1
1999 National Contests: Problems and Solutions

1.1 Belarus

National Olympiad, Fourth Round

Problem 10.1 Determine all real numbers a such that the function $f(x) = \{ax + \sin x\}$ is periodic. Here $\{y\}$ is the fractional part of y.

Solution: The solutions are $a = \frac{r}{\pi}$, $r \in \mathbb{Q}$.

First, suppose $a = \frac{r}{\pi}$ for some $r \in \mathbb{Q}$; write $r = \frac{p}{q}$ with p, $q \in \mathbb{Z}$, $q > 0$. Then

$$\begin{aligned} f(x + 2q\pi) &= \left\{ \frac{p}{q\pi}(x + 2q\pi) + \sin(x + 2q\pi) \right\} \\ &= \left\{ \frac{p}{q\pi}x + 2p + \sin x \right\} \\ &= \left\{ \frac{p}{q\pi}x + \sin x \right\} = f(x) \end{aligned}$$

so f is periodic with period $2q\pi$.

Now, suppose f is periodic; then there exists $p > 0$ such that $f(x) = f(x+p)$ for all $x \in \mathbb{R}$. Then $\{ax+\sin x\} = \{ax+ap+\sin(x+p)\}$ for all $x \in \mathbb{R}$; in other words $g(x) = ap+\sin(x+p)-\sin x$ is an integer for all x. But g is continuous, so there exists $k \in \mathbb{Z}$ such that $g(x) = k$ for all $x \in \mathbb{R}$. Rewriting this gives

$$\sin(x + p) - \sin x = k - ap \qquad \text{for all } x \in \mathbb{R}.$$

Letting $x = y$, $y + p$, $y + 2p$, $\ldots$, $y + (n - 1)p$ and summing gives

$$\sin(y + np) - \sin y = n(k - ap) \qquad \text{for all } y \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

Since the left hand side of this equation is bounded by 2, we conclude that $k = ap$ and $\sin(x + p) = \sin x$ for all $x \in \mathbb{R}$. In particular, $\sin\left(\frac{\pi}{2} + p\right) = \sin\left(\frac{\pi}{2}\right) = 1$ and hence $p = 2m\pi$ for some $m \in \mathbb{N}$. Thus $a = \frac{k}{p} = \frac{k}{2m\pi} = \frac{r}{\pi}$ with $r = \frac{k}{2m} \in \mathbb{Q}$, as desired.

Problem 10.2 Prove that for any integer $n > 1$ the sum S of all divisors of n (including 1 and n) satisfies the inequalities

$$k\sqrt{n} < S < \sqrt{2k}n,$$

where k is the number of divisors of n.

Solution: Let the divisors of n be $1 = d_1 < d_2 < \cdots < d_k = n$;

then $d_i d_{k+1-i} = n$ for each i. Thus

$$S = \sum_{i=1}^{k} d_i = \sum_{i=1}^{k} \frac{d_i + d_{k+1-i}}{2} > \sum_{i=1}^{k} \sqrt{d_i d_{k+1-i}} = k\sqrt{n},$$

giving the left inequality. (The inequality is strict because equality does not hold for $\frac{d_1+d_k}{2} \geq \sqrt{d_1 d_k}$.) For the right inequality, let $S_2 = \sum_{i=1}^{k} d_i^2$ and use the Power Mean Inequality to get

$$\frac{S}{k} = \frac{\sum_{i=1}^{k} d_i}{k} \leq \sqrt{\frac{\sum_{i=1}^{k} d_i^2}{k}} = \sqrt{\frac{S_2}{k}} \quad \text{so} \quad S \leq \sqrt{kS_2}.$$

Now

$$\frac{S_2}{n^2} = \sum_{i=1}^{k} \frac{d_i^2}{n^2} = \sum_{i=1}^{k} \frac{1}{d_{k+1-i}^2} \leq \sum_{j=1}^{n} \frac{1}{j^2} < \frac{\pi^2}{6}$$

since $d_1, \ldots, d_k$ are distinct integers between 1 and n. Therefore

$$S \leq \sqrt{kS_2} < \sqrt{\frac{kn^2\pi^2}{6}} < \sqrt{2k}n.$$

Problem 10.3 There is a 7×7 square board divided into 49 unit cells, and tiles of three types: 3×1 rectangles, 3-unit-square corners, and unit squares. Jerry has infinitely many rectangles and one corner, while Tom has only one square.

(a) Prove that Tom can put his square somewhere on the board (covering exactly one unit cell) in such a way that Jerry can not tile the rest of the board with his tiles.

(b) Now Jerry is given another corner. Prove that no matter where Tom puts his square (covering exactly one unit cell), Jerry can tile the rest of the board with his tiles.

Solution:

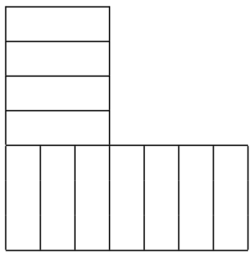
(a) Tom should place his square on the cell marked `X` in the boards below.

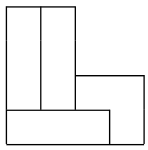
1	2	3	1	2	3	1
2	3	1	2	3	1	2
3	1	2	3	1	2	3
1	2	3	1	2	3	1
2	3	1	X	3	1	2
3	1	2	3	1	2	3
1	2	3	1	2	3	1

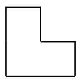
1	3	2	1	3	2	1
2	1	3	2	1	3	2
3	2	1	3	2	1	3
1	3	2	1	3	2	1
2	1	3	X	1	3	2
3	2	1	3	2	1	3
1	3	2	1	3	2	1

The grid on the left contains 17 1's, 15 2's and 16 3's; since every 3×1 rectangle contains a 1, a 2 and a 3, Jerry's corner must cover a 3 and two 1's; thus it must be oriented like a Γ. But every such corner covers a 1, a 2 and a 3 in the right grid, as does any 3×1 rectangle. Since the right grid also contains 17 1's, 15 2's and 16 3's, Jerry cannot cover the 48 remaining squares with his pieces.

(b) The following constructions suffice.







The first figure can be rotated and placed on the 7×7 board so that Tom's square falls into its blank, untiled region. Similarly, the second figure can be rotated and placed within the remaining untiled 4×4 region so that Tom's square is still uncovered; and finally, the single corner can be rotated and placed without overlapping Tom's square.

Problem 10.4 A circle is inscribed in the isosceles trapezoid $ABCD$. Let the circle meet diagonal AC at K and L (with K between

A and L). Find the value of

$$\frac{AL \cdot KC}{AK \cdot LC}.$$

First Solution:

Lemma. *Suppose we have a (not necessarily isosceles) trapezoid $ABCD$ circumscribed about a circle with radius r, where the circle touches sides AB, BC, CD, DA at points P, Q, R, S, respectively. Let line AC intersect the circle at K and L, with K between A and L. Also write $m = AP$ and $n = CR$. Then*

$$AK \cdot LC = mn + 2r^2 - \sqrt{(mn + 2r^2)^2 - (mn)^2}$$

and

$$AL \cdot KC = mn + 2r^2 + \sqrt{(mn + 2r^2)^2 - (mn)^2}.$$

Proof: Assume without loss of generality that $AB \parallel CD$, and orient the trapezoid so that lines AB and CD are horizontal. Let $t = AK$, $u = KL$, and $v = LC$; also let $\sigma = t + v$ and $\pi = tv$. By Power of a Point, we have $t(t + u) = m^2$ and $v(v + u) = n^2$; multiplying these gives $\pi(\pi + u\sigma + u^2) = m^2n^2$. Also, A and C are separated by $m + n$ horizontal distance and $2r$ vertical distance; thus $AC^2 = (m + n)^2 + (2r)^2$. Then

$$(m + n)^2 + (2r)^2 = AC^2 = (t + u + v)^2$$
$$m^2 + 2mn + n^2 + 4r^2 = t(t + u) + v(v + u) + 2\pi + u\sigma + u^2$$
$$m^2 + 2mn + n^2 + 4r^2 = m^2 + n^2 + 2\pi + u\sigma + u^2$$
$$2mn + 4r^2 - \pi = \pi + u\sigma + u^2.$$

Multiplying by π on both sides we have

$$\pi(2mn + 4r^2 - \pi) = \pi(\pi + u\sigma + u^2) = (mn)^2,$$

a quadratic in π with solutions

$$\pi = mn + 2r^2 \pm \sqrt{(mn + 2r^2)^2 - (mn)^2}.$$

But since $m^2n^2 = t(t + u)v(v + u) \geq t^2v^2$, we must have $mn \geq \pi$. Therefore $AK \cdot LC = \pi = mn + 2r^2 - \sqrt{(mn + 2r^2)^2 - (mn)^2}$. And since $(AK \cdot AL) \cdot (CK \cdot CL) = m^2 \cdot n^2$, we have $AL \cdot KC = \frac{m^2n^2}{\pi} = mn + 2r^2 + \sqrt{(mn + 2r^2)^2 - (mn)^2}$. ■

As in the lemma, assume that $AB \parallel CD$ and let the given circle be tangent to sides AB, BC, CD, DA at points P, Q, R, S, respectively. Also define $m = AP = PB = AS = BQ$ and $n = DR = RC = DS = CQ$.

Drop perpendicular $\overline{AX}$ to line CD. Then $AD = m + n$, $DX = |m-n|$, and $AX = 2r$. Then by the Pythagorean Theorem on triangle ADX, we have $(m+n)^2 = (m-n)^2 + (2r)^2$ which gives $mn = r^2$.

Using the lemma, we find that $AK \cdot LC = (3 - 2\sqrt{2})r^2$ and $AL \cdot KC = (3 + 2\sqrt{2})r^2$. Thus $\frac{AL \cdot KC}{AK \cdot LC} = 17 + 12\sqrt{2}$.

Second Solution: Suppose $A'B'C'D'$ is a square with side length s, and define K', L' analagously to K and L. Then $A'C' = s\sqrt{2}$ and $K'L' = s$, and $A'L' = K'C' = s\frac{\sqrt{2}+1}{2}$ and $A'K' = L'C' = s\frac{\sqrt{2}-1}{2}$. Thus

$$\frac{A'L' \cdot K'C'}{A'K' \cdot L'C'} = \frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)^2} = (\sqrt{2}+1)^4 = 17 + 12\sqrt{2}.$$

Consider an arbitrary isosceles trapezoid $ABCD$ with inscribed circle ω; assume $AB \parallel CD$. Since no three of A, B, C, D are collinear, there is a projective transformation τ taking $ABCD$ to a parallelogram $A'B'C'D'$. This map takes ω to a conic ω' tangent to the four sides of $A'B'C'D'$. Let $P = BC \cap AD$, and let ℓ be the line through P parallel to line AB; then τ maps ℓ to the line at ∞. Since ω does not intersect ℓ, ω' is an ellipse. Thus by composing τ with an affine transformation (which preserves parallelograms) we may assume that ω' is a circle. Let W, X, Y, Z be the tangency points of ω to sides AB, BC, CD, DA respectively, and W', X', Y', Z' their images under τ. By symmetry line WY passes through the intersection of lines BC and AD, and line XZ is parallel to lines AB and CD; thus $W'Y' \parallel B'C' \parallel A'D'$ and $X'Z' \parallel A'B' \parallel C'D'$. But ω' is tangent to the parallel lines $A'B'$ and $C'D'$ at W' and Y', so $\overline{W'Y'}$ is a diameter of ω' and $W'Y' \perp A'B'$; thus $B'C' \perp A'B'$ and $A'B'C'D'$ is a rectangle. Since $A'B'C'D'$ has an inscribed circle it must be a square. Thus we are in the case considered at the beginning of the problem; if K' and L' are the intersections of line $A'C'$ with ω', with K' between A' and L', then $\frac{A'L' \cdot K'C'}{A'K' \cdot L'C'} = 17 + 12\sqrt{2}$. Now τ maps $\{K, L\} = AC \cap \omega$ to $\{K', L'\} = A'C' \cap \omega'$ (but perhaps not in that order). If $\tau(K) = K'$ and $\tau(L) = L'$, then since projective transformations preserve cross-ratios, we would have

$$\frac{AL \cdot KC}{AK \cdot LC} = \frac{A'L' \cdot K'C'}{A'K' \cdot L'C'} = 17 + 12\sqrt{2}.$$

But if instead $\tau(K) = L'$ and $\tau(L) = K'$, then we would obtain $\frac{AL \cdot KC}{AK \cdot LC} = \frac{1}{17+12\sqrt{2}} < 1$, impossible since $AL > AK$ and $KC > LC$. It follows that $\frac{AL \cdot KC}{AK \cdot LC} = 17 + 12\sqrt{2}$, as desired.

Problem 10.5 Let P and Q be points on the side AB of the triangle ABC (with P between A and Q) such that $\angle ACP = \angle PCQ = \angle QCB$, and let $\overline{AD}$ be the angle bisector of $\angle BAC$. Line AD meets lines CP and CQ at M and N respectively. Given that $PN = CD$ and $3\angle BAC = 2\angle BCA$, prove that triangles CQD and QNB have the same area.

Solution: Since $3\angle BAC = 2\angle ACB$,

$$\angle PAN = \angle NAC = \angle ACP = \angle PCQ = \angle QCD.$$

Let θ equal this common angle measure. Thus $ACNP$ and $ACDQ$ are cyclic quadrilaterals, so

$$\theta = \angle ANP = \angle CQD = \angle CPN.$$

From angle-angle-side congruency we deduce that $\triangle NAP \cong \triangle CQD \cong \triangle PCN$. Hence $CP = CQ$, and by symmetry we have $AP = QB$. Thus, $[CQD] = [NAP] = [NQB]$.

Problem 10.6 Show that the equation

$$\{x^3\} + \{y^3\} = \{z^3\}$$

has infinitely many rational non-integer solutions. Here $\{a\}$ is the fractional part of a.

Solution: Let $x = \frac{3}{5}(125k+1)$, $y = \frac{4}{5}(125k+1)$, $z = \frac{6}{5}(125k+1)$ for any integer k. These are never integers because 5 does not divide $125k+1$. Moreover

$$125x^3 = 3^3(125k+1)^3 \equiv 3^3 \pmod{125},$$

so 125 divides $125x^3 - 3^3$ and $x^3 - \left(\frac{3}{5}\right)^3$ is an integer; thus $\{x^3\} = \frac{27}{125}$. Similarly $\{y^3\} = \frac{64}{125}$ and $\{z^3\} = \frac{216}{125} - 1 = \frac{91}{125} = \frac{27}{125} + \frac{64}{125}$, and therefore $\{x^3\} + \{y^3\} = \{z^3\}$.

Problem 10.7 Find all integers n and real numbers m such that the squares of an $n \times n$ board can be labelled 1, 2, ..., n^2 with each number appearing exactly once in such a way that

$$(m-1)a_{ij} \le (i+j)^2 - (i+j) \le m a_{ij}$$

for all $1 \le i, j \le n$, where a_{ij} is the number placed in the intersection of the ith row and jth column.

Solution: Either $n = 1$ and $2 \le m \le 3$ or $n = 2$ and $m = 3$. It is easy to check that these work using the constructions below.

$$\boxed{1} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

Now suppose we are given a labelling of the squares $\{a_{ij}\}$ which satisfies the given conditions. By assumption $a_{11} \ge 1$ so

$$m - 1 \le (m-1)a_{11} \le (1+1)^2 - (1+1) = 2$$

and $m \le 3$. On the other hand $a_{nn} \le n^2$ so

$$4n^2 - 2n = (n+n)^2 - (n+n) \le m a_{nn} \le mn^2$$

and $m \ge \frac{4n^2-2n}{n^2} = 4 - \frac{2}{n}$. Thus $4 - \frac{2}{n} \le m \le 3$ which implies the result.

Problem 11.1 Evaluate the product

$$\prod_{k=0}^{2^{1999}} \left(4\sin^2 \frac{k\pi}{2^{2000}} - 3\right).$$

Solution: For simplicity, write $f(x) = \sin\left(\frac{x\pi}{2^{2000}}\right)$.

At $k = 0$, the expression inside the parentheses equals -3. Recognizing the triple-angle formula $\sin(3\theta) = 4\sin^3\theta - 3\sin\theta$ at play, and noting that $f(k) \ne 0$ when $1 \le k \le 2^{1999}$, we can rewrite the given product as

$$-3\prod_{k=1}^{2^{1999}} \frac{\sin\left(\frac{3k\pi}{2^{2000}}\right)}{\sin\left(\frac{k\pi}{2^{2000}}\right)} \quad \text{or} \quad -3\prod_{k=1}^{2^{1999}} \frac{f(3k)}{f(k)}. \tag{1}$$

Now

$$\prod_{k=1}^{2^{1999}} f(3k) = \prod_{k=1}^{\frac{2^{1999}-2}{3}} f(3k) \cdot \prod_{k=\frac{2^{1999}+1}{3}}^{\frac{2^{2000}-1}{3}} f(3k) \cdot \prod_{\frac{2^{2000}+2}{3}}^{2^{1999}} f(3k).$$

Since $\sin\theta = \sin(\pi - \theta) = -\sin(\pi + \theta)$, we have $f(x) = f(2^{2000} - x) = -f(x - 2^{2000})$. Hence, letting $S_i = \{k \mid 1 \le k \le 2^{1999}, k \equiv i \pmod 3\}$ for $i = 0, 1, 2$, the last expression equals

$$\prod_{k=1}^{\frac{2^{1999}-2}{3}} f(3k) \cdot \prod_{k=\frac{2^{1999}+1}{3}}^{\frac{2^{2000}-1}{3}} f(2^{2000} - 3k) \cdot \prod_{\frac{2^{2000}+2}{3}}^{2^{1999}} \left(-f(3k - 2^{2000})\right)$$

$$= \prod_{k\in S_0} f(k) \cdot \prod_{k\in S_1} f(k) \cdot \prod_{k\in S_2} (-f(k))$$

$$= (-1)^{\frac{2^{1999}+1}{3}} \prod_{k=1}^{2^{1999}} f(k) = -\prod_{k=1}^{2^{1999}} f(k).$$

Combined with the expression in (1), this implies that the desired product is $(-3)(-1) = 3$.

Problem 11.2 Let m and n be positive integers. Starting with the list 1, 2, 3, ..., we can form a new list of positive integers in two different ways.

(i) We first erase every mth number in the list (always starting with the first); then, in the list obtained, we erase every nth number. We call this *the first derived list.*

(ii) We first erase every nth number in the list; then, in the list obtained, we erase every mth number. We call this *the second derived list.*

Now, we call a pair (m, n) *good* if and only if the following statement is true: if some positive integer k appears in both derived lists, then it appears in the same position in each.

(a) Prove that $(2, n)$ is good for any positive integer n.

(b) Determine if there exists any good pair (m, n) such that $2 < m < n$.

Solution: Consider whether some positive integer j is in the first derived list. If it is congruent to $1 \pmod m$, then $j + mn$ is as well so they are both erased. If not, then suppose it is the t-th number remaining after we've erased all the multiples of m. There are n multiples of m erased between j and $j+mn$, so $j+mn$ is the $(t+mn-n)$-th number remaining after we've erased all the multiples of m. But either t and $t+mn-n$ are both congruent to $1 \pmod{n}$ or both *not* congruent to $1 \pmod{n}$. Hence j is erased after our second pass if and only if $j+mn$ is as well.

A similar argument applies to the second derived list. Thus in either derived list, the locations of the erased numbers repeat with period mn; and also, among each mn consecutive numbers exactly $mn-(m+n-1)$ remain. (In the first list, $n+\left(\left\lfloor\frac{mn-n-1}{n}\right\rfloor+1\right) = n+\left(m-1+\left\lfloor\frac{-1}{n}\right\rfloor+1\right) = m+n-1$ of the first mn numbers are erased; similarly, $m+n-1$ of the first mn numbers are erased in the second list.)

These facts imply that the pair (m,n) is good if and only if when any $k \le mn$ is in both lists, it appears at the same position.

(a) Given a pair $(2,n)$, the first derived list (up to $k=2n$) is $4,6,8,\ldots,2n$. If n is even, the second derived list is $3,5,\ldots,n-1,n+2,n+4,\ldots,2n$. And if n is odd, the second derived list is $3,5,\ldots,n-2,n,n+3,n+5,\ldots,2n$. In either case the first and second lists' common elements are the even numbers between $n+2$ and $2n$ inclusive. Each such $2n-i$ (with $i<\frac{n-1}{2}$) is the $(n-1-i)$-th number on both lists, showing that $(2,n)$ is good.

(b) Such a pair exists—in fact, the simplest possible pair $(m,n) = (3,4)$ suffices. The first derived list (up to $k=12$) is $3,5,6,9,11,12$ and the second derived list is $3,4,7,8,11,12$. The common elements are $3,11,12$, and these are all in the same positions.

Problem 11.3 Let a_1, a_2, $\ldots$, a_{100} be an ordered set of numbers. At each move it is allowed to choose any two numbers a_n, a_m and change them to the numbers

$$\frac{a_n^2}{a_m}-\frac{n}{m}\left(\frac{a_m^2}{a_n}-a_m\right) \quad \text{and} \quad \frac{a_m^2}{a_n}-\frac{m}{n}\left(\frac{a_n^2}{a_m}-a_n\right)$$

respectively. Determine if it is possible, starting with the set with $a_i=\frac{1}{5}$ for $i=20$, 40, 60, 80, 100 and $a_i=1$ otherwise, to obtain a set consisting of integers only.

Solution: After transforming a_n to $a_n' = \frac{a_n^2}{a_m}-\frac{n}{m}\left(\frac{a_m^2}{a_n}-a_m\right)$ and

a_m to $a'_m = \frac{a_m^2}{a_n} - \frac{m}{n}\left(\frac{a_n^2}{a_m} - a_n\right)$, we have

$$\begin{aligned}
&\frac{a'_n}{n} + \frac{a'_m}{m} \\
&= \left[\left(\frac{1}{n}\cdot\frac{a_n^2}{a_m} - \frac{1}{m}\cdot\frac{a_m^2}{a_n}\right) + \frac{a_m}{m}\right] \\
&\quad + \left[\left(\frac{1}{m}\cdot\frac{a_m^2}{a_n} - \frac{1}{n}\cdot\frac{a_n^2}{a_m}\right) + \frac{a_n}{n}\right] \\
&= \frac{a_n}{n} + \frac{a_m}{m}.
\end{aligned}$$

Thus the quantity $\sum_{i=1}^{100} \frac{a_i}{i}$ is invariant under the given operation. At the beginning, this sum equals

$$I_1 = \sum_{i=1}^{99} \frac{a_i}{i} + \frac{1}{500}.$$

When each of the numbers $\frac{a_1}{1}, \frac{a_2}{2}, \ldots, \frac{a_{99}}{99}$ is written as a fraction in lowest terms, none of their denominators are divisible by 125; while 125 *does* divide the denominator of $\frac{1}{500}$. Thus when written as a fraction in lowest terms, I_1 must have a denominator divisible by 125.

Now suppose by way of contradiction that we could make all the numbers equal to integers $b_1, b_2, \ldots, b_{100}$ in that order. Then in $I_2 = \sum_{i=1}^{100} \frac{b_i}{i}$, the denominator of each of the fractions $\frac{b_i}{i}$ is not divisible by 125. Thus when I_2 is written as a fraction in lowest terms, its denominator is not divisible by 125 either. But then I_2 cannot possibly equal I_1, a contradiction. Therefore we can never obtain a set consisting of integers only.

Problem 11.4 A circle is inscribed in the trapezoid $ABCD$. Let K, L, M, N be the points of intersections of the circle with diagonals AC and BD respectively (K is between A and L and M is between B and N). Given that $AK \cdot LC = 16$ and $BM \cdot ND = \frac{9}{4}$, find the radius of the circle.

Solution: Let the circle touch sides AB, BC, CD, DA at P, Q, R, S, respectively, and let r be the radius of the circle. Let $w = AS = AP$; $x = BP = BQ$; $y = CQ = CR$; and $z = DR = DS$. As in problem 11.4, we have $wz = xy = r^2$ and thus $wxyz = r^4$. Also observe that from the lemma in problem 11.4, $AK \cdot LC$ depends only on r and $AP \cdot CR$; and $BM \cdot ND$ depends only on r and $BP \cdot DR$.

Now draw a parallelogram $A'B'C'D'$ circumscribed about the same circle, with points P', Q', R', S' defined analagously to P, Q, R, S, such that $A'P' = C'R' = \sqrt{wy}$. Draw points K', L', M', N' analagously to K, L, M, N. Then since $A'P' \cdot C'R' = wy$, by the observation in the first paragraph we must have $A'K' \cdot L'C' = AK \cdot LC = 16$; therefore $A'K' = L'C' = 4$. And as with quadrilateral $ABCD$, we have $A'P' \cdot B'P' \cdot C'R' \cdot D'R' = r^4 = wxyz$. Thus $B'P' \cdot D'R' = xz$ and again by the observation we must have $B'M' \cdot N'D' = BM \cdot ND = \frac{9}{4}$. Therefore $B'M' = N'D' = \frac{3}{2}$.

Then if O is the center of the circle, we have $A'O = 4 + r$ and $S'O = r$. By the Pythagorean Theorem $A'S' = \sqrt{8r + 16}$; similarly, $S'D' = \sqrt{3r + \frac{9}{4}}$. Since $A'S' \cdot S'D' = r^2$, we have

$$(8r + 16)\left(3r + \frac{9}{4}\right) = r^4,$$

which has positive solution $r = 6$ and, by Descartes' rule of signs, no other positive solutions.

Problem 11.5 Find the greatest real number k such that for any triple of positive real numbers a, b, c such that

$$kabc > a^3 + b^3 + c^3,$$

there exists a triangle with side legths a, b, c.

Solution: Equivalently, we want the greatest real number k such that for any a, b, $c > 0$ with $a + b \leq c$, we have

$$kabc \leq a^3 + b^3 + c^3.$$

First pick $b = a$ and $c = 2a$. Then we must have

$$2ka^3 \leq 10a^3 \Longrightarrow k \leq 5.$$

On the other hand, suppose $k = 5$. Then writing $c = a + b + x$, expanding $a^3 + b^3 + c^3 - 5abc$ gives

$$2a^3 + 2b^3 - 2a^2b - 2ab^2 + abx + 3(a^2 + b^2)x + 3(a + b)x^2 + x^3.$$

But $2a^3 + 2b^3 - 2a^2b - 2ab^2 \geq 0$ (either by rearrangement, by AM-GM, or from the inequality $(a + b)(a - b)^2 \geq 0$); and the other terms are nonnegative. Thus $a^3 + b^3 + c^3 - 5abc \geq 0$, as desired.

Problem 11.6 Find all integers x and y such that

$$x^6 + x^3y = y^3 + 2y^2.$$

Solution: The only solutions are (x, y) equals $(0, 0)$, $(0, -2)$, and $(2, 4)$.

If $x = 0$ then $y = 0$ or -2; if $y = 0$ then $x = 0$. Now assume that both x and y are nonzero, and rewrite the given equation as $x^3(x^3 + y) = y^2(y + 2)$.

We first show that $(x, y) = (ab, 2b^3)$, (ab, b^3), or $(ab, \frac{b^3}{2})$ for some integers a, b. Suppose some prime p divides y exactly $m > 0$ times (that is, y is divisible by p^m but not p^{m+1}). Then since $x^6 = y^3 + 2y^2 - x^3y$, p must divide x as well — say, $n > 0$ times.

First suppose $p > 2$; then it divides the right hand side $y^2(y + 2)$ exactly $2m$ times. If $3n < m$ then p divides the left hand side $x^3(x^3 + y)$ exactly $6n$ times so that $6n = 2m$, a contradiction. And if $3n > m$ then p divides the left hand side exactly $3n + m$ times so that $3n + m = 2m$ and $3n = m$, a contradiction. Therefore $3n = m$.

Now suppose $p = 2$. If $m > 1$, then 2 divides the right hand side exactly $2m + 1$ times. If $3n < m$ then 2 divides the left hand side $6n$ times so that $6n = 2m + 1 > 2m$, a contradiction. If $3n > m$, then 2 divides the left hand side $3n + m$ times so that $3n + m = 2m + 1$ and $3n = m + 1$. Or finally, we could have $3n = m$.

We wish to show that $(x, y) = (ab, 2b^3)$, (ab, b^3), or $(ab, \frac{b^3}{2})$. If 2 divides y only once, then from before (since $3n = m$ when $p > 2, m > 0$) we have $y = 2b^3$ and $x = ab$ for some a, b. And if 2 divides y more than once, then (since $3n = m$ when $p > 2, m > 0$ and since $3n = m$ or $m + 1$ when $p = 2, m > 1$) we either have $(x, y) = (ab, b^3)$ or $(x, y) = (ab, \frac{b^3}{2})$.

Now simply plug these possibilities into the equation. We then either have $a^6 + a^3 = b^3 + 2$, $a^6 + 2a^3 = 8b^3 + 8$, or $8a^6 + 4a^3 = b^3 + 4$.

In the first case, if $a > 1$ then $b^3 = a^6 + a^3 - 2$ and some algebra verifies that $(a^2 + 1)^3 > b^3 > (a^2)^3$, a contradiction; if $a < 0$ then we have $(a^2)^3 > b^3 > (a^2 - 1)^3$. Thus either $a = 0$ and $x = 0$ or $a = 1$ and $b = 0$. But we've assumed $x, y \neq 0$, so this case yields no solutions.

In the second case, if $a > 0$ then $(a^2 + 1)^3 > (2b)^3 > (a^2)^3$. If $a < -2$ then $(a^2)^3 > (2b)^3 > (a^2 - 1)^3$. Thus either $a = -2, -1$, or 0; and these yield no solutions either.

Finally, in the third case when $a > 1$ then $(2a^2+1)^3 > b^3 > (2a^2)^3$. When $a < -1$ then $(2a^2)^3 > b^3 > (2a^2-1)^3$. Thus either $a = -1, 0$, or 1; this yields both $(a, b) = (-1, 0)$ and $(a, b) = (1, 2)$. Only the latter gives a solution where $x, y \neq 0$ — namely, $(x, y) = (2, 4)$. This completes the proof.

Problem 11.7 Let O be the center of circle ω. Two equal chords AB and CD of ω intersect at L such that $AL > LB$ and $DL > LC$. Let M and N be points on AL and DL respectively such that $\angle ALC = 2\angle MON$. Prove that the chord of ω passing through M and N is equal to AB and CD.

Solution: We work backward. Suppose that P is on minor arc $\widehat{AC}$ and Q is on minor arc $\widehat{BD}$ such that $PQ = AB = CD$, where line PQ hits $\overline{AL}$ at M' and $\overline{DL}$ at N'. We prove that $\angle ALC = 2\angle M'ON'$.

Say that the midpoints of $\overline{AB}$, $\overline{PQ}$, $\overline{CD}$ are T_1, T_2, and T_3. $\overline{CD}$ is the image of $\overline{AB}$ under the rotation about O through angle $\angle T_1OT_3$; this angle also equals the measure of $\widehat{AC}$, which equals $\angle ALC$. Also, by symmetry we have $\angle T_1OM' = \angle M'OT_2$ and $\angle T_2ON' = \angle N'OT_3$. Therefore

$$\begin{aligned}\angle ALC &= \angle T_1OT_3 = \angle T_1OT_2 + \angle T_2OT_3 \\ &= 2(\angle M'OT_2 + \angle T_2ON') = 2\angle M'ON',\end{aligned}$$

as claimed.

Now back to the original problem. Since $\angle T_1OT_3 = \angle ALC$, $\angle T_1OL = \frac{1}{2}T_1OT_3 = \frac{1}{2}\angle ALC$. Then since $\angle MON = \frac{1}{2}\angle ALC = \angle T_1OL$, M must lie on $\overline{T_1L}$. Then look at the rotation about O that sends T_1 to M; it sends A to some P on $\widehat{AC}$, and B to some point Q on $\widehat{BD}$. Then $\overline{PQ}$ is a chord with length AB, passing through M on $\overline{AL}$ and N' on $\overline{DL}$. From the previous work, we know that $\angle ALC = 2\angle MON'$; and since $\angle ALC = 2\angle MON$, we must have $N = N'$. Thus the length of the chord passing through M and N indeed equals AB and CD, as desired.

IMO Selection Tests

Problem 1 Find all functions $h : \mathbb{Z} \to \mathbb{Z}$ such that

$$h(x + y) + h(xy) = h(x)h(y) + 1$$

for all x, $y \in \mathbb{Z}$.

Solution: There are three possible functions:

$$h(n) = 1;$$
$$h(2n) = 1, \quad h(2n+1) = 0;$$
$$h(n) = n + 1.$$

Plugging $(x, y) = (0, 0)$ into the functional equation, we find that

$$h(0)^2 - 2h(0) + 1 = 0$$

and hence $h(0) = 1$. Plugging in $(x, y) = (1, -1)$ then yields

$$h(0) + h(-1) = h(1)h(-1) + 1$$

and

$$h(-1) = h(1)h(-1),$$

and thus either $h(-1) = 0$ or $h(1) = 1$.

First suppose that $h(1) \neq 1$; then $h(-1) = 0$. Then plugging in $(x, y) = (2, -1)$ and $(x, y) = (-2, 1)$ yields $h(1) + h(-2) = 1$ and $h(-2) = h(-2)h(1) + 1$. Substituting $h(-2) = 1 - h(1)$ into the second equation, we find that

$$1 - h(1) = (1 - h(1))h(1) + 1,$$
$$h(1)^2 - 2h(1) = 0, \quad \text{and} \quad h(1)(h(1) - 2) = 0,$$

implying that $h(1) = 0$ or $h(1) = 2$.

Thus, $h(1) = 0$, 1, or 2. Plugging $y = 1$ into the equation for each of these cases shows that h must be one of the three functions presented.

Problem 2 Let a, b, $c \in \mathbb{Q}$, $ac \neq 0$. Given that the equation $ax^2 + bxy + cy^2 = 0$ has a non-zero solution of the form

$$(x, y) = (a_0 + a_1\sqrt[3]{2} + a_2\sqrt[3]{4}, b_0 + b_1\sqrt[3]{2} + b_2\sqrt[3]{4})$$

with a_i, $b_i \in \mathbb{Q}$, $i = 0$, 1, 2, prove that it has also has a non-zero rational solution.

Solution: Let $(\alpha, \beta) = (a_0 + a_1\sqrt[3]{2} + a_2\sqrt[3]{4}, b_0 + b_1\sqrt[3]{2} + b_2\sqrt[3]{4})$ be the given solution, and suppose without loss of generality that β is non-zero. Then $\frac{\alpha}{\beta}$ is a root to the polynomial

$$at^2 + bt + c = 0.$$

Also, $\frac{\alpha}{\beta}$ is of the form $c_0 + c_1\sqrt[3]{2} + c_2\sqrt[3]{4}$ for some rationals c_0, c_1, c_2. But because it is a root to a quadratic with rational coefficients, it must also be of the form $d + e\sqrt{f}$ for rationals d, e, f.

Thus we have $(c_0 - d) + c_1\sqrt[3]{2} + c_2\sqrt[3]{4} = e\sqrt{f}$, so the quantity $\left(c_0' + c_1\sqrt[3]{2} + c_2\sqrt[3]{4}\right)^2$ must be an integer (where we write $c_0' = c_0 - d$). After expanding this square, the coefficients of $\sqrt[3]{2}$ and $\sqrt[3]{4}$ are $2(c_2^2 + c_0'c_1)$ and $2c_0'c_2 + c_1^2$, respectively; these quantities must equal zero. From $2c_0'c_2 + c_1^2 = 0$ we have $(c_0'c_1)^2 = -2c_0'^3c_2$; and from $c_2^2 + c_0'c_1 = 0$ we have $(c_0'c_1)^2 = c_2^4$. Thus $-2c_0'^3c_2 = c_2^4$. This implies that either $c_2 = 0$ or $c_2 = -\sqrt[3]{2}c_0'$; in the latter case, since c_2 is rational we must still have $c_2 = c_0' = 0$.

Then $c_1 = 0$ as well, and $\frac{\alpha}{\beta} = c_0$ is *rational.* Thus $(x, y) = (\frac{\alpha}{\beta}, 1)$ is a non-zero rational solution to the given equation.

Problem 3 Suppose a and b are positive integers such that the product of all divisors of a (including 1 and a) is equal to the product of all divisors of b (including 1 and b). Does it follow that $a = b$?

Solution: Yes, it follows that $a = b$. Let $d(n)$ denote the number of divisors of n; then the product of all divisors of n is

$$\prod_{k|n} k = \sqrt{\prod_{k|n} k \cdot \prod_{k|n} \frac{n}{k}} = \sqrt{\prod_{k|n} n} = n^{\frac{d(n)}{2}}.$$

Thus the given condition implies that $a^{d(a)}$ and $b^{d(b)}$ equal the same number N. Since N is both a perfect $d(a)$-th power and a perfect $d(b)$-th power, it follows that it is also a perfect ℓ-th power of some number t, where $\ell = \operatorname{lcm}(d(a), d(b))$. Then $a = t^{\frac{\ell}{d(a)}}$ and $b = t^{\frac{\ell}{d(b)}}$ are both powers of the same number t as well.

Now if a is a bigger power of t than b, then it must have more divisors than b; but then $t^{\frac{\ell}{d(a)}} < t^{\frac{\ell}{d(b)}}$, a contradiction. Similarly a cannot be a smaller power of t than b. Therefore $a = b$, as claimed.

Problem 4 Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + ca} \geq \frac{3}{2}.$$

Solution: Using the AM-HM inequality or the Cauchy-Schwarz inequality, we have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{9}{x+y+z}$$

for x, y, $z\geq 0$. Also, notice that $a^2+b^2+c^2\geq ab+bc+ca$ since this inequality is equivalent to $\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2\geq 0$. Thus,

$$\begin{aligned}\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}&\geq\frac{9}{3+ab+bc+ca}\\&\geq\frac{9}{3+a^2+b^2+c^2}\geq\frac{3}{2},\end{aligned}$$

as desired.

Problem 5 Suppose triangle T_1 is similar to triangle T_2, and the lengths of two sides and the angle between them of T_1 are proportional to the lengths of two sides and the angle between them of T_2 (but not necessarily the corresponding ones). Must T_1 be congruent to T_2?

Solution: The triangles are not necessarily congruent. Say the vertices of T_1 are A, B, C with $AB=4$, $BC=6$, and $CA=9$, and say that $\angle BCA=k\angle ABC$.

Then let the vertices of T_2 be D, E, F where $DE=\frac{8k}{3}$, $EF=4k$, and $FD=6k$. Triangles ABC and DEF are similar in that order, so $\angle EFD=\angle BCA=k\angle ABC$; also, $EF=k\cdot AB$ and $FD=k\cdot BC$. Therefore these triangles satisfy the given conditions.

Now since $AB<AC$ we have $\angle BCA<\angle ABC$ and $k<1$; so $DE=\frac{8k}{3}<\frac{8}{3}<AB$. Thus triangles ABC and DEF are not congruent, as desired.

Problem 6 Two real sequences x_1, x_2, $\ldots$, and y_1, y_2, $\ldots$, are defined in the following way:

$$x_1=y_1=\sqrt{3},\quad x_{n+1}=x_n+\sqrt{1+x_n^2},\quad y_{n+1}=\frac{y_n}{1+\sqrt{1+y_n^2}}$$

for all $n\geq 1$. Prove that $2<x_ny_n<3$ for all $n>1$.

First Solution: Let $z_n=\frac{1}{y_n}$ and notice that the recursion for y_n is equivalent to

$$z_{n+1}=z_n+\sqrt{1+z_n^2}.$$

Also note that $z_2 = \sqrt{3} = x_1$; since the x_i and z_i satisfy the same recursion, this means that $z_n = x_{n-1}$ for all $n > 1$. Thus,

$$x_n y_n = \frac{x_n}{z_n} = \frac{x_n}{x_{n-1}}.$$

Because the x_i are increasing, for $n > 1$ we have $x_{n-1}^2 \geq x_1^2 = 3 > \frac{1}{3} \Rightarrow 2x_{n-1} > \sqrt{1 + x_{n-1}^2} \Rightarrow 3x_{n-1} > x_n$. Also, $\sqrt{1 + x_{n-1}^2} > x_{n-1} \Rightarrow x_n > 2x_{n-1}$. Therefore,

$$2 < x_n y_n = \frac{x_n}{x_{n-1}} < 3,$$

as desired.

Second Solution: Writing $x_n = \tan a_n$ for $0° < a_n < 90°$, we have

$$\begin{aligned} x_{n+1} &= \tan a_n + \sqrt{1 + \tan^2 a_n} = \tan a_n + \sec a_n \\ &= \frac{1 + \sin a_n}{\cos a_n} = \tan\left(\frac{90° + a_n}{2}\right). \end{aligned}$$

Since $a_1 = 60°$, we have $a_2 = 75°$, $a_3 = 82.5°$, and in general $a_n = 90° - \frac{30°}{2^{n-1}}$. Thus

$$x_n = \tan\left(90° - \frac{30°}{2^{n-1}}\right) = \cot\left(\frac{30°}{2^{n-1}}\right) = \cot\theta_n,$$

where $\theta_n = \frac{30°}{2^{n-1}}$.

Similar calculation shows that

$$y_n = \tan 2\theta_n = \frac{2\tan\theta_n}{1 - \tan^2\theta_n},$$

implying that

$$x_n y_n = \frac{2}{1 - \tan^2\theta_n}.$$

Since $0° < \theta_n < 45°$, we have $0 < \tan^2\theta_n < 1$ and $x_n y_n > 2$. And since for $n > 1$ we have $\theta_n < 30°$, we also have $\tan^2\theta_n < \frac{1}{3}$ so that $x_n y_n < 3$.

Note: From the closed forms for x_n and y_n in the second solution, we can see the relationship $y_n = \frac{1}{x_{n-1}}$ used in the first solution.

Problem 7 Let O be the center of the excircle of triangle ABC opposite A. Let M be the midpoint of $\overline{AC}$, and let P be the intersection of lines MO and BC. Prove that if $\angle BAC = 2\angle ACB$, then $AB = BP$.

First Solution: Since O is the excenter opposite A, we know that O is equidistant from lines AB, BC, and CA. We also know that line AO bisects angle BAC. Thus $\angle BAO = \angle OAC = \angle ACB$. Letting D be the intersection of $\overline{AO}$ and $\overline{BC}$, we then have $\angle DAC = \angle ACD$ and hence $DC = AD$.

Consider triangles OAC and ODC. From above their altitudes from O are equal, and their altitudes from C are also clearly equal. Thus, $OA/OD = [OAC]/[ODC] = AC/DC$.

Next, because M is the midpoint of $\overline{AC}$ we have $[OAM] = [OMC]$ and $[PAM] = [PMC]$, and hence $[OAP] = [OPC]$ as well. Then

$$\frac{OA}{OD} = \frac{[OAP]}{[ODP]} = \frac{[OPC]}{[ODP]} = \frac{PC}{DP}.$$

Thus, $\frac{AC}{DC} = \frac{OA}{OD} = \frac{PC}{DP}$, and $\frac{AC}{CP} = \frac{DC}{DP} = \frac{AD}{DP}$. By the Angle Bisector Theorem, $\overline{AP}$ bisects $\angle CAD$.

It follows that $\angle BAP = \angle BAD + \angle DAP = \angle ACP + \angle PAC = \angle APB$, and therefore $BA = BP$, as desired.

Second Solution: Let R be the midpoint of the arc BC (not containing A) of the circumcircle of triangle ABC; and let I be the incenter of triangle ABC. We have $\angle RBI = \frac{1}{2}(\angle CAB + \angle ABC) = \frac{1}{2}(180^\circ - \angle BRI)$. Thus $RB = RI$ and similarly $RC = RI$, and hence R is the circumcenter of triangle BIC. But since $\angle IBO = 90^\circ = \angle ICO$, quadrilateral $IBOC$ is cyclic and R is also the circumcenter of triangle BCO.

Let lines AO and BC intersect at Q. Since M, O, and P are collinear we may apply Menelaus' Theorem to triangle AQC to get

$$\frac{AM}{CM}\frac{CP}{QP}\frac{QO}{AO} = 1.$$

But $\frac{AM}{CM} = 1$, and therefore $\frac{CP}{PQ} = \frac{AO}{QO}$.

And since R lies on $\overline{AO}$ and $\overline{QO}$, we have

$$\frac{AO}{QO} = \frac{AR + RO}{QR + RO} = \frac{AR + RC}{CR + RQ},$$

which in turn equals $\frac{AC}{CQ}$ since triangles ARC and CRQ are similar; and $\frac{AC}{CQ} = \frac{AC}{AQ}$ since we are given that $\angle BAC = 2\angle ACB$; i.e.,

$\angle QAC = \angle QCA$ and $CQ = AQ$. Thus we have shown that $\frac{CP}{PQ} = \frac{AC}{AQ}$. By the Angle-Bisector Theorem, this implies that line AP bisects $\angle QAC$, from which it follows that $\angle BAP = \frac{3}{2}\angle ACB = \angle BPA$ and $AB = BP$.

Problem 8 Let O, O_1 be the centers of the incircle and the excircle opposite A of triangle ABC. The perpendicular bisector of $\overline{OO_1}$ meets lines AB and AC at L and N respectively. Given that the circumcircle of triangle ABC touches line LN, prove that triangle ABC is isosceles.

Solution: Let M be the midpoint of arc $\widehat{BC}$ not containing A. Angle-chasing gives $\angle OBM = \frac{1}{2}(\angle A + \angle B) = \angle BOM$ and hence $MB = MO$.

Since $\angle OBC = \frac{\angle B}{2}$ and $\angle CBO_1 = \frac{1}{2}(\pi - \angle B)$, we have $\angle OBO_1$ is a right angle. And since we know both that M lies on line AOO_1 (the angle bisector of $\angle A$) and that $MB = MO$, it follows that $\overline{BM}$ is a median to the hypotenuse of right triangle OBO_1 and thus M is the midpoint of $\overline{OO_1}$.

Therefore, the tangent to the circumcircle of ABC at M must be perpendicular to line AM. But this tangent is also parallel to line BC, implying that AM, the angle bisector of $\angle A$, is perpendicular to line BC. This can only happen if $AB = AC$, as desired.

Problem 9 Does there exist a bijection f of

(a) a plane with itself

(b) three-dimensional space with itself

such that for any distinct points A, B line AB and line $f(A)f(B)$ are perpendicular?

Solution:

(a) Yes: simply rotate the plane 90° about some axis perpendicular to it. For example, in the xy-plane we could map each point (x, y) to the point $(y, -x)$.

(b) Suppose such a bijection existed. Label the three-dimensional space with x-, y-, and z-axes; given any point $P = (x_0, y_0, z_0)$, we also view it as the vector p from $(0, 0, 0)$ to (x_0, y_0, z_0). Then the given condition says that

$$(a - b) \cdot (f(a) - f(b)) = 0$$

for any vectors a, b.

Assume without loss of generality that f maps the origin to itself; otherwise, $g(p) = f(p) - f(0)$ is still a bijection and still satisfies the above equation. Plugging $b = (0, 0, 0)$ into the equation above we have $a \cdot f(a) = 0$ for all a. Then the above equation reduces to

$$a \cdot f(b) + b \cdot f(a) = 0.$$

Given any vectors a, b, c and any reals m, n we then have

$$m\,(a \cdot f(b) + b \cdot f(a)) = 0$$
$$n\,(a \cdot f(c) + c \cdot f(a)) = 0$$
$$a \cdot f(mb + nc) + (mb + nc) \cdot f(a) = 0.$$

Adding the first two equations and subtracting the third gives

$$a \cdot (mf(b) + nf(c) - f(mb + nc)) = 0.$$

Since this must be true for any vector a, we must have $f(mb + nc) = mf(B) + nf(C)$. Therefore f is linear, and it is determined by how it transforms the unit vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$. If $f(\mathbf{i}) = (a_1, a_2, a_3)$, $f(\mathbf{j}) = (b_1, b_2, b_3)$, and $f(\mathbf{k}) = (c_1, c_2, c_3)$, then for a vector x we have

$$f(x) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{bmatrix} x.$$

Applying $f(a) \cdot a = 0$ with $a = \mathbf{i}, \mathbf{j}, \mathbf{k}$, we have $a_1 = b_2 = c_3 = 0$. Then applying $a \cdot f(b) + b \cdot f(a)$ with $(a, b) = (\mathbf{i}, \mathbf{j})$, $(\mathbf{j}, \mathbf{k})$, $(\mathbf{j}, \mathbf{k})$ we have $b_1 = -a_2$, $c_1 = -a_3$, $c_2 = -b_3$. But then the determinant of the array in the equation is

$$a_2 b_3 c_1 + a_3 b_1 c_2 = -a_2 b_3 a_3 + a_3 a_2 b_3 = 0,$$

so there exist constants k_1, k_2, k_3 not all zero such that $k_1 f(\mathbf{i}) + k_2 f(\mathbf{j}) + k_3 f(\mathbf{k}) = 0$. But then $f(k_1, k_2, k_3) = 0 = f(0, 0, 0)$, contradicting the assumption that f was a bijection!

Therefore our original assumption was false, and no such bijection exists.

Problem 10 A word is a finite sequence of two symbols a and b. The number of the symbols in the word is said to be the length of the word. A word is called 6-*aperiodic* if it does not contain a subword of the form *cccccc* for any word c. Prove that $f(n) > \left(\frac{3}{2}\right)^n$, where $f(n)$ is the total number of 6-aperiodic words of length n.

Solution: Rather than attempting to count all such words, we add some restrictions and count only some of the 6-aperiodic words. Also, instead of working with a's and b's we'll work with 0's and 1's.

The Thue-Morse sequence is defined by $t_0 = 0$, $t_1 = 1$, $t_{2n+1} = 1 - t_{2n}$, and $t_{2n} = t_n$. These properties can be used to show that the only subwords of the form $cc\ldots c$ are 00 and 11.

We restrict the 6-aperiodic words in a similar spirit. Call a word $x_1x_2\ldots x_n$ of length n *6-countable* if it satisfies the following conditions:

(i) $x_{5i} = x_i$ for $1 \le i$.

(ii) $x_{5i-1} = 1 - x_{5i}$ for $1 \le i \le \frac{n}{5}$.

(iii) If $(x_{5i+2}, x_{5i+3}, x_{5i+4}) = (1,0,1)$ [or $(0,1,0)$], then $(x_{5i+7}, x_{5i+8}, x_{5i+9}) \ne (0,1,0)$ [or $(1,0,1)$].

Lemma 1. *Every 6-countable word is 6-aperiodic.*

Proof: Suppose by way of contradiction that some 6-countable word contains a subword of the form *cccccc*, where the strings c appear in the positions x_j through $x_{j+\ell-1}$; $x_{j+\ell}$ through $x_{j+2\ell-1}$; and so on up to $x_{j+5\ell}$ through $x_{j+6\ell-1}$. Pick a word with the smallest such ℓ.

If $5 \mid \ell$, then look at the indices i between j and $j+\ell-1$ such that $5 \mid i$; say they are $5i_1, 5i_2, \ldots, 5i_{\ell/5}$. Then $x_{5i_1}x_{5i_2}\ldots x_{5i_{\ell/5}}$, $x_{5i_1+\ell}x_{5i_2+\ell}\ldots x_{5i_{\ell/5}+\ell}$, $\ldots$, $x_{5i_1+5\ell}x_{5i_2+5\ell}\ldots x_{5i_{\ell/5}+5\ell}$ all equal the same string c'; then (using the first condition of countability) the subword starting at x_{i_1} and ending on $x_{i_{\ell/5}+\ell}$ is of the form $c'c'c'c'c'c'$. But this contradicts the minimal choice of ℓ; therefore, we *can't* have $5 \mid \ell$.

Now, suppose that in the first appearance of c some two adjacent characters a_j, a_{j+1} were equal. Then since $5 \nmid \ell$, one of $j, j+\ell, j+2\ell, \ldots, j+4\ell$ is $4 \pmod 5$ — say, $j+k\ell$. Then $a_{j+k\ell}$, $a_{j+k\ell+1}$ must be the same since $a_ja_{j+1} = a_{j+k\ell}a_{j+k\ell+1}$; but they must also be different from the second condition of 6-countability! Because this is impossible, it follows that the characters in c alternate between 0 and 1.

A similar argument, though, shows that $a_{j+\ell-1}$ and $a_{j+\ell}$ must be different; hence c is of the form $1010\ldots10$ or $0101\ldots01$. But this would imply that our word violated the third condition of 6-countability—a contradiction. Therefore our original assumption was false, and any 6-countable word is 6-aperiodic. ■

Lemma 2. *Given a positive integer m, there are more than $\left(\frac{3}{2}\right)^{5m}$ 6-countable words of length $5m$.*

Proof: Let α_m be the number of length-$5m$ 6-countable words. To create a length-$5m$ 6-countable word $x_1x_2\ldots x_{5m}$, we can choose each of the "three-strings" $x_1x_2x_3$, $x_6x_7x_8$, $\ldots$, $x_{5m-4}x_{5m-3}x_{5m-2}$ to be any of the eight strings 000, 001, 010, 011, 100, 101, 110, or 111—taking care that no two adjacent strings are 010 and 101. Some quick counting then shows that $\alpha_1 = 8 > \left(\frac{3}{2}\right)^5$ and $\alpha_2 = 64 - 2 = 62 > \left(\frac{3}{2}\right)^{10}$.

Let β_m be the number of length-$5m$ 6-countable words whose last three-string is 101; by symmetry, this also equals the number of length-$5m$ 6-countable words whose last three-string is 010. Also let γ_m be the number of length-$5m$ 6-countable words whose last three-string is *not* 101; again by symmetry, this also equals the number of length-$5m$ 6-countable words whose last three-string isn't 010. Note that $\alpha_m = \gamma_m + \beta_m$.

For $m \geq 1$, observe that $\gamma_m = \beta_{m+1}$ because to any length-$5m$ word whose last three-string isn't 010, we can append the three-string 101 (as well as two other pre-determined numbers); and given a length-$5(m+1)$ word whose last three- string is 101, we can reverse this construction. Similar arguing shows that $\gamma_{m+1} = 6(\gamma_m + \beta_m) + \gamma_m$; the $6(\gamma_m + \beta_m)$ term counts the words whose last three-string is neither 010 nor 101, and the γ_m term counts the words whose last three-string is 010. Combined, these recursions give

$$\gamma_{m+2} = 7\gamma_{m+1} + 6\gamma_m$$
$$\beta_{m+2} = 7\beta_{m+1} + 6\beta_m$$
$$\alpha_{m+2} = 7\alpha_{m+1} + 6\alpha_m.$$

Now if $\alpha_{m+1} > \left(\frac{3}{2}\right)^{5m+5}$ and $\alpha_m > \left(\frac{3}{2}\right)^{5m}$, then

$$\begin{aligned}\alpha_{m+2} &= 7\alpha_{m+1} + 6\alpha_m \\ &> \left(\frac{3}{2}\right)^{5m}\left(7\cdot\left(\frac{3}{2}\right)^5 + 6\right) \\ &> \left(\frac{3}{2}\right)^{5m}\left(\frac{3}{2}\right)^{10} = \left(\frac{3}{2}\right)^{5(m+2)}.\end{aligned}$$

Then since $\alpha_m > \left(\frac{3}{2}\right)^{5m}$ is true for $m = 1, 2$, by induction it is true for all positive integers m. ■

The lemma proves the claim for $n = 5m$. Now suppose we are looking at length-$(5m + i)$ words, where $m \geq 0$ and $i = 1, 2, 3,$ or 4. Then given any length-$5m$ 6-countable word, we can form a length-$(5m+i)$ word by choosing $x_{5m+1}, x_{5m+2}, x_{5m+3}$ to be anything. (For convenience, we say there is exactly $\alpha_0 = 1 \geq \left(\frac{3}{2}\right)^0$ length-0 6-countable word: the "empty word.") Thus there are at least $2\alpha_m > \left(\frac{3}{2}\right)^{5m+1}$, $4\alpha_m > \left(\frac{3}{2}\right)^{5m+2}$, $8\alpha_m > \left(\frac{3}{2}\right)^{5m+3}$, and $8\alpha_m > \left(\frac{3}{2}\right)^{5m+4}$ 6-countable length-$(5m + 1)$, -$(5m + 2)$, -$(5m + 3)$, and -$(5m + 4)$ words, respectively. This completes the proof.

Problem 11 Determine all positive integers n, $n \geq 2$, such that $\binom{n-k}{k}$ is even for $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Solution: Lucas's Theorem states that for integers

$$n = n_r p^r + n_{r-1}p^{r-1} + \cdots + n_0$$

and

$$m = m_r p^r + m_{r-1}p^{r-1} + \cdots + m_0$$

written in base p for a prime p, we have

$$\binom{n}{m} \equiv \binom{n_r}{m_r}\binom{n_{r-1}}{m_{r-1}}\cdots\binom{n_0}{b_0} \pmod{p}.$$

With $p = 2$, the binary representation of $n = 2^s - 1$ we have $n_r = n_{r-1} = \cdots = n_0 = 1$. Then for any $0 \leq m \leq 2^s - 1$ each $\binom{n_i}{m_i} = 1$, and thus $\binom{n}{m} \equiv 1 \cdot 1 \cdot \cdots \cdot 1 \equiv 1 \pmod{2}$.

This implies that n must be one less than a power of 2, or else one of $n - k$ will equal such a number $2^s - 1$ and then $\binom{n-k}{k}$ will be odd.

In fact, all such $n = 2^s - 1$ *do* work: for $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, there is at least one 0 in the binary representation of $n - k$ (not counting leading zeros, of course). And whenever there is a 0 in the binary representation of $n - k$, there is a 1 in the corresponding digit of k. Then the corresponding $\binom{(n-k)_i}{k_i}$ equals 0, and by Lucas's Theorem $\binom{n-k}{k}$ is even.

Therefore, $n = 2^s - 1$ for integers $s \geq 2$.

Problem 12 A number of n players took part in a chess tournament. After the tournament was over, it turned out that among any four players there was one who scored differently against the other three (i.e., he got a victory, a draw, and a loss). Prove that the largest possible n satisfies the inequality $6 \leq n \leq 9$.

Solution:

Let $A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_n$ denote "A_1 beats A_2, A_2 beats A_3, ..., A_{n-1} beats A_n," and let $X \mid Y$ denote "X draws with Y."

First we show it is possible to have the desired results with $n = 6$: call the players A, B, C, D, E, F. Then let

$$A \Rightarrow B \Rightarrow C \Rightarrow D \Rightarrow E \Rightarrow A,$$

$$F \Rightarrow A,\ F \Rightarrow B,\ F \Rightarrow C,\ F \Rightarrow D,\ F \Rightarrow E,$$

and have all other games end in draws. Visually, we can view this arrangement as a regular pentagon $ABCDE$ with F at the center. There are three different types of groups of 4, represented by $ABCD$, $ABCF$, and $ABDF$; in these three respective cases, B (or C), A, and A are the players who score differently from the other three.

Alternatively, let

$$A \Rightarrow B \Rightarrow C \Rightarrow D \Rightarrow E \Rightarrow F \Rightarrow A,$$

$$B \Rightarrow D \Rightarrow F \Rightarrow B, \quad C \Rightarrow A \Rightarrow E \Rightarrow C,$$

$$A \mid D,\ B \mid E,\ C \mid F.$$

In this arrangement there are three different types of groups of four, represented by $\{A, B, C, D\}$, $\{A, B, D, E\}$, and $\{A,\ B,\ D,\ F\}$. (If the players are arranged in a regular hexagon, these correspond to a trapezoid-shaped group, a rectangle-shaped group, and a diamond-shaped group.) In these three cases, A, B (or D), and A (or D) are the players who score differently against the other three.

Now we show it is impossible to have the desired results with $n = 10$ and thus all $n \geq 10$; suppose by way of contradiction it *was* possible. First we prove that all players draw exactly 4 times.

To do this, draw a graph with n vertices representing the players, and draw an edge between two vertices if they drew in their game. If V has degree 3 or less, then look at the remaining 6 or more vertices it is not adjacent to. By Ramsey's Theorem, either three of them (call them X, Y, Z) are all adjacent or all not adjacent. But then in the group $\{V, X, Y, Z\}$, none of the players draws exactly once with the other players, a contradiction.

Thus each vertex has degree at least 4; we now prove that every vertex has degree *exactly* 4. Suppose by way of contradiction that some vertex A was adjacent to at least 5 vertices B, C, D, E, F. None of these vertices can be adjacent to two others; for example, if B was adjacent to C and D then in $\{A, B, C, D\}$ each vertex draws at least twice—but some player must draw exactly once in this group. Now in the group $\{B, C, D, E\}$ some pair must draw: without loss of generality, say B and C. In the group $\{C, D, E, F\}$ some pair must draw as well; since C can't draw with D, E, or F from our previous observation, assume without loss of generality that E and F draw.

Now in $\{A, B, C, D\}$ vertex D must beat one of B, C and lose to the other; without loss of generality, say D loses to B and beats C. Looking at $\{A, D, E, F\}$, we can similarly assume that D beats E and loses to F. Next, in $\{A, C, D, E\}$ players C and E can't draw; without loss of generality, say C beats E. And then in $\{A, C, E, F\}$, player C must lose to F. But then in $\{C, D, E, F\}$ no player scores differently against the other three players—a contradiction.

Now suppose A were adjacent to B, C, D, E, and without loss of generality assume $B \mid C$; then ABC is a triangle. For each J besides A, B, C, look at the group $\{A, B, C, J\}$: J must draw with one of A, B, C. By the Pigeonhole Principle, one of A, B, C draws with at least three of the J and thus has degree at least 5. But this is impossible from above.

It follows that it is *impossible* for n to be at least 10. But since n can be 6, the maximum n is between 6 and 9, as desired.

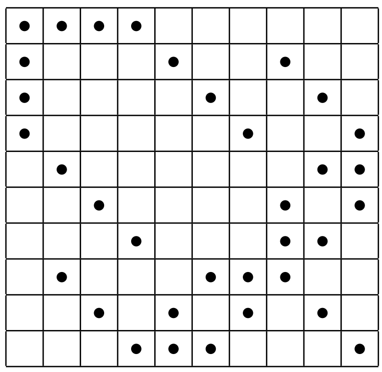
1.2 Brazil

Problem 1 Let $ABCDE$ be a regular pentagon such that the star region $ACEBD$ has area 1. Let AC and BE meet at P, and let BD and CE meet at Q. Determine $[APQD]$.

Solution: Let $R = AD \cap BE$, $S = AC \cap BD$, $T = CE \cap AD$. Now $\triangle PQR \sim \triangle CAD$ because they are corresponding triangles in regular pentagons $QTRPS$ and $ABCDE$, and since $\triangle CAD \sim \triangle PAR$ as well we have $\triangle PQR \cong \triangle PAR$. Thus, $[APQD] = \frac{[APQD]}{[ACEBD]} = \frac{2[APR]+[PQR]+[RQT]}{5[APR]+[PQR]+2[RQT]} = \frac{3[APR]+[RQT]}{6[APR]+2[RQT]} = \frac{1}{2}$.

Problem 2 Given a 10×10 board, we want to remove n of the 100 squares so that no 4 of the remaining squares form the corners of a rectangle with sides parallel to the sides of the board. Determine the minimum value of n.

Solution: The answer is 66. Consider the diagram below, in which a colored circle represents a square that has *not* been removed. The diagram demonstrates that n can be 66:



Now we proceed to show that n is at least 66. Suppose, for contradiction, that it is possible with $n = 65$. Denote by a_i the number of squares left in row i ($i = 1, 2, \ldots, 10$); in row i, there are $\binom{a_i}{2}$ pairs of remaining squares. If no four remaining squares form the corners of a rectangle, then the total number $N = \sum_{i=1}^{10} \binom{a_i}{2}$ must not exceed $\binom{10}{2} = 45$. But note that, with a fixed $\sum_{i=1}^{10} a_i = 35$, the minimum of $\sum_{i=1}^{10} \binom{a_i}{2}$ is attained when and only when no two a_i's differ by more than 1. Thus, $45 = \sum_{i=1}^{10} \binom{a_i}{2} \geq 5 \cdot \binom{4}{2} + 5 \cdot \binom{3}{2} = 45$, i.e., this minimum *is* attained here, implying that five of the a_i's equal 4 and the rest equal 3. Then it is easy to see that aside from permutations of the row and columns, the first five rows of the board must be as follows:

•	•	•	•						
•				•	•	•			
	•			•			•	•	
		•			•		•		•
			•			•		•	•

We inspect this figure and notice that it is now impossible for another row to contain at least 3 remaining squares without forming the vertices of a rectangle with sides parallel to the sides of the board. This is a contradiction, since each of the remaining 5 rows is supposed to have 3 remaining squares. Thus, it is impossible for n to be less than 66, and we are done.

Problem 3 The planet Zork is spherical and has several cities. Given any city A on Zork, there exists an antipodal city A' (i.e., symmetric with respect to the center of the sphere). In Zork, there are roads joining pairs of cities. If there is a road joining cities P and Q, then there is a road joining P' and Q'. Roads don't cross each other, and any given pair of cities is connected by some sequence of roads. Each city is assigned a value, and the difference between the values of every pair of connected cities is at most 100. Prove that there exist two antipodal cities with values differing by at most 100.

Solution: Let $[A]$ denote the value assigned to city A. Name the pairs of cities

$$(Z_1, Z_1'), (Z_2, Z_2'), (Z_3, Z_3'), \ldots, (Z_n, Z_n')$$

with

$$0 \le [Z_i] - [Z_i'] \text{ for all } i.$$

Since any given pair of cities is connected by some sequence of roads, there must exist a, b such that Z_a and Z_b' are connected by a single road. Then Z_a' and Z_b are also connected by a single road. Thus, $[Z_a] - [Z_b'] \le 100$ and $[Z_b] - [Z_a'] \le 100$. Adding, we have

$$[Z_a] - [Z_a'] + [Z_b] - [Z_b'] \le 200.$$

Hence, either $0 \le [Z_a] - [Z'_a] \le 100$ or $0 \le [Z_b] - [Z'_b] \le 100$; in either case, we are done.

Problem 4 In Tumbolia there are n soccer teams. We want to organize a championship such that each team plays exactly once with each other team. All games take place on Sundays, and a team can't play more than one game in the same day. Determine the smallest positive integer m for which it is possible to realize such a championship in m Sundays.

Solution: Let a_n be the smallest positive integer for which it is possible to realize a championship between n soccer teams in a_n Sundays. For $n > 1$, it is necessary that $a_n \ge 2\lceil \frac{n}{2} \rceil - 1$; otherwise the total number of games played would not exceed $(2\lceil \frac{n}{2} \rceil - 2) \cdot \lfloor \frac{n}{2} \rfloor \le \frac{(n-1)^2}{2} < \binom{n}{2}$, a contradiction.

On the other hand, $2\lceil \frac{n}{2} \rceil - 1$ days suffice. Suppose that $n = 2t+1$ or $2t+2$; number the teams from 1 to n and the Sundays from 1 to $2t+1$. On the i-th Sunday, let team i either sit out (if n is odd) or play team $2t+2$ (if n is even); and have any other team j play with the team $k \neq 2t+2$ such that $j + k \equiv 2i \pmod{2t+1}$. Then each team indeed plays every other team, as desired.

Problem 5 Given a triangle ABC, show how to construct, with straightedge and compass, a triangle $A'B'C'$ with minimal area such that A', B', C' lie on AB, BC, CA, respectively, $\angle B'A'C' = \angle BAC$, and $\angle A'C'B' = \angle ACB$.

Solution:

All angles are directed modulo 180°.

For convenience, call any triangle $A'B'C'$ "*zart*" if A', B', C' lie on lines AB, BC, CA, respectively, and $\triangle ABC \sim \triangle A'B'C'$. The problem is, then, to construct the *zart* triangle with minimal area.

Suppose we have any *zart* triangle, and let P be the point (different from A') where the circumcircles of triangles $AA'C'$ and $BB'A'$ meet. Then

$$\angle B'PC' = 360^\circ - \angle A'PB' - \angle C'PA'$$

$$= 360^\circ - (180^\circ - \angle CBA) - (180^\circ - \angle BAC) = 180^\circ - \angle ACB,$$

so P also lies on the circumcircle of triangle $CC'B'$.

Next,

$$\begin{aligned}\angle PAB = \angle PC'A' &= \angle B'C'A' - \angle B'C'P\\ &= \angle B'CC' - \angle B'CP' = \angle PCA,\end{aligned}$$

and with similar reasoning we have

$$\angle PAB = \angle PC'A' = \angle PCA = \angle PB'C' = \angle PBC.$$

There is a unique point P (one of the Brocard points) satisfying $\angle PAB = \angle PBC = \angle PCA$, and thus P is fixed—*independent* of the choice of triangle $A'B'C'$. And since it is the corresponding point in similar triangles ABC and $A'B'C'$, we have

$$[A'B'C'] = [ABC]\left(\frac{PA'}{PA}\right)^2.$$

Thus $[A'B'C']$ is minimal when PA' is minimal, which occurs when $PA' \perp AB$ (and analogously, when $PB' \perp PC$ and $PC' \perp PA$). Thus, the *zart* triangle with minimal area is the pedal triangle $A'B'C'$ of P to triangle ABC. This triangle is indeed similar to triangle ABC; letting $\theta = \angle PAB$ be the Brocard angle, it is the image of triangle ABC under a rotation through $\theta - 90^\circ$, followed by a homothety of ratio —$\sin\theta|$.

To construct this triangle, first draw the circles $\{X : \angle BXA = \angle BCA + \angle CAB\}$ and $\{Y : \angle CYB = \angle CAB + \angle ABC\}$ and let P' be their point of intersection (different from B); then we also have $\angle AP'C = \angle ABC + \angle BCA$. Then

$$\angle P'AB = 180^\circ - \angle ABP' - \angle BP'A =$$

$$180^\circ - (\angle ABC - \angle P'BC) - (\angle BCA + \angle CAB) = \angle P'BC,$$

and similarly $\angle P'BC = \angle P'CA$. Therefore $P = P'$. Finally, drop the perpendiculars from P' to the sides of triangle ABC to form A', B', C'. This completes the construction.

1.3 Bulgaria

National Olympiad, Third Round

Problem 1 Find all triples (x, y, z) of natural numbers such that y is a prime number, y and 3 do not divide z, and $x^3 - y^3 = z^2$.

Solution: Rewrite the equation in the form

$$(x-y)(x^2+xy+y^2) = z^2.$$

Any common divisor of $x-y$ and x^2+xy+y^2 also divides both z^2 and $(x^2+xy+y^2)-(x+2y)(x-y) = 3y^2$. But z^2 and $3y^2$ are relatively prime by assumption, hence $(x-y)$ and (x^2+xy+y^2) must be relatively prime as well. Therefore, both $(x-y)$ and (x^2+xy+y^2) are perfect squares.

Now writing $a = \sqrt{x-y}$, we have

$$x^2+xy+y^2 = (a^2+y)^2 + (a^2+y)y + y^2 = a^4 + 3a^2y + 3y^2$$

and

$$4(x^2+xy+y^2) = (2a^2+3y)^2 + 3y^2.$$

Writing $m = 2\sqrt{x^2+xy+y^2}$ and $n = 2a^2+3y$, we have

$$m^2 = n^2 + 3y^2$$

or

$$(m-n)(m+n) = 3y^2,$$

so $(m-n, m+n) = (1, 3y^2)$, $(3, y^2)$, or $(y, 3y)$.

In the first case, $2n = 3y^2 - 1$ and $4a^2 = 2n - 6y = 3y^2 - 6y - 1$ is a square, which is impossible modulo 3.

In the third case, $n = y < 2a^2 + 3y = n$, a contradiction.

In the second case, we have $4a^2 = 2n - 6y = y^2 - 6y - 3 < (y-3)^2$. And when $y \geq 10$ we have $y^2 - 6y - 3 > (y-4)^2$, hence $y = 2, 3, 5$, or 7. In this case we have $a = \frac{\sqrt{y^2-6y-3}}{2}$, which is real only when $y = 7$, $a = 1$, $x = y + a^2 = 8$, and $z = 13$. This yields the unique solution $(x, y, z) = (8, 7, 13)$.

Problem 2 A convex quadrilateral $ABCD$ is inscribed in a circle whose center O is inside the quadrilateral. Let $MNPQ$ be the quadrilateral whose vertices are the projections of the intersection point of the diagonals AC and BD onto the sides of $ABCD$. Prove that $2[MNPQ] \leq [ABCD]$.

Solution: The result actually holds even when $ABCD$ is not cyclic. We begin by proving the following result:

Lemma. *If $\overline{XW}$ is an altitude of triangle XYZ, then* $\frac{XW}{YZ} \leq \frac{1}{2}\tan\left(\frac{\angle Y + \angle Z}{2}\right)$.

Proof: X lies on an arc of a circle determined by $\angle YXZ = 180^\circ - \angle Y - \angle Z$. Its distance from $\overline{YZ}$ is maximized when it is at the center of this arc, which occurs when $\angle Y = \angle Z$; and at this point, $\frac{XW}{YZ} = \frac{1}{2}\tan\left(\frac{\angle Y + \angle Z}{2}\right)$. ■

Suppose M, N, P, Q are on sides AB, BC, CD, DA, respectively. Also let T be the intersection of $\overline{AC}$ and $\overline{BD}$.

Let $\alpha = \angle ADB$, $\beta = \angle BAC$, $\gamma = \angle CAD$, $\delta = \angle DBA$. From the lemma, $MT \leq \frac{1}{2}AB \cdot \tan\left(\frac{\beta+\delta}{2}\right)$ and $QT \leq \frac{1}{2}AD \cdot \tan\left(\frac{\alpha+\gamma}{2}\right)$; also, $\angle MTQ = 180^\circ - \angle QAM = 180^\circ - \angle DAB$. Thus $2[MTQ] = MT \cdot QT \sin \angle MTQ \leq \frac{1}{4}\tan\left(\frac{\alpha+\gamma}{2}\right)\tan\left(\frac{\beta+\delta}{2}\right) AB \cdot AD \sin \angle DAB$. But since $\frac{\alpha+\gamma}{2} + \frac{\beta+\delta}{2} = 90^\circ$, this last expression exactly equals $\frac{1}{4}AB \cdot AD \sin \angle DAB = \frac{1}{2}[ABD]$. Thus, $2[MTQ] \leq \frac{1}{2}[ABD]$.

Likewise, $2[NTM] \leq \frac{1}{2}[BCA]$, $[PTN] \leq \frac{1}{2}[CDB]$, and $[QTP] \leq \frac{1}{2}[DAC]$. Adding these four inequalities shows that $2[MNPQ]$ is at most

$$\frac{1}{2}([ABD] + [CDB]) + \frac{1}{2}([BCA] + [DAC]) = [ABCD],$$

as desired.

Problem 3 In a competition 8 judges marked the contestants by *pass* or *fail*. It is known that for any two contestants, two judges marked both with *pass*; two judges marked the first contestant with *pass* and the second contestant with *fail*; two judges marked the first contestant with *fail* and the second contestant with *pass*; and finally, two judges marked both with *fail*. What is the largest possible number of contestants?

Solution: For a rating r (either *pass* or *fail*), let $\overline{r}$ denote the opposite rating. Also, whenever a pair of judges agree on the rating for some contestant, call this an "agreement." We first prove that any two judges share at most three agreements; suppose by way of contradiction this were false.

Then assume without loss of generality that the judges (labeled with numbers) mark the first four contestants (labeled with letters) as follows in the left table:

	A	B	C	D
1	a	b	c	d
2	a	b	c	d
3	a	$\overline{b}$		
4	a	$\overline{b}$		
5	$\overline{a}$	$\overline{b}$		
6	$\overline{a}$	$\overline{b}$		
7	$\overline{a}$	b		
8	$\overline{a}$	b		

	A	B	C	D
1	a	b	c	d
2	a	b	c	d
3	a	$\overline{b}$	$\overline{c}$	$\overline{d}$
4	a	$\overline{b}$	$\overline{c}$	$\overline{d}$
5	$\overline{a}$	$\overline{b}$		
6	$\overline{a}$	$\overline{b}$		
7	$\overline{a}$	b		
8	$\overline{a}$	b		

	A	B	C	D
1	a	b	c	d
2	a	b	c	d
3	a	$\overline{b}$	$\overline{c}$	$\overline{d}$
4	a	$\overline{b}$	$\overline{c}$	$\overline{d}$
5	$\overline{a}$	$\overline{b}$	c	d
6	$\overline{a}$	$\overline{b}$	c	d
7	$\overline{a}$	b		
8	$\overline{a}$	b		

Applying the given condition to contestants A and C, judges 3 and 4 must both give C the rating $\overline{c}$; similarly, they must both give D the rating $\overline{d}$. Next, applying the condition to contestants B and C, judges 5 and 6 must both give C the rating c; similarly, they must both give D the rating d. But now the condition fails for contestants C and D, a contradiction.

