Problem 11760. [AMM, February 2014]. *Proposed by S. Siboni, Trento, Italy.* Let D be the closure of a bounded open subset of $\mathbb{R}^2$. Let n be an integer, $n > 1$, and let M be the subset of $(0,1)^n$ consisting of all points $m = (m_1, \dots, m_n)$ such that $m_1 + \cdots + m_n = 1$. Let g be a fixed point inside D. With $p = (p_1, \dots, p_n) \in D^n$, let I_g be the function from $D^n \times M$ to $\mathbb{R}$ given by

$$I_g(p, m) = \sum_{k=1}^{n} m_k \|p_k - g\|^2,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. Let C be the subset of $D^n \times M$ consisting of all points (p, m) such that $\sum_{k=1}^n m_k p_k = g$.
(a) Show that $I_g(p, m)$ is maximized on C and if the maximum is attained at (p', m'), then all entries of p' lie on the boundary of D.
(b) Restricting now to the case in which $n = 2$ and the boundary of D is an ellipse, let $((p'_1, p'_2), (m'_1, m'_2))$ be a point at which $I_g((p_1, p_2), (m_1, m_2))$ is maximized on C. Show that p'_1 and p'_2 are the intersection points of the ellipse with the straight line through g parallel to the major axis.
(c) Show that if D_r is a disk of radius r about the origin, then the maximum value of I_g on C is $r^2 - \|g\|^2$ for any $n \geq 2$.

Solution by Borislav Karaivanov, Lexington, SC. (a) Let the set M_0 be defined just like M but with $[0,1]^n$ instead of $(0,1)^n$, and let C_0 be defined just like C but with M_0 instead of M. The set C_0 is closed in $D^n \times M_0$ as the preimage of the closed set $\{g\}$ under the continuous mapping $f : D^n \times M_0 \to \mathbb{R}^2$ defined by $f(p, m) = \sum_{k=1}^n m_k p_k$. Since $D^n \times M_0$ itself is closed and bounded subset of $\mathbb{R}^{3n}$, we conclude that C_0 is such and, thus, compact. Therefore, being continuous, I_g attains maximum on C_0. However, the same maximum is attained on C as well for the range of I_g on C and C_0 is the same. Indeed, if $(p, m) \in C_0$ has a zero-mass point, we take another, non-zero-mass point of (p, m), replace them both by two copies of the non-zero-mass point, and split the non-zero mass equally between the two copies. Repeating the same act, if necessary, we eliminate all zero-mass points from p and produce an element of C at which the value of I_g is the same as that at (p, m).

For the second part of the claim, it suffices to prove that if any of the points $p_1, \dots, p_n$ is in the interior of D, then I_g does not have maximum at (p, m). The idea is to budge one such point and all masses so that for the new configuration: (i) the points are in D; (ii) the masses are positive and add up to 1; (iii) the center of mass is at g; (iv) the moment of inertia is larger.

Without loss of generality, p_1 is in the interior of D. Then there is $\varepsilon > 0$ such that the ε-neighborhood of p_1 is in D. Let $\mu = \sum_{k=2}^n m_k$, $h = \sum_{k=2}^n \frac{m_k}{\mu} p_k$, and $\tilde{\mu} = \mu + \frac{\varepsilon m_1}{\varepsilon + \|p_1 - h\|}$. We define new mass points as follows: $\tilde{p}_1 = p_1 + \frac{\varepsilon}{\|p_1 - h\|}(p_1 - h)$, $\tilde{m}_1 = \frac{\|p_1 - h\| m_1}{\varepsilon + \|p_1 - h\|}$, $\tilde{p}_k = p_k$, and $\tilde{m}_k = \frac{\tilde{\mu}}{\mu} m_k$ for $k = 2, \dots, n$. Checking (i)-(iii) is straightforward: $\|\tilde{p}_1 - p_1\| = \varepsilon$, $\sum_{k=1}^n \tilde{m}_k = \tilde{m}_1 + \tilde{\mu} = m_1 + \mu = 1$, and $\sum_{k=1}^n \tilde{m}_k \tilde{p}_k = \tilde{m}_1 \tilde{p}_1 + \tilde{\mu} h = m_1 p_1 - \frac{\varepsilon m_1}{\varepsilon + \|p_1 - h\|} h + \tilde{\mu} h = m_1 p_1 + \mu h = g$. Taking into account that $m_1 > 0$, we show (iv) as follows

