An interesting and useful inequality for math olympiad

CQT(crycry-tara1995)

10/7/2012

Problem(Vasile):Let $a,b,c \ge 0$ .prove that:

$$(a+b+c)^3 \ge \frac{27}{4}(a^2b+b^2c+c^2a+abc) \text{ (\$1)}$$

I.Solution:

Because the inequality is homogeneous so we can assume $a+b+c=3$ and we must prove:

$$a^2b+b^2c+c^2a+abc \le 4 \; (*)$$

Let x,y,z be a permutation of a,b,c sastisfy $x \ge y \ge z \Rightarrow xy \ge xz \ge yz$

Also we have $x+y+z=3, x,y,z \ge 0$.

So use arrangement inequality we have:

$$a^2b+b^2c+c^2a+abc \le xy.x+xz.y+yz.z+xyz = y(x+z)^2 \; (1)$$

Use AM-GM inequality we have:

$$y(x+z)^2 = \frac{1}{2}2y(x+z)^2 \le \frac{1}{2}\frac{(2y+x+z+x+z)^3}{27} = 4 \; (2)$$

From (1)(2) => (*) is true.so completed prove.

The inequality holds when $a=b=c$ or $c=0, a=2b$ or $b=0, c=2a$ or $a=0, b=2c$.

Note that similar we also have :

$$(a+b+c)^3 \ge \frac{27}{4}(b^2a+c^2b+a^2c+abc) \text{ (\$2)}$$

The inequality holds when $a=b=c$ or $c=0, b=2a$ or $b=0, a=2c$ or $a=0, c=2b$.

*Use ($1) we have a solution for problem 5 in Canada math Olympiad 1999
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=768&sid=da4481f13b80aeb6f70cbabd32df19ae#p768

II.Application:

1.Problem 1(unknown in a book of can_hang2007):

Let $a,b,c \ge 0, a+b+c=3$ .Prove that:

$$2(b^2a+cb^2+ac^2)+3(a^2+b^2+c^2)+4abc \ge 19$$

*Solution(me):

$$\Leftrightarrow 2(ab^2+bc^2+ca^2)+(a^2+b^2+c^2)(a+b+c)+4abc \ge 19$$

$$\Leftrightarrow a^3+b^3+c^3+4abc+3(ab^2+bc^2+ca^2)+(a^2b+b^2c+c^2a) \ge 19$$

$<=> (a+b+c)^3 \ge 19 + 2(abc + a^2b + b^2c + c^2a)$

Use $a+b+c=3$ we need prove:

$$abc + a^2b + b^2c + c^2a \le 4$$

It is inequality ($1) with $a+b+c=3$.

Completed prove.The inequality holds when $a=b=c=1$ or $c=0, b=1, a=2$ or $b=0, a=1, c=2$ or $a=0, c=1, b=2$.

2.Problem 2(nguoivn):

Let $a,b,c \ge 0, a+b+c=3$.Prove that:

$$\frac{a}{b^3+16} + \frac{b}{c^3+16} + \frac{c}{a^3+16} \ge \frac{1}{6}$$

Posted in

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=51&t=470936

*Solution(unknown):

$$\sum_{cyc} \frac{a}{b^3+16} = \frac{1}{16}\sum_{cyc}(a - \frac{ab^3}{b^3+16}) = \frac{1}{16}(3 - \sum_{cyc}\frac{ab^3}{b^3+16})$$

Use AM-GM inequality we have:

$$\frac{ab^3}{b^3+16} = \frac{ab^3}{b^3+8+8} \le \frac{ab^3}{3\sqrt[3]{64b^3}} = \frac{ab^2}{12}$$

$$\frac{bc^3}{c^3+16} = \frac{bc^3}{c^3+8+8} \le \frac{bc^3}{3\sqrt[3]{64c^3}} = \frac{bc^2}{12}$$

$$\frac{ca^3}{a^3+16} = \frac{ca^3}{a^3+8+8} \le \frac{ca^3}{3\sqrt[3]{64a^3}} = \frac{ca^2}{12}$$

So we have:

$$\sum_{cyc}\frac{a}{b^3+16} \ge \frac{1}{16}(3 - \frac{1}{12}\sum_{cyc} ab^2) \ge \frac{1}{6} <=> \sum_{cyc} ab^2 \le 4$$

It is inequality ($2) with $a+b+c=3$.

Completed prove.The inequality holds when $c=0, a=1, b=2$ or $b=0, c=1, a=2$ or $a=0, b=1, c=2$.