Thus each pair of judges agrees on at most three ratings, as claimed; thus there are at most $3 \cdot \binom{8}{2} = 84$ agreements between all the judges. On the other hand, for each contestant exactly four judges mark him with *pass* and exactly four judges mark him with *fail*, hence there are $\binom{4}{2} + \binom{4}{2} = 12$ agreements per contestant. It follows that there are at most $\frac{84}{12} = 7$ contestants; and as the following table shows (with 1 representing *pass* and 0 representing *fail*), it is indeed possible to have exactly 7 contestants:

	A	B	C	D	E	F	G
1	1	1	1	1	1	1	1
2	1	1	1	0	0	0	0
3	1	0	0	1	1	0	0
4	1	0	0	0	0	1	1
5	0	1	0	1	0	0	1
6	0	1	0	0	1	1	0
7	0	0	1	1	0	1	0
8	0	0	1	0	1	0	1

Problem 4 Find all pairs (x, y) of integers such that

$$x^3 = y^3 + 2y^2 + 1.$$

Solution: When $y^2 + 3y > 0$, $(y+1)^3 > x^3 > y^3$. Thus we must have $y^2 + 3y \leq 0$, and $y = -3, -2, -1$, or 0 — yielding the solutions $(x, y) = (1, 0)$, $(1, -2)$, and $(-2, -3)$.

Problem 5 Let B_1 and C_1 be points on the sides AC and AB of triangle ABC. Lines BB_1 and CC_1 intersect at point D. Prove that a circle can be inscribed inside quadrilateral AB_1DC_1 if and only if the incircles of the triangles ABD and ACD are tangent to each other.

Solution: Say the incircle of triangle ABD is tangent to $\overline{AD}$ at T_1 and that the incircle of triangle ACD is tangent to $\overline{AD}$ at T_2; then $DT_1 = \frac{1}{2}(DA + DB - AB)$ and $DT_2 = \frac{1}{2}(DA + DC - AC)$.

First suppose a circle can be inscribed inside AB_1DC_1. Let it be tangent to sides AB_1, B_1D, DC_1, C_1A at points E, F, G, H, respectively. Using equal tangents, we have

$$\begin{aligned} AB - BD &= (AH + HB) - (BF - DF) \\ &= (AH + BF) - (BF - DF) = AH + DF \end{aligned}$$

and similarly $AC - CD = AE + DG$. But $AH + DF = AE + DG$ by equal tangents, implying that $AB - BD = AC - CD$ and thus $DA + DB - AB = DA + DC - AC$. Therefore $DT_1 = DT_2$, $T_1 = T_2$, and the two given incircles are tangent to each other.

Next suppose the two incircles are tangent to each other. Then $DA + DB - AB = DA + DC - AC$. Let ω be the incircle of ABB_1, and let D' be the point on $\overline{BB_1}$ (different from B_1) such that line CD' is tangent to ω. Suppose by way of contradiction that $D \neq D'$. From the result in the last paragraph, we know that the incircles of triangles ABD' and ACD' are tangent and hence $D'A + D'B - AB = D'A + D'C - AC$. Then since $DB - AB = DC - AC$ and $D'B - AB = D'C - AC$, we must have $DB - D'B = DC - D'C$ by subtraction. Thus $DD' = |DB - D'B| = |DC - D'C|$. But then the triangle inequality fails in triangle $DD'C$, a contradiction. This completes the proof.

Problem 6 Each interior point of an equilateral triangle of side 1 lies in one of six congruent circles of radius r. Prove that

$$r \geq \frac{\sqrt{3}}{10}.$$

Solution: From the condition, we also know that every point inside or *on* the triangle lies inside or *on* one of the six circles.

Define $R = \frac{1}{1+\sqrt{3}}$. Orient the triangle so that A is at the top, B is at the bottom-left, and C is at the bottom-right (so that $\overline{BC}$ is horizontal). Draw point W on $\overline{AB}$ such that $WA = R$; then draw point X directly below W such that $WX = R$. Then in triangle WXB, $WB = 1 - R = \sqrt{3}R$ and $\angle BWX = 30^\circ$, implying that $XB = R$ as well. Similarly draw Y on $\overline{AC}$ such that $YA = R$, and Z directly below Y such that $YZ = ZC = R$.

In triangle AWY, $\angle A = 60^\circ$ and $AW = AY = R$, implying that $WY = R$. This in turn implies that $XZ = R$ and that $WX = YZ = R\sqrt{2}$.

Now suppose by way of contradiction that we could cover the triangle with six congruent circles of radius $r < \frac{\sqrt{3}}{10}$. The points A, B, C, W, X, Y, Z each lie on or inside one of the circles. But any two of these points are at least $R > 2r$ apart, so they must lie on or inside *different* circles. Thus there are at least seven circles, a contradiction.

National Olympiad, Fourth Round

Problem 1 A rectangular parallelepiped has integer dimensions. All of its faces of are painted green. The parallelepiped is partitioned into unit cubes by planes parallel to its faces. Find all possible measurements of the parallelepiped if the number of cubes without a green face is one third of the total number of cubes.

Solution: Let the parallelepiped's dimensions be a, b, c; they must all be at least 3 or else every cube has a green face. Then the condition is equivalent to

$$3(a-2)(b-2)(c-2) = abc,$$

or

$$3 = \frac{a}{a-2} \cdot \frac{b}{b-2} \cdot \frac{c}{c-2}.$$

If all the dimensions are at least 7, then $\frac{a}{a-2} \cdot \frac{b}{b-2} \cdot \frac{c}{c-2} \leq \left(\frac{7}{5}\right)^3 = \frac{343}{125} < 3$, a contradiction. Thus one of the dimensions — say, a — equals 3, 4, 5, or 6. Assume without loss of generality that $b \leq c$.

When $a = 3$ we have $bc = (b-2)(c-2)$, which is impossible.

When $a = 4$, rearranging the equation yields $(b-6)(c-6) = 24$. Thus $(b, c) = (7, 30)$, $(8, 18)$, $(9, 14)$, or $(10, 12)$.

When $a = 5$, rearranging the equation yields $(2b-9)(2c-9) = 45$. Thus $(b, c) = (5, 27)$, $(6, 12)$, or $(7, 9)$.

And when $a = 6$, rearranging the equation yields $(b-4)(c-4) = 8$. Thus $(b, c) = (5, 12)$ or $(6, 8)$.

Therefore the parallelepiped may measure $4 \times 7 \times 30$, $4 \times 8 \times 18$, $4 \times 9 \times 14$, $4 \times 10 \times 12$, $5 \times 5 \times 27$, $5 \times 6 \times 12$, $5 \times 7 \times 9$, or $6 \times 6 \times 8$.

Problem 2 Let $\{a_n\}$ be a sequence of integers such that for $n \geq 1$

$$(n-1)a_{n+1} = (n+1)a_n - 2(n-1).$$

If 2000 divides a_{1999}, find the smallest $n \geq 2$ such that 2000 divides a_n.

Solution: First, we note that the sequence $a_n = 2n - 2$ works. Then writing $b_n = a_n - (2n-2)$ gives the recursion

$$(n-1)b_{n+1} = (n+1)b_n.$$

Some calculations show that $b_3 = 3b_2$, $b_4 = 6b_2$, $b_5 = 10b_2$ — and in general, that $b_n = \frac{n(n-1)}{2}b_2$ for $n \geq 2$. Thus when $n \geq 2$, the solution to the original equation is of the form

$$a_n = 2(n-1) + \frac{n(n-1)}{2}c$$

for some constant c; plugging in $n = 2$ shows that $c = a_2 - 2$ is an integer.

Now, since $2000 \mid a_{1999}$ we have $2(1999-1) + \frac{1999 \cdot 1998}{2} \cdot c \equiv 0 \Longrightarrow -4 + 1001c \equiv 0 \Longrightarrow c \equiv 4 \pmod{2000}$. Then $2000 \mid a_n$ exactly when

$$\begin{aligned} 2(n-1) + 2n(n-1) &\equiv 0 \pmod{2000} \\ \Longleftrightarrow (n-1)(n+1) &\equiv 0 \pmod{1000}. \end{aligned}$$

$(n-1)(n+1)$ is divisible by 8 exactly when n is odd; and it is divisible by 125 exactly when either $n-1$ or $n+1$ is divisible by 125. The smallest $n \geq 2$ satisfying these requirements is $n = 249$.

Problem 3 The vertices of a triangle have integer coordinates and one of its sides is of length $\sqrt{n}$, where n is a square-free natural number. Prove that the ratio of the circumradius to the inradius of the triangle is an irrational number.

Solution: Label the triangle ABC; let r, R, K be the inradius, circumradius, and area of the triangle; let $a = BC$, $b = CA$, $c = AB$ and write $a = p_1\sqrt{q_1}$, $b = p_2\sqrt{q_2}$, $c = p_3\sqrt{q_3}$ for positive integers p_i, q_i with q_i square-free. By Pick's Theorem ($K = I + \frac{1}{2}B - 1$), K is rational. Also, $R = \frac{abc}{4K}$ and $r = \frac{2K}{a+b+c}$. Thus $\frac{R}{r} = \frac{abc(a+b+c)}{8K^2}$ is rational if and only if $abc(a+b+c) = a^2bc + ab^2c + abc^2$ is rational. Let this quantity equal m, and assume by way of contradiction that m *is* rational.

We have $a^2bc = m_1\sqrt{q_2q_3}$, $ab^2c = m_2\sqrt{q_3q_1}$, and $abc^2 = m_3\sqrt{q_1q_2}$ for positive integers m_1, m_2, m_3. Then $m_1\sqrt{q_2q_3} + m_2\sqrt{q_3q_1} = m - m_3\sqrt{q_1q_2}$. Squaring both sides, we find that

$$m_1^2q_2q_3 + m_2^2q_3q_1 + 2m_1m_2q_3\sqrt{q_1q_2} = m^2 + m_3^2 - 2mm_3\sqrt{q_1q_2}.$$

If $\sqrt{q_1q_2}$ is not rational, then the coefficients of $\sqrt{q_1q_2}$ must be the same on both sides; but this is impossible since $2m_1m_2q_3$ is positive while $-2mm_3$ is not.

Hence $\sqrt{q_1q_2}$ is rational. Since q_1 and q_2 are square-free, this can only be true if $q_1 = q_2$. Similarly, $q_2 = q_3$.

Assume without loss of generality that $BC = \sqrt{n}$ so that $q_1 = q_2 = q_3 = n$ and $p_1 = 1$. Also assume that A is at $(0,0)$, B is at (w,x), and C is at (y,z). By the triangle inequality, we must have $p_2 = p_3$ and hence

$$w^2 + x^2 = y^2 + z^2 = p_2^2n$$
$$(w-y)^2 + (x-z)^2 = n.$$

Notice that

$$n = (w-y)^2 + (x-z)^2 \equiv w^2 + x^2 + y^2 + z^2 = 2p_2^2n \equiv 0 \pmod{2},$$

so n is even. Thus w and x have the same parity; and y and z have the same parity. Then w, x, y, z must *all* have the same parity since $w^2 + x^2 \equiv y^2 + z^2 \pmod{4}$. But then $n = (w-y)^2 + (x-z)^2 \equiv 0 \pmod{4}$, contradicting the assumption that n is square-free.

Therefore our original assumption was false; and the ratio of the circumradius to the inradius is indeed always irrational.

Note: Without the condition that n is square-free, the ratio *can* be rational. For example, the points $(i, 2j - i)$ form a grid of points $\sqrt{2}$ apart. In this grid, we can find a $3\sqrt{2}$-$4\sqrt{2}$-$5\sqrt{2}$ right triangle by choosing, say, the points $(0,0)$, $(3,3)$, and $(7,-1)$. Then $q_1 = q_2 = q_3$, and the ratio is indeed rational.

Problem 4 Find the number of all natural numbers n, $4 \le n \le 1023$, whose binary representations do not contain three consecutive equal digits.

Solution: A *binary string* is a finite string of digits, all either 0 or 1. Call such a string (perhaps starting with zeroes) *valid* if it does not contain three consecutive equal digits. Let a_n represent the number of valid n-digit strings; let s_n be the number of valid strings starting with two equal digits; and let d_n be the number of valid strings starting with two different digits. Observe that $a_n = s_n + d_n$ for all n.

An $(n+2)$-digit string starting with 00 is valid if and only if its last n digits form a valid string starting with 1; similarly, an $(n+2)$-digit string starting with 11 is valid if and only if its last n digits form a valid string starting with 0. Thus, $s_{n+2} = a_n = s_n + d_n$.

An $(n+2)$-digit string starting with 01 is valid if and only if its last n digits form a valid string starting with 00, 01, or 10; similarly, an $(n+2)$-digit string starting with 10 is valid if and only if its last n digits form a valid string starting with 11, 01, or 10. Thus, $d_{n+2} = s_n + 2d_n$.

Solving these recursions gives

$$s_{n+4} = 3s_{n+2} - s_n \quad \text{and} \quad d_{n+4} = 3d_{n+2} - d_n,$$

which when added together yield

$$a_{n+4} = 3a_{n+2} - a_n.$$

Thus we can calculate initial values of a_n and then use the recursion to find other values:

n	1	2	3	4	5	6	7	8	9	10
a_n	2	4	6	10	16	26	42	68	110	178

Now of the a_n valid n-digit strings, only half start with 1; thus only half are binary representations of positive numbers. Therefore exactly

$$\frac{1}{2}(a_1 + a_2 + \cdots + a_{10}) = 231$$

numbers between 1 and 1023 have the desired property; and ignoring 1, 2, and 3, we find that the answer is 228.

Problem 5 The vertices A, B and C of an acute-angled triangle ABC lie on the sides B_1C_1, C_1A_1 and A_1B_1 of triangle $A_1B_1C_1$ such that $\angle ABC = \angle A_1B_1C_1$, $\angle BCA = \angle B_1C_1A_1$, and $\angle CAB = \angle C_1A_1B_1$. Prove that the orthocenters of the triangle ABC and triangle $A_1B_1C_1$ are equidistant from the circumcenter of triangle ABC.

Solution: Let H and H_1 be the orthocenters of triangles ABC and $A_1B_1C_1$, respectively; and let O, O_A, O_B, O_C be the circumcenters of triangles ABC, A_1BC, AB_1C, and ABC_1, respectively.

First note that $\angle BA_1C = \angle C_1A_1B_1 = \angle CAB = 180^\circ - \angle CHB$, showing that BA_1CH is cyclic; moreover, $O_AA_1 = \frac{BC}{2\sin\angle BA_1C} = \frac{CB}{2\sin\angle CAB} = OA$ so circles ABC and BA_1CH have the same radius. Similarly, CB_1AH and AC_1BH are cyclic with circumradius OA. Then $\angle HBC_1 = 180^\circ - \angle C_1AH = \angle HAB_1 = 180^\circ - \angle B_1CH = \angle HCA_1$; thus angles $\angle HO_CC_1$, $\angle HO_AA_1$, $\angle HO_BB_1$ are equal as well.

Let $\angle(\vec{r}_1, \vec{r}_2)$ denote the angle between rays $\vec{r}_1$ and $\vec{r}_2$. Since $O_AC = O_AB = HB = HC$, quadrilateral BO_ACH is a rhombus and hence a parallelogram. Then

$$\begin{aligned}
\angle(\overrightarrow{OA}, \overrightarrow{HO_A}) &= \angle(\overrightarrow{OA}, \overrightarrow{OB}) + \angle(\overrightarrow{OB}, \overrightarrow{HO_A}) \\
&= 2\angle ACB + \angle(\overrightarrow{CO_A}, \overrightarrow{HO_A}) \\
&= 2\angle ACB + \angle CO_AH \\
&= 2\angle ACB + 2\angle CBH \\
&= 2\angle ACB + 2(90^\circ - \angle ACB) \\
&= 180^\circ.
\end{aligned}$$

Similarly, $\angle(\overrightarrow{OB}, \overrightarrow{HO_B}) = \angle(\overrightarrow{OC}, \overrightarrow{HO_C}) = 180^\circ$. Combining this result with $\angle HO_AA_1 = \angle HO_BB_1 = \angle HO_CC_1$ from above, we find that

$$\angle(\overrightarrow{OA}, \overrightarrow{O_AA_1}) = \angle(\overrightarrow{OB}, \overrightarrow{O_BB_1}) = \angle(\overrightarrow{OC}, \overrightarrow{O_CC_1}).$$

Let this common angle be θ.

We now use complex numbers with the origin at O, letting p denote the complex number representing point P. Since HBO_AC is a parallelogram we have $o_A = b + c$ and we can write $a_1 = b + c + xa$ where $x = \operatorname{cis}\theta$. We also have $b_1 = c + a + xb$ and $c_1 = a + b + xc$ for the same x. We can rewrite these relations as

$$a_1 = a + b + c + (x - 1)a,$$

$$b_1 = a + b + c + (x - 1)b,$$

$$c_1 = a + b + c + (x - 1)c.$$

Thus the map sending z to $a + b + c + (x - 1)z = h + (x - 1)z$ is a spiral similarity taking triangle ABC into triangle $A'B'C'$. It follows that this map also takes H to H_1, so

$$h_1 = h + (x - 1)h = xh$$

and $OH_1 = |h_1| = |x||h| = |h| = OH$, as desired.

Problem 6 Prove that the equation

$$x^3 + y^3 + z^3 + t^3 = 1999$$

has infinitely many integral solutions.

Solution: Observe that $(m - n)^3 + (m + n)^3 = 2m^3 + 6mn^2$. Now suppose we want a general solution of the form

$$(x, y, z, t) = (a - b, a + b, \frac{c}{2} - \frac{d}{2}, \frac{c}{2} + \frac{d}{2})$$

for integers a, b and odd integers c, d. One simple solution to the given equation is $(x, y, z, t) = (10, 10, -1, 0)$, so try setting $a = 10$ and $c = -1$. Then

$$(x, y, z, t) = (10 - b, 10 + b, -\frac{1}{2} - \frac{d}{2}, -\frac{1}{2} + \frac{d}{2})$$

is a solution exactly when

$$(2000 + 60b^2) - \frac{1 + 3d^2}{4} = 1999 \iff d^2 - 80b^2 = 1.$$

The second equation is a Pell's equation with solution $(d_1, b_1) = (9, 1)$; and we can generate infinitely many more solutions by setting $(d_{n+1}, b_{n+1}) = (9d_n + 80b_n, 9b_n + d_n)$ for $n = 1, 2, 3, \ldots$; this follows from a general recursion $(p_{n+1}, q_{n+1}) = (p_1p_n + q_1q_nD, p_1q_n + q_1p_n)$ for generating solutions to $p^2 - Dq^2 = 1$ given a nontrivial solution (p_1, q_1).

A quick check also shows that each d_n is odd. Thus since there are infinitely many solutions (b_n, d_n) to the Pell's equation (and with each d_n odd), there are infinitely many integral solutions

$$(x_n, y_n, z_n, t_n) = (10 - b_n, 10 + b_n, -\frac{1}{2} - \frac{d_n}{2}, -\frac{1}{2} + \frac{d_n}{2})$$

to the original equation.

1.4 Canada

Problem 1 Find all real solutions to the equation $4x^2-40\lfloor x\rfloor+51=0$, where $[x]$ denotes the greatest integer less than or equal to x.

Solution: Note that $(2x-3)(2x-17) = 4x^2 - 40x + 51 \leq 4x^2 - 40\lfloor x\rfloor + 51 = 0$, which gives $1.5 \leq x \leq 8.5$ and $1 \leq \lfloor x\rfloor \leq 8$. Then

$$x = \frac{\sqrt{40\lfloor x\rfloor - 51}}{2},$$

so it is necessary that

$$\lfloor x\rfloor = \left\lfloor \frac{\sqrt{40\lfloor x\rfloor - 51}}{2} \right\rfloor.$$

Testing $\lfloor x\rfloor \in \{1,2,3,\ldots,8\}$ into this equation, we find that only $\lfloor x\rfloor = 2,6,7,$ and 8 work. Thus the only solutions for x are $\frac{\sqrt{29}}{2}, \frac{\sqrt{189}}{2}, \frac{\sqrt{229}}{2}, \frac{\sqrt{269}}{2}$.

Problem 2 Let ABC be an equilateral triangle of altitude 1. A circle, with radius 1 and center on the same side of AB as C, rolls along the segment AB; as it rolls, it always intersects both $\overline{AC}$ and $\overline{BC}$. Prove that the length of the arc of the circle that is inside the triangle remains constant.

Solution: Let ω be "the circle." Let O be the center of ω. Let ω intersect segments $\overline{AC}$ and $\overline{BC}$ at M and N, respectively. Let the circle through O, C, and M intersect $\overline{BC}$ again at P. Now $\angle PMO = 180^\circ - \angle OCP = 60^\circ = \angle MCO = \angle MPO$, so $OP = OM = 1$, and P coincides with N. Thus, $\angle MON = \angle MOP = \angle MCP = 60^\circ$. Therefore, the angle of the arc of ω that is inside the triangle ABC is constant, and hence the length of the arc must be constant as well.

Problem 3 Determine all positive integers n such that $n = d(n)^2$, where $d(n)$ denotes the number of positive divisors of n (including 1 and n).

Solution: Label the prime numbers $p_1 = 2, p_2 = 3, \ldots$. Since n is a perfect square, we have

$$n = \prod_{i=1}^{\infty} p_i^{2a_i}, \quad d(n) = \prod_{i=1}^{\infty}(2a_i + 1).$$

Then $d(n)$ is odd and so is n, whence $a_1 = 0$. Since $\frac{d(n)}{\sqrt{n}} = 1$, we have

$$\prod_{i=1}^{\infty} \frac{2a_i + 1}{p_i^{a_i}} = 1.$$

By Bernoulli's inequality, we have $p_i^{a_i} \geq (p_i - 1)a_i + 1 > 2a_i + 1$ for all primes $p_i \geq 5$ that divide n. Also, $3^{a_2} \geq 2a_2 + 1$ with equality only when $a_2 \in \{0, 1\}$. Thus, for equality to hold above, we must have $a_1 = a_3 = a_4 = a_5 = \cdots = 0$ and $a_2 \in \{0, 1\}$; therefore, $n \in \{1, 9\}$ are the only solutions.

Problem 4 Suppose $a_1, a_2, \ldots, a_8$ are eight distinct integers from the set $\mathcal{S} = \{1, 2, \ldots, 17\}$. Show that there exists an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions. Also, find a specific set of 7 distinct integers $\{b_1, b_2, \cdots, b_7\}$ from $\mathcal{S}$ such that the equation

$$b_i - b_j = k$$

does not have three distinct solutions for any $k > 0$.

Solution: For the first part of this problem, assume without loss of generality that $a_1 < a_2 < \cdots < a_8$; also assume, for the purpose of contradiction, that there does *not* exist an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions. Let $\delta_i = a_{i+1} - a_i$ for $i = 1, 2, \ldots, 7$. Then

$$16 \geq a_8 - a_1 = \delta_1 + \ldots + \delta_7 \geq 1 + 1 + 2 + 2 + 3 + 3 + 4 = 16$$

(for otherwise three of the δ_i's would be equal, a contradiction). Since equality must hold, $\Pi = (\delta_1, \delta_2, \ldots, \delta_7)$ must be a permutation of $(1, 1, 2, 2, 3, 3, 4)$.

Say we have a "m-n pair" if some $(\delta_i, \delta_{i+1}) = (m, n)$ or (n, m). Note that we cannot have any 1-1 or 1-2 pairs (δ_i, δ_{i+1}); otherwise we'd have $a_{i+2} - a_i = 2$ or 3, giving at least three solutions to $a_i - a_j = 2$ or 3. Nor can we have two 1-3 pairs because then, along with $\delta_i = 4$, we'd have three solutions to $a_i - a_j = 4$. Then considering what entries each 1 is next to, we see that we must have

$$\Pi = (1, 4, \ldots, 3, 1) \quad \text{or} \quad (1, 4, 1, 3, \ldots)$$

(or these lists backwards).

But now we can't have any 2-2 pairs; otherwise, along with the 1-3 pair and the $\delta_i = 4$, we'd have three solutions to $a_i - a_j = 4$. Thus we have either

$$\Pi = (1, 4, 2, 3, 2, 3, 1) \quad \text{or} \quad (1, 4, 1, 3, 2, 3, 2)$$

(or these lists backwards). In either case there are at least four solutions to $a_i - a_j = 5$, a contradiction.

Thus, regardless of the $\{a_1, a_2, \ldots, a_8\}$ that we choose, for some integer $k \in \{2, 3, 4, 5\}$ the equation $a_i - a_j = k$ has at least three different solutions.

For the second part of the problem, let $(b_1, b_2, \ldots, b_7) = (1, 2, 4, 9, 14, 16, 17)$. Each of 1, 2, 3, 5, 7, 8, 12, 13, and 15 is the difference of exactly two pairs of the b_i, and each of 10, 14, and 16 is the difference of exactly one pair of the b_i. But no number is the difference of more than two such pairs, and hence the set $\{b_1, b_2, \ldots, b_7\}$ suffices.

Problem 5 Let x, y, z be non-negative real numbers such that

$$x + y + z = 1.$$

Prove that

$$x^2y + y^2z + z^2x \le \frac{4}{27}.$$

and determine when equality occurs.

Solution: Assume without loss of generality that $x = \max\{x, y, z\}$.

- If $x \ge y \ge z$, then

$$\begin{aligned} x^2y + y^2z + z^2x &\le x^2y + y^2z + z^2x + z(xy + (x-y)(y-z)) \\ &= (x+z)^2y = 4\left(\frac{1}{2} - \frac{1}{2}y\right)\left(\frac{1}{2} - \frac{1}{2}y\right)y \le \frac{4}{27}, \end{aligned}$$

  where the last inequality follows from AM-GM. Equality occurs if and only if $z = 0$ (from the first inequality) and $y = \frac{1}{3}$, in which case $(x, y, z) = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$.
- If $x \ge z \ge y$, then

$$\begin{aligned} x^2y + y^2z + z^2x &= x^2z + z^2y + y^2x - (x-z)(z-y)(x-y) \\ &\le x^2z + z^2y + y^2x \le \frac{4}{27}, \end{aligned}$$

  where the second inequality is true from the result we proved for $x \ge y \ge z$ (except with y and z reversed). Equality holds in the first inequality only when two of x, y, z are equal; and in the second only when $(x, z, y) = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$. Since these conditions can't both be true, the inequality is actually strict in this case.

Therefore the inequality is indeed true, and equality holds when (x, y, z) equals $\left(\frac{2}{3}, \frac{1}{3}, 0\right)$, $\left(\frac{1}{3}, 0, \frac{2}{3}\right)$, or $\left(0, \frac{2}{3}, \frac{1}{3}\right)$.

1.5 China

Problem 1 Let ABC be an acute triangle with $\angle C > \angle B$. Let D be a point on side BC such that $\angle ADB$ is obtuse, and let H be the orthocenter of triangle ABD. Suppose that F is a point inside triangle ABC and is on the circumcircle of triangle ABD. Prove that F is the orthocenter of triangle ABC if and only if both of the following are true: $HD \parallel CF$, and H is on the circumcircle of triangle ABC.

Solution: All angles are directed modulo 180°. First observe that if P is the orthocenter of triangle UVW, then

$$\begin{aligned}\angle VPW &= (90^\circ - \angle PWV) + (90^\circ - \angle WVP) \\ &= \angle WVU + \angle UWV = 180^\circ - \angle VUW.\end{aligned}$$

First suppose that F is the orthocenter of triangle ABC. Then

$$\angle ACB = 180^\circ - \angle AFB = 180^\circ - \angle ADB = \angle AHB,$$

so $ACHB$ is cyclic. And lines CF and HD are both perpendicular to side AB, so they are parallel.

Conversely, suppose that $HD \parallel CF$ and that H is on the circumcircle of triangle ABC. Since $AFDB$ and $AHCB$ are cyclic,

$$\angle AFB = \angle ADB = 180^\circ - \angle AHB = 180^\circ - \angle ACB.$$

Thus F is an intersection point of the circle defined by $\angle AFB = 180^\circ - \angle ACB$ and the line defined by $CF \perp AB$. But there are only two such points: the orthocenter of triangle ABC and the reflection of C across line AB. The latter point lies outside of triangle ABC, and hence F must indeed be the orthocenter of triangle ABC.

Problem 2 Let a be a real number. Let $\{f_n(x)\}$ be a sequence of polynomials such that $f_0(x) = 1$ and $f_{n+1}(x) = xf_n(x) + f_n(ax)$ for $n = 0, 1, 2, \ldots$.

(a) Prove that

$$f_n(x) = x^n f_n\left(\frac{1}{x}\right)$$

for $n = 0, 1, 2, \ldots$.

(b) Find an explicit expression for $f_n(x)$.

Solution: When $a = 1$, we have $f_n(x) = (x+1)^n$ for all n, and part (a) is easily checked. Now assume that $a \neq 1$.

Observe that f_n has degree n and always has constant term 1. Write $f_n(x) = c_0 + c_1 x + \cdots + c_n x^n$; we prove by induction on n that

$$(a^i - 1)c_i = (a^{n+1-i} - 1)c_{i-1}$$

for $0 \leq i \leq n$ (where we let $c_{-1} = 0$).

The base case $n = 0$ is clear. Now suppose that $f_{n-1}(x) = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1}$ satisfies the claim: specifically, we know $(a^i - 1)b_i = (a^{n-i} - 1)b_{i-1}$ and $(a^{n+1-i} - 1)b_{i-2} = (a^{i-1} - 1)b_{i-1}$ for $i \geq 1$.

For $i = 0$, the claim states $0 = 0$. For $i \geq 1$, the given recursion gives $c_i = b_{i-1} + a^i b_i$ and $c_{i-1} = b_{i-2} + a^{i-1}b_{i-1}$. Then the claim is equivalent to

$$\begin{aligned}
&(a^i - 1)c_i = (a^{n+1-i} - 1)c_{i-1} \\
&\iff (a^i - 1)(b_{i-1} + a^i b_i) = (a^{n+1-i} - 1)(b_{i-2} + a^{i-1}b_{i-1}) \\
&\iff (a^i - 1)b_{i-1} + a^i(a^i - 1)b_i \\
&\qquad\qquad = (a^{n+1-i} - 1)b_{i-2} + (a^n - a^{i-1})b_{i-1} \\
&\iff (a^i - 1)b_{i-1} + a^i(a^{n-i} - 1)b_{i-1} \\
&\qquad\qquad = (a^{i-1} - 1)b_{i-1} + (a^n - a^{i-1})b_{i-1} \\
&\iff (a^n - 1)b_{i-1} = (a^n - 1)b_{i-1},
\end{aligned}$$

so it is true.

Now by telescoping products, we have

$$\begin{aligned}
c_i = \frac{c_i}{c_0} &= \prod_{k=1}^{i} \frac{c_k}{c_{k-1}} \\
&= \prod_{k=1}^{i} \frac{a^{n+1-k} - 1}{a^k - 1} = \frac{\prod_{k=n+1-i}^{n}(a^k - 1)}{\prod_{k=1}^{i}(a^k - 1)} \\
&= \frac{\prod_{k=i+1}^{n}(a^k - 1)}{\prod_{k=1}^{n-i}(a^k - 1)} = \prod_{k=1}^{n-i} \frac{a^{n+1-k} - 1}{a^k - 1} \\
&= \prod_{k=1}^{n-i} \frac{c_k}{c_{k-1}} = \frac{c_{n-i}}{c_0} = c_{n-i},
\end{aligned}$$

giving our explicit form. Also, $f_n(x) = x^n f_n\left(\frac{1}{x}\right)$ if and only if $c_i = c_{n-i}$ for $i = 0, 1, \ldots, n$, and from above this is indeed the case. This completes the proof.

Problem 3 There are 99 space stations. Each pair of space stations is connected by a tunnel. There are 99 two-way main tunnels, and all the other tunnels are strictly one-way tunnels. A group of 4 space stations is called *connected* if one can reach each station in the group from every other station in the group without using any tunnels other than the 6 tunnels which connect them. Determine the maximum number of connected groups.

Solution: In this solution, let $f(x) = \frac{x(x-1)(x-2)}{6}$, an extension of the definition of $\binom{x}{3}$ to all real numbers x.

In a group of 4 space stations, call a station *troublesome* if three one-way tunnels lead toward it or three one-way tunnels lead out of it. In each group there is at most one troublesome station of each type for a count of at most two troublesome stations. Also, if a station is troublesome in a group, that group is not connected.

Label the stations $1, 2, \ldots, 99$. For $i = 1, 2, \ldots, 99$, let a_i one-way tunnels point into station i and b_i one-way tunnels point out. Station i is troublesome in $\binom{a_i}{3} + \binom{b_i}{3}$ groups of four. Adding over all stations, we obtain a total count of $\sum_{i=1}^{198}\left(\binom{a_i}{3} + \binom{b_i}{3}\right)$. This equals $\sum_{i=1}^{198} f(x_i)$ for nonnegative integers $x_1, x_2, \ldots, x_{198}$ with $\sum_{i=1}^{198} x_i = 96 \cdot 99$. Without loss of generality, say that $x_1, x_2, \ldots, x_k$ are at least 1 and $x_{k+1}, x_{k+2}, \ldots, x_{198}$ are zero. Since $f(x)$ is convex as a function of x for $x \geq 1$, this is at least $k\binom{96\cdot 99/k}{2}$. Also, $mf(x) \geq f(mx)$ when $m \leq 1$ and $mx \geq 2$. Letting $m = k/198$ and $mx = 96\cdot 99/198 = 48$, we find that our total count is at least $198\binom{48}{2}$. Since each unconnected group of 4 stations has at most two troublesome stations, there are at least $99\binom{48}{3}$ unconnected groups of four and at most $\binom{99}{4} - 99\binom{48}{3}$ connected groups.

All that is left to show is that this maximum can be attained. Arrange the stations around a circle, and put a two-way tunnel between any two adjacent stations; otherwise, place a one-way tunnel running from station A to station B if and only if A is 3, 5, ..., or 97 stations away clockwise from B. In this arrangement, every station is troublesome $2\binom{48}{3}$ times. It is easy to check that under this arrangement, no unconnected group of four stations contains two adjacent stations. And suppose that station A is troublesome in a group of four stations A, B, C, D with B closest and D furthest away clockwise from A. If one-way tunnels lead from A to the other tunnels, three one-way tunnels must lead to D from the other tunnels; and if one-way tunnels lead to A from the other tunnels, three one-way tunnels must lead from B to the other tunnels. Thus every unconnected group of four stations has exactly two troublesome stations. Hence equality holds in the previous paragraph, and there are indeed exactly $\binom{99}{4} - 99\binom{48}{3}$ connected groups.

Problem 4 Let m be a positive integer. Prove that there are integers a, b, k, such that both a and b are odd, $k \geq 0$, and

$$2m = a^{19} + b^{99} + k \cdot 2^{1999}.$$

Solution: The key observation is that if $\{t_1, \cdots, t_n\}$ equals $\{1, 3, 5, \ldots, 2^n - 1\}$ modulo 2^n, then $\{t_1^s, \cdots, t_n^s\}$ does as well for any odd positive integer s. To show this, note that for $i \neq j$,

$$t_i^s - t_j^s = (t_i - t_j)(t_i^{s-1} + t_i^{s-2}t_j + \cdots + t_j^{s-1}).$$

Since $t_i^{s-1} + t_i^{s-2}t_j + \cdots + t_j^{s-1}$ is an odd number, $t_i \equiv t_j \iff t_i^s \equiv t_j^s \pmod{2^n}$.

Therefore there exists an odd number a_0 such that $2m - 1 \equiv a_0^{19} \pmod{2^{1999}}$. Hence if we pick $a \equiv a_0 \pmod{2^{1999}}$ sufficiently negative so that $2m - 1 - a^{19} > 0$, then

$$(a, b, k) = \left(a, 1, \frac{2m - 1 - a^{19}}{2^{1999}}\right)$$

is a solution to the equation.

Problem 5 Determine the maximum value of λ such that if $f(x) = x^3 + ax^2 + bx + c$ is a cubic polynomial with all its roots nonnegative, then

$$f(x) \geq \lambda(x - a)^3$$

for all $x \geq 0$. Find the equality condition.

Solution: Let α, β, γ be the three roots. Without loss of generality, suppose that $0 \leq \alpha \leq \beta \leq \gamma$. We have

$$x - a = x + \alpha + \beta + \gamma \geq 0 \quad \text{and} \quad f(x) = (x - \alpha)(x - \beta)(x - \gamma).$$

If $0 \le x \le \alpha$, then (applying the arithmetic-mean geometric mean inequality) to obtain the first inequality below)

$$\begin{aligned} -f(x) &= (\alpha - x)(\beta - x)(\gamma - x) \le \frac{1}{27}(\alpha + \beta + \gamma - 3x)^3 \\ &\le \frac{1}{27}(x + \alpha + \beta + \gamma)^3 = \frac{1}{27}(x - a)^3, \end{aligned}$$

so that $f(x) \ge -\frac{1}{27}(x - a)^3$. Equality holds exactly when $\alpha - x = \beta - x = \gamma - x$ in the first inequality and $\alpha + \beta + \gamma - 3x = x + \alpha + \beta + \gamma$ in the second; that is, when $x = 0$ and $\alpha = \beta = \gamma$.

If $\beta \le x \le \gamma$, then (again applying AM-GM to obtain the first inequality below)

$$\begin{aligned} -f(x) &= (x - \alpha)(x - \beta)(\gamma - x) \le \frac{1}{27}(x + \gamma - \alpha - \beta)^3 \\ &\le \frac{1}{27}(x + \alpha + \beta + \gamma)^3 = \frac{1}{27}(x - a)^3, \end{aligned}$$

so that again $f(x) \ge -\frac{1}{27}(x-a)^3$. Equality holds exactly when $x-\alpha = x-\beta = \gamma - x$ in the first inequality and $x+\gamma-\alpha-\beta = x+\alpha+\beta+\gamma$; that is, when $\alpha = \beta = 0$ and $\gamma = 2x$.

Finally, when $\alpha < x < \beta$ or $x > \gamma$ then

$$f(x) > 0 \ge -\frac{1}{27}(x - a)^3.$$

Thus, $\lambda = -\frac{1}{27}$ works. From the above reasoning we can find that λ must be at most $-\frac{1}{27}$ or else the inequality fails for the polynomial $f(x) = x^2(x - 1)$ at $x = \frac{1}{2}$. Equality occurs when either $\alpha = \beta = \gamma$ and $x = 0$; or $\alpha = \beta = 0$, γ any nonnegative real, and $x = \frac{\gamma}{2}$.

Problem 6 A $4\times4\times4$ cube is composed of 64 unit cubes. The faces of 16 unit cubes are to be colored red. A coloring is called *interesting* if there is exactly 1 red unit cube in every $1 \times 1 \times 4$ rectangular box composed of 4 unit cubes. Determine the number of interesting colorings. (Two colorings are different even if one can be transformed into another by a series of rotations.)

Solution: Pick one face of the cube as our bottom face. For each unit square A on the bottom face, we consider the vertical $1 \times 1 \times 4$

rectangular box with A at its bottom. Suppose the i-th unit cube up (counted from A) in the box is colored; then write the number i in A.

Each interesting coloring is mapped one-to-one to a 4×4 *Latin square* on the bottom face. (In an $n \times n$ Latin square, each row and column contains each of n symbols $a_1, \ldots, a_n$ exactly once.) Conversely, given a Latin square we can reverse this construction. Therefore, to solve the problem, we only need to count the number of distinct 4×4 Latin squares.

Note that switching rows of a Latin square will generate another Latin square. Thus if our four symbols are a, b, c, d, then each of the $4! \cdot 3!$ arrangements of the first row and column correspond to the same number of Latin squares. Therefore there are $4! \cdot 3! \cdot x$ four-by-four Latin squares, where x is the number of Latin squares whose first row and column both contain the symbols a, b, c, d in that order. The entry in the second row and second column equals either a, c, or d, yielding the Latin squares

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \quad \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & b & a \\ d & c & a & b \end{bmatrix},$$

$$\begin{bmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{bmatrix}, \quad \begin{bmatrix} a & b & c & d \\ b & d & a & c \\ c & a & d & b \\ d & c & b & a \end{bmatrix}.$$

Thus $x = 4$, and there are $4! \cdot 3! \cdot 4 = 576$ four-by-four Latin squares, and 576 interesting colorings.

1.6 Czech and Slovak Republics

Problem 1 In the fraction

$$\frac{29 \div 28 \div 27 \div \cdots \div 16}{15 \div 14 \div 13 \div \cdots \div 2}$$

parentheses may be repeatedly placed anywhere in the numerator, granted they are also placed on the identical locations in the denominator.

(a) Find the least possible integral value of the resulting expression.

(b) Find all possible integral values of the resulting expression.

Solution:

(a) The resulting expression can always be written (if we refrain from canceling terms) as a ratio $\frac{A}{B}$ of two integers A and B satisfying

$$AB = (2)(3)\cdots(29) = 29! = 2^{25}\cdot 3^{13}\cdot 5^{6}\cdot 7^{4}\cdot 11^{2}\cdot 13^{2}\cdot 17\cdot 19\cdot 23\cdot 29.$$

(To find these exponents, we could either count primes directly factor by factor, or use the rule that

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots \tag{1}$$

is the exponent of p in $n!$.)

The primes that have an odd exponent in the factorization of 29! cannot "vanish" from the ratio $\frac{A}{B}$ even after making any cancellations. For this reason no integer value of the result can be less than

$$H = 2\cdot 3\cdot 17\cdot 19\cdot 23\cdot 29 = 1,292,646.$$

On the other hand,

$$\begin{aligned}
&\frac{29 \div (28 \div 27 \div \cdots \div 16)}{15 \div (14 \div 13 \div \cdots \div 2)} \\
&\quad = \frac{29\cdot 14}{15\cdot 28}\cdot\frac{(27)(26)\cdots(16)}{(13)(12)\cdots(2)} \\
&\quad = \frac{29\cdot 14^2}{28}\cdot\frac{27!}{(15!)^2} \\
&\quad = 29\cdot 7\cdot\frac{2^{23}\cdot 3^{13}\cdot 5^{6}\cdot 7^{3}\cdot 11^{2}\cdot 13^{2}\cdot 17\cdot 19\cdot 23}{(2^{11}\cdot 3^{6}\cdot 5^{3}\cdot 7^{2}\cdot 11\cdot 13)^2} = H.
\end{aligned}$$

(Again it helps to count exponents in factorials using (1).) The number H is thus the desired least value.

(b) Let's examine the products A and B more closely. In each of the fourteen pairs of numbers

$$\{29, 15\}, \{28, 14\}, \ldots, \{16, 2\},$$

one of the numbers is a factor in A and the other is a factor in B. The resulting value V can then be written as a product

$$\left(\frac{29}{15}\right)^{\epsilon_1}\left(\frac{28}{14}\right)^{\epsilon_2}\cdots\left(\frac{16}{2}\right)^{\epsilon_{14}},$$

where each ϵ_i equals ± 1, and where $\epsilon_1 = 1$ and $\epsilon_2 = -1$ no matter how the parentheses are placed. Since the fractions $\frac{27}{13}$, $\frac{26}{12}$, $\ldots$, $\frac{16}{2}$ are greater than 1, the resulting value V (whether an integer or not) has to satisfy the estimate

$$V \le \frac{29}{15} \cdot \frac{14}{28} \cdot \frac{27}{13} \cdot \frac{26}{12} \cdot \ldots \cdot \frac{16}{2} = H,$$

where H is number determined in part (a). It follows that H is the *only* possible integer value of V!

Problem 2 In a tetrahedron $ABCD$ we denote by E and F the midpoints of the medians from the vertices A and D, respectively. (The median from a vertex of a tetrahedron is the segment connecting the vertex and the centroid of the opposite face.) Determine the ratio of the volumes of tetrahedrons $BCEF$ and $ABCD$.

Solution: Let K and L be the midpoints of the edges BC and AD, and let A_0, D_0 be the centroids of triangles BCD and ABC, respectively. Both medians AA_0 and DD_0 lie in the plane AKD, and their intersection T (the centroid of the tetrahedron) divides them in $3:1$ ratios. T is also the midpoint of $\overline{KL}$, since $\vec{T} = \frac{1}{4}(\vec{A}+\vec{B}+\vec{C}+\vec{D}) = \frac{1}{2}(\frac{1}{2}(\vec{A}+\vec{D})+\frac{1}{2}(\vec{B}+\vec{C})) = \frac{1}{2}(\vec{K}+\vec{L})$. It follows that $\frac{ET}{AT} = \frac{FT}{DT} = \frac{1}{3}$, and hence $\triangle ATD \sim \triangle ETF$ and $EF = \frac{1}{3}AD$. Since the plane BCL bisects both segments AD and EF into halves, it also divides both tetrahedrons $ABCD$ and $BCEF$ into two parts of equal volume. Let G be the midpoint of $\overline{EF}$; the corresponding volumes than satisfy

$$\frac{[BCEF]}{[ABCD]} = \frac{[BCGF]}{[BCLD]} = \frac{GF}{LD} \cdot \frac{[BCG]}{[BCL]} = \frac{1}{3}\frac{KG}{KL} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}.$$

Problem 3 Show that there exists a triangle ABC for which, with the usual labelling of sides and medians, it is true that $a \neq b$ and $a + m_a = b + m_b$. Show further that there exists a number k such that for each such triangle $a + m_a = b + m_b = k(a + b)$. Finally, find all possible ratios $a : b$ of the sides of these triangles.

Solution: We know that

$$m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2), \quad m_b^2 = \frac{1}{4}(2a^2 + 2c^2 - b^2),$$

so

$$m_a^2 - m_b^2 = \frac{3}{4}(b^2 - a^2).$$

As $m_a - m_b = b - a \neq 0$ by hypothesis, it follows that $m_a + m_b = \frac{3}{4}(b + a)$. From the system of equations

$$m_a - m_b = b - a$$
$$m_a + m_b = \frac{3}{4}(b + a)$$

we find $m_a = \frac{1}{8}(7b - a)$, $m_b = \frac{1}{8}(7a - b)$, and

$$a + m_a = b + m_b = \frac{7}{8}(a + b).$$

Thus $k = \frac{7}{8}$.

Now we examine for what $a \neq b$ there exists a triangle ABC with sides a, b and medians $m_a = \frac{1}{8}(7b - a), m_b = \frac{1}{8}(7a - b)$. We can find all three side lengths in the triangle AB_1G, where G is the centroid of the triangle ABC and B_1 is the midpoint of the side AC:

$$AB_1 = \frac{b}{2}, \quad AG = \frac{2}{3}m_a = \frac{2}{3} \cdot \frac{1}{8}(7b - a) = \frac{1}{12}(7b - a),$$
$$B_1G = \frac{1}{3}m_b = \frac{1}{3} \cdot \frac{1}{8}(7a - b) = \frac{1}{24}(7a - b).$$

Examining the triangle inequalities for these three lengths, we get the condition

$$\frac{1}{3} < \frac{a}{b} < 3,$$

from which the value $\frac{a}{b} = 1$ has to be excluded by assumption. This condition is also sufficient: once the triangle AB_1G has been constructed, it can always be completed to a triangle ABC with $b = AC, m_a = AA_1, m_b = BB_1$. Then from the equality $m_a^2 - m_b^2 = \frac{3}{4}(b^2 - a^2)$ we would also have $a = BC$.

Problem 4 In a certain language there are only two letters, A and B. The words of this language satisfy the following axioms:

(i) There are no words of length 1, and the only words of length 2 are AB and BB.

(ii) A sequence of letters of length $n > 2$ is a word if and only if it can be created from some word of length less than n by the following construction: all letters A in the existing word are left unchanged, while each letter B is replaced by some word. (While performing this operation, the B's do not all have to be replaced by the same word.)

Show that for any n the number of words of length n equals

$$\frac{2^n + 2 \cdot (-1)^n}{3}.$$

Solution: Let us call any finite sequence of letters A, B a "string." From here on, we let $\cdots$ denote a (possibly empty) string, while $***$ will stand for a string consisting of identical letters. (For example, $\underbrace{B***B}_{k}$ is a string of k B's.)

We show that an arbitrary string is a word if and only if it satisfies the following conditions: (a) the string terminates with the letter B; and (b) it either starts with the letter A, or else starts (or even wholly consists of) an even number of B's.

It is clear that these conditions are necessary: they are satisfied for both words AB and BB of length 2, and they are likewise satisfied by any new word created by the construction described in (ii) if they are satisfied by the words in which the B's are replaced.

We now show by induction on n that, conversely, any string of length n satisfying the conditions is a word. This is clearly true for $n = 1$ and $n = 2$. If $n > 2$, then a string of length n satisfying the conditions must have one of the forms

$$AA\cdots B,\ AB\cdots B,\ \underbrace{B***B}_{2k}A\cdots B,\ \underbrace{B***B}_{2k+2},$$

where $2 \le 2k \le n-2$. We have to show that these four types of strings arise from the construction in (ii) in which the B's are replaced by strings (of lengths less than n) satisfying the condition — that is, by *words* in view of the induction hypothesis.

The word $AA\cdots B$ arises as $A(A\cdots B)$ from the word AB. The word $AB\cdots B$ arises either as $A(B\cdots B)$ from the word AB, or as $(AB)(\cdots B)$ from the word BB, depending on whether its initial letter A is followed by an even or an odd number of B's. The word $\underbrace{B***B}_{2k}A\cdots B$ arises as $(B***B)(A\cdots B)$ from the word BB, and the word $\underbrace{B***B}_{2k+2}$ as $(\underbrace{B***B}_{2k})(BB)$ from the word BB. This completes the proof by induction.

Now we show that the number p_n of words of length n is indeed given by the formula

$$p_n = \frac{2^n + 2\cdot(-1)^n}{3}.$$

It is clearly true for $n = 1$ and 2 since $p_1 = 0$ and $p_2 = 2$; and the formula will then follow by induction if we can show that $p_{n+2} = 2^n + p_n$ for each n. But this recursion is obvious because each word of length $n+2$ is either of the form $A\cdots B$ where $\cdots$ is any of 2^n strings of length n; or of the form $BB\cdots$ where $\cdots$ is any of the p_n words of length n.

Problem 5 In the plane an acute angle APX is given. Show how to construct a square $ABCD$ such that P lies on side BC and P lies on the bisector of angle BAQ where Q is the intersection of ray PX with CD.

Solution: Consider the ration by 90° around the point A that maps B to D, and the points P, C, D into some points P', C', D', respectively. Since $\angle PAP' = 90^\circ$, it follows from the nature of exterior angle bisectors that AP' bisects $\angle QAD'$. Consequently, the point P' has the same distance from $\overline{AD'}$ and $\overline{AQ}$, equal to the side length s of square $ABCD$. But this distance is also the length of the altitude AD in triangle AQP'; then since the altitudes from A and P' in this triangle are equal, we have $AQ = P'Q$. Since we can construct P', we can also construct Q as the intersection of line PX with the perpendicular bisector of the segment AP'. The rest of the construction is obvious, and it is likewise clear that the resulting square $ABCD$ has the required property.

Problem 6 Find all pairs of real numbers a and b such that the system of equations

$$\frac{x+y}{x^2+y^2}=a, \qquad \frac{x^3+y^3}{x^2+y^2}=b$$

has a solution in real numbers (x,y).

Solution: If the given system has a solution (x,y) for $a=A$, $b=B$, then it clearly also has a solution (kx,ky) for $a=\frac{1}{k}A$, $b=kB$, for any $k\neq 0$. It follows that the existence of a solution of the given system depends only on the value of the product ab.

We therefore begin by examining the values of the expression

$$P(u,v)=\frac{(u+v)(u^3+v^3)}{(u^2+v^2)^2}$$

where the numbers u and v are normalized by the condition $u^2+v^2=1$. This condition implies that

$$\begin{aligned}P(u,v)&=(u+v)(u^3+v^3)=(u+v)^2(u^2-uv+v^2)\\&=(u^2+2uv+v^2)(1-uv)=(1+2uv)(1-uv).\end{aligned}$$

Under the condition $u^2+v^2=1$ the product uv can attain all values in the interval $[-\frac{1}{2},\frac{1}{2}]$ (if $u=\cos\alpha$ and $v=\sin\alpha$, then $uv=\frac{1}{2}\sin 2\alpha$). Hence it suffices to find the range of values of the function $f(t)=(1+2t)(1-t)$ on the interval $t\in[-\frac{1}{2},\frac{1}{2}]$. From the formula

$$f(t)=-2t^2+t+1=-2\left(t-\frac{1}{4}\right)^2+\frac{9}{8}$$

it follows that this range of values is the closed interval with endpoints $f\left(-\frac{1}{2}\right)=0$ and $f\left(\frac{1}{4}\right)=\frac{9}{8}$.

This means that if the given system has a solution, its parameters a and b must satisfy $0\leq ab\leq\frac{9}{8}$, where the equality $ab=0$ is possible only if $x+y=0$ (then, however, $a=b=0$).

Conversely, if a and b satisfy $0<ab\leq\frac{9}{8}$, by our proof there exist numbers u and v such that $u^2+v^2=1$ and $(u+v)(u^3+v^3)=ab$. Denoting $a'=u+v$ and $b'=u^3+v^3$, the equality $a'b'=ab\neq 0$ implies that both ratios $\frac{a'}{a}$ and $\frac{b}{b'}$ have the same value $k\neq 0$. But then $(x,y)=(ku,kv)$ is clearly a solution of the given system for the parameter values a and b.

1.7 France

Problem 1

(a) What is the maximum volume of a cylinder that is inside a given cone and has the same axis of revolution as the cone? Express your answer in terms of the radius R and height H of the cone.

(b) What is the maximum volume of a ball that is inside a given cone? Again, express your answer in terms of R and H.

(c) Given fixed values for R and H, which of the two maxima you found is bigger?

Solution: Let $\ell = \sqrt{R^2 + H^2}$ be the slant height of the given cone; also, orient the cone so that its base is horizontal and its tip is pointing upward.

(a) Intuitively, the cylinder with maximum volume rests against the base of the cone, and the center of the cylinder's base coincides with the center of the cone's base. The top face of the cylinder cuts off a smaller cone at the top of the original cone. If the cylinder has radius r, then the smaller cone has height $r \cdot \frac{H}{R}$ and the cylinder has height $h = H - r \cdot \frac{H}{R}$. Then the volume of the cylinder is

$$\pi r^2 h = \pi r^2 H \left(1 - \frac{r}{R}\right) = 4\pi R^2 H \left(\frac{r}{2R} \cdot \frac{r}{2R} \cdot \left(1 - \frac{r}{R}\right)\right).$$

And by AM-GM on $\frac{r}{2R}$, $\frac{r}{2R}$, and $1 - \frac{r}{R}$ this is at most

$$4\pi R^2 H \cdot \frac{1}{27} \left(\frac{r}{2R} + \frac{r}{2R} + \left(1 - \frac{r}{R}\right)\right)^3 = \frac{4}{27}\pi R^2 H,$$

with equality when $r/2R = 1 - r/R \iff r = \frac{2}{3}R$.

(b) Intuitively, the sphere with maximum volume is tangent to the base and lateral face of the cone; and its center lies on the cone's axis. Say the sphere has radius r.

Take a planar cross-section of the cone slicing through its axis; this cuts off a triangle from the cone and a circle from the sphere. The triangle's side lengths are ℓ, ℓ, and $2R$; and its height (from the side of length $2R$) is H. The circle has radius r and is the incircle of this triangle.

The area K of the triangle is $\frac{1}{2}(2R)(H) = RH$ and its semiperimeter is $s = R + \ell$. Then since $K = rs$ we have $r = \frac{RH}{R+\ell}$, and thus the volume of the sphere is

$$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{RH}{R+\ell}\right)^3.$$

(c) We claim that when $h/R = \sqrt{3}$ or $2\sqrt{6}$, the two volumes are equal; when $\sqrt{3} < h/R < 2$, the sphere has larger volume; and when $0 < h/R < \sqrt{3}$ or $2 < h/R$, the cylinder has larger volume.

We wish to compare $\frac{4}{27}\pi R^2 H$ and $\frac{4}{3}\pi\left(\frac{RH}{R+\ell}\right)^3$; equivalently, multiplying by $\frac{27}{4\pi R^2 H}(R+\ell)^3$, we wish to compare $(R+\ell)^3$ and $9RH^2 = 9R(\ell^2 - R^2)$. Writing $\phi = \ell/R$, this is equivalent to comparing $(1+\phi)^3$ and $9(\phi^2-1)$. Now,

$$(1+\phi)^3 - 9(\phi^2-1) = \phi^3 - 6\phi^2 + 3\phi + 10 = (\phi+1)(\phi-2)(\phi-5).$$

Thus when $\phi = 2$ or 5, the volumes are equal; when $2 < \phi < 5$, the sphere has larger volume; and when $1 < \phi < 2$ or $5 < \phi$, the cylinder has larger volume. Comparing R and H instead of R and ℓ yields the conditions stated before.

Problem 2 Find all integer solutions to $(n+3)^n = \sum_{k=3}^{n+2} k^n$.

Solution: $n = 2$ and $n = 3$ are solutions to the equations; we claim they are the only ones.

First observe that the function $f(n) = \left(\frac{n+3}{n+2}\right)^n = \left(1+\frac{1}{n+2}\right)^n$ is an increasing function for $n > 0$. To see this, note that the derivative of $\ln f(n)$ with respect to n is $\ln\left(1+\frac{1}{n+2}\right) - \frac{n}{(n+2)(n+3)}$. By the Taylor expansion,

$$\begin{aligned} \ln\left(1+\frac{1}{n+2}\right) &= \sum_{j=1}^{\infty} \frac{1}{(n+2)^{2j}}\left[\frac{1}{2j-1}(n+2) - \frac{1}{2j}\right] \\ &> \frac{2(n+2)-1}{2(n+2)^2} \end{aligned}$$

and hence

$$\begin{aligned} \frac{d}{dn}\ln f(n) &= \ln\left(\frac{n+3}{n+2}\right) - \frac{n}{(n+2)(n+3)} \\ &> \frac{2(n+2)-1}{2(n+2)^2} - \frac{n}{(n+2)^2} = \frac{3}{2(n+2)^2} > 0. \end{aligned}$$

Thus $\ln f(n)$ and therefore $f(n)$ is indeed increasing.

Now, notice that if $f(n) > 2$ then we have

$$\left(\frac{2}{1}\right)^n > \left(\frac{3}{2}\right)^n > \cdots > \left(\frac{n+3}{n+2}\right)^n > 2$$

so that

$$(n+3)^n > 2(n+2)^n > \cdots > 2^j(n+3-j)^n > \cdots > 2^n \cdot (3)^n.$$

Then

$$3^n + 4^n + \cdots + (n+2)^n < \left(\frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{2}\right)(n+3)^n$$
$$= \left(1 - \frac{1}{2^n}\right)(n+3)^n < (n+3)^n,$$

so the equality does not hold.

Then since $2 < f(6) < f(7) < \cdots$, the equality must fail for all $n \geq 6$. Quick checks show it also fails for $n = 1, 4, 5$ (in each case, one side of the equation is odd while the other is even). Therefore the only solutions are $n = 2$ and $n = 3$.

Problem 3 For which acute-angled triangle is the ratio of the shortest side to the inradius maximal?

Solution: Let the sides of the triangle have lengths $a \leq b \leq c$; let the angles opposite them be A, B, C; let the semiperimeter be $s = \frac{1}{2}(a+b+c)$; and let the inradius be r. Without loss of generality say the triangle has circumradius $R = \frac{1}{2}$ and that $a = \sin A$, $b = \sin B$, $c = \sin C$.

The area of the triangle equals both $rs = \frac{1}{2}r(\sin A + \sin B + \sin C)$ and $abc/4R = \frac{1}{2}\sin A \sin B \sin C$. Thus

$$r = \frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C}$$

and

$$\frac{a}{r} = \frac{\sin A + \sin B + \sin C}{\sin B \sin C}.$$

Since $A = 180° - B - C$, $\sin A = \sin(B+C) = \sin B \cos C + \cos B \sin C$ and we also have

$$\frac{a}{r} = \cot B + \csc B + \cot C + \csc C.$$

Note that $f(x) = \cot x + \csc x$ is a decreasing function along the interval $0° < x < 90°$. Now there are two cases: $B \leq 60°$, or $B > 60°$.

If $B \le 60^\circ$, then assume that $A = B$; otherwise the triangle with angles $A' = B' = \frac{1}{2}(A+B) \le B$ and $C' = C$ has a larger ratio a'/r'. Then since $C < 90^\circ$ we have $45^\circ < A \le 60^\circ$. Now,

$$\frac{a}{r} = \frac{\sin A + \sin B + \sin C}{\sin B \sin C} = \frac{2\sin A + \sin(2A)}{\sin A \sin(2A)} = 2\csc(2A) + \csc A.$$

Now $\csc x$ has second derivative $\csc x(\csc^2 x + \cot^2 x)$, which is strictly positive when $0^\circ < x < 180^\circ$; thus both $\csc x$ and $\csc(2x)$ are strictly convex along the interval $0^\circ < x < 90^\circ$. Therefore $g(A) = 2\csc(2A) + \csc A$, a convex function in A, is maximized in the interval $45^\circ \le A \le 60^\circ$ at one of the endpoints. Since $g(45^\circ) = 2 + \sqrt{2} < 2\sqrt{3} = g(60^\circ)$, it is maximized when $A = B = C = 60^\circ$.

As for the case when $B > 60^\circ$, since $C > B > 60^\circ$, the triangle with $A' = B' = C' = 60^\circ$ has a larger ratio a'/r'. Therefore the maximum ratio is $2\sqrt{3}$, attained with an equilateral triangle.

Problem 4 There are 1999 red candies and 6661 yellow candies on a table, made indistinguishable by their wrappers. A *gourmand* applies the following algorithm until the candies are gone:

(a) If there are candies left, he takes one at random, notes its color, eats it, and goes to (b).

(b) If there are candies left, he takes one at random, notes its color, and

  (i) if it matches the last one eaten, he eats it also and returns to (b).

  (ii) if it does not match the last one eaten, he wraps it up again, puts it back, and goes to (a).

Prove that all the candies will eventually be eaten. Find the probability that the last candy eaten is red.

Solution: If there are finitely many candies left at any point, then at the next instant the gourmand must perform either step (a), part (i) of step (b), or part (ii) of step (b). He eats a candy in the first two cases; in the third case, he returns to step (a) and eats a candy. Since there are only finitely many candies, the gourmand must eventually eat all the candies.

We now prove by induction on the total number of candies that if we start with $r > 0$ red candies and $y > 0$ yellow candies immediately before step (a), then the probability is $\frac{1}{2}$ that the last candy eaten is red.

Suppose that the claim is true for all smaller amounts of candy. After the gourmand first completes steps (a) and (b) exactly once, suppose there are r' red candies and y' yellow candies left; we must have $r' + y' < r + y$. The chances that $r' = 0$ is

$$\frac{r}{r+y} \cdot \frac{r-1}{r+y-1} \cdot \cdots \cdot \frac{1}{y+1} = \frac{1}{\binom{r+y}{r}}.$$

Similarly, the chances that $y' = 0$ is $\frac{1}{\binom{r+y}{y}} = \frac{1}{\binom{r+y}{r}}$. (In the case $r = y = 1$, this proves the claim.)

Otherwise, the probability is $1 - \frac{2}{\binom{r+y}{r}}$ that both r' and y' are still positive. By the induction hypothesis in this case the last candy is equally likely to be red as it is yellow. Thus the overall probability that the last candy eaten is red is

$$\underbrace{\frac{1}{\binom{r+y}{r}}}_{y'=0} + \underbrace{\frac{1}{2}\left(1 - \frac{2}{\binom{r+y}{r}}\right)}_{r',y'>0} = \frac{1}{2}.$$

This completes the inductive step, and the proof.

Problem 5 With a given triangle, form three new points by reflecting each vertex about the opposite side. Show that these three new points are collinear if and only if the the distance between the orthocenter and the circumcenter of the triangle is equal to the diameter of the circumcircle of the triangle.

Solution: Let the given triangle be ABC and let the reflections of A, B, C across the corresponding sides be D, E, F. Let A', B', C' be the midpoints of $\overline{BC}, \overline{CA}, \overline{AB}$, and as usual let G, H, O denote the triangle's centroid, orthocenter, and circumcenter. Let triangle $A''B''C''$ be the triangle for which A, B, C are the midpoints of $B''C'', C''A'', A''B''$, respectively. Then G is the centroid and H is the circumcenter of triangle $A''B''C''$. Let $D', E,' F'$ denote the projections of O on the lines $B''C'', C''A'', A''B''$, respectively.

Consider the homothety h with center G and ratio $-1/2$. It maps A, B, C, A'', B'', C'' into A', B', C', A, B, C, respectively. Note that $A'D' \perp BC$ since O is the orthocenter of triangle $A'B'C'$. This implies $AD : A'D' = 2 : 1 = GA : GA'$ and $\angle DAG = \angle D'A'G$. We conclude that $h(D) = D'$. Similarly, $h(E) = E'$ and $h(F) = F'$. Thus, D, E, F are collinear if and only if D', E', F' are collinear. Now D', E', F' are the projections of O on the sides $B''C'', C''A'', A''B''$, respectively. By Simson's theorem, they are collinear if and only if O lies on the circumcircle of triangle $A''B''C''$. Since the circumradius of triangle $A''B''C''$ is $2R$, O lies on its circumcircle if and only if $OH = 2R$.

1.8 Hong Kong (China)

Problem 1 Let $PQRS$ be a cyclic quadrilateral with $\angle PSR = 90°$, and let H and K be the respective feet of perpendiculars from Q to lines PR and PS. Prove that line HK bisects $\overline{QS}$.

First Solution: Since $\overline{QK}$ and $\overline{RS}$ are both perpendicular to $\overline{PS}$, $\overline{QK}$ is parallel to $\overline{RS}$ and thus $\angle KQS = \angle RSQ$. Since $PQRS$ is cyclic, $\angle RSQ = \angle RPQ$. Since $\angle PKQ = \angle PHQ = 90°$, $PKHQ$ is also cyclic and it follows that $\angle RPQ = \angle HPQ = \angle HKQ$. Thus, $\angle KQS = \angle HKQ$; since triangle KQS is right, it follows that line HK bisects $\overline{QS}$.

Second Solution: The Simson line from Q with respect to $\triangle PRS$ goes through H, K, and the foot F of the perpendicular from Q to $\overleftrightarrow{RS}$. Thus, line HK is line FK, a diagonal in rectangle $SFQK$, so it bisects the other diagonal, $\overline{QS}$.

Problem 2 The base of a pyramid is a convex nonagon. Each base diagonal and each lateral edge is colored either black or white. Both colors are used at least once. (Note that the sides of the base are not colored.) Prove that there are three segments colored the same color which form a triangle.

Solution: Let us assume the contrary. From the pigeonhole principle, 5 of the lateral edges must be of the same color; assume they are black, and say they are the segments from the vertex V to B_1, B_2, B_3, B_4, and B_5 where $B_1B_2B_3B_4B_5$ is a convex pentagon (and where the B_i's are not necessarily adjacent vertices of the nonagon). The $\overline{B_iB_{i+1}}$ (where $B_6 = B_1$) cannot all be sides of the nonagon, so without loss of generality suppose that B_1B_2 is colored. Then because triangle VB_iB_j cannot have three sides colored black, each segment B_1B_2, B_2B_4, B_4B_1 must be white; but then triangle $B_1B_2B_4$ has three sides colored white, a contradiction.

Problem 3 Let s and t be nonzero integers, and let (x, y) be any ordered pair of integers. A move changes (x, y) to $(x - t, y - s)$. The pair (x, y) is *good* if after some (possibly zero) number of moves it becomes a pair of integers that are not relatively prime.

(a) Determine if (s, t) is a good pair;

(b) Prove that for any s, t there exists a pair (x, y) which is not good.

Solution:

(a) Let us assume that (s, t) is not good. Then, after one move, we have $(s-t, t-s)$, so we may assume without loss of generality that $s - t = 1$ and $t - s = -1$ since these numbers must be relatively prime. Then $s + t$ cannot equal 0 because it is odd; also, $s + t = (s-t)+2t \neq (s-t)+0 = 1$, and $s+t = (t-s)+2s \neq (t-s)+0 = -1$. Hence some prime p divides $s+t$. After $p-1$ moves, (s, t) becomes $(s - (p-1)t, t - (p-1)s) \equiv (s + t, t + s) \equiv (0, 0) \pmod{p}$, a contradiction. Thus (s, t) is good.

(b) Let x and y be integers which satisfy $sx - ty = g$, where $g = \gcd(s, t)$. Dividing by g, we find $s'x - t'y = 1$, so $\gcd(x, y) = 1$. Now, suppose by way of contradiction that after k moves some prime p divides both $x - kt$ and $y - ks$. We then have

$$\begin{aligned} &0 \equiv x - kt \equiv y - ks \\ &\implies 0 \equiv s(x - kt) \equiv t(y - ks) \\ &\implies 0 \equiv sx - ty = g \pmod{p}. \end{aligned}$$

Thus p divides g, which divides s and t, so the first equation above becomes $0 \equiv x \equiv y \pmod{p}$; but x and y are relatively prime, a contradiction. Thus (x, y) is not good.

Problem 4 Let f be a function defined on the positive reals with the following properties:

(i) $f(1) = 1$;

(ii) $f(x + 1) = xf(x)$;

(iii) $f(x) = 10^{g(x)}$,

where $g(x)$ is a function defined on the reals satisfying

$$g(ty + (1 - t)z) \leq tg(y) + (1 - t)g(z)$$

for all real y, z and any $0 \leq t \leq 1$.

(a) Prove that

$$t[g(n) - g(n - 1)] \leq g(n + t) - g(n) \leq t[g(n + 1) - g(n)]$$

where n is an integer and $0 \leq t \leq 1$.

(b) Prove that

$$\frac{4}{3} \le f\left(\frac{1}{2}\right) \le \frac{4\sqrt{2}}{3}.$$

Solution:

(a) Setting $t = \frac{1}{2}$ in the given inequality, we find that $g(\frac{1}{2}(y+z)) \le \frac{1}{2}(g(y)+g(z))$. Now fix t (perhaps not equal to $\frac{1}{2}$) constant; letting $y = n - t$ and $z = n + t$ in $g(\frac{1}{2}(y+z)) \le \frac{1}{2}(g(y)+g(z))$ gives $g(n) \le \frac{1}{2}(g(n-t)+g(n+t))$, or

$$g(n) - g(n-t) \le g(n+t) - g(n). \tag{1}$$

Plugging in $z = n$, $y = n - 1$ into the given inequality gives $g(t(n-1)+(1-t)n) \le tg(n-1)+(1-t)g(n)$, or

$$t[g(n) - g(n-1)] \le g(n) - g(n-t).$$

Combining this with (1) proves the inequality on the left side. And the inequality on the right side follows from the given inequality with $z = n$, $y = n + 1$.

(b) From (ii), $f(\frac{3}{2}) = \frac{1}{2}f(\frac{1}{2})$, and $f(\frac{5}{2}) = \frac{3}{2}f(\frac{3}{2}) = \frac{3}{4}f(\frac{1}{2})$. Also, $f(2) = 1 \cdot f(1) = 1$, and $f(3) = 2f(2) = 2$. Now, if we let $n = 2$ and $t = \frac{1}{2}$ in the inequality in part (a), we find $\frac{1}{2}[g(2) - g(1)] \le g\left(\frac{5}{2}\right) - g(2) \le \frac{1}{2}[g(3) - g(2)]$. Exponentiating with base 10 yields $\sqrt{\frac{f(2)}{f(1)}} \le \frac{f\left(\frac{5}{2}\right)}{f(2)} \le \sqrt{\frac{f(3)}{f(2)}}$, or $1 \le f\left(\frac{5}{2}\right) \le \sqrt{2}$. Plugging in $f\left(\frac{5}{2}\right) = \frac{3}{4}f(\frac{1}{2})$ yields the desired result.

1.9 Hungary

Problem 1 I have $n \geq 5$ real numbers with the following properties:

(i) They are nonzero, but at least one of them is 1999.

(ii) Any four of them can be rearranged to form a geometric progression.

What are my numbers?

Solution: First suppose that the numbers are all nonnegative. If $x \leq y \leq z \leq w \leq v$ are any five of the numbers, then x, y, z, w; x, y, z, v; x, y, w, v; x, z, w, v; and y, z, w, v must all be geometric progressions. Comparing each two successive progressions in this list we find that $x = y = z = w = v$. Thus all our numbers are equal.

If some numbers are negative in our original list, replace each number x by $|x|$. The geometric progression property is preserved, and thus from above all the values $|x|$ are equal. Hence, each original number was 1999 or -1999. And because $n \geq 5$, some three numbers are equal. But no geometric progression can be formed from three -1999s and a 1999, or from three 1999s and a -1999. Therefore all the numbers must be equal — to 1999.

Problem 2 Let ABC be a right triangle with $\angle C = 90°$. Two squares S_1 and S_2 are inscribed in triangle ABC such that S_1 and ABC share a common vertex C, and S_2 has one of its sides on AB. Suppose that $[S_1] = 441$ and $[S_2] = 440$. Calculate $AC + BC$.

Solution: Let $S_1 = CDEF$ and $S_2 = KLMN$ with D and K on $\overline{AC}$ and N on $\overline{BC}$. Let $s_1 = 21, s_2 = \sqrt{440}$ and $a = BC, b = CA, c = AB$. Using ratios between similar triangles AED, ABC, EBF we get $c = AB = AE + EB = c(s_1/a + s_1/b)$ or $s_1(1/a + 1/b) = 1$. Since triangles ABC, AKL, NBM are similar we have $c = AB = AL + LM + MB = s_2(b/a + 1 + a/b)$ and $s_2 = abc/(ab + c^2)$. Then

$$\begin{aligned} \frac{1}{s_2^2} - \frac{1}{s_1^2} &= \left(\frac{1}{c} + \frac{c}{ab}\right)^2 - \left(\frac{1}{a} + \frac{1}{b}\right)^2 \\ &= \left(\frac{1}{c^2} + \frac{c^2}{a^2b^2} + \frac{2}{ab}\right) - \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{2}{ab}\right) = \frac{1}{c^2}. \end{aligned}$$

Thus $c = 1/\sqrt{1/s_2^2 - 1/s_1^2} = 21\sqrt{440}$. Solving $s_2 = abc/(ab + c^2)$ for ab yields $ab = s_2c^2/(c - s_2) = 21^2 \cdot 22$. Finally, $AC + BC = a + b = ab/s_1 = 21 \cdot 22 = 462$.

Problem 3 Let O and K be the centers of the respective spheres tangent to the faces, and the edges, of a right pyramid whose base is a 2 by 2 square. Determine the volume of the pyramid if O and K are equidistant from the base.

Solution: Let r, R be the spheres' respective radii. Let the pyramid have base $ABCD$, vertex P, and height h. By symmetry, O and K lie on the altitude through P.

Take a cross-section of the pyramid with a plane perpendicular to the base, cutting the base at a line through its center parallel to $\overline{AB}$. It cuts off an isosceles triangle from the pyramid with base 2 and legs $\sqrt{h^2+1}$; the triangle's incircle is the cross-section of the sphere centered at O and hence has radius r. On the one hand, the area of this triangle is the product of its inradius and semiperimeter, or $\frac{1}{2}r(2 + 2\sqrt{h^2+1})$. On the other hand, it equals half of the product of its base and height, or $\frac{1}{2} \cdot 2 \cdot h$. Setting these quantities equal, we have $r = (\sqrt{h^2+1} - 1)/h$.

Next, by symmetry the second sphere is tangent to $\overline{AB}$ at its midpoint M. Then since K must be distance r from plane $ABCD$, we have $R^2 = KM^2 = r^2 + 1$. Furthermore, if the second sphere is tangent to $\overline{AP}$ at N, then by equal tangents we have $AN = AM = 1$.

