Mixing Variables-powerful and beautiful

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Nowadays, we have many strange inequalities that have the"=" not at the cross that make us hardly solve normally.So that we must find the new way to solve the problems. And MV is one of the newest techniques we can use to proof these inequality.Let's start with some examples.

Example 1: Show that a,b,c are non-negative numbers and satisfy:ab+bc+ca=1. Proof that:

$$2(a+b+c)+abc\geqslant 4(\text{Nguyen Duy Khuong})$$

Solution:When we look at this inequality we must think about it's"=" but it's not a cross "=" so this seems a hard inequality. So we must find a powerful way to solve it and MV is the first choice. We can easily proof f(a,b,c)=2(a+b+c)+abc$\geqslant$ f(0,a+(c/2),b+(c/2))(because it$\Leftrightarrow$ abc$\geqslant$0). So all we have to do is proof the inequality at a=0.But we easily see 2(b+c)$\geqslant 4\sqrt{bc}=4$(because a=0 so bc=1). So that's all and the "=" is(0,1,1),(1,0,1),(1,1,0).

After this example we see if we must proof a 3-variable inequality by MV we only have to proof: 1) f(a,b,c)$\geqslant$f(0,g(a,b,c),t(a,b,c))(edge mixing)(g(a,b,c),t(a,b,c) follows a,b,c).If we can prove this first step we only have to prove the ineqyality in the case one variable is 0.

2)or f(a,b,c)$\geqslant$f(a,$\sqrt{bc}$,$\sqrt{bc}$),or f(a,b,c)$\geqslant$f(a,b+c/2,b+c/2),..(they call this cross mixing).After this first step we only have to prove the inquality in the case that n-1 varibles are equal.

Well follow the MV we can solve many hard inequalities. However we must know that we must have a good technique to solve the calculating work with some inequalities.

Well MV can make lessen the proof because it lessen the variables from 3,4 to only 1 variable. Next we will work with some 4 varibles inequality.

Example 2: Show that a,b,c,d are non-negative follow the condition: a+b+c+d=4.Proof that:

$$3(a^2+b^2+c^2+d^2)+abcd\leqslant 13(\text{dogvensten})$$

Solution:Here is the solution.

If we can proof that:

$f(x_1,x_2,...,x_n)\geqslant(or\leqslant)f(x_i+x_j/2,x_i+x_j,x_3,...,x_n)$

where $x_i,x_j aretheminandmaxvariablein(x_1,...,x_n)$

so we only have to proof the inequality with n-1 equal this hard problem.That's what we need for the solution.

Well apply this theorum, we see the left of the inequality

=48-6(ab+bc+cd+da+ca+bd)+abcd

=48+ca(bd-6)-6(b+d)(c+a)+bd $\leqslant f(a,b+d/2,c,b+d/2)$

Follow the SMV theorum we see now we only have to proof the inequality with b=c=d=t and a=4-3t. That's an easy work with way$\Leftrightarrow$.

At last the inequality$\Leftrightarrow$ $(1-3t)(t-1)^2(11t+1)\geqslant$0(always right)

And you can see many more hard 4 variables inequality can be solved by SMV. Now come back to 3 variables inequalities. I want to show the power of MV thourgh the VMO 1996's inequality:

Example 3(VMO 1996): Show that a,b,c are non-negative real nmbers satis the condition that:ab+bc+ca+abc=4.

Prove that a+b+c⩾ab+bc+ca.

Solution: The solution for this problem want us to have a good view about inequalities.Let's see the MV solution.Well, if we call "t" is a new variable that satisfy: $t^2 + 2at + at^2 = 4(0 \leqslant t \leqslant 2)$.

We will proof that f(a,b,c)=a+b+c-ab-bc-ca⩾f(a,t,t). That's correct because we see from the new way to create "t" satisfy that:

$t^2 + 2at + at^2 = ab + bc + ca + abc \Leftrightarrow \dfrac{a}{a+1} = \dfrac{t^2 - bc}{b + c - 2t}$

Now if we see that $t^2 - bc \leqslant 0$ and b+c-2t⩽0 so it$\Leftrightarrow t^2 - bc > 0 (nosense)$.

So that means b+c-2t⩾0.And that means f(a,b,c)-f(a,t,t)=(1-a)(b+c-2t)-(bc-t^2) $= (1-a)(b+c-2t) + (\dfrac{a}{a+1})(b+c-2t) = (b+c-2t)(1-a+\dfrac{a}{a+1})$ ⩾0(right).

So f(a,b,c)⩾f(a,t,t). And that means we only have to proof the inequality with one variable"t" and our inequality lastly$\Leftrightarrow (2-t)(\dfrac{(t-1)^2}{t^2+1})$ ⩾0(right!!!). The"=" is (0,2,2),(1,1,1),(2,0,2),...

Example 4(Nguyen Duy Khuong): Show that a,b,c are non-negative real numbers satisfy: ab+bc+ca=1. Proof that:

$$4(a+b+c)+3abc \geqslant 8$$

Solution: The solution for this problem is as same as the first example is boring but like example 4 is very intersting. Let's start: we choose a "t" variable that satisfy: 2at+t^2 = 1(0 ⩽t⩽⩽1).Well that means: ab+bc+ca=2at+t^2 $\Leftrightarrow a(b+c-2t) = t^2 - bc(1)$

Or a= $\dfrac{t^2 - bc}{b+c-2t}$.

That means like example 4 we can see that b+c-2t⩾0. After that we know f(a,b,c)-f(a,t,t)=4(b+c-2t)+3a(bc-t^2) $= (4-3a^2)(b+c-2t)$ ⩾0(we can see a<1 so we can see it's correct). So f(a,b,c)⩾f(a,t,t). Now we only have to proof a 't' variable inequality-that's an easy work.

