the equation has no solutions.

b) By the reason explained above $s(x) + s(s(x)) + s(s(s(x)))$ is always divisible by 3 and the residue after the division of any solution, x, by 3 is equal to the residue after the division of 1993 by 3, i.e., 1. Let us estimate x. Clearly, $x \le 1993$ and, therefore, $s(x) \le s(1989) = 27$, $s(s(x)) \le s(19) = 10$, $s(s(s(x))) \le 9$. The equation implies that

$$x = 1993 - s(x) - s(s(x)) - s(s(s(x))) \ge 1993 - 27 - 10 - 9 = 1947.$$

But if $x \ge 1947$ then $s(x) \ge s(1950) = 15$, $s(s(x)) \ge s(20) = 2$, $s(s(s(x))) \ge 1$; hence, $x \le 1993 - 15 - 2 - 1 = 1975$. Among the numbers from 1947 to 1975 it is only 1948, 1951, 1954, 1957, 1969, 1963, 1966, 1969, 1972, 1975 that have residue 1 after the division by 3. We directly verify that only 1963 satisfies the equation.

56.8.2. Let us make use of the identity

$$n^2 = (a^2 + b^2 + c^2)^2 = (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2.$$

If $a \ge b \ge c > 0$, then $a^2 + b^2 - c^2$ is a positive integer.

56.8.3. When a pair of chips of the same color is added one of them forms with a chip of the other color a pair spoken about in Hint if and only if the other chip forms a similar pair. Therefore the number of such pairs of chips of distinct colors with the red left chip either increases by an even number or does not vary. The same is true for the opposite operation of deleting a pair of neighboring chips of the same colors, except that the number might decrease now. But by hypothesis the number of such pairs is equal to 1 and it can not turn into 0 even for the case when the red of the two remaining chips stands to the right of the blue.

56.8.4. The looking-glass watch, LW, shows a favorable time if and only if UW shows an unfavorable time. Therefore, the whole day has as many unfavorable time as the Looking-glass has favorable one. On the other hand, at time T after midnight the hands of the LW assume the same position as the hands of the UW at time T before midnight because the watches tick uniformly. Therefore, there is as much of the favorable time as unfavorable looking-glass time, i.e., as there is the unfavorable time.

56.8.5. Let us prove by induction that the n-th word in the above sequence satisfies the condition of the problem for the alphabet consisting of the first n letters of the Russian alphabet. For $n = 1$ the solution is obvious.

Let the statement hold for any $n \le k$. Consider $n = k + 1$. First of all, there is just one n-th letter of alphabet in the n-th word, it breaks it into two words each equal to the $(n-1)$-st word. Therefore, two equal neighboring subwords of the n-th word can not have the n-th letter of alphabet and should be encountered in one of the two $(n-1)$-st words. But this is impossible by the inductive hypothesis.

Now, observe that if we ascribe to the end the n-th word the n-th letter of the alphabet, the end of the word will consist of two equal halves. Among the first and among the last segments of the n-th word we encounter all previous words of the sequence; hence, when we ascribe any of the previously encountered letters, two equal neighboring words also appear.

Remark. The length of the desired 33-rd word is equal to $2^{33} - 1 \approx 10^{10}$ letters!

56.8.6. Let α and γ be the angles at vertices A and C of $\triangle ABC$ and let O be the center of escribed circle, see Fig. 423.

Figure 423. (Sol. 56.8.6)

Then $\angle CAO = \frac{\alpha}{2}$ and $\angle CBO = \frac{1}{2}(\alpha + \gamma)$, since point O lies at the intersection of the bisectors of the angle $\angle A$ and the exterior angle at vertex B. Since $\triangle ALC$ and $\triangle BLO$ have the common exterior angle at vertex L, we have $\angle CAL + \angle ACL = \angle BOL + \angle OBL$. We know three of these angles, so the forth one is equal to

$$\angle BOL = \angle CAL + \angle ACL - \angle OBL = \frac{\alpha}{2} + \gamma - \frac{\alpha + \gamma}{2} = \frac{\gamma}{2}$$

Let D be the center of the circumscribed circle of triangle ABO. Since the inscribed angle $\angle BOA$ and the central one, $\angle BDA$, intercept the same arc, $\smile AB$, we have $\angle BDA = 2\angle BOA = \gamma = \angle BCA$, cf. Hint.

56.9.2. Observe that

$$x_1 = 4,\ \ x_2 = 6,\ \ x_3 = 9,\ \ x_4 = 14 = x_3 + 5,\ \ x_5 = 20 = x_4 + 6,\ \ x_6 = 27 = x_5 + 7,\ \dots$$

In other words, we make a guess that

$$x_n = x_{n-1} + n + 1 \quad \text{for } n \ge 4. \tag{$*$}$$

Let us prove $(*)$, together with the statement that $x_n = \dfrac{n(n+3)}{2}$, by induction . The latter formula can be observed by consecutively inserting the preceding formula in the subsequent one in the following chain:

$$x_3 = 9,\ \ x_4 = x_3 + 5,\ \ x_5 = x_4 + 6,\ \ x_6 = x_5 + 7,\ \ \dots,\ \ x_n = x_{n-1} + n + 1$$

which yields

$$x_n = 9 + (5 + 6 + 7 + \dots + (n+1)) = 9 - 1 - 2 - 3 - 4 + \frac{1}{2}(n+1)(n+2)$$

$$= \frac{1}{2}(n+1)(n+2) - 1 = \tfrac{1}{2}n(n+3).$$

For $n = 4$ the statement is true. Let it be true for all $n \le k$. Then for $n = k + 1$ we have $2x_{n-1} - x_{n-2} = 2 \cdot \dfrac{k(k+3)}{2} - \dfrac{(k-1)(k+3)}{2}$ and the next integer, $\dfrac{(k+1)(k+4)}{2}$ is non-prime for $k \ge 2$. Thus, $x_{1000} = \dfrac{1000 \cdot 1003}{2}$.

56.9.3. Suppose that at some step we got a triangle similar to the initial one. Observe that all its angles are multiples of 20°, hence, as is easy to see, all angles of the preceding triangle, and, more generally, of all preceding triangles are multiples of 20°. This, however, can not be true even after the first cut of the initial triangle: if we start with an angle of 20° we get an angle of 10°, and if we start with the angle of 140° we get an angle of 70°. The contradiction shows that it is impossible to get a triangle similar to the initial one.

56.9.4. Let us solve the problem by replacing 28 by an arbitrary even number $2k$. Let us prove by induction that Pete has k friends. For $n = 1$ the statement is obviously true. Let us assume that it holds for $n = k$ and consider the case when $n = k + 1$. Since each of Pete's classmates may have 0 to $2k + 2$ friends, there are $2k{+}2$ of Pete's classmates altogether, and each two of them make friends with distinct number of classmates, then the following two cases are possible:

If there is a student in the class who is friendly with all classmates, then the number of friends Pete's classmates might have are: 0, 1, 2, ..., $2k + 1$. By the same argument as in Hint (assuming that we transfer the most and the least friendly classmates in the other class) we see that Pete has $k + 1$ friend. Thus, the statement is true for any n.

56.9.5.

$$x * y = (x * y) + y - y = (x * y * y)) - y = (x * 0) - y$$
$$= (x * (x * x)) - y = (x * x) + x - y = 0 + x - y = x - y.$$

56.9.6. Let K be the point symmetric to B through line AM. Since then $AB = AK$ and $\angle BAK = 2\angle BAM = 60°$, it follows that triangle $\triangle ABK$ is an equilateral one. Hence, points A, C, K lie on the circle centered at B with radius AB. Since the inscribed angle is equal to the half of the central one intercepting the same arc, $\angle ACK = 30°$. In other words, since $\angle MCA = 150°$, points C, K, M lie on one line. By construction, AM is the bisector of $\angle BMK$, hence, of $\angle BMC$.

Figure 424. (Sol. 56.9.6)

56.10.1. Let us show that if k is the length of a period of two infinite fractions, P and Q, then k is also the length of a period of $P+Q$. Indeed, if a fraction is a periodic decimal fraction with the length of a period equal to k then the fraction can be represented as the usual fraction (of the form a/b) with the denominator 10^{k-1}. (Make use, for example, of the formula for the sum of infinite geometric progression.) The sum of two fractions with the same denominator can be written as a fraction with the same denominator.

Let us prove by the rule of contraries that *Each infinite periodic fraction has the minimal period whose length divides that of any other period of the given fraction.* Indeed, if this statement is false and a fraction has a period of minimal length m and another period of length l not divisible by m. Then the residue after the division of l by m is also a period. Since this residue is smaller than m we get a contradiction thus having justified the arguments of Hint.

56.10.2. It is not difficult to construct a closed rout circumventing 8 rooms of each hall; 64 routes altogether. Observe an important property of this collection of routes: *they go through all the rooms and two different routes in the halls with a common wall run in parallel with the wall.*

Now, two such routes can be united into one by replacing two of their parts parallel to the wall with two perpendicular ones, see Fig. 425.

Figure 425. (Sol. 56.10.2)

As a result, the important property remains preserved while the number of routes diminishes by 1. By repeating the operation we can reduce the number of routes to 1. Baron is right, as always!

56.10.3. 56.10.3. In heading a) the answer "no" follows from Fig. 426. Indeed, the segment AB divides the river into two parts and, therefore, the route of the boat that connects the two lakes should intersect segment AB at some point. The distance from any point of AB to one of the banks is > 700 m.

Let us show that under conditions of heading b) such a route can always be found. Let us consider the set of points of the river distant from the right bank more than by 800 m. The boundary of this set consists of the points of the river whose distance from the right bank is equal to 800 m (the inner points of the river) and points of the left bank.

FIGURE 426. (Sol. 56.10.3a)

FIGURE 427. (Sol. 56.10.3b)

Let us progress from one lake to the other one along the boundary of this region consisting of the inner points or, if this is impossible to perform, along the left bank. Each point of this route is not more than 800 m away from the right bank. Let us show that its distance from the left bank also does not exceed 800 m.

Indeed, assuming the contrary we find a point on the rout with the distance $\leq$ 800 m away from the left bank. Therefore, we may construct a disc centered at this point and of radius of 800 m lying completely inside the river. Let us construct all possible rays with vertices at the center of the disc and paint the points of the circle that bounds it blue if the ray through this point first intersects the right bank and red if it first intersects the left bank. Let us prove that such a coloration divides the circle into not more than 2 arcs each painted one color. Indeed, if this were not true, we could select on the circle four points placed in the following order: red R_1^0, blue B_1^0, red R_2^0, blue B_2^0, see Fig. 427.

Let R_1, B_1, R_2, B_2 be the corresponding points of the first intersection of the rays with the banks. The coastline connecting R_1 with R_2 should intersect the ray passing through one of the points B_1^0 and B_2^0 further from the origin of the ray than one of the points B_1 or B_2 is situated on it.