Then $PN = PA - 1 = \sqrt{h^2+2} - 1$. Also, $PK = h + r$ if K is on the opposite side of plane $ABCD$ as O, and it equals $h - r$ otherwise. Thus

$$\begin{aligned} PK^2 &= PN^2 + NK^2 \\ (h \pm r)^2 &= (\sqrt{h^2+2} - 1)^2 + (r^2 + 1) \\ \pm\, 2rh &= 4 - 2\sqrt{h^2+2}. \end{aligned}$$

Recalling that $r = (\sqrt{h^2+1} - 1)/h$, this gives

$$\pm\,(\sqrt{h^2+1} - 1) = 2 - \sqrt{h^2+2}.$$

This equation has the unique solution $h = \sqrt{7}/3$. Thus the volume of the pyramid is $\frac{1}{3} \cdot 4 \cdot \frac{\sqrt{7}}{3} = 4\sqrt{7}/9$.

Problem 4 For any given positive integer n, determine (as a function of n) the number of ordered pairs (x, y) of positive integers such that

$$x^2 - y^2 = 10^2 \cdot 30^{2n}.$$

Further prove that the number of such pairs is never a perfect square.

Solution: Since $10^2 \cdot 30^{2n}$ is even, x and y must have the same parity. Then (x, y) is a valid solution if and only if $(u, v) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ is a pair of positive integers that satisfies $u > v$ and $uv = 5^2 \cdot 30^{2n}$. Now $5^2 \cdot 30^{2n} = 2^{2n} \cdot 3^{2n} \cdot 5^{2n+2}$ has exactly $(2n+1)^2(2n+3)$ factors; thus without the condition $u > v$ there are exactly $(2n+1)^2(2n+3)$ such pairs (u, v). Exactly one pair has $u = v$, and by symmetry half of the remaining pairs have $u > v$; and it follows that there are $\frac{1}{2}\left((2n+1)^2(2n+3) - 1\right) = (n+1)(4n^2 + 6n + 1)$ valid pairs.

Now suppose that $(n+1)(4n^2+6n+1)$ were a square. Since $n+1$ and $4n^2 + 6n + 1 = (4n+2)(n+1) - 1$ are coprime, $4n^2 + 6n + 1$ must be a square as well; but $(2n+1)^2 < 4n^2 + 6n + 1 < (2n+2)^2$, a contradiction.

Problem 5 For $0 \leq x, y, z \leq 1$, find all solutions to the equation

$$\frac{x}{1+y+zx} + \frac{y}{1+z+xy} + \frac{z}{1+x+yz} = \frac{3}{x+y+z}.$$

Solution: Assume $x + y + z > 0$, since otherwise the equation is meaningless. $(1-z)(1-x) \geq 0 \Rightarrow 1 + zx \geq x + z$, and hence $x/(1+y+zx) \leq x/(x+y+z)$. Doing this for the other two fractions yields that the left hand side is at most $(x+y+z)/(x+y+z) \leq 3/(x+y+z)$. If equality holds, we must have in particular that $x + y + z = 3 \Rightarrow x = y = z = 1$. We then verify that this is indeed a solution.

Problem 6 The midpoints of the edges of a tetrahedron lie on a sphere. What is the maximum volume of the tetrahedron?

Solution: Let the sphere have center O. First let A, B, C be any points on its surface. Then $[OAB] = \frac{1}{2} OA \cdot OB \sin\angle AOB \leq \frac{1}{2}r^2$. Likewise, the height from C to plane OAB is at most $CO = r$, whence tetrahedron $OABC$ has maximum volume $r^3/6$. Now, if $\{A, A'\}, \{B, B'\}, \{C, C'\}$ are pairs of antipodal points on the sphere, the octahedron $ABCA'B'C'$ can be broken up into 8 such tetrahedra with vertex O and therefore has maximum volume $4r^3/3$. Equality holds for a regular octahedron.

In the situation of the problem, shrink the tetrahedron T (with volume V) by a factor of $1/2$ about each vertex to obtain four tetrahedra, each with volume $V/8$. Then the six midpoints form an octahedron with volume $V/2$. Moreover, the segment connecting two opposite vertices C and D of this octahedron has T's centroid P as its midpoint. If $O \neq P$ then line OP is a perpendicular bisector of each segment, and then all these segments must lie in the plane through P perpendicular to line OP; but then $V/2 = 0$. Otherwise, the midpoints form three pairs of antipodal points, whose volume is at most $4r^3/3$ from the last paragraph. Therefore $V \leq 8r^3/3$, with equality for a regular tetrahedron.

Problem 7 A positive integer is written in each square of an n^2 by n^2 chess board. The difference between the numbers in any two adjacent squares (sharing an edge) is less than or equal to n. Prove that at least $\lfloor n/2 \rfloor + 1$ squares contain the same number.

Solution: Consider the smallest and largest numbers a and b on the board. They are separated by at most $n^2 - 1$ squares horizontally and $n^2 - 1$ vertically, so there is a path from one to the other with length at most $2(n^2 - 1)$. Then since any two successive squares differ by at most n, we have $b - a \leq 2(n^2 - 1)n$. But since all numbers on the board are integers lying between a and b, only $2(n^2 - 1)n + 1$ distinct numbers can exist; and because $n^4 > (2(n^2 - 1)n + 1)(n/2)$, more than $n/2$ squares contain the same number, as needed.

Problem 8 One year in the 20th century, Alex noticed on his birthday that adding the four digits of the year of his birth gave his actual age. That same day, Bernath—who shared Alex's birthday but was not the same age as him—also noticed this about his own birth year and age. That day, both were under 99. By how many years do their ages differ?

Solution: Let c be the given year. Alex's year of birth was either $18\underline{uv}$ or $19\underline{uv}$ respectively (where u and v are digits), and thus either $c = 18\underline{uv} + (9 + u + v) = 1809 + 11u + 2v$ or $c = 19\underline{uv} + (10 + u + v) = 1910 + 11u + 2v$.

Similarly, let Bernath's year of birth end in the digits u', v'. Alex and Bernath could not have been born in the same century. Otherwise, we would have $11u + 2v = 11u' + 2v' \Rightarrow 2(v - v') = 11(u' - u)$; thus either $(u, v) = (u', v')$ or else $|v - v'| \geq 11$, which are both impossible. Then without loss of generality say Alex was born in the 1800s, and that $1809 + 11u + 2v = 1910 + 11u' + 2v' \Rightarrow 11(u - u') + 2(v - v') = 101 \Rightarrow u - u' = 9, v - v' = 1$. The difference between their ages then equals $19\underline{u'v'} - 18\underline{uv} = 100 + 10(u' - u) + (v' - v) = 9$.

Problem 9 Let ABC be a triangle and D a point on the side AB. The incircles of the triangles ACD and CDB touch each other on $\overline{CD}$. Prove that the incircle of ABC touches $\overline{AB}$ at D.

Solution: Suppose that the incircle of a triangle XYZ touches sides YZ, ZX, XY at U, V, W. Then (using equal tangents) $XY + YZ + ZX = (YW + YU) + (XW + ZU) + XZ = (2YU) + (XZ) + XZ$, and $YU = \frac{1}{2}(XY + YZ - ZX)$.

Thus if the incircles of triangles ACD and CDB touch each other at E, then $AD + DC - CA = 2DE = BD + DC - CB \Rightarrow AD - CA = (AB - AD) - BC \Rightarrow AD = \frac{1}{2}(CA + AB - BC)$. But if the incircle of ABC is tangent to $\overline{AB}$ at D', then $AD' = \frac{1}{2}(CA + AB - BC)$ as well—so $D = D'$, as desired.

Problem 10 Let R be the circumradius of a right pyramid with a square base. Let r be the radius of the sphere touching the four lateral faces and the circumsphere. Suppose that $2R = (1 + \sqrt{2})r$. Determine the angle between adjacent faces of the pyramid.

Solution: Let P be the pyramid's vertex, $ABCD$ the base, and M, N the midpoints of sides AB, CD. By symmetry, both spheres are centered along the altitude from P. Plane PMN intersects the pyramid in triangle PMN and meets the spheres in great circles. Let the smaller circle have center O; it is tangent to $\overline{PM}$, $\overline{PN}$, and the large circle at some points U, V, W. Again by symmetry W is lies on the altitude from P, implying that it is diametrically opposite P on the larger circle. Thus $OP = 2R - r = \sqrt{2}r$, triangle OUP is a $45°$-$45°$-$90°$ triangle, and $\angle OPU = \angle OPV = 45°$. Therefore triangle NPM is isosceles right, and the distance from P to plane $ABCD$ equals $BC/2$.

Hence one can construct a cube with P as its center and $ABCD$ as a face; this cube can be decomposed into six pyramids congruent to $PABCD$. In particular, three such pyramids have a vertex at A; so three times the dihedral angle between faces PAB, PAD forms one revolution, and this angle is $2\pi/3$. Stated differently, say the three pyramids are $PABD$, $PADE$, $PAEB$; let P' be the midpoint of $\overline{AP}$, and let B', D', E' be points on planes PAB, PAD, PAE such that lines $B'P'$, $D'P'$, $E'P'$ are all perpendicular to line AP. The desired angle is the angle between any two of these lines. But since these three lines all lie in one plane (perpendicular to line AP), this angle must be $2\pi/3$.

Problem 11 Is there a polynomial $P(x)$ with integer coefficients such that $P(10) = 400, P(14) = 440$, and $P(18) = 520$?

Solution: If P exists, then by taking the remainder modulo $(x - 10)(x - 14)(x - 18)$ we may assume P is quadratic. Writing $P(x) = ax^2 + bx + c$, direct computation reveals $P(x+4)+P(x-4)-2P(x) = 32a$ for all x. Plugging in $x = 14$ gives $40 = 32a$, which is impossible since a must be an integer. Therefore no such polynomial exists.

Problem 12 Let a, b, c be positive numbers and $n \geq 2$ be an integer such that $a^n + b^n = c^n$. For which k is it possible to construct an obtuse triangle with sides a^k, b^k, c^k?

Solution: First, $a, b < c$. Thus for $m > n$ we have $c^m = c^{m-n}(a^n + b^n) > a^{m-n}a^n + b^{m-n}b^n = a^m + b^m$, while for $m < n$ we have $c^m = c^{m-n}(a^n + b^n) < a^{m-n}a^n + b^{m-n}b^n = a^m + b^m$. Now, a triangle with sides a^k, b^k, c^k exists iff $a^k + b^k > c^k$, and it is then obtuse iff $(a^k)^2 + (b^k)^2 < (c^k)^2$, i.e. $a^{2k} + b^{2k} < c^{2k}$. From our first observation, these correspond to $k < n$ and $2k > n$, respectively; and hence $n/2 < k < n$.

Problem 13 Let $n > 1$ be an arbitrary real number and k be the number of positive primes less than or equal to n. Select $k+1$ positive integers such that none of them divides the product of all the others. Prove that there exists a number among the $k + 1$ chosen numbers which is bigger than n.

Solution: Suppose otherwise; then our chosen numbers $a_1, \ldots, a_{k+1}$ have a total of at most k distinct prime factors (i.e. the primes less than or equal to n). Let $o_p(a)$ denote the highest value of d such that $p^d \mid a$. Also let $q = a_1 a_2 \cdots a_{k+1}$. Then for each prime p, $o_p(q) = \sum_{i=1}^{k+1} o_p(a_i)$, and it follows that there can be at most one "hostile" value of i for which $o_p(a_i) > o_p(q)/2$. Since there are at most k primes which divide q, there is some i which is not hostile for any such prime. Then $2o_p(a_i) \le o_p(q) \Rightarrow o_p(a_i) \le o_p(q/a_i)$ for each prime p dividing q, implying that $a_i \mid q/a_i$, a contradiction.

Problem 14 The polynomial $x^4 - 2x^2 + ax + b$ has four distinct real roots. Show that the absolute value of each root is smaller than $\sqrt{3}$.

Solution: Let the roots be p, q, r, s. We have $p+q+r+s = 0, pq + pr + ps + qr + qs + rs = -2$, and hence $p^2+q^2+r^2+s^2 = 0^2 - 2(-2) = 4$. But by Cauchy-Schwarz, $(1+1+1)(q^2+r^2+s^2) \ge (q+r+s)^2$ for any real q, r, s; furthermore, since q, r, s must be distinct, the inequality becomes strict. Thus $4 = p^2 + q^2 + r^2 + s^2 > p^2 + (-p)^2/3 = 4p^2/3$ or $|p| < \sqrt{3}$, and the same argument holds for q, r, s.

Problem 15 Each side of a convex polygon has integral length and the perimeter is odd. Prove that the area of the polygon is at least $\sqrt{3}/4$.

Solution:

Lemma 1. *If $0 \le x, y \le 1$, then*

$$\sqrt{1-x^2} + \sqrt{1-y^2} \ge \sqrt{1-(x+y-1)^2}.$$

Proof: Squaring and subtracting $2 - x^2 - y^2$ from both sides gives the equivalent inequality $2\sqrt{(1-x^2)(1-y^2)} \ge -2(1-x)(1-y)$, which is true since the left side is nonnegative and the right is at most 0. ■

Lemma 2. *If $x_1 + \cdots + x_n \le n - 1/2$ and $0 \le x_i \le 1$ for each i, then* $\sum_{i=1}^{n} \sqrt{1-x_i^2} \ge \sqrt{3}/2$.

Proof: Use induction on n. In the case $n = 1$, the statement is clear. If $n > 1$, then either $\min(x_1, x_2) \le 1/2$ or $x_1 + x_2 > 1$. In the first case we immediately have $\max(\sqrt{1-x_1^2}, \sqrt{1-x_2^2}) \ge \sqrt{3}/2$; in the second case we can replace x_1, x_2 by the single number $x_1 + x_2 - 1$ and use the induction hypothesis together with the previous lemma. ■

Now consider our polygon. Let P, Q be vertices such that $l = PQ$ is maximal. The polygon consists of two paths from P to Q, each of integer length $\geq l$; these lengths are distinct since the perimeter is odd. Then the greater of the two lengths is $m \geq l + 1$. Position the polygon in the coordinate plane with $P = (0,0), Q = (l,0)$ and the longer path in the upper half-plane. Since each side of the polygon has integer length, we can divide this path into line segments of length 1. Let the endpoints of these segments, in order, be $P_0 = P, P_1 = (x_1, y_1), P_2 = (x_2, y_2), \ldots, P_m = Q$. There exists some r such that y_r is maximal; then either $r \geq x_r + 1/2$ or $(m - r) \geq (\ell - x_r) + 1/2$. Assume the former (otherwise, just reverse the choices of P and Q). We already know that $y_1 \geq 0$, and by the maximal definition of l we must have $x_1 \geq 0$ as well; then since the polygon is convex we must have $y_1 \leq y_2 \leq \ldots \leq y_r$ and $x_1 \leq x_2 \leq \ldots \leq x_r$. But $y_{i+1} - y_i = \sqrt{1 - (x_{i+1} - x_i)^2}$, so

$$y_r = \sum_{i=0}^{r-1} (y_{i+1} - y_i) = \sum_{i=0}^{r-1} \sqrt{1 - (x_{i+1} - x_i)^2} \geq \sqrt{3}/2$$

by the second lemma. And we must have $l \geq 1$, implying that triangle PP_rQ has area at least $\sqrt{3}/4$. Since this triangle lies within the polygon (by convexity), we are done.

Problem 16 Determine if there exists an infinite sequence of positive integers such that

(i) no term divides any other term;

(ii) every pair of terms has a common divisor greater than 1, but no integer greater than 1 divides all of the terms.

Solution: The desired sequence exists. Let $p_0, p_1, \ldots$ be the primes greater than 5 in order, and let $q_{3i} = 6, q_{3i+1} = 10, q_{3i+2} = 15$ for each nonnegative integer i. Then let $s_i = p_i q_i$ for all $i \geq 0$. The sequence $s_0, s_1, s_2, \ldots$ clearly satisfies (i) since s_i is not even divisible by p_j for $i \neq j$. For the first part of (ii), any two terms have their indices both in $\{0,1\}$, both in $\{0,2\}$, or both in $\{1,2\}$ (mod 3), so they have a common divisor of 2, 3, or 5, respectively. For the second part, we just need to check that no prime divides all the s_i; this holds since $2 \not| s_2, 3 \not| s_1, 5 \not| s_0$, and no prime greater than 5 divides more than one s_i.

Problem 17 Prove that, for every positive integer n, there exists a polynomial with integer coefficients whose values at $1, 2, \ldots, n$ are different powers of 2.

Solution: We may assume $n \geq 4$. For each $i = 1, 2, \ldots, n$, write $\prod_{j=1, j\neq i}^{n}(i-j) = 2^{q_i} m_i$ for positive integers q_i, m_i with m_i odd. Let L be the least common multiple of all the q_i, and let $r_i = L/q_i$. For each i, there are infinitely many powers of 2 which are congruent to 1 modulo $|m_i^{r_i}|$. (Specifically, by Euler's theorem, $2^{\phi(|m_i^{r_i}|)j} \equiv 1 \pmod{|m_i^{r_i}|}$ for all $j \geq 0$.) Thus there are infinitely many integers c_i such that $c_i m_i^{r_i} + 1$ is a power of 2; choose one. Then define

$$P(x) = \sum_{i=1}^{n} c_i \left(\prod_{\substack{j=1\\ j\neq i}}^{n} (x-j) \right)^{r_i} + 2^L.$$

For each $k, 1 \leq k \leq n$, in the sum each term $\left(\prod_{j=1, j\neq i}^{n}(x-j)\right)^{r_i}$ vanishes for all $i \neq k$. Then

$$P(k) = c_k \left(\prod_{\substack{j=1\\ j\neq k}}^{n} (k-j) \right)^{r_i} + 2^L = 2^L(c_i m_i^{r_i} + 1),$$

a power of 2. Moreover, by choosing the c_i appropriately, we can guarantee that these values are all distinct, as needed.

Problem 18 Find all integers $N \geq 3$ for which it is possible to choose N points in the plane (no three collinear) such that each triangle formed by three vertices on the convex hull of the points contains exactly one of the points in its interior.

Solution: First, if the convex hull is a k-gon, then it can be divided into $k-2$ triangles each containing exactly one chosen point; and since no three of the points are collinear, the sides and diagonals of the convex hull contain no chosen points on their interiors, giving $N = 2k - 2$.

Now we construct, by induction on $k \geq 3$, a convex k-gon with a set S of $k-2$ points inside such that each triangle formed by vertices of the k-gon contains exactly one point of S in its interior. The case $k = 3$ is easy. Now, assume we have a k-gon $P_1P_2\ldots P_k$ and a set S. Certainly we can choose Q such that $P_1P_2\ldots P_kQ$ is a convex $(k+1)$-gon. Let

R move along the line segment from P_k to Q. Initially (at $R = P_k$), for any indices $1 \leq i < j < k$, the triangle P_iP_jR internally contains a point of S by assumption; if R is moved a sufficiently small distance d_{ij}, this point still lies inside triangle P_iP_jR. Now fix a position of R such that P_kR is less than the minimum d_{ij}; $P_1P_2 \ldots P_kR$ is a convex $(k+1)$-gon. Let P be an interior point of the triangle bounded by lines P_1P_k, RP_{k-1}, P_kR. We claim the polygon $P_1P_2 \ldots P_kR$ and the set $S \cup \{P\}$ satisfy our condition. If we choose three of the P_i, they form a triangle containing a point of S by hypothesis, and no others; any triangle P_iP_jR $(i, j < k)$ contains only the same internal point as triangle $P_iP_jP_k$; and each triangle P_iP_kR contains only P. This completes the induction step.

1.10 Iran

First Round

Problem 1 Suppose that $a_1 < a_2 < \cdots < a_n$ are real numbers. Prove that

$$a_1 a_2^4 + a_2 a_3^4 + \cdots + a_n a_1^4 \geq a_2 a_1^4 + a_3 a_2^4 + \cdots + a_1 a_n^4.$$

First Solution: We prove the claim by induction on n. For $n = 2$, the two sides are equal; now suppose the claim is true for $n - 1$, i.e.,

$$a_1 a_2^4 + a_2 a_3^4 + \cdots + a_{n-1} a_1^4 \geq a_2 a_1^4 + a_3 a_2^4 + \cdots + a_1 a_{n-1}^4.$$

Then the claim for n will follow from the inequality

$$a_{n-1} a_n^4 + a_n a_1^4 - a_{n-1} a_1^4 \geq a_n a_{n-1}^4 + a_1 a_n^4 - a_1 a_{n-1}^4$$

(Notice that this is precisely the case for $n = 3$.) Without loss of generality, suppose $a_n - a_1 = 1$; otherwise, we can divide each of a_1, a_{n-1}, a_n by $a_n - a_1 > 0$ without affecting the truth of the inequality. Then by Jensen's inequality for the convex function x^4, we have

$$a_1^4(a_n - a_{n-1}) + a_n^4(a_{n-1} - a_1) \geq (a_1(a_n - a_{n-1}) + a_n(a_{n-1} - a_1))^4$$
$$= (a_{n-1}(a_n - a_1))^4 = a_{n-1}^4(a_n - a_1),$$

which rearranges to yield our desired inequality.

Second Solution: We use an elementary method to prove the case $n = 3$. Define

$$p(x, y, z) = xy^4 + yz^4 + zx^4 - yx^4 - zy^4 - xz^4.$$

We wish to prove that $p(x, y, z) \geq 0$ when $x \leq y \leq z$. Since $p(x, x, z) = p(x, y, y) = p(z, y, z) = 0$, we know that $(y - x)(z - y)(z - x)$ divides $p(x, y, z)$. In fact,

$$p(x, y, z) = yz^4 - zy^4 + zx^4 - xz^4 + xy^4 - yx^4$$

$$= zy(z^3 - y^3) + xz(x^3 - z^3) + xy(y^3 - x^3)$$

$$= zy(z^3 - y^3) + xz(y^3 - z^3) + xz(x^3 - y^3) + xy(y^3 - x^3)$$

$$= z(y - x)(z^3 - y^3) + x(z - y)(x^3 - y^3)$$

$$= (y-x)(z-y)\left(z(z^2+zy+y^2) - x(x^2+xy+y^2)\right)$$

$$= (y-x)(z-y)\left((z^3-x^3) + y^2(z-x) + y(z^2-x^2)\right)$$

$$= (y-x)(z-y)(z-x)\left(z^2+zx+x^2+y^2+yz+yx\right)$$

$$= \frac{1}{2}(y-x)(z-y)(z-x)\left((x+y)^2+(y+z)^2+(z+x)^2\right) \geq 0,$$

as desired.

Problem 2 Suppose that n is a positive integer. The n-tuple $(a_1, \ldots, a_n)$ of positive integers is said to be *good* if $a_1 + \cdots + a_n = 2n$ if for every k between 1 and n, no k of the n integers add up to n. Find all n-tuples that are good.

First Solution: Call an n-tuple of positive integers *proper* if the integers add up to $2n$.

Without loss of generality, we suppose that $a_1 \leq a_2 \leq \cdots \leq a_n = M$. If $M \leq 2$ then $a_1 = \cdots = a_n = 2$ and this leads to the solution $(2, 2, \ldots, 2)$ for odd n. Now we suppose that $M \geq 3$. Since the average of $\{a_1, \ldots, a_n\}$ is 2, we must have $a_1 = 1$. Now say we have a proper n-tuple $S = (a_1, \ldots, a_n)$, where

$$1 = a_1 = \cdots = a_i < a_{i+1} \leq a_{i+2} \leq \cdots \leq a_n = M.$$

Lemma. *If $i \geq \max\{a_n - a_{i+1}, a_{i+1}\}$, then S is not good.*

Proof: Suppose that we have a balance and weights $a_1, a_2, \ldots, a_n$. We put a_n on the left hand side of the balance, then put a_{n-1} on the right hand side, and so on — adding the heaviest available weight to the lighter side (or if the sides are balanced, we add it to the left hand side). Before we put x_{i+1} on the balance, the difference between the two sides is between 0 and x_n. After we put x_{i+1} on the balance, the difference between the two sides is no greater than $\max\{x_n - x_{i+1}, x_{i+1}\}$. Now we have enough 1's to put on the lighter side to balance the two sides. Since the the total weight is $2n$ an even number, there will have even number of 1's left and we can split them to balance the sides, i.e., there is a subtuple A with its sum equal to n and thus S is not good. ■

Note that

$$2n = a_1 + \cdots + a_n \geq i + 2(n-i-1) + M \iff i \geq M-2. \quad (1)$$

Now we consider the following cases:

(i) $M = 3$ and $i = 1$. Then $S = (1, \underbrace{2, \ldots, 2}_{n-2}, 3)$. If $n = 2m$ and $m \geq 2$, then $A = (\underbrace{2, \ldots, 2}_{m})$ has sum $2m = n$ and thus S is not good; if $n = 2m+1$ and $m \geq 1$, then $A = (\underbrace{2, \ldots, 2}_{m-1}, 3)$ has sum $2m+1 = n$ and thus S is not good. Therefore $(1, 3)$ is the only good tuple for $M = 3, i = 1$.

(ii) $M = 3$ and $i \geq 2$. Then $a_{i+1} = 2$ or 3 and $i \geq \max\{a_n - a_{i+1}, a_{i+1}\}$ — implying that S is not good from our lemma — unless $i = 2$ and $a_{i+1} = 3$. But then $S = (1, 1, 3, 3)$, and $(1, 3)$ has sum $4 = n$.

(iii) $M \geq 4$ and $a_{i+1} = 2$. Then from (1), $i \geq M - 2 = \max\{a_n - a_{i+1}, a_{i+1}\}$. By the lemma S is not good.

(iv) $M \geq 4$, $a_{i+1} > 2$, and $i + 1 \neq n$. Since $a_{i+1} \neq 2$ equality does not hold in (1), and thus $i \geq M - 1 \geq \max\{a_n - a_{i+1}, a_{i+1}\}$ (and S is not good) unless $a_{i+1} = M$. In this case, $S = (1, \ldots, 1, M, \ldots, M)$. Note that

$$2n = i + (n-i)M \geq i + 4(n-i)$$

so that $i \geq \frac{2}{3}n$. Then the remaining $n - i \geq 2$ values M have sum at most $\frac{4}{3}n$, and hence $M \leq \frac{2}{3}n$. Thus

$$i \geq 2n/3 \geq M = \max\{a_n - a_{i+1}, a_{i+1}\},$$

and by the lemma S is not good.

(v) $M \geq 4$ and $i+1 = n$. Then we have the good n-tuple $(1, \ldots, 1, n+1)$.

Therefore the only possible good n-tuples are $(1, 1, \ldots, 1, n+1)$ and (2, 2, …, 2), and the second n-tuple is good if and only if n is odd.

Second Solution: Say a proper n-tuple has "subsum" m if some k of the integers $(0 \leq k \leq n)$ add up to m.

Lemma. *Every proper n-tuple besides* $(2,\ 2,\ \ldots,\ 2)$ *has subsums* 0, $1,\ \ldots,\ n-1$ *(and possibly others). Furthermore, if a proper n-tuple besides* $(2, 2, \ldots, 2)$ *contains a* 2, *it has subsum n as well.*

Proof: If $n = 1$ the claim is trivial. Now assume the claims are true for $n-1$; we prove each is true for n as well. Suppose we have an n-tuple N besides $(2, 2, \ldots, 2)$.

If there is a 2 in the n-tuple, then the other $n-1$ integers form a proper $(n-1)$-tuple besides $(2, 2, \ldots, 2)$. By the induction hypothesis, this $(n-1)$-tuple has subsums $0, \ldots, n-2$. Remembering the original 2, our complete n-tuple N has subsums $0, \ldots, n$.

Otherwise, suppose there is no 2 in N. We prove by induction on $k < n$ that there is a subtuple A_k of k numbers with subsums $0, \ldots, k$. Since the average value of the integers in N is 2, we must have at least one 1 in N, proving the case for $k = 1$. Now assume the claim is true for $k-1$; the elements of A_{k-1} are each at least 1, so they add up to at least $k-1$. Then the inequality

$$(k-1)(k-n) < 0$$

implies

$$2n < (k+1)(n-k+1) + (k-1),$$

so that the average value of $N \setminus A_{k-1}$ is less than $k+1$. Therefore at least one of the other $n-k+1$ integers x is at most k, implying that $A_{k-1} \cup (x)$ has subsums $0, \ldots, k$. This completes the inductive step and the proof of our lemma. ■

From our lemma, the only possible good n-tuple with a 2 is $(2, 2, \ldots, 2)$, and this is good if and only if n is odd. Every other n-tuple N has a subtuple A_{n-1} of $n-1$ integers with subsums $0, \ldots, n-1$. If these integers are not all 1, then the remaining integer is at most n and N must have subsum n. Therefore, the only other possible good n-tuple is $(1, 1, \ldots, 1, n+1)$, which is indeed always a good n-tuple.

Third Solution: Suppose we have a good n-tuple $(a_1, \ldots, a_n)$, and consider the sums $a_1,\ a_1 + a_2,\ \ldots,\ a_1 + a_2 + \cdots + a_{n-1}$. All these sums are between 0 and $2n$ exclusive; thus if any of the sums is 0 (mod n), it equals n and we have a contradiction. Also, if any two are congruent modulo n, we can subtract these two sums to obtain another partial sum that equals n, a contradiction again. Therefore, the sums must all be nonzero and distinct modulo n.

Specifically, $a_2 \equiv a_1 + \cdots + a_k \pmod{n}$ for some $k \geq 1$. If $k > 1$ then we can subtract a_2 from both sides to find a partial sum that equals n. Therefore $k = 1$ and $a_1 \equiv a_2 \pmod{n}$. Similarly, all the a_i are congruent modulo n. From here, easy algebra shows that the presented solutions are the only ones possible.

Problem 3 Let I be the incenter of triangle ABC and let AI meet the circumcircle of ABC at D. Denote the feet of the perpendiculars from I to BD and CD by E and F, respectively. If $IE+IF = AD/2$, calculate $\angle BAC$.

Solution: A well-known fact we will use in this proof is that $DB = DI = DC$. In fact, $\angle BDI = \angle C$ gives $\angle DIB = (\angle A + \angle B)/2$ while $\angle IBD = (\angle A + \angle B)/2$. Thus $DB = DI$, and similarly $DC = DI$.

Let $\theta = \angle BAD$. Then

$$\begin{aligned}\frac{1}{4} ID \cdot AD &= \frac{1}{2} ID \cdot (IE + IF) \\ &= \frac{1}{2} BD \cdot IE + \frac{1}{2} CD \cdot IF = [BID] + [DIC] \\ &= \frac{ID}{AD}([BAD] + [DAC]) = \frac{1}{2} ID \cdot (AB + AC) \cdot \sin\theta,\end{aligned}$$

whence $\frac{AD}{AB+AC} = 2\sin\theta$.

Let X be the point on $\overrightarrow{AB}$ different from A such that $DX = DA$. Since $\angle XBD = \angle DCA$ and $\angle DXB = \angle XAD = \angle DAC$, we have $\triangle XBD \cong \triangle ACD$, and $BX = AC$. Then $2\sin\theta = \frac{AD}{AB+AC} = \frac{AD}{AB+BX} = \frac{AD}{AX} = \frac{1}{2\cos\theta}$, so that $2\sin\theta\cos\theta = \frac{1}{2}$, and $\angle BAC = 2\theta = 30^\circ$ or 150°.

Problem 4 Let ABC be a triangle with $BC > CA > AB$. Choose points D on $\overline{BC}$ and E on $\overrightarrow{BA}$ such that

$$BD = BE = AC.$$

The circumcircle of triangle BED intersects $\overline{AC}$ at P and the line BP intersects the circumcircle of triangle ABC again at Q. Prove that $AQ + QC = BP$.

First Solution: Except where indicated, all angles are directed modulo 180°.

Let Q' be the point on line BP such that $\angle BEQ' = \angle DEP$. Then

$$\angle Q'EP = \angle AED - \angle BEQ' + \angle DEP = \angle BED.$$

Since $BE = BD$, $\angle BED = \angle EDB$; since $BEPD$ is cyclic, $\angle EDB = \angle EPB$. Therefore $\angle Q'EP = \angle EPB = \angle EPQ'$ and $Q'P = Q'E$.

Since $BEPD$ and $BAQC$ are cyclic, we have

$$\angle BEQ' = \angle DEP = \angle DBP = \angle CAQ,$$

$$\angle Q'BE = \angle QBA = \angle QCA.$$

Combining this with $BE = AC$ yields that triangles EBQ' and ACQ are congruent. Thus $BQ' = QC$ and $EQ' = AQ$. Therefore

$$AQ + QC = EQ' + BQ' = PQ' + BQ',$$

which equals BP if Q' is between B and P.

Since E is on $\overrightarrow{BA}$ and P is on $\overrightarrow{CA}$, E and P are on the same side of $\overline{BC}$ and thus $\overline{BD}$. And since D is on $\overrightarrow{BC}$ and P is on $\overrightarrow{AC}$, D and P are on the same side of $\overline{BA}$ and thus $\overline{BE}$. Thus, $BEPD$ is cyclic in that order and (using undirected angles) $\angle BEQ' = \angle DEP < \angle BEP$. It follows that Q' lies on segment BP, as desired.

Second Solution: Since $BEPD$ and $BAQC$ are cyclic, we have

$$\angle PED = \angle PBD = \angle QBC = \angle QAC$$

and

$$\angle EPD = \pi - \angle DBE = \pi - \angle CBA = \angle AQC,$$

which together imply $\triangle PED \sim \triangle QAC$. Then

$$\frac{AC \cdot EP}{DE} = AQ$$

and

$$\frac{AC \cdot PD}{DE} = QC.$$

As in the first solution, $BEPD$ is cyclic in that order, so Ptolemy's Theorem implies that

$$BD \cdot EP + BE \cdot PD = BP \cdot DE.$$

Substituting $BD = BE = AC$ we have

$$\frac{AC \cdot EP}{DE} + \frac{AC \cdot PD}{DE} = BP,$$

or $AQ + QC = BP$, as desired.

Problem 5 Suppose that n is a positive integer and let

$$d_1 < d_2 < d_3 < d_4$$

be the four smallest positive integer divisors of n. Find all integers n such that

$$n = d_1^2 + d_2^2 + d_3^2 + d_4^2.$$

Solution: The answer is $n = 130$. Note that $x^2 \equiv 0 \pmod 4$ when x is even, and $1 \pmod 4$ when x is odd.

If n is odd, then all the d_i are odd and $n \equiv d_1^2 + d_2^2 + d_3^2 + d_4^2 \equiv 1 + 1 + 1 + 1 \equiv 0 \pmod 4$, a contradiction. Thus $2 \mid n$.

If $4 \mid n$ then $d_1 = 1$ and $d_2 = 2$, and $n \equiv 1 + 0 + d_3^2 + d_4^2 \not\equiv 0 \pmod 4$, a contradiction. Thus $4 \nmid n$.

Therefore $\{d_1, d_2, d_3, d_4\} = \{1, 2, p, q\}$ or $\{1, 2, p, 2p\}$ for some odd primes p, q. In the first case $n \equiv 3 \pmod 4$, a contradiction. Thus $n = 5(1 + p^2)$ and $5 \mid n$, so $p = d_3 = 5$ and $n = 130$.

Problem 6 Suppose that $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ are two 0-1 sequences. The difference $d(A, B)$ between A and B is defined to be the number of i's for which $a_i \neq b_i$ $(1 \leq i \leq n)$. Suppose that A, B, C are three 0-1 sequences and that $d(A, B) = d(A, C) = d(B, C) = d$.

(a) Prove that d is even.

(b) Prove that there exists an 0-1 sequence D such that

$$d(D, A) = d(D, B) = d(D, C) = \frac{d}{2}.$$

Solution:

(a) Modulo 2, we have

$$\begin{aligned} d(A, B) &= (a_1 - b_1) + (a_2 - b_2) + \cdots + (a_n - b_n) \\ &\equiv (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n). \end{aligned}$$

Thus,

$$3d \equiv d(A, B) + d(B, C) + d(C, A) = 2\left(\sum a_i + \sum b_i + \sum c_i\right),$$

so d must be divisible by 2.

(b) Define D as follows: for each i, if $a_i = b_i = c_i$, then let $d_i = a_i = b_i = c_i$. Otherwise, two of a_i, b_i, c_i are equal; let d_i equal that value. We claim this sequence D satisfies the requirements.

Let α be the number of i for which $a_i \neq b_i$ and $a_i \neq c_i$ (that is, for which a_i is "unique"). Define β and γ similarly, and note that $d(A, D) = \alpha$, $d(B, D) = \beta$, and $d(C, D) = \gamma$. We also have

$$d = d(A, B) = \alpha + \beta$$

$$d = d(B, C) = \beta + \gamma$$

$$d = d(C, A) = \gamma + \alpha.$$

Thus, $\alpha = \beta = \gamma = \frac{d}{2}$, as desired.

Second Round

Problem 1 Define the sequence $\{x_n\}_{n \geq 0}$ by $x_0 = 0$ and

$$x_n = \begin{cases} x_{n-1} + \dfrac{3^{r+1} - 1}{2}, & \text{if } n = 3^r(3k+1), \\ \\ x_{n-1} - \dfrac{3^{r+1} + 1}{2}, & \text{if } n = 3^r(3k+2), \end{cases}$$

where k and r are nonnegative integers. Prove that every integer appears exactly once in this sequence.

First Solution: We prove by induction on $t \geq 1$ that

(i) $\{x_0, x_1, \ldots, x_{3^t-2}\} = \left\{-\frac{3^t-3}{2}, -\frac{3^t-1}{2}, \ldots, \frac{3^t-1}{2}\right\}$.

(ii) $x_{3^t-1} = -\frac{3^t-1}{2}$.

These claims imply the desired result, and they are easily verified for $t = 1$. Now supposing they are true for t, we show they are true for $t + 1$.

For any positive integer m, write $m = 3^r(3k + s)$ for nonnegative integers r, k, s, with $s \in \{1, 2\}$; and define $r_m = r$ and $s_m = s$.

Then for $m < 3^t$, observe that

$$r_m = r_{m+3^t} = r_{m+2\cdot 3^t}$$

$$s_m = s_{m+3^t} = r_{m+2\cdot 3^t},$$

so that

$$x_m - x_{m-1} = x_{3^t+m} - x_{3^t+m-1} = x_{2\cdot 3^t+m} - x_{2\cdot 3^t+m-1}.$$

Adding these equations from $m = 1$ to $m = k < 3^t$, we have

$$x_k = x_{3^t+k} - x_{3^t}$$

$$x_k = x_{2\cdot 3^t+k} - x_{2\cdot 3^t}.$$

Now, setting $n = 3^t$ in the recursion and using (ii) from the induction hypothesis, we have $x_{3^t} = 3^t$ — and

$$\{x_{3^t}, \ldots, x_{2\cdot 3^t-2}\} = \left\{\tfrac{3^t+3}{2}, \ldots, \tfrac{3^{t+1}-1}{2}\right\}$$

$$x_{2\cdot 3^t-1} = \tfrac{3^t+1}{2}.$$

Then setting $n = 2\cdot 3^t$ in the recursion we have $x_{2\cdot 3^t} = -3^t$ — giving

$$\{x_{2\cdot 3^t}, \ldots, x_{3^{t+1}-2}\} = \left\{-\tfrac{3^{t+1}-3}{2}, \ldots, -\tfrac{3^t+1}{2}\right\}$$

$$x_{2\cdot 3^{t+1}-1} = -\tfrac{3^{t+1}-1}{2}.$$

Combining this with (i) and (ii) from the induction hypothesis proves the claims for $t + 1$. This completes the proof.

Second Solution: For $n_i \in \{-1, 0, 1\}$, let the number

$$[n_m n_{m-1} \cdots n_0]$$

in "base $\bar{3}$" equal $\sum_{i=0}^m n_i \cdot 3^i$. It is simple to prove by induction on k that the base $\bar{3}$ numbers with at most k digits equal

$$\left\{-\frac{3^k-1}{2}, -\frac{3^k-3}{2}, \ldots, \frac{3^k-1}{2}\right\},$$

which implies every integer has a unique representation in base $\bar{3}$.

Now we prove by induction on n that if $n = a_m a_{m-1} \ldots a_0$ in base 3, then $x_n = [b_m b_{m-1} \ldots b_0]$ in base $\bar{3}$, where $b_i = -1$ if $a_i = 2$ and $b_i = a_i$ for all other cases.

For the base case, $x_0 = 0 = [0]$. Now assume the claim is true for $n - 1$. First suppose that $n = 3^r(3k + 1)$. Then

$$n = a_m a_{m-1} \ldots a_i \; 1 \; \underbrace{0\,0 \ldots\ 0}_{r}$$

$$\frac{3^{r+1}-1}{2} = \underbrace{1\,1 \ldots 1}_{r+1} = [\underbrace{1\,1 \ldots 1}_{r+1}]$$

$$n - 1 = a_m a_{m-1} \ldots a_i \; 0 \; \underbrace{2\,2 \ldots 2}_{r}$$

$$x_{n-1} = [b_m b_{m-1} \dots b_i \; 0 \; \underbrace{-1 \, -1 \; \dots \; -1}_{r}],$$

so that

$$\begin{aligned} x_n &= [b_m b_{m-1} \dots b_i \; 0 \; \underbrace{-1 \, -1 \; \dots \; -1}_{r}] + [\underbrace{1 \; 1 \dots 1}_{r+1}] \\ &= [b_m b_{m-1} \dots b_i \; 1 \; \underbrace{0 \; 0 \; \dots \; 0}_{r}]. \end{aligned}$$

Now suppose that $n = 3^r(3k+2)$. Then

$$\begin{aligned} n &= a_m a_{m-1} \dots a_i \; 2 \; \underbrace{0 \; 0 \dots 0}_{r} \\ n - 1 &= a_m a_{m-1} \dots a_i \; 1 \; \underbrace{2 \; 2 \dots 2}_{r} \\ x_{n-1} &= [b_m b_{m-1} \dots b_i \; 1 \; \underbrace{-1 \, -1 \; \dots \; -1}_{r}]. \end{aligned}$$

Also,

$$\begin{aligned} -\frac{3^{r+1}+1}{2} &= -(\underbrace{1 \; 1 \dots 1}_{r} \; 2) \\ &= -3^r - 3^{r-1} - \dots - 3 - 2 \\ &= -3^{r+1} + 3^r + 3^{r-1} + \dots + 3 + 1 \\ &= [-1 \; \underbrace{1 \; 1 \dots 1}_{r+1}]. \end{aligned}$$

Therefore

$$\begin{aligned} x_n &= [b_m b_{m-1} \dots b_i \; 1 \; \underbrace{-1 \, -1 \; \dots \; -1}_{r}] + [-1 \; \underbrace{1 \; 1 \dots 1}_{r+1}] \\ &= [b_m b_{m-1} \dots b_i \; -1 \; \underbrace{0 \; 0 \; \dots \; 0}_{r}]. \end{aligned}$$

In either case, the claim is true for n, completing the induction.

And since all integers appear exactly once in base $\overline{3}$, they appear exactly once in $\{x_n\}_{n \geq 0}$, as desired.

Problem 2 Suppose that $n(r)$ denotes the number of points with integer coordinates on a circle of radius $r > 1$. Prove that

$$n(r) < 6\sqrt[3]{\pi r^2}.$$

Solution: Consider a circle of radius r containing n lattice points; we must prove that $n < 6\sqrt[3]{\pi r^2}$.

Since $r > 1$ and $6\sqrt[3]{\pi} > 8$, we may assume $n > 8$. Label the n lattice points on the circle $P_1, P_2, \ldots, P_n$ in counterclockwise order. Since the sum of the (counterclockwise) arcs P_1P_3, P_2P_4, P_nP_2 is 4π, one of the arcs P_iP_{i+2} has measure at most $\frac{4\pi}{n}$; assume without loss of generality it is arc P_1P_3.

Consider a triangle ABC inscribed in an arc of angle $\frac{4\pi}{n}$; clearly its area is maximized by moving A and C to the endpoints of the arc and then moving B to the midpoint (where the distance to line AC is greatest). Then $\angle CAB = \angle BCA = \frac{\pi}{n}$ and $\angle ABC = 180^\circ - \frac{2\pi}{n}$, so

$$\begin{aligned}
[ABC] &= \frac{abc}{4r} = \frac{(2r\sin\frac{\pi}{n})(2r\sin\frac{2\pi}{n})(2r\sin\frac{\pi}{n})}{4r} \\
&\leq \frac{(2r\frac{\pi}{n})(2r\frac{2\pi}{n})(2r\frac{\pi}{n})}{4r} \\
&= \frac{4r^2\pi^3}{n^3}.
\end{aligned}$$

Since triangle $P_1P_2P_3$ is inscribed in an arc of measure $\frac{4\pi}{n}$, by the preceding argument, $[P_1P_2P_3] \leq \frac{4r^2\pi^3}{n^3}$. But since P_1, P_2, and P_3 are lattice points, the area $[P_1P_2P_3]$ is at least $\frac{1}{2}$ (this can be proven by either Pick's Formula $K = I + \frac{1}{2}B - I$ or the "determinant formula" $K = \frac{1}{2}\left|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\right|$). Therefore,

$$\begin{aligned}
&\frac{1}{2} \leq [P_1P_2P_3] \leq \frac{4r^2\pi^3}{n^3} \\
&\Longrightarrow n^3 \leq 8r^2\pi^3 \\
&\Longrightarrow n \leq \sqrt[3]{8r^2\pi^3} = 2\pi\sqrt[3]{r^2} < 6\sqrt[3]{\pi r^2},
\end{aligned}$$

as desired.

Problem 3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

Solution: Let $(x, y) = (x, x^2)$. Then

$$f(f(x) + x^2) = f(0) + 4x^2f(x). \tag{1}$$

Let $(x, y) = (x, -f(x))$. Then

$$f(0) = f(x^2 + f(x)) - 4f(x)^2. \tag{2}$$

Adding (1) and (2) gives $4f(x)(f(x) - x^2) = 0$. This implies that for each individual x, either $f(x) = 0$ or $f(x) = x^2$. (Alternatively, plugging $y = \frac{x^2 - f(x)}{2}$ into the original equation also yields this result.) Clearly $f(x) = 0$ and $f(x) = x^2$ satisfy the given equation; we now show that f cannot equal some combination of the two functions.

Suppose that there is an $a \neq 0$ such that $f(a) = 0$. Plugging in $x = a$ into the original equation, we have

$$f(y) = f(a^2 - y).$$

If $y \neq \frac{a^2}{2}$, then $y^2 \neq (a^2 - y)^2$ so $f(y) = f(a^2 - y) = 0$. Thus $f(y) = 0$ for all $y \neq \frac{a^2}{2}$. And by choosing $x = 2a$ or some other value in the original equation, we can similarly show that $f(\frac{a^2}{2}) = 0$.

Therefore $f(x) = 0$ for all x or $f(x) = x^2$ for all x, as claimed.

Problem 4 In triangle ABC, the angle bisector of $\angle BAC$ meets BC at D. Suppose that ω is the circle which is tangent to BC at D and passes through A. Let M be the second point of intersection of ω and AC. Let P be the second point of intersection of ω and BM. Prove that P lies on a median of triangle ABD.

Solution: Extend $\overline{AP}$ to meet $\overline{BD}$ at E. We claim that $BE = ED$ and thus $\overline{AP}$ is a median of triangle ABD, as desired. In fact,

$$BE = ED \iff BE^2 = ED^2 = EP \cdot EA$$

$$\iff \triangle BEP \sim \triangle AEB \iff \angle EBP = \angle BAE.$$

Let N be the second intersection of ω with AB. Using directed angles and arc measures, since $\overline{AD}$ bisects the angle between lines AN and AC, we have $\widehat{DM} = \widehat{ND}$ and

$$\angle BAE = \angle NAP = \frac{\widehat{ND} - \widehat{PD}}{2} = \frac{\widehat{DM} - \widehat{PD}}{2} = \angle DBM = \angle EBP,$$

as desired.

Problem 5 Let ABC be a triangle. If we paint the points of the plane in red and green, prove that either there exist two red points which are one unit apart or three green points forming a triangle congruent to ABC.

First Solution: We call a polygon or a segment green (red) if the vertices of the polygon or the segment are all green (red).

Suppose that there is no red unit segment. We prove that there is a green triangle congruent to triangle ABC. If the whole plane is green, the proof is trivial.

Now we further suppose that there is a red point R on the plane. We claim that there is a green equilateral triangle with unit side length. In fact, let ω be the circle with center R and radius $\sqrt{3}$. Then ω is not all red, since otherwise we could find a red unit segment. Let G be a green point on ω. Let ω_1 and ω_2 be two unit circles centered at R and G, respectively, and let ω_1 and ω_2 meet at P and Q. Then both P and Q must be green and triangle PQG is a green unit equilateral triangle.

Let $G_1G_2G_3$ be a green unit equilateral triangle. Construct a triangle $G_1X_1Y_1$ that is congruent to triangle ABC. If both X_1 and Y_1 are green, we are done. Without loss of generality, we assume that X_1 Y_1 is red. Translate triangle $G_1G_2G_3$ by $\overrightarrow{G_1Y_1}$ to obtain triangle $Y_1Y_2Y_3$. Then both Y_2 and Y_3 are green. Similarly, translate triangle $G_1G_2G_3$ by $\overrightarrow{G_1X_1}$ to obtain triangle $X_1X_2X_3$. Then at least one of X_2 and X_3 is green (since X_2X_3 cannot be a red unit segment). Without loss of generality, say X_2 is green. Now triangle $G_2X_2Y_2$ is a green triangle and congruent to triangle $G_1X_1Y_1$ (translated by $\overrightarrow{G_1G_2}$) and thus congruent to triangle ABC, as desired.

Second Solution: Suppose by way of contradiction there were no such red or green points, and say the sides of triangle ABC are a, b, and c.

First we prove no red segment has length a. If XY were a red segment of length a, then the unit circles around X and Y must be completely green. Now draw Z so that $\triangle XYZ \cong \triangle ABC$; the unit circle around Z must be completely *red*, or else it would form an illegal triangle with the corresponding points around X and Y. But on this unit circle we can find a red unit segment, a contradiction.

Now, the whole plane cannot be green so there must be some red point R. The circle ω around R with radius a must be completely green. Then pick two points D, E on ω with $DE = a$, and construct F outside ω so that $\triangle DEF \cong \triangle ABC$ (we can do this since $a \le b, c$); F must be red. Thus if we rotate DE around R, F forms a completely red circle of radius greater than a — and on this circle we can find two red points distance a apart, a contradiction.

Third Round

Problem 1 Suppose that $S = \{1, 2, \ldots, n\}$ and that $A_1, A_2, \ldots, A_k$ are subsets of S such that for every $1 \le i_1, i_2, i_3, i_4 \le k$, we have

$$|A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4}| \le n - 2.$$

Prove that $k \le 2^{n-2}$.

Solution: For a set T, let $|T|$ denote the numbers of elements in T. We call a set $T \subset S$ *2-coverable* if $T \subseteq A_i \cup A_j$ for some i and j (not necessarily distinct). Among the subsets of S that are not 2-coverable, let A be a subset with minimum $|A|$.

Consider the family of sets $S_1 = \{A \cap A_1, A \cap A_2, \ldots, A \cap A_k\}$. ($A \cap A_i$ might equal $A \cap A_j$, but we ignore any duplicate sets.) Since A is not 2-coverable, if $X \in S_1$, then $A - X \notin S_1$. Thus at most half the subsets of $|A|$ are in S_1, and $|S_1| \le 2^{|A|-1}$.

On the other hand, let $B = S - A$ and consider the family of sets $S_2 = \{B \cap A_1, B \cap A_2, \ldots, B \cap A_k\}$. We claim that if $X \in S_2$, then $B - X \notin S_2$. Suppose on the contrary that both $X, B - X \in S_2$ for some $X = B \cap A_\ell$ and $B - X = B \cap A_{\ell'}$. By the minimal definiton of A there are A_i and A_j such that $A_i \cup A_j = A \setminus \{m\}$ for some i, j, and m. Then

$$|A_\ell \cup A_{\ell'} \cup A_i \cup A_j| = n - 1,$$

a contradiction. Thus we assumption is false and $|S_2| \le 2^{|B|} - 1 = 2^{n-|A|-1}$.

Since every set A_i is uniquely determined by its intersection with sets A and $B = S - A$, it follows that $|A| \le |B| \cdot |C| \le 2^{n-2}$.

Problem 2 Let ABC be a triangle and let ω be a circle passing through A and C. Sides AB and BC meet ω again at D and E, respectively. Let γ be the incircle of the circular triangle EBD and let S be its center. Suppose that γ touches the arc DE at M. Prove that the angle bisector of $\angle AMC$ passes through the incenter of triangle ABC.

First Solution: We work backward. Let I be the incenter of triangle ABC. Let N be the midpoint of arc AC on ω that is opposite to B, and let $\overrightarrow{NI}$ meet with ω again at M'. Then $\overline{M'N}$ bisects $\angle AM'C$. We claim that γ is tangent to ω at M', and our desired results follows.

To prove our claim, we are going to do some heavy trigonometry calculations. Let ℓ be the line tangent to ω at M', and let ℓ meet AB and AC at P and Q, respectively. Let ω' be the incircle of triangle PBQ. We are reduced to proving that ω' is tangent to $\overline{PQ}$ at M'.

Let $\angle IAM' = a, \angle M'AB = x, \angle M'CI = b, \angle BCM' = y$. Then $\angle CAI = \angle IAB = x + a$ and $\angle ICA = \angle BCI = y + b$. Since PM' is tangent to ω, we have $\angle PM'A = \angle M'CA = 2b + y$ and thus $\angle BPQ = \angle PM'A + \angle M'AP = 2b + x + y$. Applying the law of sines to triangle PAM', we have

$$\frac{PM'}{\sin x} = \frac{AM'}{\sin(2b + x + y)} \iff PM' = \frac{AM' \sin x}{\sin(2b + x + y)}. \qquad (1)$$

Similarly,

$$M'Q = \frac{M'C \sin y}{\sin(2a + x + y)}. \qquad (2)$$

And, applying the law of sines to triangle $AM'C$ gives

$$\frac{AM'}{\sin(2b + y)} = \frac{M'C}{\sin(2a + x)}. \qquad (3)$$

Combining (1), (2), and (3) we have

$$\frac{PM'}{M'Q} = \frac{\sin x \sin(2b + y) \sin(2a + x + y)}{\sin y \sin(2a + x) \sin(2b + x + y)}. \qquad (4)$$

Now, observe that for a triangle XYZ with inradius r, and with incircle Γ touching XY at T, we have

$$\frac{XT}{TY} = \frac{r \cot \frac{\angle X}{2}}{r \cot \frac{\angle Y}{2}} = \frac{\cot \frac{\angle X}{2}}{\cot \frac{\angle Y}{2}}.$$

Thus it suffices to prove that $\frac{PM'}{M'Q} = \frac{\cot \angle BPQ}{\cot \angle BQP}$, or equivalently (from (4)) any of the following statements:

$$\frac{\sin x \sin(2b + y) \sin(2a + x + y)}{\sin y \sin(2a + x) \sin(2b + x + y)} = \frac{\cot \frac{2b+x+y}{2}}{\cot \frac{2a+x+y}{2}}$$

$$\frac{\sin x \sin(2b + y) \sin \frac{2a+x+y}{2} \cos \frac{2a+x+y}{2}}{\sin y \sin(2a + x) \sin \frac{2b+x+y}{2} \cos \frac{2b+x+y}{2}} = \frac{\cos \frac{2b+x+y}{2} \sin \frac{2a+x+y}{2}}{\cos \frac{2a+x+y}{2} \sin \frac{2b+x+y}{2}}$$

$$\sin x \sin(2b+y)\cos^2\frac{2a+x+y}{2} = \sin y \sin(2a+x)\cos^2\frac{2b+x+y}{2}$$

$$\begin{aligned}&\sin x\sin(2b+y)(\cos(2a+x+y)+1)\\ &= \sin y\sin(2a+x)(\cos(2b+x+y)+1),\end{aligned}$$

or equivalently that $L = R$ where

$$\begin{aligned}R &= \sin x\sin(2b+y)\cos(2a+x+y)\\ &\quad - \sin y\sin(2a+x)\cos(2b+x+y),\\ L &= \sin y\sin(2a+x) - \sin x\sin(2b+y).\end{aligned}$$

Note that

$$\begin{aligned}R &= 1/2[\ \sin x(\sin(2b+2y+x+2a)+\sin(2b-2a-x))\ ]\\ &\quad -1/2[\ \sin y(\sin(2b+2x+y+2a)+\sin(2a-2b-y))\ ]\\ &= -1/4[\ \cos(2a+2b+2x+2y)-\cos(2a+2b+2y)\\ &\qquad +\cos(2b-2a)-\cos(2b-2a-2x)\\ &\qquad -\cos(2a+2b+2x+2y)+\cos(2b+2a+2x)\\ &\qquad -\cos(2b-2a)+\cos(2a-2b-2y)\ ]\\ &= -1/4[\ \cos(2a+2b+2x)-\cos(2b-2a-2x)\\ &\qquad +\cos(2a-2b-2y)-\cos(2a+2b+2y)\ ]\\ &= 1/2[\ \sin(2b)\sin(2a+2x)-\sin(2a)\sin(2b+2y)\ ]\\ &= 2[\ \underbrace{\sin b\sin(a+x)}_{(i)}\cos b\cos(a+x)\\ &\qquad -\underbrace{\sin a\sin(b+y)}_{(ii)}\cos a\cos(b+y)\ ].\end{aligned}$$

To simplify this expression, note that $\overline{MI}, \overline{AI}$, and $\overline{CI}$ meet at I and that $\angle IMA = \angle IMC$. Then the trigonometric form of Ceva's Theorem gives

$$\begin{aligned}&\frac{\sin\angle IMA\sin\angle IAC\sin\angle ICM}{\sin\angle IAM\sin\angle ICA\sin\angle IMC} = 1\\ &\iff \sin a\sin(b+y) = \sin b\sin(a+x).\end{aligned}$$

Swapping quantities (i) and (ii) above thus yields

$$\begin{aligned}
R &= 2\left(\sin a \sin(b+y)\cos b\cos(a+x)\right.\\
&\qquad \left.-\sin b\sin(a+x)\cos a\cos(b+y)\right)\\
&= \cos b\sin(b+y)[\sin(2a+x)-\sin x]\\
&\qquad -\sin b\cos(b+y)[\sin(2a+x)+\sin x]\\
&= \sin(2a+x)[\ \sin(b+y)\cos b-\sin b\cos(b+y)\ ]\\
&\quad -\sin x[\ \sin(b+y)\cos b+\sin b\cos(b+y)\ ]\\
&= \sin(2a+x)\sin y-\sin x\sin(2b+y)=L,
\end{aligned}$$

as desired.

Second Solution: Let O and R be the center and radius of ω, r be the radius of γ, and I be the incenter of triangle ABC. Extend lines AI and CI to hit ω at M_A and M_C respectively; also let line AD be tangent to γ at F, and let line CE be tangent to γ at G. Finally, let d be the length of the exterior tangent from M_A to ω. Notice that since line AM_A bisects $\angle DAC$, we have $DM_A = M_AC$; similarly, $EM_C = M_CA$.

Applying Generalized Ptolemy's Theorem to the "circles" M_A, C, D, and γ externally tangent to ω gives

$$CG \cdot DM_A = M_AC \cdot DF + d \cdot CD$$
$$d^2 = M_AC^2\left(\tfrac{CG-DF}{CD}\right)^2.$$

Note that d^2 equals the power of M_A with respect to γ, so $d^2 = M_AS^2 - r^2$.

By Stewart's Theorem on cevian M_AM in triangle SOM_A, we also have

$$M_AS^2 \cdot OM + M_AO^2 \cdot MS = M_AM^2 \cdot SO + SM \cdot MO \cdot SO$$
$$M_AS^2 \cdot R + R^2 \cdot r = M_AM^2 \cdot (R+r) + r \cdot R \cdot (R+r)$$
$$M_AM^2(R+r) = (M_AS^2 - r^2)R = d^2R,$$

Combining the two equations involving d^2, we find

$$M_AC^2\left(\frac{CG-DF}{CD}\right)^2 = \frac{M_AM^2(R+r)}{R}$$

$$\left(\frac{M_A M}{M_A C}\right)^2 = \left(\frac{R}{R+r}\right)\left(\frac{CG-DF}{CD}\right)^2.$$

Similarly,

$$\left(\frac{M_C M}{M_C A}\right)^2 = \left(\frac{R}{R+r}\right)\left(\frac{AF-EG}{AE}\right)^2.$$

But

$$CG - DF = (CG + GB) - (DF + FB) = CB - DB$$

and similarly

$$AF - EG = (AF + FB) - (EG + GB) = AB - BE.$$

Furthermore, because $ACDE$ is cyclic some angle-chasing gives $\angle BDC = \angle AEC$ and $\angle DCB = \angle BAE$, so $\triangle CBD \sim \triangle ABE$ and

$$\frac{CG-DF}{CD} = \frac{CB-DB}{CD} = \frac{AB-BE}{EA} = \frac{AF-EG}{AE}.$$

Therefore we have $\frac{M_A M}{M_A C} = \frac{M_C M}{M_C A} \Longrightarrow \frac{\sin\angle MAM_A}{\sin\angle M_A AC} = \frac{\sin\angle M_C CM}{\sin\angle ACM_C}$. But by the trigonometric form of Ceva's theorem in triangle AMC applied to lines AM_A, CM_C, and MI, we have

$$\frac{\sin\angle MAM_A}{\sin\angle M_A AC} \cdot \frac{\sin\angle ACM_C}{\sin\angle M_C CM} \cdot \frac{\sin\angle CMI}{\sin\angle IMA} = 1$$

so that

$$\sin\angle CMI = \sin\angle IMA \quad\Longrightarrow\quad \angle CMI = \angle IMA$$

since $\angle AMC < 180°$. Therefore, line MI bisects $\angle AMC$, so the angle bisector of $\angle AMC$ indeed passes through the incenter I of triangle ABC.

Problem 3 Suppose that $C_1, C_2, \ldots, C_n$ are circles of radius 1 in the plane such that no two of them are tangent and the subset of the plane formed by the union of these circles is connected (i.e., for any partition of $\{1, 2, \ldots, n\}$ into nonempty subsets A and B, $\bigcup_{a\in A} C_a$ and $\bigcup_{b\in B} C_b$ are not disjoint). Prove that $|S| \geq n$, where

$$S = \bigcup_{1\leq i<j\leq n} C_i \cap C_j,$$

the intersection points of the circles. (Each circle is viewed as the set of points on its circumference, not including its interior.)

Solution: Let $T = \{C_1, C_2, \ldots, C_n\}$. For every $s \in S$ and $C \in T$ define

$$f(s, C) = \begin{cases} 0, & \text{if } s \notin C, \\ \frac{1}{k}, & \text{if } s \in C, \end{cases}$$

where k is the number of circles passing through s (including C). Thus

$$\sum_{C \in T} f(s, C) = 1$$

for every $s \in S$.

On the other hand, for a fixed circle $C \in T$ let $s_0 \in S \cap C$ be a point such that

$$f(s_0, C) = \min\{f(s, C) \mid s \in S \cap C\}.$$

Suppose that $C, C_2, \ldots C_k$ be the circles which pass through s_0. Then C meets $C_2, \ldots, C_k$ again in distinct points $s_2, \ldots, s_k$. Therefore

$$\sum_{s \in C} f(s, C) \geq \frac{1}{k} + \frac{k-1}{k} = 1.$$

We have

$$|S| = \sum_{s \in S} \sum_{C \in T} f(s, C) = \sum_{C \in T} \sum_{s \in S} f(s, C) \geq n,$$

as desired.

Problem 4 Suppose that $-1 \leq x_1, x_2, \ldots, x_n \leq 1$ are real numbers such that $x_1 + x_2 + \ldots + x_n = 0$. Prove that there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that, for every $1 \leq p \leq q \leq n$,

$$|x_{\sigma(p)} + \cdots + x_{\sigma(q)}| < 2 - \frac{1}{n}.$$

Also prove that the expression on the right hand side cannot be replaced by $2 - \frac{4}{n}$.

Solution: If $n = 1$ then $x_1 = 0$, and the permutation $\sigma(1) = 1$ suffices; if $n = 2$ then $|x_1|, |x_2| \leq 1$ and $|x_1 + x_2| = 0$, and the permutation $(\sigma(1), \sigma(2)) = (1, 2)$ suffices. Now assume $n \geq 3$.

View the x_i as vectors; the problem is equivalent to saying that if we start at a point on the number line, we can travel along the n vectors $x_1, x_2, \ldots, x_n$ in some order so that we stay within an interval $(m, m + 2 - \frac{1}{n}]$.

Call x_i "long" if $|x_i| \geq 1 - \frac{1}{n}$ and call it "short" otherwise. Also call x_i "positive" if $x_i \geq 0$ and "negative" if $x_i < 0$. Suppose without loss of generality that there are at least as many long positive vectors as long negative vectors — otherwise, we could replace each x_i by $-x_i$. We make our trip in two phases:

(i) First, travel alternating along long positive vectors and long negative vectors until no long negative vectors remain. Suppose at some time we are at a point P. Observe that during this leg of our trip, traveling along a pair of vectors changes our position by at most $\frac{1}{n}$ in either direction. Thus if we travel along $2t \leq n$ vectors after P, we stay within $\frac{t}{n} \leq \frac{1}{2}$ of P; and if we travel along $2t+1$ vectors after P, we stay within $\frac{t}{n} + 1 \leq \frac{3}{2} < 2 - \frac{1}{n}$ of P. Therefore, during this phase, we indeed stay within an interval $I = \left(m, m + 2 - \frac{1}{n}\right]$ of length $2 - \frac{1}{n}$.

(ii) After phase (i), we claim that as long as vectors remain unused and we are inside I, there is an unused vector we can travel along while remaining in I; this implies we can finish the trip while staying in I.

If there are no positive vectors, then we can travel along any negative vector, and vice versa. Thus assume there are positive *and* negative vectors remaining; since all the long negative vectors were used in phase (i), only short negative vectors remain.

Now if we are to the right of $m + 1 - \frac{1}{n}$, we can travel along a short negative vector without reaching or passing m. And if we are on or to the left of $m + 1 - \frac{1}{n}$, we can travel along a positive vector (short or long) without passing $m + 2 - \frac{1}{n}$.

Therefore it is possible to complete our journey, and it follows that the desired permutation σ indeed exists.

However, suppose $\frac{1}{n}$ is changed to $\frac{4}{n}$. This bound is never attainable for $n = 1$, and it is not always attainable when $n = 2$ (when $x_1 = 1$, $x_2 = -1$, for example).

And if $n = 2k+1 \geq 3$ or $2k+2 \geq 4$, suppose that $x_1 = x_2 = \cdots = x_k = 1$ and $x_{k+1} = x_{k+2} = \cdots = x_{2k+1} = -\frac{k}{k+1}$ — if n is even, we can let $x_n = 0$ and ignore this term in the permutation.

If two adjacent numbers in the permutation are equal then their sum is either $2 \geq 2 - \frac{4}{n}$ or $-2 \cdot \frac{k}{k+1} \leq -2 + \frac{4}{n}$. Therefore in the permutation, the vectors must alternate between $-\frac{k}{k+1}$ and 1, starting and ending with $-\frac{k}{k+1}$.

But then the outer two vectors add up to $-2 \cdot \frac{k}{k+1}$, so the middle $2k-1$ vectors add up to $2 \cdot \frac{k}{k+1} \geq 2 - \frac{4}{n}$, a contradiction. Therefore, $\frac{1}{n}$ cannot be replaced by $\frac{4}{n}$.

Problem 5 Suppose that $r_1, \ldots, r_n$ are real numbers. Prove that there exists $S \subseteq \{1, 2, \ldots, n\}$ such that

$$1 \leq |S \cap \{i, i+1, i+2\}| \leq 2,$$

for $1 \leq i \leq n-2$, and

$$\left| \sum_{i \in S} r_i \right| \geq \frac{1}{6} \sum_{i=1}^{n} |r_i|.$$

Solution: Let $S = \sum_{i=1}^{n} |r_i|$ and for $i = 0, 1, 2$, define

$$s_i = \sum_{r_j \geq 0,\, j \equiv i} r_j,$$

$$t_i = \sum_{r_j < 0,\, j \equiv i} r_j,$$

where congruences are taken modulo 3. Then we have $S = s_1 + s_2 + s_3 - t_1 - t_2 - t_3$, and $2S$ equals

$$(s_1 + s_2) + (s_2 + s_3) + (s_3 + s_1) - (t_1 + t_2) - (t_2 + t_3) - (t_3 + t_1).$$

Therefore there are $i_1 \neq i_2$ such that either $s_{i_1} + s_{i_2} \geq s/3$ or $t_{i_1} + t_{i_2} \leq -s/3$ or both. Without loss of generality, we assume that $s_{i_1} + s_{i_2} \geq s/3$ and $|s_{i_1} + s_{i_2}| \geq |t_{i_1} + t_{i_2}|$. Thus $s_{i_1} + s_{i_2} + t_{i_1} + t_{i_2} \geq 0$. We have

$$[s_{i_1} + s_{i_2} + t_{i_1}] + [s_{i_1} + s_{i_2} + t_{i_2}] \geq s_{i_1} + s_{i_2} \geq s/3.$$

Therefore at least one of $s_{i_1} + s_{i_2} + t_{i_1}$ and $s_{i_1} + s_{i_2} + t_{i_2}$ is bigger or equal to $s/6$ and we are done.

1.11 Ireland

Problem 1 Find all the real values of x which satisfy

$$\frac{x^2}{(x+1-\sqrt{x+1})^2} < \frac{x^2+3x+18}{(x+1)^2}.$$

Solution: We must have $x \in (-1,0) \cup (0,\infty)$ for the quantities above to be defined. Make the substitution $y = \sqrt{x+1}$, so that $y \in (0,1) \cup (1,\infty)$ and $x = y^2 - 1$. Then the inequality is equivalent to

$$\begin{aligned} & \frac{(y^2-1)^2}{(y^2-y)^2} < \frac{(y^2-1)^2+3(y^2-1)+18}{y^4} \\ \iff & \frac{(y+1)^2}{y^2} < \frac{y^4+y^2+16}{y^4} \\ \iff & (y+1)^2y^2 < y^4+y^2+16 \\ \iff & 2y^3 < 16, \end{aligned}$$

so the condition is satisfied exactly when $y < 2$; i.e., exactly when $y \in (0,1) \cup (1,2)$, which is equivalent to $x \in (-1,0) \cup (0,3)$.

Problem 2 Show that there is a positive number in the Fibonacci sequence which is divisible by 1000.

Solution: In fact, for any natural number n, there exist infinitely many positive Fibonacci numbers divisible by n.

The Fibonacci sequence is defined thus: $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for all $k \geq 0$. Consider ordered pairs of consecutive Fibonacci numbers $(F_0, F_1), (F_1, F_2), \ldots$ taken modulo n. Since the Fibonacci sequence is infinite and there are only n^2 possible ordered pairs of integers modulo n, two such pairs (F_j, F_{j+1}) must be congruent: $F_i \equiv F_{i+m}$ and $F_{i+1} \equiv F_{i+m+1} \pmod{n}$ for some i and m.

If $i \geq 1$ then $F_{i-1} \equiv F_{i+1} - F_i \equiv F_{i+m+1} - F_{i+m} \equiv F_{i+m-1} \pmod{n}$; similarly $F_{i+2} \equiv F_{i+1} + F_i \equiv F_{i+m+1} + F_{i+m} \equiv F_{i+2+m} \pmod{n}$. Continuing similarly, we have $F_j \equiv F_{j+m} \pmod{n}$ for all $j \geq 0$. In particular, $0 = F_0 \equiv F_m \equiv F_{2m} \equiv \cdots \pmod{n}$, so the numbers F_m, $F_{2m}, \ldots$ are all positive Fibonacci numbers divisible by n. Applying this to $n = 1000$, we are done. (In fact, the smallest such m is 750.)

Problem 3 Let D, E, F be points on the sides BC, CA, AB, respectively, of triangle ABC such that $AD \perp BC$, $AF = FB$, and BE is the angle bisector of $\angle B$. Prove that AD, BE, CF are concurrent if and only if

$$a^2(a-c) = (b^2-c^2)(a+c),$$

where $a = BC$, $b = CA$, $c = AB$.

Solution: By Ceva's Theorem, the cevians AD, BE, CF in $\triangle ABC$ are concurrent if and only if (using directed line segments)

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

In this problem, $\frac{AF}{FB} = 1$, and $\frac{CE}{EA} = \frac{a}{c}$. Thus AD, BE, CF are concurrent if and only if $\frac{BD}{DC} = \frac{c}{a}$.

This in turn is true if and only if $BD = \frac{ac}{a+c}$ and $DC = \frac{a^2}{a+c}$. Because $AB^2 - BD^2 = BD^2 = AC^2 - CD^2$, this last condition hold exactly when the following equations are true:

$$\begin{aligned} AB^2 - \left(\frac{ac}{a+c}\right)^2 &= AC^2 - \left(\frac{a^2}{a+c}\right)^2 \\ (a+c)^2c^2 - a^2c^2 &= (a+c)^2b^2 - a^4 \\ a^4 - a^2c^2 &= (b^2-c^2)(a+c)^2 \\ a^2(a-c) &= (b^2-c^2)(a+c). \end{aligned}$$

Therefore the three lines concur if and only if the given equation holds, as desired.

Alternatively, applying the law of cosines gives

$$\frac{BD}{DC} = \frac{c\cos B}{b\cos C} = \frac{c}{b} \cdot \frac{a^2+c^2-b^2}{2ac} \cdot \frac{2ab}{a^2+b^2-c^2} = \frac{a^2+c^2-b^2}{a^2+b^2-c^2}.$$

Again, this equals $\frac{c}{a}$ exactly when the given equation holds.

Problem 4 A 100 by 100 square floor is to be tiled. The only available tiles are rectangular 1 by 3 tiles, fitting exactly over three squares of the floor.

(a) If a 2 by 2 square is removed from the center of the floor, prove that the remaining part of the floor can be tiled with available tiles.

(b) If, instead, a 2 by 2 square is removed from the corner, prove that the remaining part of the floor cannot be tiled with the available tiles.

Solution: Choose a coordinate system so that the corners of the square floor lie along the lattice points $\{(x, y) \mid 0 \leq x, y \leq 100, x, y \in \mathbb{Z}\}$. Denote the rectanglar region $\{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ by $[(a, c) - (b, d)]$.

(a) It is evident that any rectangle with at least one dimension divisible by 3 can be tiled. First tile the four rectangles

$$[(0,0)-(48,52)], \quad [(0,52)-(52,100)],$$
$$[(52,48)-(100,100)], \quad \text{and} \quad [(48,0)-(100,52)].$$

The only part of the board left untiled is $[(48, 48) - (52, 52)]$. But recall that the central region $[(49, 49) - (51, 51)]$ has been removed. It is obvious that the remaining portion can be tiled.

(b) Assume without loss of generality that $[(0, 0) - (2, 2)]$ is the 2 by 2 square which is removed. Label each remaining square $[(x, y) - (x + 1, y + 1)]$ with the number $L(x, y) \in \{0, 1, 2\}$ such that $L(x, y) \equiv x + y \pmod 3$. There are 3333 squares labeled 0, 3331 squares labeled 1, and 3332 squares labeled 2. However, each 1 by 3 tile covers an equal number of squares of each label. Therefore, the floor cannot be tiled.

Problem 5 Define a sequence u_n, $n = 0, 1, 2, \ldots$ as follows: $u_0 = 0$, $u_1 = 1$, and for each $n \geq 1$, u_{n+1} is the smallest positive integer such that $u_{n+1} > u_n$ and $\{u_0, u_1, \ldots, u_{n+1}\}$ contains no three elements which are in arithmetic progression. Find u_{100}.

Solution: Take any nonnegative integer n (e.g., 100) and express it in base-2 (e.g., $100 = 1100100_2$). Now interpret that sequence of 1's and 0's as an integer in base-3 (e.g., $1100100_3 = 981$). Call that integer t_n (e.g., $t_{100} = 981$).

We now prove that $t_n = u_n$ by strong induction on n. It is obvious that $t_0 = u_0$ and that $t_1 = u_1$. Now assume that $t_k = u_k$ for all $k < n$. We shall show that $t_n = u_n$.

First we show that $u_n \leq t_n$ by proving that, in the sequence $t_0, t_1, t_2, \ldots, t_n$, no three numbers form an arithmetic progression.

