Undergraduate Competitions Putnam

2007

Α

- 1 Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.
- 2 Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola xy = 1 and both branches of the hyperbola xy = -1. (A set S in the plane is called *convex* if for any two points in S the line segment connecting them is contained in S.)
- 3 Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k + 1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.
- 4 A repunit is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is f(n).
- 5 Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either n = 0 or p divides n + 1.
- 6 A triangulation \mathcal{T} of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side of P is a side of exactly one triangle in \mathcal{T} . Say that \mathcal{T} is admissible if every internal vertex is shared by 6 or more triangles. For example

[The included figure shows a heptagon. The first and third vertices are connected by a diagonal. Every vertex except the second (the one "outside" the diagonal) is connected by an edge to a single internal vertex; that internal vertex has 6 edges connected to it. There are 7 triangles in the picture; 6 around the internal vertex and one outside the diagonal.]

Prove that there is an integer M_n , depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.

Β

- 1 Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then f(n) divides f(f(n) + 1) if and only if n = 1.
- 2 Suppose that $f:[0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x) dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left| \int_{0}^{\alpha} f(x) \, dx \right| \le \frac{1}{8} \max_{0 \le x \le 1} |f'(x)|$$

- 3 Let $x_0 = 1$ and for $n \ge 0$, let $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007} . ($\lfloor a \rfloor$ means the largest integer $\le a$.)
- 4 Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^{2} + (Q(X))^{2} = X^{2n} + 1$$

and $\deg P < \deg Q$.

5 Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n), \ldots, P_{k-1}(n)$ (which may depend on k) such that for any integer n,

$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \dots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}$$

 $(\lfloor a \rfloor$ means the largest integer $\leq a$.)

6 For each positive integer n, let f(n) be the number of ways to make n! cents using an unordered collection of coins, each worth k! cents for some $k, 1 \le k \le n$. Prove that for some constant C, independent of n,

$$n^{n^2/2 - Cn} e^{-n^2/4} \le f(n) \le n^{n^2/2 + Cn} e^{-n^2/4}$$