Let us consider the contour that consists of the coastline that connects points R_1 with R_2 and the segments that connect the center of O with R_1 and R_2. This contour has no selfintersections; one of the points B_1 or B_2 is inside the contour and the other one is outside it. Thus, the coastline that connects B_1 with B_2 should at some moment intersect the contour. It can not intersect the right bank, since the banks do not intersect; it can not intersect neither of segments R_1O or R_2O, since this would contradict the rule of coloration.

Thus, there exists an arc of the circle painted one (let us say, red) color and corresponding to the angle of measure greater than 180°. Let us take points R_r^0 and R_1^0, the distance between which along the red arc is equal to 180° and the center R_b^0 of arc $\smile R_r^0 R_1^0$. Let us construct the contour consisting of the coastline that connects point R_1 with R_r and segments that connect the center of O with R_1 and R_2. The shortest way that connects point R_b with the opposite bank intersects the given contour at one more point not belonging to the coastline. Therefore, this way is seen from the center of the disk under the angle of more than 90°. But the rout seen from the center of the disc under the angle of more than 90° and with endpoints outside the disc of radius of 800 m should be longer than 1 km. This contradicts the hypothesis on the width of the river.

Hence, it is impossible to place the disk of radius 800 m inside the river and the pilot can lead the boat at a distance not further than 800 m away from each of the banks.

56.10.4. a) Let us make use of the geometric interpretation of the sequence p_n given in Hint. Let us construct sequences p_n in which all four-letter words beginning with 0 are encountered. There are 8 such subwords; the other 8 subwords are encountered in the sequences corresponding to pairs $(-a, -b)$ instead of (a, b).

EXAMPLES. A regular octagon; a square; a "2-gon".

1) $b = 0$, $a = 1/8$. Subwords: 0000, 0001, 0011, 0111.

2) $b = 0$, $a = 1/4$. A new subword: 0110.

3) $b = 0$, $a = 1/2$. Subword: 0101.

An equilateral triangle: $b = 0$, $a = \frac{1}{3}$. Subword: 0010.

b) Observe that if in the formulas that determine the sequences the coefficients of n differ by any positive integer, then such sequences coincide thanks to the properties of the function $y = \{x\}$. Therefore, we may assume that $0 \leq a < 1$. It is easy to see that for a fixed a the number of successive points that occupy a semicircle can only vary by 1 (from $\left[\frac{1}{2a}\right] + 1$ to $\left[\frac{1}{2a}\right]$) depending on the position of the first point that got to the semicircle (if the angle between the beginning point of the semicircle and the first point on it lies on the segment $[0, \frac{1}{2} - a\left[\frac{1}{2a}\right]]$ of full revolutions, then the number of points is by one more than in the opposite case). Thus, the number of consecutive zeroes confined between two units can not change by more than one. The same is true for consequtive units confined between zeroes. The set 00010 can not be a word of such a sequence, since it contains tree zeroes in a row and, therefore, the number of units confined between zeroes can not be less than 2.

56.10.5. Suppose that a good classifier can describe more than 50 plants. Let m_i be the number of plants with a i-th feature and R the total number of plants. Then the number of pairs of plants that can be distinguished by the i-th feature is equal to $(R - m_i)m_i$ and the total number S of differences between the plants is equal to

$$\sum_{i=1}^{100}(R - m_i)m_i.$$

Let us now evaluate S by another method. The total number of pairs of plants is equal to $R(R-1)/2$ and by the hypothesis each pair is distinguished by at least 51 feature. Therefore, the total number of differences is not less than $51 \cdot 51 \cdot 50/2 = 51^2 \cdot 25$ because by the inductive hypothesis $R \geq 51$. Observe that

$$(51 - m_i)m_i \leq 51^2/4.$$

The equality is attained for $m_i = 51/2$. Therefore,

$$\sum_{i=1}^{100}(51 - m_i)m_i \leq \sum_{i=1}^{100}\frac{51^2}{4} = 51^2 \cdot 25.$$

We have, therefore, reached a contradiction:

$$51^2 \cdot 25 = S = \sum_{i=1}^{100}(51 - m_i)m_i < 51^2 \cdot 25.$$

Thus, a good classifier can not describe more than 50 plants.

56.10.6. Denote the other constructed vertices of the square $ABDE$ by D and E, let $\angle CAB = \alpha$, $\angle ABC = \beta$, $\angle ACB = \gamma$. Let $AB = c$, $CD = d$, $CE = e$. Then by the theorem on the midline applied to triangles $\triangle EBC$ and $\triangle ADC$ we get $ON = 0.5e$ and $OM = 0.5d$. Adding up these equalities we get: $OM + ON = \frac{d+e}{2}$. By the law of sines, for $\triangle ABC$ we have

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta} = \frac{a}{\sin\alpha}.$$

The the law of cosines for $\triangle ABC$ gives $c^2 = a^2 + b^2 - 2ab\cos\gamma$. Let us now apply the law of cosines to $\triangle AEC$; then express c^2 with the help of the the law of cosines for $\triangle ABC$ and $\cos(\alpha + 90°)$ with the help of the the law of sines for $\triangle ABC$:

$$e^2 = b^2 + c^2 - 2bc\cos(\alpha + 90°) = 2b^2 + a^2 - 2ab\cos\gamma + 2ab\sin\gamma.$$

Similar arguments applied to $\triangle CBD$ and $\triangle ABC$ imply:

$$d^2 = a^2 - c^2 + 2ac\cos(\beta + 90°) = b^2 + 2a^2 - 2ab\cos\gamma + 2ab\sin\gamma.$$

Hence, the maximum of $e + d$ is attained simultaneously with the maximum of $\sin\gamma - \cos\gamma$, i.e., at $\gamma = 135°$.

56.11.1. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta} = \dfrac{p}{1 - \tan\alpha\cdot\tan\beta}$; $\dfrac{1}{\tan\alpha} + \dfrac{1}{\tan\beta} = q$. Therefore, $\dfrac{\tan\alpha + \tan\beta}{\tan\alpha\cdot\tan\beta} = q$; hence, $\tan\alpha\cdot\tan\beta = \frac{p}{q}$. Finally, $\tan(\alpha+\beta) = \dfrac{p\cdot q}{q-p}$.

56.11.2. The sum of perimeters of the squares in the lower half of the great square marked by symbol k is equal to $4\cdot 2^{-k}\cdot 2^{k-1} = 2^{-1}\cdot 4$. This sum in the whole square is equal to 4. Thus, the total perimeter of the marked squares only is equal to $4\cdot 1993 > 1993$.

FIGURE 428. (Sol. 56.11.2) FIGURE 429. (Sol. 56.11.3)

REMARKS. a) The statement proved in the problem can be sharpened. *There exists a partition of the square into lesser squares such that the sum of perimeters of the squares having a common point with the main diagonal of the given square is* > 1993. An idea of construction of such a partition is as follows: We take a partition of the lower half of the square, shown on Fig. 428, then *slightly enlarge* the sides of the little squares so that their lengths become rational numbers. This rationality is needed to enable us to decompose the remaining uppertriangular part of the square into equal squares.

b) This problem appeared from the first mathematical work of Acad. A. N. Kolmogorov, performed yet at school. The problem posed for him was to prove that a curve of bounded length contained in a unit square can only intersect the squares of a partition that has a bounded total perimeter. Kolmogorov has constructed a counterexample.

56.11.3. First, let us show that there exists a position of points at which this estimate is attained, namely, the set of vertices of a regular n-gon. Let us show that the number of directions of its sides and diagonals is equal to n. Let us assign to each side and diagonal the axis of symmetry of the polygon — its midperpendicular. The parallel chords (i.e. sides or diagonals) have the same midperpendicular; the converse is also true. Therefore, it suffices to count the number of the axes of symmetry of a regular n-gon.

For an odd n each axis of symmetry passes through a vertex, and there is just one such vertex. Hence, the total number of axes of symmetry is equal to n.

For an even n an axis of symmetry passes either through a pair of opposite vertices (there are $n/2$ such axes) or through the midpoints of opposite sides (there are also $n/2$ such axes). Hence, the total number of axes of symmetry is again equal to n.

Now, it remains to show that for any disposition of n points no 3 of which lie on one line we can always select n pair-wise non-parallel lines. The convex hull of a set of n points is a polygon with vertices at some of these points. Then $n-1$ lines passing through *one* of the vertices, A, see Fig. 429, and the diagonal connecting *neighboring* vertices, B and C, are not pair-wise parallel.

56.11.4. 1) Let all possible sets of stones 1 to n lie in n boxes and we choose a number k (from Hint). Then after one move the boxes will contain 1 to n_1 stones. Here

$$n_1 = \max(q+r, q+k-2),\ \text{where } n = kq+r, 0\le r<q. \qquad (*)$$

Indeed, in the box with n stones there remains $q+r$ stones. By decreasing r from its initial value to 0 we can ensure that the box contains any number of stones, $q+r-1$ to q. By decreasing now q from its initial value to 1 we can get in the box any number of stones $q-1$ to 1. Further, in the box that originally contained $m = k(q-1)+(k-1) = kq-1 < n$ stones there will remain $q+k-2$ stones.

Let us prove that n_1 is the maximal possible number of stones left. Let the box initially contain $kq_1 + r_1 < n$ stones. Then $r_1 < r$ for $q_1 = q$ and there remains $q_1 + r_1 \le q+k-2$ stones. For $q_1 < q$ we have: $r_1 \le k-1$ and $q + r_1 < q + r$. This shows that n_1 is maximal.

2) The method indicated in Table allows us to leave 1 stone in each box that initially contained not more than 460 stones. This follows from step 1).

3) Let us prove that for any $k' \ne [\sqrt{n}]$ there is a box with not less than n_1 stones left, where n_1 is given by formula $(*)$ for $k = [\sqrt{n}]$. Note that

$$n = k^2 + r, \quad r \le k-1.$$

There are subcases: a) $n_1 = 2k-2$ for $r < k-1$; b) $n_1 = 2k-1$ for $r = k-1$.

a) There remains $n_1 = 2k-2$ stones in the box that initially contained $m = k'(2k-k'-1)+(k'-1)$ stones. But $m = k'(2k-k')-1 < k^2-1 < n$.

b) Let $k' = k+l$ for a nonzero integer l. Then there remains $n_1 = 2k-1$ stones in the box that initially contained $m = k'(2k-k')+(k'-1)$ stones. But

$$m = (k+l)(k-l)+(k+l-1)-1 = k^2+k-1-l^2+l = n-l^2+l \le n,$$

since $l-l^2 \le 0$ for any integer l.

Thus, we have proved that the move with $k = [\sqrt{n}]$ is the best. Therefore, if at the beginning of the play the boxes contained all sets 1 to 461, then for any strategy there will remain a box with 41 stones after the first move, with 11, 5, 3 and 2 stones after the second, third, fourth and fifth moves, respectively. Thus, it is impossible to leave one stone in each box after 5 moves in case b).