We can see the effective and powerful of making the mixing variable "t". When we come up a hard conditional inequality we should always remember about this technique in MV-method.Remember that the 't' variable is borned to satisfy the condition of the exercise!!!

Now we come to a hard extension of this example:

Extension of ex4:

Show that a,b,c are non-negative real numbers satisfy:ab+bc+ca=1. Find the best k that satisfy this inequality correctly: k(a+b+c)+3abc⩾2

Next we come to example 5

Example 5(Nguyen Duy Khuong): Show that a,b,c are non-negative numbers that satisfy:a+b+c+3abc=6.

Prove that: ab+bc+ca+3abc⩽ 6.

Solution: Well, in this case and many other cases, we should always make a new mixing variable-that's avery important work at all. Some how after seeing the solution of ex 5,6, we find that it's important to make a new variable "t". That's for what, that's for satisfying the condition and also the "=". Again we gonna choose a new "t" variable satisfy: a+2t+3at^2 = $6(0 \leqslant t \leqslant 4.5)$.

That leads to a+2t+3at$^2 = a+b+c+3abc \Leftrightarrow (b+c-2t)+3a(bc-t^2) = 0 \Leftrightarrow 3a = \dfrac{t^2-bc}{b+c-2t} \Rightarrow b+c-2t \geqslant 0$

(like the example 3 and 4).We can see now f(a,b,c)-f(a,t,t)=(b+c-2t)(-18a$^2 - 2a) \leqslant 0(correct)$.

So we only have to prove one "t" variable(easy).

Hint: The "=" is where a=0(!!!) and a=1

After this example we see that if a+b+c+3abc=6 and a,b,c are non-negative real numbers so a+b+c$\geqslant ab+bc+ca$

Example 6(Vietnam IMO Shortlist): Show that a,b,c,d are non-negative real numbers:a+b+c+d=1. Prove that:

abc+bcd+cda+dab$\leqslant \dfrac{1}{27}+\dfrac{176abcd}{27}$

Solution: The way to prove this hard and cool inequality is to use SMV theorum. We call f(a,b,c,d)=abc+bcd+cda+dab- $\dfrac{176abcd}{27}$

=ac(b+d)+bd(a+c- $\dfrac{176ca}{27}$).

We consider a$\leqslant b \leqslant c \leqslant d. So a+c \leqslant 1/2$.

That means $\dfrac{1}{a}+\dfrac{1}{c} \geqslant \dfrac{4}{a+c} \geqslant 8 > 176/27$.

It means that f(a,b,c,d)$\leqslant f(a,b+d/2,c,b+d/2)$.

So we only have to prove the inquality in the case

that b=c=d=t and a=4-3t(follow the SMV theorum)

At last the inequality$\Leftrightarrow (1-3t)(4t-1)^2(11t+1) \geqslant 0(correct)$

Example 7(Nguyen Anh Cuong): Show that a,b,c are non-negative numbers: a+b+c+d=4.

Prove that $3(\sum a^3)+4abcd \geqslant 16$.

Solution: We can easily prove that

f(a,b,c,d)=$3(\sum a^3)+4abcd \geqslant f(a,bcd^{1/3},bcd^{1/3},bcd^{1/3})$.

because it $\Leftrightarrow$

the AM-GM inequality.

So we only have to prove the inequality

in the case such that b=c=d=t and a=4-3t(0$\leqslant t \leqslant 4/3$)

(Follow the MV)

But it's an easy work now.

So we must stop here to pay time for others powerful techniques. Now let's end this topic with some exercises:

Problem 1: Show that x,y,z in R that satisfy x$^2+y^2+z^2=9$.

Prove that 2(x+y+z)-xyz$\leqslant$10(VMO 2002)

Problem 2: Show that a,b,c are non-negative real nunbers satisfy: ab+bc+ca=1.Prove that:

$\Sigma\dfrac{1}{a+b}\geqslant \dfrac{5}{2}$

Problem 3: Show that x,y,z are non-negative real numbers satisfy:xy+yz+zx+xyz=4. Prove that:

$$x^2+y^2+z^2+5xyz \geqslant 8$$

Problem 4: Show that a,b,c are non-negative numbers.

Prove that: $\Sigma\frac{1}{(a+b)^2} \geqslant \frac{9}{4(ab+bc+ca)}(Iran96)$

Problem 5(dogvensten): Show that a,b,c,d,e are non-negative real numbers satisfy:a+b+c+d+e=4.

$$\text{Prove that:}4(a^2+b^2+c^2+d^2+e^2)+5abcde \leqslant 25$$

Problem 6(Nguyen Duy Khuong): Show that a,b,c,d are non-negative real numbers satisfy: a+b+c+d=1.

Find the Min of P=$5(a^3+b^3+c^3+d^3)+7abcd$

Problem 7: Show that a,b,c are non-negative real numbers.

Prove that: $a^3+b^3+c^3+3abc \geqslant ab(a+b)+bc(b+c)+ca(c+a)(SchurInquality)$

Problem 8: Show that a,b,c,d are non-negative satisfy: a+b+c+d=4.

Prove that: (1+3a)(1+3b)(1+3c)(1+3d)$\leqslant 125+131abcd$

We see that by SMV and the 'edge' or 'cross' mixing, we can solve almost hard inequalities with 4 variables.

This makes the method MV strongful than any other methods. Besides, we can see that MV make shorten our solution, that also make the solution with MV always interesting.

Hope that you'll always be excited about maths and always find out new techniques to simplize the excersises.We should always remember that:

'Mathematic only simplize things and never complicate things'

Thanks you for reading this topic and hope you'll complete it more