Pick any three numbers $0 \leq \alpha < \beta < \gamma \leq n$, and consider t_α, t_β, and t_γ in base-3. In the addition of t_α and t_γ, since both t_α and t_γ consist of only 1's and 0's, no carrying can occur. But $t_\alpha \neq t_\gamma$, so they must differ in at least one digit. In that digit in the sum $t_\alpha + t_\gamma$ must lie a "1." On the other hand, t_β consists of only 1's and 0's, so $2t_\beta$ consists of only 2's and 0's. Thus, the base-3 representations of $t_\alpha + t_\gamma$ and $2t_\beta$ are different: the former contains a "1" while the latter does not. Thus, $t_\alpha + t_\gamma \neq 2t_\beta$ for any choice of α, β, γ, so among $t_0, t_1, t_2, \ldots, t_n$, no three numbers are in arithmetic progression. Hence $u_n \leq t_n$.

Next we show that $u_n \geq t_n$ by showing that for all $k \in \{t_{n-1} + 1, t_{n-1} + 2, \ldots, t_n - 1\}$, there exist numbers a and b such that $t_a + k = 2t_b$. First note that k must contain a 2 in its base-3 representation, because the t_i are the only nonnegative integers consisting of only 1's and 0's in base-3. Therefore, we can find two numbers a and b with $0 \leq t_a < t_b < k$ such that:

- whenever k has a "0" or a "1" in its base-3 representation, t_a and t_b each also have the same digit in the corresponding positions in their base-3 representations;
- whenever k has a "2" in its base-3 representation, t_a has a "0" in the corresponding position in its base-3 representation, but t_b has a "1" in the corresponding position in its base-3 representation.

The t_a and t_b, thus constructed, satisfy $t_a < t_b < k$ while $t_a + k = 2t_b$, so t_a, t_b, k form an arithmetic progression. Thus, $u_n \geq t_n$. Putting this together with the previous result, we have forced $u_n = t_n$; hence $u_{100} = t_{100} = 981$.

Problem 6 Solve the system of equations

$$y^2 - (x+8)(x^2+2) = 0$$
$$y^2 - (8+4x)y + (16+16x-5x^2) = 0.$$

Solution: We first check that the solutions $(x, y) = (-2, -6)$ and $(-2, 6)$ both work and are the only solutions with $x = -2$.

We substitute $y^2 = (x+8)(x^2+2)$ into $y^2 + 16 + 16x - 5x^2 = 4(x+2)y$ to get $4(x+2)y = x^3 + 3x^2 + 18x + 32 = (x+2)(x^2+x+16)$. The case $x = -2$ has already been finished, so to deal with the case $x \neq -2$, we write

$$4y = x^2 + x + 16.$$

Squaring both sides, we have

$$16y^2 = x^4 + 2x^3 + 33x^2 + 32x + 256,$$

but from the first original equation we have

$$16y^2 = 16x^3 + 128x^2 + 32x + 256;$$

subtracting these two equations, we have $x^4 - 14x^3 - 95x^2 = 0$, or $x^2(x+5)(x-19) = 0$. Thus, $x \in \{0, -5, 19\}$. We use the equation $4y = x^2 + x + 16$ to find the corresponding y's.

In this way we find that the only solutions (x, y) are $(-2, -6)$, $(-2, 6)$, $(0, 4)$, $(-5, 9)$, and $(19, 99)$; it can be checked that each of these pairs works.

Problem 7 A function $f : \mathbb{N} \to \mathbb{N}$ satisfies

(i) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1;

(ii) $f(p + q) = f(p) + f(q)$ for all prime numbers p and q.

Prove that $f(2) = 2$, $f(3) = 3$, and $f(1999) = 1999$.

Solution: Let us agree on the following notation: we shall write $(\mathrm{i})_{a,b}$ when we plug (a, b) (where a and b are relatively prime) into (i), and $(\mathrm{ii})_{p,q}$ when we plug (p, q) (where p and q are primes) into (ii).

First we find $f(1)$, $f(2)$, and $f(4)$. By $(\mathrm{i})_{1,b}$ we find $f(1) = 1$. By $(\mathrm{i})_{2,3}$ we find $f(6) = f(2)f(3)$; by $(\mathrm{ii})_{3,3}$ we get $f(6) = 2f(3)$; thus,

$$f(2) = 2.$$

Now by $(\mathrm{ii})_{2,2}$ we have

$$f(4) = 4.$$

Next we find some useful facts. From $(\mathrm{ii})_{3,2}$ and $(\mathrm{ii})_{5,2}$, respectively, we obtain

$$f(5) = f(3) + 2,\ f(7) = f(5) + 2 = f(3) + 4.$$

Now we can find $f(3)$. By $(\mathrm{ii})_{5,7}$ we have $f(12) = f(5) + f(7) = 2f(3) + 6$; from $(\mathrm{i})_{4,3}$ we have $f(12) = 4f(3)$; we solve for $f(3)$ to find

$$f(3) = 3.$$

Then using the facts from the previous paragraph, we find

$$f(5) = 5,\ f(7) = 7.$$

We proceed to find $f(13)$ and $f(11)$. By $(\mathrm{i})_{3,5}$, we have $f(15) = 15$. From $(\mathrm{ii})_{13,2}$ and $(\mathrm{ii})_{11,2}$, respectively, we find

$$f(13) = f(15) - f(2) = 13,\ f(11) = f(13) - f(2) = 11.$$

Finally, we can calculate $f(1999)$. By using (i) repeatedly with 2, 7, 11, and 13, we find $f(2002) = f(2\cdot 7\cdot 11\cdot 13) = f(2)f(7)f(11)f(13) = 2002$. Noting that 1999 is a prime number; from $(\mathrm{ii})_{1999,3}$ we obtain

$$f(1999) = f(2002) - f(3) = 1999,$$

and we have finished.

Problem 8 Let a, b, c, d be positive real numbers whose sum is 1. Prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$$

with equality if and only if $a = b = c = d = 1/4$.

Solution: Apply the Cauchy-Schwarz inequality to find $[(a+b) + (b+c)+(c+d)+(d+a)]\left(\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a}\right) \geq (a+b+c+d)^2$, which is equivalent to

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}(a+b+c+d) = \frac{1}{2},$$

with equality if and only if $\frac{a+b}{a} = \frac{b+c}{b} = \frac{c+d}{d} = \frac{d+a}{a}$, i.e., if and only if $a = b = c = d = \frac{1}{4}$.

Problem 9 Find all positive integers m such that the fourth power of the number of positive divisors of m equals m.

Solution: If the given condition holds for some integer m, then m must be a perfect fourth power and we may write its prime factorization as $m = 2^{4a_2}3^{4a_3}5^{4a_5}7^{4a_7}\cdots$ for nonnegative integers $a_2, a_3, a_5, a_7, \ldots$. Now the number of positive divisors of m equals $(4a_2+1)(4a_3+1)(4a_5+1)(4a_7+1)\cdots$; this is odd, so m is odd and $a_2 = 0$. Thus,

$$1 = \frac{4a_3+1}{3^{a_3}} \cdot \frac{4a_5+1}{5^{a_5}} \cdot \frac{4a_7+1}{7^{a_7}} \cdot \cdots = x_3x_5x_7\cdots,$$

where we write $x_p = \frac{4a_p+1}{p^{a_p}}$ for each p.

When $a_3 = 1$, $x_3 = \frac{5}{3}$; when $a_3 = 0$ or 2, $x_3 = 1$. And by Bernoulli's inequality, when $a_3 > 2$ we have

$$3_3^a = (8+1)^{a_3/2} > 8(a_3/2) + 1 = 4a_3 + 1$$

so that $x_3 < 1$.

When $a_5 = 0$ or 1, $x_5 = 1$; and by Bernoulli's inequality, when $a_5 \geq 2$ we have

$$5^{a_5} = (24+1)^{a_5/2} \geq 24a_5/2 + 1 = 12a_5 + 1$$

so that $x_5 \leq \frac{4a_5+1}{12a_5+1} \leq \frac{9}{25}$.

Finally, for any $p > 5$ when $a_p = 0$ we have $x_p = 1$; when $a_p = 1$ we have $p^{a_p} = p > 5 = 4a_p + 1$ so that $x_p < 1$; and when $a_p > 0$ then again by Bernoulli's inequality we have

$$p^{a_p} > 5^{a_p} > 12a_p + 1$$

so that as above $x_p < \frac{9}{25}$.

Now if $a_3 \neq 1$ then we have $x_p \leq 1$ for all p; but since $1 = x_2 x_3 x_5 \cdots$ we must actually have $x_p = 1$ for all p. This means that $a_3 \in \{0, 2\}$, $a_5 \in \{0, 1\}$, and $a_7 = a_{11} = \cdots = 0$; so that $m = 1^4$, $(3^2)^4$, 5^4, or $(3^2 \cdot 5)^4$.

Otherwise, if $a_3 = 1$ then $3 \mid m = 5^4(4a_5+1)^4(4a_7+1)^4 \cdots$. Then for some prime $p' \geq 5$, $3 \mid 4a_{p'} + 1$ so that $a_{p'} \geq 2$; from above we have $x_{p'} \leq \frac{9}{25}$. But then $x_3 x_5 x_7 \cdots \leq \frac{5}{3}\frac{9}{27} < 1$, a contradiction.

Thus the only solutions are 1, 5^4, 3^8, and $3^8 \cdot 5^4$; and these can be easily verified by inspection.

Problem 10 Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$, $EF = FA$, and

$$\angle ABC + \angle CDE + \angle EFA = 360^\circ.$$

Prove that the respective perpendiculars from A, C, E to FB, BD, DF are concurrent.

First Solution: The result actually holds even without the given angle condition. Let $\mathcal{C}_1$ be the circle with center B and radius $AB = BC$, $\mathcal{C}_2$ the circle with center D and radius $CD = DE$, and $\mathcal{C}_3$ the circle with center F and radius $EF = FA$. The line through A and perpendicular to line FB is the radical axis of circles $\mathcal{C}_3$ and $\mathcal{C}_1$, the line through C and perpendicular to line BD is the radical axis of circles $\mathcal{C}_1$ and $\mathcal{C}_2$, and the line through E and perpendicular to line DF is the radical axis of circles $\mathcal{C}_2$ and $\mathcal{C}_3$. The result follows because these three radical axes meet at the radical center of the three circles.

Second Solution: We first establish two lemmas:

Lemma 1. *Given points $W \neq Y$ and $X \neq Z$, lines WY and XZ are perpendicular if and only if*

$$XW^2 - WZ^2 = XY^2 - YZ^2. \qquad (*)$$

Proof: Introduce Cartesian coordinates such that $W = (0.0)$, $X = (1, 0)$, $Y = (x_1, y_1)$, and $Z = (x_2, y_2)$. Then (*) becomes

$$x_1^2 + y_1^2 - x_2^2 - y_2^2 = (x_1 - 1)^2 + y_1^2 - (x_2 - 1)^2 - y_2^2,$$

which upon cancellation yields $x_1 = x_2$. This is true if and only if line YZ is perpendicular to the x-axis WX. ■

If P is the intersection of the perpendiculars from A and C to lines FB and BD, respectively, then the lemma implies that

$$PF^2 - PB^2 = AF^2 - AB^2$$

and

$$PB^2 - PD^2 = CB^2 - CD^2.$$

From the given isosceles triangles, we have $EF = FA$, $AB = BC$, and $CD = DE$. Subtracting the first equation from the second then gives

$$PD^2 - PF^2 = ED^2 - EF^2.$$

Hence line PE is also perpendicular to line DF, which completes the proof.

1.12 Italy

Problem 1 Given a rectangular sheet with sides a and b, with $a > b$, fold it along a diagonal. Determine the area of the triangle that passes over the edge of the paper.

Solution: Let $ABCD$ be a rectangle with $AD = a$ and $AB = b$. Let D' be the reflection of D across line AC, and let $E = AD' \cap BC$. We wish to find $[CD'E]$. Since $AB = CD'$, $\angle ABE = \angle CD'E = 90°$, and $\angle BEA = \angle D'EC$, triangles ABE and $CD'E$ are congruent. Thus $AE = EC$ and $CE^2 = AE^2 = AB^2 + BE^2 = b^2 + (a - CE)^2$. Hence $CE = \frac{a^2+b^2}{2a}$. It follows that

$$[CD'E] = [ACD'] - [ACE] = \frac{ab}{2} - \frac{b}{2} \cdot CE = \frac{b(a^2 - b^2)}{4a}.$$

Problem 2 A positive integer is said to be *balanced* if the number of its decimal digits equals the number of its distinct prime factors (for instance 15 is balanced, while 49 is not). Prove that there are only finitely many balanced numbers.

Solution: For $n > 15$, consider the product of the first n primes. The first sixteen primes have product

$$(2{\cdot}53)(3{\cdot}47)(5{\cdot}43)(7{\cdot}41){\cdot}11{\cdot}13{\cdot}17{\cdot}19{\cdot}23{\cdot}29{\cdot}31{\cdot}37 > 100^4{\cdot}10^8 = 10^{16},$$

while the other $n - 16$ primes are each at least 10. Thus the product of the first n primes is greater than 10^n.

Then if x has n digits and is balanced, then it it as at least the product of the first n primes. If $n \geq 16$ then from the previous paragraph x would be greater than 10^n and would have at least $n+1$ digits, a contradiction. Thus x can have at most 15 digits, implying that the number of balanced numbers is finite.

Problem 3 Let $\omega, \omega_1, \omega_2$ be three circles with radii r, r_1, r_2, respectively, with $0 < r_1 < r_2 < r$. The circles ω_1 and ω_2 are internally tangent to ω at two distinct points A and B and meet in two distinct points. Prove that $\overline{AB}$ contains an intersection point of ω_1 and ω_2 if and only if $r_1 + r_2 = r$.

Solution: Let O be the center of ω, and note that the centers C, D of ω_1, ω_2 lie on $\overline{OA}$ and $\overline{OB}$, respectively. Let E be a point on $\overline{AB}$

such that $CE \parallel OB$. Then $\triangle ACE \sim \triangle AOB$. Hence $AE = CE$ and E is on ω_1. We need to prove that $r = r_1 + r_2$ if and only if E is on ω_2.

Note that $r = r_1 + r_2$ is equivalent to

$$OD = OB - BD = r - r_2 = r_1 = AC,$$

that is $CEDO$ is a parallelogram or $DE \parallel AO$. Hence $r = r_1 + r_2$ if and only if $\triangle BDO \sim \triangle BOA$ or $BD = DE$, that is, E is on ω_2.

Problem 4 Albert and Barbara play the following game. On a table there are 1999 sticks: each player in turn must remove from the table some sticks, provided that the player removes at least one stick and at most half of the sticks remaining on the table. The player who leaves just one stick on the table loses the game. Barbara moves first. Determine for which of the players there exists a winning strategy.

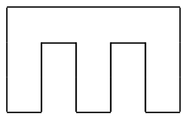
Solution: Call a number k *hopeless* if a player faced with k sticks has no winning strategy. If k is hopeless, then so is $2k+1$: a player faced with $2k+1$ sticks can only leave a pile of $k+1, k+2, \ldots,$ or $2k$ sticks, from which the other player can leave k sticks. Then since 2 is hopeless, so are $5, 11, \ldots, 3 \cdot 2^n - 1$ for all $n \geq 0$. Conversely, if $3 \cdot 2^n - 1 < k < 3 \cdot 2^{n+1} - 1$, then given k sticks a player can leave $3 \cdot 2^n - 1$ sticks and force a win. Since 1999 is not of the form $3 \cdot 2^n - 1$, it is not hopeless and hence Barbara has a winning strategy.

Problem 5 On a lake there is a village of pile-built dwellings, set on the nodes of an $m \times n$ rectangular array. Each dwelling is an endpoint of exactly p bridges which connect the dwelling with one or more of the adjacent dwellings (here adjacent means with respect to the array, hence diagonal connection is not allowed). Determine for which values of m, n, p it is possible to place the bridges so that from any dwelling one can reach any other dwelling. (Clearly, two adjacent dwellings can be connected by more than one bridge).

Solution: Suppose it is possible to place the bridges in this manner, and set the villages along the lattice points $\{(a, b) \mid 1 \leq a \leq m, 1 \leq b \leq n\}$. Color the dwellings cyan and magenta in a checkerboard fashion, so that every bridge connects a cyan dwelling with a magenta dwelling. Since each dwelling is at the end of the same number of bridges (exactly p of them), the number of cyan dwellings must equal the number of magenta dwellings; thus $2 \mid mn$.

Obviously $mn = 2$ works for all values of p. And for $p = 1$, we cannot have $mn > 2$ because otherwise if any two dwellings A and B are connected, then they cannot be connected to any other dwellings. Similarly, if $m = 1, n > 2$ (or $n = 1, m > 2$) then p bridges must connect $(1, 1)$ and $(1, 2)$ (or $(1, 1)$ and $(2, 1)$); but then neither of these dwellings is connected to any other dwellings, a contradiction.

Now assume that $2 \mid mn$ with $m, n > 1$ and $p > 1$; assume without loss of generality that $2 \mid m$. Build a sequence of bridges starting at $(1, 1)$, going up to $(1, n)$, right to (m, n), down to $(m, 1)$, and left to $(m - 1, 1)$; and then weaving back to $(1, 1)$ by repeatedly going from $(k, 1)$ up to $(k, n - 1)$ left to $(k - 1, n - 1)$ down to $(k - 1, 1)$ and left to $(k - 2, 1)$ for $k = m - 1, m - 3, \ldots, 3$. (The sideways E below shows this construction for $m = 6$, $n = 4$.)



So far we have built two bridges leading out of every dwelling, and any dwelling can be reached from any other dwelling. For the remaining $p - 2$ bridges needed for each dwelling, note that our sequence contains exactly mn bridges, an even number; so if we build *every other* bridge in our sequence, and do this $p - 2$ times, then exactly p bridges come out of every dwelling.

Thus either $mn = 2$ and p equals any value; or $2 \mid mn$ with $m, n, p > 1$.

Problem 6 Determine all triples (x, k, n) of positive integers such that

$$3^k - 1 = x^n.$$

Solution: $(3^k - 1, k, 1)$ for all positive integers k, and $(2, 2, 3)$.

The case of $n = 1$ is obvious. Now, n cannot be even because then 3 could not divide $3^k = \left(x^{\frac{n}{2}}\right)^2 + 1$ (since no square is congruent to 2 modulo 3); and also, we must have $x \neq 1$.

Assume that $n > 1$ is odd and $x \geq 2$. Then $3^k = (x+1) \sum_{i=0}^{n-1} (-x)^i$ implying that both $x + 1$ and $\sum_{i=0}^{n-1} (-x)^i$ are powers of 3. Then since

$x+1 \le x^2-x+1 \le \sum_{i=0}^{n-1}(-x)^i$, we must have $0 \equiv \sum_{i=0}^{n-1}(-x)^i \equiv n \pmod{x+1}$, so that $x+1 \mid n$. Specifically, this means that $3 \mid n$.

Writing $x' = x^{\frac{n}{3}}$, we have $3^k = x'^3 + 1 = (x'+1)(x'^2 - x' + 1)$. As before $x'+1$ must equal some power of 3, say 3^t. But then $3^k = (3^t-1)^3+1 = 3^{3t} - 3^{2t+1} + 3^{t+1}$, which is strictly between 3^{3t-1} and 3^{3t} for $t > 1$. Therefore we must have $t = 1$, $x' = 2$, and $k = 2$, giving the solution $(x, k, n) = (2, 2, 3)$.

Problem 7 Prove that for each prime p the equation

$$2^p + 3^p = a^n$$

has no integer solutions (a, n) with $a, n > 1$.

Solution: When $p = 2$ we have $a^n = 13$, which is impossible. When p is odd, then $5 \mid 2^p + 3^p$; then since $n > 1$, we must have $25 \mid 2^p + 3^p$. Then

$$2^p + (5-2)^p \equiv 2^p + \left(\binom{p}{1} 5 \cdot (-2)^{p-1} + (-2)^p\right) \equiv 5p \cdot 2^{p-1} \pmod{25},$$

so $5 \mid p$. Thus we must have $p = 5$, but then $a^n = 2^5 + 3^5 = 5^2 \cdot 11$ has no solutions.

Problem 8 Points D and E are given on the sides AB and AC of triangle ABC such that $DE \parallel BC$ and $\overline{DE}$ is tangent to the incircle of ABC. Prove that

$$DE \le \frac{AB + BC + CA}{8}.$$

Solution: Let $BC = a$, $CA = b$, $AB = c$. Also let $h = \frac{2[ABC]}{a}$ be the distance from A to line BC and let $r = \frac{2[ABC]}{a+b+c}$ the inradius of triangle ABC; note that $\frac{h-2r}{h} = \frac{b+c-a}{a+b+c}$.

Let $x = b + c - a$, $y = c + a - b$, $z = a + b - c$. Then

$$(x + y + z)^2 \ge (2\sqrt{x(y + z)})^2 = 4x(y + z)$$

by AM-GM, which implies that $(a + b + c)^2 \ge 8(b + c - a)a$, or

$$\frac{b+c-a}{a+b+c} \cdot a \le \frac{a+b+c}{8} \implies \frac{h-2r}{h} \cdot BC \le \frac{AB+BC+CA}{8}.$$

But since $DE \parallel BC$, we have $\frac{DE}{BC} = \frac{h-2r}{h}$; substituting this into the above inequality gives the desired result.

Problem 9

(a) Find all the strictly monotonic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y, \qquad \text{for all } x, y \in \mathbb{R}.$$

(b) Prove that for every integer $n > 1$ there do not exist strictly monotonic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y^n, \qquad \text{for all } x, y \in \mathbb{R}.$$

Solution:

(a) The only such functions are $f(x) = x$ and $f(x) = -x$. Setting $x = y = 0$ gives $f(f(0)) = f(0)$, while setting $x = -f(0), y = 0$ gives $f(-f(0)) = f(0)$. Since f is strictly monotonic it is injective, so $f(0) = -f(0)$ and thus $f(0) = 0$. Next, setting $x = 0$ gives $f(f(y)) = y$ for all y.

Suppose f is increasing. If $f(x) > x$ then $x = f(f(x)) > f(x)$, a contradiction; if $f(x) < x$ then $x = f(f(x)) < f(x)$, a contradiction. Thus $f(x) = x$ for all x.

Next suppose that f is decreasing. Plugging in $x = -f(t)$, $y = t$, and then $x = 0$, $y = -t$ shows that $f(-f(t)) = f(f(-t)) = -t$, so $f(t) = -f(-t)$ for all t. Now given x, if $f(x) < -x$ then $x = f(f(x)) > f(-x) = -f(x)$, a contradiction. And if $f(x) > -x$ then $x = f(f(x)) < f(-x) = -f(x)$, a contradiction. Hence we must have $f(x) = -x$ for all x.

Therefore either $f(x) = x$ for all x or $f(x) = -x$ for all x; and it is easy to check that these two functions work.

(b) Since f is strictly monotonic, it is injective. Then for $y \neq 0$ we have $f(y) \neq f(-y)$ so that $f(x+f(y)) \neq f(x+f(-y))$ and hence $f(x) + y^n \neq f(x) + (-y)^n$; thus, n can't be even.

Now suppose there is such an f for odd n; then by arguments similar to those in part (a), we find that $f(0) = 0$ and $f(f(y)) = y^n$. Specifically, $f(f(1)) = 1$. If f is increasing then as in part (a) we have $f(1) = 1$; then $f(2) = f(1 + f(1)) = f(1) + 1^n = 2$ and $2^n = f(f(2)) = f(2) = 2$, a contradiction. If f is decreasing, then as in part (a) we have $f(1) = -1$; then $f(2) = f(1 + f(-1)) = f(1) + (-1)^n = -2$ and $2^n = f(f(2)) = f(-2) = -f(2) = 2$, a contradiction.

Problem 10 Let X be a set with $|X| = n$, and let $A_1, A_2, \ldots, A_m$ be subsets of X such that

(a) $|A_i| = 3$ for $i = 1, 2, \ldots, m$.

(b) $|A_i \cap A_j| \le 1$ for all $i \ne j$.

Prove that there exists a subset of X with at least $\lfloor \sqrt{2n} \rfloor$ elements, which does not contain A_i for $i = 1, 2, \ldots, m$.

Solution: Let A be a subset of X containing no A_i, and having the maximum number of elements subject to this condition. Let k be the size of A. By assumption, for each $x \in X - A$, there exists $i(x) \in \{1, \ldots, m\}$ such that $A_{i(x)} \subseteq A \cup \{x\}$. Let $L_x = A \cap A_{i(x)}$, which by the previous observation must have 2 elements. Since $|A_i \cap A_j| \le 1$ for $i \ne j$, the L_x must all be distinct. Now there are $\binom{k}{2}$ 2-element subsets of A, so there can be at most $\binom{k}{2}$ sets L_x. Thus $n - k \le \binom{k}{2}$ or $k^2 + k \ge 2n$. It follows that

$$k \ge \frac{1}{2}(-1 + \sqrt{1 + 8n}) > \sqrt{2n} - 1,$$

that is, $k \ge \lfloor \sqrt{2n} \rfloor$.

1.13 Japan

Problem 1 You can place a stone at each of 1999×1999 squares on a grid pattern. Find the minimum number of stones you must place such that, when an arbitrary blank square is selected, the total number of stones placed in the corresponding row and column is at least 1999.

Solution: Place stones in a checkerboard pattern on the grid, so that stones are placed on the four corner squares. This placement satisfies the condition and contains $1000 \times 1000 + 999 \times 999 = 1998001$ stones. We now prove this number is minimal.

Suppose the condition is satisfied. Assume without loss of generality that the j-th column contains k stones, and every other row or column also contains at least k stones. For each of the k stones in the j-th column, the row containing that stone must contain at least k stones by our minimal choice of k. And for each of the $1999 - k$ blank squares in the j-th column, to satisfy the given condition there must be at least $1999 - k$ stones in the row containing that square. Thus total number of stones is at least

$$k^2 + (1999 - k)^2 = 2\left(k - \frac{1999}{2}\right)^2 + \frac{1999^2}{2} \geq \frac{1999^2}{2} = 1998000.5,$$

and it follows that there indeed must be at least 1998001 stones.

Problem 2 Let $f(x) = x^3 + 17$. Prove that for each natural number n, $n \geq 2$, there is a natural number x for which $f(x)$ is divisible by 3^n but not by 3^{n+1}.

Solution: We prove the result by induction on n. If $n = 2$, then $x = 1$ suffices. Now suppose that the claim is true for $n \geq 2$ — that is, there is a natural number y such that $y^3 + 17$ is divisible by 3^n but not 3^{n+1}. We prove that the claim is true for $n + 1$.

Suppose we have integers a, m such that a is not divisible by 3 and $m \geq 2$. Then $a^2 \equiv 1 \pmod 3$ and thus $3^m a^2 \equiv 3^m \pmod{3^{m+1}}$. Also, since $m \geq 2$ we have $3m - 3 \geq 2m - 1 \geq m + 1$. Hence

$$(a + 3^{m-1})^3 \equiv a^3 + 3^m a^2 + 3^{2m-1} a + 3^{3m-3} \equiv a^3 + 3^m \pmod{3^{m+1}}.$$

Since $y^3 + 17$ is divisible by 3^n, it is congruent to either 0, 3^n, or $2 \cdot 3^n$ modulo 3^{n+1}. Since 3 does not divide 17, 3 cannot divide y either.

Hence applying our result from the previous paragraph twice — once with $(a, m) = (y, n)$ and once with $(a, m) = (y + 3^{n-1}, n)$ — we find that 3^{n+1} must divide either $(y + 3^{n-1})^3 + 17$ or $(y + 2 \cdot 3^{n-1})^3 + 17$.

Hence there exists a natural number x' not divisible by 3 such that $3^{n+1} \mid x'^3 + 17$. If 3^{n+2} does not divide $x'^3 + 17$, we are done. Otherwise, we claim the number $x = x' + 3^n$ suffices. Since $x = x' + 3^{n-1} + 3^{n-1} + 3^{n-1}$, the result from two paragraphs ago tells us that $x^3 \equiv x'^3 + 3^n + 3^n + 3^n \equiv x'^3 \pmod{3^{n+1}}$. Thus $3^{n+1} \mid x^3 + 17$ as well. On the other hand, since $x = x' + 3^n$, we have $x^3 \equiv x'^3 + 3^{n+1} \not\equiv x'^3 \pmod{3^{n+2}}$. It follows that 3^{n+2} does *not* divide $x^3 + 17$, as desired. This completes the inductive step.

Problem 3 From a set of $2n + 1$ weights (where n is a natural number), if any one weight is excluded, then the remaining $2n$ weights can be divided into two sets of n weights that balance each other. Prove that all the weights are equal.

Solution: Label the weights $a_1, a_2, \ldots, a_{2n+1}$. Then for each j, $1 \le j \le 2n$, we have

$$c_1^{(j)} a_1 + c_2^{(j)} a_2 + \cdots + c_{2n}^{(j)} a_{2n} = a_{2n+1}$$

where $c_j^{(j)} = 0$, n of the other $c_i^{(j)}$ equal 1, and the remaining $c_i^{(j)}$ equal -1.

Thus we have $2n$ equations in the variables $a_1, a_2, \ldots, a_{2n}$. Clearly $(a_1, a_2, \ldots, a_{2n}) = (a_{2n+1}, a_{2n+1}, \ldots, a_{2n+1})$ is a solution to this system of equations. By Kramer's Rule, this solution is unique if and only if the determinant of the matrix

$$\begin{bmatrix} c_1^{(1)} & c_2^{(1)} & \cdots & c_{2n}^{(1)} \\ c_1^{(2)} & c_2^{(2)} & \cdots & c_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{(2n)} & c_2^{(2n)} & \cdots & c_{2n}^{(2n)} \end{bmatrix}$$

is nonzero. We show this is true by proving that this determinant is odd.

If we add an integer m to any single integer in the matrix, its determinant changes by m multiplied by the corresponding cofactor. Specifically, if m is even then the parity of the determinant does not change. Thus the parity of the presented determinant is the same as the parity of the determinant

$$\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{vmatrix}.$$

This matrix has eigenvector $(1, 1, \ldots, 1)$ corresponding to the eigenvalue $2n-1$; and eigenvectors $(1, -1, 0, \ldots, 0, 0)$, $(0, 1, -1, \ldots, 0, 0)$, $\ldots$, $(0, 0, 0, \ldots, 1, -1)$ corresponding to eigenvalue -1. These $2n$ eigenvectors are linearly independent, so the matrix's characteristic polynomial is $p(x) = (x-(2n-1))(x+1)^{2n-1}$. Hence its determinant $p(0) = -(2n-1)$ is odd, as desired.

Problem 4 Prove that

$$f(x) = (x^2+1^2)(x^2+2^2)(x^2+3^2)\cdots(x^2+n^2)+1$$

cannot be expressed as a product of two integral-coefficient polynomials with degree greater than 0.

Solution: The claim is obvious when $n = 1$. Now assume $n \geq 2$ and suppose by way of contradiction that $f(x)$ *could* be expressed as such a product $g(x)h(x)$ with

$$g(x) = a_0 + a_1x + \cdots + a_\ell x^\ell,$$
$$h(x) = b_0 + b_1x + \cdots + b_{\ell'} x^{\ell'},$$

where $\ell, \ell' > 0$ and the coefficients a_i and b_i are integers.

For $m = \pm1, \pm2, \ldots, \pm n$, since $(mi)^2 + m^2 = 0$ we have $1 = f(mi) = g(mi)h(mi)$. But since g and h have integer coefficients, $g(mi)$ equals either $1, -1, i$, or $-i$. Moreover, since the imaginary part of

$$g(mi) = (a_0 - a_2m^2 + a_4m^4 - \cdots) + m(a_1 - a_3m^2 + a_5m^4 - \cdots)i$$

is a multiple of m, $g(mi)$ must equal ±1 for $m \neq \pm1$. Going further, since $1 = g(mi)h(mi)$ we have $g(mi) = h(mi) = \pm1$ for $m \neq \pm1$.

Then by the factor theorem,

$$g(x) - h(x) = (x^2+2^2)(x^2+3^2)\cdots(x^2+n^2)k(x)$$

for some integer-coefficient polynomial $k(x)$ with degree at most 1. Since $(g(i), h(i))$ equals $(1, -1), (-1, 1), (i, -i)$, or $(-i, i)$, we have

$$2 \geq |g(i) - h(i)| = (-1 + 2^2)(-1 + 3^2) \cdots (-1 + n^2)|k(i)|,$$

and hence we must have $k(i) = 0$. Since $k(x)$ has degree at most 1, this implies that $k(x) = 0$ for *all* x and that $g(x) = h(x)$ for all x. But then $a_0^2 = g(0)h(0) = f(0) = (1^2)(2^2) \cdots (n^2) + 1$, which is impossible.

Problem 5 For a convex hexagon $ABCDEF$ whose side lengths are all 1, let M and m be the maximum and minimum values of the three diagonals AD, BE, and CF. Find all possible values of m and M.

Solution: We claim that the possible values are $\sqrt{3} \leq M \leq 3$ and $1 \leq m \leq 2$.

First we show all such values are attainable. Continuously transform $ABCDEF$ from an equilateral triangle ACE of side length 2, into a regular hexagon of side length 1, and finally into a segment of length 3 (say, by enlarging the diagonal AD of the regular hexagon while bringing B, C, E, F closer to line AD). Then M continuously varies from $\sqrt{3}$ to 2 to 3. Similarly, by continuously transforming $ABCDEF$ from a 1×2 rectangle into a regular hexagon, we can make m vary continuously from 1 to 2.

Now we prove no other values are attainble. First, we have $AD \leq AB + BC + CD = 3$, and similarly $BE, CF \leq 3$ so that $M \leq 3$.

Next, suppose by way of contradiction that $m < 1$ and say without loss of generality that $AD < 1$. Since $AD < AB = BC = CD = 1$,

$$\angle DCA < \angle DAC,\ \angle ABD < \angle ADB,$$
$$\angle CBD = \angle CDB,\ \angle BCA = \angle BAC.$$

Therefore,

$$\angle CDA + \angle BAD = \angle CDB + \angle BDA + \angle BAC + \angle CAD$$
$$> \angle CBD + \angle DBA + \angle BCA + \angle ACD = \angle CBA + \angle BCD.$$

Consequently $\angle CDA + \angle BAD > 180^\circ$ and likewise $\angle EDA + \angle FAD > 180^\circ$. But then

$$\angle CDE + \angle BAF = \angle CDA + \angle EDA + \angle BAD + \angle FAD > 360^\circ,$$

which is impossible since $ABCDEF$ is convex. Hence $m \geq 1$.

Next we demonstrate that $M \geq \sqrt{3}$ and $m \leq 2$. Since the sum of the six interior angles in $ABCDEF$ is 720°, some pair of adjacent angles has sum greater than or equal to 240° and some pair has sum less than or equal to 240°. Thus it suffices to prove that $CF \geq \sqrt{3}$ when $\angle A + \angle B \geq 240^\circ$, and that $CF \leq 2$ when $\angle A + \angle B \leq 240^\circ$.

Suppose by way of contradiction that $\angle A + \angle B \geq 240^\circ$ and $CF < \sqrt{3}$. By the law of cosines, $CF^2 = BC^2 + BF^2 - 2BC \cdot BF \cos \angle FBC$. Thus if we fix A, B, F and decrease $\angle ABC$, we decrease $\angle FBC$ and CF; similarly, by fixing A, B, C and decreasing $\angle BAF$, we decrease CF. Therefore, it suffices to prove that $\sqrt{3} \geq CF$ when $\angle A + \angle B = 240^\circ$. And likewise, it suffices to prove that $CF \leq 2$ when $\angle A + \angle B = 240^\circ$.

Now suppose that $\angle A + \angle B$ *does* equal 240°. Let lines AF and BC intersect at P, and set $x = PA$ and $y = PB$. Since $\angle A + \angle B = 240^\circ$, $\angle P = 60^\circ$. Then applying the law of cosines to triangles PAB and PCF yields

$$1 = AB^2 = x^2 + y^2 - xy$$

and

$$CF^2 = (x+1)^2 + (y+1)^2 - (x+1)(y+1) = 2 + x + y.$$

Therefore, we need only find the possible values of $x + y$ given that $x^2 + y^2 - xy = 1$ and $x, y \geq 0$. These conditions imply that $(x+y)^2 + 3(x-y)^2 = 4$, $x + y \geq 0$, and $|x - y| \leq x + y$. Hence

$$1 = \frac{1}{4}(x+y)^2 + \frac{3}{4}(x-y)^2 \leq (x+y)^2 \leq (x+y)^2 + 3(x-y)^2 = 4,$$

so $1 \leq x + y \leq 2$ and $\sqrt{3} \leq CF \leq 2$. This completes the proof.

1.14 Korea

Problem 1 Let R and r be the circumradius and inradius of triangle ABC respectively, and let R' and r' be the circumradius and inradius of triangle $A'B'C'$ respectively. Prove that if $\angle C = \angle C'$ and $Rr' = R'r$, then the triangles are similar.

Solution: Let ω be the circumcircle of triangle ABC. By scaling, rotating, and translating, we may assume that $A = A'$, $B = B'$, $R = R'$, $r = r'$ and that C, C' lie on the same arc $\widehat{AB}$ of ω. If the triangles were similar before these transformations, they still remain similar; so it suffices to prove they are now congruent.

Since $r = \frac{1}{2}(AC + BC - AB)\cot(\angle C)$ and $r' = \frac{1}{2}(A'C' + B'C' - A'B')\cot(\angle C') = \frac{1}{2}(A'C' + B'C' - AB)\cot(\angle C)$, we must have $AC + BC = A'C' + B'C'$ and hence $AB + BC + CA = A'B' + B'C' + C'A'$. Then the area of triangle ABC is $\frac{1}{2}r(AB + BC + CA)$, which thus equals the area of triangle $A'B'C'$, $\frac{1}{2}r'(A'B' + B'C' + C'A')$. Since these triangles share the same base $\overline{AB}$, we know that the altitudes from C and C' to $\overline{AB}$ are equal. This implies that $\triangle ABC$ is congruent to either $\triangle A'B'C'$ or $\triangle B'A'C'$, as desired.

Problem 2 Suppose $f : \mathbb{Q} \to \mathbb{R}$ is a function satisfying

$$|f(m+n) - f(m)| \leq \frac{n}{m}$$

for all positive rational numbers n and m. Show that for all positive integers k,

$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \leq \frac{k(k-1)}{2}.$$

Solution: It follows from the condition $|f(m+n) - f(m)| \leq \frac{n}{m}$ that

$$\left|f(2^{i+1}) - f(2^i)\right| \leq \frac{2^{i+1} - 2^i}{2^i} = 1.$$

Therefore, for $k > i$,

$$\left|f(2^k) - f(2^i)\right| \leq \sum_{j=i}^{k-1} \left|f(2^{j+1}) - f(2^j)\right| \leq k - i.$$

From the above inequality, we obtain

$$\sum_{i=1}^{k}\left|f(2^k)-f(2^i)\right|=\sum_{i=1}^{k-1}\left|f(2^k)-f(2^i)\right|\le\sum_{i=1}^{k-1}(k-i)=\frac{k(k-1)}{2}.$$

This completes the proof.

Problem 3 Find all positive integers n such that 2^n-1 is a multiple of 3 and $\frac{2^n-1}{3}$ is a divisor of $4m^2+1$ for some integer m.

Solution: The answer is all 2^k where $k=1,2,\ldots$.

First, it is easy to conclude (using Fermat's Little Theorem, or by simple observation) that if $3 \mid 2^n-1$, then n must be even.

Suppose by way of contradiction that $\ell \ge 3$ is a positive odd divisor of n. Then $2^\ell-1$ is not divisible by 3 but it *is* a divisor of 2^n-1, so it is a divisor of $4m^2+1$ as well. On the other hand, $2^\ell-1$ has a prime divisor p of the form $4r+3$. Then $(2m)^2\equiv -1 \pmod{4r+3}$; but a standard number theory result states that a square cannot equal -1 modulo a prime of the form $4r+3$.

Therefore n is indeed of the form 2^k for $k\ge 1$. For such n, we have

$$\frac{2^n-1}{3}=(2^{2^1}+1)(2^{2^2}+1)(2^{2^3}+1)\cdots(2^{2^{k-1}}+1).$$

The factors on the right side are all relatively prime to 2 since they are all odd. They are also Fermat numbers, and another result from number theory states that they are relatively prime. (Suppose that some prime p divided both $2^{2^a}+1$ and $2^{2^b}+1$ for $a<b$. Then $2^{2^a}\equiv 2^{2^b}\equiv -1 \pmod p$. But then $-1\equiv 2^{2^b}=\left(2^{2^a}\right)^{2^{b-a}}\equiv \left((-1)^2\right)^{2^{b-a-1}}\equiv 1 \pmod p$, implying that $p=2$; again, this is impossible.) Therefore by the Chinese Remainder Theorem, there is a positive integer c simultaneously satisfying

$$c\equiv 2^{2^{i-1}} \pmod{2^{2^i}+1} \qquad \text{for all } i=1,2,\ldots,k-1$$

and $c\equiv 0 \pmod 2$. Putting $c=2m$, $4m^2+1$ is a multiple of $\frac{2^n-1}{3}$, as desired.

Problem 4 Suppose that for any real x with $|x|\ne 1$, a function $f(x)$ satisfies

$$f\left(\frac{x-3}{x+1}\right)+f\left(\frac{3+x}{1-x}\right)=x.$$

Find all possible $f(x)$.

Solution: Set $t = \frac{x-3}{x+1}$ so that $x = \frac{3+t}{1-t}$. Then the given equation can be rewritten as

$$f(t) + f\left(\frac{t-3}{t+1}\right) = \frac{3+t}{1-t}.$$

Similarly, set $t = \frac{3+x}{1-x}$ so that $x = \frac{t-3}{t+1}$ and $\frac{x-3}{x+1} = \frac{3+t}{1-t}$. Again we can rewrite the given equation, this time as

$$f\left(\frac{3+t}{1-t}\right) + f(t) = \frac{t-3}{t+1}.$$

Adding these two equations we have

$$\frac{8t}{1-t^2} = 2f(t) + f\left(\frac{t-3}{t+1}\right) + f\left(\frac{3+t}{1-t}\right) = 2f(t) + t,$$

so that

$$f(t) = \frac{4t}{1-t^2} - \frac{t}{2},$$

and some algebra verifies that this solution works.

Problem 5 Consider a permutation $(a_1, a_2, \ldots, a_6)$ of $1, 2, \ldots, 6$ such that the minimum number of transpositions needed to transform $(a_1, a_2, a_3, a_4, a_5, a_6)$ to $(1, 2, 3, 4, 5, 6)$ is four. Find the number of such permutations.

Solution: Given distinct numbers $b_1, b_2, \ldots, b_k$ between 1 and n, in a k-cycle with these numbers b_1 is mapped to one of the other $k-1$ numbers; its image is mapped to one of the $k-2$ remaining numbers; and so on until the remaining number is mapped to b_1. Hence there are $(k-1)(k-2)\cdots(1) = (k-1)!$ cycles of length k involving these numbers.

Any permutation which can be achieved with four transpositions is even, so a permutation satisfying the given conditions must be either (i) the identity permutation, (ii) a composition of two transpositions, (iii) a 3-cycle, (iv) a composition of a 2-cycle and a 4-cycle, (v) a composition of two 3-cycles, or (vi) a 5-cycle. Permutations of type (i), (ii), and (iii) can be attained with fewer transpositions from our observations above. Conversely, any even permutation that can be achieved with zero or two transpositions is of these three types. Hence the permutations described in the problem statement are precisely those of types (iv), (v), and (vi). For type-(iv) permutations, there are $\binom{6}{2} = 15$ ways to assign which cycle each of $1, 2, \ldots, 6$ belongs; and there are $(2-1)!(4-1)! = 6$ ways to rearrange them within the cycles, for a total of $15 \cdot 6 = 90$ permutations. For type-(v) permutations, there are $\frac{1}{2}\binom{6}{3} = 10$ ways to assign which cycle each number belongs to (since $\binom{6}{3}$ counts each such permutation twice, once in the form $(a\,b\,c)(d\,e\,f)$ and again in the form $(d\,e\,f)(a\,b\,c)$). And there are $(3-1)!(3-1)! = 4$ ways to rearrange the numbers within these two cycles for a total of $10 \cdot 4 = 40$ type-(v) permutations. Finally, for type-(v) permutations there are $\binom{6}{5} = 6$ ways to choose which five numbers are cycled, and $(5-1)! = 24$ different cycles among any five numbers. This gives a total of $6 \cdot 24 = 144$ type-(v) permutations, and altogether

$$90 + 40 + 144 = 274$$

permutations which can be attained with four permutations, but no less.

Problem 6 Let $a_1, a_2, \cdots, a_{1999}$ be nonnegative real numbers satisfying the following two conditions:

(a) $a_1 + a_2 + \cdots + a_{1999} = 2$;

(b) $a_1a_2 + a_2a_3 + \cdots + a_{1998}a_{1999} + a_{1999}a_1 = 1$.

Let $S = a_1^2 + a_2^2 + \cdots + a_{1999}^2$. Find the maximum and minimum possible values of S.

Solution: Without loss of generality assume that a_{1999} is the minimum a_i. We may also assume that $a_1 > 0$. From the given equations we have

$$\begin{aligned}
4 &= (a_1 + a_2 + \cdots + a_{1999})^2 \\
&\geq (a_1 + a_2 + \cdots + a_{1999})^2 - (a_1 - a_2 + a_3 - \cdots - a_{1998} + a_{1999})^2 \\
&= 4(a_1 + a_3 + \cdots + a_{1999})(a_2 + a_4 + \cdots + a_{1998}) \\
&\geq 4(a_1a_2 + a_2a_3 + \cdots + a_{1998}a_{1999}) \\
&\qquad +4(a_1a_4 + a_2a_5 + \cdots + a_{1996}a_{1999}) \\
&\qquad +4a_1(a_6 + a_8 + \cdots + a_{1998}) \\
&= 4(1 - a_{1999}a_1) + 4(a_1a_4 + a_2a_5 \cdots + a_{1996}a_{1999}) \\
&\qquad +4a_1(a_6 + a_8 + \cdots + a_{1998})
\end{aligned}$$

$$= 4 + 4(a_1a_4 + a_2a_5 + \cdots + a_{1996}a_{1999})$$
$$+4a_1(a_6 + a_8 + \cdots + a_{1998} - a_{1999})$$
$$\geq 4.$$

Hence equality must hold in the first and third inequality. Thus we must have

(i) $a_1 + a_3 + \cdots + a_{1999} = a_2 + a_4 + \cdots + a_{1998} = 1$

(ii) $a_1a_4 = a_2a_5 = \cdots = a_{1996}a_{1999} = 0$

(iii) $a_6 + a_8 + \cdots + a_{1998} = a_{1999}$.

Condition (ii) implies $a_4 = 0$; from (iii) we get $a_6 = a_8 = \cdots = a_{1998} = 0$. Thus from (i), we have $a_2 = 1$, and from (b), we have $a_1 + a_3 = 1$. Applying these to the first given condition (a), we have

$$a_4 + a_5 + \cdots + a_{1999} = 0,$$

so that $a_4 = a_5 = \cdots = a_{1999} = 0$. Therefore

$$\begin{aligned} S &= a_1^2 + a_2^2 + a_3^2 \\ &= a_1^2 + 1 + (1 - a_1)^2 \qquad \text{since } a_2 = a_1 + a_3 = 1 \\ &= 2(a_1^2 - a_1 + 1) \\ &= 2\left(a_1 - \frac{1}{2}\right)^2 + \frac{3}{2}. \end{aligned}$$

Thus S has maximum value 2 attained when $a_1 = 1$, and minimum value $\frac{3}{2}$ when $a_1 = \frac{1}{2}$.

1.15 Poland

Problem 1 Let D be a point on side BC of triangle ABC such that $AD > BC$. Point E on side AC is defined by the equation

$$\frac{AE}{EC} = \frac{BD}{AD - BC}.$$

Show that $AD > BE$.

First Solution: Fix the points B, C, D and the distance AD, and let A vary; its locus is a circle with center D. From the equation, the ratio $\frac{AE}{EC}$ is fixed; therefore, $\lambda = \frac{EC}{AC}$ is also fixed. Since E is the image of A under a homothety about C with ratio λ, the locus of all points E is the image of the locus of A under this homothety — a circle centered on $\overline{BC}$. Then BE has its unique maximum when E is the intersection of the circle with line BC farther from B. If we show that $AD = BE$ in this case then we are done (the original inequality $AD > BE$ will be strict because equality can only hold in this degenerate case). Indeed, in this case the points B, D, C, E, A are collinear in that order; our equation gives

$$AE \cdot (AC - BD) = AE \cdot (AD - BC) = EC \cdot BD$$

$$\Rightarrow AE \cdot AC = (AE + EC) \cdot BD = AC \cdot BD$$

$$\Rightarrow AE = BD \Rightarrow AD = BE.$$

Second Solution: Let F be the point on $\overline{AD}$ such that $FA = BC$, and let line BF hit side AC at E'. By the law of sines we have $AE' = FA \cdot \frac{\sin \angle AFE'}{\sin \angle FE'A} = CB \cdot \frac{\sin \angle DFB}{\sin \angle CE'F}$ and $E'C = CB \cdot \frac{\sin \angle E'BC}{\sin \angle CE'B} = CB \cdot \frac{\sin \angle FBD}{\sin \angle CE'F}$. Hence $\frac{AE'}{E'C} = \frac{\sin \angle DFB}{\sin \angle FBD} = \frac{DB}{FD} = \frac{BD}{AD - BC} = \frac{AE}{EC}$, and $E' = E$.

Let ℓ be the line passing through A parallel to side BC. Draw G on ray BC such that $BG = AD$ and $CG = FD$; and let lines GE and ℓ intersect at H. Triangles ECG and EAH are similar, so $AH = CG \cdot \frac{AE}{EC} = FD \cdot \frac{AE}{EC}$.

By Menelaus' Theorem applied to triangle CAD and line EFB, we have

$$\frac{CE \cdot AF \cdot DB}{EA \cdot FD \cdot BC} = 1.$$

Thus $AH = FD \cdot \frac{AE}{EC} = FD \cdot \frac{AF \cdot DB}{FD \cdot BC} = DB \cdot \frac{AF}{BC} = DB$, implying that quadrilateral $BDAH$ is a parallelogram and that $BH = AD$. It follows that triangle BHG is isosceles with $BH = BG = AD$; and since $\overline{BE}$ in a cevian in this triangle, we must have $BE < AD$, as desired.

Problem 2 Given are nonnegative integers $a_1 < a_2 < \cdots < a_{101}$ smaller than 5050. Show that one can choose four distinct integers a_k, a_l, a_m, a_n such that

$$5050 \mid (a_k + a_l - a_m - a_n).$$

Solution: First observe that the a_i are all distinct modulo 5050 since they are all between 0 and 5050. Now consider all sums $a_i + a_j, i < j$; there are $\binom{101}{2} = 5050$ such sums. If any two such sums, $a_k + a_l$ and $a_m + a_n$, are congruent mod 5050, we are done. (In this case, all four indices would indeed be distinct: if, for example, $k = m$, then we would also have $l = n$ since all a_i are different mod 5050, but we chose the pairs $\{k, l\}$ and $\{m, n\}$ to be distinct.) The only other possibility is that these sums occupy every possible congruence class mod 5050. Then, adding all such sums gives $100(a_1 + a_2 + \cdots + a_{101}) \equiv 0 + 1 + \cdots + 5049 = 2525 \cdot 5049 \pmod{5050}$. Since the number on the left side is even but $2525 \cdot 5049$ is odd, we get a contradiction.

Problem 3 For a positive integer n, let $S(n)$ denote the sum of its digits. Prove that there exist distinct positive integers $\{n_i\}_{1 \le i \le 50}$ such that

$$n_1 + S(n_1) = n_2 + S(n_2) = \cdots = n_{50} + S(n_{50}).$$

Solution: We show by induction on k that there exist positive integers $n_1, \ldots, n_k$ with the desired property. For $k = 1$ the statement is obvious. For $k > 1$, we have (by induction) numbers $m_1 < \cdots < m_{k-1}$ with the desired property. Note that we can make all m_i arbitrarily large, e.g. by adding some large power of 10 to all of them (which preserves our property). Then, choose m with $1 \le m \le 9$ and $m \equiv m_1 + 1 \pmod 9$; recall that $S(x) \equiv x \pmod 9$. Then we have $m_1 - m + S(m_1) - S(m) + 11 = 9\ell$ for some integer ℓ; by choosing the m_i large enough we can ensure $10^\ell > m_{k-1}$. Now let $n_i = 10^{\ell+1} + m_i$ for $i < k$ and $n_k = m + 10^{\ell+1} - 10$. Now it is obvious that $n_i + S(n_i) = n_j + S(n_j)$ for $i, j < k$, and

$$\begin{aligned} n_1 + S(n_1) &= (10^{l+1} + m_1) + (1 + S(m_1)) \\ &= (m_1 + S(m_1) + 1) + 10^{l+1} \\ &= (9\ell + S(m) + m - 10) + 10^{\ell+1} \\ &= (m + 10^{l+1} - 10) + (9\ell + S(m)) \\ &= n_k + S(n_k), \end{aligned}$$

as needed.

Problem 4 Find all integers $n \geq 2$ for which the system of equations

$$\begin{aligned} x_1^2 + x_2^2 + 50 &= 16x_1 + 12x_2 \\ x_2^2 + x_3^2 + 50 &= 16x_2 + 12x_3 \\ &\cdots\cdots\cdots \\ x_{n-1}^2 + x_n^2 + 50 &= 16x_{n-1} + 12x_n \\ x_n^2 + x_1^2 + 50 &= 16x_n + 12x_1 \end{aligned}$$

has a solution in integers $(x_1, x_2, \dots, x_n)$.

Solution: Answer: $3 \mid n$.

We rewrite the equation $x^2 + y^2 + 50 = 16x + 12y$ as $(x-8)^2 + (y-6)^2 = 50$, whose integer solutions are

$$(7,-1),\ (7,13),\ (9,-1),\ (9,13),\ (3,1),\ (3,11)$$
$$(13,1),\ (13,11),\ (1,5),\ (1,7),\ (15,5),\ (15,7).$$

Thus every pair (x_i, x_{i+1}) (where $x_{n+1} = x_1$) must be one of these. If $3 \mid n$ then just let $x_{3i} = 1, x_{3i+1} = 7, x_{3i+2} = 13$ for each i. Conversely, if a solution exists, consider the pairs (x_i, x_{i+1}) which occur; every pair's first coordinate is the second coordinate of another pair, and vice versa, which reduces the above possibilities to $(1,7), (7,13), (13,1)$. It follows that the x_i must form a repeating sequence $1, 7, 13, 1, 7, 13, \dots$, which is only possible when $3 \mid n$.

Problem 5 Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be integers. Prove that

$$\sum_{1 \leq i < j \leq n} (|a_i - a_j| + |b_i - b_j|) \leq \sum_{1 \leq i,j \leq n} |a_i - b_j|.$$

Solution: Define $f_{\{a,b\}}(x) = 1$ if either $a \leq x < b$ or $b \leq x < a$, and 0 otherwise. Observe that when a, b are integers, $|a-b| = \sum_x f_{\{a,b\}}(x)$ where the sum is over all integers (the sum is valid since only finitely many terms are nonzero). Now suppose $a_\leq$ is the number of values of i for which $a_i \leq x$, and $a_>, b_\leq, b_>$ are defined analogously. We have $(a_\leq - b_\leq) + (a_> - b_>) = (a_\leq + a_>) - (b_\leq + b_>) = n - n = 0 \Rightarrow (a_\leq - b_\leq)(a_> - b_>) \leq 0$. Thus $a_\leq a_> + b_\leq b_> \leq a_\leq b_> + a_> b_\leq$. But $a_\leq a_> = \sum_{i<j} f_{\{a_i,a_j\}}(x)$ since both sides count the same set of pairs, and the other terms reduce similarly, giving

$$\sum_{1 \leq i < j \leq n} f_{\{a_i,a_j\}}(x) + f_{\{b_i,b_j\}}(x) \leq \sum_{1 \leq i,j \leq n} f_{\{a_i,b_j\}}(x).$$

Now summing over all integers x and using our first observation, we get the desired inequality. Equality holds iff the above inequality is an equality for all x, which is true precisely when the a_i equal the b_i in some order.

Problem 6 In a convex hexagon $ABCDEF$, $\angle A + \angle C + \angle E = 360°$ and

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA.$$

Prove that $AB \cdot FD \cdot EC = BF \cdot DE \cdot CA$.

First Solution: Construct point G so that triangle GBC is similar to triangle FBA (and with the same orientation). Then $\angle DCG = 360° - (\angle GCB + \angle BCD) = \angle DEF$ and $\frac{GC}{CD} = \frac{FA \cdot \frac{BC}{AB}}{CD} = \frac{FE}{ED}$, so triangles DCG, DEF are similar.

Now $\frac{AB}{BF} = \frac{CB}{BG}$ by similar triangles, and $\angle ABC = \angle ABF + \angle FBC = \angle CBG + \angle FBC = \angle FBG$; thus $\triangle ABC \sim \triangle FBG$, and likewise $\triangle EDC \sim \triangle FDG$. Then

$$\frac{AB}{CA} \cdot \frac{EC}{DE} \cdot \frac{FD}{BF} = \frac{FB}{GF} \cdot \frac{FG}{DF} \cdot \frac{FD}{BF} = 1$$

as needed.

Second Solution: Invert about F with some radius r. The original equality becomes

$$\frac{A'B' \cdot r^2}{A'F \cdot B'F} \cdot \frac{C'D' \cdot r^2}{C'F \cdot D'F} \cdot \frac{r^2}{E'F} = \frac{B'C' \cdot r^2}{B'F \cdot C'F} \cdot \frac{D'E' \cdot r^2}{D'F \cdot E'F} \cdot \frac{r^2}{A'F}$$

or $\frac{A'B'}{B'C'} = \frac{E'D'}{D'C'}$. The original angle condition is $\angle FAB + \angle BCF + \angle FCD + \angle DEF = 360°$; using directed angles, this turns into $\angle A'B'F + \angle FB'C' + \angle C'D'F + \angle FD'E' = 360°$, or $\angle A'B'C' = \angle E'D'C'$. Thus triangles $A'B'C', E'D'C'$ are similar, giving $\frac{A'B'}{A'C'} = \frac{E'D'}{E'C'}$ or, equivalently,

$$\frac{A'B' \cdot r^2}{A'F \cdot B'F} \cdot \frac{r^2}{D'F} \cdot \frac{E'C' \cdot r^2}{C'F \cdot E'F} = \frac{r^2}{B'F} \cdot \frac{D'E' \cdot r^2}{D'F \cdot E'F} \cdot \frac{C'A' \cdot r^2}{A'F \cdot C'F}.$$

Inverting back, we see that we are done.

Third Solution: Position the hexagon in the complex plane and let $a = B - A, b = C - B, \ldots, f = A - F$. The product identity implies that $|ace| = |bdf|$, and the angle equality implies $\frac{-b}{a} \cdot \frac{-d}{c} \cdot \frac{-f}{e}$ is positive real; hence $ace = -bdf$. Also $a + b + c + d + e + f = 0$; Multiplying this by ad and adding $ace + bdf = 0$ gives

$$a^2d + abd + acd + ad^2 + ade + adf + ace + bdf = 0$$

which factors to $a(d + e)(c + d) + d(a + b)(f + a) = 0$. Thus $|a(d + e)(c + d)| = |d(a + b)(f + a)|$, which is what we wanted.

1.16 Romania

National Olympiad

Problem 7.1 Determine the side lengths of a right triangle if they are integers and the product of the leg lengths is equal to three times the perimeter.

Solution: One of the leg lengths must be divisible by 3; let the legs have lengths $3a$ and b and let the hypotenuse have length c, where a, b, and c are positive integers. From the given condition we have $3ab = 3(3a+b+c)$, or $c = ab - 3a - b$. By the Pythagorean theorem, we have $(3a)^2 + b^2 = c^2 = (ab - 3a - b)^2$, which simplifies to

$$ab[(a-2)(b-6)-6] = 0.$$

Since $a, b > 0$, we have $(a, b) \in \{(3, 12), (4, 9), (5, 8), (8, 7)\}$, and therefore the side lengths of the triangle are either $(9, 12, 15)$, $(8, 15, 17)$, or $(7, 24, 25)$.

Problem 7.2 Let a, b, c be nonzero integers, $a \neq c$, such that

$$\frac{a}{c} = \frac{a^2+b^2}{c^2+b^2}.$$

Prove that $a^2 + b^2 + c^2$ cannot be a prime number.

Solution: Cross-multiplying and factoring, we have $(a-c)(b^2 - ac) = 0$. Since $a \neq c$, we have $ac = b^2$. Now, $a^2 + b^2 + c^2 = a^2 + (2ac - b^2) + c^2 = (a+c)^2 - b^2 = (a+b+c)(a-b+c)$. Also, $|a|, |c|$ cannot both be 1. Then $a^2+b^2+c^2 > |a|+|b|+|c| \geq |a+b+c|, |a-b+c|$, whence $a^2 + b^2 + c^2$ cannot be a prime number.

Problem 7.3 Let $ABCD$ be a convex quadrilateral with $\angle BAC = \angle CAD$ and $\angle ABC = \angle ACD$. Rays AD and BC meet at E and rays AB and DC meet at F. Prove that

(a) $AB \cdot DE = BC \cdot CE$;

(b) $AC^2 < \frac{1}{2}(AD \cdot AF + AB \cdot AE)$.

Solution:

(a) Because $\angle BAC + \angle CBA = \angle ECA$, we have $\angle ECD = \angle BAC$. Then $\triangle CDE \sim \triangle ACE$, and $\frac{CE}{DE} = \frac{AE}{CE}$. But since $\overline{AC}$ is the angle bisector of $\angle A$ in triangle ABE, we also have $\frac{AE}{CE} = \frac{AB}{BC}$. Thus $\frac{CE}{DE} = \frac{AB}{BC}$, whence $AB \cdot DE = BC \cdot CE$.

(b) Note that $\overline{AC}$ is an angle bisector of both triangle ADF and triangle AEB. Thus it is enough to prove that if $\overline{XL}$ is an angle bisector in an arbitrary triangle XYZ, then $XL^2 < XY \cdot XZ$. Let M be the intersection of $\overrightarrow{XL}$ and the circumcircle of triangle XYZ. Because $\triangle XYL \sim \triangle XMZ$, we have $XL^2 < XL \cdot XM = XY \cdot XZ$, as desired.

Problem 7.4 In triangle ABC, D and E lie on sides BC and AB, respectively, F lies on side AC such that $EF \parallel BC$, G lies on side BC such that $EG \parallel AD$. Let M and N be the midpoints of $\overline{AD}$ and $\overline{BC}$, respectively. Prove that

(a) $\dfrac{EF}{BC} + \dfrac{EG}{AD} = 1$;

(b) the midpoint of $\overline{FG}$ lies on line MN.

Solution:

(a) Since $EF \parallel BC$, $\triangle AEF \sim \triangle ABC$ and $\frac{EF}{BC} = \frac{AE}{AB}$. Similarly, since $EG \parallel AD$, $\triangle BEG \sim \triangle BAD$ and $\frac{EG}{AD} = \frac{EB}{AB}$. Hence $\frac{EF}{BC} + \frac{EG}{AD} = 1$.

(b) Let lines AN, EF intersect at point P, and let Q be the point on line BC such that $PQ \parallel AD$. Since $BC \parallel EF$ and N is the midpoint of $\overline{BC}$, P is the midpoint of $\overline{EF}$. Then vector EP equals both vectors PF and GQ, and $PFQG$ is a parallelogram. Thus the midpoint X of $\overline{FG}$ must also be the midpoint of $\overline{PQ}$. But then since M is the midpoint of $\overline{AD}$ and $AD \parallel PQ$, points M, X, N must be collinear.

Problem 8.1 Let $p(x) = 2x^3 - 3x^2 + 2$, and let

$$S = \{p(n) \mid n \in \mathbb{N}, n \leq 1999\},$$
$$T = \{n^2 + 1 \mid n \in \mathbb{N}\},$$
$$U = \{n^2 + 2 \mid n \in \mathbb{N}\}.$$

Prove that $S \cap T$ and $S \cap U$ have the same number of elements.

Solution: Note that $|S \cap T|$ is the number of squares of the form $2n^3 - 3n^2 + 1 = (n-1)^2(2n+1)$ where $n \in \mathbb{N}, n \leq 1999$. And for

$n \le 1999$, $(n-1)^2(2n+1)$ is a square precisely when either $n = 1$ or when $n \in \{\frac{1}{2}(k^2-1) \mid k = 1, 3, 5, \ldots, 63\}$. Thus, $|S \cap T| = 33$.

Next, $|S \cap U|$ is the number of squares of the form $2n^3 - 3n^2 = n^2(2n-3)$ where $n \in \mathbb{N}, n \le 1999$. And for $n \le 1999$, $n^2(2n-3)$ is a square precisely when either $n = 0$ or when $n \in \{\frac{1}{2}(k^2+3) \mid k = 1, 3, 5, \ldots, 63\}$. Thus $|S \cap U| = 33$ as well, and we are done.

Problem 8.2

(a) Let $n \ge 2$ be a positive integer and

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n$$

be positive real numbers such that

$$x_1 + x_2 + \cdots + x_n \ge x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Prove that

$$x_1 + x_2 + \cdots + x_n \le \frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n}.$$

(b) Let a, b, c be positive real numbers such that

$$ab + bc + ca \le 3abc.$$

Prove that

$$a^3 + b^3 + c^3 \ge a + b + c.$$

Solution:

(a) Applying the Cauchy-Schwarz inequality and then the given inequality, we have

$$\left(\sum_{i=1}^{n} x_i\right)^2 \le \sum_{i=1}^{n} x_iy_i \cdot \sum_{i=1}^{n} \frac{x_i}{y_i} \le \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} \frac{x_i}{y_i}.$$

Dividing both sides by $\sum_{i=1}^{n} x_i$ yields the desired inequality.

(b) By the AM-HM inequality on a, b, c we have

$$a + b + c \ge \frac{9abc}{ab + bc + ca} \ge \frac{9abc}{3abc} = 3.$$

Then, since the given condition is equivalent to $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le 3$, we have $a + b + c \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Hence setting $x_1 = a, x_2 = b, x_3 = c$ and $y_1 = \frac{1}{a^2}, y_2 = \frac{1}{b^2}, y_3 = \frac{1}{c^2}$ in the result from part (a) gives $a + b + c \le a^3 + b^3 + c^3$, as desired.

Problem 8.3 Let $ABCDA'B'C'D'$ be a rectangular box, let E and F be the feet of perpendiculars from A to lines $A'D$ and $A'C$ respectively, and let P and Q be the feet of perpendiculars from B' to lines $A'C'$ and $A'C$ respectively. Prove that

(a) planes AEF and $B'PQ$ are parallel;

(b) triangles AEF and $B'PQ$ are similar.

Solution:

(a) Let $(P_1P_2\dots P_k)$ denote the plane containing points $P_1, P_2, \dots, P_k$. First observe that quadrilateral $A'B'CD$ is a parallelogram and thus lies in a single plane.

We are given that $AE \perp A'D$. Also, line AE is contained in plane $(ADD'A)$, which is perpendicular to line CD. Hence $AE \perp CD$ as well, and therefore $AE \perp (A'B'CD)$ and $AE \perp A'C$. And since we know that $A'C \perp AF$, we have $A'C \perp (AEF)$ and $A'C \perp EF$.

Likewise, $B'Q \perp A'C$. And since lines $EF, B'Q$, and $A'C$ all lie in plane $(A'B'CD)$, it follows that $EF \parallel B'Q$. In a similar way we deduce that $AF \parallel PQ$. Hence the planes (AEF) and $(B'PQ)$ are parallel, as desired.

(b) Since $EF \parallel B'Q$ and $FA \parallel QP$, we have $\angle EFA = \angle PQB'$. Furthermore, from above $AE \perp EF$ and likewise $B'P \perp PQ$, implying that $\angle AEF = \angle B'PQ = 90^\circ$ as well. Therefore $\triangle AEF \sim \triangle B'PQ$, as desired.

Problem 8.4 Let $SABC$ be a right pyramid with equilateral base ABC, let O be the center of ABC, and let M be the midpoint of $\overline{BC}$. If $AM = 2SO$ and N is a point on edge SA such that $SA = 25SN$, prove that planes ABP and SBC are perpendicular, where P is the intersection of lines SO and MN.

Solution: Let $AB = BC = CA = s$. Then some quick calculations show that $AO = \frac{\sqrt{3}}{3}s$, $AM = \frac{\sqrt{3}}{2}s$, $AS = \frac{5}{\sqrt{48}}s$, and $AN = \frac{24}{5\sqrt{48}}s$. Then $AO \cdot AM = AN \cdot AS = \frac{1}{2}s^2$, whence $MONS$ is a cyclic quadrilateral. Thus, $\angle MNS = 90^\circ$, and P is the orthocenter of triangle AMS. Let Q be the intersection of lines AP and MS. Note that $\angle AMB = \angle AQM = \angle QMB = 90^\circ$. From repeated applications of the Pythagorean theorem, we have $AB^2 = AM^2 +$

$MB^2 = AQ^2 + QM^2 + MB^2 = AQ^2 + QB^2$, whence $\angle AQB = 90°$. Now $AQ \perp QB$ and $AQ \perp QM$, so line AQ must be perpendicular to plane SBC. Then since plane ABP contains line AQ, planes ABP and SBC must be perpendicular.

Problem 9.1 Let ABC be a triangle with angle bisector $\overline{AD}$. One considers the points M, N on rays AB and AC respectively, such that $\angle MDA = \angle ABC$ and $\angle NDA = \angle BCA$. Lines AD and MN meet at P. Prove that

$$AD^3 = AB \cdot AC \cdot AP.$$

Solution: Since $\triangle ADB \sim \triangle AMD$, $\frac{AD}{AB} = \frac{AM}{AD}$. Also, $\angle MAN + \angle NDM = \pi$, whence $AMDN$ is cyclic. Since $\angle DCA = \angle ADN = \angle AMN$, $\triangle ADC \sim \triangle APM$, and $\frac{AD}{AP} = \frac{AC}{AM}$. Therefore,

$$\frac{AD}{AB}\frac{AD}{AC}\frac{AD}{AP} = \frac{AM}{AD}\frac{AD}{AC}\frac{AC}{AM} = 1.$$

Problem 9.2 For $a, b > 0$, denote by $t(a, b)$ the positive root of the equation

$$(a+b)x^2 - 2(ab-1)x - (a+b) = 0.$$

Let $M = \{(a, b) \mid a \neq b,\ t(a, b) \leq \sqrt{ab}\}$. Determine, for $(a, b) \in M$, the minimum value of $t(a, b)$.

Solution: Consider the polynomial $P(x) = (a+b)x^2 - 2(ab-1)x - (a+b) = 0$. Since $a + b \neq 0$, the product of its roots is $-\frac{a+b}{a+b} = -1$. Hence P must have a unique positive root $t(a, b)$ and a unique negative root. Since the leading coefficient of $P(x)$ is positive, the graph of $P(x)$ is positive for $x > t(a, b)$ and negative for $0 \leq x < t(a, b)$ (since in the latter case, x is between the two roots). Thus, the condition $t(a, b) \leq \sqrt{ab}$ is equivalent to $P(\sqrt{ab}) \geq 0$, or

$$(ab-1)(a+b-2\sqrt{ab}) \geq 0.$$

But $a + b > 2\sqrt{ab}$ by AM-GM, where the inequality is sharp since $a \neq b$. Thus $t(a, b) \leq \sqrt{ab}$ exactly when $ab \geq 1$.

Now using the quadratic formula, we find that

$$t(a, b) = \frac{ab-1}{a+b} + \sqrt{\left(\frac{ab-1}{a+b}\right)^2 + 1}.$$

Thus given $ab \geq 1$, we have $t(a, b) \geq 1$ with equality when $ab = 1$.

Problem 9.3 In the convex quadrilateral $ABCD$ the bisectors of angles A and C meet at I. Prove that there exists a circle inscribed in $ABCD$ if and only if

$$[AIB] + [CID] = [AID] + [BIC].$$

Solution: It is well known that a circle can be inscribed in a convex quadrilateral $ABCD$ if and only if $AB + CD = AD + BC$. The bisector of angle A consists of those points lying inside $\angle BAD$ equidistant from lines AB and AD; similarly, the bisector of angle C consists of those points lying inside $\angle BCD$ equidistant from lines BC and BD.

Suppose $ABCD$ has an incircle. Then its center is equidistant from all four sides of the quadrilateral, so it lies on both bisectors and hence equals I. If we let r denote the radius of the incircle, then we have

$$[AIB] + [CID] = r(AB + CD) = r(AD + BC) = [AID] + [BIC].$$

Conversely, suppose that $[AIB] + [CID] = [AID] + [BIC]$. Let $d(I, \ell)$ denote the distance from I to any line ℓ, and write $x = d(I, AB) = d(I, AD)$ and $y = d(I, BC) = d(I, CD)$. Then

$$[AIB] + [CID] = [AID] + [BIC]$$
$$AB \cdot x + CD \cdot y = AD \cdot x + BC \cdot y$$
$$x\,(AB - AD) = y\,(BC - CD).$$

If $AB = AD$, then $BC = CD$ and it follows that $AB + CD = AD + BC$. Otherwise, suppose that $AB > AD$; then $BC > CD$ as well. Consider the points $A' \in \overline{AB}$ and $C' \in \overline{BC}$ such that $AD = AA'$ and $CD = CC'$. By SAS, we have $\triangle AIA' \cong \triangle AID$ and $\triangle DCI \cong \triangle C'IC$. Hence $IA' = ID = IC'$. Furthermore, subtracting $[AIA'] + [DCI] = [AID] + [C'IC]$ from both sides of our given condition, we have $[A'IB] = [C'IB]$ or $IA' \cdot IB \cdot \sin\angle A'IB = IC' \cdot IB \cdot \sin\angle CIB$. Thus $\angle A'IB = \angle C'IB$, and hence $\triangle A'IB \cong C'IB$ by SAS.

Thus $\angle IBA' = \angle IBC'$, implying that I lies on the angle bisector of $\angle ABC$. Therefore $x = d(I, AB) = d(I, BC) = y$, and the circle centered at I with radius $x = y$ is tangent to all four sides of the quadrilateral.

Problem 9.4

(a) Let $a, b \in \mathbb{R}$, $a < b$. Prove that $a < x < b$ if and only if there exists $0 < \lambda < 1$ such that $x = \lambda a + (1 - \lambda)b$.

(b) The function $f : \mathbb{R} \to \mathbb{R}$ has the property:

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in \mathbb{R}$, $x \neq y$, and all $0 < \lambda < 1$. Prove that one cannot find four points on the function's graph that are the vertices of a parallelogram.

Solution:

(a) No matter what x is, there is a unique value $\lambda = \frac{b-x}{b-a}$ such that $x = \lambda a + (1 - \lambda)b$; and $0 < \frac{b-x}{b-a} < 1 \iff a < x < b$, which proves the claim.

(b) The condition is Jensen's inequality and shows that the function f is strictly convex. Stated geometrically, whenever $x < t < y$ the point $(t, f(t))$ lies strictly below the line joining $(x, f(x))$ and $(y, f(y))$. Suppose there were a parallelogram on the graph of f whose vertices, from left to right, have x-coordinates a, b, d, c. Then either $(b, f(d))$ or $(d, f(d))$ must lie on or above the line joining $(a, f(a))$ and $(c, f(c))$, a contradiction.

Problem 10.1 Find all real numbers x and y satisfying

$$\begin{aligned} \frac{1}{4^x} + \frac{1}{27^y} &= \frac{5}{6} \\ \log_{27} y - \log_4 x &\geq \frac{1}{6} \\ 27^y - 4^x &\leq 1. \end{aligned}$$

Solution: First, for the second equation to make sense we must have $x, y > 0$ and thus $27^y > 1$. Now from the third equation we have

$$\frac{1}{27^y} \geq \frac{1}{4^x + 1},$$

which combined with the first equation gives

$$\frac{1}{4^x} + \frac{1}{4^x + 1} \leq \frac{5}{6},$$

whence $x \geq \frac{1}{2}$. Similarly, the first and third equations also give

$$\frac{5}{6} \leq \frac{1}{27^y - 1} + \frac{1}{27^y},$$

whence $y \leq \frac{1}{3}$. If either $x > \frac{1}{2}$ or $y < \frac{1}{3}$, we would have $\log_{27} y - \log_4 x < \frac{1}{6}$, contradicting the second given equation. Thus, the only solution is $(x, y) = \left(\frac{1}{2}, \frac{1}{3}\right)$, which indeed satisfies all three equations.