56.11.5. a) Divide segment $[-1,1]$ into 4 half-segments and point 0: $[-1,-\frac{1}{2})$, $[-\frac{1}{2},0)$, $(0,\frac{1}{2}]$, $(\frac{1}{2},1]$. Define the function f by the formula:

$$f(x) = \begin{cases} -\frac{1}{2}-x & \text{for } -1 \le x < -\frac{1}{2} \\ x-\frac{1}{2} & \text{for } -\frac{1}{2} \le x < 0 \\ 0 & \text{for } x = 0 \\ x+\frac{1}{2} & \text{for } 0 < x \le \frac{1}{2} \\ \frac{1}{2}-x & \text{for } \frac{1}{2} < x \le 1. \end{cases}$$

REMARKS. 1) Actually, the statement of the problem is true for functions f with finite number of discontinuity points but otherwise arbitrary. The solution is also applicable to functions with finitely many intervals of monotonity.

2) The general statement reads: *Let $g(x)$ be a monotonously decreasing transformation of the whole real line. Then there is no "superpositional root" of g (i.e., a function f such that $f(f(x)) = g(x)$ for all x) in the class of functions with a finite number of discontinuity points*. The same is true for roots of any even order.

56.11.6. Lemma. *Let P and Q be the midpoints of sides KL and MN, respectively, of quadrilateral $KLMN$. Then $PQ \le \frac{KN+LM}{2}$.*

PROOF. Denote by R the midpoint of diagonal LN, see Fig. 430 a). We have $PR = \frac{KN}{2}$, $PQ = \frac{LM}{2}$. Hence, $PQ \le PL + LQ = \frac{KN+LM}{2}$, as required. Q.E.D.

FIGURE 430. (Sol. 56.11.6)

Let us turn to our problem. Consider tetrahedron $ABCD$ and a spatial quadrilateral $EFGH$ whose respective vertices lie on faces ABC, BCD, DAB, and ACD, see Fig. 430 b). Through DC draw the plane perpendicular to AB (the plane of symmetry of tetrahedron $ABCD$) and consider quadrilateral $E_1F_1G_1H_1$

symmetric to $EFGH$ through this plane. (The vertices E_1 and G_1 lie on "their own" faces ABC and ABD, vertex F_1 lies on ACD, vertex H_1 lies on BCD.) The perimeters of quadrilaterals $EFGH$ and $E_1F_1G_1H_1$ are equal.

Denote the midpoints of segments EE_1, FH_1, GC_1, and HF_1 by E_2, F_2, G_2, and H_2, respectively. By Lemma the perimeter of $E_2F_2G_2H_2$ is not greater than that of $EFGH$. Moreover, vertices E_2 and G_2, the midpoints of sides EE_1 and GG_1, respectively, lie on a symmetry plane of the tetrahedron. The plane passes through CD; hence, points E_2 and G_2 lie on the medians of faces ABC and ABD drawn to AB.

Denote the midpoint of segment AB by T. Now, starting with $E_2F_2G_2H_2$, we will similarly construct, first, quadrilateral $E_3F_3G_3H_3$ symmetric to $E_2F_2G_2H_2$ through the passing through AB plane of symmetry of the tetrahedron. Next, take the midpoints of segments connecting the vertices of the quadrilaterals lying on one face; we get quadrilateral $E_4F_4G_4H_4$ all whose vertices belong to the union of two planes of passing through AB and CD planes of symmetry of tetrahedron $ABCD$.

In other words, vertices E_4 and G_4 lie on the medians of faces ABC and ABD drawn to AB; vertices F_4 and H_4 lie on the medians of faces BCD and ACD drawn to CD. Denote these medians by AS and BS, see Fig. 430 c).

We see that the perimeter of $E_4F_4G_4H_4$ does not surpass that of $EFGH$. Hence, the perimeter of $EFGH$ is not less than $4d$, where d is the distance between CT and BS.

Taking quadrilateral $E_0F_0G_0H_0$ so that its sides E_0F_0, F_0G_0, G_0H_0, and H_0E_0 are common perpendiculars to CT and BS, BS and DT, DT and AS, AS and CT, respectively, we get a quadrilateral with perimeter $4d$. Thus, we have proved that the least perimeter of quadrilaterals $EFGH$ is equal to $4d$. Let us find d.

Let us draw the plane through AB perpendicular to CT and take projection of the tetrahedron onto the plane, see Fig. 430 d). We obtain triangle ABD' in which $AB = a$, $D'T = a\sqrt{\frac{2}{3}}$. Point S passes under projection into point S', the midpoint of $D'T$. Hence, d is equal to the distance between $T(= C')$ and straight line BS'. In right triangle BTS' the legs $BT = \frac{a}{2}$ and $TS' = \frac{1}{2}a\sqrt{\frac{2}{3}}$ are known. Hence, $BS' = \frac{1}{2}a\sqrt{\frac{5}{3}}$, and $d = \frac{BT \cdot TS'}{BS'} = \frac{a}{\sqrt{10}}$.

57.6.O. bserve that $2 = 1 \cdot 2$, $6 = 2 \cdot 3$, $12 = 3 \cdot 4$, etc. Therefore, it is most probable (unless the authors have something increadible up their sleeves) that the 6-th number is $6 \cdot 7 = 42$ and $1994 \cdot 1995 = 3978030$.

57.6.3. Let each of t teams have g guards who slept n nights. By the hypothesis

$$t > n > g > 1$$

(well, though this was not given explicitely, we can assume that $g > 1$, can't we?) and $t \cdot n \cdot g = 1001$. The prime factorisation of 1001 is $13 \cdot 11 \cdot 7$. Since there are only 3 prime factors, it is impossible to factor 1001 into the product of other natural factors each > 1. The inequality implies that $t = 13$, $n = 11$, $g = 7$. (Cf. Problem 7.2.)

57.6.5. Let us take square $ABCD$ with side 3. Mark point F on CD at distance 1 from D and draw BF; mark the midpoint E of BF and draw AE. Let us rearrange the parts obtained as shown on Fig. 431.

FIGURE 431. (Sol. 57.6.5)

To find the lengths of the sides of $\triangle PQR$ the students of lower grades can measure with a ruler; the sudents of higher grades can apply the Pythagorus' theorem: $PR = 6$, $QR = \sqrt{3^2 + 4^2} = 5$, $PQ = \sqrt{2^2 + 3^2} = \sqrt{13} < 4$. Thus, $\triangle PQR$ is scalene.

Further, observe that segment AF is longer than the perpendicular $AD = 3$; therefore, $AF > AB$ and the median AE in $\triangle FAB$ can not coincide with the hight: $\angle AEB \neq 90°$. Since the angles $\angle BAE$ and $\angle ABE$ are also less than $90°$, it follows that $\triangle ABE$ is not a right one. But $\triangle ABE \sim \triangle PQR$ in two angles; hence, $\triangle PQR$ is not right, Q.E.D.

57.6.6. If there are less than 6 persons in the family, the total of the liquid Kate drank does not exceed $\frac{1}{6}$ of the milk plus $\frac{1}{6}$ of coffee. However, Kate drank more, hence, there are less than 6 members in the family. If there would have been not more than 4 of them, Kate would have drank not less than $\frac{1}{4}$ of the milk plus $\frac{1}{4}$ of the coffee. But she drank less than that; hence, there are 5 people in the family.

57.6.7. a) Let us divide the kids into groups of piers (that is to say, from the same grade). Call the groups of one person *small*, the other groups *big*. Let there are b big groups and s small ones. If we select 2 persons from each big group and add all from the small ones, then there will be no piers among the selected students. By hypothesis this means that there are selected less than 10 students, i.e., $2b + s \leq 9$. Hence, $b \leq 4$, $s \leq 9$.

The total population of small groups does not ecceed 9; hence, there are not less than 51 students in the big groups. If $b \leq 3$, then one of the big groups has 17 students (by Dirichlet's principle!). If $b = 4$, then $s \leq 9 - 2b \leq 1$. The total of small groups ≤ 1; hence, the total of big groups ≥ 59. By Dirichlet's principle one of the big groups has not less than 15 students.

b) For example, 15 students of each of 4 grades. Dividing any 10 students into 4 groups of piers we deduce thanks to Dirichlet's principle that one of the groups has not less than 3 students.

57.6.6. Starting from point A the pedestrian circumvent the streets in the following order: 1, 1, 4, 2, 2, 5, 3, 3, 6, 4, 5, 6. It is impossible to cross(?) each street just once because the pedestrian has to go twice along one of the streets 1, 2, or 3.

57.7.2. Let there be s staircases, f floors and a appartmentts on the floor in the house. Then

$$f > a > s > 1$$

and $f \cdot a \cdot s = 105$. The prime factorization of 105: $3 \cdot 5 \cdot 7$. Since there are just 3 prime factors of 105, it is impossible to factor in a different way into the product with factors > 1 each. The inequality implies: $s = 3$, $a = 5$, $f = 7$. (Cf. Problem 6.3.)

FIGURE 432. (Sol. 57.6.8)

FIGURE 433. (Sol. 57.8.3)

57.7.3. There are even several solutions, for example:

$$\begin{array}{r} +\begin{array}{r}657\\432\end{array}\\ \hline 1089\end{array} \qquad \begin{array}{r} +\begin{array}{r}475\\623\end{array}\\ \hline 1098\end{array}$$

57.7.5. Let us call the center of the cell the chip occupies the *position of the chip*. Let the *mark* be the midpoint of the segment connecting the positions of the white chips. It is clear that only the left black chip can be killed if it steps on the mark.

Observe that after each move of the white the mark moves a half-cell up and the black chip moves one cell down. So if the black chip occures below the mark it will survive.

Ivan has to make the first 4 moves with the left chip. Then after the first 3 moves of both boys this chip stands a half-cell above the mark and after the 4-th move a half-cell below the mark.

57.7.6. There was $5 \cdot 20 = 100$ man-meetings of the club altogether. Suppose that less than 20 students attended the circle. Then there exists a student that came to not less than 6 meetings (otherwise the number of meetings is less than $5 \cdot 20$). Obvioously, (s)he saw 4 new faces each time, $6 \cdot 4 = 24$ altogether. Controdiction proves the statement.

57.8.1. To manufacture 30 jars of the drink according to the old recipe it required 5 cans of the apple juice and 3 jars of the grape juice; the total of 8 cans.

After the recipe had been changed 30 jars required 6 cans of apple juice; hence, $8 - 6 = 2$ cans of grape juice. Now, one can of the grape juice lasts $30 : 2 = 15$ jars.

57.8.3. Let O be the intersection point of the bisectors; then BO is also a bisector. Set $\alpha = \angle OAC$, $\beta = \angle OBC$, $\gamma = \angle OCB$. Since these angles are halves of the triangles angles, $\alpha + \beta + \gamma = 90^\circ$. In the right triangle CQB, see Fig. 433, we have $\angle QBC = 90^\circ - \gamma = \alpha + \beta$, hence, $\angle QBO = \angle QBC - \angle OBC = (\alpha + \beta) - \beta = \alpha$.

In quadrilateral $QBPO$, we have $\angle Q + \angle P = 180^\circ$, therefore, the quadrilateral is an inscribed one. Clearly, $\angle QPO = \angle QBO = \alpha$, as angles that subtend the same arc. The *crosswise angles* $\angle QPA$ and $\angle PAC$ at the secant AP are equal, as is known, $\angle QPA = \angle PAC = \alpha$; hence, $PQ \parallel AC$. Q.E.D.

