Maths Monthly Problems

July 2012*

AMM problem 11653 *Solution by TCDmath problem group, Mathematics, Trinity College, Dublin 2, Ireland.*

Let n be a positive integer. Determine all entire functions f that satisfy, for all complex s and t, the functional equation

$$f(s+t) = \sum_{k=0}^{n-1} f^{(n-1-k)}(s) f^{(k)}(t), \qquad (1)$$

Here, $f^{(m)}$ denotes the mth derivative of f.

Answer: *It is clear that $f = 0$ is a solution for all n. Let us suppose therefore that $f(s)$ does not vanish identically.*

We have

$$\frac{\partial f(s+t)}{\partial s} = \frac{\partial f(s+t)}{\partial t} = f'(s+t).$$

Thus, on differentiating the functional equation with respect to s and t, and equating,

$$\sum_{k=0}^{n-1} f^{(n-k)}(s) f^{(k)}(t) = \sum_{k=0}^{n-1} f^{(n-1-k)}(s) f^{(k+1)}(t),$$

ie, cancelling common terms,

$$f(s)f^{(n)}(t) = f^{(n)}(s)f(t).$$

*Solutions should be submitted by 31 October

Hence

$$\frac{f^{(n)}(s)}{f(s)} = \frac{f^{(n)}(t)}{f(t)} = c,$$

where c is constant.

If $c = 0$ then $f^{(n)}(s)$ vanishes identically, ie $f(s)$ is a polynomial of degree $m < n$. Suppose

$$f(s) = a_m s^m + \cdots a_0.$$

If $m < n - 1$ then $f^{(n-1)}(t) = 0$, and there is no term on the right-hand side of (1) in s^m. Hence $m = n - 1$.

By Taylor's Theorem,

$$f(s+t) = f^{(0)}(s) + f^{(1)}(s)t + f^{(2)}(s)t^2/2! + \cdots + f^{(m)}(s)t^m/m!.$$

By degree arguments, the polynomials $f^{(0)}(s), f^{(1)}(s), \cdots, f^{(m)}(s)$ are linearly independent. It follows, by comparison with (1), that

$$f^{(m-i)}(t) = t^i/i!$$

for $i = 0, 1, \ldots, m$. In particular

$$f(t) = t^m/m!,$$

from which the other equations follow.

It follows that, for each $n \geq 1$,

$$f(s) = \frac{s^{n-1}}{(n-1)!}$$

is a solution of (1).

Now suppose $c \neq 0$. Then

$$f^{(n)}(s) = cf(s).$$

Hence

$$f(s) = \sum_i b_i e^{\lambda_i s},$$

where λ_i runs over the roots of

$$t^n = c.$$

If we isolate one of the summands

$$b_i e^{\lambda_i s}$$

then all terms involving $e^{\lambda_i s} e^{\lambda_i t}$ *on both sides of (1) arise from this summand, and hence the summand itself will also be a solution to (1).*

So let us consider

$$f(s) = b e^{\lambda s}.$$

We have

$$f^{(k)}(s) = b \lambda^k e^{\lambda s},$$

so (1) gives

$$b e^{\lambda(s+t)} = n b^2 \lambda^{n-1} e^{\lambda(s+t)}.$$

Thus the equation will hold provided

$$b = \frac{1}{n\lambda^{n-1}} = \frac{\lambda}{nc},$$

where $c = \lambda^n$*.*

Finally, we must consider if a sum of such solutions, with different λ_i*, can satisfy (1).*

Let

$$f(s) = \sum_i b_i e^{\lambda_i s}$$

where the λ_i *are distinct roots of* $t^n - c$ *and each summand satisfies (1).*

On the left-hand side of (1) we have

$$\sum_i b_i e^{\lambda_i (s+t)}$$

and on the right

$$\sum_{i,j,k} b_i \lambda_i^{n-1-k} e^{\lambda_i s} b_j \lambda_j^k e^{\lambda_j t}$$

For $i = j$ *we get* $b_i^2 n \lambda_i^{n-1} e^{\lambda_i (s+t)}$*, as previously, which fits (1), and for* $i \neq j$*, the terms cancel:*

$$b_i b_j e^{\lambda_i s + \lambda_j t} \sum_k \lambda_i^{n-1-k} \lambda_j^k = 0,$$

because the sum can be written

$$\lambda_i^{n-1}\sum_k(\lambda_j/\lambda_i)^k =$$
$$\lambda_i^{n-1}\frac{1-(\lambda_j/\lambda_i)^n}{1-\lambda_j/\lambda_i} =$$
$$\lambda_i^{n-1}\frac{1-c/c}{1-\lambda_j/\lambda_i} = 0.$$

We conclude that the only solutions are:

1. *any of the* 2^n *combinations*

$$f(s) = \frac{1}{nc}(\sum_i \lambda_i e^{\lambda_i s}),$$

   where the λ_i *are some or all of the distinct roots of* $\lambda^n = c$*;*

2. $f(s) = s^{n-1}/(n-1)!$ *for each* $n \geq 1$*.*