Problem 10.2 A plane intersects edges AB, BC, CD, DA of the regular tetrahedron $ABCD$ at points M, N, P, Q, respectively. Prove that

$$MN \cdot NP \cdot PQ \cdot QM \geq AM \cdot BN \cdot CP \cdot DQ.$$

Solution: By the law of cosines in triangle MBN, we have

$$MN^2 = MB^2 + BN^2 - MB \cdot BN \geq MB \cdot BN.$$

Similarly, $NP^2 \geq CN \cdot CP$, $PN^2 \geq DP \cdot DQ$, and $MQ^2 \geq AQ \cdot AM$. Multiplying these inequalities yields

$$(MN \cdot NP \cdot PQ \cdot MQ)^2 \geq$$
$$(BM \cdot CN \cdot DP \cdot AQ) \cdot (AM \cdot BN \cdot CP \cdot DQ).$$

Now the given plane is different from plane (ABC) and (ADC). Thus if it intersects line AC at some point T, then points M, N, T must be collinear—because otherwise, the only plane containing M, N, T would be plane (ABC). Therefore it intersects line AC at most one point T, and by Menelaus' Theorem applied to triangle ABC and line MNT we have

$$\frac{AM \cdot BN \cdot CT}{MB \cdot NC \cdot TA} = 1.$$

Similarly, P, Q, T are collinear and

$$\frac{AQ \cdot DP \cdot CT}{QD \cdot PC \cdot TA} = 1.$$

Equating these two fractions and cross-multiplying, we find that

$$AM \cdot BN \cdot CP \cdot DQ = BM \cdot CN \cdot DP \cdot AQ.$$

This is true even if the plane does not actually intersect line AC: in this case, we must have $MN \parallel AC$ and $PQ \parallel AC$, in which case ratios of similar triangles show that $AM \cdot BN = BM \cdot CN$ and $CP \cdot DQ = DP \cdot AQ$.

Combining this last equality with the inequality from the first paragraph, we find that

$$(MN \cdot NP \cdot PQ \cdot QM)^2 \geq (AM \cdot BN \cdot CP \cdot DQ)^2,$$

which implies the desired result.

Problem 10.3 Let a, b, c ($a \neq 0$) be complex numbers. Let z_1 and z_2 be the roots of the equation $az^2 + bz + c = 0$, and let w_1 and w_2 be the roots of the equation

$$(a + \overline{c})z^2 + (b + \overline{b})z + (\overline{a} + c) = 0.$$

Prove that if $|z_1|, |z_2| < 1$, then $|w_1| = |w_2| = 1$.

Solution: We begin by proving that $\operatorname{Re}(b)^2 \leq |a+\overline{c}|^2$. If $z_1 = z_2 = 0$, then $b = 0$ and the claim is obvious. Otherwise, write $a = m + ni$ and $c = r + si$; and write $z_1 = x + yi$ where $t = |z_1| = \sqrt{x^2 + y^2} < 1$. Also note that

$$r^2 + s^2 = |c|^2 = |az_1z_2|^2 < |a|^2|z_1|^2 = (m^2 + n^2)t^2. \qquad (1)$$

Assume WLOG that $z_1 \neq 0$. Then $|\operatorname{Re}(b)| = |\operatorname{Re}(-b)| = |\operatorname{Re}(az_1 + c/z_1)| = |\operatorname{Re}(az_1) + \operatorname{Re}(c/z_1)|$; that is,

$$\begin{aligned} |\operatorname{Re}(b)| &= |(mx - ny) + (rx + sy)/t^2| \\ &= |x(m + r/t^2) + y(s/t^2 - n)| \\ &\leq \sqrt{x^2 + y^2}\sqrt{(m + r/t^2)^2 + (s/t^2 - n)^2} \\ &= t\sqrt{(m + r/t^2)^2 + (s/t^2 - n)^2}, \end{aligned}$$

where the inequality follows from Cauchy-Schwarz. Proving our claim then reduces to showing that

$$\begin{aligned} &t^2\left((m + r/t^2)^2 + (s/t^2 - n)^2\right) \leq (m + r)^2 + (n - s)^2 \\ &\iff (mt^2 + r)^2 + (st^2 - n)^2 \leq t^2\left((m + r)^2 + (n - s)^2\right) \\ &\iff (r^2 + s^2)(1 - t^2) < (m^2 + n^2)(t^4 - t^2) \\ &\iff (1 - t^2)\left((m^2 + n^2)t^2 - (r^2 + s^2)\right). \end{aligned}$$

But $1-t^2>0$ by assumption, and $(m^2+n^2)t^2-(r^2+s^2)>0$ from (1); therefore our claim is true.

Now since $|c/a|=|z_1z_2|<1$, we have $|c|<|a|$ and $a+\overline{c}\neq 0$. Then by the quadratic equation, the roots to $(a+\overline{c})z^2+(b+\overline{b})z+(\overline{a}+c)=0$ are given by

$$\frac{-(b+\overline{b})\pm\sqrt{(b+\overline{b})^2-4(a+\overline{c})(\overline{a}+c)}}{2(a+\overline{c})},$$

or (dividing the numerator and denominator by 2)

$$\frac{-\operatorname{Re}(b)\pm\sqrt{\operatorname{Re}(b)^2-|a+\overline{c}|^2}}{a+\overline{c}}=\frac{-\operatorname{Re}(b)\pm i\sqrt{|a+\overline{c}|^2-\operatorname{Re}(b)^2}}{a+\overline{c}}.$$

When evaluating either root, the absolute value of the numerator is $\sqrt{\operatorname{Re}(b)^2+(|a+\overline{c}|^2-\operatorname{Re}(b)^2)}=|a+\overline{c}|$; and the absolute value of the denominator is clearly $|a+\overline{c}|$ as well. Therefore indeed $|w_1|=|w_2|=1$, as desired.

Problem 10.4

(a) Let $x_1,y_1,x_2,y_2,\ldots,x_n,y_n$ be positive real numbers such that

(i) $x_1y_1<x_2y_2<\cdots<x_ny_n$;

(ii) $x_1+x_2+\cdots+x_k\geq y_1+y_2+\cdots+y_k$ for all $k=1,2,\ldots,n$.

Prove that

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\leq\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_n}.$$

(b) Let $A=\{a_1,a_2,\ldots,a_n\}\subset\mathbb{N}$ be a set such that for all distinct subsets $B,C\subseteq A$, $\displaystyle\sum_{x\in B}x\neq\sum_{x\in C}x$. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<2.$$

Solution:

(a) Let $\pi_i=\frac{1}{x_iy_i}$, $\delta_i=x_i-y_i$ for all $1\leq i\leq n$. We are given that $\pi_1>\pi_2>\cdots>\pi_n>0$ and that $\sum_{i=1}^k\delta_i\geq 0$ for all $1\leq k\leq n$. Note that

$$\sum_{k=1}^n\left(\frac{1}{y_k}-\frac{1}{x_k}\right)=\sum_{k=1}^n\pi_k\delta_k$$

$$= \pi_n \sum_{i=1}^{n} \delta_i + \sum_{k=1}^{n-1} (\pi_k - \pi_{k+1}) (\delta_1 + \delta_2 + \cdots + \delta_k) \geq 0,$$

as desired.

(b) Assume without loss of generality that $a_1 < a_2 < \cdots < a_n$, and let $y_i = 2^{i-1}$ for all i. Clearly,

$$a_1 y_1 < a_2 y_2 < \cdots < a_n y_n.$$

For any k, the $2^k - 1$ sums made by choosing at least one of the numbers $a_1, a_2, \ldots, a_k$ are all distinct. Hence the largest of them, $\sum_{i=1}^{k} a_i$, must be at least $2^k - 1$. Thus for all $k = 1, 2, \ldots, n$ we have

$$a_1 + a_2 + \cdots + a_k \geq 2^k - 1 = y_1 + y_2 + \cdots + y_k.$$

Then by part (a), we must have

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n} = 2 - \frac{1}{2^{n-1}} < 2,$$

as desired.

IMO Selection Tests

Problem 1

(a) Show that out of any 39 consecutive positive integers, it is possible to choose one number with the sum of its digits divisible by 11.

(b) Find the first 38 consecutive positive integers, none with the sum of its digits divisible by 11.

Solution: Call an integer "deadly" if its sum of digits is divisible by 11, and let $d(n)$ equal the sum of the digits of a positive integer n.

If n ends in a 0, then the numbers $n, n+1, \ldots, n+9$ differ only in their units digits, which range from 0 to 9; hence $d(n), d(n+1), \ldots, d(n+9)$ is an arithmetic progression with common difference 1. Thus if $d(n) \not\equiv 1 \pmod{11}$, then one of these numbers is deadly.

Next suppose that if n ends in $k \geq 0$ nines. Then $d(n+1) = d(n) + 1 - 9k$: the last k digits of $n+1$ are 0's instead of 9's, and the next digit to the left is 1 greater than the corresponding digit in n.

Finally, suppose that n ends in a 0 and that $d(n) \equiv d(n+10) \equiv 1 \pmod{11}$. Since $d(n) \equiv 1 \pmod{11}$, we must have $d(n+9) \equiv$

$10 \pmod{11}$. If $n+9$ ends in k 9's, then we have $2 \equiv d(n+10) - d(n+9) \equiv 1 - 9k \Longrightarrow k \equiv 6 \pmod{11}$.

(a) Suppose we had 39 consecutive integers, none of them deadly. One of the first ten must end in a 0: call it n. Since none of $n, n+1, \ldots, n+9$ are deadly, we must have $d(n) \equiv 1 \pmod{11}$. Similarly, $d(n+10) \equiv 1 \pmod{11}$ and $d(n+20) \equiv 1 \pmod{11}$. From our third observation above, this implies that both $n+9$ and $n+19$ must end in at least six 9's. But this is impossible, because $n+10$ and $n+20$ can't both be multiples of one million!

(b) Suppose we have 38 consecutive numbers $N, N+1, \ldots, N+37$, none of which is deadly. By an analysis similar to that in part (a), none of the first nine can end in a 0; hence, $N+9$ must end in a 0, as must $N+19$ and $N+29$. Then we must have $d(N+9) \equiv d(N+19) \equiv 1 \pmod{11}$. Therefore $d(N+18) \equiv 10 \pmod{11}$; and furthermore, if $N+18$ ends in k 9's we must have $k \equiv 6 \pmod{11}$. The smallest possible such number is 999999, yielding the 38 consecutive numbers $999981, 999982, \ldots, 1000018$. And indeed, none of these numbers is deadly: their sums of digits are congruent to $1, 2, \ldots, 10, 1, 2, \ldots, 10, 1, 2, \ldots, 10, 2, 3, \ldots, 9$, and $10 \pmod{11}$, respectively.

Problem 2 Let ABC be an acute triangle with angle bisectors $\overline{BL}$ and $\overline{CM}$. Prove that $\angle A = 60^\circ$ if and only if there exists a point K on $\overline{BC}$ ($K \neq B, C$) such that triangle KLM is equilateral.

Solution: Let I be the intersection of lines BL and CM. Then $\angle BIC = 180^\circ - \angle ICB - \angle CBI = 180^\circ - \frac{1}{2}(\angle C + \angle B) = 180^\circ - \frac{1}{2}(180^\circ - \angle A) = 90^\circ + \angle A$, and thus $\angle BIC = 120^\circ$ if and only if $\angle A = 60^\circ$.

For the "only if" direction, suppose that $\angle A = 60^\circ$. Then let K be the intersection of $\overline{BC}$ and the internal angle bisector of $\angle BIC$; we claim that triangle KLM is equilateral. Since $\angle BIC = 120^\circ$, we know that $\angle MIB = \angle KIB = 60^\circ$. And since $\angle IBM = \angle IBK$ and $IB = IB$, by ASA congruency we have $\triangle IBM \cong \triangle IBK$; in particular, $IM = IK$. Similarly, $IL = IK$; and since $\angle KIL = \angle LIM = \angle MIK = 120^\circ$, we know that triangle KLM is equilateral.

For the "if" direction, suppose that K is on $\overline{BC}$ and triangle KLM is equilateral. Consider triangles BLK and BLM: $BL = BL$, $LM = LK$, and $\angle MBL = \angle KBL$. There is no SSA congruency, but we do then know that either $\angle LKB + \angle BML = 180°$ or $\angle LKB = \angle BML$. But since $\angle KBM < 90°$ and $\angle MLK = 60°$, we know that $\angle LKB + \angle BML > 210°$. Thus $\angle LKB = \angle BML$, whence $\triangle BLK \cong \triangle BLM$, and $BK = BM$. It follows that $IK = IM$. Similarly, $IL = IK$, and I is the circumcenter of triangle KLM. Thus $\angle LIM = 2\angle LKM = 120°$, giving $\angle BIC = \angle LIM = 120°$ and $\angle A = 60°$.

Problem 3 Show that for any positive integer n, the number

$$S_n = \binom{2n+1}{0} \cdot 2^{2n} + \binom{2n+1}{2} \cdot 2^{2n-2} \cdot 3 + \cdots + \binom{2n+1}{2n} \cdot 3^n$$

is the sum of two consecutive perfect squares.

Solution: Let $\alpha = 1+\sqrt{3}$, $\beta = 1-\sqrt{3}$, and $T_n = \frac{1}{2}\left(\alpha^{2n+1} + \beta^{2n+1}\right)$. Note that $\alpha\beta = -2$, $\frac{\alpha^2}{2} = 2 + \sqrt{3}$, and $\frac{\beta^2}{2} = 2 - \sqrt{3}$. Also, applying the binomial expansion to $(1+\sqrt{3})^n$ and $(1-\sqrt{3})^n$, we find that $T_n = \sum_{k=0}^{n} \binom{2n+1}{2k} 3^k$—which is an integer for all n.

Applying the binomial expansion to $(2+\sqrt{3})^{2n+1}$ and $(2-\sqrt{3})^{2n+1}$ instead, we find that

$$\begin{aligned} S_n &= \frac{\left(\frac{\alpha^2}{2}\right)^{2n+1} + \left(\frac{\beta^2}{2}\right)^{2n+1}}{4} \\ &= \frac{\alpha^{4n+2} + \beta^{4n+2}}{2^{2n+3}} \\ &= \frac{\alpha^{4n+2} + 2(\alpha\beta)^{2n+1} + \beta^{4n+2}}{2^{2n+3}} + \frac{1}{2} \\ &= \frac{\left(\alpha^{2n+1} + \beta^{2n+1}\right)^2}{2^{2n+3}} + \frac{1}{2} \\ &= \frac{T_n^2}{2^{2n+1}} + \frac{1}{2}. \end{aligned}$$

Thus $2^{2n+1}S_n = T_n^2 + 2^{2n}$. Then $2^{2n} \mid T_n^2$ but $2^{2n+1} \nmid T_n^2$, and hence $T_n \equiv 2^n \pmod{2^{n+1}}$. Therefore

$$S_n = \frac{T_n^2}{2^{2n+1}} + \frac{1}{2} = \left(\frac{T_n - 2^n}{2^{n+1}}\right)^2 + \left(\frac{T_n + 2^n}{2^{n+1}}\right)^2$$

is indeed the sum of two consecutive perfect squares.

Problem 4 Show that for all positive real numbers $x_1, x_2, \cdots, x_n$ such that

$$x_1 x_2 \cdots x_n = 1,$$

the following inequality holds:

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1.$$

First Solution: Let $a_1 = \sqrt[n]{x_1}$, $a_2 = \sqrt[n]{x_2}$, $\ldots$, $a_n = \sqrt[n]{x_n}$. Then $a_1 a_2 \cdots a_n = 1$ and

$$\frac{1}{n-1+x_k} = \frac{1}{n-1+a_k^n} = \frac{1}{n-1+\frac{a_k^{n-1}}{a_1\cdots a_{k-1}a_{k+1}\cdots a_n}}$$

$$\leq \frac{1}{n-1+\frac{(n-1)a_k^{n-1}}{a_1^{n-1}+\cdots+a_{k-1}^{n-1}+a_{k+1}^{n-1}+\cdots+a_n^{n-1}}}$$

by the AM-GM Inequality. It follows that

$$\frac{1}{n-1+x_k} \leq \frac{a_1^{n-1}+\cdots+a_{k-1}^{n-1}+a_{k+1}^{n-1}+\cdots+a_n^{n-1}}{(n-1)(a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1})}.$$

Summing up yields $\sum_{k=1}^{n} \frac{1}{n-1+x_k} \leq 1$, as desired.

Second Solution: Let $f(x) = \frac{1}{n-1-x}$; we wish to prove that $\sum_{i=1}^{n} f(x_i) \leq 1$. Note that

$$f(y) + f(z) = \frac{2(n-1)+y+z}{(n-1)^2+yz+(y+z)(n-1)}.$$

Suppose that any of our x_i does not equal 1; then we have $x_j < 1 < x_k$ for some j, k. If $f(x_j) + f(x_k) \leq \frac{1}{n-1}$, then all the other $f(x_i)$ are less than $\frac{1}{n-1}$. But then $\sum_{i=1}^{n} f(x_i) < 1$ and we are done.

Otherwise, $f(x_j) + f(x_k) > \frac{1}{n-1}$. Now set $x_j' = 1$ and $x_k' = x_j x_k$; then $x_j' x_k' = x_j x_k$, while $x_j < 1 < x_k \Rightarrow (1-x_j)(x_k-1) > 0 \Rightarrow x_j + x_k > x_j' + x_k'$. Let $a = 2(n-1)$, $b = (n-1)^2 + x_j x_k = (n-1)^2 + x_j' x_k'$, and $c = \frac{1}{n-1}$; also let $m = x_j + x_k$ and $m' = x_j' + x_k'$. Then we have

$$f(x_j) + f(x_k) = \frac{a+cm}{b+m} \quad \text{and} \quad f(x_j') + f(x_k') = \frac{a+cm'}{b+m'}.$$

Now $\frac{a+cm}{b+m} > c \Rightarrow a + cm > (b+m)c \Rightarrow \frac{a}{b} > c$; and from here,

$$(a-bc)(m-m') > 0 \Rightarrow \frac{a+cm'}{b+m'} > \frac{a+cm}{b+m}$$

$$\Rightarrow f(x'_j) + f(x'_k) = f(x_j) + f(x_k).$$

Hence as long as no pair $f(x_j) + f(x_k) \leq \frac{1}{n-1}$ and the x_i do not all equal 1, we can continually replace pairs x_j and x_k (neither equal to 1) by 1 and $x_j x_k$. This keeps the product $x_1 x_2 \cdots x_n$ equal to 1 while increasing $\sum_{i=1}^n f(x_i)$. Then eventually our new $\sum_{i=1}^n f(x_i) \leq 1$, which implies that our original $\sum_{i=1}^n f(x_i)$ was also at most 1. This completes the proof.

Third Solution: Suppose, for the sake of contradiction, that $\dfrac{1}{n-1+x_1} + \dfrac{1}{n-1+x_2} + \cdots + \dfrac{1}{n-1+x_n} > 1$. Letting $y_i = x_i/(n-1)$ for $i = 1, 2, \ldots, n$, we have

$$\frac{1}{1+y_1} + \frac{1}{1+y_2} + \cdots + \frac{1}{1+y_n} > n-1$$

and hence

$$\begin{aligned}\frac{1}{1+y_1} &> \left(1 - \frac{1}{1+y_2}\right) + \left(1 - \frac{1}{1+y_3}\right) + \cdots + \left(1 - \frac{1}{1+y_n}\right) \\ &= \frac{y_2}{1+y_2} + \frac{y_3}{1+y_3} + \cdots + \frac{y_n}{1+y_n} \\ &> (n-1) \sqrt[n-1]{\frac{y_2 y_3 \cdots y_n}{(1+y_2)(1+y_3)\cdots(1+y_n)}}.\end{aligned}$$

We have analagous inequalities with $\frac{1}{1+y_2}, \frac{1}{1+y_n}, \ldots, \frac{1}{1+y_n}$ on the left hand side; multiplying these n inequalities together gives

$$\begin{aligned}&\prod_{k=1}^n \frac{1}{1+y_k} > (n-1)^n \frac{y_1 y_2 \cdots y_n}{(1+y_1)(1+y_2)\cdots(1+y_n)} \\ &1 > ((n-1)y_1)((n-1)y_2)\cdots((n-1)y_n) = x_1 x_2 \cdots x_n,\end{aligned}$$

a contradiction.

Problem 5 Let $x_1, x_2, \ldots, x_n$ be distinct positive integers. Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq \frac{(2n+1)(x_1 + x_2 + \cdots + x_n)}{3}.$$

Solution: Assume without loss of generality that $x_1 < x_2 < \cdots < x_n$. We will prove that $3x_k^2 \geq 2(x_1 + x_2 + \cdots + x_{k-1}) + (2k+1)x_k$; then, summing this inequality over $k = 1, 2, \ldots, n$, we will have the desired inequality.

First, $x_1 + x_2 + \cdots + x_{k-1} \le (x_k - (k-1)) + (x_k - (k-2)) + \cdots + (x_k - 1) = (k-1)x_k - \frac{k(k-1)}{2}$. Thus,

$$2(x_1 + x_2 + \cdots + x_{k-1}) + (2k+1)x_k \le (4k-1)x_k - k(k-1).$$

Now

$$3x_k^2 - [(4k-1)x_k - k(k-1)] = x_k(3x_k - 4k + 1) + k(k-1),$$

which is minimized at $x_k = \frac{2}{3}k$. Then since $x_k \ge k$,

$$x_k(3x_k - 4k + 1) + k(k-1) \ge k(3k - 4k + 1) + k(k-1) = 0$$

so

$$3x_k^2 \ge (4k-1)x_k - k(k-1) \ge 2(x_1 + x_2 + \cdots + x_{k-1}) + (2k+1)x_k,$$

and we have finished.

Problem 6 Prove that for any integer n, $n \ge 3$, there exist n positive integers $a_1, a_2, \ldots, a_n$ in arithmetic progression, and n positive integers $b_1, b_2, \ldots, b_n$ in geometric progression, such that

$$b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n.$$

Give one example of such progressions $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ each having at least 5 terms.

Solution: Our strategy is to find progressions where $b_n = a_{n-1} + 1$ and $b_{n-1} = a_{n-2} + 1$. Write $d = a_{n-1} - a_{n-2}$. Then for all $2 \le i, j \le n-1$ we have $b_{i+1} - b_i \le b_n - b_{n-1} = d$, so that $b_j = b_n + \sum_{i=j}^{n-1}(b_i - b_{i+1}) > a_{n-1} + (n-j)d = a_{j-1}$.

And if we ensure that $b_1 < a_1$, then $b_j = b_1 + \sum_{i=1}^{j-1}(b_{i+1} - b_i) \le a_1 + (j-1)d = a_j$ for all j, so the chain of inequalities is satisfied.

Let $b_1, b_2, \ldots, b_n$ equal $k^{n-1}, k^{n-2}(k+1), \ldots, k^0(k+1)^{n-1}$, where k is a value to be determined later. Also set $a_{n-1} = b_n - 1$ and $a_{n-2} = b_{n-1} - 1$, and define the other a_i accordingly. Then $d = a_n - a_{n-1} = b_n - b_{n-1} = (k+1)^{n-2}$, and $a_1 = (k+1)^{n-2}(k+3-n) - 1$. Thus, we need only pick k such that

$$(k+1)^{n-2}(k+3-n) - 1 - k^{n-1} > 0.$$

Viewing the left hand side as a polynomial in k, the coefficient of k^{n-1} is zero but the coefficient of k^{n-2} is 1. Therefore, it is positive for sufficiently large k and we can indeed find satisfactory sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$.

For $n = 5$, we seek k such that

$$(k+1)^3(k-2) - 1 - k^4 > 0.$$

Computation shows that $k = 5$ works, yielding

$$625 < 647 < 750 < 863 < 900 < 1079 < 1080 < 1295 < 1296 < 1511.$$

Problem 7 Let a be a positive real number and $\{x_n\}$ $(n \geq 1)$ be a sequence of real numbers such that $x_1 = a$ and

$$x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} kx_k,$$

for all $n \geq 1$. Show that there exists a positive integer n such that $x_n > 1999!$.

Solution: We will prove by induction on $n \geq 1$ that

$$x_{n+1} > \sum_{k=1}^{n} kx_k > a \cdot n!.$$

For $n = 1$, we have $x_2 \geq 3x_1 > x_1 = a$.

Now suppose that the claim holds for all values up through n. Then

$$\begin{aligned} x_{n+2} &\geq (n+3)x_{n+1} - \sum_{k=1}^{n} kx_k \\ &= (n+1)x_{n+1} + 2x_{n+1} - \sum_{k=1}^{n} kx_k \\ &> (n+1)x_{n+1} + 2\sum_{k=1}^{n} kx_k - \sum_{k=1}^{n} kx_k = \sum_{k=1}^{n+1} kx_k, \end{aligned}$$

as desired. Furthermore, $x_1 > 0$ by definition and $x_2, x_3, \ldots, x_n$ are also positive by the induction hypothesis; thus $x_{n+2} > (n+1)x_{n+1} > (n+1)(a \cdot n!) = a \cdot (n+1)!$. This completes the inductive step.

Therefore for sufficiently large n, we have $x_{n+1} > n! \cdot a > 1999!$.

Problem 8 Let O, A, B, C be variable points in the plane such that $OA = 4, OB = 2\sqrt{3}$ and $OC = \sqrt{22}$. Find the maximum possible area of triangle ABC.

Solution: We first look for a tetrahedron $MNPQ$ with the following properties: (i) if H is the foot of the perpendicular from M to plane (NPQ), then $HN = 4$, $HP = 2\sqrt{3}$, and $HQ = \sqrt{22}$; and (ii) lines MN, MP, MQ are pairwise perpendicular.

If such a tetrahedron exists, then let $O = H$ and draw triangle ABC in plane (NPQ). We have $MA = \sqrt{MO^2 + OA^2} = \sqrt{MH^2 + HN^2} = MN$, and similarly $MB = MP$ and $MC = MQ$. Hence

$$[ABCM] \leq \frac{1}{3}[ABM] \cdot MC \leq \frac{1}{3} \cdot \left(\frac{1}{2}MA \cdot MB\right) \cdot MC$$
$$= \frac{1}{3} \cdot \left(\frac{1}{2}MN \cdot MP\right) \cdot MQ = [MNPQ],$$

and therefore the maximum possible area of triangle $[ABC]$ is $[NPQ]$.

It remains to find tetrahedron $MNPQ$. Let $x = MH$; then $MN = \sqrt{x^2+16}$, $MP = \sqrt{x^2+12}$, and $MQ = \sqrt{x^2+22}$. By the Pythagorean Theorem on triangle MHN, we have $NH = 4$. Next let lines NH and PQ intersect at R; then in similar right triangles MHN and MRN, we have $MR = MH \cdot \frac{MN}{NH} = \frac{1}{4}(x^2+16)$.

Since $MN \perp (MPQ)$ we have $MN \perp PQ$; and since $MH \perp (NPQ)$ we have $MH \perp PQ$ as well. Hence $PQ \perp (MNHR)$, so that $\overline{MR}$ is an altitude in the right triangle MPQ. Therefore $MR \cdot PQ = 2[MPQ] = MP \cdot MQ$, or (after squaring both sides)

$$\sqrt{\frac{(x^2+16)^2}{16} - (x^2+16)}\sqrt{x^2+12+x^2+22} = \sqrt{x^2+12}\sqrt{x^2+22}.$$

Setting $4y = x^2 + 16$ and squaring both sides, we obtain

$$(y^2 - 4y)(8y+2) = (4y-4)(4y+6)(y-6)(4y^2+y-2) = 0.$$

Since $y = \frac{1}{4}(x^2+16) > 4$, the only solution is $y = 6 \Longrightarrow x = \sqrt{8}$. Then by taking $MN = \sqrt{24}, MP = \sqrt{20}$, $MQ = \sqrt{30}$, we get the required tetrahedron.

Then $[MNPQ]$ equals both $\frac{1}{3}MH \cdot [MPQ]$ and $\frac{1}{6}MN \cdot MP \cdot MQ$. Setting these two expressions equal, we find that the maximum area of $[ABC]$ is

$$[NPQ] = \frac{MN \cdot MP \cdot MQ}{2 \cdot MH} = 15\sqrt{2}.$$

Problem 9 Let a, n be integers and let p be a prime such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be represented as a product of two nonconstant polynomials with integer coefficients.

Solution: Let z be a complex root of the polynomial. We shall prove that $|z| > 1$. Suppose $|z| \le 1$. Then, $z^n + az = -p$, we deduce that

$$p = |z^n + az| = |z||z^{n-1} + a| \le |z^{n-1}| + |a| \le 1 + |a|,$$

which contradicts the hypothesis.

Now, suppose $f = gh$ is a decomposition of f into nonconstant polynomials with integer coefficients. Then $p = f(0) = g(0)h(0)$, and either $|g(0)| = 1$ or $|h(0)| = 1$. Assume without loss of generality that $|g(0)| = 1$. If $z_1, z_2, \ldots, z_k$ are the roots of g then they are also roots of f. Therefore

$$1 = |g(0)| = |z_1 z_2 \cdots z_k| = |z_1||z_2| \cdots |z_k| > 1,$$

a contradiction.

Problem 10 Two circles meet at A and B. Line ℓ passes through A and meets the circles again at C and D respectively. Let M and N be the midpoints of arcs $\widehat{BC}$ and $\widehat{BD}$, which do not contain A, and let K be the midpoint of $\overline{CD}$. Prove that $\angle MKN = 90^\circ$.

Solution: All angles are directed modulo 180°. Let M' be the reflection of M across K. Then triangles MKC and $M'KD$ are congruent in that order, and $M'D = MC$. Because M is the midpoint of $\widehat{BC}$, we have $M'D = MC = MB$; and similarly, because N is the midpoint of $\widehat{BD}$ we have $BN = DN$. Next, $\angle MBN = (180^\circ - \angle ABM) + (180^\circ - \angle NBA) = \angle MCA + \angle ADN = \angle M'DA + \angle ADN = \angle M'DN$. Hence $\triangle M'DN \cong \triangle MBN$, and $MN = M'N$. Therefore $\overline{NK}$ is the median to the base of isosceles triangle MNM', so it is also an altitude and $NK \perp MK$.

Problem 11 Let $n \ge 3$ and $A_1, A_2, \ldots, A_n$ be points on a circle. Find the greatest number of acute triangles having vertices in these points.

Solution: Without loss of generality assume the points $A_1, A_2, \ldots, A_n$ are ordered in that order counterclockwise; also, take indices modulo n os that $A_{n+1} = A_1, A_{n+2} = A_2$, and so on. Denote by $A_i A_j$

the arc of the circle starting from A_i and ending in A_j in the counterclockwise direction; let $m(A_iA_j)$ denote the angle measure of the arc; and call an arc A_iA_j obtuse if $m(A_iA_j) \geq 180°$. Obviously, $m(A_iA_j) + m(A_jA_i) = 360°$, and thus at least one of the arcs A_iA_j and A_jA_i is obtuse. Let x_s be the number of obtuse arcs each having exactly $s-1$ points along their interiors. If $s \neq \frac{n}{2}$, then for each i at least one of the arcs A_iA_{i+s} or $A_{i+s}A_i$ is obtuse; summing over all i, we deduce that

$$x_s + x_{n-s} \geq n \tag{1}$$

for every $s \neq \frac{n}{2}$; and similar reasoning shows that this inequality also holds even when $s = \frac{n}{2}$. For all s, equality holds if and only if there are no diametrically opposite points A_i, A_{i+s}.

The number of non-acute triangles $A_iA_jA_k$ equals the number of non-acute angles $\angle A_iA_jA_k$. And for each obtuse arc A_iA_k containing $s-1$ points in its interior, there are $n-s-1$ non-acute angles $A_iA_jA_k$: namely, with those A_j in the interior of arc A_kA_i. It follows that the number N of non-acute triangles is

$$N = x_1(n-2) + x_2(n-3) + \ldots + x_{n-3} \cdot 2 + x_{n-2} \cdot 1 + x_{n-1} \cdot 0.$$

By regrouping terms and using (1) we obtain

$$\begin{aligned} N &\geq \sum_{s=1}^{\frac{n-1}{2}} (s-1) \cdot (x_{n-s} + x_s) \\ &\geq n\left(1 + 2 + \cdots + \frac{n-3}{2}\right) = \frac{n(n-1)(n-3)}{8} \end{aligned}$$

if n is odd, and

$$\begin{aligned} N &\geq \sum_{s=1}^{\frac{n-2}{2}} (s-1) \cdot (x_{n-s} + x_s) + \frac{n-2}{2} x_{n/2} \\ &\geq n\left(1 + 2 + \cdots + \frac{n-4}{2}\right) + \frac{n-2}{2} \cdot \frac{n}{2} = \frac{n(n-2)^2}{8} \end{aligned}$$

if n is even.

Equality is obtained when there are no diametrically opposite points, and when $x_k = 0$ for $k < \frac{n}{2}$. When n is odd, for instance, this happens when the points form a regular n-gon; and when n is even, equality occurs when $m(A_1A_2) = m(A_2A_3) = \cdots = m(A_{n-1}A_n) = \frac{360°}{n} + \epsilon$ where $0 < \epsilon < \frac{360°}{n^2}$.

Finally, note that the total number of triangles having vertices in the n points is $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$. Subtracting the minimum values of N found above, we find that the maximum number of acute angles is $\frac{(n-1)n(n+1)}{24}$ if n is odd, and $\frac{(n-2)n(n+2)}{24}$ if n is even.

Problem 12 The scientists at an international conference are either *native* or *foreign*. Each native scientist sends exactly one message to a foreign scientist and each foreign scientist sends exactly one message to a native scientist, although at least one native scientist does not receive a message. Prove that there exists a set S of native scientists and a set T of foreign scientists such that the following conditions hold: (i) the scientists in S sent messages to exactly those foreign scientists who were not in T (that is, every foreign scientist not in T received at least one message from somebody in S, but none of the scientists in T received any messages from scientists in S); and (ii) the scientists in T sent messages to exactly those native scientists not in S.

Solution: Let A be the set of native scientists and B be the set of foreign scientists. Let $f\colon A \to B$ and $g\colon B \to A$ be the functions defined as follows: $f(a)$ is the foreign scientist receiving a message from a, and $g(b)$ is the native scientist receiving a message from b. If such subsets S, T exist we must have $T = B - f(S)$; hence we have to prove that there exists a subset $S \subseteq A$ such that $A - S = g(B - f(S))$.

For each subset $X \subseteq A$, let $h(X) = A - g(B - f(X))$. If $X \subseteq Y$, then $f(X) \subseteq f(Y) \Longrightarrow B - f(Y) \subseteq B - f(X) \Longrightarrow g(B - f(Y)) \subseteq g(B - f(X)) \Longrightarrow A - g(B - f(X)) \subseteq A - g(B - f(Y)) \Longrightarrow h(X) \subseteq h(Y)$.

Let $M = \{X \subseteq A \mid h(X) \subseteq X\}$. The set M is nonempty, since $A \in M$. Furthermore, it is given that g is not surjective, so that some native scientist a_0 is never in $g(B - f(X))$ and thus always in $h(X)$ for all $X \subseteq A$. Thus every subset in M contains a_0, so that $S = \bigcap_{X \in M} X$ is nonempty.

From the definition of S we have $h(S) \subseteq S$. And from the monotony of h it follows that $h(h(S)) \subseteq h(S)$; thus, $h(S) \in M$ and $S \subset h(S)$. Combining these results, we have $S = h(S)$, as desired.

Problem 13 A polyhedron P is given in space. Determine whether there must exist three edges of P that can be the sides of a triangle.

Solution: The answer is "yes." Assume, for the purpose of contradiction, that there exists a polyhedron P in which no three edges can form the sides of a triangle. Let the edges of P be $E_1, E_2, E_3, \dots, E_n$, in non-increasing order of length; let e_i be the length of E_i. Consider the two faces that share E_1: for each of those faces, the sum of the lengths of all its edges except E_1 is greater than e_1. Therefore,

$$e_2 + e_3 + \cdots + e_n > 2e_1.$$

But, since we are assuming that no three edges of P can form the sides of a triangle, we have $e_{i+1} + e_{i+2} \le e_i$ for $i = 1, 2, \dots, n-2$. Hence,

$$\begin{aligned}
&2(e_2 + e_3 + \cdots + e_n) \\
&\quad = e_2 + (e_2 + e_3) + (e_3 + e_4) + \cdots + (e_{n-1} + e_n) + e_n \\
&\quad \le e_2 + (e_1) + (e_2) + \cdots + (e_{n-2}) + e_n,
\end{aligned}$$

so

$$e_2 + e_3 + \cdots + e_n \le e_1 + e_2 - e_{n-1} < e_1 + e_1 + 0 = 2e_1,$$

a contradiction. Thus, our assumption was incorrect and some three edges *can* be the sides of a triangle.

1.17 Russia

Fourth round

Problem 8.1 A father wishes to take his two sons to visit their grandmother, who lives 33 kilometers away. He owns a motorcycle whose maximum speed is 25 km/h. With one passenger, its maximum speed drops to 20 km/h. (He cannot carry two passengers.) Each brother walks at a speed of 5 km/h. Show that all three of them can reach the grandmother's house in 3 hours.

Solution: Have the father drive his first son 24 kilometers, which takes $\frac{6}{5}$ hours; then drive back to meet his second son 9 kilometers from home, which takes $\frac{3}{5}$ hours; and finally drive his second son $\frac{6}{5}$ more hours.

Each son spends $\frac{6}{5}$ hours riding 24 kilometers, and $\frac{9}{5}$ hours walking 9 kilometers. Thus they reach their grandmother's house in exactly 3 hours — as does the father, who arrives at the same time as his second son.

Problem 8.2 The natural number A has the following property: the sum of the integers from 1 to A, inclusive, has decimal expansion equal to that of A followed by three digits. Find A.

Solution: We know that

$$\begin{aligned} k &= (1 + 2 + \cdots + A) - 1000A \\ &= \frac{A(A+1)}{2} - 1000A = A\left(\frac{A+1}{2} - 1000\right) \end{aligned}$$

is between 0 and 999, inclusive. If $A < 1999$ then k is negative. If $A \geq 2000$ then $\frac{A+1}{2} - 1000 \geq \frac{1}{2}$ and $k \geq 1000$. Therefore $A = 1999$, and indeed $1 + 2 + \cdots + 1999 = 1999000$.

Problem 8.3 On sides BC, CA, AB of triangle ABC lie points A_1, B_1, C_1 such that the medians A_1A_2, B_1B_2, C_1C_2 of triangle $A_1B_1C_1$ are parallel to AB, BC, CA, respectively. Determine in what ratios the points A_1, B_1, C_1 divide the sides of ABC.

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[1] Problems are numbered as they appeared in the contests. Problems that appeared more than once in the contests are only printed once in this book.

First Solution: A_1, B_1, C_1 divide sides BC, CA, AB in $1:2$ ratios (so that $\frac{BA_1}{A_1C} = \frac{1}{2}$, and so on).

Lemma. *In any triangle XYZ, the medians can be translated to form a triangle. Furthermore, the medians of this new triangle are parallel to the sides of triangle XYZ.*

Proof: Let x, y, z denote the vectors $\overrightarrow{YZ}$, $\overrightarrow{ZX}$, $\overrightarrow{XY}$ respectively; then $x + y + z = \overrightarrow{0}$. Also, the vectors representing the medians of triangle XYZ are $m_x = z + \frac{x}{2}$, $m_y = x + \frac{y}{2}$, $m_z = y + \frac{z}{2}$. These vectors add up to $\frac{3}{2}(x + y + z) = \overrightarrow{0}$, so the medians indeed form a triangle.

Furthermore, the vectors representing the medians of the new triangle are $m_x + \frac{m_y}{2} = x + y + z - \frac{3}{4}y = -\frac{3}{4}y$, and similarly $-\frac{3}{4}z$ and $-\frac{3}{4}x$. Therefore, these medians are parallel to XZ, YX, and ZY. ■

Let D, E, F be the midpoints of sides BC, CA, AB, and let l_1, l_2, l_3 be the segments A_1A_2, B_1B_2, C_1C_2.

Since l_1, l_2, l_3 are parallel to AB, BC, CA, the medians of the triangle formed by l_1, l_2, l_3 are parallel to CF, AD, BE. But from the lemma, they are also parallel to B_1C_1, C_1A_1, A_1B_1.

Therefore, $BE \parallel A_1B_1$, and hence $\triangle BCE \sim \triangle A_1CB_1$. Then

$$\frac{B_1C}{AC} = \frac{1}{2} \cdot \frac{B_1C}{EC} = \frac{1}{2} \cdot \frac{A_1C}{BC} = \frac{1}{2}\left(1 - \frac{A_1B}{CB}\right).$$

Similarly

$$\frac{C_1A}{BA} = \frac{1}{2}\left(1 - \frac{B_1C}{AC}\right)$$
$$\frac{A_1B}{CB} = \frac{1}{2}\left(1 - \frac{C_1A}{BA}\right).$$

Solving these three equations gives

$$\frac{B_1C}{AC} = \frac{C_1A}{BA} = \frac{A_1B}{CB} = \frac{1}{3},$$

as claimed; and it is straightforward to verify with the above equations that these ratio indeed work.

Second Solution: As above, we know that $A_1B_1 \parallel BE$, $B_1C_1 \parallel AD$, $C_1A_1 \parallel CF$.

Let A', B', C' be the points dividing the sides BC, CA, AB in $1:2$ ratios — since $\frac{CA'}{CB} = \frac{CB'}{\frac{1}{2}CA}$, we know $A'B' \parallel BE \parallel A_1B_1$, and so on.

Suppose by way of contradiction that A_1 were closer to B than A'. Then since $A_1B_1 \parallel A'B'$, B_1 is farther from C than B'. Similarly, C_1 is closer to A than C', and A_1 is *farther* from B than A' — a contradiction.

Likewise, A_1 cannot be farther from B than A'. Thus $A_1 = A'$, $B_1 = B'$, and $C_1 = C'$.

Problem 8.4 We are given 40 balloons, the air pressure inside each of which is unknown and may differ from balloon to balloon. It is permitted to choose up to k of the balloons and equalize the pressure in them (to the arithmetic mean of their respective original pressures). What is the smallest k for which it is always possible to equalize the pressures in all of the balloons?

Solution: $k = 5$ is the smallest such value.

First suppose that $k = 5$. Note that we can equalize the pressure in any 8 balloons: first divide them into two groups of four $\{A, B, C, D\}$ and $\{E, F, G, H\}$ and equalize the pressure in each group; then equalize the pressure in $\{A, B, E, F\}$ and $\{C, D, G, H\}$.

Then divide the 40 balloons into eight "5-groups" of five and equalize the pressure in each group. Then form five new groups of eight — containing one balloon from each "5-group" — and equalize the pressure in each of these new groups.

Now suppose that $k \leq 4$. Let $b_1, b_2, \ldots, b_{40}$ denote the original air pressures inside the balloons. It is simple to verify that the pressure in each balloon can always be written as a linear combination $a_1b_1 + \cdots + a_{40}b_{40}$, where the a_i are rational with denominators not divisible by any primes except 2 and 3. Thus if we the b_j are linearly independent over the rationals (say, if $b_j = e^j$), we can never obtain

$$\frac{1}{40}b_1 + \frac{1}{40}b_2 + \cdots + \frac{1}{40}b_{40}$$

in a balloon. In this case, we can never equalize the pressures in all 40 balloons.

Problem 8.5 Show that the numbers from 1 to 15 cannot be divided into a group A of 2 numbers and a group B of 13 numbers in such a way that the sum of the numbers in B is equal to the product of the numbers in A.

Solution: Suppose by way of contradiction this were possible, and let a and b be the two numbers in A. Then we have

$$(1 + 2 + \cdots + 15) - a - b = ab$$
$$120 = ab + a + b$$
$$121 = (a + 1)(b + 1),$$

Since a and b are integers between 1 and 15, the only possible solution to this equation is $(a, b) = (10, 10)$. But a and b must be distinct, a contradiction.

Problem 8.6 Given an acute triangle ABC, let A_1 be the reflection of A across the line BC, and let C_1 be the reflection of C across the line AB. Show that if A_1, B, C_1 lie on a line and $C_1B = 2A_1B$, then $\angle CA_1B$ is a right angle.

Solution: By the given reflections, we have $\triangle ABC \cong \triangle ABC_1 \cong \triangle A_1BC$.

Since $\angle B$ is acute, C_1 and A lie on the same side of BC. Thus C_1 and A_1 lie on opposite sides of BC as well.

Then since C_1, B, A_1 lie on a line we have

$$\begin{aligned} 180^\circ &= \angle C_1BA + \angle ABC + \angle CBA_1 \\ &= \angle ABC + \angle ABC + \angle ABC, \end{aligned}$$

so that $\angle ABC = 60^\circ$. Also we know that

$$C_1B = 2A_1B \Longrightarrow CB = 2AB,$$

implying that triangle ABC is a 30°-60°-90° triangle and $\angle CA_1B = \angle BAC = 90^\circ$.

Problem 8.7 In a box lies a complete set of 1×2 dominoes. (That is, for each pair of integers i, j with $0 \le i \le j \le n$, there is one domino with i on one square and j on the other.) Two players take turns selecting one domino from the box and adding it to one end of an open (straight) chain on the table, so that adjacent dominoes have the same numbers on their adjacent squares. (The first player's move may be any domino.) The first player unable to move loses. Which player wins with correct play?

Solution: The first player has a winning strategy. If $n = 0$, this is clear. Otherwise, have the first player play the domino $(0, 0)$ and suppose the second player plays $(0, a)$; then have the first player play (a, a).

At this point, the second player faces a chain whose ends are either 0 or a; also, the domino $(0, k)$ is on the table if and only if the domino (a, k) is on the table. In such a "good" situation, if the second player plays $(0, k)$ the first player can play (k, a) next to it; and if the second player plays (a, k) the first player can play $(k, 0)$. In both cases, the same conditions for a "good" situation occur.

Therefore the first player can always play a domino with this strategy, forcing the second player to lose.

Problem 8.8 An open chain of 54 squares of side length 1 is made so that each pair of consecutive squares is joined at a single vertex, and each square is joined to its two neighbors at opposite vertices. Is it possible to cover the surface of a $3 \times 3 \times 3$ cube with this chain?

Solution: It is not possible; suppose by way of contradiction it were.

Create axes so that the cube has corners at $(3i, 3j, 3k)$ for $i, j, k \in \{0, 1\}$, and place the chain onto the cube. Imagine that every two adjacent squares in the chain are connected by pivots, and also let the start and end vertices of the chain be "pivots."

Consider some pivot P at (x, y, z); then the next pivot Q in the chain is either at $(x, y \pm 1, z \pm 1)$, $(x \pm 1, y, z \pm 1)$, or $(x \pm 1, y \pm 1, z)$. In any case, the sum of the coordinates of P has the same parity as the sum of the coordinates of Q — and hence *all* the pivots' sums of coordinates have the same parity. Suppose without loss of generality the sums are even.

Form a graph whose vertices are the lattice points on the cube with even sums of coordinates; and join two vertices with an edge if the two lattice points are opposite corners of a unit square. Every square in our chain contains one of these edges — but since there are exactly 54 such edges (one across each unit square on the cube's surface), and 54 squares in our chain, every edge is used exactly once. Then as we travel from pivot to pivot along our chain, we create an Eulerian path visiting all the edges. But four vertices — at $(0, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$ — have odd degree 3, so this is impossible.

Problem 9.1 Around a circle are written all of the positive integers from 1 to N, $N \geq 2$, in such a way that any two adjacent integers have at least one common digit in their decimal expansions. Find the smallest N for which this is possible.

Solution: $N = 29$. Since 1 must be adjacent to two numbers, we must have $N \geq 11$. But then 9 must be adjacent to two numbers, and the next smallest numbers containing 9 as a digit are 19 and 29. Therefore $N \geq 29$, and indeed $N = 29$ suffices:

$$19, 9, 29, 28, 8, 18, 17, 7, 27, \ldots, 13, 3, 23, 2, 22, 21, 20, 12, 11, 10, 1.$$

Problem 9.2 In triangle ABC, points D and E are chosen on side CA such that $AB = AD$ and $BE = EC$ (E lying between A and D). Let F be the midpoint of the arc BC of the circumcircle of ABC. Show that B, E, D, F lie on a circle.

Solution: Let I be the incenter of triangle ABC, and notice that

$$\begin{aligned}
\angle BIC &= 180^\circ - \angle ICB - \angle CBI \\
&= 180^\circ - \frac{\angle B}{2} - \frac{\angle C}{2} \\
&= 90^\circ + \frac{\angle A}{2}.
\end{aligned}$$

Also, since $AD = AB$ we have $\angle ADB = 90^\circ - \frac{\angle A}{2}$ and $\angle BDC = 180^\circ - \angle ADB = 90^\circ + \frac{\angle A}{2}$. Therefore, $BIDC$ is cyclic.

Some angle-chasing shows that that B, I, and C lie on a circle with center F. Thus D lies on this circle, $FD = FC$, and $\angle FDC = \angle DCF$.

Also, since $BE = EC$, we have $\angle CBE = \angle C$. Combining these facts, we have

$$\begin{aligned}
180^\circ - \angle EDF &= \angle FDC \\
&= \angle DCF \\
&= \angle ACF \\
&= \angle C + \frac{\angle A}{2} \\
&= \angle CBE + \angle FBC \\
&= \angle FBE.
\end{aligned}$$

Therefore $BEDF$ is cyclic, as desired.

Problem 9.3 The product of the positive real numbers x, y, z is 1. Show that if

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq x+y+z,$$

then

$$\frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k}\geq x^k+y^k+z^k$$

for all positive integers k.

First Solution: Write $x=\frac{a}{b}$, $y=\frac{b}{c}$, $z=\frac{c}{a}$ for some positive numbers a,b,c. (For example, we could take $a=1, b=\frac{1}{x}, c=\frac{1}{xy}$.) The given equation becomes

$$\begin{aligned}&\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\\ &\iff a^2b+b^2c+c^2a\geq ab^2+bc^2+ca^2\\ &\iff 0\geq(a-b)(b-c)(c-a).\end{aligned}$$

For any positive integer k, write $A=a^k$, $B=b^k$, $C=c^k$. Then $a>b\iff A>B$ and $a<b\iff A<B$, and so on. Thus we also know that $0\geq(A-B)(B-C)(C-A)$, and

$$\begin{aligned}&0\geq(A-B)(B-C)(C-A)\\ &\iff \frac{B}{A}+\frac{C}{B}+\frac{A}{C}\geq\frac{A}{B}+\frac{B}{C}+\frac{C}{A}\\ &\iff \frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k}\geq x^k+y^k+z^k,\end{aligned}$$

as desired.

Second Solution: The inequality

$$0\geq(a-b)(b-c)(c-a)$$

might spark this realization: dividing through by abc we have

$$0\geq(x-1)(y-1)(z-1).$$

Indeed,

$$\begin{aligned}&(x-1)(y-1)(z-1)=xyz+x+y+z-xy-yz-zx-1\\ &=x+y+z-\frac{1}{z}-\frac{1}{x}-\frac{1}{y}\leq 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned} &(x-1)(y-1)(z-1) \le 0 \\ \Rightarrow\ &(x^k-1)(y^k-1)(z^k-1) \le 0 \\ \Rightarrow\ &x^k+y^k+z^k \ge \frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k}, \end{aligned}$$

as desired.

Problem 9.4 A maze consists of an 8×8 grid, in each 1×1 cell of which is drawn an arrow pointing up, down, left or right. The top edge of the top right square is the exit from the maze. A token is placed on the bottom left square, and then is moved in a sequence of turns. On each turn, the token is moved one square in the direction of the arrow. Then the arrow in the square the token moved from is rotated 90° clockwise. If the arrow points off of the board (and not through the exit), the token stays put and the arrow is rotated 90° clockwise. Prove that sooner or later the token will leave the maze.

Solution: Suppose by way of contradiction the token did not leave the maze. Let *position* denote the set-up of the board, including both the token's location and the directions of all the arrows. Since the token moves infinitely many times inside the maze, and there are only finitely many positions, some position must repeat.

During the "cycle time" between two occurrences of this position, suppose the token visits some square S. Then the arrow on S must make at least four 90° rotations: thus at some point during the cycle time, the token must visit all the squares adjacent to S. It follows that the token visits *all* the squares on the board during the cycle time.

Specifically, the token visits the upper-right square during the cycle time; but at some point, this square's arrow will point out of the maze. Then when the token lands on this square it will exit — a contradiction.

Problem 9.5 Each square of an infinite grid is colored in one of 5 colors, in such a way that every 5-square (Greek) cross contains one square of each color. Show that every 1×5 rectangle also contains one square of each color.

Solution: Label the centers of the grid squares with coordinates, and suppose that square $(0, 0)$ is colored maroon. The Greek cross centered at $(1, 1)$ must contain a maroon-colored square. However, the squares $(0, 1)$, $(1, 0)$, and $(1, 1)$ cannot be maroon because each of these squares is in a Greek cross with $(0, 0)$. Thus either $(1, 2)$ or $(2, 1)$ is maroon — without loss of generality, say $(1, 2)$.

Then by a similar analysis on square $(1, 2)$ and the Greek cross centered at $(2, 1)$, one of the squares $(2, 0)$ and $(3, 1)$ must be maroon. $(2, 0)$ is in a Greek cross with $(0, 0)$ though, so $(3, 1)$ is maroon.

Repeating the analysis on square $(2, 0)$ shows that $(2, -1)$ is maroon; and spreading outward, every square of the form $(i+2j, 2i-j)$ is maroon. But since these squares are the centers of Greek crosses that tile the plane, no other squares can be maroon. And since no two of these squares are in the same 1×5 rectangle, no two maroon squares can be in the same 1×5 rectangle.

The same argument applies to all the other colors — lavender, tickle-me-pink, green, neon orange. Therefore the five squares in each 1×5 rectangle have distinct colors, as desired.

Problem 9.7 Show that each natural number can be written as the difference of two natural numbers having the same number of prime factors.

Solution: If n is even, then we can write it as $(2n) - (n)$.

Now suppose n is odd, and let d be the smallest odd prime that does not divide n. Then write $n = (dn) - ((d-1)n)$. The number dn contains exactly one more prime factor than n. As for $(d-1)n$, it is divisible by 2 since $d-1$ is even; but its odd factors are less than d so they all divide n. Therefore $(d-1)n$ also contains exactly one more prime factor than n, and dn and $(d-1)n$ have the same number of prime factors.

Problem 9.8 In triangle ABC, with $AB > BC$, points K and M are the midpoints of sides AB and CA, and I is the incenter. Let P be the intersection of the lines KM and CI, and Q the point such that $QP \perp KM$ and $QM \parallel BI$. Prove that $QI \perp AC$.

Solution: Draw point S on ray CB such that $CS = CA$. Let P' be the midpoint of AS. Since triangle ACS is isosceles, P' lies on CI;

and since P' and M are midpoints of AS and AC, we have $P'M \parallel SC$. It follows that $P = P'$.

Let the incircle touch BC, CA, AB at D, E, F respectively. Writing $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$, we have

$$SD = SC - DC = b - (s-c) = \frac{1}{2}(b+c-a) = FA,$$
$$BF = s - b = DB,$$
$$AP = PS.$$

Therefore
$$\frac{SD}{DB}\frac{BF}{FA}\frac{AP}{PS} = 1,$$
and by Menelaus' Theorem applied to triangle ABS, P lies on line DF.

Then triangle PDE is isosceles, and $\angle DEP = \angle PDE = \angle FEA = 90^\circ - \frac{\angle A}{2}$ while $\angle CED = 90^\circ - \frac{\angle C}{2}$. Therefore

$$\angle PEA = 180^\circ - \angle DEP - \angle CED = 90^\circ - \frac{\angle B}{2}.$$

Now let Q' be the point such that $Q'I \perp AC$, $Q'M \parallel BI$. Then $\angle Q'EP = 90^\circ - \angle PEA = \frac{\angle B}{2}$.

But we also know that $\angle Q'MP = \angle IBC$ (from parallel lines $BC \parallel MP$ and $IB \parallel Q'M$), and $\angle IBC = \frac{\angle B}{2}$ as well. Therefore $\angle Q'MP = \angle Q'EP$, quadrilateral $Q'EMP$ is cyclic, and $\angle Q'PM = \angle Q'EM = 90^\circ$. Therefore $Q = Q'$, and QI is indeed perpendicular to AC.

Problem 10.2 In the plane is given a circle ω, a point A inside ω, and a point B not equal to A. Consider all possible triangles BXY such that X and Y lie on ω and A lies on the chord XY. Show that the circumcenters of these triangles all lie on a line.

Solution: We use directed distances. Let O be the circumcenter and R be the circumradius of triangle BXY. Drop the perpendicular OO' to line AB.

The power of A with respect to circle BXY equals both $AX \cdot AY$ and $AO^2 - R^2$. Therefore

$$BO' - O'A = \frac{BO'^2 - O'A^2}{BO' + O'A}$$

$$= \frac{(BO^2 - O'O^2) - (OA^2 - OO'^2)}{AB}$$
$$= \frac{XA \cdot AY}{AB}$$

which is constant since $AX \cdot AY$ also equals the power of A with respect to ω.

Since $BO' - O'A$ and $BO' + O'A = AB$ are constant, BO' and $O'A$ are constant as well. Thus O' is fixed regardless of the choice of X and Y. Therefore O lies on the line through O' perpendicular to AB, as desired.

Problem 10.3 In space are given n points in general position (no three points are collinear and no four are coplanar). Through any three of them is drawn a plane. Show that for any $n-3$ points in space, there exists one of the drawn planes not passing through any of these points.

Solution: Call the given n points *given* and the $n-3$ points *random*, and call all these points "level-0." Since there are more given points than random points, one of the given points is not random: say, A. Draw a plane not passing through A, and for each of the other points P let (P) be the intersection of AP with this plane. Call these points (P) level-1.

Since no four given points were coplanar, no three of the level-1 given points map to collinear points on this plane; and since no three given points were collinear, no two of the level-1 given points map to the same point on this plane. Thus we have $n-1$ level-1 given points and at most $n-3$ level-1 random points.

Now perform a similar operation — since there are more level-1 given points than random points, one of them is not random: say, (B). Draw a line not passing through (B), and for each of the other points (P) let $((P))$ be the intersection of $B(P)$ with this plane. Call these points $((P))$ level-2.

Since no three level-1 given points were collinear, all of the level-2 given points are distinct. Thus we have $n-2$ level-2 given points but at most $n-3$ level-2 random points. Therefore one of these given points $((C))$ is not random.

Consider the drawn plane ABC. If it contained some level-0 random point — say, Q — then (Q) would be collinear with (B) and (C), and thus $((Q)) = ((C))$, a contradiction. Therefore plane ABC does not pass through any of the level-0 random points, as desired.

Problem 10.5 Do there exist 10 distinct integers, the sum of any 9 of which is a perfect square?

Solution: Yes, there do exist 10 such integers. Write $S = a_1 + \cdots + a_{10}$, and consider the linear system of equations

$$\begin{aligned} S - a_1 &= 9 \cdot 1^2 \\ S - a_2 &= 9 \cdot 2^2 \\ &\vdots \\ S - a_{10} &= 9 \cdot 10^2. \end{aligned}$$

Adding all these gives

$$9S = 9 \cdot (1^2 + 2^2 + \cdots + 10^2)$$

so that

$$a_i = S - 9i^2 = 1^2 + 2^2 + \cdots + 10^2 - 9i^2.$$

Then all the a_i's are distinct integers, and any nine of them add up to perfect square.

Problem 10.6 The incircle of triangle ABC touches sides BC, CA, AB at A_1, B_1, C_1, respectively. Let K be the point on the circle diametrically opposite C_1, and D the intersection of the lines B_1C_1 and A_1K. Prove that $CD = CB_1$.

Solution: Draw D' on B_1C_1 such that $CD' \parallel AB$. Then $\angle D'CB_1 = \angle C_1AB_1$ and $\angle CD'B_1 = \angle AC_1B_1$, implying that $\triangle AB_1C_1 \sim \triangle CB_1D'$.

Thus triangle CB_1D' is isosceles and $CD' = CB_1$. But $CB_1 = CA_1$, so that triangle CA_1D' is isosceles also. And since $\angle D'CA_1 = 180^\circ - \angle B$, we have $\angle CA_1D' = \frac{\angle B}{2}$.

But note that

$$\begin{aligned} \angle CA_1K &= \angle A_1C_1K \\ &= 90^\circ - \angle C_1KA_1 \end{aligned}$$

$$= 90^\circ - \angle C_1A_1B$$
$$= \frac{\angle B}{2}$$

also. Therefore D' lies on A_1K and by definition it lies on B_1C_1. Hence $D' = D$.

But from before $CD' = CB_1$; thus $CD = CB_1$, as desired.

Problem 10.7 Each voter in an election marks on a ballot the names of n candidates. Each ballot is placed into one of $n+1$ boxes. After the election, it is observed that each box contains at least one ballot, and that for any $n+1$ ballots, one in each box, there exists a name which is marked on all of these ballots. Show that for at least one box, there exists a name which is marked on all ballots in the box.

Solution: Suppose by way of contradiction that in every box, no name is marked on all the ballots. Label the boxes $1, 2, \ldots, n$, and look at an arbitrary ballot from the first box.

Suppose it has n "chosen" names Al, Bob, ..., Zed. By assumption, some ballot in the second box does not have the name Al on it; some ballot in the third box does not have the name Bob on it; and so on, so some ballot in the $(i+1)$-th box does not have the i-th chosen name on it. But then on these $n+1$ ballots, one from each box, there is no name marked on all the ballots — a contradiction.

Problem 10.8 A set of natural numbers is chosen so that among any 1999 consecutive natural numbers, there is a chosen number. Show that there exist two chosen numbers, one of which divides the other.

Solution: Draw a large table with 1999 columns and 2000 rows. In the first row write $1, 2, \ldots, 1999$.

Define the entries in future rows recursively as follows: suppose the entries in row i are $k+1$, $k+2$, $\ldots$, $k+1999$, and that their product is M. Then fill row $i+1$ with $M+k+1$, $M+k+2$, $\ldots$, $M+k+1999$. All the entries in row $i+1$ are bigger than the entries in row i; furthermore, every entry divides the entry immediately below it (and therefore *all* the entries directly below it).

In each row there are 1999 consecutive numbers, and hence each row contains a chosen number. Then since we have 2000 rows, there are two chosen numbers in the same column — and one of them divides another, as desired.

Problem 11.1 The function $f(x)$ is defined on all real numbers. It is known that for all $a > 1$, the function $f(x) + f(ax)$ is continuous. Show that $f(x)$ is continuous.

First Solution: We know that for $a > 1$, the functions

$$P(x) = f(x) + f(ax),$$
$$Q(x) = f(x) + f(a^2x),$$
$$P(ax) = f(ax) + f(a^2x)$$

are all continuous. Thus the function

$$\frac{1}{2}\left(P(x) + Q(x) - P(ax)\right) = f(x)$$

is continuous as well.

Problem 11.3 In a class, each boy is friends with at least one girl. Show that there exists a group of at least half of the students, such that each boy in the group is friends with an odd number of the girls in the group.

Solution: We perform strong induction on the total number of students. The base case of zero students is obvious.

Now suppose that we know the claim is true for any number of students less than n (where $n > 0$), and we wish to prove it for n. Since there must be at least one girl, pick any girl from the n students. We now partition the class into three subsets: $A =$ this girl, $B =$ this girl's male friends, and $C =$ everybody else.

Because we are using strong induction, the induction hypothesis states that there must be a subset C' of C, with at least $\frac{|C|}{2}$ students, such that any boy in C' is friends with an odd number of girls in C'.

Let B_O be the set of boys in B who are friends with an odd number of girls in C', and let B_E be the set of boys in B who are friends with an even number of girls in C'. Then there are two possible cases:

(i) $|B_O| \geq \frac{|A \cup B|}{2}$.

The set $S = B_O \cup C'$ will realize the claim, i.e., S will have at least $\frac{n}{2}$ elements, and each boy in S will be friends with an odd number of girls in S.

(ii) $|A \cup B_E| \geq \frac{|A \cup B|}{2}$.

The set $T = A \cup B_E \cup C'$ will realize the claim. T will have at least $\frac{n}{2}$ elements; each boy in C' will be friends with an odd number of girls in C' but not the girl in A; and each boy in B_E will be friends with an even number of girls in C' and the girl in A — making a total of an odd number of girls.

Thus the induction is complete.

Note: With a similar proof, it is possible to prove a slightly stronger result: suppose each boy in a class is friends with at least one girl, and that every boy has a parity, either "even" or "odd." Then there is a group of at least half the students, such that each boy in the group is friends with the same parity of girls as his own parity. (By letting all the boys' parity be "odd," we have the original result.)

Problem 11.4 A polyhedron is circumscribed about a sphere. We call a face big if the projection of the sphere onto the plane of the face lies entirely within the face. Show that there are at most 6 big faces.

Solution:

Lemma. *Given a sphere of radius R, let a "slice" of the sphere be a portion cut off by two parallel planes. The surface area of the sphere contained in this slice is $2\pi RW$, where W is the distance between the planes.*

Proof: Orient the sphere so that the slice is horizontal. Take an infinitesimal horizontal piece of this slice, shaped like a frustrum (a small sliver from the bottom of a radially symmetric cone). Say it has width w, radius r, and slant height ℓ; then its lateral surface area (for infinitesimal w) is $2\pi r\ell$. But if the side of the cone makes an angle θ with the horizontal, then we have $\ell \sin\theta = w$ and $R \sin\theta = r$ so that the surface area also equals $2\pi Rw$. Adding over all infinitesimal pieces, the complete slice has lateral surface area $2\pi RW$, as desired. ■

Say that the inscribed sphere has radius R and center O. For each big face F in the polyhedron, project the sphere onto F to form a circle k. Then connect k with O to form a cone. Because these cones don't share any volume, they hit the sphere's surface in several non-overlapping circular regions.

Each circular region is a slice of the sphere with width $R(1-\frac{1}{2}\sqrt{2})$, and it contains $2\pi R^2(1-\frac{1}{2}\sqrt{2}) > \frac{1}{7}(4\pi R^2)$ of the sphere's surface area. Thus each circular region takes up more than $\frac{1}{7}$ of the surface area of the sphere, implying there must be less than 7 such regions and therefore at most six big faces.

Problem 11.5 Do there exist real numbers a, b, c such that for all real numbers x, y,

$$|x+a|+|x+y+b|+|y+c|>|x|+|x+y|+|y|?$$

Solution: No such numbers exist; suppose they did. Let $y=-b-x$. Then for all real x we have

$$|x+a|+|-b-x+c|>|x|+|-b|+|-b-x|.$$

If we pick x sufficiently negative, this gives

$$\begin{aligned}&(-x-a)+(-b-x+c)>(-x)+|b|+(-b-x)\\ &\Rightarrow -a+c>|b|\geq 0,\end{aligned}$$

so $c>a$. On the other hand, if we pick x sufficiently positive, this gives

$$\begin{aligned}&(x+a)+(b+x-c)>(x)+|b|+(b+x)\\ &\Rightarrow a-c>|b|\geq 0,\end{aligned}$$

so $c<a$ as well — a contradiction.

Problem 11.6 Each cell of a 50×50 square is colored in one of four colors. Show that there exists a cell which has cells of the same color directly above, directly below, directly to the left, and directly to the right of it (though not necessarily adjacent to it).

Solution: By the pigeonhole principle, at least one-quarter of the squares (625) are the same color: say, red.

Of these red squares, at most 50 are the topmost red squares of their columns, and at most 50 are the bottommost red squares of their columns. Similarly, at most 50 are the leftmost red squares in their rows and at most 50 are the rightmost red squares in their rows. This gives at most 200 squares; the remaining 425 or more red squares then have red squares directly above, directly below, directly to the left, and directly to the right of them.

Problem 11.8 A polynomial with integer coefficients has the property that there exist infinitely many integers which are the value of the polynomial evaluated at more than one integer. Prove that there exists at most one integer which is the value of the polynomial at exactly one integer.

Solution: First observe that the polynomial cannot be constant. Now let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be the polynomial with $c_n \neq 0$. The problem conditions imply that n is even and at least 2, and we can assume without loss of generality that $c_n > 0$.

Since $P(x)$ has positive leading coefficient and it is not constant, there exists a value N such that $P(x)$ is decreasing for all $x < N$. Also consider the pairs of integers (s, t) with $s < t$ and $P(s) = P(t)$; since there are infinitely many pairs, there must be infinitely many with $s < N$.

Now for any integer k, look at the polynomial $P(x) - P(k - x)$. Some algebra shows that the coefficient of x^n is zero and that the coefficient of x^{n-1} is $f(k) = 2c_{n-1} + c_n(nk)$.

Let K be the largest integer such that $f(K) < 0$ (such an integer exists because from assumptions made above, $c_n \cdot n > 0$). Then for sufficiently large t we have

$$P(t) < P(K - t) < P(K - 1 - t) < \cdots$$

and

$$P(t) \geq P(K + 1 - t) > P(K + 2 - t) > P(K + 3 - t) > \cdots > P(N).$$

Therefore we must have $s = K+1-t$ and $P(t) - P(K+1-t) = 0$ for infinitely many values of t. But since P has finite degree, this implies that $P(x) - P(K + 1 - x)$ is *identically* zero.

Then if $P(a) = b$ for some integers a, b, we also have $P(K+1-a) = b$. Therefore there is at most one value b that could possibly be the value of $P(x)$ at exactly one integer x — specifically, $b = P(\frac{K+1}{2})$.

Fifth round

Problem 9.1 In the decimal expansion of A, the digits occur in increasing order from left to right. What is the sum of the digits of

$9A$?

Solution: Write $A = \underline{a_1 a_2 \ldots a_k}$. Then since $9A = 10A - A$, by performing the subtraction

$$\begin{array}{rcccccc} & a_1 & a_2 & a_3 & \cdots & a_k & 0 \\ - & & a_1 & a_2 & \cdots & a_{k-1} & a_k \\ \hline \end{array}$$

we find that the digits of $9A$ are

$$a_1,\ a_2 - a_1,\ a_3 - a_2,\ \ldots,\ a_{k-1} - a_{k-2},\ a_k - a_{k-1} - 1,\ 10 - a_k,$$

and that these digits add up to $10 - 1 = 9$.

Problem 9.3 Let S be the circumcircle of triangle ABC. Let A_0 be the midpoint of the arc BC of S not containing A, and C_0 the midpoint of the arc AB of S not containing C. Let S_1 be the circle with center A_0 tangent to BC, and let S_2 be the circle with center C_0 tangent to AB. Show that the incenter I of ABC lies on a common external tangent to S_1 and S_2.

Solution: We prove a more general result: I lies on a common external tangent to S_1 and S_2, parallel to AC.

Drop the perpendicular from A_0 to BC, hitting at P, and drop the perpendicular from A_0 to AC, hitting circle S_1 at Q (with Q closer to AC than A_0).

Note that A, I, and A_0 are collinear. Then

$$\angle CIA_0 = \angle CAI + \angle ICA = \frac{\angle A + \angle C}{2} = \angle A_0CI,$$

so that $IA_0 = CA_0$.

Next, from circle S_1 we know that $A_0Q = A_0P$.

Finally,

$$\angle IA_0Q = \angle AA_0Q = 90^\circ - \angle CAA_0 = \frac{1}{2}(180^\circ - \angle CAB) = \angle CA_0P.$$

Thus, triangles IA_0Q and CA_0P are congruent. Then

$$\angle IQA_0 = \angle CPA_0 = 90^\circ,$$

so that IQ is tangent to S_1 at Q. Furthermore, since A_0Q is perpendicular to both IQ and AC, we have $IQ \parallel AC$.

Therefore, the line through I parallel to AC is tangent to S_1; by a similar argument it is tangent to S_2; and thus it is a common external tangent to S_1 and S_2, as claimed.

Problem 9.4 The numbers from 1 to 1000000 can be colored black or white. A permissible move consists of selecting a number from 1 to 1000000 and changing the color of that number and each number not relatively prime to it. Initially all of the numbers are black. Is it possible to make a sequence of moves after which all of the numbers are colored white?

First Solution: It is possible. We begin by proving the following lemma:

Lemma. *Given a set S of positive integers, there is a subset $T \subseteq S$ such that every element of S divides an odd number of elements in T.*

Proof: We prove the claim by induction on $|S|$, the number of elements in S. If $|S| = 1$ then let $T = S$.

If $|S| > 1$, then say the smallest element of S is a. Look at the set $S' = S \setminus \{a\}$ — the set of the largest $|S| - 1$ elements in S. By induction there is a subset $T' \subseteq S'$ such that every element in S' divides an odd number of elements in T'.

If a also divides an odd number of elements in T', then the set $T = T'$ suffices. Otherwise, the set $T = T' \cup \{a\}$ suffices: a divides an odd number of elements in T; the other elements are bigger than a and can't divide it, and therefore still divide an odd number of elements in T. This completes the induction and the proof of the lemma. ∎

Now, write each number $n > 1$ in its prime factorization

$$p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

for distinct primes p_i and positive integers a_i. Then notice that the color of n will always be the same as the color of $P(n) = p_1 p_2 \cdots p_k$.

Apply the lemma to the set $S = \bigcup_{i=2}^{1000000} P(i)$ to find a subset $T \subseteq S$ such that every element of S divides an odd number of elements in T. For each $q \in S$, let $t(q)$ equal the number of elements in T that q divides, and let $u(q)$ equal the number of primes dividing q.

Select all the numbers in T, and consider how the color of a number $n > 1$ changes. The number of elements in T not relatively prime to

n equals

$$\sum_{q \mid P(n),\, q>1} (-1)^{u(q)+1} t(q)$$

by the Inclusion-Exclusion Principle: if $q \mid P(n)$ is divisible by exactly $m > 0$ primes, then it is counted $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots = 1$ time in the sum. (For example, if $n = 6$ then the number of elements in T divisible by 2 or 3 equals $t(2) + t(3) - t(6)$.)

But by the definition of T, each of the values $t(q)$ is odd. Then since there are $2^k - 1$ divisors $q > 1$ of $P(n)$, the above quantity is the sum of $2^k - 1$ odd numbers and is odd itself. Therefore after selecting T, every number $n > 1$ will switch color an odd number of times and will turn white.

Finally, select 1 to turn 1 white, and we are done.

Note: In fact, a slight modification of the above proof shows that T is unique, which with some work implies that there is *exactly* one way to make all the numbers white by only selecting square-free numbers at most once each (other methods are different only trivially, either by selecting a number twice or by selecting numbers that aren't square-free).

Second Solution: Yes, it is possible. We prove a more general statement, where we replace 1000000 in the problem by some arbitrary positive integer m, and where we focus on the numbers divisible by just a few primes instead of all the primes.

Lemma. *For a finite set of distinct primes $S = \{p_1, p_2, \ldots, p_n\}$, let $Q_m(S)$ be the set of numbers between 2 and m divisible only by primes in S. The elements of $Q_m(S)$ can be colored black or white; a permissible move consists of selecting a number in $Q_m(S)$ and changing the color of that number and each number not relatively prime to it. Then it is possible to reverse the coloring of $Q_m(S)$ by selecting several numbers in a subset $R_m(S) \subseteq Q_m(S)$.*

Proof: We prove the lemma by induction on n. If $n = 1$, then selecting p_1 suffices. Now suppose $n > 1$, and assume without loss of generality that the numbers are all black to start with.

Let $T = \{p_1, p_2, \ldots, p_{n-1}\}$, and define t to be the largest integer such that $tp_n \leq m$. We can assume $t \geq 1$ because otherwise we could ignore p_n and just use use the smaller set T, and we'd be done by our induction hypothesis.