57.8.4. Suppose the grasshoppers can increase the square. But then hopping in the reverse order they could decrease the larger square. Likewise they could diminish the initial square.

Let us draw the infinite lattice so that the grasshoppers' places were at the nodes of a unit square. Notice that if all the grasshoppers were at some moment at nodes of the lattice (not necessarily forming a square), then after a hop (all together) they will again occupy nodes. But they can not remain at the nodes and form vertices of a square smaller than a unit one. Contradiction.

57.9.2. 1) Let $\frac{k}{l} \le 1$, i.e., $k \le l$. Here is the winning strategy for Leo. Let Nick divide his segment into parts of lengths a, b, c so that $a \ge b \ge c$. Then it suffices that Leo divides his segment into parts of lengths a, $\frac{l-a}{2}$ and $\frac{l-a}{2}$. Since $a \ge b$, it is possible to construct a triangle from segments of lengths a, a, b. It is only possible to construct a triangle of the remaining parts if $\frac{l-a}{2} + \frac{l-a}{2} > c$. But this inequality holds since $l - a \ge k - a = b + c > c$.

2) Let $\frac{k}{l} > 1$, i.e., $k > l$. Here is a winning strategy for Nick. Take a positive number smaller than $\frac{k}{4}$ and let $\frac{k-l}{3}$. It suffices for Nick to divide his segment into parts of length $k - 2d$, d and d.

It is easy to see that the part of length $k-2d$ is so long (longer than $l+d$ and $2d$) that whatever division Leo uses for his segment, the sum of any two of the remaining parts is shorter than $k-2d$. Therefore, it is impossible to construct a triangle with side $k-2d$.

57.9.3. Here is an infinite series of solutions:

$$x = 2u^2+1, \quad y = u(2u^2+1), \quad z = -u(2u^2+1), \text{ for } u = 0, 1, 2, \dots$$

Verification is straightforward. How to guess this answer?

Since we only have to find infinitely many solutions, NOT all of them, let us impose some additional constraints. Observe that for $x, y, z > 1$ the right hand side is greater than the left hand side, so it is natural to assume that some of the roots are negative, say, $x, y > 0$ whereas $z = -v < 0$. We have $x^2+y^2+v^2 = x^3+y^3-v^3$. Setting $y = v$ we simplify the equation: $x^2+2v^2 = x^3$. Let us divide the equation by x^2; we get $x-1 = 2(\frac{v}{x})^2$. The left hand side is integer, hence, so is the right hand side. This means that $u = \frac{v}{x}$ is an integer. But then $x-1 = 2u^2$ or $x = 2u^2+1$; $v = ux$ and, therefore, $y = u(2u^2+1)$ and $z = -y$.

Remark. There certainly exist other series, e.g.,

$$(\frac{1}{2}n(n^2-1)+1, -\frac{1}{2}n(n^2-1)+1, -n^2+1) \text{ for any } n \in \mathbb{Z}.$$

P. Grozman used a computer to see (just for fun) if there are some singular solutions, i.e., solutions that do not belong to a series. He found $(2, -1, -1)$, $(99, -98, -21)$, $(385, -330, -275)$ and these are not all.

57.9.4. Denote: $\alpha = \angle MAB$, $\beta = \angle NAB$. The angle that subtends the chord and the angle between the tangent and the chord is measured by the half of the arc it subtends, hence, $\angle ABP = \angle ANB = \angle MAB = \alpha$ and $\angle AQB = \angle AMB = \angle NAB = \beta$; we also have $\angle MAB = \angle MQB = \alpha$ and $\angle QMA = \angle QBA$ as subtending the same arc; $\angle QBA = \angle ANB + \angle NAB = \alpha + \beta$ as the outer angle of $\triangle ABN$. Therefore, $\angle QMA = \angle QBA = \alpha+\beta$ and $\angle AQM = \angle AQB + \angle MQB = \beta + \alpha = \angle QMA$ and, therefore, triangle QMA is an isosceles one: $AQ = AM$.

Consider triangles AQN and AMP. Then $\angle AQN = \angle AMP$ as subtending the same arc, similarly, $\angle APM = \angle ANB$; hence, the respective remaining angles are also equal. But $AQ = AM$, hence, these triangles are equal and $NQ = MP$, as required.

57.9.5. Let we strike b out of $N = \overline{a_1 \dots a_i b c_1 \dots c_k}$ and get a number N' that is L times less than N. Denote: $A = \overline{a_1 \dots a_i}$ and $C = \overline{c_1 \dots c_k}$ (if it is the last figure that is crossed out, then $k = 0$ and $C = 0$).

Then $N = 10^{k+1}A + 10^k b + C$, $N' = 10^k A + C$ and the condition $N = L \cdot N'$ is equivalent to the equation

$$10^k(10-L)A + 10^k b + (1-L)C = 0. \qquad (*)$$

1) Observe that $L \geq 2$, $L \neq 10$ (otherwise $N = LN'$ ends with a 0) and $A > 0$, $C < 10^k$.

2) Let us prove that the crossed out figure is the second from the left. To this end, let us express A from $(*)$:

$$A = \frac{(L-1)C - 10^k b}{10^k(10-L)}.$$

If $L < 10$, then

$$A < \frac{(L-1)C}{10^k(10-L)} < \frac{(L-1)10^k}{10^k(10-L)} = \frac{L-1}{10-L} = \frac{9}{10-L} \leq 9-1 = 8.$$

If $L > 10$, then

$$A = \frac{10^k b - (L-1)C}{10^k(L-10)} \leq \frac{10^k b}{10^k(L-10)} = \frac{b}{L-10} \leq b \leq 9.$$

In both cases A is a one-digit number, i.e., $A = a_1$; hence, b is the second figure.

3) Let us prove that

$$L \leq 19. \qquad (**)$$

Let us express L from $(*)$:

$$L = \frac{10^{k+1}a_1 + 10^k b + C}{10^k a_1 + C} = 10 + \frac{10^k b - 9C}{10^k a_1 + C} \leq 10 + \frac{10^k b}{10^k a_1} \leq 10 + \frac{9}{1} = 19.$$

4) The greatest N corresponds to the greatest k. Let us find this k. If $k > 0$, then $C > 0$ and C is not divisible by 10 (otherwise N would have ended with a 0). Then C is not divisible by 2 or 5. Let us rewrite $(*)$ as follows:

$$(L-1)C = 10^k(10A - LA + b),$$

hence, $(L-1)C \vdots 10^k$. If C is not divisible by 2, then $L-1$ is divisible by 2^k. Due to $(**)$ $L-1$ can be an integer 1 to 18. Among these integers, 16 is divisible by the greatest power of 2, hence, $k \le 4$ and $k = 4$ only for $L = 17$. If C is not divisible by 5, then, similarly, $k \le 1$.

5) Let us find an example when $k = 4$. It is unique. Thus, $L = 17$.

But since $(17-1) \vdots 10^4$, it follows that $C \vdots 625$, i.e., $C = 625 \cdot D$, where D is odd! Let us rewrite $(*)$:

$$10^4(10-17)a_1 + 10^4 b + (1-17)625 \cdot D = 0,$$

or, equivalently,

$$b = 7a_1 + D.$$

Taking into account that $a_1 \ge 1$, $b \le 9$, D is odd, we find: $a_1 = 1$, $b = 8$, $D = 1$; hence, the greatest N is $17 \times 10625 = 180625$.

57.9.6. a) Let us reduce the problem to a simpler one: prove that there is no way to place 5 domino pieces on the 4×4 square board so that it is impossible to place the 6-th piece. It is not difficult to solve the new problem case-by-case. Indeed, if such a position would have existed, then among the 6 free cells (3 white, 3 black) there would be no neighboring ones, i.e., 3 black cells would have had no more than 5 white neighbors.

By considering all combinations of 3 black cells (there are only 56 of them, or even 17 if symmetric constructions are considered as equal), we find 2 variants, see Fig. 434. a). Both variants do not fit: the cell marked with a "?" is isolated and can not be covered.

FIGURE 434. (Sol. 57.9.6)

Now, let us show how to reduce our problem to the above one. Let us divide the 10×10 field into 16 cells as shown on Fig. 434 b). The larger battleships occupy 2 cells each, as domino chips, the small ones (of size 1×1) just one cell. We need $(1+2+3) \cdot 2 + 4 \cdot 1 = 16$ cells altogether, we have exactly that many.

While we place large ships we need to have at each move two neighboring empty cells; but we always have them, see above. After this the small ships can be squeezzed into the remaining 4 (or more) cells.

b) It suffices to place 8 ships at the centers of shaded cells to fasten all of them and then the battleship (4×1) is impossible to fit into.

57.10.1. Denote the left 7-digit number by x, the right one by y. Then

$$10^7 x + y = 3xy. \qquad (*)$$

Therefore, $y = (3y - 10^7)x$, $y \vdots x$, $y = kx$, where $k \in \mathbb{N}$. Let us insert kx instead of y in $(*)$ and divide the equation by x. We get

$$10^7 + k = 3kx. \qquad (**)$$

If $k \ge 10$, then $y = kx$ has more than 7 digits, hence, $k < 10$. Moreover,

$$10^7 + 9 \ge 10^7 + k = 3kx \ge 3k \cdot 10^6,$$

wherefrom $k \le \frac{10^7+9}{3 \cdot 10^6} < 4$.

It follows from $(**)$ that $10^7 + k \vdots 3$. Checking $k = 1, 2, 3$ we see that this is only possible for $k = 2$. Hence, $x = \frac{1}{3}10000002 = 1666667$, $y = 2x = 3333334$ and $16666673333334 = 3 \times 1666667 \times 3333334$.

57.10.2. a) By induction on n it is not difficult to prove that

(1) $0 \le x_n \le 1$ for all n;

(2) if x_1 is representable as a fraction with denominator q, then so is x_n.

By (1) the numerators of such fractions only attain integer values 0 to q; hence, some x_n would coincide with one of the previous, x_m. But then $x_{n+1} = x_{m+1}$, $x_{n+2} = x_{m+2}$, etc., i.e., the sequence is periodic.

b) The formula $x_{n+1} = 1 - |1 - 2x_n|$ can be expressed as follows:

$$x_{n+1} = \begin{cases} 2x_n & \text{for } x_n < \frac{1}{2} \\ 2 - 2x_n & \text{for } x_n \ge \frac{1}{2}. \end{cases}$$

To express x_{n+2} via x_n we have four possibilities:

$$x_{n+2} = \begin{cases} 4x_n & \text{for } x_n < \frac{1}{4} \\ 2 - 4x_n & \text{for} \frac{1}{4} \le x_n < \frac{1}{2} \\ 4x_n - 2 & \text{for } \frac{1}{2} \le x_n < \frac{3}{4} \\ 4 - 4x_n & \text{for } \frac{3}{4} \le x_n. \end{cases}$$

By induction on n we can prove that for every natural k there exists a finite set (of cardinality not greater than 2^k) of formulas $x_{n+k} = a_i x_n + b_i$, where $a_i = \pm 2^k$ and b_i are integers, and the choice of the formula only depends on the value of x_n.