3.Problem 3(hungkhtn):

Let $a,b,c \ge 0, a+b+c \le 3$.Prove that:

$$a\sqrt{1+b^3} + b\sqrt{1+c^3} + c\sqrt{1+a^3} \le 5$$

*Solution(unknown):

Use AM-GM inequality we have:

$$a\sqrt{1+b^3}=a\sqrt{(1+b)(1-b+b^2)}\le\frac{a(1+b+1-b+b^2)}{2}=\frac{ab^2+2a}{2}$$
$$b\sqrt{1+c^3}=b\sqrt{(1+c)(1-c+c^2)}\le\frac{b(1+c+1-c+c^2)}{2}=\frac{bc^2+2b}{2}$$
$$c\sqrt{1+a^3}=c\sqrt{(1+a)(1-a+a^2)}\le\frac{c(1+a+1-a+a^2)}{2}=\frac{ca^2+2c}{2}$$

So we need prove:

$$\frac{ab^2+bc^2+ca^2}{2}+a+b+c\le 5 <=> ab^2+bc^2+ca^2\le 4$$

It is inequality ($2) with $a+b+c=3$.

Completed prove.the inequality holds when $c=0,a=1,b=2$ or $b=0,c=1,a=2$ or $a=0,b=1,c=2$.

4.Problem 4(lilteevn):

Let $a,b,c>0,a+b+c=3$.Prove that:

$$\frac{a}{b^2+\sqrt{c}}+\frac{b}{c^2+\sqrt{a}}+\frac{c}{a^2+\sqrt{b}}\ge\frac{3}{3-\sqrt[3]{abc}}$$

Posted in http://boxmath.vn/4rum/f22/bat-dang-thuc-chao-mung-ngay-30-4-a-29346/

*Solution(songvuive):

Use Cauchy inequality we have:

$$\frac{a}{b^2+\sqrt{c}}+\frac{b}{c^2+\sqrt{a}}+\frac{c}{a^2+\sqrt{b}}=\frac{a^2}{ab^2+a\sqrt{c}}+\frac{b^2}{bc^2+b\sqrt{a}}+\frac{c^2}{ca^2+c\sqrt{b}}$$
$$\ge\frac{(a+b+c)^2}{ab^2+a\sqrt{c}+bc^2+b\sqrt{a}+ca^2+c\sqrt{b}}=\frac{9}{ab^2+a\sqrt{c}+bc^2+b\sqrt{a}+ca^2+c\sqrt{b}}$$

So we need prove:

$$3(3-\sqrt[3]{abc})\ge ab^2+a\sqrt{c}+bc^2+b\sqrt{a}+ca^2+c\sqrt{b}$$
$$<=> 9\ge 3\sqrt[3]{abc}+ab^2+a\sqrt{c}+bc^2+b\sqrt{a}+ca^2+c\sqrt{b}$$

Use AM-GM inequality we have: $3\sqrt[3]{abc}\le abc+1+1$

Use Cauchy inequality we have:

$$a\sqrt{c}+b\sqrt{a}+c\sqrt{b}\le\sqrt{(a+b+c)(ab+ac+bc)}\le\sqrt{\frac{(a+b+c)^3}{3}}=3$$

So we need prove:

$$4\ge ab^2+bc^2+ca^2+abc$$

it is inequality ($2) with $a+b+c=3$.

Completed prove.The inequality holds when $a=b=c=1$.

5.Problem 5(nguoivn):

Let $a,b,c\ge 0,a+b+c=3$.Prove that:

$$(ab^3+bc^3+ca^3)(ab+ac+bc)\le 16$$

*Solution(unknown):

$$<=>(ab^3+bc^3+ca^3)^2(ab+ac+bc)^2\le 256$$

Use AM-GM inequality we have:

$$(ab^3+bc^3+ca^3)^2(ab+ac+bc)^2=\frac{1}{2}2(ab+ac+bc)^2(ab^3+bc^3+ca^3)$$

$$\le\frac{1}{2}\frac{[(2(ab+ac+bc)^2+2(ab^3+bc^3+ca^3)]^3}{27}=\frac{4}{27}[(ab+ac+bc)^2+(ab^3+bc^3+ca^3)]^3$$

So we need to prove:

$$(ab+ac+bc)^2+(ab^3+bc^3+ca^3)\le 12$$

We have:

$$(ab+ac+bc)^2+(ab^3+bc^3+ca^3)=(a^2b^2+ab^3)+(b^2c^2+bc^3)+(a^2c^2+ca^3)+2abc(a+b+c)$$

$$=ab^2(a+b)+bc^2(b+c)+ca^2(c+a)+6abc=ab^2(3-c)+bc^2(3-a)+ca^2(3-b)+6abc$$

$$=3(ab^2+bc^2+ca^2)-abc(a+b+c)+6abc=3(ab^2+bc^2+ca^2+abc)$$

So we need prove:

$$ab^2+bc^2+ca^2+abc\le 4$$

It is inequality ($2) with $a+b+c=3$.