Now select the numbers in $R_m(T)$, $R_t(T)$, and $p_nR_t(T) = \{p_nx \mid x \in R_t(T)\}$, and consider the effect of this action on a number y:

- *y is not a multiple of p_n.* Selecting the numbers in $R_m(T)$ makes y white. Then if selecting $x \in R_t(T)$ changes y's color, selecting xp_n will change it back so that y will become white.
- *y is a power of p_n.* Selecting the numbers in $R_m(T)$ and $R_t(T)$ has no effect on y, but each of the $|R_t(T)|$ numbers in $xR_t(T)$ changes y's color.
- *$p_n \mid y$ but y is not a power of p_n.* Selecting the numbers in $R_m(T)$ makes y white. Since $y \neq p_n^i$, it is divisible by some prime in T so selecting the numbers in $R_t(T)$ makes y black again. Finally, each of the $|R_t(T)|$ numbers in $xR_t(T)$ changes y's color.

Therefore, all the multiples of p_n are the same color (black if $|R_t(T)|$ is even, white if $|R_t(T)|$ is odd), while all the other numbers in $Q_m(S)$ are white. If the multiples of p_n are still black, we can select p_n to make them white, and we are done. ■

Now back to the original problem: set $m = 1000000$ and let S be the set of all primes under 1000000. Then from the lemma, we can select numbers between 2 and 1000000 so that all the numbers $2, 3, \ldots, 1000000$ are white. And finally, we finish off by selecting 1.

Problem 9.5 An equilateral triangle of side length n is drawn with sides along a triangular grid of side length 1. What is the maximum number of grid segments on or inside the triangle that can be marked so that no three marked segments form a triangle?

Solution: The grid is made up of $\frac{n(n+1)}{2}$ small equilateral triangles of side length 1. In each of these triangles, at most 2 segments can be marked so we can mark at most $\frac{2}{3} \cdot \frac{3n(n+1)}{2} = n(n+1)$ segments in all. Every segment points in one of three directions, so we can achieve the maximum $n(n+1)$ by marking all the segments pointing in two of the directions.

Problem 9.6 Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Prove that for every natural number n,

$$\sum_{k=1}^{n^2} \{\sqrt{k}\} \le \frac{n^2-1}{2}.$$

Solution: We prove the claim by induction on n. For $n = 1$, we have $0 \le 0$. Now supposing that the claim is true for n, we prove it is true for $n+1$.

Each of the numbers $\sqrt{n^2+1}, \sqrt{n^2+2}, \ldots, \sqrt{n^2+2n}$ is between n and $n+1$, and thus

$$\begin{aligned}\{\sqrt{n^2+i}\} &= \sqrt{n^2+i} - n \\ &< \sqrt{n^2+i+\frac{i^2}{4n^2}} - n \\ &= \frac{i}{2n}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\sum_{k=1}^{(n+1)^2} \{\sqrt{k}\} &= \sum_{k=1}^{n^2}\{\sqrt{k}\} + \sum_{k=n^2+1}^{(n+1)^2} \{\sqrt{k}\} \\ &< \frac{n^2-1}{2} + \frac{1}{2n}\sum_{i=1}^{2n} i + 0 \\ &= \frac{n^2-1}{2} + \frac{2n+1}{2} \\ &= \frac{(n+1)^2-1}{2},\end{aligned}$$

completing the inductive step and the proof.

Problem 9.7 A circle passing through vertices A and B of triangle ABC intersects side BC again at D. A circle passing through vertices B and C intersects side AB again at E, and intersects the first circle again at F. Suppose that the points A, E, D, C lie on a circle centered at O. Show that $\angle BFO$ is a right angle.

Solution: Since $AEDC$ is cyclic with O as its center,

$$\begin{aligned}\angle COA &= 2\angle CDA = \angle CDA + \angle CEA \\ &= (180^\circ - \angle ADB) + (180^\circ - \angle BEC).\end{aligned}$$

Since $BDFA$ and $BEFC$ are cyclic, $\angle ADB = \angle AFB$ and $\angle BEC = \angle BFC$. Hence

$$\angle COA = 360^\circ - \angle AFB - \angle BFC = \angle CFA.$$

Hence $AFOC$ is cyclic. Therefore

$$\angle OFA = 180^\circ - \angle ACO = 180^\circ - \frac{180^\circ - \angle COA}{2} = 90^\circ + \angle CDA.$$

Since $ABDF$ is cyclic,

$$\angle OFA + \angle AFB = 90^\circ + \angle CDA + \angle ADB = 270^\circ.$$

Hence $\angle BFO = 90^\circ$, as desired.

Problem 9.8 A circuit board has 2000 contacts, any two of which are connected by a lead. The hooligans Vasya and Petya take turns cutting leads: Vasya (who goes first) always cuts one lead, while Petya cuts either one or three leads. The first person to cut the last lead from some contact loses. Who wins with correct play?

Solution: Petya wins with correct play; arrange the contacts in a circle and label them 1, 2, ..., 2000, and let (x, y) denote the lead between contacts x and y (where labels are taken modulo 2000).

If Vasya disconnects $(a, 1000 + a)$, Petya can disconnect $(500 + a, 1500 + a)$; otherwise, if Vasya disconnects (a, b), Petya can disconnect the three leads $(a + 500, b + 500)$, $(a + 1000, b + 1000)$, and $(a+1500, b+1500)$. Notice that in each case, Petya and Vasya tamper with different contacts.

Using this strategy, after each of Petya's turns the circuit board is symmetrical under 90°, 180°, and 270° rotations, ensuring that he can always make the above moves — for example, if $(a + 1500, b + 1500)$ were already disconnected during Petya's turn, then (a, b) must have been as well before Vasya's turn.

Also, Petya can never lose, because if he disconnected the last lead (x, y) from some contact x, then Vasya must have already disconnected the last lead $(x - 1500, y - 1500)$, $(x - 1000, y - 1000)$, or $(x - 500, y - 500)$ from some other contact, a contradiction.

Problem 10.1 Three empty bowls are placed on a table. Three players A, B, C, whose order of play is determined randomly, take turns putting one token into a bowl. A can place a token in the first or second bowl, B in the second or third bowl, and C in the third or first bowl. The first player to put the 1999th token into a bowl loses. Show that players A and B can work together to ensure that C will lose.

Solution: Suppose A plays only in the first bowl until it contains 1998 tokens, then always plays in the second bowl; and suppose B plays only in the third bowl until it contains 1998 tokens, then always plays in the second bowl as well.

Suppose by way of contradiction that C doesn't lose. Without loss of generality, say the first bowl fills up to 1998 tokens before the third bowl does — call this point in time the "critical point."

First suppose the third bowl never contains 1998 tokens. Then at most 999 round pass after the critical point since during each round, the third bowl gains 2 tokens (one from B, one from C). But then A plays at most 999 tokens into the second bowl and doesn't lose; thus nobody loses, a contradiction.

Thus the third bowl does contain 1998 tokens some $k \le 999$ more rounds after the critical point. After this k-th round A has played at most k tokens into the second bowl, and B has possibly played at most one token into the second bowl during the k-th round; so the second bowl has at most 1000 tokens. However, the first and third bowls each have 1998 tokens, so during the next round C will lose.

Problem 10.2 Find all infinite bounded sequences $a_1, a_2, \ldots$ of positive integers such that for all $n > 2$,

$$a_n = \frac{a_{n-1} + a_{n-2}}{\gcd(a_{n-1}, a_{n-2})}.$$

Solution: The only such sequence is $2, 2, 2, \ldots$.

Let $g_n = \gcd(a_n, a_{n+1})$. Then g_{n+1} divides both a_{n+1} and a_{n+2}, so it divides $g_n a_{n+2} - a_{n+1} = a_n$ as well. Thus g_{n+1} divides both a_n and a_{n+1}, and it divides their greatest common divisor g_n.

Therefore, the g_i form a nonincreasing sequence of positive integers and eventually equal some positive constant g. At this point, the a_i satisfy the recursion

$$ga_n = a_{n-1} + a_{n-2}.$$

If $g = 1$, then $a_n = a_{n-1} + a_{n-2} > a_{n-1}$ so the sequence is increasing and unbounded.

If $g \geq 3$, then $a_n = \frac{a_{n-1}+a_{n-2}}{g} < \frac{a_{n-1}+a_{n-2}}{2} \leq \max\{a_{n-1}, a_{n-2}\}$. Similarly, $a_{n+1} < \max\{a_{n-1}, a_n\} \leq \max\{a_{n-2}, a_{n-1}\}$, so that $\max\{a_n, a_{n+1}\} < \max\{a_{n-2}, a_{n-1}\}$. Therefore the maximum values of successive pairs of terms form an infinite decreasing sequence of positive integers, a contradiction.

Thus $g = 2$ and eventually we have $2a_n = a_{n-1} + a_{n-2}$ or $a_n - a_{n-1} = -\frac{1}{2}(a_{n-1} - a_{n-2})$. This implies that $a_i - a_{i-1}$ converges to 0 and that the a_i are eventually constant as well. From $2a_n = a_{n-1} + a_{n-2}$, this constant must be 2.

Now if $a_n = a_{n+1} = 2$ for $n > 1$, then $\gcd(a_{n-1}, a_n) = \gcd(a_{n-1}, 2)$ either equals 1 or 2. Since

$$2 = a_{n+1} = \frac{a_{n-1} + a_n}{\gcd(a_{n-1}, 2)},$$

this either implies $a_{n-1} = 0$ — which is impossible — or $a_{n-1} = 2$. Therefore all the a_i equal 2, and this sequence indeed works.

Problem 10.3 The incircle of triangle ABC touches sides AB, BC, CA at K, L, M, respectively. For each two of the incircles of AMK, BKL, CLM is drawn the common external tangent not lying along a side of ABC. Show that these three tangents pass through a single point.

Solution: Let D, E, F be the midpoints of minor arcs MK, KL, LM of the incircle, respectively; and let S_1, S_2, S_3 be the incircles of triangles AMK, BKL, and CLM, respectively.

Since AK is tangent to the incircle, $\angle AKD = \angle KLD = \angle KMD = \angle DKM$; similarly, $\angle AMD = \angle DMK$. Thus, D is the incenter of AMK and the center of S_1.

Likewise, E is center of S_2 and F is the center of S_3. By the result proved in Problem 9.3, the incenter I of triangle KLM lies on a common external tangent to S_1 and S_2. But it does not lie on AB, so it must lie on the other external tangent. Similarly, the common external tangent to S_2 and S_3 (not lying on BC) passes through I, as does the common external tangent to S_3 and S_1 (not lying on CA); so the three tangents all pass through I, as desired.

Problem 10.4 An $n \times n$ square is drawn on an infinite checkerboard. Each of the n^2 cells contained in the square initially contains a token. A move consists of jumping a token over an adjacent token (horizontally or vertically) into an empty square; the token jumped over is removed. A sequence of moves is carried out in such a way that at the end, no further moves are possible. Show that at least $\frac{n^2}{3}$ moves have been made.

Solution: At the end of the game no two adjacent squares contain tokens: otherwise (since no more jumps are possible) they would have to be in an infinitely long line of tokens, which is not allowed. Then during the game, each time a token on square A jumps over another token on square B, imagine putting a 1×2 domino over squares A and B. At the end, every tokenless square on the checkerboard is covered by a tile; so no two uncovered squares are adjacent. We now prove there must be at least $\frac{n^2}{3}$ dominoes, implying that at least $\frac{n^2}{3}$ moves have been made:

Lemma. *If an $n \times n$ square board is covered with 1×2 rectangular dominoes (possibly overlapping, and possibly with one square off the board) in such a way that no two uncovered squares are adjacent, then at least $\frac{n^2}{3}$ tiles are on the board.*

Proof: Call a pair of adjacent squares on the checkerboard a "tile." If a tile contains two squares on the border of the checkerboard, call it an "outer tile"; otherwise, call it an "inner tile."

Now for each domino D, consider any tile it partly covers. If this tile is partly covered by exactly m dominoes, say D *destroys* $\frac{1}{m}$ of that tile. Adding over all the tiles that D lies on, we find the total quantity a of outer tiles that D destroys, and the total quantity b of inner tiles that D destroys. Then say that D scores $1.5a + b$ points.

Consider a vertical domino D wedged in the upper-left corner of the chessboard; it partly destroys two horizontal tiles. But one of the two squares immediately to D's right must be covered; so if D destroys all of one horizontal tile, it can only destroy at most half of the other.

Armed with this type of analysis, some quick checking shows that any domino scores at most 6 points; and that any domino scoring 6 points must lie completely on the board, not be wedged in a corner, not overlap any other dominoes, and not have either length-1 edge hit another domino.

Now in a valid arrangement of dominoes, every tile is destroyed completely; since there are $4(n-1)$ outer tiles and $2(n-1)(n-2)$ inner tiles, this means that a total of $1.5 \cdot 4(n-1)+2(n-1)(n-2) = 2(n^2-1)$ points are scored. Therefore, there must be at least $\lceil \frac{2(n^2-1)}{6} \rceil = \lceil \frac{n^2-1}{3} \rceil$ dominoes.

Suppose by way of contradiction that we have *exactly* $\frac{n^2-1}{3}$ dominoes. First, for this to be an integer 3 cannot divide n. Second, the restrictions described two paragraphs ago must hold for every domino.

Suppose we have any horizontal domino not at the bottom of the chessboard; one of the two squares directly below it must be covered. But to satisfy our restrictions, it must be covered by a horizontal domino (not a vertical one). Thus we can find a chain of horizontal dominoes stretching to the bottom of the board, and similarly we can follow this chain to the top of the board.

Similarly, if there is any vertical domino then some chain of vertical dominoes stretches across the board. But we can't have both a horizontal *and* a vertical chain, so all the dominoes must have the same orientation: say, horizontal.

Now to cover the tiles in any given row while satisfying the restrictions, we must alternate between blank squares and horizontal dominoes. In the top row, since no dominoes are wedged in a corner we must start and end with blank squares; thus we must have $n \equiv 1 \pmod 3$. But then in the second row, we must start with a horizontal domino (to cover the top-left vertical tiles); then after alternating between dominoes and blank squares, the end of the row will contain two blank squares—a contradiction. Thus it is impossible to cover the chessboard with exactly $\frac{n^2-1}{3}$ dominoes, and indeed at least $\frac{n^2}{3}$ dominoes are needed. ■

Note: When n is even, there is a simpler proof of the main result: split the n^2 squares of the board into 2×2 mini-boards, each containing four (overlapping) 1×2 tiles. At the end of the game, none of these n^2 tiles can contain two checkers (since no two checkers can be adjacent at the end of the game). But any jump removes a checker from at most three full tiles; therefore, there must be at least $\frac{n^2}{3}$ moves.

Sadly, a similar approach for odd n yields a lower bound of only $\frac{n^2-n-1}{3}$ moves. For large enough n though, we can count the number of tokens that end up completely *outside* the $(n+2) \times (n+2)$ area around the checkerboard — each made a jump that freed at most two full tiles, and from here we can show that $\frac{n^2}{3}$ moves are necessary.

Problem 10.5 The sum of the decimal digits of the natural number n is 100, and that of $44n$ is 800. What is the sum of the digits of $3n$?

Solution: The sum of the digits of $3n$ is 300.

Let $S(x)$ denote the sum of the digits of x. Then $S(a+b)$ equals $S(a)+S(b)$, minus nine times the number of carries in the addition $a+b$. Therefore, $S(a+b) \le S(a)+S(b)$; applying this repeatedly, we have $S(a_1+\cdots+a_k) \le S(a_1)+\cdots+S(a_k)$.

Also note that for a digit $d \le 2$ we have $S(44d) = 8d$; for $d = 3$ we have $S(8d) = 6 < 8d$; and for $d \ge 4$, $44d \le 44(9)$ has at most 3 digits so its sum is at most $27 < 8d$.

Now write $n = \sum n_i \cdot 10^i$, so that the n_i are the digits of n in base 10. Then

$$\sum 8n_i = S(44n) \le \sum S(44n_i \cdot 10^i) = \sum S(44n_i) \le \sum 8n_i,$$

so equality must occur in the second inequality — that is, each of the n_i must equal 0, 1, or 2. But then each digit of $3n$ is simply three times the corresponding digit of n, and $S(3n) = 3S(n) = 300$, as claimed.

Problem 10.7 The positive real numbers x and y satisfy

$$x^2+y^3 \ge x^3+y^4.$$

Show that $x^3+y^3 \le 2$.

Solution: Equivalently we can prove that if $x^3+y^3 > 2$, then

$$x^2+y^3 < x^3+y^4.$$

First notice that $\sqrt{\frac{x^2+y^2}{2}} \le \sqrt[3]{\frac{x^3+y^3}{2}}$ by the Power-Mean Inequality, implying that

$$\begin{aligned} x^2+y^2 &\le (x^3+y^3)^{2/3} \cdot 2^{1/3} \\ &< (x^3+y^3)^{2/3}(x^3+y^3)^{1/3} \\ &= x^3+y^3, \end{aligned}$$

or $x^2-x^3 < y^3-y^2$. But $0 \le y^2(y-1)^2 \Rightarrow y^3-y^2 \le y^4-y^3$, so that

$$\begin{aligned} x^2-x^3 &< y^4-y^3 \\ \Rightarrow x^2+y^3 &< x^3+y^4, \end{aligned}$$

as desired.

Problem 10.8 In a group of 12 people, among every 9 people one can find 5 people, any two of whom know each other. Show that there exist 6 people in the group, any two of whom know each other.

Solution: Suppose by way of contradiction that no 6 people know each other. Draw a complete graph with twelve vertices corresponding to the people, labeling the people (and their corresponding vertices) $A, B, \ldots, L$. Color the edge between two people red if they know each other, and blue otherwise. Then among every nine vertices there is at least one red K_5; and among any six vertices there is at least one blue edge.

We prove that there are no blue cycles of odd length in this graph. Suppose, for sake of contradiction, that there is a blue cycle of length (i) 3 or 5, (ii) 7, (iii) 9, or (iv) 11.

(i) First suppose there is a blue 3-cycle (say, ABC) or a blue 5-cycle (without loss of generality, $ABCDE$). In the first case, there is a blue edge among $DEFGHI$ (sat, DE); then any red K_5 contains at most one vertex from $\{A, B, C\}$ and at most one vertex from $\{D, E\}$. In the second case, any K_5 still contains at most two vertices from $\{A, B, C, D, E\}$.

Now, $FGHIJK$ contains some other blue edge: without loss of generality, say FG is blue. Now for each edge V_1V_2 in $HIJKL$, there must be a red K_5 among $ABCDEFGV_1V_2$. From before, this K_5 can contain at most two vertices from $\{A, B, C, D, E\}$; and it contains at most one vertex from each of $\{F, G\}$, $\{V_1\}$, and $\{V_2\}$. Therefore V_1 and V_2 must be connected by a red edge, so $HIJKL$ is a red K_5. Now $FHIJKL$ cannot be a red K_6, so without loss of generality suppose FH is blue. Similarly, $GHIJKL$ cannot be a red K_6, so without loss of generality either GH or GI is blue. In either case, $ABCDEFGHI$ must contain some red K_5. If GH is blue then this K_5 contains at most four vertices, two from $\{A, B, C, D, E\}$ and one from each of $\{F, G, H\}$ and $\{I\}$; and if GI is blue then this K_5 again contains at most four vertices, two from $\{A, B, C, D, E\}$ and one from each of $\{F, H\}$ and $\{G, I\}$. Either possibility yields a contradiction.

(ii) If there is some blue 7-cycle, say without loss of generality it is $ABCDEFG$. As before, any K_5 contains at most three vertices from $\{A, B, \ldots, G\}$, so $HIJKL$ must be a red K_5. Now for each of the $\binom{5}{2} = 10$ choices of pairs $\{V_1, V_2\} \subset \{H, I, J, K, L\}$, there must be a red K_5 among $ABCDEFGV_1V_2$; so for each edge in $HIJKL$, some red triangle in $ABCDEFG$ forms a red K_5 with that edge. But $ABCDEFG$ contains at most 7 red triangles: ACE, BDF, $\ldots$, and GBD. Thus some triangle corresponds to two edges. Without loss of generality, either ACE corresponds to both HI and HJ; or ACE corresponds to both HI and JK. In either case, $ACEHIJ$ is a red K_6, a contradiction.

(iii) Next suppose that there is some blue 9-cycle; then among these nine vertices there can be no red K_5, a contradiction.

(iv) Finally, suppose that there is some blue 11-cycle; without loss of generality, say it is $ABCDEFGHIJK$. There is a red K_5 among $\{A, B, C, D, E, F, G, H, I\}$, which must be $ACEGI$. Likewise, $DFHJA$ must be a red K_5, so AC, AD, $\ldots$, AH are all red. Similarly, *every* edge in $ABCDEFGHI$ is red except for those in the blue 11-cycle.

Now among $\{A, B, C, D, E, F, G, H, L\}$ there is some red K_5, either $ACEGL$ or $BDFHL$. Without loss of generality, assume the former. Then since $ACEGLI$ and $ACEGLJ$ can't be red 6-cycles, AI and AJ must be blue. But then AIJ is a blue 3-cycle, a contradiction. ■

Thus there are indeed no blue cycles of odd length, so the blue edges form a bipartite graph: that is, the twelve vertices can be partitioned into two groups $\mathcal{G}_1$ and $\mathcal{G}_2$ containing no blue edges. One of these groups, say $\mathcal{G}_1$, has at least 6 vertices; but then $\mathcal{G}_1$ is a red K_6, a contradiction. Therefore our original assumption was false; there *is* some red K_6, so some six people do indeed know each other.

Problem 11.1 Do there exist 19 distinct natural numbers which add to 1999 and which have the same sum of digits?

Solution: No such integers exist; suppose by way of contradiction they did.

The average of the numbers is $\frac{1999}{19} < 106$, so one number is at most 105 and has digit sum at most 18.

Every number is congruent to its digit sum modulo 9, so all the numbers and their digit sums are congruent modulo 9 — say, congruent to k. Then $19k \equiv 1999 \Rightarrow k \equiv 1 \pmod 9$, so the common digit sum is either 1 or 10.

If it is 1 then all the numbers equal 1, 10, 100, or 1000 so that some two are equal — which is not allowed. Thus the common digit sum is 10. Note that the twenty smallest numbers with digit sum 10 are:

$$19, 28, 37, \ldots, 91, 109, 118, 127, \ldots, 190, 208.$$

The sum of the first nine numbers is $(10 + 20 + \cdots + 90) + (9 + 8 + \cdots + 1) = 450 + 45 = 495$, while the sum of the next nine numbers is $(900)+(10+20+\cdots+80)+(9+8+7+\cdots+1) = 900+360+45 = 1305$, so the first eighteen numbers add up to 1800.

Since $1800 + 190 \neq 1999$, the largest number among the nineteen must be at least 208. But then the smallest eighteen numbers add up to at least 1800, giving a total sum of at least $2028 > 1999$, a contradiction.

Problem 11.2 At each rational point on the real line is written an integer. Show that there exists a segment with rational endpoints, such that the sum of the numbers at the endpoints does not exceed twice the number at the midpoint.

First Solution: Let $f : \mathbb{Q} \to \mathbb{Z}$ be the function that maps each rational point to the integer written at that point. Suppose by way of contradiction that for all $q, r \in \mathbb{Q}$,

$$f(q) + f(r) > 2f\left(\frac{q+r}{2}\right).$$

For $i \geq 0$, let $a_i = \frac{1}{2^i}$ and $b_i = -\frac{1}{2^i}$. We shall prove that for some k, $f(a_k)$ and $f(b_k)$ are both less than $f(0)$. Suppose that for some i, $f(a_i) \geq f(0)$. Now we apply the condition:

$$f(a_{i+1}) < \frac{f(a_i) + f(0)}{2} \leq f(a_i).$$

Since the range of f is the integers, $f(a_{i+1}) \leq f(a_i) - 1$ as long as $f(a_i) \geq f(0)$. Therefore, there exists some m such that $f(a_m) < f(0)$. Then

$$f(a_{m+1}) < \frac{f(a_m) + f(0)}{2} < \frac{2f(0)}{2},$$

so $f(a_i) < f(0)$ for $i \geq m$.

Similarly, there exists n such that $f(b_i) < f(0)$ for $i \geq n$. Now if we just take $k = \max\{m, n\}$, we have a contradiction:

$$f(a_k) + f(b_k) < 2f(0).$$

Second Solution: Define f as in the first solution, and suppose by way of contradiction that there was no such segment; rewrite the inequality in the first solution as

$$f(p) - f\left(\frac{p+q}{2}\right) > f\left(\frac{p+q}{2}\right) - f(q).$$

For a continuous function, this would be equivalent to saying that f is strictly convex; however, f is not continuous. But we can still show a similar result for the set $\mathbb{F} = \left\{\frac{i}{2^j} \mid i, j \in \mathbb{Z}, j \geq 0\right\}$, fractions whose denominators are powers of 2. For convenience, write P_x to represent $(x, f(x))$ on the graph of f in the xy-plane. Then we have the following result:

Lemma. *For all $a, b, c \in \mathbb{F}$ with b between a and c, P_b is below the segment connecting P_a and P_c.*

Proof: Equivalently we can prove that the average rate of change of f in the interval $[a, b]$ is smaller than the average rate of change of f in the interval $[b, c]$ — that is,

$$\frac{f(b) - f(a)}{b - a} < \frac{f(c) - f(b)}{c - a}.$$

Partition $[a, b]$ and $[b, c]$ into sub-intervals of equal length δ. For example, if $a = \frac{\alpha}{2^j}$, $b = \frac{\beta}{2^j}$, and $c = \frac{\gamma}{2^j}$, we could use $\delta = \frac{1}{2^j}$.

Let $\Delta_x = \frac{f(x+\delta) - f(x)}{\delta}$, the average rate of change of f in the interval $[x, x + \delta]$. Then apply our inequality to find that

$$\Delta_a < \Delta_{a+\delta} < \cdots < \Delta_{c-\delta}.$$

Thus,

$$\begin{aligned}
&\frac{f(b) - f(a)}{b - a} \leq \max\left\{\Delta_a, \Delta_{a+\delta}, \ldots, \Delta_{b-\delta}\right\} \\
&= \Delta_{b-\delta} \\
&< \Delta_b \\
&= \min\left\{\Delta_b, \Delta_{b+\delta}, \ldots, \Delta_{c-\delta}\right\}
\end{aligned}$$

$$\leq \frac{f(c) - f(b)}{c - b},$$

as desired. ∎

Now consider all numbers $x \in \mathbb{F}$ between 0 and 1. Since P_x lies below the segment connecting P_0 and P_1, we have $f(x) \leq \max\{f(0), f(1)\}$.

Pick some number $k \in \mathbb{F}$ between 0 and 1. For $k < x < 1$, P_x must lie above the line connecting P_0 and P_k; otherwise, P_k would be above the segment connecting P_0 and P_x, contradicting our lemma. Similarly, for $0 < x < k$, P_x must lie above the line connecting P_k and P_1.

Since there are infinitely many values $x \in \mathbb{F}$ in the interval $(0, 1)$ but $f(x)$ is bounded from above and below in this interval, some three points have the same y-coordinate – contradicting our lemma. Therefore our original assumption was false and the segment described in the problem *does* exist.

Problem 11.3 A circle inscribed in quadrilateral $ABCD$ touches sides DA, AB, BC, CD at K, L, M, N, respectively. Let S_1, S_2, S_3, S_4 be the incircles of triangles AKL, BLM, CMN, DNK, respectively. The common external tangents to S_1 and S_2, to S_2 and S_3, to S_3 and S_4, and to S_4 and S_1, not lying on the sides of $ABCD$, are drawn. Show that the quadrilateral formed by these tangents is a rhombus.

Solution: Let P be the intersection of the two common external tangents involving S_1, and let Q, R, S be the intersections of the pairs of tangents involving S_2, S_3, S_4, respectively.

As in problem 10.3, the centers of S_1, S_2, S_3, S_4 are the midpoints of arcs KL, LM, MN, NK, respectively. AB does not pass through the incenter I of triangle KLM, so by the result proved in problem 9.3 the other external tangent PQ must pass through I and be parallel to KM. Likewise, $RS \parallel KM$ so we have $PQ \parallel RS$.

Similarly, $QR \parallel LN \parallel SP$, so $PQRS$ is a parallelogram.

Let $\langle X \mid \omega \rangle$ denote the length of the tangent from point X to circle ω, and let $\langle \omega_1 \mid \omega_2 \rangle$ denote the length of the external tangent to circles ω_1 and ω_2. Then we also know

$$AB = \langle A \mid S_1 \rangle + \langle S_1 \mid S_2 \rangle + \langle S_2 \mid B \rangle$$

$$= \langle A \mid S_1 \rangle + \langle S_1 \mid P \rangle + PQ + \langle Q \mid S_2 \rangle + \langle S_2 \mid B \rangle$$

and three analogous equations. Substituting these into $AB + CD = BC + DA$, which is true since $ABCD$ is circumscribed about a circle, we find that $PQ + RS = QR + SP$.

But since $PQRS$ is a parallelogram, $PQ = RS$ and $QR = SP$, implying that $PQ = QR = RS = SP$ and that $PQRS$ is a rhombus.

Problem 11.5 Four natural numbers have the property that the square of the sum of any two of the numbers is divisible by the product of the other two. Show that at least three of the four numbers are equal.

Solution: Suppose by way of contradiction four such numbers did exist, and pick a counterexample a, b, c, d with minimum sum $a+b+c+d$. If some prime p divided both a and b, then from $a \mid (b+c)^2$ and $a \mid (b+d)^2$ we know that p divides c and d as well: but then $\frac{a}{p}$, $\frac{b}{p}$, $\frac{c}{p}$, $\frac{d}{p}$ are a counter-example with smaller sum. Therefore, the four numbers are pairwise relatively prime.

Suppose that some prime $p > 2$ divided a. Then since a divides each of $(b+c)^2, (c+d)^2, (d+b)^2$, we know that p divides $b+c$, $c+d$, $d+b$. Hence p divides $(b+c)+(c+d)+(d+b)$ and thus $b+c+d$. Therefore $p \mid (b+c+d)-(b+c) = d$, and similarly $p \mid c$ and $p \mid b$, a contradiction.

Thus each of a, b, c, d are powers of 2. But since they are pairwise relatively prime, three of them must equal 1 — a contradiction. Therefore our original assumption was false, and no such counterexample exists.

Problem 11.6 Show that three convex polygons in the plane cannot be intersected by a single line if and only if for each of the polygons, there exists a line intersecting none of the polygons, such that the given polygon lies on the opposite side of the line from the other two.

Solution: In this proof, "polygon" refers to both the border and interior of a polygon – the problem statement is not affected by this assumption, because a line hitting the interior of a polygon must hit its border as well.

Suppose that some line ℓ intersects all three polygons; orient the figure to make ℓ horizontal, and say it hits the polygons (from left to right) at A, B, and C. Any line m not hitting any of the polygons is either parallel to ℓ; hits ℓ to the left of B; or hits ℓ to the right of B. In all of these cases, m does not separate B from both A and C, so m cannot separate the polygon containing B from the other polygons. (In the first two cases B and C are not separated; and in the first and third cases A and B are not separated.)

To prove the other direction, we begin by proving an intuitively obvious but nontrivial lemma:

Lemma. *Given two non-intersecting polygons, there is a line that separates them.*

Let V be the convex hull of the two polygons. If all its vertices are in one polygon, then this polygon contains the other — a contradiction. Also, for any four vertices A, B, C, D in that order on V (not necessarily adjacent), since AC and BD intersect we cannot have A and C in one polygon and B and D in the other. Thus one run of adjacent vertices $V_1, \ldots, V_m$ is in one polygon P; and the remaining vertices $W_1, \ldots, W_n$ are in the other polygon Q.

Then V_1V_m is contained in polygon P, so line V_1V_m does not intersect Q; therefore we can simply choose a line extremely close to V_1V_m that doesn't hit P, and separates P and Q. ■

Now call the polygons T, U, V, and suppose no line intersects all three. Then every two polygons are disjoint — if M was in $T \cup U$ and $N \neq M$ was in V, then the line MN hits all three polygons.

Triangulate the convex hull H of T and U (that is, divide it into triangles whose vertices are vertices of H). If V intersects H at some point M, then M is on or inside of these triangles, XYZ. Without loss of generality say $X \in T$ and $Y, Z \in U$ (otherwise both triangle XYZ and M are inside either T or U, so this polygon intersects V). Then line XM intersects both T and V; and since it hits YZ, it intersects U as well – a contradiction.

Thus H is disjoint from V, and from the lemma we can draw a line separating the two — and thus separating T and U from V, as desired. We can repeat this construction for T and U, so we are done.

Problem 11.7 Through vertex A of tetrahedron $ABCD$ passes a plane tangent to the circumscribed sphere of the tetrahedron. Show that the lines of intersection of the plane with the planes ABC, ACD,

ABD form six equal angles if and only if

$$AB \cdot CD = AC \cdot BD = AD \cdot BC.$$

Solution: Perform an inversion about A with arbitrary radius r. Since the given plane P is tangent to the circumscribed sphere of $ABCD$, the sphere maps to a plane parallel to P containing B', C', D', the images of B, C, D under inversion. Planes P, ABC, ACD, and ABD stay fixed under the inversion since they all contain A.

Now, since $C'D'$ is in a plane parallel to P, plane $ACD = AC'D'$ intersects P in a line parallel to $C'D'$. More rigorously, complete parallelogram $C'D'AX$. Then X is both in plane $AC'D' = ACD$ and in plane P (since $PX \parallel C'D'$), so the intersection of ACD and P is the line PX, parallel to $C'D'$.

Similarly, plane ADB intersects P in a line parallel to $D'B'$, and plane ABC intersects P in a line parallel to $B'C'$. These lines form six equal angles if and only if $C'D'$, $D'B'$, $B'C'$ form equal angles: that is, if triangle $C'D'B'$ is equilateral and $C'D' = D'B' = B'C'$. Under the inversion distance formula, this is true if and only if

$$\frac{CD \cdot r^2}{AC \cdot AD} = \frac{DB \cdot r^2}{AD \cdot AB} = \frac{BC \cdot r^2}{AB \cdot AC},$$

which (multiplying by $\frac{AB \cdot AC \cdot AD}{r^2}$) is equivalent to

$$AB \cdot CD = AC \cdot BD = AD \cdot BC,$$

as desired.

1.18 Slovenia

Problem 1 The sequence of real numbers $a_1, a_2, a_3, \ldots$ satisfies the initial conditions $a_1 = 2, a_2 = 500, a_3 = 2000$ as well as the relation

$$\frac{a_{n+2} + a_{n+1}}{a_{n+1} + a_{n-1}} = \frac{a_{n+1}}{a_{n-1}}$$

for $n = 2, 3, 4, \ldots$. Prove that all the terms of this sequence are positive integers and that 2^{2000} divides the number a_{2000}.

Solution: From the recursive relation it follows that $a_{n+2}a_{n-1} = a_{n+1}^2$ for $n = 2, 3, \ldots$. No term of our sequence can equal 0, and hence it is possible to write

$$\frac{a_{n+2}}{a_{n+1}a_n} = \frac{a_{n+1}}{a_n a_{n-1}}$$

for $n = 2, 3, \ldots$. It follows by induction that the value of the expression $\frac{a_{n+1}}{a_n a_{n-1}}$ is constant, namely equal to $\frac{a_3}{a_2 a_1} = 2$. Thus $a_{n+2} = 2a_n a_{n+1}$ and all terms of the sequence are positive integers.

From this new relation, we also know that $\frac{a_{n+1}}{a_n}$ is an even integer for all positive integers n. Write $a_{2000} = \frac{a_{2000}}{a_{1999}} \frac{a_{1999}}{a_{1998}} \cdots \frac{a_2}{a_1} \cdot a_1$. In this product each of the 1999 fractions is divisible by 2, and $a_1 = 2$ is even as well. Thus a_{2000} is indeed divisible by 2^{2000}.

Problem 2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the condition

$$f(x - f(y)) = 1 - x - y$$

for all $x, y \in \mathbb{R}$.

Solution: For $x = 0, y = 1$ we get $f(-f(1)) = 0$. For $y = -f(1)$ it follows that $f(x) = 1 + f(1) - x$. Writing $a = 1 + f(1)$ and $f(x) = a - x$, we have

$$1 - x - y = f(x - f(y)) = a - x + f(y) = 2a - x - y$$

so that $a = \frac{1}{2}$. And indeed, the function $f(x) = \frac{1}{2} - x$ satisfies the functional equation.

Problem 3 Let E be the intersection of the diagonals in cyclic quadrilateral $ABCD$, and let F and G be the midpoints of sides AB and CD, respectively. Prove that the three lines through G, F, E perpendicular to $\overline{AC}, \overline{BD}, \overline{AD}$, respectively, intersect at one point.

Solution: All angles are directed modulo 180°. Drop perpendicular $\overline{GP}$ to diagonal AC and perpendicular $\overline{FQ}$ to diagonal BD. Let R be the intersection of lines PG and FQ, and let H be the foot of the perpendicular from E to side AD. We wish to prove that H, E, R are collinear.

Since F and G are midpoints of corresponding sides in similar triangles DEC and ABE (with opposite orientations), triangles DPE and AQE are similar with opposite orientations as well. Thus $\angle DPE = \angle EQA$ and therefore $AQPD$ is a cyclic quadrilateral. And because $\angle EQR = 90^\circ = \angle EPR$, the quadrilateral $EQRP$ is cyclic, too. So

$$\angle ADQ = \angle APQ = \angle EPQ = \angle ERQ.$$

It follows that $\angle DEH = 90^\circ - \angle ADQ = 90^\circ - \angle ERQ = \angle QER$; and since D, E, Q are collinear then H, E, R must be as well.

Problem 4 Three boxes with at least one marble in each are given. In a *step* we choose two of the boxes, doubling the number of marbles in one of the boxes by taking the required number of marbles from the other box. Is it always possible to empty one of the boxes after a finite number of steps?

Solution: Without loss of generality suppose that the number of marbles in the boxes are a, b, and c with $a \leq b \leq c$. Write $b = qa + r$ where $0 \leq r < a$ and $q \geq 1$. Then express q in binary:

$$q = m_0 + 2m_1 + \cdots + 2^k m_k,$$

where each $m_i \in \{0, 1\}$ and $m_k = 1$. Now for each $i = 0, 1, \ldots, k$, add $2^i a$ marbles to the first box: if $m_i = 1$ take these marbles from the second box; otherwise take them from the third box. In this way we take at most $(2^k - 1)a < qa \leq b \leq c$ marbles from the third box and exactly qa marbles from the second box altogether.

In the second box there are now $r < a$ marbles left. Thus the box with the least number of marbles now contains less than a marbles. Then by repeating the described procedure, we will eventually empty one of the boxes.

1.19 Taiwan

Problem 1 Determine all solutions (x, y, z) of positive integers such that

$$(x+1)^{y+1} + 1 = (x+2)^{z+1}.$$

Solution: Let $a = x+1$, $b = y+1$, $c = z+1$. Then $a, b, c \geq 2$ and

$$a^b + 1 = (a+1)^c$$
$$((a+1)-1)^b + 1 = (a+1)^c.$$

Taking either equation mod $(a+1)$ yields $(-1)^b + 1 \equiv 0$, so b is odd. Then taking the second equation mod $(a+1)^2$ after applying the binomial expansion yields

$$\binom{b}{1}(a+1)(-1)^{b-1} + (-1)^b + 1 \equiv 0 \pmod{(a+1)^2}$$

so $a+1 \mid b$ and a is even.

On the other hand, taking the first equation mod a^2 after applying the binomial expansion yields

$$1 \equiv \binom{c}{1}a + 1 \pmod{a^2}$$

so c is divisible by a and is even as well. Write $a = 2a_1$ and $c = 2c_1$. Then

$$2^b a_1^b = a^b = (a+1)^c - 1 = ((a+1)^{c_1} - 1)((a+1)^{c_1} + 1).$$

It follows that $\gcd((a+1)^{c_1} - 1, (a+1)^{c_1} + 1) = 2$. Therefore, using the fact that $2a_1$ is a divisor of $(a+1)^{c_1} - 1$, we may conclude that

$$(a+1)^{c_1} - 1 = 2a_1^b$$
$$(a+1)^{c_1} + 1 = 2^{b-1}.$$

We must have $2^{b-1} > 2a_1^b \Rightarrow a_1 = 1$. Then these equations give $c_1 = 1$ and $b = 3$, and therefore the only solution is $(x, y, z) = (1, 2, 1)$.

Problem 2 There are 1999 people participating in an exhibition. Out of any 50 people, at least 2 do not know each other. Prove that we can find at least 41 people who each know at most 1958 other people.

Solution: Let Y be the set of people who know at least 1959 other people, and let $N(p)$ denote the set of people whom p knows. Assume by way of contradiction that less than 41 people each know at most 1958 people; then $|Y| \geq 1959$. We now show that some 50 people all know each other, a contradiction.

Pick a person $y_1 \in Y$ and write $B_1 = N(y_1)$ with $|B_1| \geq 1959$. Then $|B_1| + |Y| > 1999$, and there is a person $y_2 \in B_1 \cap Y$.

Now write $B_2 = N(y_1) \cap N(y_2)$ with $|B_2| = |B_1| + |N(y_2)| - |B_1 \cup N(y_2)| \geq 1959 + 1959 - 1999 = 1999 - 40 \cdot 2$. Then $|B_2| + |Y| > 1999$, and there is a person $y_3 \in B_2 \cap Y$.

Now continue similarly: suppose we have $j \leq 48$ different people $y_1, y_2, \ldots, y_j$ in Y who all know each other; and suppose that $B_j = N(y_1) \cap N(y_2) \cap \cdots \cap N(y_j)$ has at least $1999 - 40j \geq 79 > 40$ elements. Then $|B_j| + |Y| > 1999$, and there is a person $y_{j+1} \in B_j \cap Y$; and $B_{j+1} = B_j \cap N(y_{j+1})$ has at least $|B_j| + |N(y_{j+1})| - |B_j \cup N(y_{j+1})| \geq (1999 - 40j) + 1959 - 1999 = 1959 - 40(j+1) > 0$ elements, and we can continue onward.

Thus we can find 49 people $y_1, y_2, \ldots, y_{49}$ such that $B_{49} = N(y_1) \cap N(y_2) \cap \cdots \cap N(y_{49})$ is nonempty. Thus there is a person $y_{50} \in B_{49}$; but then any two people from $y_1, y_2, \ldots, y_{50}$ know each other, a contradiction.

Problem 3 Let P^* denote all the odd primes less than 10000, and suppose $p \in P^*$. For each subset $S = \{p_1, p_2, \cdots, p_k\}$ of P^*, with $k \geq 2$ and not including p, there exists a $q \in P^* \setminus S$ such that

$$q + 1 \mid (p_1 + 1)(p_2 + 1) \cdots (p_k + 1).$$

Find all such possible values of p.

Solution: A "Mersenne prime" is a prime of the form $2^n - 1$ for some positive integer n. Notice that if $2^n - 1$ is prime then $n > 1$ and n is prime because otherwise we could either write (if n were even) $n = 2m$ and $2^n - 1 = (2^m - 1)(2^m + 1)$, or (if n were odd) $n = ab$ for odd a, b and $2^n - 1 = (2^a - 1)(2^{(b-1)a} + 2^{(b-2)a} + \cdots + 2^a + 1)$. Then some calculations show that the set T of Mersenne primes less than 10000 is

$$\{M_2, M_3, M_5, M_7, M_{13}\} = \{3, 7, 31, 127, 8191\},$$

where $M_p = 2^p - 1$. ($2^{11} - 1$ is not prime: it equals $23 \cdot 89$.) We claim this is the set of all possible values of p.

If some prime p is *not* in T, then look at the set $S = T$. Then there must be some prime $q \notin S$ less than 10000 such that

$$q + 1 \mid (M_2 + 1)(M_3 + 1)(M_5 + 1)(M_7 + 1)(M_{13} + 1) = 2^{30}.$$

Thus, $q+1$ is a power of 2 and q is a Mersenne prime less than 10000 — and therefore $q \in T = S$, a contradiction.

On the other hand, suppose p is in T. Suppose we have a set $S = \{p_1, p_2, \ldots, p_k\} \subset P^*$ not including p, with $k \geq 2$ and $p_1 < p_2 < \cdots < p_k$. Suppose by way of contradiction that for all $q \in P^*$ such that $q + 1 \mid (p_1 + 1) \cdots (p_k + 1)$, we have $q \in S$. Then

$$4 \mid (p_1 + 1)(p_2 + 1) \Longrightarrow M_2 \in S$$

$$8 \mid (M_2 + 1)(p_2 + 1) \Longrightarrow M_3 \in S$$

$$32 \mid (M_2 + 1)(M_3 + 1) \Longrightarrow M_5 \in S$$

$$128 \mid (M_2 + 1)(M_5 + 1) \Longrightarrow M_7 \in S$$

$$8192 \mid (M_3 + 1)(M_5 + 1)(M_7 + 1) \Longrightarrow M_{13} \in S.$$

Then p, a Mersenne prime under 10000, must be in S — a contradiction. Therefore there *is* some prime $q < 10000$ not in S with $q + 1 \mid (p_1 + 1) \cdots (p_k + 1)$, as desired. This completes the proof.

Problem 4 The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F, respectively, and $AB > AC$. The line EF meets BC at P, and the line through D parallel to EF meets the lines AC and AB at Q and R, respectively. Let N be a point on the side BC such that $\angle NQP + \angle NRP < 180^\circ$. Prove that $BN > CN$.

Solution: Let M be the midpoint of BC. We claim that P, Q, M, R are concyclic. Given this, we would have

$$\angle MQP + \angle MRP = 180^\circ > \angle NQP + \angle NRP.$$

This can only be true if N is between M and C; then $BN > CN$, as desired.

Since $\angle BEC = \angle BFC = 90^\circ$, we observe that the points B, C, E, F are concyclic and thus $PB \cdot PC = PE \cdot PF$. Also, the points D, E, F, M lie on the nine-point circle of triangle ABC so that

$PE \cdot PF = PD \cdot PM$. (Alternatively, it's easy to show that $DEFM$ is cyclic with some angle-chasing). These two equations yield

$$PB \cdot PC = PD \cdot PM. \tag{1}$$

On the other hand, since $\triangle AEF \sim \triangle ABC$ and $QR \parallel EF$, we have $\angle RBC = \angle AEF = \angle CQR$. Thus $CQBR$ is cyclic and

$$DQ \cdot DR = DB \cdot DC. \tag{2}$$

Now let $MB = MC = a$, $MD = d$, $MP = p$. Then we have $PB = p + a$, $DB = a + d$, $PC = p - a$, $CD = a - d$, $DP = p - d$. Then equation (1) implies

$$\begin{gathered} (p+a)(p-a) = (p-d)p \\ \Longrightarrow a^2 = dp \\ \Longrightarrow (a+d)(a-d) = (p-d)d, \end{gathered}$$

or equivalently

$$DB \cdot DC = DP \cdot DM. \tag{3}$$

Combining (2) and (3) yields $DQ \cdot DR = DP \cdot DM$, so that the points P, Q, M, R are concyclic, as claimed.

Problem 5 There are 8 different symbols designed on n different T-shirts, where $n \geq 2$. It is known that each shirt contains at least one symbol, and for any two shirts, the symbols on them are not all the same. Also, for any k symbols, $1 \leq k \leq 7$, the number of shirts containing at least one of the k symbols is even. Find the value of n.

Solution: Let X be the set of 8 different symbols, and call a subset S of X "stylish" if some shirt contains exactly those symbols in S. Look at a stylish set A with the minimal number of symbols $|A| \geq 1$; since $n \geq 2$, we must have $|A| \leq 7$. Then all the other $n-1$ stylish sets contain at least one of the $k = 8 - |A|$ symbols in $X \setminus A$, so $n-1$ is even and n is odd.

Observe that any nonempty subset $S \subseteq X$ contains an odd number of stylish subsets: For $S = X$ this number is n; and for $|S| \leq 7$, an even number t of stylish sets contain some element of $X \setminus S$, so the remaining *odd* number $n - t$ of stylish sets are contained in S.

Then every nonempty subset of X is stylish. Otherwise, pick a minimal non-stylish subset $S \subseteq X$. Its only stylish subsets are its

$2^{|S|}-2$ proper subsets, which are all stylish by the minimal definition of S; but this is an *even* number, which is impossible. Thus there must be $2^8 - 1 = 255$ T-shirts; and indeed, given any k symbols $(1 \le k \le 7)$, an even number $2^8 - 2^{8-k}$ of T-shirts contain at least one of these k symbols.

1.20 Turkey

Problem 1 Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on $\overline{BC}$ such that $BD = 2DC$, and let P be a point on $\overline{AD}$ such that $\angle BAC = \angle BPD$. Prove that

$$\angle BAC = 2\angle DPC.$$

Solution: Draw X on $\overline{BP}$ such that $BX = AP$. Then $\angle ABX = \angle ABP = \angle DPB - \angle PAB = \angle CAB - \angle PAB = \angle CAP$. And since $AB = CA$ and $BX = AP$, by SAS we have $\triangle ABX \cong \triangle CAP$. Hence $[ABX] = [CAP]$, and also $\angle DPC = 180^\circ - \angle CPA = 180^\circ - \angle AXB = \angle PXA$.

Next, since $BD = 2CD$, the distance from B to line AD is twice the distance from C to line AD. Therefore $[ABP] = 2[CAP] \Longrightarrow [ABX] + [AXP] = 2[ABX]$. Hence $[AXP] = [ABX]$ and $XP = BX = AP$. Hence $\angle PXA = \angle XAP$, and $\angle BAC = \angle BPD = \angle PXA + \angle XAP = 2\angle PXA = 2\angle DPC$, as desired.

Problem 2 Prove that

$$(a + 3b)(b + 4c)(c + 2a) \geq 60abc$$

for all real numbers $0 \leq a \leq b \leq c$.

Solution: By AM-GM we have $a + b + b \geq 3\sqrt[3]{ab^2}$; multiplying this and the analagous inequalities yields $(a+2b)(b+2c)(c+2a) \geq 27abc$. Then

$$\begin{aligned}
&(a + 3b)(b + 4c)(c + 2a)\\
&\geq \left(a + \frac{1}{3}a + \frac{8}{3}b\right)\left(b + \frac{2}{3}b + \frac{10}{3}c\right)(c + 2a)\\
&= \frac{20}{9}(a + 2b)(b + 2c)(c + 2a) \geq 60abc,
\end{aligned}$$

as desired.

Problem 3 The points on a circle are colored in three different colors. Prove that there exist infinitely many isosceles triangles with vertices on the circle and of the same color.

First Solution: Partition the points on the circle into infinitely many regular 13-gons. In each 13-gon, by the Pigeonhole Principle there are at least 5 vertices of the same color: say, red. Later we use some extensive case analysis to show that among these 5 vertices, some three form an isosceles triangle. Then for each 13-gon there is a monochrome isosceles triangle; so there are infinitely many monochrome isosceles triangles, as desired.

It suffices now to prove the following claim:

Claim *Suppose 5 vertices of a regular 13-gon are colored red. Then some three red vertices form an isosceles triangle.*

Proof: Suppose none of these 5 vertices did form an isosceles triangle. Label the vertices $P_0, \ldots, P_{12}$ (with indices taken modulo 13); first we prove that P_i and P_{i+2} cannot both be red. Assume they could be, and say without loss of generality that P_{12} and P_1 were red; then P_{10}, P_0, and P_3 cannot be red. Furthermore, at most one vertex from each pair (P_{11}, P_4), (P_4, P_7), and (P_7, P_8) is red since each of these pairs forms an isosceles triangle with P_1. Similarly, at most one vertex from each pair (P_2, P_9), (P_9, P_6), and (P_6, P_5) is red. Now three vertices from $\{P_{11}, P_4, P_7, P_8\} \cup \{P_2, P_9, P_6, P_5\}$ are red; assume without loss of generality that two vertices from $\{P_{11}, P_4, P_7, P_8\}$ are. Vertices P_4 and P_8 can't both be red because they form an isosceles triangle with P_{12}; so vertices P_{11} and P_7 must be red. But then any remaining vertex forms an isosceles triangle with some two of P_1, P_7, P_{11}, P_{12}, so we can't have five red vertices, a contradiction.

Next we prove that P_i and P_{i+1} can't be red. If so, suppose without loss of generality that P_6 and P_7 are red. Then P_4, P_5, P_8, and P_9 cannot be red from the result in the last paragraph. P_0 cannot be red either, because triangle $P_0P_6P_7$ is isosceles. Now each pair (P_3, P_{11}) and (P_{11}, P_1) contains at most one red vertex because triangles $P_3P_7P_{11}$ and $P_1P_6P_{11}$ are isosceles. Also, P_1 and P_3 can't both be red from the result in the last paragraph. Thus at most one of $\{P_1, P_3, P_{11}\}$ can be red; similarly, at most one of $\{P_{12}, P_{10}, P_2\}$ can be red. But then we have at most four red vertices, again a contradiction.

Thus if P_i is red then $P_{i-2}, P_{i-1}, P_{i+1}, P_{i+2}$ cannot be red; but then we can have at most four red vertices, a contradiction. ■

Second Solution: Suppose we have $k \geq 1$ colors and a number $n \geq 3$. Then Van der Warden's theorem states that we can find N such that for any coloring of the numbers $1, 2, \ldots, N$ in the k colors, there are n numbers in arithmetic progression which are colored the same. Apply this theorem with $k = n = 3$ to find such an N, and partition the points on the circle into infinitely many regular N-gons rather than 13-gons. For each N-gon $P_1 P_2 \ldots P_N$, there exist i, j, k (between 1 and N) in arithmetic progression such that P_i, P_j, P_k are all the same color. Hence triangle $P_i P_j P_k$ is a monochrome isosceles triangle. It follows that since we have infinitely many such N-gons, there are infinitely many monochrome isosceles triangles.

Problem 4 Let $\angle XOY$ be a given angle, and let M and N be two points on the rays OX and OY, respectively. Determine the locus of the midpoint of $\overline{MN}$ as M and N varies along the rays OX and OY such that $OM + ON$ is constant.

Solution: Let $\hat{x}$ and $\hat{y}$ be the unit vectors pointing along rays OX and OY. Suppose we want $OM + ON$ to equal the constant k; then when $OM = c$ we have $ON = k - c$, and thus the midpoint of $\overline{MN}$ is $\frac{1}{2}(c\hat{x} + (k - c)\hat{y})$. As c varies from 0 to k, this traces out the line segment connecting $\frac{1}{2}k\hat{x}$ with $\frac{1}{2}k\hat{y}$; that is, the segment $\overline{M'N'}$ where $OM' = ON' = \frac{1}{2}k$, $M' \in \overrightarrow{OX}$, and $N' \in \overrightarrow{OY}$.

Problem 5 Some of the vertices of the unit squares of an $n \times n$ chessboard are colored such that any $k \times k$ square formed by these unit squares has a colored point on at least one of its sides. If $l(n)$ denotes the minimum number of colored points required to ensure the above condition, prove that

$$\lim_{n \to \infty} \frac{l(n)}{n^2} = \frac{2}{7}.$$

Solution: For each colored point P, consider any 1×1 square of the board it lies on. If this square contains m colored points, say that P gains $\frac{1}{m}$ points from that square. Adding over all the 1×1 squares that P lies on, we find the total number of points that P accrues.

Any colored point on the edge of the chessboard gains at most 2 points. As for a colored point P on the chessboard's interior, the 2×2 square centered at P must have a colored point Q on its border. Then P and Q both lie on some unit square, which P gains at most half a point from; thus P accrues at most $\frac{7}{2}$ points.

Therefore any colored point collects at most $\frac{7}{2}$ points, and $l(n)$ colored points collectively accrue at most $\frac{7}{2}l(n)$ points. But for the given condition to hold, the total number of points accrued must be n^2. It follows that $\frac{7}{2}l(n) \geq n^2$ and thus $\frac{l(n)}{n^2} \geq \frac{2}{7}$.

Now, given some $n \times n$ board, embed it as the corner of an $n' \times n'$ board where $7 \mid n' + 1$ and $n \leq n' \leq n + 6$. To each 7×7 grid of vertices on the $n' \times n'$ board, color the vertices as below:

$$\begin{array}{ccccccc}
\bullet & \circ & \bullet & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \bullet & \circ & \bullet \\
\circ & \bullet & \circ & \bullet & \circ & \circ & \circ \\
\bullet & \circ & \circ & \circ & \circ & \bullet & \circ \\
\circ & \circ & \bullet & \circ & \bullet & \circ & \circ \\
\circ & \bullet & \circ & \circ & \circ & \circ & \bullet \\
\circ & \circ & \circ & \bullet & \circ & \bullet & \circ
\end{array}$$

Then any $k \times k$ square on the chessboard has a colored point on at least one of its sides. Since we color $\frac{2}{7}(n'+1)^2$ vertices in this coloring, we have

$$l(n) \leq \frac{2}{7}(n' + 1)^2 \leq \frac{2}{7}(n + 7)^2$$

so that

$$\frac{l(n)}{n^2} \leq \frac{2}{7}\left(\frac{n + 7}{n}\right)^2.$$

As $n \to \infty$, the right hand side becomes arbitrarily close to $\frac{2}{7}$. Since from before $\frac{l(n)}{n^2} \geq \frac{2}{7}$ for all n, this implies that $\lim_{n\to\infty} \frac{l(n)}{n^2}$ exists and equals $\frac{2}{7}$.

Problem 6 Let $ABCD$ be a cyclic quadrilateral, and let L and N be the midpoints of diagonals AC and BD, respectively. Suppose that $\overline{BD}$ bisects $\angle ANC$. Prove that $\overline{AC}$ bisects $\angle BLD$.

Solution: Suppose we have *any* cyclic quadrilateral $ABCD$ where L and N are the midpoints of $\overline{AC}$ and $\overline{BD}$. Perform an inversion about B with arbitrary radius; A, D, C map to collinear points A', D', C', while N maps to the point N' such that D' is the midpoint of $\overline{BN'}$.

There are only two points X on line $A'D'$ such that $\angle BXN' = \angle BA'N'$: the point A' itself, and the reflection of A' across D'. Then $\angle ANB = \angle BNC \iff \angle BA'N' = \angle BC'N' \iff A'D' = D'C' \iff \frac{AD}{BA \cdot BD} = \frac{DC}{BD \cdot BC} \iff AD \cdot BC = BA \cdot DC$.

Similarly, $\angle BLA = \angle DLA \iff AD \cdot BC = BA \cdot DC$. Therefore $\angle ANB = \angle BNC \iff \angle BLA = \angle DLA$; that is, $\overline{BD}$ bisects $\angle ANC$ if and only if $\overline{AC}$ bisects $\angle BLD$, which implies the claim.

Problem 7 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the set

$$\left\{ \frac{f(x)}{x} \mid x \in \mathbb{R} \text{ and } x \neq 0 \right\}$$

is finite and

$$f(x - 1 - f(x)) = f(x) - x - 1$$

for all $x \in \mathbb{R}$.

Solution: First we show that the set $\{x - f(x) \mid x \in \mathbb{R}\}$ is finite. If not, there exist infinitely many $k \neq 1$ such that for some x_k, $x_k - f(x_k) = k$. But then

$$\frac{f(k-1)}{k-1} = \frac{f(x_k - 1 - f(x_k))}{k-1} = \frac{f(x_k) - x_k - 1}{k-1} = -1 - \frac{2}{k-1}.$$

Since k takes on infinitely many values, $\frac{f(k-1)}{k-1}$ does as well—a contradiction.

Now choose x_0 so that $|x - f(x)|$ is maximal for $x = x_0$. Then for $y = x_0 - 1 - f(x_0)$ we have

$$y - f(y) = y - (f(x_0) - x_0 - 1) = 2(x_0 - f(x_0)).$$

Then because of the maximal definition of x_0, we must have $y - f(y) = x_0 - f(x_0) = 0$. Therefore $f(x) = x$ for all x, and this function indeed satisfies the given conditions.

Problem 8 Let the area and the perimeter of a cyclic quadrilateral C be A_C and P_C, respectively. If the area and the perimeter of the quadrilateral which is tangent to the circumcircle of C at the vertices of C are A_T and P_T, respectively, prove that

$$\frac{A_C}{A_T} \geq \left(\frac{P_C}{P_T}\right)^2.$$

Solution: Let the outer quadrilateral be $EFGH$ with angles $\angle E = 2\alpha_1$, $\angle F = 2\alpha_2$, $\angle G = 2\alpha_3$, $\angle H = 2\alpha_4$; also let the circumcircle of C have radius r and center O. Say that sides EF, FG, GH, HE are tangent to C at I, J, K, L.

In right triangle EIO, we have $IO = r$ and $\angle OEI = \alpha_1$ so that $EI = r\cot\alpha_1$. After finding $IF, FJ, \ldots, LE$ similarly, we find that $P_T = 2r\sum_{i=1}^4 \cot\alpha_i$. Also, $[EFO] = \frac{1}{2}EF \cdot IO = \frac{1}{2}EF \cdot r$; finding $[FGO], [GHO], [HEO]$ similarly shows that $A_T = \frac{1}{2}P_T \cdot r$.

As for quadrilateral $IJKL$, note that $IJ = 2r\sin\angle IKJ = 2r\sin\angle FIJ = 2r\sin(90^\circ - \alpha_2) = 2r\cos\alpha_2$. After finding JK, KL, LI in a similar manner we have $P_C = 2r\sum_{i=1}^4 \cos\alpha_i$. Also note that $\angle IOJ = 180^\circ - \angle JFI = 180^\circ - 2\alpha_2$, and hence $[IOJ] = \frac{1}{2}OI \cdot OJ \sin\angle IOJ = \frac{1}{2}r^2\sin(2\alpha_2) = r^2\sin\alpha_2\cos\alpha_2$. Adding this to the analogous expressions for $[JOK], [KOL], [LOI]$, we find that $A_C = r^2\sum_{i=1}^4 \sin\alpha_i\cos\alpha_i$.

Therefore the inequality we wish to prove is

$$\begin{aligned}
&A_C \cdot P_T^2 \geq A_T \cdot P_C^2 \\
&\iff r^2 \sum_{i=1}^4 \sin\alpha_i\cos\alpha_i \cdot P_T^2 \geq \left(\frac{1}{2}P_T \cdot r\right) \cdot 4r^2 \left(\sum_{i=1}^4 \cos\alpha_i\right)^2 \\
&\iff P_T \cdot \sum_{i=1}^4 \sin\alpha_i\cos\alpha_i \geq 2r \cdot \left(\sum_{i=1}^4 \cos\alpha_i\right)^2 \\
&\iff \sum_{i=1}^4 \cot\alpha_i \cdot \sum_{i=1}^4 \sin\alpha_i\cos\alpha_i \geq \left(\sum_{i=1}^4 \cos\alpha_i\right)^2.
\end{aligned}$$

But this is true by the Cauchy-Schwarz inequality $\sum a_i^2 \sum b_i^2 \geq \left(\sum a_i b_i\right)^2$ applied with each $a_i = \sqrt{\cot\alpha_i}$ and $b_i = \sqrt{\sin\alpha_i\cos\alpha_i}$.

Problem 9 Prove that the plane is not a union of the inner regions of finitely many parabolas. (The outer region of a parabola is the union of the lines on the plane not intersecting the parabola. The inner region of a parabola is the set of points on the plane that do not belong to the outer region of the parabola.)

Solution: Suppose by way of contradiction we could cover the plane with the inner regions of finitely many parabolas — say, n of them. Choose some fixed positive acute angle $\theta < \left(\frac{360}{2n}\right)^\circ$.

Take any of the parabolas and (temporarily) choose a coordinate system so that it satisfies the equation $y = ax^2$ with $a \geq 0$ (and where our coordinates are chosen to scale, so that one unit along the y-axis has the same length as one unit along the x-axis). Draw the tangents to the parabola at $x = \pm\frac{\cot\theta}{2a}$; these lines have slopes $2ax = \pm\cot\theta$. These lines meet on the y-axis at an angle of 2θ, forming a V-shaped region in the plane that contains the inner region of the parabola.

Performing the above procedure with all the parabolas, we obtain n V-shaped regions covering the entire plane. Again choose an x-axis, and say the rays bordering these regions make angles ϕ_j and $\phi_j + 2\theta$ with the positive x-axis (with angles taken modulo $360°$). Then since $2n\theta < 360°$, there is some angle ϕ' not in any of the intervals $[\phi_j, \phi_j + 2\theta]$. Then consider the line passing through the origin and making angle of ϕ' with the positive x-axis; far enough out, the points on this line cannot lie in any of the V-shaped regions, a contradiction. Thus our original assumption was false, and we *cannot* cover the plane with the inner regions of finitely many parabolas.

1.21 Ukraine

Problem 1 Let $P(x)$ be a polynomial with integer coefficients. The sequence $\{x_n\}_{n\geq 1}$ satisfies the conditions $x_1 = x_{2000} = 1999$, and $x_{n+1} = P(x_n)$ for $n \geq 1$. Calculate

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{1999}}{x_{2000}}.$$

Solution: Write $a_i = x_i - x_{i-1}$ for each i, where we take subscripts (of both the x_i and a_i) modulo 1999. Since $c-d$ divides $P(c)-P(d)$ for integers c and d, we have that $a_i = x_i - x_{i-1}$ divides $P(x_i) - P(x_{i-1}) = a_{i+1}$ for all i.

First suppose that all the $a_i \neq 0$. Then $|a_{i+1}| \geq |a_i|$ for all i but also $|a_1| = |a_{2000}|$; hence all the $|a_i|$ equal the same value $m > 0$. But if n of the $a_1, a_2, \ldots, a_{1999}$ equal $m \neq 0$ and the other $1999 - n$ equal $-m$, then their sum $0 = x_{1999} - x_0 = a_1 + a_2 + \cdots + a_{1999}$ equals $m(2n - 1999) \neq 0$, a contradiction.

Thus for some k we have $a_k = 0$; then since a_k divides a_{k+1}, we have $a_{k+1} = 0$ and similarly $a_{k+2} = 0$, and so on. Thus all the x_i are equal and the given expression equals 1999.

Problem 2 For real numbers $0 \leq x_1, x_2, \ldots, x_6 \leq 1$ prove the inequality

$$\frac{x_1^3}{x_2^5 + x_3^5 + x_4^5 + x_5^5 + x_6^5 + 5} + \frac{x_2^3}{x_1^5 + x_3^5 + x_4^5 + x_5^5 + x_6^5 + 5} + \cdots + \frac{x_6^3}{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + 5} \leq \frac{3}{5}.$$

Solution: The condition $0 \leq x_1, x_2, \ldots, x_6 \leq 1$ implies that the left hand side of the inequality is at most

$$\sum_{i=1}^{6} \frac{x_i^3}{x_1^5 + x_2^5 + \cdots + x_6^5 + 4} = \frac{x_1^3 + x_2^3 + \cdots + x_6^3}{x_1^5 + x_2^5 + \cdots + x_6^5 + 4}.$$

For $t \geq 0$ we have $\frac{t^5+t^5+t^5+1+1}{5} \geq t^3$ by AM-GM. Adding up the six resulting inequalities for $t = x_1, x_2, \ldots, x_6$ and dividing by $(x_1^5 + x_2^5 + \cdots + x_6^5 + 4)$ shows that the above expression is at most $\frac{3}{5}$.

Problem 3 Let $\overline{AA_1}, \overline{BB_1}, \overline{CC_1}$ be the altitudes of an acute triangle ABC, and let O be an arbitrary point inside the triangle $A_1B_1C_1$.

Let M, N, P, Q, R, S be the orthogonal projections of O onto lines AA_1, BC, BB_1, CA, CC_1, AB, respectively. Prove that lines MN, PQ, RS are concurrent.

Solution: Observe that three lines passing through different vertices of a triangle are concurrent if and only if their reflections across the corresponding angle bisectors are also concurrent; this is easily proved using the trigonometric form of Ceva's Theorem.

Let A_0, B_0, C_0 be the centers of rectangles OMA_1N, OPB_1Q, OSC_1R, respectively. Under the homothety with center O and ratio $\frac{1}{2}$, triangle $A_1B_1C_1$ maps to triangle $A_0B_0C_0$. Then since lines AA_1, BB_1, CC_1 are the angle bisectors of triangle $A_1B_1C_1$ (easily proved with angle-chasing), the angle bisectors of triangle $A_0B_0C_0$ are parallel to lines AA_1, BB_1, CC_1.

Because OMA_1N is a rectangle, diagonals OA_1 and MN are reflections of each across the line through A_0 parallel to line AA_1. From above, this line is precisely the angle bisector of $\angle C_0A_0B_0$ in triangle $A_0B_0C_0$. Similarly, lines OB_1 and OC_1 are reflections of lines PQ and RS across the other angle bisectors. Then since lines OA_1, OB_1, OC_1 concur at O, from our initial observation lines MN, PQ, RS concur as well.

1.22 United Kingdom

Problem 1 I have four children. The age in years of each child is a positive integer between 2 and 16 inclusive and all four ages are distinct. A year ago the square of the age of the oldest child was equal to sum of the squares of the ages of the other three. In one year's time the sum of the squares of the ages of the oldest and the youngest children will be equal to the sum of the squares of the other two children. Decide whether this information is sufficient to determine their ages uniquely, and find all possibilities for their ages.

Solution: Let the children's present ages be $a+1$, $b+1$, $c+1$, and $d+1$. We are given that $1 \le a < b < c < d \le 15$; note that $b \le 13$ so that $b - a \le 12$. We are also given

$$d^2 = a^2 + b^2 + c^2 \tag{1}$$

and

$$(d+2)^2 + (a+2)^2 = (b+2)^2 + (c+2)^2. \tag{2}$$

Subtracting (1) from (2) gives $4(a+d) + a^2 = 4(b+c) - a^2$, or

$$a^2 = 2(b + c - a - d). \tag{3}$$

Then a must be even since its square is even. Furthermore, since $d > c$,

$$a^2 = 2(b - a + (c - d)) < 2(b - a) < 24,$$

and hence either $a = 2$ and $a = 4$.

If $a = 4$ then, since $a^2 < 2(b - a)$, we have $2b > a^2 + 2a = 24$ so that $b > 12$. This forces $b = 13$, $c = 14$, and $d = 15$, which contradicts the given conditions.

Thus $a = 2$. Equation (3) gives $b + c - d = 4$, so substituting $a = 2$ and $d = b + c - 4$ into (1) and simplifying yields

$$(b - 4)(c - 4) = 10 = 1 \cdot 10 = 2 \cdot 5.$$

Therefore we have $(b, c) = (5, 14)$ or $(6, 9)$, in which cases $d = 15$ and $d = 11$ respectively.

Hence the only possible solutions are $(a, b, c, d) = (2, 5, 14, 15)$ or $(2, 6, 9, 11)$, and these indeed satisfy (1) and (2). It follows that there is no unique solution, and it is not possible to determine the childrens' ages.

Problem 2 A circle has diameter $\overline{AB}$ and X is a fixed point on the segment AB. A point P, distinct from A and B, lies on the circle. Prove that, for all possible positions of P,

$$\frac{\tan \angle APX}{\tan \angle PAX}$$

is a constant.

Solution: Let Q be the projection of X onto $\overline{AP}$. Note that $\angle APB = 90^\circ$, and thus $\tan \angle PAX = \frac{PB}{PA}$. Also, $XQ \parallel PB$ so $\triangle AQX \sim \triangle APB$. Therefore,

$$\tan \angle APX = \frac{QX}{QP} = \frac{\frac{AX \cdot BP}{AB}}{\frac{BX \cdot AP}{AB}} = \frac{AX \cdot BP}{BX \cdot AP},$$

and

$$\frac{\tan \angle APX}{\tan \angle PAX} = \frac{AX}{BX}$$

is fixed.

Problem 3 Determine a positive constant c such that the equation

$$xy^2 - y^2 - x + y = c$$

has exactly three solutions (x, y) in positive integers.

Solution: When $y = 1$ the left hand side is 0. Thus we can rewrite our equation as

$$x = \frac{y(y-1) + c}{(y+1)(y-1)}.$$

The numerator is congruent to $-1(-2) + c$ modulo $y+1$, and it is also congruent to c modulo $y-1$. Hence we must have $c \equiv -2 \pmod{y+1}$ and $c \equiv 0 \pmod{y-1}$. Since $c = y - 1$ satisfies these congruences, we must have $c \equiv y - 1 \,(\mathrm{mod}\ \mathrm{lcm}(y-1, y+1))$. When y is even, $\mathrm{lcm}(y-1, y+1) = y^2 - 1$; when y is odd, $\mathrm{lcm}(y-1, y+1) = \frac{1}{2}(y^2 - 1)$.

Then for $y = 2, 3, 11$ we have $c \equiv 1 \pmod 3$, $c \equiv 2 \pmod 4$, $c \equiv 10 \pmod{60}$. Hence, we try setting $c = 10$. For x to be an integer we must have $y - 1 \mid 10 \Rightarrow y = 2, 3, 6,$ or 11; these values give $x = 4, 2, \frac{2}{7}$, and 1 respectively. Thus there *are* exactly three solutions in positive integers, namely $(x, y) = (4, 2), (2, 3)$, and $(1, 11)$.

Problem 4 Any positive integer m can be written uniquely in base 3 form as a string of 0's, 1's and 2's (not beginning with a zero). For example,

$$98 = 81 + 9 + 2 \times 3 + 2 \times 1 = (10122)_3.$$

Let $c(m)$ denote the sum of the cubes of the digits of the base 3 form of m; thus, for instance

$$c(98) = 1^3 + 0^3 + 1^3 + 2^3 + 2^3 = 18.$$

Let n be any fixed positive integer. Define the sequence $\{u_r\}$ as

$$u_1 = n, \text{ and } u_r = c(u_{r-1}) \text{ for } r \geq 2.$$

Show that there is a positive integer r such that $u_r = 1, 2$, or 17.

Solution: If m has $d \geq 5$ digits then we have $m \geq 3^{d-1} = (80+1)^{(d-1)/4} \geq 80 \cdot \frac{d-1}{4} + 1 > 8d$ by Bernoulli's inequality. Thus $m > c(m)$.

If $m > 32$ has 4 digits in base 3, then $c(m) \leq 2^3 + 2^3 + 2^3 + 2^3 = 32 < m$. And if $27 \leq m \leq 32$, then m starts with the digits 10 in base 3 and $c(m) < 1^3 + 0^3 + 2^3 + 2^3 = 17 < m$.

Therefore $0 < c(m) < m$ for all $m \geq 27$, and hence eventually we have some positive $u_s < 27$. Since u_s has at most three digits, u_{s+1} can only equal $8, 16, 24, 1, 9, 17, 2, 10$, or 3. If it equals 1, 2, or 17 we are already done; if it equals 3 or 9 then $u_{s+2} = 1$; and otherwise a simple check shows that u_r will eventually equal 2:

$$\left.\begin{array}{r} 8 = (22)_3 \\ 24 = (220)_3 \end{array}\right\} \rightarrow 16 = (121)_3 \rightarrow 10 = (101)_3 \rightarrow 2.$$

Problem 5 Consider all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

(i) for each positive integer m, there is a unique positive integer n such that $f(n) = m$;

(ii) for each positive integer n, $f(n+1)$ is either $4f(n)-1$ or $f(n)-1$.

Find the set of positive integers p such that $f(1999) = p$ for some function f with properties (i) and (ii).

Solution: Imagine hopping along a sidewalk whose blocks are marked from left to right with the positive integers, where at time n we stand on the block marked $f(n)$. Note that if $f(n) - f(n+1) > 0$ then $f(n) - f(n+1) = 1$; that is, whenever we move to the left we move exactly one block. And whenever we move to the right from $f(n)$ we must move to $4f(n) - 1$.

Then suppose that we are at block $f(a)$ and that $f(a) - 1$ is unvisited; if we move to the right (that is, if $f(a+1) > f(a)$) then at some point we must pass through block $f(a)$ to reach block $f(a) - 1$ again— which is not allowed. Thus we must have $f(a+1) = f(a) - 1$.

Therefore our path is completely determined by the value of $f(1)$: because whenever we are at $f(n)$, if $f(n) - 1 > 0$ is unvisited we must have $f(n+1) = f(n) - 1$. And otherwise, we must have $f(n+1) = 4f(n) - 1$.

If $f(1) = 1$ then consider the function f defined as follows: whenever $2^k \le n < 2^{k+1}$, set $f(n) = (3 \cdot 2^k - 1) - n$. It is bijective since for $n = 2^k, 2^k + 1, \ldots, 2^{k+1} - 1$ we have $f(n) = 2^{k+1} - 1, 2^{k+1} - 2, \ldots, 2^k$; and a quick check shows it satisfies condition (ii) as well. Thus from the previous paragraph this is the *only* function with $f(1) = 1$, and in this case sice $2^{10} \le 1999 < 2^{11}$ we have $f(1999) = (3 \cdot 2^{10} - 1) - 1999 = 1072$.