Let now be known that the sequence is periodic. Then $x_n = x_{n+k} = a_i x_n + b_i$ for some n and k. Since $a_i \ne 1$ in the equation $x_n = a_i x_n + b_i$, then, $x_n = \frac{b_i}{1-a_i}$ is a rational number. Furthermore, $x_n = a_j x_1 + b_j$ and, therefore, $x_1 = \frac{x_n - b_j}{a_j}$ is also rational.

57.10.3. Let us prove by induction on n that for $n \le 665$ it is possible to compose a committee of n delegates such that none of its members hit anybody.

The base of induction: $n = 1$. Take any delegate: (s)he did not hit self.

The inductive step: suppose there is a desired committee of $n \le 664$ members. Let us divide the Parliament into 3 groups: 1) the committee, 2) non-members of the committee, hit by the members of the committee, 3) others.

There are not more than n delegates in the 2-nd group; hence the 3-rd group has not less than $1994 - 2n = 666$. If the 3-rd group has a delegate who did not hit any member of the committee, let us introduce him/her in the committee, thus enlarging its number to $n + 1$ persons.

If all members of the 3-rd group hit somebody from the committee, they did not hit each other and we can compose of them a new committee taking any 665 persons.

57.10.5. 7.10.4. Let us make use of the fact that the segments of the tangents drawn from one point to the given circle are equal. Besides, $G_1G_2 = I_1I_2$ as segments of two common outer tangents to the two given circles. We also have

$$\begin{aligned} AK &= AH_1 - KH_1 = AE_1 - KG_1 = (AB - BE_1) - KG_1 = AB - BI_1 - KG_1 \\ &= AB - (BD - DI_1) - KG_1 = AB - BD + DI_1 - KG_1. \end{aligned}$$

Analogously, $AK = AC - CD + DI_2 - KG_2$. Let us add these two equalities:

$$\begin{aligned} 2AK &= AB + AC - (BD + CD) + (DI_1 + DI_2) - (KG_1 + KG_2) \\ &= AB + AC - BC + II_1 - G_1G_2 = AB + AC - BC, \end{aligned}$$

hence, AK does not depend on D.

57.10.5. a) For example, consider the following polygon composed of the 3 rooms of area 1 and a sufficiently thin corridor of total area $\varepsilon < 0.1$:

FIGURE 435. (Sol. 57.10.5)

It is clear that any chord cuts off it a part consisting of not more than one room and all corridors. The area of such a part is $S < 1 + \varepsilon < \frac{1}{2}(3 + \varepsilon)$.

b) Let us select a direction in the general position (not coinciding with the directions of sides and diagonals of the polygon) and consider cords of this direction only; let it be vertical. Take an arbitrary chord and let us move it so as to equate the areas.

If nothing happens, the areas will become equal. It is, however, possible that at some moment there will be not two possibilities (ahead or back) but three, but even in this case we get not less than one third of the area along one of the directions.

There can not be more than three ways for movement due to the generality of the position of the chord (it never passes through two vertices).

57.11.3. Clearly, the lower boundary of the disc required is given by equation $y = r - \sqrt{r^2 - x^2}$ and to touch the bottom it is necessary and sufficient that

$$r - \sqrt{r^2 - x^2} \geq x^4 \text{ for } |x| \leq r.$$

Squaring we get an equivalent system of inequalities for $|x| \leq r$:

$$\begin{cases} x^6 - 2rx^2 + 1 \geq 0 & (*) \\ x^4 \leq r & (**) \end{cases}$$

The inequality $(*)$ is satisfied on the segment $[-r, r]$ if and only if the least value of the function $y(x) = x^6 - 2rx^2 + 1$ on this segment is nonnegative. With the help of the derivative we easily find:

$$y_{\min} = \begin{cases} y(\sqrt[4]{\frac{2r}{3}}) & \text{for } \sqrt[4]{\frac{2r}{3}} \leq r \\ y(r) & \text{for } \sqrt[4]{\frac{2r}{3}} > r \end{cases} = \begin{cases} 1 - 2(\frac{2r}{3})^{2/3} & \text{for } \sqrt[3]{\frac{2}{3}} \leq r \\ r^6 - 2r^3 + 1 & \text{for } \sqrt[3]{\frac{2r}{3}} > r \end{cases}$$

We also see that $r^6 - 2r^3 + 1 \geq 0$ for all r and $1 - 2(\frac{2r}{3})^{2/3} \geq 0$ for $r \leq \frac{3}{4}\sqrt[3]{2}$.

Since $\sqrt[3]{\frac{2}{3}} < \frac{3}{4}\sqrt[3]{2}$ (this follows from an obvious inequality $\frac{2}{3} < (\frac{3}{4})^3 \cdot 2$), the inequality $(*)$ holds on the whole segment $[-r, r]$ for $r \leq \frac{3}{4}\sqrt[3]{2}$.

Similarly, the inequality $(**)$ is equivalent to $r^4 \leq r$ or $r \leq 1$.

Since $\frac{3}{4}\sqrt[3]{2} < 1$, the final answer is $r \leq \frac{3}{4}\sqrt[3]{2}$.

57.11.4. The homothety with center A and coefficient 2 sends the initial polyhedron V into a polyhedron W of 8 times greater volume. By convexity all 8 transported polyhedra lie inside of W.

Their union does not, however, cover a neighborhood of vertex A in W. Indeed, let A_i be the vertex of V distinct from A, l the line AA_i. The intersection of the convex polyhedron V with l is the segment AA_i. Under the translation by vector $\overrightarrow{AA_i}$ the intersection with l will also be traslated by $\overrightarrow{AA_i}$, hence, A will remain non-covered and its distance from the translated polyhedron will be equal to some $r_i > 0$. Now, if $r > 0$ but less than all the r_i, the ball of radius r centered in A does not intersect with any of the 8 translated polyhedra.

The sum of volumes of translated polyhedra is equal to the volume of W while the volume of their union is less than that; hence, the intersection of at least two of these polyhedra contains their inner points.

57.11.5. All the three points indicated satisfy the equation of a line

$$\overrightarrow{AX} \times \vec{a} + \overrightarrow{BX} \times \vec{b} + \overrightarrow{CX} \times \vec{c} + \overrightarrow{DX} \times \vec{d} = 0.$$

In this equation the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are unit ones and instead of A, B, C, D we can take any points on the lines a, b, c, d, respectively; the vector product does not change, see Fig. 436.

FIGURE 436. (Sol. 57.11.5)

For example, the intersection point of the bisectors B and D is

$$\begin{cases} \overrightarrow{DX} \times (\vec{c} + \vec{d}) = 0 \\ \overrightarrow{BX} \times (\vec{b} + \vec{a}) = 0 \end{cases}$$

or, which is the same,

$$\begin{cases} \overrightarrow{CX} \times \vec{c} + \overrightarrow{DX} \times \vec{d} = 0 \\ \overrightarrow{BX} \times \vec{b} + \overrightarrow{AX} \times \vec{a} = 0. \end{cases}$$

Here we have made use of the fact that $\overrightarrow{DX} \times \vec{c} = \overrightarrow{CX} \times \vec{c}$ (since $\overrightarrow{DC} \parallel \vec{c}$) and $\overrightarrow{BX} \times \vec{a} = \overrightarrow{AX} \times \vec{a}$.

57.11.6. It suffices to find for every k an m such that $(2^m + 1) \vdots 5^k$ (we will find such an m later).

If $2^m + 1 = 5^k b$, where $b \in \mathbb{Z}$, then 2^{m+k} satisfies the conditions of the problem.

To prove it, observe that

$$2^{m+k} = (2^{m+k} + 2^k) - 2^k = (2^m + 1)2^k - 2^k = b \cdot 5^k \cdot 2^k - 2^k = 10^k b - 2^k.$$

We have represented 2^{m+k} as the difference of a number that ends with k zeroes and 2^k.

Let 2^k be expressed as an l-digit number ($l \leq k$). Then among the last k digits of the difference there are not less than $k-l$ nines; namely, the $(l+1)$-st to the k-th counting backwards. It suffices to prove that $l \leq \frac{k}{2}$. For $k = 2, 3, 4, 5$ this is subject to an easy check. In the general case $2^k \geq 10^{l-1}$, $10^k > 8^k = (2^k)^3 \geq 10^{3(l-1)}$, $k > 3(l-1)$, $l < \frac{k}{3} + 1$. For $k \geq 6$ we have $l < \frac{k}{3} + 1 \leq \frac{k}{3} + \frac{k}{6} = \frac{k}{2}$.

It remains to find m.

Lemma. *If p is odd and $(a+1) \vdots p^n$, then $(a^p + 1) \vdots p^{n+1}$.*

Proof. In the product

$$a^p + 1 = (a+1)(a^{p-1} - a^{p-2} + a^{p-3} - \cdots + 1) = (a+1)((-a)^{p-1} + (-a)^{p-2} + \cdots + 1)$$

the left factor is divisible by p^n, while the right one has p summands each of which is congruent to 1 modulo p (since $-a \equiv 1 \pmod p$); hence, it is divisible by p. Q.E.D.

Corollary. *If p is odd and $(a+1) \vdots p$ then $(a^{p^n} + 1) \vdots p^{n+1}$.*

Prooffollows from Lemma by induction on n. Q.E.D.

Now, setting $a = 2^2$, $n = k-1$, $p = 5$ and applying Corollary, we see that $((2^2)^{5^{k-1}} + 1) \vdots 5^k$ wherefrom $m = 2 \cdot 5^{k-1}$.

58.8.1. Let denezhka be the unit of the then monetary system (actually, it was a half of it); let b and kv be the initial prices of bread and kvas, respectively. Then

$$1 = b + kv.$$

After the first inflation the prices satisfy

$$1 = (\frac{b}{2} + kv) \cdot 1.2.$$

The equations imply:

$$2b = kv.$$

Let us express denezhka in terms of kvas: $1 = 1.5 \cdot kv$. The price of kvas afterthe second inflation becomes $kv \cdot 1.2 \cdot 1.2 = 1.44 \cdot kv$. Thus, kvas became too expensive after the second inflation.

58.8.2. Let $a_n = 100\underbrace{1\ldots1}_{n\text{ times}}7$. Clearly, $a_n - a_{n-1} \vdots 53$ and $a_1 = 1007$ is also divisibly by 53. This implies that all our numbers are divisible by 53.

58.8.3. Consider the case when $\angle COB < 60°$. Then quadrilaterals $ABCO$ and $BCDO$ are convex, see Fig. 437. Let N and P be the midpoints of BO and CO, respectively. Then $\triangle KNL = \triangle LPM$ since $KN = \frac{1}{2}AO = \frac{1}{2}BO = LP$ and $LN = \frac{1}{2}CO = \frac{1}{2}DO = MP$ and, moreover, $\angle KNL = \angle AOC = 120° + \angle BOC = \angle BOD = \angle LPM$. Therefore, $KL = LM$.