$$\begin{aligned}
\tilde{I}_g - I_g &= \tilde{m}_1 (\|p_1 - g\| + \varepsilon)^2 + \frac{\tilde{\mu}}{\mu}(I_g - m_1 \|p_1 - g\|^2) - I_g \\
&> \tilde{m}_1 \big(\|p_1 - g\|^2 + 2\varepsilon \|p_1 - g\| + \varepsilon^2 - \frac{\varepsilon + \mu \|p_1 - h\|}{\mu(\varepsilon + \|p_1 - h\|)} \frac{m_1}{\tilde{m}_1} \|p_1 - g\|^2\big) \\
&> \frac{\tilde{m}_1 \|p_1 - g\|}{\varepsilon + \|p_1 - h\|} \big(\|p_1 - g\| + 2\varepsilon - \frac{\varepsilon + \mu \|p_1 - h\|}{\mu \|p_1 - h\|} \|p_1 - g\|\big) \\
&= \frac{\tilde{m}_1 \varepsilon \|p_1 - g\|}{\mu \|p_1 - h\| (\varepsilon + \|p_1 - h\|)} \big(2\mu(\|g - h\| + \|p_1 - g\|) - \|p_1 - g\|\big),
\end{aligned}$$

where for the last equality we used collinearity of p_1, g, and h. Using $\mu\|g-h\| = m_1\|p_1-g\|$, we find that the expression in the last set of parenthesis above equals $\|p_1 - g\|$. Therefore, $\tilde{I_g} > I_g$.

(b) After solving the equations $m_1 + m_2 = 1$ and $m_1p_1 + m_2p_2 = g$ for m_1 and m_2, we plug the found masses in $I_g = m_1\|p_1 - g\|^2 + m_2\|p_2 - g\|^2$ and obtain

$$I_g = \|p_1 - g\| \cdot \|p_2 - g\|. \tag{1}$$

By part (a), the maximum is attained for the endpoints of a chord. Squeezing the ellipse along its major axis to a circle reduces the major-axis projection of any segment while preserving its minor-axis projection. Thus, a segment gets reduced most if it is parallel to the major axis. Hence, by (1), the moment of inertia of a chord gets reduced most if that chord is parallel to the major axis. (Note that squeezing maps the center of mass of two points to the center of mass of their images.) Since after squeezing the moments of inertia of all chords end up being the same, as shown in part (c), the moment that gets reduced most is the largest one.

(c) For any point h, we have $I_g = I_h - \|hg\|^2 \sum_{k=1}^n m_i$. Taking h to be the origin o, we obtain $I_g = I_o - \|og\|^2 = \sum_{k=1}^n m_i\|op_i\|^2 - \|g\|^2 \le r^2 - \|g\|^2$ with equality attained exactly when all $p_1, \dots, p_n$ are on the bounding circle of D_r. $\square$

Remark. *One can consider the same problem with the set C replaced by its subset C_2 defined by further requiring that the points $p_1, \dots, p_n$ be distinct. Maximum moment of inertia on C_2 may no longer exist for C_2 is not guaranteed to be compact. For example, take D to be a regular $(n-1)$-gon with g placed at its center. The maximum moment of inertia on C is attained by placing two points with masses of $\frac{1}{2(n-1)}$ at one vertex and points with mass of $\frac{1}{n-1}$ at each of the remaining $(n-2)$ vertices. However, if repeated points are not allowed, the maximum on C becomes unattainable supremum on C_2. Although the existence of maximum can not be claimed in all cases, it is still true that if maximum is attained at (p', m'), then all entries of p' lie on the boundary of D. The proof we gave above remains valid with the minor adjustment that when choosing ε one has to make sure that the new point $\tilde{p}_1$ does not duplicate any of the other points.*

Moreover, the claims and our proofs of (b) and (c) remain valid when C is replaced by C_2: for (b), the $2n$-tuple at which the maximum on C is attained clearly belongs to C_2 as well; as for (c), there are plenty of distinct points on the circle at which to attain the maximum.

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