Completed prove.The inequality holds when $a=0,b=1,c=2$ or $b=0,c=1,a=2$ or $c=0,a=1,b=2$.

6.Problem 6(sieubebuvietnam):

Let $a,b,c>0,a+b+c=1$.Prove that:

$$\frac{ab}{b+c}+\frac{bc}{c+a}+\frac{ca}{a+b}\le\frac{1}{6(ab+ac+bc)}$$

Posted in

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=52&t=469347

*Solution(me):

We have:

$$(\frac{ab}{b+c}+\frac{bc}{c+a}+\frac{ca}{a+b})(ab+ac+bc)=a^2b+b^2c+c^2a+abc(\frac{b}{b+c}+\frac{c}{a+c}+\frac{a}{b+a})$$

$$=a^2b+b^2c+c^2a+abc(3-\frac{c}{b+c}-\frac{a}{a+c}-\frac{b}{b+a})$$

Use Cauchy inequality we have:

$$\frac{c}{c+b}+\frac{a}{a+c}+\frac{b}{b+a}=\frac{c^2}{c^2+bc}+\frac{a^2}{a^2+ac}+\frac{b^2}{b^2+ab}\ge\frac{(a+b+c)^2}{a^2+b^2+c^2+ab+ac+bc}$$

So:

$$\left(\frac{ab}{b+c}+\frac{bc}{c+a}+\frac{ca}{a+b}\right)(ab+ac+bc)$$
$$\le a^2b+b^2c+c^2a + abc\left(3-\frac{(a+b+c)^2}{a^2+b^2+c^2+ab+ac+bc}\right)$$
$$= a^2b+b^2c+c^2a+abc+\frac{abc(a^2+b^2+c^2)}{a^2+b^2+c^2+ab+ac+bc} \quad (*)$$

Lemma (unknown): $a,b,c>0 \Rightarrow abc(a^2+b^2+c^2)\le \frac{1}{81}(a+b+c)^5$

*Solution(unknown)
Posted inhttp://www.artofproblemsolving.com/Forum/viewtopic.php?f=52&t=449336
Because the inequality is homogeneous so we can assume $a+b+c=3$ and we must prove:

$$abc(a^2+b^2+c^2)\le 3$$

Assume a is $\max(a,b,c)$
let $t=\frac{b+c}{2}$

we have:
$$abc(a^2+b^2+c^2)-at^2(a^2+2t^2)=-a^3\frac{(b-c)^2}{4}-a\frac{(b-c)^4}{8}\le 0$$
So we need prove:
$at^2(a^2+2t^2)\le 3$
with $a+2t=1, t\le 1$
$\Leftrightarrow (t-1)^2(4t^3-6t^2+2t+1)\ge 0$ (it is true with $t\le 1$)
Now return our problem,use lemma with $a+b+c=1$ we have :

$$abc(a^2+b^2+c^2)\le \frac{1}{81}$$

But:

$$a^2+b^2+c^2+ab+ac+bc\ge \frac{2(a+b+c)^2}{3}=\frac{2}{3}$$

So:

$$\frac{abc(a^2+b^2+c^2)}{a^2+b^2+c^2+ab+ac+bc}\le \frac{\frac{1}{81}}{\frac{2}{3}}=\frac{1}{54}$$

So if we want to prove (*) we must prove:

$$a^2b+b^2c+c^2a+abc\le\frac{4}{27}$$

It is inequality ($1) with $a+b+c=1$.