Moreover, $\angle KLM = \angle KLN + \angle NLP + \angle PLM = \angle KLN + \angle BNL + \angle NKL = 180° - \angle BNK = 60°$. Thus, in $\triangle KLM$ we have: $KL = LM$; $\angle KLM = 60°$, hence, $\triangle KLM$ is an equilateral one.

The case when $\angle COB > 60°$ is treated similarly, and the case when $\angle COB = 60°$ is much simpler than these two cases.

58.8.4. The volume of the box of size $11 \times 13 \times 14$ is equal to 2002, i.e., it suffices. The total area of its faces is equal to $2 \cdot (11 \times 13 + 11 \times 14 + 13 \times 14) = 958$.

Figure 437. (Sol. 58.8.3)

Figure 438. (Sol. 58.8.6)

58.8.5. It suffices to prove that the price of the delivery does not vary under a transposition of two consecutive routes. Indeed, applying such transpositions we can ensure any order of the routes.

Let the bus visited consequtevely villages M and N and delivered goods weighting m and n, respectively. For all the other loads the transposition of routes $M \longleftrightarrow N$ does not affect the price of delivery; the load $m+n$ will be carried over the distance m and load n over the distance $m+n$ (from M to the town, then to N). Hence, the price — $(m+n)m+n(m+n)=(m+n)^2$ — does not depend on the order the bus visited M and N.

58.8.6. Let us assume that N lies on AB, A lies on AF. Observe that $FK=AN$. Let su select points P, Q, R and Ton BC, CD, DE and EF, respectively, so that $FK=AN=BP=CR=DS=ET$, see Fig. 438. Then $\angle KBN=\angle TAK$, $\angle KCN=\angle SAT$, $\angle KDN=\angle RAS$; $\angle KEN=\angle PAR$, $\angle KFN=\angle NAP$. This implies

$$\begin{aligned}&\angle KAN+\angle KBN+\angle KCN+\angle KDN+\angle KEN+\angle KFN=\\&\angle KAN+\angle TAK+\angle SAT+\angle RAS+\angle PAR+\angle NAP=\\&\angle KAN+\angle KAN=120^\circ+120^\circ=240^\circ.\end{aligned}$$

58.9.1. Observe that

$$19\times 6\underbrace{3\dots3}_{n\text{ times}}2=20\times 6\underbrace{3\dots3}_{n\text{ times}}2-6\underbrace{3\dots3}_{n\text{ times}}2=12\underbrace{6\dots6}_{n\text{ times}}40-6\underbrace{3\dots3}_{n\text{ times}}2=120\underbrace{3\dots3}_{n-1\text{ times}}08.$$

Hence, the number with 3's inserted is divisible by 19 for any natural n.

58.9.2 and **58.10.2.** (Cf. Problem 18.1.7.3.) Select an arbitrary point A on BC and draw segment AA'. On AB, let us mark points C' and C'' such that $BC'=BA'$ and $AC''=BA'$. From equality of triangles $AA'B$, $CC'B$ and $CC''A$ it follows that $CC'=CC''=AA'$. The Pythagorus theorem easily implies that there is no other segments of the same length as AA' with one endpoint C and the other one on AB.

Figure 439. (Sol. 58.9.2)

The intersection point of AA' with CC' belongs to the height of $\triangle ABC$ dropped from B. It is easy to see that all points on this height satisfy the condition of the problem. Let P be the intersection point of AA' with CC''. Since $\triangle AA'B=\triangle CC''A$, it follows that $\angle PAC''=\angle PAC$; this implies (see Fig. 439)

$$\angle APC=180^\circ-(\angle PAC+\angle PCA)=180^\circ-(\angle PAC+\angle PAC'')=180^\circ-\angle CAB=120^\circ.$$

This means that P belongs to the circle through points A, C and the intersection point of heights of $\triangle ABC$. The other way round, all points on the arc of this circle that lies inside $\triangle ABC$ satisfies the conditions of the problem. (Prove this on your own.)

58.9.3. It is clear that the sides of the strips should be parallel to the sides of the given rectangle. Let us assume that the dimensions of rectangles are given in cm and consider two cases.

$1^\circ: n < 1995$. Observe that in this case the length of the longest strip that fits into the rectengle is 1995 cm. But different strips each shorter than 1995 cm can cover not more than $1 + 2 + \cdots + 1995 = \frac{1995 \cdot 1996}{2}$ cm^2 of the rectangle's area. Since the area of the given rectangle is equal to $1995 \cdot n$ cm^2, it follows that $\frac{1995 \cdot 1996}{2} \geq 1995 \cdot n$ implying $n < \frac{1996}{2}$.

It is easy to construct an example of cutting for $n < 998$: first, we cut the rectangle into n strips of length 1995 cm; then cut the k-th strip into two bits of lengths $k-1$ and $1995 - k + 1$ for $k = 1, 2, \dots, 1995$ (i.e., the first and the last strips are actually not cut at all).

$2^\circ: n \geq 1995$. Let us argue similarly; in this case the length of the longest strip that fits into the rectengle is n cm. Since different strips each shorter than n cm can cover not more than $\frac{n(n+1)}{2}$ cm^2, it follows that $\frac{n(n+1)}{2} \geq 1995 \cdot n$, i.e., $n > 2 \cdot 1995 + 1$.

We similarly to case 1° construct an example of cutting for $n > 2 \cdot 1995 + 1$: first, we cut the rectangle into 1995 strips of length n cm; then cut the k-th strip into two bits of lengths $k - 1$ and $n - k + 1$ for $k = 1$, 2, ... , n.

58.9.4. Suppose the contrary. Then $p = a + b + c + d$ for a prime p. Let us multiply the equality by a, we get $ap = a^2 + ab + ac + ad = (a + c)(a + d)$. The product on the right is divisible by p, hence one of the factors, say, $a + c$, is divisible by p. But then $a + c \geq p$ and the formula $p = (a + c) + (b + c) > p$ is self contradictory.

58.9.5 and **58.10.4.** Assume the contrary and consider the least number n such that having made n cuts we can we can obtain pairwise nonequal triangles from a foursome of identical right triangles. Let us assume that the given triangles are just these ones.

Observe for the future that the order in which we cut the triangles is irrelevant (in the sence that the final result does not depend on the order of cuting).

Since initially we had four *equal* triangles, we have to cut three of them. Assume that the first three cuts go across these three triangles. As a result we get two triples of identical triangles. In each triple we have to cut two of the triangles. Having done the cuts we again get on our hands a foursome of equal triangles, as is easy to see.

We have $n - 7$ cuts at our disposal to cut the triangles into paiwise nonequal but this contradicts to the minimality of n. The contradiction implies that at every move there are two equal triangles.

58.9.6. a) The cook introduced for her convenience an imaginary weightless can. She then numbered the real cans 1 trough 81 in order of increase of their weight (she did this while it was known which can contains what). To convince us that she correctly numbered the cans she divided the cans into three groups of 27 cans each. The first group — cans 1 to 27 — the lightest; the second one — cans 28 to 54 — intermedeate; the third one — cans 55 to 81 — the heaviest. Then she placed the first group on the left pan and the third group on the right pan. The balance shows the greatest difference in weight between groups of 27 cans.

This difference is easy to get from the inventory and it is obtained in one case only. So if we get the greatest difference of weights between two sets of 27 cans, this would mean that we have found the above three groups: two extremes are on the balance pans, the remaining cans constitute the intermedeate group.

Then the cook divided each group into three piles, 9 cans in each pile. She then placed the lightest piles from each group (i.e., the cans 1 – 9, 28 – 36 and 55 – 63) on the left pan and the heavyest piles from each group (i.e., the cans 19 – 27, 46 – 54 and 73 – 81) on the right pan. This yields the greatest differens of weights for collections of 9 cans from each group. So we should get the greatest difference of weights between two unions of three sets of 9 cans.

Thus, after the second weighing we got 9 piles ordered in order of increase of their weight, each pile with 9 cans.

In the third weighing the cook divided the cans into 27 piles and placed the lightest three cans from each pile on the left pan, while the heaviest cans on the right pan. Thus she distinguished in each pile the lightest three can, the heaviest three and the internediate three.

In the fourth weighing the cook placed the lightest can from each 3-can pile on the left pan, while the heaviest cans on the right pan. Upon viewing the balace's scale the brightest geologists got convinced that the cook had numbered the cans correctly and then spent the whole season trying to convince the others.

b) Observe that each weighing divideds the cans into three groups: those on the left pan, those on the right pan and the other cans. After the first weighing one of such groups contains not less than a third of all cans (≥ 27 cans); for the geologists (except the cook) the cans are indistinguishable. Under the second weighing this group also splits into 3 groups each with not less than 9 cans (also indistinguishable for all except the cook).

Under the third weighing not less than 3 of these cans will enter one of the new groups. Therefore, 3 weighings is not enough.

58.10.1. a) Let $\sin a = x$. All such a are of the form $(-1)^k \arcsin x + k\pi$ for $k \in \mathbb{Z}$. Hence, $\frac{a}{2} = \frac{1}{2}((-1)^k \arcsin x + k\pi)$ corresponding to 4 points on the circle:

$$\frac{1}{2}\arcsin x, \quad \frac{1}{2}\arcsin x + \pi, \quad \frac{1}{2}(\pi - \arcsin x), \quad \frac{1}{2}(3\pi - \arcsin x).$$

Therefore, $\sin \frac{a}{2}$ can not assume more than 4 distinct values; e.g., for $x = \frac{\sqrt{2}}{2}$ we have $a = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$.

b) Similarly to heading a): we see that $\frac{a}{3} = \frac{1}{3}((-1)^k \arcsin x + k\pi)$ for $k \in \mathbb{Z}$. The 6 points on the circle correspond to these values of $\frac{a}{3}$:

$$\frac{1}{3}\arcsin x,\ \frac{1}{3}\arcsin x + \frac{2\pi}{3},\ \frac{1}{3}\arcsin x + \frac{4\pi}{3},\ \frac{1}{3}(\pi - \arcsin x),\ \frac{2\pi}{3} + \frac{1}{3}(\pi - \arcsin x),\ \frac{4\pi}{3} + \frac{1}{3}(\pi - \arcsin x).$$

Since the sum of the anles of the $1 - J$ and $5 - J$ points is equal to π, then sines of these points coincide; similarly, the sines of the $2 - J$ and $4 - J$ points with the sines of the $3 - J$ and $6 - J$ points. Hence, $\sin \frac{a}{3}$ can not assume more than 3 distinct values; whereas it can assume 3 distinct values: e.g., for $x = 0$ we have $a = 0, 2\pi, 4\pi$.

58.10.3. Let the sides of the trapezoid are AB and CD. Denote by M and N the other intersection points of AC and BD, respectively, with the circles, see Fig. 440. Without loss of generality we can concider the case when $AB > CD$, $M \in KA$, $N \in KB$.

