Problem 11800. [AMM, October 2014]. *Proposed by O. Klurman, Montreal, Canada.* Let f be a continuous function from $[0,1]$ into $\mathbb{R}^+$. Prove that

$$\int_0^1 f(x)dx - \exp\Big[\int_0^1 \log f(x)dx\Big] \le \max_{0\le x,y\le 1}\Big(\sqrt{f(x)}-\sqrt{f(y)}\Big)^2.$$

Solution by Borislav Karaivanov, Lexington, SC, and Tzvetalin S. Vassilev, Nipissing University, North Bay, Ontario, Canada. Let $x_k = \frac{k}{n}$ and $y_k = f(x_k)$ for $k = 0,1,\dots,n$. Using the uniform continuity of f on $[0,1]$, we find

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\sum_{k=1}^n f(x_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\frac{y_1+y_2+\cdots+y_n}{n},$$

$$\begin{aligned}\exp\Big[\int_0^1 \log f(x)dx\Big] &= \exp\Big[\lim_{n\to\infty}\sum_{k=1}^n \log f(x_k)(x_k-x_{k-1})\Big]\\ &= \lim_{n\to\infty}\exp\Big[\sum_{k=1}^n \log f(x_k)(x_k-x_{k-1})\Big] = \lim_{n\to\infty}\sqrt[n]{y_1y_2\dots y_n},\end{aligned}$$

and

$$\max_{0\le x,y\le 1}\Big(\sqrt{f(x)}-\sqrt{f(y)}\Big)^2 = \lim_{n\to\infty}\Big(\sqrt{\max_{1\le k\le n} y_k}-\sqrt{\min_{1\le k\le n} y_k}\Big)^2 = \lim_{n\to\infty}\big(\sqrt{M}-\sqrt{m}\big)^2,$$

where $M = \max_{1\le k\le n} y_k$ and $m = \min_{1\le k\le n} y_k$. Therefore, it suffices to show that

$$\frac{y_1+y_2+\cdots+y_n}{n} - \sqrt[n]{y_1y_2\dots y_n} \le \big(\sqrt{M}-\sqrt{m}\big)^2. \tag{1}$$

Regarding the left-hand side of (1) as a function of $\sqrt[n]{y_k}$ for any one of the y_k's, we obtain a function of the form $g(t) = \frac{1}{n}t^n - At + B$ with $A \ge 0$. Since $g'(t) = t^{n-1} - A$, we conclude that g attains its global maximum on $[\sqrt[n]{m}, \sqrt[n]{M}]$ at one of the endpoints. Hence, the left-hand side of (1) is majorized by $\frac{kM+(n-k)m}{n} - \sqrt[n]{M^k m^{n-k}}$ for some $0 \le k \le n$, and it suffices to prove that

$$\frac{kM+(n-k)m}{n} - \sqrt[n]{M^k m^{n-k}} \le \big(\sqrt{M}-\sqrt{m}\big)^2,$$

for any $0 \le k \le n$, or equivalently

$$t^a + (1-a)t - 2\sqrt{t} + a \ge 0, \tag{2}$$

where $a := \frac{k}{n} \in [0,1]$ and $t := \frac{M}{m} \ge 1$. Denoting the left-hand side of (2) by $h(t)$, we find

$$h'(t) = at^{a-1} + (1-a) - \frac{1}{\sqrt{t}} \quad \ge at^{a-1} + (1-a)t^{-a} - \frac{1}{\sqrt{t}} = t^{-a}(t^{a-\frac12}-1)(at^{a-\frac12}-(1-a)) \ge 0,$$

for the last two factors are non-negative when $a \ge 1/2$ and non-positive when $a \le 1/2$. Therefore, h is non-decreasing for $t \ge 1$ and $h(t) \ge h(1) = 0$ which proves (2). $\square$