Problem 11781
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Proposed by R. Tauraso (Italia).

*For $n \geq 2$, call a positive integer n-*smooth *if none of its prime factors is larger than n. Let S_n be the set of all n-smooth positive integers. Let C be a finite, nonempty set of the nonnegative integers, and let a and d be positive integers. Let M be the set of all positive integers of the form $m = \sum_{k=1}^{d} c_k s_k$ where $c_k \in C$ and $s_k \in S_n$ for $k = 1, \ldots, d$. Prove that there are infinitely many primes p such that $p^a \notin M$.*

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If M is finite then the property trivially holds. Let us assume that M is infinite and let $m_1 < m_2 < m_3 < \cdots$ be the sequence of positive integers of its elements. It suffices to show that for $\alpha > 0$,

$$\sum_{k=1}^{\infty} \frac{1}{m_k^{\alpha}} < +\infty.$$

In fact, if $p_k^a \in M$ for any $k \geq k_0$, where p_k is the kth prime, then, for $\alpha = 1/a$, we get a contradiction

$$+\infty = \sum_{k=k_0}^{\infty} \frac{1}{p_k} = \sum_{k=k_0}^{\infty} \frac{1}{(p_k^a)^{\alpha}} \leq \sum_{k=1}^{\infty} \frac{1}{m_k^{\alpha}} < +\infty.$$

Let $f(x) = |M \cap [1, x]|$ then $f(x)$ is a non-decreasing function such that

$$f(x) \leq (|C|+1)^d |S_n \cap [1, x]|^d \leq (|C|+1)^d \prod_{i=1}^{\pi(n)} \left(1 + \frac{\ln(x)}{\ln(p_i)}\right)^d .$$

Hence

$$\lim_{j \to +\infty} \frac{f(j)/j^{1+\alpha}}{1/j^{1+\alpha/2}} = \lim_{j \to +\infty} \frac{f(j)}{j^{\alpha/2}} = 0$$

and therefore

$$\begin{aligned} +\infty > \sum_{j=1}^{\infty} \frac{f(j)}{j^{1+\alpha}} &= \sum_{k=1}^{\infty} \sum_{j=m_k}^{m_{k+1}-1} \frac{f(j)}{j^{1+\alpha}} = \sum_{k=1}^{\infty} k \sum_{j=m_k}^{m_{k+1}-1} \frac{1}{j^{1+\alpha}} \\ &\geq \sum_{k=1}^{\infty} k \int_{m_k}^{m_{k+1}} \frac{dx}{x^{1+\alpha}} = \frac{1}{\alpha} \sum_{k=1}^{\infty} k \left(\frac{1}{m_k^{\alpha}} - \frac{1}{m_{k+1}^{\alpha}} \right) \\ &= \frac{1}{\alpha} \left(\sum_{k=1}^{\infty} \frac{1}{m_k^{\alpha}} - \lim_{k \to +\infty} \frac{k}{m_{k+1}^{\alpha}} \right) = \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{1}{m_k^{\alpha}} \end{aligned}$$

because

$$\lim_{k \to +\infty} \frac{k}{m_{k+1}^{\alpha}} = \lim_{k \to +\infty} \frac{k}{k+1} \cdot \frac{f(m_{k+1})}{m_{k+1}^{\alpha}} = \lim_{x \to +\infty} \frac{f(x)}{x^{\alpha}} = 0.$$

$\square$