Recall a theorem on the tangent and a secant: *the squared lengths of the tangents from point K to the circle are equal to $KM \cdot KA$ and $KN \cdot KD$.* Hence, we have to prove that $KM \cdot KA = KN \cdot KD$.

Since $\angle AMB$ subtends the diameter, we see that $\angle AMB = 90^\circ$. Hence, $\angle BMC = 90^\circ$. Similarly, $\angle BNC = 90^\circ$. Therefore, points B, N, M and C lie on the circle with diameter BC; this implies that $\angle CMN = \angle CBN = \angle BDA$ because $BC \parallel AD$.

But then $\angle AMN + \angle NDA = 180^\circ$, therefore, points A, M, N and D lie on one circle. By a theorem on the product of the whole section by its outer part, $KM \cdot KA = KN \cdot KD$, as required.

FIGURE 440. (Sol. 58.10.3) FIGURE 441. (Sol. 58.11.7)

58.10.5. Assume the contrary. If the numbers a, b, c have a common divisor d, we can replace them by $\frac{a}{d}, \frac{b}{d}, \frac{c}{d}$; hence, without loss of generality we can assume that the common divisor of a, b and c is equal to 1 or -1. One of the numbers a, b, c is not equal to 1 or -1; let it be a.

Let p be a prime divisor of a. Then $abc(\frac{a}{d} + \frac{b}{d} + \frac{c}{d}) = a^2c + b^2a + c^2b \,\vdots\, p$. Therefore, p divides c^2b which means that p divides either c or b. Let, for example, $b \,\vdots\, p$. Then by hypothesis $c \not\vdots\, p$.

Let p^r is the maximal power of p that divides a, let p^s is the maximal power of p that divides b. We may assume that $r \leq s$. Then p^{r+s} is the maximal power of p that divides a; moreover, $a^2b \,\vdots\, p^{r+s}$.

Since $r \leq s$, it follows that $b^2c \,\vdots\, p^{r+s}$. Hence, $c^2a \,\vdots\, p^{r+s}$. But $c \not\vdots\, p$ and, therefore, $a \,\vdots\, p^{r+s}$.

But by hypothesis p^r is the greatest power of p that divides a: contradiction.

ANOTHER SOLUTION. Observe that

$$(x - \frac{a}{b})(x - \frac{b}{c})(x - \frac{c}{a}) = x^3 - (\frac{a}{b} + \frac{b}{c} + \frac{c}{a})x^2 + (\frac{a}{b} + \frac{b}{c} + \frac{c}{a})x - 1$$

is a polynomial with integer coefficients whose roots are rational numbers $\frac{a}{b}$, $\frac{b}{c}$ and $\frac{c}{a}$. It is known that (prove it!) *the rational numbers that are roots of a polynomial with integer coefficients and leading coefficient* 1 *are integers*. Hence, $\frac{a}{b}$, $\frac{b}{c}$ and $\frac{c}{a}$ are integers. But their product is equal to 1, hence, $\frac{a}{b} = \frac{b}{c} = \frac{c}{a} = 1$. Thus, $a = b = c = 1$, as was required.

58.10.6. First of all, observe that the result does not depend on the order in which we push the buttons. Let us carry out an induction on n, where n is the number of bulbs. For $n = 1$ the statement of the problem is obvious.

Let the statement is true for $n - 1$ bulbs. Let us prove it for n bulbs. By inductive hypothsis we can switch off all of them, except, perhaps, one bulb assigned before. Denote the set of buttons we have to push for this by S_i, where i is the number of the assigned bulb (it will, perhaps, continue to glow).

If for some set S_i all the bulbs go out, we are done. Otherwise, after each S_i only the i-th bulb is glowing.

Let B be the set of bulbs glowing initially. If we now apply S_i this will switch out the set B and light the i-th bulb. If we apply S_i in another situation, this would mean that we changed the state of each bulb in the set B and than changed the state of i.

We can, however, change the the order of the operations: first change the state of i and then change B: the result will be the same. (Observe that it does not matter if $i \in B$ or not.)

Let us see what will happen if we first push all the buttons from S_i and then from S_j. This will change first the state of i, then B becomes B', then B' becomes B again and, lastly, we change the state of j. Thus, application of S_i and S_j (in either order) changes the states of the i-th and j-th bulbs.

By hypothesis, there is a button T connected with an odd number of bulbs; let their number be $i_1, \dots, i_{2k-1}$. Let us push the buttons form S_{i_1}: exactly the i_1-th bulb is glowing. If now we push the buttons from S_{i_2} and then those from S_{i_3} (this will light the bulbs i_2 and i_3); next we push the buttons from S_{i_4} and S_{i_5}, etc. We thus switch on exactly the bulbs the button T is connected with. Now, push T. All the lights will go out.

58.11.1. $(x + y - z) + (x - y + z) = 2x$, implying $\frac{(x+y-z)}{2} + (x - y + z) \geq x$, since $|a + b| \leq |a| + |b|$. Consider similar inequalities for x replaced with y and z and add the three of them. We get the statement desired.

58.11.2. a) Let us paint the edges of the base clockwise: the first edge first color, the second one second color, the third one third color; then the fourtht one the first color again, etc. Since $1995 \,\vdots\, 3$ our painting terminates.

Let us paint each edge of the top face the same color the edge of the base under it. Let us paint each lateral edge the color different from the colors of the edges connected with it.

Clearly, the colors of lateral edges repeat with period 3; hence, in each lateral face the lateral edges are of two distinct colors, the edges of the top and bottom faces of the third color, as was required.

b) Suppose the prizm is painted as required. If the base has no three consequtive edges of different colors, then the neighboring edges of each edge from the base are of the same color. This means that the colors of base edges are alternating two colors, the third solor does not enter into play: contradiction.

If the base has three consequtive edges of different colors, say, the first of the first color,the second one second color, the third one third color; then the lateral edge through vertex 12 is of color 3, that through vertex 23 is of color 1. This means that the color of the second edge of the top face is of color 2 (since it can not be either color 1 or 3).

The lateral edge through vertex 34 is of color 2 (since there should be 3 colors in each face) and, therefore, the fourth edges of the top and bottom faces are of color 1 as distinct edges at each vertex.

Repeat this argument for the next edges: 234, 345, etc. We see that the bottom face should be painted with period 3, as in heading a) but 1996 is not divisible by 3 so our process will never terminate.

58.11.3. Denote vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ by $\vec{b}$ and $\vec{c}$, respectively. Then

$$\overrightarrow{AM} = \frac{\vec{b} + \vec{c}}{2}; \quad \overrightarrow{AL} = \frac{\vec{b} \times \vec{c} + \vec{c} \times \vec{b}}{\vec{b} + \vec{c}}$$

implying that $\frac{AK}{AM} = \frac{LC}{MC} = 2\frac{c}{\vec{b}+\vec{c}}$ since $LK \parallel CA$ and, therefore, $AK = \frac{c\times(b+c)}{\vec{b}+\vec{c}}$. Then $CK = \frac{\vec{b}\times\vec{c}+\vec{c}\times\vec{b}}{\vec{b}+\vec{c}}$. For the scalar product of vectors we have

$$CK \cdot AL = \frac{(\vec{b} \times \vec{c} - \vec{c} \times \vec{b})(\vec{b} \times \vec{c} + \vec{c} \times \vec{b})}{(\vec{b} + \vec{c})^2} = \frac{(\vec{b} \times \vec{c})^2 - (\vec{c} \times \vec{b})^2}{(\vec{b} + \vec{c})^2} = 0.$$

Hence, $AK \perp AL$.

58.11.4. Let us prove by induction that it is possible to divide the left segment into wight and black subsegments so that the sum of integrals of the left polynomial of degree not higher than n over wight segments was equal to that over black segments.

For $n = 0$ divide the segment in halves. Suppose, we did it for some n. Observe that the segment already divided can be translated or uniformly zoomed with coeffitient either < 1 or < 1 without violating the equality of integrals. Indeed, such operations are described by a linear change of variables; it does not alter the degree of the polynomial.

For $n + 1$ we divide the segment in halves and paint each half as for n. Now, leave the left half intact and repaint the right half-segment other way round: wight$\leftrightarrow$black.

For the polynomial $f(x)$ the difference of its integral over the wight and black segments is equal to the difference of integrals of $f(x) - f(x - \frac{a}{2})$, where a is the length of the initial segment, over the wight and black subsegments of the left half of the segment.

But if $\deg f(x) \leq n + 1$, then $\deg f(x) - f(x - \frac{a}{2}) \leq n - 1$ and the difference is equal to 0.

58.11.5. Let us prove that $n > 1994$ is impossible. Observe that each next symbol in A is uniquely recovered from the preceiding 1994 ones. Indeed: 1995 symbols constitute the period A and the sets of symbols in all periods are identical (only the order can change). Hence, from the known 1994 symbols of the set we uniquely recover the remaining symbol. The same applies to the preceiding symbol.

Suppose that all the segments of length 1995 from B are contained in A. Then each symbol from B can also be recovered from 1994 symbols that preceed or follow it according the same law as for A. Let us cut from A a bit corresponding to a piece of length 1995 from B and translate B so that the corresponding symbols from A and B coincided. Then by the above their immediate symbols should coincide. Hence, *their* neighbors coincide, etc. Thus, the sequences completely coincide and have period 1995

58.11.6. Set $n = 3^{2^t} - 2^{2^t}$, where $t > 1$. We see then that n is not a prime:

$$n = (3^{2^{t-1}} - 2^{2^{t-1}}) \cdot (3^{2^{t-1}} - 2^{2^{t-1}}).$$

Make use of the fact that

$$x^k - y^l \,\vdots\, x^l - y^l \qquad \text{for natural } k \text{ and } l \text{ and distinct integers } x \text{ and } y.$$

Indeed, if $k = lm$ for an integer m, the statement is equivalent to divisibility of $u^m - v^m$ by $u - v$ for $u = x^l$ and $v = y^l$.

Now, if we prove that $n - 1$ is divisible by $l = 2^t$ this will show that $3^{n-1} - 2^{n-1}$ is divisible by $3^{2^t} - 2^{2^t}$; this is just what we need.

Since the summand 2^{2^t} is divisible by 2^t, it suffices to prove that $3^{2^t} - 1 \,\vdots\, 2^t$. Let us prove this by induction on t.

For $t = 1$ this is obvious. Suppose the statement holds for $t = r$ and consider $3^{2^{r+1}} - 1 = (3^{2^r} - 1)(3^{2^r} + 1)$. The first factor is divisible by 2^r and the second one by 2; hence the product is divisible by 2^{r+1}, as was required.