If $f(1) = 2$ then consider instead the function f defined as follows: whenever $4^k \le n < 3 \cdot 4^k$, set $f(n) = (4^{k+1} - 1) - n$; and if $3 \cdot 4^k \le n < 4^{k+1}$ set $f(n) = (7 \cdot 4^k - 1) - n$. Again, we can check that this function satisfies the conditions; and again, this must be the only function with $f(1) = 2$. In this case since $4^5 \le 1999 < 3 \cdot 4^5$, we have $f(1999) = (4^6 - 1) - 1999 = 2096$.

Finally, suppose that $f(1) \ge 3$; first we must visit $f(1) - 1, f(1) - 2, \ldots, 1$. It follows that $f(n) = 3$ and $f(n+2) = 1$ for some n. But then $f(n+3) = 4 \cdot 1 - 1 = 3 = f(n)$, a contradiction.

Therefore the only possible values of $f(1999)$ are 1072 and 2096.

Problem 6 For each positive integer n, let $S_n = \{1, 2, \ldots, n\}$.

(a) For which values of n is it possible to express S_n as the union of two non-empty disjoint subsets so that the elements in the two subsets have equal sum?

(b) For which values of n is it possible to express S_n as the union of three non-empty disjoint subsets so that the elements in the three subsets have equal sum?

Solution:

(a) Let $\sigma(T)$ denote the sum of the elements in a set T. For the condition to hold $\sigma(S_n) = \frac{n(n+1)}{2}$ must be even, and hence we must have $n = 4k - 1$ or $4k$ where $k \in \mathbb{N}$. For such n, let A consist of the second and third elements of each of the sets

$\{n, n-1, n-2, n-3\}$, $\{n-4, n-5, n-6, n-7\}$, ..., $\{4, 3, 2, 1\}$ (or if $n = 4k-1$, the last set in this grouping will be $\{3, 2, 1\}$); and let $B = S_n \setminus A$. Then $\sigma(A) = \sigma(B)$, as desired.

(b) For the condition to hold, $\sigma(S_n) = \frac{n(n+1)}{2}$ must be divisible by 3; furthermore, the construction is impossible for $n = 3$. Thus n must be of the form $3k+2$ or $3k+3$ where $k \in \mathbb{N}$. We prove all such n work by induction on n. We have $S_5 = \{5\} \cup \{1, 4\} \cup \{2, 3\}$, $S_6 = \{1, 6\} \cup \{2, 5\} \cup \{3, 4\}$, $S_8 = \{8, 4\} \cup \{7, 5\} \cup \{1, 2, 3, 6\}$, and $S_9 = \{9, 6\} \cup \{8, 7\} \cup \{1, 2, 3, 4, 5\}$. Now suppose that we can partition S_{n-6} into $A \cup B \cup C$ with $\sigma(A) = \sigma(B) = \sigma(C)$; then $\sigma(A \cup \{n-5, n\}) = \sigma(B \cup \{n-4, n-1\}) = \sigma(C \cup \{n-3, n-2\})$, completing the inductive step and the proof of our claim.

Problem 7 Let $ABCDEF$ be a hexagon which circumscribes a circle ω. The circle ω touches sides AB, CD, EF at their respective midpoints P, Q, R. Let ω touch sides BC, DE, FA at X, Y, Z respectively. Prove that lines PY, QZ, RX are concurrent.

Solution: Let O be the center of ω. Since P is the midpoint of $\overline{AB}$, $AP = PB$; then by equal tangents, $ZA = AP = PB = BX$. Thus $\angle ZOA = \angle AOP = \angle POB = \angle BOX$. It follows that $\angle ZOP = \angle POX$, and hence $\angle ZYP = \angle PYX$. Therefore line YP is the angle bisector of $\angle XYZ$. Similarly lines XR and ZQ are the angle bisectors of $\angle ZXY$ and $\angle YZX$, and therefore lines PY, QZ, RX meet at the incenter of triangle XYZ.

Problem 8 Some three non-negative real numbers p, q, r satisfy

$$p + q + r = 1.$$

Prove that

$$7(pq + qr + rp) \le 2 + 9pqr.$$

Solution: Given a function f of three variables, let $\sum_{\text{cyc}} f(p, q, r)$ denote the "cyclic sum" $f(p, q, r) + f(q, r, p) + f(r, p, q)$; for example, $\sum_{\text{cyc}}(pqr + p) = 3pqr + p + q + r$. Since $p + q + r = 1$ the inequality is equivalent to

$$7(pq + qr + rp)(p + q + r) \le 2(p + q + r)^3 + 9pqr$$
$$\iff 7 \sum_{\text{cyc}} \left(p^2 q + pq^2 + pqr\right)$$

$$\leq 9pqr + \sum_{\text{cyc}} \left(2p^3 + 6p^2q + 6pq^2 + 4pqr\right)$$

$$\iff \sum_{\text{cyc}} p^2q + \sum_{\text{cyc}} pq^2 \leq \sum_{\text{cyc}} 2p^3 = \sum_{\text{cyc}} \frac{2p^3+q^3}{3} + \sum_{\text{cyc}} \frac{p^3+2q^3}{3},$$

and this last inequality is true by weighted AM-GM.

Problem 9 Consider all numbers of the form $3n^2+n+1$, where n is a positive integer.

(a) How small can the sum of the digits (in base 10) of such a number be?

(b) Can such a number have the sum of its digits (in base 10) equal to 1999?

Solution:

(a) Let $f(n) = 3n^2+n+1$. When $n = 8$, the sum of the digits of $f(8) = 201$ is 3. Suppose that some $f(m)$ had a smaller sum of digits; then the last digit of $f(m)$ must be either 0, 1, or 2. However, for any n, $f(n) = n(n+3)+1 \equiv 1 \pmod 2$; thus $f(m)$ must have units digit 1.

Because $f(n)$ can never equal 1, this means we must have $3m^2+m+1 = 10^k+1$ for some positive integer k, and $m(3m+1) = 10^k$. Since m and $3m+1$ are relatively prime, and $m < 3m+1$, we must either have $(m, 3m+1) = (1, 10^k)$—which is impossible—or $(m, 3m+1) = (2^k, 5^k)$. For $k = 1$, $5^k \neq 3\cdot 2^k+1$; and for $k > 1$, we have $5^k = 5^{k-2}\cdot 25 > 2^{k-2}\cdot(12+1) \geq 3\cdot 2^k+1$. Therefore, $f(m)$ *can't* equal 10^k+1, and 3 is indeed the minimum value for the sum of digits.

(b) Consider $n = 10^{222}-1$. $f(n) = 3\cdot 10^{444} - 6\cdot 10^{222} + 3 + 10^{222}$. Thus, its decimal expansion is

$$2\underbrace{9\ldots9}_{221}5\underbrace{0\ldots0}_{221}3,$$

and the sum of the digits in $f(10^{222}-1)$ is 1999.

1.23 United States of America

Problem 1 Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

(i) every square that does not contain a checker shares a side with one that does;

(ii) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $\frac{n^2-2}{3}$ checkers have been placed on the board.

Solution: It suffices to show that if m checkers are placed so as to satisfy condition (b), then the number of squares they either cover or are adjacent to is at most $3m+2$. But this is easily seen by induction: it is obvious for $m = 1$, and if m checkers are so placed, some checker can be removed so that the remaining checkers still satisfy (b); they cover at most $3m-1$ squares, and the new checker allows us to count at most 3 new squares (since the square it occupies was already counted, and one of its neighbors is occupied).

Note. The exact number of checkers required is known for $m \times n$ checkerboards with m small, but only partial results are known in the general case. Contact the authors for more information.

Problem 2 Let $ABCD$ be a convex cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

First Solution: Let E be the intersection of $\overline{AC}$ and $\overline{BD}$. Then the triangles ABE and DCE are similar, so if we let $x = AE, y = BE, z = AB$, then there exists k such that $kx = DE, ky = CE, kz = CD$. Now

$$|AB - CD| = |k-1|z$$

and

$$|AC - BD| = |(kx + y) - (ky + x)| = |k-1| \cdot |x - y|.$$

Since $|x-y| \le z$ by the triangle inequality, we conclude $|AB-CD| \ge |AC-BD|$, and similarly $|AD-BC| \ge |AC-BD|$. These two inequalities imply the desired result.

Second Solution: Let $2\alpha, 2\beta, 2\gamma, 2\delta$ be the measures of the arcs subtended by AB, BC, CD, DA, respectively, and take the radius of the circumcircle of $ABCD$ to be 1. Assume without loss of generality that $\beta \le \delta$. Then $\alpha+\beta+\gamma+\delta = \pi$, and (by the Extended Law of Sines)

$$|AB-CD| = 2\left|\sin\alpha - \sin\gamma\right| = 4\left|\sin\frac{\alpha-\gamma}{2}\right|\left|\cos\frac{\alpha+\gamma}{2}\right|$$

and

$$\begin{aligned}|AC-BD| &= 2\left|\sin(\alpha+\beta) - \sin(\beta+\gamma)\right| \\ &= 4\left|\sin\frac{\alpha-\gamma}{2}\right|\left|\cos\left(\frac{\alpha+\gamma}{2}+\beta\right)\right|.\end{aligned}$$

Since $0 \le \frac{1}{2}(\alpha+\gamma) \le \frac{1}{2}(\alpha+\gamma)+\beta \le \frac{\pi}{2}$ (by the assumption $\beta \le \delta$) and the cosine function is nonnegative and decreasing on $[0, \frac{\pi}{2}]$, we conclude that $|AB-CD| \ge |AC-BD|$, and similarly $|AD-BC| \ge |AC-BD|$.

Problem 3 Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p, such that

$$\left\{\frac{ra}{p}\right\} + \left\{\frac{rb}{p}\right\} + \left\{\frac{rc}{p}\right\} + \left\{\frac{rd}{p}\right\} = 2$$

for any integer r not divisible by p. Prove that at least two of the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ are divisible by p. Here, for real numbers x, $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

Solution: For convenience, we write $[x]$ for the unique integer in $\{0, \ldots, p-1\}$ congruent to x modulo p. In this notation, the given condition can be written

$$[ra] + [rb] + [rc] + [rd] = 2p \quad \text{for all } r \text{ not divisible by } p. \qquad (1)$$

The conditions of the problem are preserved by replacing a, b, c, d with ma, mb, mc, md for any integer m relatively prime to p. If we choose m so that $ma \equiv 1 \pmod{p}$ and then replace a, b, c, d with $[ma], [mb], [mc], [md]$, respectively, we end up in the case $a = 1$ and

$b, c, d \in \{1, \dots, p-1\}$. Applying (1) with $r = 1$, we see moreover that $a + b + c + d = 2p$.

Now observe that

$$[(r+1)x] - [rx] = \begin{cases} [x] & [rx] < p - [x] \\ -p + [x] & [rx] \geq p - [x]. \end{cases}$$

Comparing (1) applied to two consecutive values of r and using the observation, we see that for each $r = 1, \dots, p-2$, two of the quantities

$$p - a - [ra], p - b - [rb], p - c - [rc], p - d - [rd]$$

are positive and two are negative. We say that a pair (r, x) is *positive* if $[rx] < p - [x]$ and *negative* otherwise; then for each $r < p - 1$, $(r, 1)$ is positive, so exactly one of $(r, b), (r, c), (r, d)$ is also positive.

Lemma. *If $r_1, r_2, x \in \{1, \dots, p-1\}$ have the property that (r_1, x) and (r_2, x) are negative but (r, x) is positive for all $r_1 < r < r_2$, then*

$$r_2 - r_1 = \left\lfloor \frac{p}{x} \right\rfloor \quad \textit{or} \quad r_2 - r_1 = \left\lfloor \frac{p}{x} \right\rfloor + 1.$$

Proof: Note that (r', x) is negative if and only if $\{r'x + 1, r'x + 2, \dots, (r'+1)x\}$ contains a multiple of p. In particular, exactly one multiple of p lies in $\{r_1 x, r_1 x + 1, \dots, r_2 x\}$. Since $[r_1 x]$ and $[r_2 x]$ are distinct elements of $\{p - [x], \dots, p - 1\}$, we have

$$p - x + 1 < r_2 x - r_1 x < p + x - 1,$$

from which the lemma follows. ■

$[rx]$	9	10	**0**	1	2	3	4	5	6	7	8	9	10	**0**
is (r, x) + or −?	−			+			+			+			−	
r	**<u>3</u>**			4			5			6			**<u>7</u>**	

(The above diagram illustrates the meanings of *positive* and *negative* in the case $x = 3$ and $p = 11$. Note that the difference between 7 and 3 here is $\lfloor \frac{p}{x} \rfloor + 1$. The next r such that (r, x) is negative is $r = 10$; $10 - 7 = \lfloor \frac{p}{x} \rfloor$.)

Recall that exactly one of $(1, b)$, $(1, c)$, $(1, d)$ is positive; we may as well assume $(1, b)$ is positive, which is to say $b < \frac{p}{2}$ and $c, d > \frac{p}{2}$. Put $s_1 = \lfloor \frac{p}{b} \rfloor$, so that s_1 is the smallest positive integer such that (s_1, b) is negative. Then exactly one of (s_1, c) and (s_1, d) is positive, say the former. Since s_1 is also the smallest positive integer such that (s_1, c) is positive, or equivalently such that $(s_1, p - c)$ is negative, we have

$s_1 = \lfloor \frac{p}{p-c} \rfloor$. The lemma states that consecutive values of r for which (r, b) is negative differ by either s_1 or $s_1 + 1$. It also states (when applied with $x = p - c$) that consecutive values of r for which (r, c) is positive differ by either s_1 or $s_1 + 1$. From these observations we will show that (r, d) is always negative.

r	1		s_1	s_1+1		s'	$s'+1$		s	$s+1 \stackrel{?}{=} t$
(r,b)	$+$		$-$	$+$		$-$	$+$		$-$	$-_?$
(r,c)	$-$	$\ldots$	$+$	$-$	$\ldots$	$+$	$-$	$\ldots$	$-$	$+_?$
(r,d)	$-$		$-$	$-$		$-$	$-$		$+$	$-_?$

Indeed, if this were not the case, there would exist a smallest positive integer $s > s_1$ such that (s, d) is positive; then (s, b) and (s, c) are both negative. If s' is the last integer before s such that (s', b) is negative (possibly equal to s_1), then (s', d) is negative as well (by the minimal definition of s). Also,

$$s - s' = s_1 \quad \text{or} \quad s - s' = s_1 + 1.$$

Likewise, if t were the next integer after s' such that (t, c) were positive, then

$$t - s' = s_1 \quad \text{or} \quad t - s' = s_1 + 1.$$

From these we deduce that $|t - s| \leq 1$. However, we can't have $t \neq s$ because then both (s, b) and (t, b) would be negative—and any two values of r for which (r, b) is negative differ by at least $s_1 \geq 2$, a contradiction. (The above diagram shows the hypothetical case when $t = s+1$.) But nor can we have $t = s$ because we already assumed that (s, c) is negative. Therefore we *can't* have $|t - s| \leq 1$, contradicting our findings and thus proving that (r, d) is indeed always negative.

Now if $d \neq p - 1$, then the unique $s \in \{1, \ldots, p-1\}$ such that $[ds] = 1$ is not equal to $p - 1$; and (s, d) is positive, a contradiction. Thus $d = p - 1$ and $a + d$ and $b + c$ are divisible by p, as desired.

Problem 4 Let $a_1, a_2, \ldots, a_n$ $(n > 3)$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \qquad \text{and} \qquad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$.

Solution: Let $b_i = 2 - a_i$, and let $S = \sum b_i$ and $T = \sum b_i^2$. Then the given conditions are that

$$(2-b_1)+\cdots+(2-b_n)\geq n$$

and

$$(4-4b_1+b_1^2)+\cdots+(4-4b_n+b_n^2)\geq n^2,$$

which is to say $S\leq n$ and $T\geq n^2-4n+4S$.

From these inequalities, we obtain

$$T\geq n^2-4n+4S\geq(n-4)S+4S=nS.$$

On the other hand, if $b_i>0$ for $i=1,\ldots,n$, then certainly $b_i<\sum b_i=S\leq n$, and so

$$T=b_1^2+\cdots+b_n^2<nb_1+\cdots+nb_n=nS.$$

Thus we cannot have $b_i>0$ for $i=1,\ldots,n$, so $b_i\leq 0$ for some i; then $a_i\geq 2$ for that i, proving the claim.

Note: The statement is false when $n\leq 3$. The example $a_1=a_2=\cdots=a_{n-1}=2$, $a_n=2-n$ shows that the bound cannot be improved. Also, an alternate approach is to show that if $a_i\leq 2$ and $\sum a_i\geq n$, then $\sum a_i^2\leq n^2$ (with the equality case just mentioned), by noticing that replacing a pair a_i,a_j with $2,a_i+a_j-2$ increases the sum of squares.

Problem 5 The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

Solution: Call a partially filled board *stable* if there is no SOS and no single move can produce an SOS; otherwise call it *unstable*. For a stable board call an empty square *bad* if either an S or an O played in that square produces an unstable board. Thus a player will lose if the only empty squares available to him are bad, but otherwise he can at least be guaranteed another turn with a correct play.

Claim: A square is bad if and only if it is in a block of 4 consecutive squares of the form S – – S.

Proof: If a square is bad, then an O played there must give an unstable board. Thus the bad square must have an S on one side and an empty square on the other side. An S played there must also give an unstable board, so there must be another S on the other side of the empty square. ∎

From the claim it follows that there are always an even number of bad squares. Thus the second player has the following winning strategy:

(a) If the board is unstable at any time, play the winning move; otherwise continue as below.

(b) On the first move, play an S at least four squares away from either end and at least seven squares from the first player's first move. (The board is long enough that this is possible.)

(c) On the second move, play an S three squares away from the second player's first move, so that the squares in between are empty and so that the board remains stable. (Regardless of the first player's second move, this must be possible on at least one side.) This produces two bad squares; whoever plays in one of them first will lose. Thus the game will not be a draw.

(d) On any subsequent move, play in a square which is not bad—keeping the board stable, of course. Such a square will always exist because if the board is stable, there will be an odd number of empty squares and an even number of bad squares.

Since there exist bad squares after the second player's second move, the game cannot end in a draw; and since the second player can always leave the board stable, the first player cannot win. Therefore eventually the second player will win.

Note: Some other names for the S – – S block, from submitted solutions, included arrangement, combo, configuration, formation, pattern, sandwich, segment, situation, and trap. (Thanks to Alexander Soifer and Zvezdelina Stankova-Frenkel for passing these along.)

Problem 6 Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and

G. Prove that the triangle AFG is isosceles.

Solution: We will show that $FA = FG$. Let H be the center of the escribed circle of triangle ACD opposite vertex A. Then H lies on the angle bisector AF. Let K be the point where this escribed circle touches CD. By a standard computation using equal tangents, we see that $CK = \frac{1}{2}(AD + CD - AC)$. By a similar computation in triangle BCD, we see that $CE = \frac{1}{2}(BC + CD - BD) = CK$. Therefore $E = K$ and $F = H$.

Since F is now known to be an excenter, we have that FC is the external angle bisector of $\angle DCA = \angle GCA$. Therefore

$$\angle GAF = \angle GCF = \frac{\pi}{2} - \frac{1}{2}\angle GCA = \frac{\pi}{2} - \frac{1}{2}\angle GFA.$$

We conclude that the triangle GAF is isosceles with $FA = FG$, as desired.

1.24 Vietnam

Problem 1 Solve the system of equations

$$(1+4^{2x-y})\cdot 5^{1-2x+y} = 1+2^{2x-y+1}$$
$$y^3+4x+1+\ln(y^2+2x) = 0.$$

Solution: The only solution is $(x,y)=(0,-1)$.

First solve the first equation for $t=2x-y$. Multiplying the equation by 5^{t-1} yields

$$(1-5^{t-1})+4(4^{t-1}-10^{t-1})=0.$$

This has the obvious solution $t=1$. There are no other solutions: if $t>1$ then both $1-5^{t-1}$ and $4^{t-1}-10^{t-1}$ are negative; and if $t<1$ then both these terms are positive. Therefore, $2x-y=1$.

Substitute $2x=y+1$ into the second equation to get

$$y^3+2y+3+\ln(y^2+y+1)=0.$$

This has the not-so-obvious solution $y=-1$. To prove this is the only solution, it suffices to show that $f(y)=y^3+2y+3+\ln(y^2+y+1)$ is always increasing. Its derivative is

$$f'(y)=3y^2+2+\frac{2y+1}{y^2+y+1}.$$

But we know that

$$\begin{aligned} &2(y+1)^2+1>0 \\ &\Rightarrow 2y+1>-2(y^2+y+1) \\ &\Rightarrow \frac{2y+1}{y^2+y+1}>-2, \end{aligned}$$

where we can safely divide by $y^2+y+1=(y+\frac{1}{2})^2+\frac{3}{4}>0$. Thus $f'(y)>3y^2>0$ for all y, as desired.

Problem 2 Let A', B', C' be the respective midpoints of the arcs BC, CA, AB, not containing points A, B, C, respectively, of the circumcircle of the triangle ABC. The sides BC, CA, AB meet the pairs of segments

$$\{C'A', A'B'\},\ \{A'B', B'C'\},\ \{B'C', C'A'\}$$

at the pairs of points

$$\{M, N\},\ \{P, Q\},\ \{R, S\},$$

respectively. Prove that $MN = PQ = RS$ if and only if the triangle ABC is equilateral.

Solution: If ABC is equilateral then $MN = PQ = RS$ by symmetry.

Now suppose that $MN = PQ = RS$. Observe that $\angle NMA' = \angle BMS = \frac{1}{2}(\widehat{BC'} + \widehat{CA'}) = \frac{1}{2}(\angle C + \angle A)$ and similarly $\angle C'SR = \angle MSB = \frac{1}{2}(\angle A + \angle C)$. Furthermore, $\angle A'B'C' = \angle A'B'B + \angle BB'C' = \frac{1}{2}(\angle A + \angle C)$ as well.

Thus $MB = SB$, and also $\triangle C'RS \sim \triangle C'A'B' \sim \triangle NA'M$. Next, by the law of sines in triangles $C'SB$ and $A'MB$ we have

$$C'S = SB \cdot \frac{\sin \angle C'BS}{\sin \angle SC'B} = SB \cdot \frac{\sin \frac{\angle C}{2}}{\sin \frac{\angle A}{2}}$$

and

$$MA' = MB \cdot \frac{\sin \angle A'BM}{\sin \angle MA'B} = MB \cdot \frac{\sin \frac{\angle A}{2}}{\sin \frac{\angle C}{2}},$$

giving $\frac{C'S}{MA'} = \left(\frac{\sin \frac{\angle C}{2}}{\sin \frac{\angle A}{2}}\right)^2$.

Next, because $\triangle C'RS \sim \triangle C'A'B'$ we have $RS = A'B' \cdot \frac{C'S}{C'B'}$; and because $\triangle NA'M \sim \triangle C'A'B'$ we have $MN = B'C' \cdot \frac{MA'}{B'A'}$. Therefore since $RS = MN$ we have

$$\begin{aligned}
&A'B' \cdot \frac{C'S}{C'B'} = B'C' \cdot \frac{MA'}{B'A'} \\
&\Longrightarrow \frac{C'S}{MA'} = \left(\frac{B'C'}{A'B'}\right)^2 = \left(\frac{\sin \frac{1}{2}(\angle B + \angle C)}{\sin \frac{1}{2}(\angle B + \angle A)}\right)^2 = \left(\frac{\cos \frac{\angle A}{2}}{\cos \frac{\angle C}{2}}\right)^2 \\
&\Longrightarrow \left(\frac{\sin \frac{\angle C}{2}}{\sin \frac{\angle A}{2}}\right)^2 = \left(\frac{\cos \frac{\angle A}{2}}{\cos \frac{\angle C}{2}}\right)^2 \\
&\Longrightarrow \left(\sin \frac{\angle C}{2} \cos \frac{\angle C}{2}\right)^2 = \left(\sin \frac{\angle A}{2} \cos \frac{\angle A}{2}\right)^2 \\
&\Longrightarrow \frac{1}{4} \sin^2 \angle C = \frac{1}{4} \sin^2 \angle A \\
&\Longrightarrow \sin \angle C = \sin \angle A.
\end{aligned}$$

Since $\angle A + \angle C < 180°$, we must have $\angle A = \angle C$. Similarly $\angle A = \angle B$, and therefore triangle ABC is equilateral.

Problem 3 For $n = 0, 1, 2, \ldots$, let $\{x_n\}$ and $\{y_n\}$ be two sequences defined recursively as follows:

$$x_0 = 1,\ x_1 = 4,\ x_{n+2} = 3x_{n+1} - x_n;$$

$$y_0 = 1,\ y_1 = 2,\ y_{n+2} = 3y_{n+1} - y_n.$$

(a) Prove that $x_n^2 - 5y_n^2 + 4 = 0$ for all non-negative integers n.

(b) Suppose that a, b are two positive integers such that $a^2 - 5b^2 + 4 = 0$. Prove that there exists a non-negative integer k such that $x_k = a$ and $y_k = b$.

Solution: We first prove by induction on k that for $k \geq 0$, we have $(x_{k+1}, y_{k+1}) = (\frac{3x_k+5y_k}{2}, \frac{x_k+3y_k}{2})$. For $k = 0$ we have $(4, 2) = (\frac{3+5}{2}, \frac{1+3}{2})$, and for $k = 1$ we have $(11, 5) = (\frac{12+10}{2}, \frac{4+6}{2})$. Now assuming it's true for k and $k+1$, we know that

$$(x_{k+3}, y_{k+3}) = (3x_{k+2} - x_{k+1}, 3y_{k+2} - y_{k+1}).$$

Substituting the expressions for $x_{k+2}, x_{k+1}, y_{k+2}, y_{k+1}$ from the induction hypothesis, this equals

$$\left(\frac{3(3x_{k+1} - x_k) + 5(3y_{k+1} - y_k)}{2}, \frac{(3x_{k+1} - x_k) + 3(3y_{k+1} - y_k)}{2}\right)$$
$$= \left(\frac{3x_{k+2} + 5y_{k+2}}{2}, \frac{x_{k+2} + 3y_{k+2}}{2}\right),$$

completing the induction and the proof of our claim.

(a) We prove the claim by induction; for $n = 0$ we have $1 - 5 + 4 = 0$. Now assuming it is true for n, we prove (with the help of our above observation) that it is true for $n + 1$:

$$\begin{aligned}
&x_{n+1}^2 - 5y_{n+1}^2 \\
&= \left(\frac{3x_n + 5y_n}{2}\right)^2 - 5\left(\frac{x_n + 3y_n}{2}\right)^2 \\
&= \frac{9x_n^2 + 30x_ny_n + 25y_n^2}{4} - 5 \cdot \frac{x_n^2 + 6x_ny_n + 9y_n^2}{4} \\
&= \frac{4x_n^2 - 20y_n^2}{4} = x_n^2 - 5y_n^2 = -4,
\end{aligned}$$

as desired.

(b) Suppose by way of contradiction that $a^2 - 5b^2 + 4 = 0$ for integers $a, b > 0$, and that there did *not* exist k such that $(x_k, y_k) = (a, b)$. Choose a counterexample that minimizes $a + b$.

Note that $0 \equiv a^2 - 5b^2 + 4 \equiv a - b \pmod 2$. Next, $a^2 = 5b^2 - 4 < 9b^2 \Rightarrow a < 3b$. And there are no counterexamples with $a = 1$ or 2; thus $a^2 > 5$ and $0 = 5a^2 - 25b^2 + 20 < 5a^2 - 25b^2 + 4a^2 \Rightarrow 3a > 5b$.

Therefore $a' = \frac{3a-5b}{2}$ and $b' = \frac{3b-a}{2}$ are positive integers. Then since $a^2 - 5b^2 = -4$, some quick algebra shows that $a'^2 - 5b'^2 = -4$ as well; but $a' + b' = \frac{3a-5b}{2} + \frac{3b-a}{2} = a - b < a + b$. It follows from the minimal definition of (a, b) that there must exist some (a_k, b_k) equal to (a', b').

But then it is easy to verify that $(a, b) = (\frac{3a'+5b'}{2}, \frac{a'+3b'}{2}) = (a_{k+1}, b_{k+1})$, a contradiction! This completes the proof.

Problem 4 Let a, b, c be real numbers such that $abc + a + c = b$. Determine the greatest possible value of the expression

$$P = \frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1}.$$

Solution: The condition is equivalent to $b = \frac{a+c}{1-ac}$, which suggests making the substitutions $A = \mathrm{Tan}^{-1} a$ and $C = \mathrm{Tan}^{-1} c$; then we have $b = \tan(A + C)$ and

$$\begin{aligned} P &= \frac{2}{\tan^2 A + 1} - \frac{2}{\tan^2(A+C) + 1} + \frac{3}{\tan^2 C + 1} \\ &= 2\cos^2 A - 2\cos^2(A+C) + 3\cos^2 C \\ &= (2\cos^2 A - 1) - (2\cos^2(A+C) - 1) + 3\cos^2 C \\ &= \cos(2A) - \cos(2A + 2C) + 3\cos^2 C \\ &= 2\sin(2A + C)\sin C + 3\cos^2 C. \end{aligned}$$

Letting $u = |\sin C|$, this expression is at most

$$\begin{aligned} 2u + 3(1 - u^2) &= -3u^2 + 2u + 3 \\ &= -3\left(u - \frac{1}{3}\right)^2 + \frac{10}{3} \leq \frac{10}{3}. \end{aligned}$$

Equality can be achieved when $\sin(2A+C) = 1$ and $\sin C = \frac{1}{3}$, which gives $(a, b, c) = (\frac{\sqrt{2}}{2}, \sqrt{2}, \frac{\sqrt{2}}{4})$. Thus the maximum value of P is $\frac{10}{3}$.

Problem 5 In the three-dimensional space let Ox, Oy, Oz, Ot be four nonplanar distinct rays such that the angles between any two of them have the same measure.

(a) Determine this common measure.

(b) Let Or be another ray different from the above four rays. let $\alpha, \beta, \gamma, \delta$ be the angles formed by Or with Ox, Oy, Oz, Ot, respectively. Put

$$p = \cos\alpha + \cos\beta + \cos\gamma + \cos\delta,$$
$$q = \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta.$$

Prove that p and q remain constant as Or rotates about the point O.

Solution: Put O at the origin, and say the four rays hit the unit sphere at A, B, C, D. Then $ABCD$ is a regular tetrahedron, and (letting X also represent the vector $\overrightarrow{OX}$) we have $A+B+C+D = 0$.

(a) Say the common angle is ϕ. Then

$0 = A \cdot (A+B+C+D) = A \cdot A + A \cdot (B+C+D) = 1 + 3\cos\phi,$

so $\phi = \cos^{-1}\left(-\frac{1}{3}\right)$.

(b) Without loss of generality say that Or hits the unit sphere at $U = (1, 0, 0)$; also write $A = (x_1, y_1, z_1)$, and so on. Then

$$\begin{aligned} p &= U \cdot A + U \cdot B + U \cdot C + U \cdot D \\ &= U \cdot (A+B+C+D) \\ &= U \cdot \vec{0} = 0, \end{aligned}$$

a constant. Also, $(x_1, x_2, x_3, x_4) = (\cos\alpha, \cos\beta, \cos\gamma, \cos\delta)$ and $q = \sum x_i^2$. The following lemma then implies that q will always equal $\frac{4}{3}$:

Lemma. *Suppose we are given a regular tetrahedron T inscribed in the unit sphere and with vertices (x_i, y_i, z_i) for $1 \le i \le 4$. Then we have $\sum x_i^2 = \sum y_i^2 = \sum z_i^2 = \frac{4}{3}$ and $\sum x_i y_i = \sum y_i z_i = \sum z_i x_i = 0$.*

Proof: This is easily verified when the vertices are at

$$A_0 = (0,0,1), B_0 = (\tfrac{2\sqrt{2}}{3}, 0, -\tfrac{1}{3}),$$
$$C_0 = (-\tfrac{\sqrt{2}}{3}, \tfrac{\sqrt{6}}{3}, -\tfrac{1}{3}), D_0 = (-\tfrac{\sqrt{2}}{3}, -\tfrac{\sqrt{6}}{3}, -\tfrac{1}{3}).$$

Now assume these equations are true for a tetrahedron $ABCD$, and rotate it about the z-axis through an angle θ. Then each (x_i, y_i, z_i) becomes $(x_i', y_i', z_i') = (x_i \cos\theta - y_i \sin\theta, x_i \sin\theta + y_i \cos\theta, z_i)$, and

$$\sum x_i'^2 = \cos^2\theta \sum x_i^2 - 2\sin\theta\cos\theta \sum x_i y_i + \sin^2\theta \sum y_i^2 = \frac{4}{3}$$
$$\sum y_i'^2 = \sin^2\theta \sum x_i^2 + 2\sin\theta\cos\theta \sum x_i y_i + \cos^2\theta \sum y_i^2 = \frac{4}{3}$$
$$\sum z_i'^2 = \sum z_i^2 = \frac{4}{3}$$
$$\sum x_i' y_i' = \sin\theta\cos\theta \sum (x_i^2 - y_i^2) + (\cos^2\theta - \sin^2\theta) \sum x_i y_i = 0$$
$$\sum y_i' z_i' = \sin\theta \sum x_i z_i + \cos\theta \sum y_i z_i = 0$$
$$\sum z_i' x_i' = \cos\theta \sum z_i x_i - \sin\theta \sum z_i y_i = 0.$$

Similarly, the equations remain true after rotating $ABCD$ about the y- and z-axes.

Now, first rotate our given tetrahedron T about the z-axis until one vertex is in the yz-plane; next rotate it about the x-axis until this vertex is at $(0,0,1)$; and finally rotate it about the z-axis again until the tetrahedron corresponds with the initial tetrahedron $A_0B_0C_0D_0$ described above. Since we *know* the above equations are true for $A_0B_0C_0D_0$, if we reverse the rotations to return to T the equations will remain true, as claimed. ■

Problem 6 Let $\mathcal{S} = \{0, 1, 2, \ldots, 1999\}$ and $\mathcal{T} = \{0, 1, 2, \ldots\}$. Find all functions $f : \mathcal{T} \to \mathcal{S}$ such that

(i) $f(s) = s$ for all $s \in \mathcal{S}$.

(ii) $f(m+n) = f(f(m) + f(n))$ for all $m, n \in \mathcal{T}$.

Solution: Suppose that $f(2000) = 2000 - t$, where $1 \le t \le 2000$. We prove by induction on n that for all $n \ge 2000$, we have $f(n) = f(n-t)$. By assumption it is true for $n = 2000$. Then assuming it is true for

n, we have

$$f(n+1) = f(f(n) + f(1)) = f(f(n-t) + f(1)) = f(n-t+1),$$

completing the inductive step. Therefore the function is completely determined by the value of $f(2000)$, and it follows that there are at most 2000 such functions.

Conversely, given any $c = 2000 - t \in S$, let f be the function such that $f(s) = s$ for all $s \in \mathcal{S}$ while $f(n) = f(n-t)$ for all $n \geq 2000$. We prove by induction on $m+n$ that condition (ii) holds. If $m+n \leq 2000$ then $m, n \in \mathcal{S}$ and the claim is obvious. Otherwise, $m + n > 2000$. Again, if $m, n \in \mathcal{S}$ the claim is obvious; otherwise assume without loss of generality that $n \geq 2000$. Then

$$f(m+n) = f(m+n-t) = f(f(m) + f(n-t)) = f(f(m) + f(n)),$$

where the first and third equalities come from our periodic definition of f, and the second equality comes from the induction hypothesis. Therefore there are exactly 2000 functions f.

Problem 7 For $n = 1, 2, \ldots$, let $\{u_n\}$ be a sequence defined by

$$u_1 = 1,\ u_2 = 2,\ u_{n+2} = 3u_{n+1} - u_n.$$

Prove that

$$u_{n+2} + u_n \geq 2 + \frac{u_{n+1}^2}{u_n}$$

for all n.

Solution: We first prove by induction that for $n \geq 1$, we have $u_n u_{n+2} = u_{n+1}^2 + 1$. Since $u_3 = 5$, for $n = 1$ we have $1 \cdot 5 = 2^2 + 1$, as desired.

Now assuming our claim is true for n, we have

$$\begin{aligned} u_{n+2}^2 + 1 &= u_{n+2}(3u_{n+1} - u_n) + 1 \\ &= 3u_{n+1}u_{n+2} - (u_n u_{n+2} - 1) \\ &= 3u_{n+1}u_{n+2} - u_{n+1}^2 \\ &= u_{n+1}(3u_{n+2} - u_{n+1}) = u_{n+1}u_{n+3}, \end{aligned}$$

so it is true for $n + 1$ as well and thus all $n \geq 1$.

Therefore, for all $n \geq 1$ we have

$$u_{n+2} + u_n = \frac{u_{n+1}^2 + 1}{u_n} + u_n = \frac{u_{n+1}^2}{u_n} + \left(u_n + \frac{1}{u_n}\right) \geq \frac{u_{n+1}^2}{u_n} + 2,$$

where $u_n + \frac{1}{u_n} \geq 2$ by AM-GM. This proves the inequality.

Problem 8 Let ABC be a triangle inscribed in circle ω. Construct all points P in the plane (ABC) and not lying on ω, with the property that the lines PA, PB, PC meet ω again at points A', B', C' such that $A'B' \perp A'C'$ and $A'B' = A'C'$.

Solution: All angles are directed modulo $180°$. We solve a more general problem: suppose we have some fixed triangle DEF and we want to find all points P such that when $A' = PA \cap \omega$, $B' = PB \cap \omega$, $C' = PC \cap \omega$ then triangles $A'B'C'$ and DEF are similar with the same orientations. (In other words, we want $\angle B'C'A' = \angle EFD$ and $\angle C'A'B' = \angle FDE$.)

Given X, Y on ω, let $\angle \widehat{XY}$ denote the angle $\angle XZY$ for any other point Z on ω. Now given a point P, we have $\angle BPA = \angle \widehat{BA} + \angle \widehat{B'A'} = \angle BCA + \angle B'C'A'$ and $\angle CPB = \angle \widehat{CB} + \angle \widehat{C'B'} = \angle CAB + \angle C'A'B'$. Thus $\angle B'C'A' = \angle EFD$ if and only if $\angle BPA = \angle BCA + \angle EFD$, while $\angle C'A'B' = \angle FDE$ if and only if $\angle CPB = \angle CAB + \angle FDE$. Therefore our desired point P is the intersection point, different than B, of the two circles $\{P' \mid \angle BP'A = \angle BCA + \angle EFD\}$ and $\{P' \mid \angle CP'B = \angle CAB + \angle FDE\}$.

Now back to our original problem: we wish to find P such that triangle $A'B'C'$ is a $45°$-$45°$-$90°$ triangle with $\angle C'A'B' = 90°$. Because our angles are directed, there are two possible orientations for such a triangle: either $\angle A'B'C' = 45°$ or $\angle A'B'C' = -45°$. Applying the above construction twice with triangle DEF defined appropriately yields the two desired possible locations of P.

Problem 9 Consider real numbers a, b such that $a \neq 0, a \neq b$ and all roots of the equation

$$ax^3 - x^2 + bx - 1 = 0$$

are real and positive. Determine the smallest possible value of the expression

$$P = \frac{5a^2 - 3ab + 2}{a^2(b - a)}.$$

Solution: When the roots of the equation are all $\sqrt{3}$, we have $a = \frac{1}{3\sqrt{3}}$, $b = \sqrt{3}$, and $P = 12\sqrt{3}$. We prove that $12\sqrt{3}$ is minimal.

Say the roots to $ax^3 - x^2 + bx - 1$ are $p = \tan A$, $q = \tan B$, and $r = \tan C$ with $0 < A, B, C < 90°$. Then

$$\begin{aligned} ax^3 - x^2 + bx - 1 &= a(x-p)(x-q)(x-r) \\ &= ax^3 - a(p+q+r)x^2 + a(pq+qr+rp)x - a(pqr). \end{aligned}$$

Thus $a = \frac{1}{p+q+r} = \frac{1}{pqr} > 0$ and $p + q + r = pqr$. Then

$$\begin{aligned} r &= \frac{p+q}{pq-1} \\ &= -\tan(A+B) \\ &= \tan(180° - A - B), \end{aligned}$$

so $A + B + C = 180°$. Then since $\tan x$ is convex for $0 < x < 90°$, we have

$$\frac{1}{a} = \tan A + \tan B + \tan C \geq 3 \tan 60° = 3\sqrt{3},$$

so $a \leq \frac{1}{3\sqrt{3}}$.

Also notice that

$$\frac{b}{a} = pq + qr + rp \geq 3\sqrt[3]{p^2q^2r^2} = 3\sqrt[3]{\frac{1}{a^2}} \geq 9 > 1,$$

so $b > a$. Thus the denominator of P is always positive and is an increasing function of b, while the numerator of P is a decreasing function of b. Therefore, for a constant a, P is a decreasing function of b.

Furthermore,

$$\begin{aligned} &(p-q)^2 + (q-r)^2 + (r-p)^2 \geq 0 \\ &\Longrightarrow (p+q+r)^2 \geq 3(pq+qr+rp) \\ &\Longrightarrow \frac{1}{a^2} \geq \frac{3b}{a} \Longrightarrow b \leq \frac{1}{3a}, \end{aligned}$$

and

$$P \geq \frac{5a^2 - 3a(\frac{1}{3a}) + 2}{a^2(\frac{1}{3a} - a)} = \frac{5a^2 + 1}{\frac{a}{3} - a^3}.$$

Thus for $0 < a \le \frac{1}{3\sqrt{3}}$, it suffices to show that

$$\begin{aligned} & 5a^2 + 1 \ge 12\sqrt{3}\left(\frac{a}{3} - a^3\right) = 4\sqrt{3}a - 12\sqrt{3}a^3 \\ \iff & 12\sqrt{3}a^3 + 5a^2 - 4\sqrt{3}a + 1 \ge 0 \\ \iff & 3\left(a - \frac{1}{3\sqrt{3}}\right)(4\sqrt{3}a^2 + 3a - \sqrt{3}) \ge 0 \\ \iff & 4\sqrt{3}a^2 + 3a - \sqrt{3} \le 0. \end{aligned}$$

But the last quadratic has one positive root

$$\frac{-3 + \sqrt{57}}{8\sqrt{3}} \ge \frac{-3 + 7}{8\sqrt{3}} = \frac{1}{2\sqrt{3}} > \frac{1}{3\sqrt{3}},$$

so it is indeed negative when $0 < a \le \frac{1}{3\sqrt{3}}$. This completes the proof.

Problem 10 Let $f(x)$ be a continuous function defined on $[0,1]$ such that

(i) $f(0) = f(1) = 0$;

(ii) $2f(x) + f(y) = 3f\left(\dfrac{2x + y}{3}\right)$ for all $x, y \in [0,1]$.

Prove that $f(x) = 0$ for all $x \in [0,1]$.

Solution: We prove by induction on k that $f\left(\frac{m}{3^k}\right) = 0$ for all integers $k \ge 0$ and all integers $0 \le m \le 3^k$. The given conditions show this claim is true for $k = 0$; now assuming it is true for $k-1$, we prove it is true for k.

If $m \equiv 0 \pmod 3$ then $f\left(\frac{m}{3^k}\right) = f\left(\frac{\frac{m}{3}}{3^{k-1}}\right) = 0$ by the induction hypothesis.

If $m \equiv 1 \pmod 3$, then $1 \le m \le 3^k - 2$ and

$$3f\left(\frac{m}{3^k}\right) = 2f\left(\frac{\frac{m-1}{3}}{3^{k-1}}\right) + f\left(\frac{\frac{m+2}{3}}{3^{k-1}}\right) = 0 + 0 = 0.$$

Thus $f\left(\frac{m}{3^k}\right) = 0$.

And if $m \equiv 2 \pmod 3$, then $2 \le m \le 3^k - 1$ and

$$3f\left(\frac{m}{3^k}\right) = 2f\left(\frac{\frac{m+1}{3}}{3^{k-1}}\right) + f\left(\frac{\frac{m-2}{3}}{3^{k-1}}\right) = 0 + 0 = 0.$$

Hence $f\left(\frac{m}{3^k}\right) = 0$, finishing our induction.

Now, for any $x \in [0,1]$ we can form a sequence of numbers of the form $\frac{m}{3^k}$ whose limit is x; then since f is continuous, it follows that $f(x) = 0$ for all $x \in [0,1]$, as desired.

Problem 11 The base side and the altitude of a right regular hexagonal prism $ABCDEF - A'B'C'D'E'F'$ are equal to a and h respectively.

(a) Prove that six planes

$$(AB'F'), (CD'B'), (EF'D'), (D'EC), (F'AE), (B'CA)$$

touch the same sphere.

(b) Determine the center and the radius of the sphere.

Solution:

(a) Let O be the center of the prism. $(AB'F')$ is tangent to a unique sphere centered at O. Now the other five planes are simply reflections and rotations of $(AB'F')$ with respect to O; and since the sphere remains fixed under these transformations, it follows that all six planes are tangent to this same sphere.

(b) From part (a), the center of the sphere is the center O of the prism. Let P be the midpoint of $\overline{AE}$ and let P' be the midpoint of $\overline{A'E'}$. Also let Q be the midpoint of $\overline{PF'}$, and let $\overline{OR}$ be the perpendicular from O to line PF'. Note that P, P', Q, R, O, F' all lie in one plane.

It is straightforward to calculate that $F'P' = \frac{a}{2}$ and $QO = \frac{3a}{4}$. Also, since $QO \parallel F'P'$ we have $\angle RQO = \angle PF'P'$; combined with $\angle ORQ = \angle PP'F' = 90^\circ$, this gives $\triangle ORQ \sim \triangle PP'F'$. Hence the radius of the sphere is

$$OR = PP' \cdot \frac{OQ}{PF'} = h \cdot \frac{\frac{3a}{4}}{\sqrt{\left(\frac{a}{2}\right)^2 + h^2}} = \frac{3ah}{2\sqrt{a^2 + 4h^2}}.$$

Problem 12 For $n = 1, 2, \ldots$, two sequences $\{x_n\}$ and $\{y_n\}$ are defined recursively by

$$x_1 = 1,\ y_1 = 2,\ x_{n+1} = 22y_n - 15x_n,\ y_{n+1} = 17y_n - 12x_n.$$

(a) Prove that x_n and y_n are not equal to zero for all $n = 1, 2, \ldots$.

(b) Prove that each sequence contains infinitely many positive terms and infinitely many negative terms.

(c) For $n = 1999^{1945}$, determine whether x_n and y_n are divisible by 7 or not.

Solution:

(a) The recursive equation for x_{n+1} gives $y_n = \frac{1}{22}(15x_n + x_{n+1})$ and thus also $y_{n+1} = \frac{1}{22}(15x_{n+1} + x_{n+2})$. Substituting these expressions into the other recursive equation gives

$$x_{n+2} = 2x_{n+1} - 9x_n$$

and similarly we can also find

$$y_{n+2} = 2y_{n+1} - 9y_n.$$

Quick computation gives $x_2 = 29$ and $y_2 = 22$. Then x_1, x_2 are odd, and if x_n, x_{n+1} are odd then x_{n+2} must be as well; thus all the x_n are odd and hence nozero. Similarly, all the y_n are congruent to $2 \pmod 4$ and thus nonzero as well.

(b) Note that $x_{n+3} = 2(2x_{n+1} - 9x_n) - 9x_{n+1} = -5x_{n+1} - 18x_n$. Thus if x_n, x_{n+1} are positive (or negative) then x_{n+3} is negative (or positive). Hence among every four consecutive terms x_n there is some positive number and some negative number. Therefore there are infinitely many positive terms and infinitely many negative terms in this sequence; and a similar proof holds for the y_n.

(c) All congruences are taken modulo 7 unless stated otherwise. Note that $x_1 \equiv x_2 \equiv 1$. Now if $x_n \equiv x_{n+1}$ and $x_n \not\equiv 0$, then we have $(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}) \equiv (0, 5x_n, 3x_n, 3x_n)$ and $5x_n \not\equiv 0$, $3x_n \not\equiv 0$. This implies that $x_3, x_7, x_{11}, \ldots$ are all divisible by 7 but no other x_n are. Since $1999^{1945} \equiv 3^{1945} \equiv 3 \cdot 9^{1944/2} \equiv 3 \pmod 4$, we indeed have $7 \mid x_n$ when $n = 1999^{1945}$.

Now suppose by way of contradiction that $7 \mid y_{n'}$ for some n', and choose the minimal such n'. From the recursion for y_n, we have $y_n \equiv y_{n+1} + 3y_{n+2}$. Now if $n' \geq 5$ then $y_{n'-2} \equiv y_{n'-1}$, $y_{n'-3} \equiv 4y_{n'-1}$, and $y_{n'-4} \equiv 0$ — contradicting the minimal choice of n'. Thus we have $n' \leq 4$, but we have $(y_1, y_2, y_3, y_4) \equiv (2, 1, 5, 1)$. Therefore no term is divisible by 7; and specifically, $7 \nmid y_n$ when $n = 1999^{1945}$.

1
1999 Regional Contests: Problems and Solutions

1.1 Asian Pacific Mathematical Olympiad

Problem 1 Find the smallest positive integer n with the following property: There does not exist an arithmetic progression of 1999 real numbers containing exactly n integers.

Solution: Look at the 1999-term arithmetic progression with common difference $\frac{1}{q}$ and beginning term $\frac{p}{q}$, where p and q are integers with $1 < q < 1999$; without loss of generality assume that $1 \le p \le q$. When $p = 1$, the progression contains the integers $1, 2, \ldots, \left\lfloor \frac{1999}{q} \right\rfloor$; when $p = q$, the progression contains the integers $1, 2, \ldots, 1 + \left\lfloor \frac{1998}{q} \right\rfloor$. Since 1999 is prime, q does not divide 1999 and hence $\left\lfloor \frac{1999}{q} \right\rfloor = \left\lfloor \frac{1998}{q} \right\rfloor$. Thus the progression contains either $\left\lfloor \frac{1999}{q} \right\rfloor$ or $\left\lfloor \frac{1999}{q} \right\rfloor + 1$ integers, and any k of this form can be attained. Call such numbers *good*.

Conversely, suppose we have an arithmetic progression containing exactly k integers, where $1 < k < 1999$; without loss of generality, say its common difference is positive and say it contains 0 as its t-th term. Its common difference cannot be irrational, so it is of the form $\frac{p}{q}$ for some positive, relatively prime integers p, q. And since $1 < k < 1999$, $1 < q < 1999$. But then look at the arithmetic progression with common difference $\frac{1}{q}$ and 0 as its t-th term. It, too, contains exactly k integers; so from our previous argument, k is good.

Now, for $q \ge 32$ we have $1999 < 2q(q+1) \Longrightarrow \frac{1999}{q} - \frac{1999}{q+1} < 2$. Since $\left\lfloor \frac{1999}{32} \right\rfloor = 62$ and $\left\lfloor \frac{1999}{1998} \right\rfloor = 1$, this implies that every integer k between 1 and 63 is good. Also,

$$\left\lfloor \frac{1999}{31} \right\rfloor = 64, \quad \left\lfloor \frac{1999}{30} \right\rfloor = 66, \quad \left\lfloor \frac{1999}{29} \right\rfloor = 68,$$

$$\left\lfloor \frac{1999}{q} \right\rfloor \le 62 \text{ when } q \ge 32, \quad \left\lfloor \frac{1999}{q} \right\rfloor \ge 71 \text{ when } q \le 28.$$

Thus 70 is the smallest integer that k cannot equal, and $n = 70$.

Problem 2 Let a_1, a_2, $\ldots$ be a sequence of real numbers satisfying

$$a_{i+j} \le a_i + a_j$$

for all i, $j = 1, 2, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for all positive integers n.

Solution:

Lemma. *If m, n are positive integers with $m \geq n$, then $a_1 + a_2 + \cdots + a_n \geq \frac{n(n+1)}{2m} \cdot a_m$.*

Proof: The result for $m = n$ comes from adding the inequalities $a_1 + a_{n-1} \geq a_n$, $a_2 + a_{n-2} \geq a_n$, $\ldots$, $a_{n-1} + a_1 \geq a_n$, $2a_n \geq 2a_n$, and then dividing by two. Now for positive integers j, write $\beta_j = \frac{a_1+a_2+\cdots+a_j}{1+2+\cdots+j}$. Then the inequality for $m = n = j = k+1$ is equivalent to both $\beta_j \geq \frac{a_j}{j}$ and $\beta_k \geq \beta_{k+1}$; so when $m \geq n$ we have $\beta_n \geq \beta_{n+1} \geq \cdots \geq \beta_m \geq \frac{a_m}{m}$, as desired. ■

The desired inequality now follows from expressing $a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n}$ as an Abel sum and then applying the lemma multiple times:

$$\begin{aligned}
&a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} \\
&= \frac{1}{n}(a_1 + a_2 + \cdots + a_n) + \sum_{j=1}^{n-1} \left(\frac{1}{j} - \frac{1}{j+1}\right)(a_1 + a_2 + \cdots + a_j) \\
&\geq \frac{1}{n}\frac{n(n+1)}{2n}a_n + \sum_{j=1}^{n-1} \frac{1}{j(j+1)} \cdot \frac{j(j+1)}{2n}a_n \\
&= a_n,
\end{aligned}$$

as desired.

Problem 3 Let ω_1 and ω_2 be two circles intersecting at P and Q. The common tangent, closer to P, of ω_1 and ω_2, touches ω_1 at A and ω_2 at B. The tangent to ω_1 at P meets ω_2 again at C, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Solution: We shall use directed angles. Using tangents and cyclic quadrilaterals, we have $\angle QAR = \angle QAP = \angle QPC = \angle QBC = \angle QBR$, so $QABR$ is cyclic.

Since $\angle BPR$ is an exterior angle to triangle ABP, we have $\angle BPR = \angle BAP + \angle PBA$. Then again using tangents and cyclic quadrilaterals, we have $\angle BAP + \angle PBA = \angle BAR + \angle PCB = \angle BQR + \angle PQB = \angle PQR$. Thus $\angle BPR = \angle PQR$, which implies that line BP is tangent to the circumcircle of triangle PQR.

Next, $\angle PRB$ is an exterior angle to triangle CRP so $\angle PRB = \angle PCR + \angle RPC$. We know that $\angle PCR = \angle PCB = \angle PQB$; and letting T be the intersection of lines CP and AB, we have $\angle RPC = \angle APT = \angle AQP = \angle BAP = \angle BAR = \angle BQR$. Therefore $\angle PRB = \angle PQB + \angle BQR = \angle PQR$, which implies that line BR is tangent to the circumcircle of triangle PQR as well.

Problem 4 Determine all pairs (a, b) of integers for which the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

Solution: If $a = 0$ then b must be a perfect square, and vice versa. Now assume both a and b are nonzero. Also observe that $a^2 + 4b$ and a^2 have the same parity, and similarly $b^2 + 4a$ and b^2 have the same parity.

If b is positive then $a^2+4b \geq (|a|+2)^2 = a^2+4|a|+4$ so $|b| \geq |a|+1$. If b is negative then $a^2+4b \leq (|a|-2)^2 = a^2-4|a|+4$ so $|b| \geq |a|-1$. Similarly, $a > 0 \Longrightarrow |a| \geq |b| + 1$ and $a < 0 \Longrightarrow |a| \geq |b| - 1$.

Assume without loss of generality that $b > a$. If a and b are positive, then from the last paragraph we have $b \geq a + 1$ and $a \geq b + 1$, a contradiction.

If a and b are negative, then we have either $a = b$ or $a = b - 1$. For $b \geq -5$, only $(a, b) = (-4, -4)$ and $(-6, -5)$ work. Otherwise, we have $(b + 4)^2 < b^2 + 4a < (b + 2)^2$, a contradiction.

Finally, if a is negative and b is positive, then we have both $|b| \geq |a| + 1$ and $|a| \geq |b| - 1$. Then we must have $|b| = |a| + 1$ and hence $a + b = 1$. Any such pair works, since then $a^2 + 4b = (a - 2)^2$ and $b^2 + 4a = (b - 2)^2$ are both perfect squares.

Therefore the possible pairs (a, b) are $(-4, -4)$, $(-6, -5)$, $(-5, -6)$; and $(0, n^2)$, $(n^2, 0)$, and $(n, 1 - n)$ where n is any integer.

Problem 5 Let $\mathcal{S}$ be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of $\mathcal{S}$ on its circumference, $n-1$ points in its interior, and $n - 1$ in its exterior. Prove that the number of good circles has the same parity as n.

Solution:

Lemma. *Suppose we have $2n \geq 1$ given points in the plane with no three collinear, and one distinguished point A among them. Call a line "good" if it passes through A and one other given point, and if it separates the remaining $2n-2$ points: that is, half of them lie on one side of the line, and the other half lie on the other. Then there are an odd number of good lines.*

Proof: We prove the claim by induction. It is trivial for $n = 1$; now assuming it is true for $n-1$, we prove it is true for n.

Without loss of generality, arrange the $2n-1$ points different from A on a circle centered at A. From those $2n-1$ points, choose two points, B and C, that are the greatest distance apart. Then if $P \neq A, B, C$ is a given point, B and C lie on different sides of line AP. Thus line AP is good if and only if it separates the other $2n-3$ points; and by the induction hypothesis, there are an odd number of such lines. Finally, line AB is good if and only if line AC is good — adding either 0 or 2 good lines to our count, so that our total count remains odd. This completes the inductive step. ■

Suppose we have a pair of points $\{A, B\}$ in $\mathcal{S}$. Perform an inversion about A with arbitrary radius. B and the other $2n-1$ points $C_1, C_2, \ldots, C_{2n-1}$ map to $2n$ distinct points $B', C_1', C_2', \ldots, C_{2n-1}'$ (no three collinear); and the circle passing through A, B, C_k is good if and only if the corresponding line $B'C_k'$ separates the other C_i'. By the lemma, there are an odd number of such lines; so an odd number of good circles pass through any pair of given points.

For each pair of points count the number of good circles passing through these points; each good circle is counted three times in this manner, so if there are k good circles then our count will be $3k$. And there are $\binom{2n+1}{2} = n(2n+1)$ pairs of points, each contributing an odd amount to our count. Therefore $3k \equiv n(2n+1) \implies k \equiv n \pmod{2}$, as desired.

1.2 Austrian-Polish Mathematics Competition

Problem 1 Let n be a positive integer and $M = \{1, 2, \ldots, n\}$. Find the number of ordered 6-tuples $(A_1, A_2, A_3, A_4, A_5, A_6)$ which satisfy the following two conditions:

(a) A_1, A_2, A_3, A_4, A_5, A_6 (not necessarily distinct) are subsets of M;

(b) each element of M belongs to either 0, 3, or 6 of the sets A_1, A_2, A_3, A_4, A_5, A_6.

Solution: Given $k \in M$, there are $\binom{6}{0}$ ways to put k into exactly 0 of the 6 sets; $\binom{6}{3}$ ways to put k into exactly 3 of the sets; and $\binom{6}{6}$ ways to put k into all 6 sets. This adds up to $1 + 20 + 1 = 22$ ways to put k into the sets. Every admissible 6-tuple is uniquely determined by where each k is placed; therefore, there are 22^n possible distributions.

Problem 2 Find the largest real number C_1 and the smallest real number C_2 such that for all positive real numbers a, b, c, d, e the following inequalities hold:

$$C_1 < \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+e} + \frac{e}{e+a} < C_2.$$

Solution: Let $f(a,b,c,d,e) = \frac{a}{a+b} + \cdots + \frac{e}{e+a}$. Note that $f(e,d,c,b,a) = 5 - f(a,b,c,d,e)$. Hence $f(a,b,c,d,e)$ can attain the value x if and only if it can attain the value $5 - x$. Thus if we find C_1, then $C_2 = 5 - C_1$.

If $a = \epsilon^4$, $b = \epsilon^3$, $c = \epsilon^2$, $d = \epsilon$, $e = 1$, then

$$f(a,b,c,d,e) = \frac{4\epsilon}{1+\epsilon} + \frac{1}{1+\epsilon^4},$$

which for small ϵ can become arbitrarily close to 1. We now prove that $f(a,b,c,d,e)$ is *always* bigger than 1. Since f is homogenous, we may assume without loss of generality that $a+b+c+d+e = 1$. Now $g(x) = \frac{1}{x}$ is convex for all positive x; so by Jensen's inequality, $ag(x_1) + bg(x_2) + \cdots + eg(x_5) \geq g(ax_1 + bx_2 + \cdots + ex_5)$. Applying this inequality with $x_1 = a+b$, $x_2 = b+c$, …, $x_5 = e+a$, we find that

$$
\begin{aligned}
f(a,b,c,d,e) &\geq \frac{1}{\sum_{\text{cyc}} a(a+b)} \\
&= \frac{(a+b+c+d+e)^2}{\sum_{\text{cyc}} a(a+b)} > \frac{\sum_{\text{cyc}} a(a+2b)}{\sum_{\text{cyc}} a(a+b)} > 1,
\end{aligned}
$$

as desired. (Here $\sum_{\text{cyc}} h(a,b,c,d,e)$ means $h(a,b,c,d,e)+h(b,c,d,e,a)+\dots+h(e,a,b,c,d)$.) Hence $C_1 = 1$; and from our initial arguments, $C_2 = 4$.

Problem 3 Let $n \geq 2$ be a given integer. Determine all systems of n functions $(f_1, \dots, f_n)$ where $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \dots, n$, such that for all $x, y \in \mathbb{R}$ the following equalities hold:

$$
\begin{aligned}
f_1(x) - f_2(x)f_2(y) + f_1(y) &= 0 \\
f_2(x^2) - f_3(x)f_3(y) + f_2(y^2) &= 0 \\
&\vdots \\
f_{n-1}(x^{n-1}) - f_n(x)f_n(y) + f_{n-1}(y^{n-1}) &= 0 \\
f_n(x^n) - f_1(x)f_1(y) + f_n(y^n) &= 0.
\end{aligned}
$$

Solution: Setting $x = y$ into the k-th equation gives $2f_k(x^k) = f_{k+1}(x)^2$ (where we write $f_{n+1} = f_1$). Thus if $f_k(x) = 0$ for all $x \in \mathbb{R}$, then $f_{k+1}(x)$ is also always zero; and similarly, *all* the f_i's are identically zero. Now assume that no f_k is identically zero.

For odd k, given any real value r let $x = \sqrt[k]{r}$. Then $2f_k(r) = 2f_k(x^k) = f_{k+1}(x)^2 \geq 0$ for all r. Similarly, for even k, $f_k(r) \geq 0$ whenever $r \geq 0$. Also observe that for even k, we cannot have some $f_k(x) < 0$ and some $f_k(y) > 0$ because then we'd have $f_{k-1}(x^{k-1}) - f_k(x)f_k(y) + f_{k-1}(y^{k-1}) > 0$, contradicting the $(k-1)$-th equation. And since $f_{k+1}(x) \neq 0$ for some x, we have $f_k(x^k) = \frac{1}{2}f_{k+1}(x)^2 > 0$. Therefore, we must have $f_k(x) \geq 0$ for all x and for *all* k.

We now prove by induction on k that f_k is a constant function. For $k = 1$, plugging in $f_2(x) = \sqrt{2f_1(x)}$ and $f_2(y) = \sqrt{2f_1(y)}$ into the first equation gives $f_1(x) - 2\sqrt{f_1(x)f_1(y)} + f_1(y) = 0$ for all $x, y \in \mathbb{R}$. By the arithmetic mean-geometric mean inequality, this is true only when $f_1(x) = f_1(y)$ for all $x, y \in \mathbb{R}$.

Now assume that $f_k(x) = f_k(y)$ for all x, y. Then $f_{k+1}(x) = \sqrt{2f_k(x^k)} = \sqrt{2f_k(y^k)} = f_{k+1}(y)$ for any $x, y \in \mathbb{R}$, completing the inductive step.

Writing $f_k(x) = c_k$ (where we write $c_{n+1} = c_1$), observe that $c_k = \frac{1}{2}c_{k+1}^2$ for all k; since $f_k(x) \geq 0$ for all x but f_k is not identically zero, each c_k is positive. If $0 < c_{k+1} < 2$, then $0 < c_k < c_{k+1} < 2$. Thus $c_n > c_{n-1} > \cdots > c_1 > c_n$, a contradiction. Similarly, if $c_{k+1} > 2$ then $c_k > c_{k+1} > 2$; so that $c_n < c_{n-1} < \cdots < c_1 < c_n$, a contradiction. Hence, we must have $c_k = 2$ for all k.

Therefore, the only possible functions are $f_k(x) = 0$ for all x, k; and $f_k(x) = 2$ for all x, k.

Problem 4 Three straight lines k, l, and m are drawn through some fixed point P inside a triangle ABC such that:

(a) k meets the lines AB and AC in A_1 and A_2 ($A_1 \neq A_2$) respectively and $PA_1 = PA_2$;

(b) l meets the lines BC and BA in B_1 and B_2 ($B_1 \neq B_2$) respectively and $PB_1 = PB_2$;

(c) m meets the lines CA and CB in C_1 and C_2 ($C_1 \neq C_2$) respectively and $PC_1 = PC_2$.

Prove that the lines k, l, m are uniquely determined by the above conditions. Find the point P (and prove that there exists exactly one such point) for which the triangles AA_1A_2, BB_1B_2, and CC_1C_2 have the same area.

Solution: Let ℓ be the reflection of line AB about P; since A_1 and A_2 are also mirror images of each other across P, A_2 must lie on ℓ. Thus A_2 must be the intersection of ℓ and line AC, and this intersection point is unique since lines AB and AC are not parallel. Therefore k must be the line passing through P and this point, and conversely this line satisfies condition (a). Similarly, lines l and m are also uniquely determined.

Now suppose that triangles AA_1A_2 and BB_1B_2 have the same area. Let Q be the midpoint of $\overline{AA_1}$; then since P is the midpoint of $\overline{A_1A_2}$, we have $[AQP] = \frac{1}{2}[AA_1P] = \frac{1}{4}[AA_1A_2]$. Similarly, let R be the midpoint of $\overline{BB_2}$; then $[BRP] = \frac{1}{2}[BB_2P] = \frac{1}{4}[BB_1B_2]$. Therefore $[AQP] = [BRP]$; and since the heights from P in triangles AQP and BRP are equal, we must have $AQ = BR$.

Now since P and Q are the midpoints of $\overline{A_1A_2}$ and $\overline{A_1A}$, we have $PQ \parallel AA_2$ and hence $PQ \parallel AC$. This implies that Q lies between A and B. Similarly, R lies between A and B as well. Then since $AQ = BR$, Q and R are equidistant from the midpoint C' of $\overline{AB}$. Therefore the homothety about C' that maps A to Q also maps B to R. This homothety also maps line AC to QP since $AC \parallel QP$; and it maps line BC to RP. Hence it maps C to P, so that C', P, C are collinear and P lies on the median from C in triangle ABC.

Similarly, P must lie on the medians from A and B in triangle ABC, so it must be the centroid G of triangle ABC. And conversely, if $P = G$ then k, l, m are parallel to lines BC, CA, AB respectively, and $[AA_1A_2] = [BB_1B_2] = [CC_1C_2] = \frac{4}{9}[ABC]$.

Problem 5 A sequence of integers $\{a_n\}_{n\geq 1}$ satisfies the following recursive relation

$$a_{n+1} = a_n^3 + 1999 \quad \text{for } n = 1,\, 2,\, \ldots.$$

Prove that there exists at most one n for which a_n is a perfect square.

Solution: Consider the possible values of (a_n, a_{n+1}) modulo 4:

a_n	0	1	2	3
a_{n+1}	3	0	3	2

No matter what a_1 is, the terms $a_3, a_4, \ldots$ are all 2 or 3 (mod 4); but all perfect squares are 0 or 1 (mod 4), so at most two terms (a_1 and a_2) can be perfect squares. But if a_1 and a_2 are both perfect squares, then writing $a_1 = a^2$, $a_2 = b^2$ we have $a^6 + 1999 = b^2$ or $1999 = b^2 - (a^3)^2 = (b+a^3)(b-a^3)$. But since 1999 is prime, $b - a^3 = 1$ and $b + a^3 = 1999$. Thus $a^3 = \frac{1999-1}{2} = 999$, which is impossible. Hence at most one term of the sequence is a perfect square.

Problem 6 Find all real numbers $x_0, x_1, x_2, \ldots, x_{1998} \geq 0$ which satisfy

$$x_{i+1}^2 + x_{i+1}x_i + x_i^4 = 1$$

for $i = 0, 1, \ldots, 1998$, where $x_{1999} = x_0$.

Solution: Let R be the positive real root of $x^4 + 2x^2 - 1 = 0$, found using the quadratic formula:

$$R^2 = \frac{-2+\sqrt{8}}{2} = -1 + \sqrt{2} \quad \Longrightarrow \quad R = \sqrt{-1+\sqrt{2}}.$$

If $x_n, x_{n+1} \geq R$ then $1 = x_{n+1}^2 + x_{n+1}x_n + x_n^4 \geq R^2 + R^2 + R^4 = 1$, with equality when $x_n = x_{n+1} = R$. Similarly, if $x_n, x_{n+1} \leq R$ then we must have $x_n = x_{n+1} = R$. Hence either $x_n = x_{n+1} = R$, $x_n < R < x_{n+1}$, or $x_n > R > x_{n+1}$.

Now if $x_0 < R$ then $x_1 > R$, $x_2 < R$, …, $x_0 = x_{1999} > R$, a contradiction. Similarly, we cannot have $x_0 > R$. Therefore $x_0 = R$ and the only solution is

$$x_0 = x_1 = \cdots = x_{1999} = R = \sqrt{-1 + \sqrt{2}}.$$

Problem 7 Find all pairs (x, y) of positive integers such that

$$x^{x+y} = y^{y-x}.$$

Solution: Let $\gcd(x, y) = c$, and write $a = \frac{x}{c}$, $b = \frac{y}{c}$. Then the equation becomes

$$\begin{aligned}(ac)^{c(a+b)} &= (bc)^{c(b-a)}\\ (ac)^{a+b} &= (bc)^{b-a}\\ a^{a+b}c^{2a} &= b^{b-a}.\end{aligned}$$

Thus $a^{a+b} \mid b^{b-a}$; then since $\gcd(a, b) = 1$, a must equal 1. Therefore

$$b^{b-1} = c^2$$

is a perfect square. This is true exactly when b is odd, or when b is a perfect square. If $b = 2n + 1$ then $c = (2n + 1)^n$; and if $b = n^2$ then $c = n^{n^2-1}$. Therefore (x, y) equals either

$$((2n + 1)^n, (2n + 1)^{n+1}) \quad \text{or} \quad (n^{n^2-1}, n^{n^2+1})$$

for some positive integer n, and any such pair indeed satisfies the given equations.

Problem 8 Let ℓ be a given straight line and let the points P and Q lie on the same side of the line ℓ. The points M, N lie on the line ℓ and satisfy $PM \perp \ell$ and $QN \perp \ell$. The point S lies between the lines PM and QN such that $PM = PS$ and $QN = QS$. The perpendicular bisectors of $\overline{SM}$ and $\overline{SN}$ meet at R. Let T be the second intersection of the line RS and the circumcircle of triangle PQR. Prove that $RS = ST$.

Solution: All angles are directed modulo $180°$. Let T' be the reflection of R across S; we wish to prove that $PRQT'$ is cyclic.

Since $RM = RS = RN$, $\angle RMN = \angle MNR = x$. Note that $PMRS$ is a kite which is symmetric with respect to line PR. Hence $\angle PRM = \angle SRP = y$, $\angle MPR = \angle RPS = z$, and $\angle PSR = \angle RMP = 90° + x$. Similarly, $\angle QRS = \angle NRQ = u$, $\angle SQR = \angle RQN = v$, and $\angle RSQ = \angle QNR = 90° + x$. In triangle MNR, $2(x+y+u) = 180°$. In triangle PMR, $y+z+90°+x = 180°$ so that $2(x+y+z) = 180°$. Hence $2u = 2z$.

Orient our diagram so that line MN is horizontal and so that P and Q are above line MN. From the given information, S is above line MN and between lines PM and QN. Also, since since R is the circumcenter of triangle MSN, R lies below between lines PM and QN and below S. This information allows us to safely conclude that $u = z$, or $\angle SPR = \angle SRQ$.

Similarly, $y = v$ and $\angle PRS = \angle SQR$. Thus $\triangle PSR \sim \triangle RSQ$. Let A and B be the midpoints of $\overline{PR}$ and $\overline{QR}$, respectively. Then $\overline{SA}$ and $\overline{SB}$ are corresponding medians in similar triangles PRS and RQS. Hence $\triangle ASR \sim \triangle BSQ$. It follows that

$$\angle ASB = \angle ASR + \angle RSB = \angle BSQ + \angle RSB = \angle RSQ = 90° + x.$$

Thus $\angle ASB + \angle ARB = \angle ASB + \angle PRQ = 90° + x + y + z = 180°$ and $ASRB$ is cyclic. Since $PRQT'$ is the image of $ARBS$ under a homothety about R with ratio 2, it follows that $PQRT'$ is cyclic as well.

Problem 9 Consider the following one player game. On the plane, a finite set of selected lattice points and segments is called a *position* in this game if the following hold:

(i) the endpoints of each selected segment are lattice points;

(ii) each selected segment is parallel to a coordinate axis, or to the line $y = x$, or to the line $y = -x$;

(iii) each selected segment contains exactly 5 lattice points and all of them are selected;

(iv) any two selected segments have at most one common point.

A move in this game consists of selecting a new lattice point and a new segment such that the new set of selected lattice points and segments is a position. Prove or disprove that there exists an initial position such that the game has infinitely many moves.

Solution: There exists no position so that the game can last for infinitely many moves.

Given any segment, let its "extreme point" be its leftmost, upper-leftmost, highest, or upper-rightmost point (depending on whether the segment is parallel to $y = 0$, $y = -x$, $x = 0$, or $y = x$); let its other four lattice points form the segment's "mini-segment." Observe that no two mini-segments pointing in the same direction can intersect.

Also, given any position, call a lattice point a "missing point" if it is the extreme point of a selected segment in one direction, but does not lie on any other selected segment pointing in the same direction. (Notice that during the game, a lattice point might switch between being missing and not-missing.)

Lemma. *Given an integer $N > 0$, if the game continues forever then at some point at least N missing points will exist at the same time.*

Proof: Suppose we have k lines that contain at least one selected segment. Some $\lceil \frac{k}{4} \rceil$ of them must point in the same direction; then each of these lines contains at least one different missing point: its leftmost, upper-leftmost, highest, or upper- rightmost extreme point. Therefore it is enough to show that k gets arbitrarily large.

For sake of contradiction, suppose that k is bounded. Then all the selected segments will lie on some finite number of lines; these lines have only a finite set S of t intersection points, so from some position onward no new selected point will be in S. At this point say we have s selected segments, and p selected points outside of S.

After n more moves there will be $s + n$ mini-segments, and $p + n$ selected points outside of S. Each mini-segment contains 4 points for a total count of $4(s + n)$. On the other hand, each of the t points lies on at most 4 mini-segments; and each of the $p + n$ other points lies on at most 1 mini-segment, for a total count of at most $4t + (p + n)$. Thus $4(s + n) \leq 4t + (p + n)$, but this is false for large enough n — a contradiction. ∎

Now suppose in our original position we have s selected segments and p selected points. From the lemma, eventually we will have more than $4(p - s)$ missing points. Say this happens after n moves, when we have $s + n$ selected segments and $p + n$ selected points. Then as in the lemma, we will have $s + n$ mini-segments each containing 4 points—for a total count of $4(s + n)$. On the other hand, each of the missing points lies on at most 3 mini-segments; and all the other selected points lie on at most 4 mini-segments each. Thus our total count is less than $3 \cdot 4(p - s) + 4 \cdot (p + n - 4(p - s)) = 4(s + n)$, a contradiction. This completes the proof.

1.3 Balkan Mathematical Olympiad

Problem 1 Given an acute-angled triangle ABC, let D be the midpoint of minor arc $\widehat{BC}$ of circumcircle ABC. Let E and F be the respective images of D under reflections about BC and the center of the circumcircle. Finally, let K be the midpoint of AE. Prove that:

(a) the circle passing through the midpoints of the sides of the triangle ABC also passes through K.

