Problem 11148
(American Mathematical Monthly, Vol.112, April 2005)

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Show that

$$\int_0^{+\infty} \frac{x^8 - 4x^6 + 9x^4 - 5x^2 + 1}{x^{12} - 10x^{10} + 37x^8 - 42x^6 + 26x^4 - 8x^2 + 1}\,dx = \frac{\pi}{2}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $f(x)$ be the rational function to be integrated and let

$$g(x) = \frac{(x+1)^2}{x^6 + 4x^5 + 3x^4 - 4x^3 - 2x^2 + 2x + 1}.$$

Then $f(x) = (g(x) + g(-x))/2$ and

$$\int_0^{+\infty} f(x)\,dx = \frac{1}{2}\int_0^{+\infty} g(x)\,dx + \frac{1}{2}\int_0^{+\infty} g(-x)\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} g(x)\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} g(x-1)\,dx.$$

Therefore it suffices to show that

$$\int_{-\infty}^{+\infty} g(x-1)\,dx = \int_{-\infty}^{+\infty} \frac{x^2}{x^6 - 2x^5 - 2x^4 + 4x^3 + 3x^2 - 4x + 1}\,dx = \pi.$$

Consider the complex polynomials

$$q(z) = z^3 - (1+i)z^2 - (2-i)z + 1 \quad \text{and} \quad q^*(z) = \overline{q(\overline{z})} = z^3 - (1-i)z^2 - (2+i)z + 1.$$

It is easy to see that $q(z)$ has three distinct complex roots, say w_1, w_2, w_3, in the upper half plane and that the conjugates $\overline{w_1}, \overline{w_2}, \overline{w_3}$ are the roots of $q^*(z)$. Moreover, for $z \in \mathbb{C}$

$$q(z)q^*(z) = z^6 - 2z^5 - 2z^4 + 4z^3 + 3z^2 - 4z + 1.$$

For $0 \le j \le 4$, let

$$\begin{aligned} I_j &= \int_{-\infty}^{+\infty} \frac{x^j}{x^6 - 2x^5 - 2x^4 + 4x^3 + 3x^2 - 4x + 1}\,dx \\ &= \lim_{R\to+\infty} \int_{\Gamma_R} \frac{z^j}{q(z)q^*(z)}\,dz = 2\pi i \sum_{k=1}^{3} \mathrm{Res}\left(\frac{z^j}{q(z)q^*(z)}, w_k\right) \end{aligned}$$

where Γ_R is counter-clockwise upper semicircle with diameter $[-R, R]$ (note that $|z^j/(q(z)q^*(z))| \le M_j/R^2$ for $|z| = R$ sufficiently large).
We will show that $I_2 = \pi$. Since

$$\frac{1}{2i}\left(\frac{1}{q(z)} - \frac{1}{q^*(z)}\right) = \frac{z^2 - z}{q(z)q^*(z)}$$

then

$$\begin{aligned} I_2 - I_1 &= 2\pi i \sum_{k=1}^{3} \mathrm{Res}\left(\frac{z^2 - z}{q(z)q^*(z)}, w_k\right) \\ &= \pi \sum_{k=1}^{3} \mathrm{Res}\left(\frac{1}{q(z)} - \frac{1}{q^*(z)}, w_k\right) = \pi \sum_{k=1}^{3} \lim_{z\to w_k} \left(\frac{z - w_k}{q(z)}\right) \\ &= \pi \left(\frac{1}{(w_1 - w_2)(w_1 - w_3)} + \frac{1}{(w_2 - w_1)(w_2 - w_3)} + \frac{1}{(w_3 - w_1)(w_3 - w_2)}\right) = 0. \end{aligned}$$

In a similar way, we find that

$$\begin{aligned}
I_3 - I_2 &= 2\pi i \sum_{k=1}^{3} \operatorname{Res}\left(\frac{z^3 - z^2}{q(z)q^*(z)}, w_k\right) \\
&= \pi \sum_{k=1}^{3} \operatorname{Res}\left(\frac{z}{q(z)} - \frac{z}{q^*(z)}, w_k\right) = \pi \sum_{k=1}^{3} \lim_{z \to w_k} \left(\frac{z(z-w_k)}{q(z)}\right) \\
&= \pi \left(\frac{w_1}{(w_1 - w_2)(w_1 - w_3)} + \frac{w_2}{(w_2 - w_1)(w_2 - w_3)} + \frac{w_3}{(w_3 - w_1)(w_3 - w_2)}\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
I_1 - I_0 &= 2\pi i \sum_{k=1}^{3} \operatorname{Res}\left(\frac{z - 1}{q(z)q^*(z)}, w_k\right) \\
&= \pi \sum_{k=1}^{3} \operatorname{Res}\left(\frac{z^{-1}}{q(z)} - \frac{z^{-1}}{q^*(z)}, w_k\right) = \pi \sum_{k=1}^{3} \lim_{z \to w_k} \left(\frac{z^{-1}(z-w_k)}{q(z)}\right) \\
&= \pi \left(\frac{w_1^{-1}}{(w_1 - w_2)(w_1 - w_3)} + \frac{w_2^{-1}}{(w_2 - w_1)(w_2 - w_3)} + \frac{w_3^{-1}}{(w_3 - w_1)(w_3 - w_2)}\right) \\
&= \frac{\pi}{w_1 w_2 w_3} = -\frac{\pi}{q(0)} = -\pi.
\end{aligned}$$

So we have that $I_1 = I_2$, $I_3 = I_2$ and $I_0 = I_1 + \pi = I_2 + \pi$. Moreover, since

$$\frac{1}{2}\left(\frac{1}{q(z)} + \frac{1}{q^*(z)}\right) = \frac{z^3 - z^2 - 2z + 1}{q(z)q^*(z)}$$

then

$$\begin{aligned}
I_3 - I_2 - 2I_1 + I_0 &= 2\pi i \sum_{k=1}^{3} \operatorname{Res}\left(\frac{z^3 - z^2 - 2z + 1}{q(z)q^*(z)}, w_k\right) \\
&= \pi i \sum_{k=1}^{3} \operatorname{Res}\left(\frac{1}{q(z)} + \frac{1}{q^*(z)}, w_k\right) = \pi i \sum_{k=1}^{3} \lim_{z \to w_k} \left(\frac{z - w_k}{q(z)}\right) = 0
\end{aligned}$$

that is $0 = I_3 - I_2 - 2I_1 + I_0 = -I_2 + \pi$ and $I_2 = \pi$.