58.11.7. Take 6 right papallelepipeds with two edges of length 1 and the thrid one of length 10. Let us paint the unit square in the middle of one of rectangular faces of each parallelepiped. In space, consider a transparent unit cube with edges parallel to the coordinate axes. Let Ox be directed straight ahead, Oy to the right, Oz upwards. Let us align the painted faces of the parallelepipeds with the cube's faces, so that the first pair of parallelepipeds were on top and under the cube and the longer edges of the parallelepipeds were parallel to Ox, the second pair of parallelepipeds in front of and behind the cube and with the longer edges of the parallelepipeds parallel to Oy, the third pair of parallelepipeds on the left and on th eright of the cube and with the longer edges of the parallelepipeds parallel to Oz. We get a "cross" as on Fig. 441. From the center of the cube one can only see the painted squares and can not see the vertices of the papallelepipeds. If we push the papallelepipeds away from the cube perpendicularly to its faces by 0.1, we slightly widen the perspective, but still can not see the vertices.

By bridging the unseen parts, we make a polygon from the papallelepipeds.

60.8.1. The conditions imply that there is a horizontal on which there stands exactly one figure, there is another horizontal on which there stands exactly two figures, etc., and there is a horizontal completely filled with 8 figures. Let us number the diagonals so that the diagonal's number is equal to the number of the figures standing on it. Mark the figure on the first diagonal. Since there are two figures on the second diagonal, at least one of them can be marked, do it. Since there are three figures on the third diagonal, at least one of them can be marked, do it; and so on.

60.8.2. The trip along the road and the passway (tour and retour) takes 16 hours. Hence, if one departs immediately after the first crater finished eruption, it will not be dangerous. The trip along thethe passway (there and back) takes 8 hours. Hence, if one departs immediately after the second crater finished eruption, it will not be dangerous.

To ensure safety, it suffices to Vanya to find a moment of time when by the time he takes the road the first crater stopped to erupt, and 4 hours later, by the time he takes the passwalk, the second crater stopped to erupt. Let us find such a moment.

The first crater starts to erupt at the beginning of the 1-st, 19-th, 37-th, etc., hours. The second crater starts to erupt at the beginning of the 1-st, 11-th, 21-th, 31-th, 41-th, etc., hours. Hence, Vanya can depart at the beginning of the 38-th hour.

60.8.3. Let points L and K be symmetric to point M with respect to lines OX and OY, respectively. Then the points K, P and N lie on one straight line and $NK = NP + PK = NP + PM$. Similarly, Points the points N, Q and L lie on one line and $NL = NQ + QL = NQ + QM$.

It remains to prove the equality of triangles KON and LON in two legs and the angle between them. Indeed, $KO = OM = OL$, the side ON is the common one, while

$$\angle KON = \angle KOP + \angle PON = \angle POM + \angle MOQ = \angle NOQ + \angle QOL = \angle NOL.$$

60.8.4. Such a number does exist. Let us multiply the odd numbers 1001 to 1999. Since there are 500 of them and each of them is < 2000, their product is smaller than

$$2000^{500} = 1^{500} \cdot 10^{1500} = 32^{100} \cdot 101500 < 100^{100} \cdot 10^{1500} = 10^{1700}.$$

Let us write to the end of this number several 0's, a 1 and three more 0's so that the total number of digits were equal to 1997.

If we do not replace the last digit of the number obtained, call it N, the replaced number remains even, hence, non-prime. If we replace the last three 0's with an odd number $\overline{abc}$, then the last four digits form the number $\overline{1abc}$ which divides N.

60.8.5. Let straight lines DE and AB meet at G. Thentriangles DEC and BEG are equal. Hence, $BG = CD = BA$. The triangle GFA is a right one. Therefore, $BF = BA = BG$.

The triangles CBF and FBA are isosceles ones. Therefore, $\angle BCF + \angle BAF = \angle CFA$.

The sum of the angles in quadrilateral $ABCF$ is equal to $360°$. Therefore,and FBA are isosceles ones. Therefore, $\angle CFA = (360° - 40°) : 2 = 160°$. Hence,

$$\angle CFD = 360° - \angle AFC - \angle AFD = 360° - 160° - 90° = 110°.$$

60.8.6. First, let us solve a simpler problem. Let Banker allows the expert to lay coins on the pans not oftener than once. What is the largest number of coins needed to determine the counterfite one using k weighings?

If at a weighing a pan has more than one coin, one can not distinguish the counterfite mong them, since it is forbidden to use the coin twice. Therefore, at each weighing one puts one coin at a pan. If the balance is not in equilibrium, the counterfeit is clearly found, while the equilibrium state of the balance diminishes the number of coins to be examined by 2. Hence, k weighings are sufficient to determine the counterfeit coin of $2k + 1$ coins.

In the initial problem, let at the first weighing there were s coins on each pan. If the balance is not in equilibrium, than, as we have shown, $s \leq 2(n-1) + 1 = 2n - 1$.

If the balance is in equilibrium, we get a problem similar to the initial one for the coins not on the pans and $n - 1$ weighing. Therefore, if $f(n)$ is the answer of the inintial problem, we get a restriction:

$$f(n) = f(n-1) + 2(2n-1).$$

Therefore,

$$f(n) = 2(2n-1) + 2(2n-3+) + \dots + 2 \cdot 3 + f(1).$$

Since, as is easy to verify, $f(1) = 3$, we get $f(n) = 2n^2 + 1$.

60.9.1. Since the shorter side of the triangle is opposit eto the lesser angle, it suffices to prove that if a, b and c are the lengths of the triangle's sides and $a = \frac{1}{3}(b+c)$, then a is the (length of the) shortest side. Indeed, if for example $a \geq b$, then we wouold have had $\frac{1}{3}(b+c) = a \geq b$, implying $c \geq 2b$.

On the other hand, by the triangle inequality $a + b > c$, hence, $\frac{1}{3}(b+c) + b > c$, i.e., $2b >$; contradiction.

60.9.2. Denote the masses of the pieces in the increasing order: $m_1, m_2, \dots, m_9$. Put the 1-st, 3-rd, 5-th and 7-th pieces to the left and the 2-nd, 4-th, 6-th and 8-th to the right. Then

$$m_1 + m_3 + m_5 + m_7 \le m_2 + m_4 + m_6 + m_8,$$

whereas

$$m_1 + m_3 + m_5 + m_7 + m_9 > m_2 + m_4 + m_6 + m_8.$$

Hence, it suffices to cut the 9-th piece.

60.9.3. It turns out that one can compose $\triangle ABC$ of $\triangle AB_1C$, $\triangle BCA_1$ and $\triangle AC_1B$. To acheave this, turn $\triangle AB_1C$ about point C until the image of vertex B_1 coincides with vertex A_1 of $\triangle BCA_1$. Let the image of A under this rotation is A'. Verify that $\triangle A'A_1B \cong \triangle ABC_1$ in two legs and the contained angle.

60.9.4. Denote the length of the distance between the trains by l. Suppose a point P lies in the forest. then all the points at the distance equal to an integer multiple of l (along the circle) also lie in the forest. Therefore, it suffices to determine which points on the segment of length l belong to the forest, since the structure of the forest periodically reproduces itself.

Consider the moment, when a train departs from station B. Let the train nearest to station A be at this moment at distance x from A. Then the whole segment of length x is covered by the forest. Let us show that none of the points of the complemantary segment of length $l - x$ is in the forest. Indeed, when the nearest to A train reaches A, the train nearest to station B occures at distance $l - x$ to B.

The value of x is equal to the residue after the division of the arc $\smile AB$ by l. Since $\overset{\frown}{\to} BA = 240°$, while $l = \frac{360°}{n}$, then $x = \left\{\frac{240°}{360°/n}\right\} = \left\{\frac{2n}{3}\right\}$, where $\{\cdot\}$ denot the fractional part of the number. Hence the answer:

If n is divisible by 3, there is no forest at all; if the residue after division of $n 4by3isequalto1, thentheforestconstitutes2/$ by 3 is equal to 2, then the forest constitutes 1/3 of the road.

60.9.5. The participants can be divided into two groups: those who increased the sum of points (and then not less than by n) and those who decreased it. At least one of these groups contains not less than n sportspersons. Let, for example, such be the first group and x the number of persons in it. If the sum total of their points has increased by D, then

$$D \ge x \cdot n.$$

This sum increased thanks to the encounters of these x sportspersons with the other $2n - x$ sportspersons. Each of these encounters added not more than 1 point, hence,

$$D \le x \cdot (2n - x).$$

By comparing the inequalities, we see that $2n - x \ge n$. But by assumption $x \ge n$. Hence, $x = n$. Therefore, $D = n \cdot n$ and each member of the first group increased the sum of the points (s)he scored exactly by n. The same argument is applicable to the second group.

60.9.6. Let

$$\begin{aligned} F(x) &= a_0 + a_1x + a_2x^2 + \dots, \\ G(x) &= b_0 + b_1x + b_2x^2 + \dots, \\ F(x)G(x) &= c_0 + c_1x + c_2x^2 + \dots. \end{aligned}$$

Clearly, $a_0 = b_0 = 1$. Assume that $b_1 = \dots = b_{t-1} = 1$ and $b_t = 0$. Then $a_1 = \dots = a_{t-1} = 1$ and $a_t = 0$.

Let us show that there are no maximal segments of nonzero coefficients of length shorter than t. Assume the contrary and consider the left-most of such segments: $b_i, \dots b_{i+k}$, where $k + 1 < t$. We get: $c_i = a_0 \cdot b_i$, $\dots, c_{i+k} = a_0 \cdot b_{i+k}$; further, $c_{i+t} = a_t \cdot b_i$. Consider the segment $c_{i+k+1}, \dots, c_{i+t-1}$ of length $\delta = t - k - 1$. Clearly, $0 < \delta < t$.

Let su study how do we get $c_{i+k+1} = a_z \cdot b_y$. Since $z \ge t$, it follows that $y \le i + k + 1 - t < i$. Therefore, b_y lies in one of the segments of length t. By multiplying a_z by all the units from this segment we get a segment of length t containing c_{i+k+1}. Hence, this segment contains at least one of the numbers c_{i+k}, c_{i+t}. Contradiction.

Therefore, the polynomila $G(x)$ (with coefficients b_i) can be represented in the form of the product of the polynomial $1 + x + x^2 + \dots + x^{t-1}$ by the polynomial all of whose coefficients are either zeros or units.

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60.1.1. Consider the ball and its projections on three pairwise orthogonal planes. Let the projection of a point A of the sphere not belong to none of the circles that bound the projections. Then a small circle with the center at A also has this property. Let us cut off the ball the appropriate bit. We get a solid body wich is not the ball but whose projections on the same planes are the same as those of the ball!

One could apply a different argument. Take a ball and its projections in the directions indicated. These projections are discs; while the set of points whose projection is a disc is a cylinder. It suffices then to show tha the intersection of a finite number of cylinders can not be a ball.

In fact, each part of the side surface of the cylinder contains a segment and we know that the intersection of any straight line with the sphere, the surface of the ball, is not a segment.

60.1.2. Let us translate the quadrilateral $ABCD$ by vector $\vec{AC}$. We get the quadrilateral $A * B'C'D'$, where $A' = c$ and the quadrilateral $BB'D'D$ is a parallelogram. By the triangle inequality $BC + CD \geq BD'$ and $BC' + CD \geq B'D$. The equalitie sin the last two equations are only obtained for the parallelogram.

60.1.3.