Completed prove.The inequality holds when $a=b=c=\frac{1}{3}$

7.Problem 7(a problem in Inequalities with Beautiful Solutions):

Let $a,b,c\ge 0$, $a+b+c=3$.Prove that:

$$\frac{1}{ab^2+8}+\frac{1}{bc^2+8}+\frac{1}{ca^2+8}\ge\frac{1}{3}$$

*Solution(can_hang or Vasile):

After expanding,this simplifies to:

$$64\ge r^3+(16-5r)A+5r(A+B)$$

With $r=abc, A=ab^2+bc^2+ca^2, B=a^2b+b^2c+c^2a$

Use AM-GM inequality we have:

$$3=a+b+c\ge 3\sqrt[3]{abc} => r\le 1$$

Use Schur inequality we have:

$$27=(a+b+c)^3\ge 3abc+4(a^2b+b^2c+c^2a+b^2a+c^2b+a^2c) => A+B\le\frac{9(3-r)}{4}$$

Use inequality ($2) with $a+b+c=3$,we have:

$$A\le 4-r$$

So we need prove:

$$64\ge r^3+(16-5r)(4-r)+\frac{15r(9-r)}{4} <=> r(r-1)(4r+9)\le 0$$

It is true because $r\le 1$.

Completed prove.The inequality holds when $a=b=c=1$ or $a=0,b=1,c=2$ or $b=0,c=1,a=2$ or $c=0,a=1,b=2$.

8.Problem 8(nguoivn):

Let $a,b,c\ge 0, a+b+c=3$.Prove that:

$$\frac{ab}{\sqrt{b+c}}+\frac{bc}{\sqrt{c+a}}+\frac{ca}{\sqrt{a+b}}+\frac{1}{\sqrt{2}}\ge\frac{(a+b)(a+c)(b+c)}{2\sqrt{2}}$$

*Solution(in a book of can_hang2007 and nguoivn):

Use AM-GM inequality we have:

$$\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{b+c}}+\frac{b+c}{2\sqrt{2}}\ge\frac{3}{\sqrt{2}} => \frac{1}{\sqrt{b+c}}\ge\frac{6-b-c}{4\sqrt{2}} => \frac{ab}{\sqrt{b+c}}\ge\frac{6ab-ab(b+c)}{4\sqrt{2}}$$

Similar we have:

$$\frac{bc}{\sqrt{c+a}}\ge\frac{6bc-bc(c+a)}{4\sqrt{2}} \text{ and } \frac{ca}{\sqrt{a+b}}\ge\frac{6ca-ca(a+b)}{4\sqrt{2}}$$

So we need prove:

$$6\sum ab+4-3abc-\sum_{cyc}ab^2\ge 2(a+b)(a+c)(b+c)$$

We have $2(a+b)(a+c)(b+c)=2[(a+b+c)(ab+ac+bc)-abc]=6\sum ab-2abc$

So we need prove:

$$4\ge ab^2+bc^2+ca^2+abc$$

It is ($2) inequality with $a+b+c=3$.

Completed prove.the inequality holds when $a=b=c=1$ or $a=0,b=1,c=2$ or $b=0,c=1,a=2$ or $c=0,a=1,b=2$.

III.Proposed problem:

1.Problem 1(nguoivn):

Let $a,b,c\ge 0,a+b+c=3$.Prove that:

$$(2+ab^2)^2(2+bc^2)^3(2+ca^2)^2\le 3456$$

2.Problem 2(hungkhtn):

Let $a,b,c\ge 0,a+b+c>0$.Prove that:

$$\frac{a}{4a+4a+c}+\frac{b}{4b+4c+a}+\frac{c}{4c+4a+b}\le\frac{1}{3}$$

Hint:assume $a+b+c=3$ and expand

3.Problem 3(a problem in Inequalities with Beautiful Solutions):

Let $0\le a,b,c\le 1$.Prove that:

$$a(1-b^2)+b(1-c^2)+c(1-a^2)\le\frac{5}{4}$$

4.Problem 4(sieubebuvietnam):

Let $a,b,c>0,a+b+c=1$.Prove that:

$$\frac{ab}{b+c}+\frac{bc}{c+a}+\frac{ca}{a+b}\le\frac{1}{162abc(a^2+b^2+c^2)}$$

Hint(me):Posted in http://www.artofproblemsolving.com/Forum/viewtopic.php?f=52&t=469347

5.Problem 5(unknown):

Let $a,b,c\ge 0,a+b+c=3$.Prove that:

$$\sum_{cyc}\frac{a^2b}{a+b+1}\le 1$$

Hint(me):posted in
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=52&t=468841

6.Problem 6(nguoivn+quykhtn-qa1):
Let $a,b,c\ge 0,a+b+c=3$.Prove that:

$$(a+bc^3)(b+ca^3)(c+ab^3)\le 16$$

Hint(quykhtn-qa1):posted in
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=51&t=213849

Please contact me if there are some mistakes.
Email:crycry.tara1995@yahoo.com
Thank you and best regards.

Hoang Sa,Truong Sa belong to Vietnam