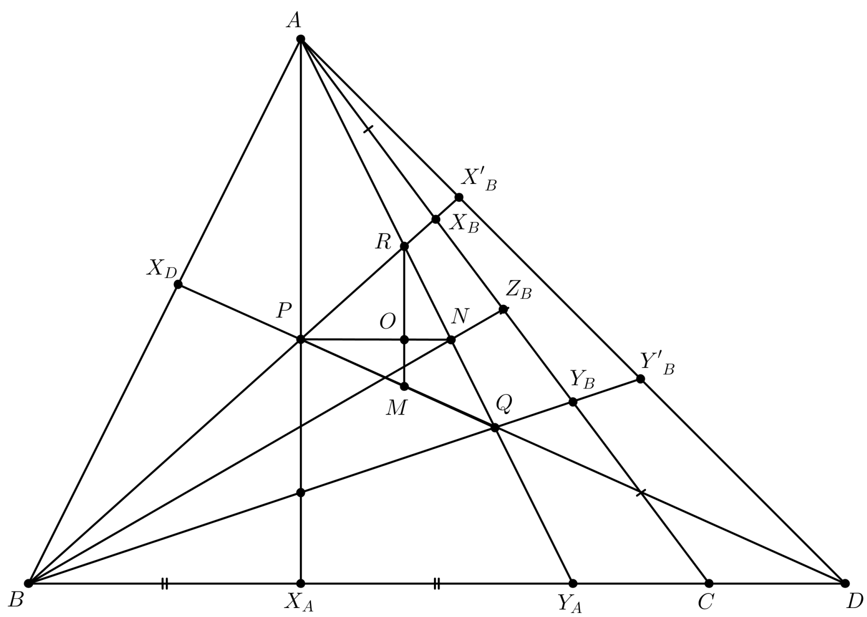
Problem 11758. [AMM, February 2014]. *Proposed by B. Brzycki, Whittier, CA.* Acute triangle ABC has several ancillary points and properties shown in the figure. Segment AX_A is perpendicular to BC, and segments marked with the same symbol have the same length. The angle at C is less than the angle at B, and lines PN and RM are parallel and perpendicular, respectively, to BC.



(a) Prove that $|RO|/|OM| = 2$.
(b) Show that PQ is not parallel to BC.
(c) Letting D be the intersection of PQ and BC, show that if AD is perpendicular to BX_B, then P, R, N, and M lie on a common circle.
(d) For fixed B and C, describe the set of A such that AD is perpendicular to BX_B.

Solution by Borislav Karaivanov, Lexington, SC, and Tzvetalin S. Vassilev, Nipissing University, North Bay, Ontario, Canada. In parts (a) and (d) we repeatedly use the fact that cross-ratio is a projective invariant, meaning that for any four lines passing through a common point and a line l, not passing through that common point but intersecting the four lines in points A, B, C, and D, the cross-ratio defined as $(A, B; C, D) := \frac{|AC|}{|BC|}\frac{|BD|}{|AD|}$ is independent of l. We refer to this fact as Pappus's lemma for, in some form, it can be traced back as far as the original proof of Pappus's hexagon theorem.

(a) Similarity of triangles BY_AR and PNR yields $|RY_A|/|RN| = |BY_A|/|PN|$, and similarity of triangles PNA and X_AY_AA yields $|AN|/|AY_A| = |PN|/|X_AY_A|$. Hence, $(A, R; N, Y_A) = 2$. Pappus's lemma applied to lines BA, BR, BN, and BY_A crossed by AY_A and AC yields $(A, X_B; Z_B, C) = 2$, where Z_B is the intersection of AC and BN. Using $|AX_B| = |X_BY_B| = |Y_BC|$ in the last cross-ratio, we find $|X_BZ_B| = |AC|/6$. Hence, by direct calculation, $(A, X_B; Z_B, Y_B) = 3/2$. By Pappus's lemma applied to lines BA, BX_B, BZ_B, and BY_B crossed by AC and AY_A, we get $(A,R;N,Q) = 3/2$. Hence, $\frac{3}{2} = \frac{|AN|}{|RN|}\frac{|RQ|}{|AQ|} = \frac{|AP|}{|RO|}\frac{|RM|}{|AP|} = \frac{|RM|}{|RO|}$, where we used that $\triangle ANP \sim \triangle RNO$ and $\triangle RQM \sim \triangle AQP$. From the last ratio, we derive that $|RO|/|OM| = 2$.

(b) If PQ were parallel to BC, then O and M would coincide, both being points on the line through R perpendicular to BC and on the line through P parallel to BC. Hence, by (a), $|RO| = 2|OM| = 0$, i.e., R and O would coincide too. Thus, the line PR, being the same as PM, would be parallel to BC, and, in the same time, meeting BC in B. The contradiction indicates that PQ and BC can not be parallel.

(c) If AD is perpendicular to BX_B, then P is the orthocenter of triangle ABD. Hence, $\angle DPX_A = \angle DBA$. Also $\angle DPX_A = \angle PMR$ since RM is parallel to AX_A. Therefore, $\angle PMR = \angle DBA$. On the other hand, $\angle PNA = \angle X_AY_AA = \angle DBA$ for PN is parallel to BC and triangle BY_AA is isosceles. Thus, points M and N see segment PR at the same angle, and therefore P, R, N, and M lie on a common circle.

(d) Let X_B' and Y_B' be the intersections of BX_B and BY_B with AD correspondingly, and let X_D be the intersection of PQ and AB. By direct calculation, $(A, X_B; Y_B, C) = 4/3$. Hence, by Pappus's lemma applied to lines BA, BX_B, BY_B, and BC crossed by AC and AD, we get $(A, X_B'; Y_B', D) = 4/3$. From here, by Pappus's lemma applied to BA, BX_B, BY_B, and BD crossed by DX_D and DA , we get $(X_D, P; Q, D) = 4/3$. Finally, by Pappus's lemma applied to AB, AX_A, AY_A, and AD crossed by DX_D and DB, we obtain $(B, X_A; Y_A, D) = 4/3$. Since $|BX_A| = |X_AY_A|$, from the last cross-ratio we find $|BD| = 3|BX_A|$.

To find the locus of A, we consider the Cartesian coordinate system with origin at B and positive x-axis along the ray BC. Let $c := |BC|$, and let $A = (x, y)$. Then $B = (0,0)$, $C = (c, 0)$, $X_B = \frac{1}{3}C + \frac{2}{3}A = \frac{1}{3}(c + 2x, 2y)$, and $D = (3x, 0)$. Therefore, $\overrightarrow{AD} = (2x, -y)$ and $\overrightarrow{BX_B} = \frac{1}{3}(c+2x, 2y)$. Setting the dot product of the last vectors to zero, we find that AD and BX_B are perpendicular if and only if

$$\frac{(x + c/2)^2}{(c/2)^2} - \frac{y^2}{c^2} = 1.$$

Since it is required that triangle ABC be acute and the angle at C be less than the angle at B, we conclude that the locus of A is the part of the hyperbola corresponding to $0 < x < c/2$.