(b) the line passing through K and the midpoint of BC is perpendicular to AF.

Solution:

(a) Let M, B_1, and C_1 denote the midpoints of sides BC, CA, and AB, respectively; then $\triangle ABC \sim \triangle MB_1C_1$ and $\angle C_1MB_1 = \angle A$. Also, $BECD$ is a rhombus, with $\angle BEC = \angle CDB = 180° - \angle A$. The homothety centered at A with ratio $\frac{1}{2}$ maps triangle BEC to triangle C_1KB_1. Thus, $\angle C_1KB_1 + \angle C_1MB_1 = \angle BEC + \angle A = 180°$, so MC_1KB_1 is cyclic.

(b) Since $ED = 2EM$ and $EA = 2EK$, $MK \parallel AD$. But $\overline{DF}$ is a diameter, so $AD \perp AF$. Hence also $MK \perp AF$.

Problem 2 Let $p > 2$ be a prime number such that $3 \mid (p-2)$. Let

$$\mathcal{S} = \{y^2 - x^3 - 1 \mid x \text{ and } y \text{ are integers, } 0 \le x, y \le p-1\}.$$

Prove that at most p elements of $\mathcal{S}$ are divisible by p.

Solution:

Lemma. *Given a prime p and a positive integer $k > 1$, if k and $p-1$ are relatively prime then $x^k \equiv y^k \Rightarrow x \equiv y \,(mod\, p)$ for all x, y.*

Proof: If $y \equiv 0 \pmod p$ the claim is obvious. Otherwise, note that $x^k \equiv y^k \implies (xy^{-1})^k \equiv 1 \pmod p$, so it suffices to prove that $a^k \equiv 1 \implies a \equiv 1 \pmod p$.

Since $\gcd(p-1, k) = 1$, there exist integers b and c such that $b(p-1)+ck = 1$. Thus, $a^k \equiv 1 \implies a^c \equiv 1 \implies a^{1-b(p-1)} \equiv 1 \pmod p$. If $a = 0$ this is impossible; otherwise, by Fermat's Little Theorem, $(a^{-b})^{p-1} \equiv 1 \pmod p$ so that $a \equiv 1 \pmod p$, as desired.

Alternatively, again note that clearly $a \not\equiv 1 \pmod p$. Then let d be the order of a, the smallest positive integer such that $a^d \equiv 1 \pmod p$;

we have $d \mid k$. Take the set $\{1, a, a^2, \ldots, a^{d-1}\}$; if it does not contain all of $1, 2, \ldots, p-1$ then pick some other element b and consider the set $\{b, ba, ba^2, \ldots, ba^{d-1}\}$. These two sets are disjoint, since otherwise $ba^i \equiv a^j \Rightarrow b \equiv a^{j-1} \pmod{p}$, a contradiction. Continuing similarly, we can partition $\{1, 2, \ldots, p-1\}$ into d-element subsets, and hence $d \mid p-1$. But since $d \mid k$ and $\gcd(k, p-1) = 1$, we must have $d = 1$. Therefore $a \equiv a^d \equiv 1 \pmod{p}$, as desired. ■

Since $3 \mid p-2$, $\gcd(3, p-1) = 1$. Then from the claim, it follows that the set of elements $\{1^3, 2^3, \ldots, p^3\}$ equals $\{1, 2, \ldots, p\}$ modulo p. Hence, for each y with $0 \le y \le p-1$, there is exactly one x between 0 and $p-1$ such that $x^3 \equiv y^2 - 1 \pmod{p}$: that is, such that $p \mid y^2 - x^3 - 1$. Therefore $\mathcal{S}$ contains at most p elements divisible by p, as desired.

Problem 3 Let ABC be an acute triangle, and let M, N, and P be the feet of the perpendiculars from the centroid to the three sides. Prove that

$$\frac{4}{27} < \frac{[MNP]}{[ABC]} \le \frac{1}{4}.$$

Solution: We begin by proving that $\frac{9[MNP]}{[ABC]} = \sin^2 A + \sin^2 B + \sin^2 C$. Let G be the centroid of triangle ABC, and let M, N, and P be on sides BC, AC, and AB, respectively. Also let $AB = c$, $BC = a$, $CA = b$, and $K = [ABC]$.

We have $[ABG] = \frac{K}{3} = \frac{1}{2}c \cdot GP \implies GP = \frac{2K}{3c}$. Similarly, $GN = \frac{2K}{3b}$, so

$$[PGN] = \frac{1}{2}GP \cdot GN \sin A = \frac{2K^2 \sin A}{9bc} = \frac{K^2 a^2}{9Rabc}.$$

Summing this formula with the analogous ones for $[NGM]$ and $[MGP]$ yields

$$[MNP] = \frac{K^2(a^2 + b^2 + c^2)}{9Rabc}.$$

Dividing this by $[ABC] = K$ and then substituting $K = \frac{abc}{4R}$, $a = 2R\sin A$, $b = 2R\sin B$, and $c = 2R\sin C$ on the right yields $\frac{[MNP]}{[ABC]} = \frac{1}{9}\left(\sin^2 A + \sin^2 B + \sin^2 C\right)$, as desired.

Hence the problem reduces to proving $\frac{4}{3} < \sin^2 A + \sin^2 B + \sin^2 C \le \frac{9}{4}$. Assume without loss of generality that $A \ge B \ge C$.

To prove the right inequality, first note that $A < \frac{\pi}{2} \Rightarrow B > \frac{\pi}{4}$. The function $\sin^2 x$ is concave on $[\frac{\pi}{4}, \frac{\pi}{2}]$; applying Jensen's Inequality gives $\sin^2 A + \sin^2 B \leq 2\sin^2\left(\frac{A}{2} + \frac{B}{2}\right) = 2\cos^2\left(\frac{C}{2}\right)$. Thus it suffices to prove $2\cos^2\left(\frac{C}{2}\right) + \sin^2 C \leq \frac{9}{4} \iff 1 + \cos C + 1 - \cos^2 C \leq \frac{9}{4} \iff -(\cos C + \frac{1}{2})^2 \leq 0$, which is true.

For the left inequality, note that $\sin^2 x$ is an increasing function on $[0, \frac{\pi}{2}]$. We have $B \geq \frac{\pi - A}{2}$, so $\sin^2 A + \sin^2 B + \sin^2 C > \sin^2 A + \cos^2\left(\frac{A}{2}\right) = -\cos^2 A + \frac{1}{2}\cos A + \frac{3}{2}$. But since A is the largest angle, we have $\frac{\pi}{2} > A \geq \frac{\pi}{3}$ so $\frac{1}{2} \geq \cos A > 0$; then $-\cos^2 A + \frac{1}{2}\cos A + \frac{3}{2} \geq \frac{3}{2} > \frac{4}{3}$, as desired.

Problem 4 Let $\{x_n\}_{n\geq 0}$ be a nondecreasing sequence of nonnegative integers such that for every $k \geq 0$ the number of terms of the sequence which are less than or equal to k is finite; let this number be y_k. Prove that for all positive integers m and n,

$$\sum_{i=0}^{n} x_i + \sum_{j=0}^{m} y_j \geq (n+1)(m+1).$$

Solution: Under the given construction, $y_s \leq t$ if and only if $x_t > s$. But this condition is equivalent to saying that $y_s > t$ if and only if $x_t \leq s$. Thus the sequences $\{x_i\}$ and $\{y_j\}$ are dual, meaning that applying the given algorithm to $\{y_j\}$ will restore the original $\{x_i\}$.

To find $\sum_{i=0}^{n} x_i$, note that among $x_0, x_1, \ldots, x_n$ there are exactly y_0 terms equal to 0, $y_1 - y_0$ terms equal to 1, $\ldots$, and $y_{x_{n-1}} - y_{x_{n-2}}$ terms equal to x_{n-1}; and the remaining $n+1-x_{n-1}$ terms equal x_n. Hence,

$$\begin{aligned}\sum_{i=0}^{n} x_i &= 0\cdot(y_0) + 1\cdot(y_1 - y_0) + \cdots \\ &\qquad + (x_n - 1)\cdot(y_{x_n-1} - y_{x_n-2}) + x_n \cdot (n + 1 - y_{x_n-1}) \\ &= -y_0 - y_1 - \cdots - y_{x_n-1} + (n+1)x_n.\end{aligned}$$

First suppose that $x_n - 1 \geq m$, and write $x_n - 1 = m + k$ for $k \geq 0$. Since $x_n > m + k$, from our initial observations we have $y_{m+k} \leq n$. But then $n + 1 \geq y_{m+k} \geq y_{m+k-1} \geq \cdots \geq y_m$, so

$$\sum_{i=0}^{n} x_i + \sum_{j=0}^{m} y_j = (n+1)x_n - \left(\sum_{j=0}^{x_{n-1}} y_j - \sum_{i=0}^{m} y_i\right)$$

$$
\begin{aligned}
&= (n+1)x_n - \sum_{i=m+1}^{m+k} y_i \\
&\geq (n+1)(m+k+1) - k \cdot (n+1) \\
&= (n+1)(m+1),
\end{aligned}
$$

as desired.

Next suppose that $x_n - 1 < m$. Then $x_n \leq m \Longrightarrow y_m > n \Longrightarrow y_m - 1 \geq n$. Since $\{x_i\}$ and $\{y_j\}$ are dual, we may therefore apply the same argument with the roles of the two sequences reversed. This completes the proof.

1.4 Czech and Slovak Match

Problem 1 For arbitrary positive numbers a, b, c, prove the inequality
$$\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}\geq 1.$$

First Solution: Set $x = b + 2c$, $y = c + 2a$, $z = a + 2b$. Then $a = \frac{1}{9}(4y + z - 2x)$, $b = \frac{1}{9}(4z + x - 2y)$, $c = \frac{1}{9}(4x + y - 2z)$, so the desired inequality becomes
$$\frac{4y+z-2x}{9x}+\frac{4z+x-2y}{9y}+\frac{4x+y-2z}{9z}\geq 1,$$
which is equivalent to
$$\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)+\left(\frac{z}{x}+\frac{x}{z}\right)+3\cdot\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)\geq 15.$$
But this is true because by AM-GM, the quantities in parentheses are at least 2, 2, 2, and 3, respectively; or alternatively, it is true by the AM-GM inequality on all 15 fractions of the form $\frac{x}{y}$ on the left-hand side (where $3\cdot\left(\frac{y}{x}+\frac{x}{z}+\frac{z}{y}\right)$ contributes nine such fractions).

Second Solution: By the Cauchy-Schwarz inequality
$$(u_1v_1+u_2v_2+u_3v_3)^2\leq(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2),$$
the quantity $(a+b+c)^2$ is at most
$$\left(\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}\right)[a(b+2c)+b(c+2a)+c(a+2b)].$$
On the other hand, from the inequality $(a-b)^2+(b-c)^2+(c-a)^2\geq 0$ we have
$$a(b+2c)+b(c+2a)+c(a+2b)\leq(a+b+c)^2.$$
Combining these gives
$$\begin{aligned}&a(b+2c)+b(c+2a)+c(a+2b)\\&\quad\leq\left(\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}\right)[a(b+2c)+b(c+2a)+c(a+2b)],\end{aligned}$$
which yields our desired inequality upon division by the (positive) expression $a(b+2c)+b(c+2a)+c(a+2b)$.

Problem 2 Let ABC be a nonisosceles acute triangle with altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Let ℓ be the line through D parallel to line EF. Let $P = \overleftrightarrow{BC} \cap \overleftrightarrow{EF}$, $Q = \ell \cap \overleftrightarrow{AC}$, and $R = \ell \cap \overleftrightarrow{AB}$. Prove that the circumcircle of triangle PQR passes through the midpoint of $\overline{BC}$.

Solution: Let M be the midpoint of $\overline{BC}$. Without loss of generality we may assume that $AB > AC$. Then the order of the points in question on line BC is B, M, D, C, P. Since $\angle BEC = \angle BFC = 90^\circ$, $BCEF$ is cyclic so that $\angle QCB = 180^\circ - \angle BCE = \angle EFB = \angle QRB$. Thus $RCQB$ is cyclic as well and

$$DB \cdot DC = DQ \cdot DR.$$

Quadrilateral $MRPQ$ is cyclic if and only if $DM \cdot DP = DQ \cdot DR$, so it remains to prove that $DB \cdot DC = DP \cdot DM$. Since the points B, C, E, F are concyclic, we have $PB \cdot PC = PE \cdot PF$. The circumcircle of triangle DEF (the so-called nine-point circle of triangle ABC) also passes through the midpoints of the sides of triangle ABC, which implies that $PE \cdot PF = PD \cdot PM$. Comparing both equalities shows that $PB \cdot PC = PD \cdot PM$. Denoting $MB = MC = u$, $MD = d$, $MP = p$, the last equality reads $(p+u)(p-u) = (p-d)p \iff u^2 = dp \iff (u+d)(u-d) = (p-d)d \iff DB \cdot DC = DP \cdot DM$. This completes the proof.

Problem 3 Find all positive integers k for which there exists a ten-element set M of positive numbers such that there are exactly k different triangles whose side lengths are three (not necessarily distinct) elements of M. (Triangles are considered different if they are not congruent.)

Solution: Given any 10-element set M of positive integers, there are exactly $\binom{12}{3}$ triples x, y, z $(x \le y \le z)$ chosen from the numbers in M; thus, we must have $k \le \binom{12}{3} = 220$. On the other hand, for each of the $\binom{11}{2}$ pairs x, y from M (with $x \le y$) we can form the triangle with side lengths x, y, y; hence $k \ge \binom{11}{2} = 55$. Then applying the following lemma, the possible values of k are then $55, 56, \ldots, 220$.

Lemma. *Suppose we have some positive integers n and k, with*

$$\binom{n+1}{2} \le k \le \binom{n+2}{3}.$$

Then there exists an n-element set M of positive numbers, such that there are exactly k triangles whose side lengths are three (not necessarily distinct) elements of M.

Furthermore, there exists an n-element set S_n with numbers $x_1 < x_2 < \cdots < x_n$ such that $x_n < 2x_1$ and such that all the $\binom{n+1}{2}$ pairwise sums

$$x_i + x_j \quad (1 \leq i \leq j \leq n)$$

are distinct. Exactly $\binom{n+2}{3}$ triangles can be formed from the elements of S_n.

Proof: We prove the claim by induction on n; it clearly holds for $n = 1$. Now suppose the claim is true for n and that $\binom{n+2}{2} \leq k \leq \binom{n+3}{3}$.

If $\binom{n+2}{2} \leq k \leq \binom{n+2}{3} + (n+1)$, then first find the n-element set $M' = \{x_1, x_2, \ldots, x_n\}$ from which exactly $k-(n+1)$ triangles can be formed. Choose $x_{n+1} > 2 \cdot \max\{M'\}$, and write $M = M' \cup \{x_{n+1}\}$. Then exactly k triangles can be formed from the elements of M: $k-(n+1)$ triangles from $\{x_1, x_2, \ldots, x_n\}$; and an additional $n+1$ triangles with side lengths x_i, x_{n+1}, x_{n+1} for $i = 1, 2, \ldots, n+1$.

Otherwise $k = \binom{n+2}{3} + n + 1 + q$, where $q \in \{1, 2, \ldots, \binom{n+1}{2}\}$. To the set S_n described in the lemma, add an element x_{n+1} which is greater than x_n but smaller than precisely q of the $\binom{n+1}{2}$ original pairwise sums from S_n. This gives a set M from which exactly k triangles can be formed: $\binom{n+2}{3}$ triangles from $\{x_1, x_2, \ldots, x_n\}$; $n+1$ additional triangles with side lengths x_i, x_{n+1}, x_{n+1}; and exactly q more triangles with side lengths x_i, x_j, x_{n+1} (where $i, j \neq n+1$).

None of the numbers $x_i + x_{n+1}$ equals any of the original pairwise sums. Thus we can construct S_{n+1}, completing the proof of the lemma. ■

Problem 4 Find all positive integers k for which the following assertion holds: if $F(x)$ is a polynomial with integer coefficients which satisfies $0 \leq F(c) \leq k$ for all $c \in \{0, 1, \ldots, k+1\}$, then

$$F(0) = F(1) = \cdots = F(k+1).$$

Solution: The claim is false for $k < 4$; we have the counterexamples $F(x) = x(2-x)$ for $k = 1$, $F(x) = x(3-x)$ for $k = 2$, and $F(x) = x(4-x)(x-2)^2$ for $k = 3$.

Now suppose $k \geq 4$ is fixed and that $F(x)$ has the described property. First of all $F(k+1) - F(0) = 0$, because it is a multiple of $k+1$ whose absolute value is at most k. Hence $F(x) - F(0) = x(x-k-1)G(x)$, where $G(x)$ is another polynomial with integer coefficients. Then we have

$$k \geq |F(c) - F(0)| = c(k+1-c)|G(c)|$$

for each $c = 1, 2, \ldots, k$. If $2 \leq c \leq k-1$ (such numbers c exist since $k \geq 4$) then

$$c(k+1-c) = 2(k-1) + (c-2)(k-1-c) \geq 2(k-1) > k,$$

which in view of our first inequality means that $|G(c)| < 1 \Longrightarrow G(c) = 0$. Thus $2, 3, \ldots, k-1$ are roots of the polynomial $G(x)$, so

$$F(x) - F(0) = x(x-2)(x-3)\cdots(x-k+1)(x-k-1)H(x),$$

where $H(x)$ is again a polynomial with integer coefficients. It remains to explain why $H(1) = H(k) = 0$. But this is easy: both values $c = 1$ and $c = k$ satisfy $k \geq |F(c) - F(0)| = (k-2)! \cdot k \cdot |H(c)|$; and since $(k-2)! > 1$, we must have $H(1) = H(k) = 0$.

Problem 5 Find all functions $f : (1, \infty) \to \mathbb{R}$ such that

$$f(x) - f(y) = (y - x)f(xy)$$

for all $x, y > 1$.

Solution: For every $t > 1$ we use the equation in turn for $(x, y) = (t, 2)$, $(t, 4)$ and $(2t, 2)$:

$$\begin{aligned} f(t) - f(2) &= (2-t)f(2t), \\ f(t) - f(4) &= (4-t)f(4t), \\ f(2t) - f(2) &= (2-2t)f(4t). \end{aligned}$$

We eliminate $f(t)$ by subtracting the second equation from the first, and then substitute for $f(2t)$ from the third. This yields the equality

$$f(4) + (t-3)f(2) = t(2t-5)f(4t)$$

for any $t > 1$. Taking $t = \frac{5}{2}$ we get $f(4) = \frac{1}{2}f(2)$, and feeding this back gives

$$\left(t - \frac{5}{2}\right)f(2) = t(2t-5)f(4t).$$

It follows that for any $t > 1$, $t \neq \frac{5}{2}$,

$$f(4t) = \frac{f(2)}{2t},$$

so by the middle of the three equations in the beginning of this solution we obtain

$$f(t) = f(4) + (4-t)f(4t) = \frac{1}{2}f(2) + \frac{(4-t)f(2)}{2t} = \frac{2f(2)}{t}.$$

This formula for $f(t)$ holds even for $t = \frac{5}{2}$, as can now be checked directly by applying the original equation to $x = \frac{5}{2}$ and $y = 2$ (and using $f(5) = \frac{2f(2)}{5}$). Setting $c = 2f(2)$, we must have

$$f(x) = \frac{c}{x}$$

for all x; and this function has the required property for any choice of the real constant c.

Problem 6 Show that for any positive integer $n \geq 3$, the least common multiple of the numbers 1, 2, $\ldots$, n is greater than 2^{n-1}.

Solution: Since for any $n \geq 3$ we have

$$2^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} < \sum_{k=0}^{n-1} \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} = n \cdot \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor},$$

it suffices to show that the number $n \cdot \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$ divides the least common multiple of $1, 2, \ldots, n$. Using a prime factorization argument, we will prove the more general assertion that for each $k < n$ the least common multiple of the numbers $n, n-1, \ldots, n-k$ is divisible by $n \cdot \binom{n-1}{k}$.

Let k and n be fixed natural numbers with $k < n$, and let $p \leq n$ be an arbitrary prime. Let p^α be the highest power of p which divides $\operatorname{lcm}(n, n-1, \ldots, n-k)$, where $p^\alpha \mid n - \ell$ for some ℓ. Then for each $i \leq \alpha$, we know that $p^i \mid n - \ell$. Thus exactly $\left\lfloor \frac{\ell}{p^i} \right\rfloor$ of $\{n-\ell+1, n-\ell+2, \ldots, n\}$ and exactly $\left\lfloor \frac{k-\ell}{p^i} \right\rfloor$ of $\{n-\ell-1, n-\ell-2, \ldots, n-k\}$ are multiples of p^i, so p^i divides $\left\lfloor \frac{\ell}{p^i} \right\rfloor + \left\lfloor \frac{k-\ell}{p^i} \right\rfloor \leq \left\lfloor \frac{k}{p^i} \right\rfloor$ of the remaining k numbers — that is, at most the number of multiples of p^i between 1 and k. It follows that p divides $n \cdot \binom{n-1}{k} = \frac{n(n-1)\cdots(n-\ell+1)(n-\ell-1)\cdots(n-k)}{k!} \cdot (n-\ell)$ at most α times, so that indeed $n \cdot \binom{n-1}{k} \mid \operatorname{lcm}(n, n-1, \ldots, n-k)$.

1.5 Hungary-Israel Binational Mathematical Competition

Individual Round

Problem 1 Let S be the set of all partitions $(a_1, a_2, \ldots, a_k)$ of the number 2000, where $1 \le a_1 \le a_2 \le \cdots \le a_k$ and $a_1 + a_2 + \cdots + a_k = 2000$. Compute the smallest value that $k + a_k$ attains over all such partitions.

Solution: AM-GM gives

$$k + a_k \ge 2\sqrt{ka_k} \ge 2(\sqrt{2000}) > 89.$$

Thus since $k + a_k$ is an integer, it must be at least 90. And 90 is attainable, since $k + a_k = 90$ for the partition $(\underbrace{40, 40, \cdots, 40}_{50})$.

Problem 2 Prove or disprove the following claim: For any positive integer k, there exists a positive integer $n > 1$ such that the binomial coefficient $\binom{n}{i}$ is divisible by k for any $1 \le i \le n - 1$.

First Solution: The statement is false. To prove this, take $k = 4$ and assume by contradiction that there exists a positive integer $n > 1$ for which $\binom{n}{i}$ is divisible by 4 for every $1 \le i \le n - 1$. Then

$$0 \equiv \sum_{i=1}^{n-1} \binom{n}{i} = 2^n - 2 \equiv -2 \pmod 4,$$

a contradiction.

Second Solution: The claim is obviously true for $k = 1$; we prove that the set of positive integers $k > 1$ for which the claim holds is exactly the set of primes. First suppose that k is prime; then express n in base k, writing $n = n_0 + n_1 k + \cdots + n_m k^m$ where $0 \le n_0, n_1, \ldots, n_m \le k - 1$ and $n_m \ne 0$. Also suppose we have $1 \le i \le n - 1$, and write $i = i_0 + i_1 k + \cdots + i_m k^m$ where $0 \le i_0, i_1, \ldots, i_m \le k-1$ (although perhaps $i_m = 0$). Lucas's Theorem states that

$$\binom{n}{i} \equiv \prod_{j=0}^{m} \binom{n_j}{i_j} \pmod k.$$

Now if $n = k^m$, then $n_0 = n_1 = \cdots = n_{m-1} = 0$; and $1 \le i \le n-1$ so that $i_m = 0$ but some other $i_{j'}$ is nonzero. Then $\binom{n_{j'}}{i_{j'}} = 0$, and indeed $\binom{n}{i} \equiv 0 \pmod{k}$ for all $1 \le i \le n-1$.

However, suppose that $n \ne k^m$. If $n_m > 1$ then letting $i = k^m < n$ we have $\binom{n}{i} \equiv (n_m)(1)(1)\cdots(1) \equiv n_m \not\equiv 0 \pmod{k}$. Otherwise, some other $n_{j'} \ne 0$; but then setting $i = n_{j'}k^{j'} < n$ we have $\binom{n}{i} \equiv (1)(1)\cdots(1) \equiv 1 \not\equiv 0 \pmod{k}$. Therefore the claim holds for prime k exactly when $n = k^m$.

Now suppose the claim holds for some $k > 1$ with the number n. If some prime p divides k, the claim must also hold for p with the number n. Thus n must equal a prime power p^m where $m \ge 1$. Then $k = p^r$ for some $r \ge 1$ as well, because if two primes p and q divided k then n would equal a perfect power of both p and q, which is impossible.

Choose $i = p^{m-1}$. Kummer's Theorem then states that $p^t \mid \binom{n}{i}$ if and only if t is less than or equal to the number of carries in the addition $(n-i)+i$ in base p. But there is only one such carry, between the p^{m-1} and p^m places:

$$\begin{array}{cccccccc} & 1 & & & & & & \\ & & & 1 & 0 & 0 & \ldots & 0 \\ + & & & p-1 & 0 & 0 & \ldots & 0 \\ \hline & 1 & & 0 & 0 & 0 & \ldots & 0 \end{array}$$

Thus, we must have $r \le 1$ and k must be prime, as claimed.

(Alternatively, for $n = p^m$ and $i = p^{m-1}$ we have

$$\binom{n}{i} = \prod_{j=0}^{p^{m-1}-1} \frac{p^m - j}{p^{m-1} - j}.$$

When $j = 0$ then $\frac{p^m-j}{p^{m-1}-j} = p$. Otherwise, $0 < j < p^{m-1}$ so that if $p^t < p^{m-1}$ is the highest power of p dividing j, then it is also the highest power of p dividing both $p^m - j$ and $p^{m-1} - j$. Therefore $\frac{p^m-j}{p^{m-1}-j}$ contributes one factor of p to $\binom{n}{i}$ when $j = 0$ and zero factors of p when $j > 0$. Thus $p^2 \nmid \binom{n}{i}$, and hence again $r \le 1$.)

Problem 3 Let ABC be a non-equilateral triangle with its incircle touching $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at A_1, B_1, and C_1, respectively, and let H_1 be the orthocenter of triangle $A_1B_1C_1$. Prove that H_1 is on the line passing through the incenter and circumcenter of triangle ABC.

First Solution: Let ω_1, I, ω_2, and O be the incircle, incenter, circumcircle, and circumcenter of triangle ABC, respectively. Since $I \neq O$, line IO is well defined.

Let T be the center of the homothety with positive ratio that sends ω_1 to ω_2 and hence I to O. Also let A_2, B_2, C_2 be the midpoints of the arcs BC, CA, AB of ω_2 not containing A, B, C, respectively. Since rays IA_1 and OA_2 point in the same direction, T must send A_1 to A_2 and similarly B_1 to B_2 and C_1 to C_2.

Also, because the measures of arcs AC_2 and A_2B_2 add up to $180°$, we know that $\overline{AA_2} \perp \overline{C_2B_2}$. Similarly, $\overline{BB_2} \perp \overline{C_2A_2}$ and $\overline{CC_2} \perp \overline{A_2B_2}$. Then since lines AA_2, BB_2, CC_2 intersect at I, I is the orthocenter of triangle $A_2B_2C_2$. Hence I is the image of H_1 under the defined homothety. Therefore T, H_1, I are collinear; and from before T, I, O are collinear. It follows that H_1, I, O are collinear, as desired.

Second Solution: Define ω_1, I, ω_2, and O as before. Let ω_3 be the nine-point circle of triangle $A_1B_1C_1$, and let S be its center. Since I is the circumcenter of triangle $A_1B_1C_1$, S is the midpoint of $\overline{IH_1}$ and I, H_1, S are collinear.

Now invert the figure with respect to ω_1. The midpoints of $\overline{A_1B_1}, \overline{B_1C_1}, \overline{C_1A_1}$ are mapped to C, A, B, and thus ω_3 is mapped to ω_2. Thus I, O, S are collinear; and so I, H_1, O, S are collinear, as desired.

Problem 4 Given a set X, define

$$X' = \{s - t \mid s, t \in X, s \neq t\}.$$

Let $S = \{1, 2, \ldots, 2000\}$. Consider two sets $A, B \subseteq S$, such that $|A||B| \geq 3999$. Prove that $A' \cap B' \neq \emptyset$.

Solution: Consider all $|A| \cdot |B| \geq 3999$ sums, not necessarily distinct, of the form $a + b$ where $a \in A$, $b \in B$. If both 2 and 4000 are of this form, then both A and B contain 1 and 2000 so that $2000 - 1 \in A' \cap B'$. Otherwise, each sum $a + b$ takes on one of at most 3998 values either between 2 and 3999, or between 3 and 4000. Thus by the pigeonhole principle, two of our $|A| \cdot |B|$ sums $a_1 + b_1$ and $a_2 + b_2$ are equal with $a_1, a_2 \in A$, $b_1, b_2 \in B$, and $(a_1, b_1) \neq (a_2, b_2)$. Then $a_1 \neq a_2$ (since otherwise we would have $b_1 = b_2$ and $(a_1, b_1) = (a_2, b_2)$), and therefore $A' \cap B'$ is nonempty because it contains $a_1 - a_2 = b_2 - b_1$.

Problem 5 Given an integer d, let

$$S = \{m^2 + dn^2 \mid m, n \in \mathbb{Z}\}.$$

Let $p, q \in S$ be such that p is a prime and $r = \frac{q}{p}$ is an integer. Prove that $r \in S$.

Solution: Note that

$$(x^2 + dy^2)(u^2 + dv^2) = (xu \pm dyv)^2 + d(xv \mp yu)^2.$$

Write $q = a^2+db^2$ and $p = x^2+dy^2$ for integers a, b, x, y. Reversing the above construction yields the desired result. Indeed, solving for u and v after setting $a = xu+dyv, b = xv-yu$ and $a = xu-dyv, b = xv+yu$ gives

$$u_1 = \frac{ax - dby}{p}, v_1 = \frac{ay + bx}{p}, u_2 = \frac{ax + dby}{p}, v_2 = \frac{ay - bx}{p}.$$

Note that

$$(ay + bx)(ay - bx) = (a^2 + db^2)y^2 - (x^2 + dy^2)b^2 \equiv 0 \pmod{p}.$$

Hence p divides one of $ay+bx, ay-bx$ so that one of v_1, v_2 is an integer. Without loss of generality, assume that v_1 is an integer. Then since $r = u_1^2 + dv_1^2$ is an integer and u_1 is rational, u_1 is an integer as well and $r \in S$, as desired.

Problem 6 Let k and ℓ be two given positive integers, and let a_{ij}, $1 \le i \le k$ and $1 \le j \le \ell$, be $k\ell$ given positive numbers. Prove that if $q \ge p > 0$ then

$$\left(\sum_{j=1}^{\ell} \left(\sum_{i=1}^{k} a_{ij}^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \le \left(\sum_{i=1}^{k} \left(\sum_{j=1}^{\ell} a_{ij}^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}.$$

First Solution: Define

$$b_j = \sum_{i=1}^{k} a_{ij}^p,$$

and denote the left hand side of the required inequality by L and its right hand side by R. Then

$$L^q = \sum_{j=1}^{\ell} b_j^{\frac{q}{p}} = \sum_{j=1}^{\ell} \left(b_j^{\frac{q-p}{p}} \left(\sum_{i=1}^{k} a_{ij}^p \right) \right) = \sum_{i=1}^{k} \left(\sum_{j=1}^{\ell} b_j^{\frac{q-p}{p}} a_{ij}^p \right).$$

Using Hölder's inequality it follows that

$$L^q \le \sum_{i=1}^{k} \left[\left(\sum_{j=1}^{\ell} \left(b_j^{\frac{q-p}{p}} \right)^{\frac{q}{q-p}} \right)^{\frac{q-p}{q}} \left(\sum_{j=1}^{\ell} \left(a_{ij}^p\right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \right]$$

$$= \sum_{i=1}^{k} \left[\left(\sum_{j=1}^{\ell} b_j^{\frac{q}{p}} \right)^{\frac{q-p}{q}} \left(\sum_{j=1}^{\ell} a_{ij}^q \right)^{\frac{p}{q}} \right]$$

$$= \left(\sum_{j=1}^{\ell} b_j^{\frac{q}{p}} \right)^{\frac{q-p}{q}} \cdot \left[\sum_{i=1}^{k} \left(\sum_{j=1}^{\ell} a_{ij}^q \right)^{\frac{p}{q}} \right] = L^{q-p} R^p.$$

The inequality $L \le R$ follows by dividing both sides of $L^q \le L^{q-p}R^p$ by L^{q-p} and taking the p-th root.

Second Solution: Let $r = \frac{q}{p}$, $c_{ij} = a_{ij}^p$. Then $r \ge 1$, and the given inequality is equivalent to the following inequality:

$$\left(\sum_{j=1}^{\ell} \left(\sum_{i=1}^{k} c_{ij} \right)^r \right)^{\frac{1}{r}} \le \sum_{i=1}^{k} \left(\sum_{j=1}^{\ell} c_{ij}^r \right)^{\frac{1}{r}}$$

We shall prove this inequality by induction on k. For $k = 1$, we have equality. For $k = 2$, the inequality becomes Minkowski's inequality.

Suppose that $k \ge 3$ and the inequality holds for $k - 1$. Then by the induction assumption for $k - 1$ we have

$$\sum_{i=1}^{k-1} \left(\sum_{j=1}^{\ell} c_{ij}^r \right)^{\frac{1}{r}} + \left(\sum_{j=1}^{\ell} c_{kj}^r \right)^{\frac{1}{r}} \ge \left(\sum_{j=1}^{\ell} \left(\sum_{i=1}^{k-1} c_{ij} \right)^r \right)^{\frac{1}{r}} + \left(\sum_{j=1}^{\ell} c_{kj}^r \right)^{\frac{1}{r}}$$

Using Minkowski's inequality with $\tilde{c}_{1j} = \sum_{i=1}^{k-1} c_{ij}$, $\tilde{c}_{2j} = c_{kj}$, we have

$$\left(\sum_{j=1}^{\ell} \left(\sum_{i=1}^{k-1} c_{ij} \right)^r \right)^{\frac{1}{r}} + \left(\sum_{j=1}^{\ell} c_{kj}^r \right)^{\frac{1}{r}} \ge \left(\sum_{j=1}^{\ell} \left(\sum_{i=1}^{k} c_{ij} \right)^r \right)^{\frac{1}{r}},$$

completing the inductive step.

Team Round

Problem 1 Let ABC be a triangle and let P_1 be a point inside triangle ABC.

(a) Prove that the lines obtained by reflecting lines P_1A, P_1B, P_1C through the angle bisectors of $\angle A$, $\angle B$, $\angle C$, respectively, meet at a common point P_2.

(b) Let A_1, B_1, C_1 be the feet of the perpendiculars from P_1 to sides BC, CA and AB, respectively. Let A_2, B_2, C_2 be the feet of the perpendiculars from P_2 to sides BC, CA and AB, respectively. Prove that these six points A_1, B_1, C_1, A_2, B_2, C_2 lie on a circle.

(c) Prove that the circle in part (b) touches the nine point circle (Feuerbach's circle) of triangle ABC if and only if P_1, P_2, and the circumcenter of triangle ABC are collinear.

First Solution:

(a) By the trigonometric form of Ceva's Theorem, we have

$$\frac{\sin\angle ABP_1 \sin\angle BCP_1 \sin\angle CAP_1}{\sin\angle P_1BC \sin\angle P_1CA \sin\angle P_1AB} = 1.$$

Now suppose that the given reflections of lines P_1A, P_1B, P_1C meet sides BC, CA, AB at points D, E, F, respectively. Then $\angle ABE = \angle P_1BC$, $\angle EBC = \angle ABP_1$, and so on; thus

$$\frac{\sin\angle EBC \sin\angle FCA \sin\angle DAB}{\sin\angle BCF \sin\angle CAD \sin\angle ABE} = 1$$

as well. Therefore, again by the trigonometric form of Ceva's Theorem, the three new lines also concur.

(b) Note that $P_1A_1CB_1$ and $P_2A_2CB_2$ are both cyclic quadrilaterals because they each have two right angles opposite each other. Since $\angle P_1CB_1 = \angle A_2CP_2$, we have $\angle B_1P_1C = \angle CP_2A_2$, and, by the previous statement, that implies $\angle B_1A_1C = \angle CB_2A_2$, whence A_1, A_2, B_1, B_2 are concyclic. For similar reasons, A_1, A_2, C_1, C_2 are concyclic. Then all six points $A_1, B_1, C_1, A_2, B_2, C_2$ must be concyclic, or else the radical axes of circles $A_1A_2B_1B_2$, $B_1B_2C_1C_2$, $C_1C_2A_1A_2$ would not concur, contradicting the radical axis theorem.

Problem 2 An ant is walking inside the region bounded by the curve whose equation is $x^2 + y^2 + xy = 6$. Its path is formed by straight segments parallel to the coordinate axes. It starts at an arbitrary point on the curve and takes off inside the region. When reaching the boundary, it turns by $90°$ and continues its walk inside the region. When arriving at a point on the boundary which it has already visited, or where it cannot continue its walk according to the given rule, the ant stops. Prove that, sooner or later, and regardless of the starting point, the ant will stop.

First Solution: If the ant moves from (a, b) to (c, b), then a and c are the roots to $f(t) = t^2 + bt + b^2 - 6$. Thus $c = -a - b$. Similarily, if the ant moves from (a, b) in a direction parallel to the y-axis, it meets the curve at $(a, -a - b)$.

Let (a, b) be the starting point of the ant, and assume that the ant starts walking in a direction parallel to the x-axis; the case when it starts walking parallel to the y-axis is analogous. If after five moves the ant is still walking, then it will return to its original position after six moves:

$$(a, b) \to (-a - b, b) \to (-a - b, a) \to (b, a)$$
$$\to (b, -a - b) \to (a, -a - b) \to (a, b).$$

Therefore, the ant stops moving after at most six steps.

Second Solution: Rotate the curve by $45°$, where (x, y) is on our new curve $\mathcal{C}_1$ when $\frac{1}{\sqrt{2}}(x - y, x + y)$ is on the original curve. The equation of the image of the curve under the rotation is

$$3x^2 + y^2 = 12.$$

Hence the curve is an ellipse, while the new directions of the ant's motion are inclined by $\pm 45°$ with respect to the x-axis. Next apply an affine transformation so that (x, y) is on our new curve $\mathcal{C}_2$ when $(x, \sqrt{3}y)$ is on $\mathcal{C}_1$. The ant's curve then becomes

$$x^2 + y^2 = 4,$$

a circle with radius 2, and the directions of the ant's paths are now inclined by $\pm 30°$ with respect to the x-axis.

Thus if the ant moves from P_1 to P_2 to P_3, then $\angle P_1P_2P_3$ will either equal 0°, 60°, or 120°. Thus as long as it continues moving, every two moves the ant travels to the other end of an arc measuring 0°, 120°, or 240° along the circle; thus after at most six moves, he must return to a position he visited earlier.

Problem 3

(a) In the plane, we are given a circle ω with unknown center, and an arbitrary point P. Is it possible to construct, using only a straightedge, the line through P and the center of the circle?

(b) In the plane, we are given a circle ω with unknown center, and a point Q on the circle. Construct the tangent to ω at the point Q, using only a straightedge.

(c) In the plane, we are given two circles ω_1 and ω_2 with unknown centers. Construct, with a straightedge only, the line through their centers when:

  (i) the two circles intersect;

  (ii) the two circles touch each other, and their point of contact T is marked;

  (iii) the two circles have no common point. This is an open question, and the construction might even be impossible; the problem does not belong to the official team contest, but any progress will be appreciated.

Solution:

(a) It is not possible. First we prove that given a circle with unknown center O, it is impossible to construct its center using only a straightedge. Assume by contradiction that this construction *is* possible; then perform a projective transformation on the figure, taking O to O' and ω to another circle ω'. The drawn lines remain lines, and thus they still yield the point O'. On the other hand, those lines compose a construction which gives the center of ω'. But O is not mapped to the center of ω', a contradiction.

Then the described construction must also be impossible. Otherwise, given a circle ω with unknown center O, we could mark a point P_1 and draw the line P_1O; and then mark another point P_2 not on ℓ_1, and draw the line P_2O. Then the intersection of these lines would yield O, which is impossible from above.

(b) Given a point A not on ω, suppose line ℓ_1 passes through A and hits ω at B_1 and C_1; and suppose another line ℓ_2 also passes through A and hits ω at B_2 and C_2. Then the line connecting $B_1B_2 \cap C_1C_2$ and $B_1C_2 \cap C_1B_2$ is the pole of A with respect to ω. Conversely, given any line ℓ we can pick two points on it that are not on ω, and construct the poles of these two points. If ℓ passes through the center of ω then these two poles are parallel; otherwise, they intersect at the polar of ℓ.

Now given ω and Q, mark another point R on the circle and then construct the polar T of line QR. (If the polar is the point at infinity, pick another point for R.) But then line TQ is the pole of Q—and this pole is tangent to ω at Q. Hence line QT is our desired line.

(c) (i) Draw the lines tangent to ω_1 at P and Q; they intersect at a point X_1, which by symmetry must lie on our desired line. Next draw the lines tangent to ω_2 at P and Q; they also intersect at a point X_2 that lies on our desired line. Then line X_1X_2 passes through both circles' centers, as desired.

(ii)

Lemma. *Suppose we have a trapezoid $ABCD$ with $AB \parallel CD$. Suppose that lines AD and BC intersect at M and that lines AC and BD intersect at N. Then line MN passes through the midpoints of $\overline{AB}$ and $\overline{CD}$.*

Proof: Let line MN hit lines AB and CD at Y_1 and Y_2, respectively. Perform an affine transformation that sends $ABCD$ into isosceles trapezoid $A'B'C'D'$ (with $A'D' = B'C'$) and sends points M, N, Y_1, Y_2 to M', N', Y_1', Y_2'. Then line $M'N'$ still passes through Y_1' and Y_2', and by symmetry these points are the midpoints of $\overline{A'B'}$ and $\overline{C'D'}$. But since (using directed lengths) $\frac{AY_1}{Y_1B} = \frac{A'Y_1'}{Y_1'B'}$, Y_1 must be the midpoint of $\overline{AB}$; and similarly, Y_2 is the midpoint of $\overline{CD}$. ■

Now construct a line through T intersecting ω_1 at A and ω_2 at C; construct a different line through T hitting ω_1 at B and ω_2 at D. Under the homothety about T that maps ω_1 to ω_2, segment AB gets mapped to segment CD; thus, $AB \parallel CD$. If lines AD and BC are parallel, pick different A and C; otherwise, using the construction in the lemma we

can find the midpoint F_1 of $\overline{CD}$. Next, from the construction described in part (b), we can find the polar F_2 of line CD; then line F_1F_2 passes through the center O_2 of ω_2.
Similarly, we can find a different line G_1G_2 passing through O_2; and hence O_2 is the intersection of lines F_1F_2 and G_1G_2.
We can likewise find the center O_1 of ω_1; then drawing the line O_1O_2, we are done.

1.6 Iberoamerican Math Olympiad

Problem 1 Find all positive integers n less than 1000 such that n^2 is equal to the cube of the sum of n's digits.

Solution: In order for n^2 to be a cube, n must be a cube itself; and since $n < 1000$ we must have $n = 1^3, 2^3, \ldots$, or 9^3. Quick checks show that $n = 1$ and $n = 27$ work while $n = 8, 64$, and 125 don't. And for $n \geq 6^3 = 216$, we have $n^2 \geq 6^6 > 27^3$; but since the sum of n's digits is at most $9 + 9 + 9 = 27$, this implies that no $n \geq 6^3$ works. Thus $n = 1, 27$ are the only answers.

Problem 2 Given two circles ω_1 and ω_2, we say that ω_1 bisects ω_2 if they intersect and their common chord is a diameter of ω_2. (If ω_1 and ω_2 are identical, we still say that they bisect each other.) Consider two non-concentric fixed circles ω_1 and ω_2.

(a) Show that there are infinitely many circles ω that bisect both ω_1 and ω_2.

(b) Find the locus of the center of ω.

Solution:

Suppose we have any circle ω with center O and radius r. Then we show that given any point P, there is a unique circle centered at P bisecting ω; and that the radius of this circle is $\sqrt{r^2 + OP^2}$. If $O = P$ the claim is obvious; otherwise, let $\overline{AB}$ be the diameter perpendicular to OP, so that $PA = PB = \sqrt{r^2 + OP^2}$.

Since $PA = PB$, there is a circle centered at P and passing through both A and B; this circle indeed bisects ω. Conversely, if circle Γ centered at P bisects ω along diameter $A'B'$, then both O and P lie on the perpendicular bisector of $\overline{A'B'}$. Thus $A'B' \perp OP$, and we must have $\overline{AB} = \overline{A'B'}$ and hence indeed $PA' = PB' = PA = PB = \sqrt{r^2 + OP^2}$.

Now back to the original problem. Set up a coordinate system where ω_1 is centered at the origin $O_1 = (0, 0)$ with radius r_1; and ω_2 is centered at $O_2 = (a, 0)$ with radius r_2. Given any point $P = (x, y)$, the circle centered at P bisecting ω_1 is the *same* as the circle centered at P bisecting ω_2 if and only if $\sqrt{r_1^2 + O_1P^2} = \sqrt{r_2^2 + O_2P^2}$; that is, if and only if

$$r_1^2 + x^2 + y^2 = r_2^2 + (x-a)^2 + y^2$$
$$2ax = r_2^2 - r_1^2 + a^2.$$

Therefore given any point P along the line $x = \frac{r_2^2 - r_1^2 + a^2}{2a}$, some circle ω centered at P bisects both ω_1 and ω_2. Since there are infinitely many such points, there are infinitely many such circles.

And conversely, from above if ω does bisect both circles then it must be centered at a point on the given line—which is a line perpendicular to the line passing through the centers of ω_1 and ω_2. In fact, recall that the radical axis of ω_1 and ω_2 is the line $2a(a-x) = r_2^2 - r_1^2 + a^2$; therefore the desired locus is the radical axis of ω_1 and ω_2, reflected across the perpendicular bisector of the segment joining the centers of the circles.

Problem 3 Let $P_1, P_2, \ldots, P_n$ $(n \geq 2)$ be n distinct collinear points. Circles with diameter P_iP_j $(1 \leq i < j \leq n)$ are drawn and each circle is colored in one of k given colors. All points that belong to more than one circle are not colored. Such a configuration is called a *(n,k)-cover.* For any given k, find all n such that for any (n,k)-cover there exist two lines externally tangent to two circles of the same color.

Solution: Without loss of generality label the points so that $P_1, P_2, \ldots, P_n$ lie on the line in that order from left to right. If $n \leq k+1$ points, then color any circle P_iP_j $(1 \leq i < j \leq n)$ with the i-th color. Then any two circles sharing the same i-th color are internally tangent at P_i, so there do not exist two lines externally tangent to them.

However, if $n \geq k+2$ then some two of the circles with diameters P_1P_2, P_2P_3, …, $P_{k+1}P_{k+2}$ must have the same color. Then there *do* exist two lines externally tangent to them. Therefore the solution is $n \geq k+2$.

Problem 4 Let n be an integer greater than 10 such that each of its digits belongs to the set $S = \{1,3,7,9\}$. Prove that n has some prime divisor greater than or equal to 11.

Solution: Note that any product of any two numbers from $\{1,3,7,9\}$ taken modulo 20 is still in $\{1,3,7,9\}$. Therefore *any* finite product of such numbers is still in this set; specifically, any number of the form 3^j7^k is congruent to $1, 3, 7$, or $9 \pmod{20}$.

Now if all the digits of $n \geq 10$ are in S, then its tens digit is odd and we cannot have $n \equiv 1, 3, 7$, or $9 \pmod{20}$. Thus, n cannot be of the form 3^j7^k. Nor can n be divisible by 2 or 5 (otherwise, its last digit would not be 1, 3, 7, or 9); so it must be divisible by some prime greater than or equal to 11, as desired.

Problem 5 Let ABC be an acute triangle with circumcircle ω centered at O. Let $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ be the altitudes of ABC. Let line EF meet ω at P and Q.

(a) Prove that $AO \perp PQ$.

(b) If M is the midpoint of $\overline{BC}$, prove that

$$AP^2 = 2AD \cdot OM.$$

Solution: Let H be the orthocenter of triangle ABC, so that $AEHF$, $BFHD$, $CDHE$ are cyclic. Then $\angle AFE = \angle AHE = \angle DHB = 90^\circ - \angle HBD = 90^\circ - \angle EBC = \angle BCE = C$, while $\angle OAF = \angle OAB = 90^\circ - C$. Therefore $AO \perp EF$ and thus $AO \perp PQ$, as desired.

Say the circumradius of triangle ABC is R. Draw diameter AA' perpendicular to $\overline{PQ}$, intersecting $\overline{PQ}$ at T. Then $AT = AF\cos(90^\circ - C) = AF\sin C = AC\cos A\sin C = 2R\cos A\sin B\sin C$. By symmetry, $PT = TQ$; then by Power of a Point, $PT^2 = PT \cdot TQ = AT \cdot TA' = AT(2R - AT)$. Thus $AP^2 = AT^2 + PT^2 = AT^2 + AT(2R - AT) = 2R \cdot AT = 4R^2\cos A\sin B\sin C$.

On the other hand, $AD = AC\sin C = 2R\sin B\sin C$, while $OM = OC\sin\angle OCM = R\sin(90^\circ - A) = R\cos A$. Thus

$$AP^2 = 4R^2\cos A\sin B\sin C = 2AD \cdot OM,$$

as desired.

Problem 6 Let AB be a segment and C a point on its perpendicular bisector. Construct $C_1, C_2, \ldots, C_n, \ldots$ as follows: $C_1 = C$, and for $n \geq 1$, if C_n is not on $\overline{AB}$, then C_{n+1} is the circumcenter of triangle ABC_n. Find all points C such that the sequence $\{C_n\}_{n\geq 1}$ is well defined for all n and such that the sequence eventually becomes periodic.

Solution: All angles are directed modulo 180° unless otherwise indicated. Then C_n is uniquely determined by $\theta_n = \angle AC_nB$; and furthermore, we have $\theta_{n+1} = 2\theta_n$ for all n. For this to be eventually periodic we must have $\theta_{j+1} = \theta_{k+1}$ or $2^j\theta_1 = 2^k\theta_1$ for some j, k; that is, 180° must divide $(2^k - 2^j)\theta$ for some k and (not using directed angles) $180^\circ \cdot r = (2^k - 2^j)\theta_1$ for some integers p, k.

Therefore $\theta_1 = 180^\circ \cdot \frac{r}{2^k - 2^j}$ must be a rational multiple $\frac{p}{q} \cdot 180^\circ$ with p, q relatively prime. And for the sequence to be well-defined, q must not be a power of two; because if $q = 2^n$ then $\theta_{n+1} = 180^\circ \cdot p = 180^\circ$, which we cannot have.

Conversely, suppose we have such an angle $\frac{p}{q} \cdot 180^\circ$ where p, q are relatively prime and q is not a power of 2. First, the sequence of points is well-defined because $\frac{2^n p}{q}$ will always have a nontrivial odd divisor in its denominator; so it will never be an integer and $\theta_{n+1} = \frac{2^n p}{q} \cdot 180^\circ$ will never equal 180°.

Next write $q = 2^j t$ for odd t, and let $k = \phi(t) + j$. Then since $t \mid 2^{\phi(t)} - 1$ we have

$$\theta_{k+1} = 2^k \cdot \theta_1 = 2^{\phi(t)+j} \cdot \frac{p}{q} \cdot 180^\circ = 2^{\phi(t)} \cdot \frac{p}{t} \cdot 180^\circ \equiv \frac{p}{t} \cdot 180^\circ,$$

while

$$\theta_{j+1} \equiv 2^j \frac{p}{q} \cdot 180^\circ = \frac{p}{t} \cdot 180^\circ.$$

Thus $\theta_{j+1} \equiv \theta_{k+1}$, so the sequence is indeed periodic.

Therefore the set of valid points C is all points C such that $\angle ACB$ (no longer directed) equals $\frac{p}{q} \cdot 180^\circ$ for relatively prime positive integers p, q, where q is not a power of 2.

1.7 Olimpiada de mayo

Problem 1 Find the smallest positive integer n such that the 73 fractions

$$\frac{19}{n+21}, \frac{20}{n+22}, \frac{21}{n+23}, \ldots, \frac{91}{n+93}$$

are all irreducible.

Solution: Note that the difference between the numerator and denominator of each fraction is $n+2$. Then $n+2$ must be relatively prime to each of the integers from 19 to 91. Since this list contains a multiple of each prime p less than or equal to 91, $n+2$ must only have prime factors greater than 91. The smallest such number is 97, so $n = 95$.

Problem 2 Let ABC be a triangle with $\angle A = 90°$. Construct point P on $\overline{BC}$ such that if Q is the foot of the perpendicular from P to $\overline{AC}$ then $PQ^2 = PB \cdot PC$.

Solution: Draw D on ray AB such that $AB = BD$, and draw E on ray AC such that $AC = CE$. Next draw F on $\overline{DE}$ such that $\angle FBD = \angle BCA$. Finally, draw the line through F perpendicular to $\overline{AC}$; we claim that it intersects $\overline{BC}$ and $\overline{AC}$ at our desired points P and Q.

Since $BD \parallel FQ$ we have $\angle BFQ = \angle FBD = \angle BCA = \angle BCQ$, so $BFCQ$ is cyclic. And since $BD = BA$ and $DA \parallel FQ$, we have $PF = PQ$. Thus by Power of a Point, we have $PB \cdot PC = PF \cdot PQ = PQ^2$, as desired.

Problem 3 There are 1999 balls in a row. Each ball is colored either red or blue. Underneath each ball we write a number equal to the sum of the number of red balls to its right and blue balls to its left. Exactly three numbers each appear on an odd number of balls; determine these three numbers.

First Solution: Call of a coloring of $4n-1$ balls "good" if exactly 3 numbers each appear on an odd number of balls; we claim that these three numbers will then be $2n-2$, $2n-1$, and $2n$. Let $b_1, b_2, \ldots$ represent the ball colors (either B or R), and let $x_1, x_2, \ldots$ represent the numbers under the respective balls. Then $x_{k+1} - x_k = 1$ if

$b_k = b_{k+1} = B$; $x_{k+1} - x_k = -1$ if $b_k = b_{k+1} = R$; and $x_k = x_{k+1}$ if $b_k \neq b_{k+1}$. Now, when $n = 1$, the only good colorings are RRR and BBB, which both satisfy the claim. For the sake of contradiction, let $n_0 > 1$ be the least n for which the claim no longer holds. Now, let a "couple" be a pair of adjacent, different-colored balls; then for our coloring of n_0 balls, one of the following cases is true:

(i) *There exist two or more disjoint couples in the coloring.* Removing the two couples decreases all the other x_i by exactly 2, while the numbers originally on the four removed balls are removed in pairs. Thus we have constructed a good coloring of $4(n_0 - 1) - 1$ balls for which the claim does not hold, a contradiction.

(ii) *The balls are colored $RR \cdots RBRR \cdots R$ or $BB \cdots BRBB \cdots B$.* Then $\{x_i\}$ is a nondecreasing series, with equality only under the balls BRB or RBR. Then $(4n_0 - 1) - 2$ numbers appear an odd number of times, but this cannot equal 3.

(iii) *There exists exactly one couple in the coloring.* Suppose, without loss of generality, that we have a string of m blue balls followed by a string of n red balls. It is trivial to check that m and n cannot equal 1. Then by removing the final two blue balls and the first two red balls, we construct an impossible coloring of $4(n_0 - 1) - 1$ balls as in case (i).

(iv) *All the balls are of the same color.* We then have $4n_0 - 1 > 3$ distinct numbers, a contradiction.

Thus, all good colorings satisfy our claim; so for $n_0 = 500$, we find that the three numbers must be 998, 999, and 1000.

Second Solution: If a ball has the number m on it, call it an "m-ball." Also let $x_1, x_2, \ldots, x_{1999}$ represent both the balls and the numbers written on them; and let $\langle i, j \rangle$ denote the i-th through j-th balls, inclusive.

Read from left to right, the numbers on the balls increase by 1 when two blue balls are adjacent, decrease by 1 when two red balls are adjacent, and otherwise remain constant. This implies that when viewed from left to right, the m-balls in our line alternate in color. Also, if $m < x_i$ (or $m > x_i$) then the first m-ball after x_i must be red (or blue) while the last m-ball before it must be blue (or red). It follows that in $\langle i, j \rangle$, there are an odd number of m-balls if $\min(x_i, x_j) < m < \max(x_i, x_j)$, and an even number of m-balls if

$m < \min(x_i, x_j)$ or $m > \max(x_i, x_j)$.

This result implies that the three given numbers are consecutive: say they are $k-1, k, k+1$. Without loss of generality say that $x_1 \le k$, and suppose that x_r is the rightmost $(k+1)$-ball. If any $x_j \le k+1$ for $j > r$ then in $\langle 1, j\rangle$ there are an even number of $(k+1)$-balls, a contradiction.

Then in $\langle 1, r-1\rangle$, for $m \neq k-1, k$ there are an even number of m-balls so that the number of red m-balls equals the number of blue m-balls. As for $m = k-1, k$, there are an odd number of m-balls in $\langle 1, r-1\rangle$ and the last such m-ball is blue. It follows that there must be $\frac{r+2}{2}$ blue balls and $\frac{r-2}{2}$ red balls in $\langle 1, r-1\rangle$.

And in $\langle r+1, 1999\rangle$, the number of red m-balls equals the number of blue m-balls for *all* m; this is because there are no such m-balls for $m \le k+1$, and for $m \ge k+2$ an even number of m-balls are in $\langle 1, r-1\rangle$ so in $\langle r+1, 1999\rangle$ an even number of m-balls must remain. Thus there are $\frac{1998-r}{2}$ blue balls and $\frac{1998-r}{2}$ red balls in $\langle r, 1999\rangle$. Hence $k+1 = x_r = \frac{r+2}{2} + \frac{1998-r}{2} = 1000$, and the three numbers are 998, 999, and 1000.

Problem 4 Let A be a number with six digits, three of which are colored and are equal to $1, 2, 4$. Prove that it is always possible to obtain a multiple of 7 by doing one of the following:

(1) eliminate the three colored numbers;

(2) write the digits of A in a different order.

Solution: Note that modulo 7, the six numbers 421, 142, 241, 214, 124, 412 are congruent to 1, 2, 3, 4, 5, 6, respectively. Let the other digits besides 1, 2, and 4 be x, y, and z, appearing in that order from left to right. If the 3-digit number xyz is divisible by 7, we may eliminate the three colored numbers. If not, the 6-digit number $xyz000$ is also not divisible by 7, and we may add the appropriate permutation abc of 124 to $xyz000$ to make $xyzabc$ divisible by 7.

Problem 5 Consider a square of side length 1. Let S be a set of finitely many points on the sides of the square. Prove that there is a vertex of the square such that the arithmetic mean of the squares of the distances from the vertex to all the points in S is no less than $\frac{3}{4}$.

Solution: Let the four vertices of the square be V_1, V_2, V_3, and V_4, and let the set of points be $\{P_1, P_2, \ldots, P_n\}$. For a given P_k, we may assume without loss of generality that P_k lies on side V_1V_2. Writing $x = P_kV_1$, we have

$$\sum_{i=1}^{4} P_kV_i^2 = x^2+(1-x)^2+(1+x^2)+(1+(1-x)^2) = 4\left(x-\frac{1}{2}\right)^2+3 \geq 3.$$

Hence $\sum_{i=1}^4 \sum_{j=1}^n P_jV_i^2 \geq 3n$, or $\sum_{i=1}^4 \left(\frac{1}{n}\sum_{j=1}^n P_jV_i^2\right) \geq 3$. Thus the average of $\frac{1}{n}\sum_{j=1}^n P_jV_i^2$ (for $i = 1, 2, 3, 4$) is at least $\frac{3}{4}$; so if we select the V_i for which $\frac{1}{n}\sum_{j=1}^n P_jV_i^2$ is maximized, we are guaranteed it will be at least $\frac{3}{4}$.

Problem 6 An ant crosses a circular disc of radius r and it advances in a straight line, but sometimes it stops. Whenever it stops, it turns $60°$, each time in the opposite direction. (If the last time it turned $60°$ clockwise, this time it will turn $60°$ counterclockwise, and vice versa.) Find the maximum length of the ant's path.

Solution: Suppose the ant begins its path at P_0, stops at P_1, P_2, ..., P_{n-1} and ends at P_n. Note that all the segments $P_{2i}P_{2i+1}$ are parallel to each other and that all the segments $P_{2i+1}P_{2i+2}$ are parallel to each other. We may then translate all the segments so as to form two segments P_0Q and QP_n where $\angle P_0QP_n = 120°$. Then $P_0P_n \leq 2r$, and the length of the initial path is equal to $P_0Q + QP_n$. Let $P_0P_n = c$, $P_0Q = a$, and $QP_n = b$. Then

$$(2r)^2 \geq c^2 = a^2 + b^2 + ab = (a+b)^2 - ab \geq (a+b)^2 - \frac{1}{4}(a+b)^2,$$

so $\frac{4}{\sqrt{3}}r \geq a + b$ with equality iff $a = b$. The maximum is therefore $\frac{4}{\sqrt{3}}r$; this can be attained with the path where $\overline{P_0P_2}$ is a diameter of the circle, and $P_0P_1 = P_1P_2 = \frac{2}{\sqrt{3}}r$.

1.8 St. Petersburg City Mathematical Olympiad (Russia)

Problem 9.1 Let $x_0 > x_1 > \cdots > x_n$ be real numbers. Prove that

$$x_0 + \frac{1}{x_0 - x_1} + \frac{1}{x_1 - x_2} + \cdots + \frac{1}{x_{n-1} - x_n} \geq x_n + 2n.$$

Solution: For $i = 0, 1, \ldots, n-1$, we have $x_i - x_{i+1} > 0$ so that by AM-GM,

$$(x_i - x_{i+1}) + \frac{1}{(x_i - x_{i+1})} \geq 2.$$

Adding up these inequalities for $i = 0, 1, \ldots, n-1$ gives the desired result.

Problem 9.2 Let $f(x) = x^2 + ax + b$ be a quadratic trinomial with integral coefficients and $|b| \leq 800$. It is known also that $f(120)$ is prime. Prove that $f(x) = 0$ has no integer roots.

Solution: Suppose by way of contradiction $f(x)$ had an integer root r_1; then writing $f(x) = (x - r_1)(x - r_2)$, we see that its other root must be $r_2 = -a - r_1$, also an integer.

Since $f(120) = (120 - r_1)(120 - r_2)$ is prime, one of $|120 - r_1|$ and $|120 - r_2|$ equals 1, and the other equals some prime p.

Without loss of generality say $|120 - r_1| = 1$, so that $r_1 = 119$ or 121, and $|r_1| \geq 119$. Then $|120 - r_2|$ is a prime, but the numbers $114, 115, \ldots, 126$ are all composite: $119 = 7 \cdot 17$, and all the other numbers are clearly divisible by 2, 3, 5, or 11. Therefore $|r_2| \geq 7$, and $|b| = |r_1 r_2| \geq |119 \cdot 7| > 800$, a contradiction.

Problem 9.3 The vertices of a regular n-gon ($n \geq 3$) are labeled with distinct integers from $\{1, 2, \ldots, n\}$. For any three vertices A, B, C with $AB = AC$, the number at A is either larger than the numbers at B and C, or less than both of them. Find all possible values of n.

Solution: Suppose that $n = 2^s t$, where $t \geq 3$ is odd. Look at the regular t-gon $P_1 \cdots P_t$ formed by every 2^s-th point. In this t-gon, some

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[1] Problems are numbered as they appeared in the contests. Problems that appeared more than once in the contests are only printed once in this book.

vertex A has the second-smallest number, and some vertex B has the smallest number. But then at the third vertex C with $AB = AC$ (which exists since t is odd), the number must also be smaller than A's number — a contradiction.

Alternatively, if P_1 has a bigger number than P_t and P_2, then P_2 has a smaller number than P_3, P_3 has a bigger number than P_4, and so on around the circle — so that P_t has a bigger number than P_1, a contradiction.

Now we prove by induction on $s \geq 0$ that we can satisfy the conditions for $n = 2^s$. For $s = 1$, label the single point 1. And if we can label a regular 2^s-gon with the numbers $a_1, \ldots, a_{2^s}$ in that order, then we can label a regular 2^{s+1}-gon with the numbers $a_1, a_1 + 2^s, a_2, a_2 + 2^s, \ldots, a_{2^s}, a_{2^s} + 2^s$, as desired.

(Alternatively, when $n = 2^s$ one could label the vertices as follows. For $i = 1, 2, \ldots, 2^s$, reverse each digit of the s-bit binary expansion of $i - 1$ and then add 1 to the result. Label the i-th vertex with this number.)

Problem 9.4 Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of an isosceles triangle ABC ($AB = BC$). It is known that $\angle BC_1A_1 = \angle CA_1B_1 = \angle A$. Let BB_1 and CC_1 meet at P. Prove that AB_1PC_1 is cyclic.

Solution: All angles are directed modulo $180°$.

Let the circumcircles of triangles AB_1C_1 and A_1B_1C intersect at P', so that $AB_1P'C_1$ and $CB_1P'A_1$ are cyclic. Then

$$\begin{aligned}\angle A_1P'C_1 &= (180° - \angle C_1P'B_1) + (180° - \angle B_1P'A_1)\\ &= \angle B_1AC_1 + \angle A_1CB_1 = \angle CAB + \angle BCA\\ &= 180° - \angle ABC = 180° - \angle C_1BA_1,\end{aligned}$$

so $BA_1P'C_1$ is cyclic as well.

Now, $\angle BC_1A_1 = \angle A = \angle C$ implies that $\triangle BC_1A_1 \sim \triangle BCA$, so $BC_1 \cdot BA = BA_1 \cdot BC$. Thus B has equal power with respect to the circumcircles of triangles AB_1C_1 and A_1B_1C, so it lies on their radical axis B_1P'.

Similarly, $\angle CA_1B_1 = \angle A$ implies that $\triangle ABC \sim \triangle A_1B_1C$, so $CA \cdot CB_1 = CB \cdot CA_1$. Thus C has equal power with respect to the circumcircles of triangles B_1AC_1 and A_1BC_1, so it lies on their radical axis C_1P'.

Then P' lies on both CC_1 and BB_1, so it must *equal* P. Therefore AB_1PC_1 is indeed cyclic, as desired.

Problem 9.5 Find the set of possible values of the expression

$$f(x,y,z) = \left\{ \frac{xyz}{xy+yz+zx} \right\},$$

for positive integers x, y, z. Here $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.

Solution: Clearly $f(x,y,z)$ must be a nonnegative rational number below 1; we claim all such numbers are in the range of f. Suppose we have nonnegative integers p and q with $p < q$; let $x_1 = y_1 = 2(q-1)$ and let $z_1 = 1$. Then

$$f(x_1, y_1, z_1) = \left\{ \frac{4(q-1)^2}{4(q-1)^2 + 4(q-1)} \right\} = \frac{q-1}{q}.$$

Writing $X = \frac{xyz}{xy+yz+zx}$, notice that for any nonzero integer k we have

$$f(kx, ky, kz) = \{ kX \} = \{ k\lfloor X \rfloor + k\{X\} \} = \{ k \cdot f(x,y,z) \}.$$

Then $f(p(q-1) \cdot x_1, p(q-1) \cdot y_1, p(q-1) \cdot z_1) = \frac{p}{q}$, so every nonnegative rational $\frac{p}{q} < 1$ is indeed in the range of f.

Problem 9.6 Let $\overline{AL}$ be the angle bisector of triangle ABC. Parallel lines ℓ_1 and ℓ_2 equidistant from A are drawn through B and C respectively. Points M and N are chosen on ℓ_1 and ℓ_2 respectively such that lines AB and AC meet lines LM and LN at the midpoints of $\overline{LM}$ and $\overline{LN}$ respectively. Prove that $LM = LN$.

Solution: Let A, B, C also represent the angles at those points in triangle ABC.

Let line ℓ pass through A perpendicular to AL. Next, draw M' and N' on ℓ so that $\angle ALM' = \angle ALN' = \frac{A}{2}$ (with M' and N' on the same sides of line AL as B and C, respectively). Finally, let ℓ hit ℓ_1 at Q.

We claim that M' lies on ℓ_1. Orient the figure so that ℓ_1 and ℓ_2 are vertical, and let $x = \angle QBA$. Note that $AM' = AL \tan \angle ALM' = AL \tan \frac{A}{2}$, so the horizontal distance between A and M' is

$$AM' \sin(180^\circ - \angle AQB)$$

$$= AL \tan\frac{A}{2} \cdot \sin(\angle QBA + \angle BAQ)$$
$$= AL \tan\frac{A}{2} \cdot \sin\left(x + 90^\circ - \frac{A}{2}\right)$$
$$= AL \tan\frac{A}{2} \cdot \cos\left(\frac{A}{2} - x\right).$$

On the other hand, the horizontal distance between A and ℓ_1 is $AB \sin x$. Thus we need only prove that

$$AB \sin x = AL \tan\frac{A}{2} \cdot \cos\left(\frac{A}{2} - x\right).$$

By the law of sines on triangle ABL, we know that $AL \sin\left(\frac{A}{2} + B\right) = AB \sin B$. Hence we must prove

$$\sin\left(\frac{A}{2} + B\right) \cdot \sin x = \sin B \cdot \tan\frac{A}{2} \cdot \cos\left(\frac{A}{2} - x\right)$$
$$\iff \sin\left(\frac{A}{2} + B\right) \cdot \cos\frac{A}{2} \cdot \sin x = \sin B \cdot \sin\frac{A}{2} \cdot \cos\left(\frac{A}{2} - x\right)$$
$$\iff (\sin(A + B) + \sin B) \cdot \sin x = \sin B \cdot (\sin(A - x) + \sin x)$$
$$\iff \sin C \cdot \sin x = \sin B \cdot \sin(A - x)$$
$$\iff AB \sin x = AC \sin(A - x).$$

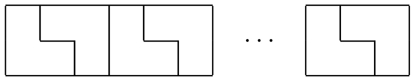
But $AB \sin x$ is the distance between A and ℓ_1, and $AC \sin(A - x)$ is the distance between A and ℓ_2 — and these distances are equal. Therefore, M' indeed lies on ℓ_1.

Now say that lines AB and LM' intersect at P. Then since $\angle LAP = \angle PLA = \frac{A}{2}$, $\overline{AP}$ is the median to the hypotenuse of right triangle $M'AL$. Thus line AB hits the midpoint of $\overline{LM'}$, so $M = M'$.

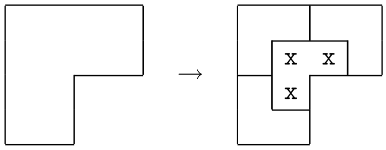
Similarly, $N = N'$. But then $\angle LMN = 90^\circ - \frac{A}{2} = \angle LNM$, so $LM = LN$, as desired.

Problem 9.7 A *corner* is the figure resulted by removing 1 unit square from a 2×2 square. Prove that the number of ways to cut a 998×999 rectangle into corners, where two corners *can* form a 2×3 rectangle, does not exceed the number of ways to cut a 1998×2000 rectangle into corners, so that *no* two form a 2×3 rectangle.

Solution: Take any tiling of a 998×999 rectangle with corners, and add a 2×999 block underneath also tiled with corners:



Next, enlarge this 1000×999 board to twice its size, and replace each large corner by four normal-sized corners as follows:



For each corner in the tiling of the 1000×999 board, none of the four new corners can be half of a 2×3 rectangle. Also, the "central" corners (like the one marked with x's above) have the same orientations as the original corners tiling the 1000×999 rectangle.

Thus, different tilings of a 998×999 rectangle turn into different tilings of a 2000×1998 board where no two corners form a 2×3 rectangle — which implies the desired result.

Problem 9.8 A convex $n-$gon ($n > 3$) is divided into triangles by non-intersecting diagonals. Prove that one can mark $n-1$ segments among these diagonals and sides of the polygon so that no set of marked segments forms a closed polygon and no vertex belongs to exactly two segments.

Solution: After we mark some segments, let the *degree* $d(V)$ of a vertex V be the number of marked segments it is on. Also, say we *mark up* a triangulated n-gon with respect to side AB if we mark $n-1$ segments, no marked segments form a closed polygon, and none of the $n-2$ vertices besides A and B have degree 2.

Lemma. *Given any triangulated convex n-gon ($n \geq 3$) and any two adjacent vertices A and B, we can mark up the n-gon with respect to side AB in three ways (i), (ii), (iii), each satisfying the corresponding condition from the following list:*

(i) $\overline{AB}$ is marked, and $d(A) \geq 2$.

(ii) $\overline{AB}$ is marked, and $d(A) \neq 2$.

(iii) $(d(A), d(B)) \neq (1,1)$ or $(2,2)$.

Proof: For $n = 3$, given a triangle ABC we can mark sides AB and (i) AC, (ii) BC, and (iii) AC. Now suppose that $n \geq 4$ and that the claims are true for all smaller n; we prove that they are true for n as well.

$\overline{AB}$ must be part of some triangle ABC of drawn segments. Then either (a) $\overline{AC}$ is a side of the polygon but $\overline{BC}$ is not; (b) $\overline{BC}$ is a side but $\overline{AC}$ is not; or (c) neither $\overline{AC}$ nor $\overline{BC}$ is a side.

(a) Let P be the $(n-1)$-gon formed by the vertices not including A.

  (i) Apply (ii) to P so that $d(C) \neq 2$ and $\overline{BC}$ is marked; then unmark $\overline{BC}$, and mark $\overline{AC}$ and $\overline{AB}$.

  (ii) Apply (ii) to P so that $d(C) \neq 2$ and $\overline{BC}$ is marked; then mark $\overline{AB}$.

  (iii) Apply (i) to P so that $d(B) \geq 2$ and $\overline{BC}$ is marked. If $d(C) \neq 2$, then mark $\overline{AB}$; otherwise mark $\overline{AC}$.

(b) Let P be the $(n-1)$-gon formed by the vertices besides B.

  (i) Apply (ii) to P so that $d(C) \neq 2$ and $\overline{AC}$ is marked; then mark $\overline{AB}$.

  (ii) Apply (ii) to P so that $d(C) \neq 2$ and $\overline{AC}$ is marked. If $d(A) = 2$, then mark $\overline{AB}$; otherwise unmark $\overline{AC}$ and mark $\overline{AB}$ and $\overline{BC}$.

  (iii) Repeat the construction in (i).

(c) Let P be the polygon formed by A, C, and the vertices in between (not on the same side of line AC as B); and let Q be the polygon formed by B, C, and the vertices in between (not on the same side of line BC as A).

  (i) Apply (i) to P and Q so that $d(C) \geq 2 + 2 = 4$ and $\overline{AC}$, $\overline{BC}$ are marked; then unmark $\overline{BC}$ and mark $\overline{AB}$.

  (ii) Apply (i) to P and Q so that $d(C) \geq 2 + 2 = 4$ and $\overline{AC}$, $\overline{BC}$ are marked. If $d(A) = 2$, unmark $\overline{BC}$ and mark $\overline{AB}$; otherwise, unmark $\overline{AC}$ and mark $\overline{AB}$.

  (iii) Repeat the construction in (ii). ■

Now to the main result. Since there are $n \geq 4$ sides but only $n - 2$ triangles, some triangle contains two adjacent sides XY and YZ. Let P be the $(n-1)$-gon formed by the vertices not including Y, and apply (iii) to P and vertices X, Z.

Then if $d(X)$ or $d(Z)$ equals 2, mark $\overline{XY}$ or $\overline{YZ}$, respectively. Otherwise, if $d(X)$ or $d(Z)$ equals 1, mark $\overline{YZ}$ or $\overline{XY}$, respectively. Otherwise, both $d(X)$ and $d(Z)$ are at least 3, and we can mark either $\overline{XY}$ or $\overline{YZ}$ to finish off.

Problem 10.1 The sequence $\{x_n\}$ of positive integers is formed by the following rule: $x_1 = 10^{999} + 1$, and for every $n \geq 2$, the number x_n is obtained from the number $11x_{n-1}$ by rubbing out its first digit. Is the sequence bounded?

Solution: If x_{n-1} has k digits, then $x_{n-1} < 10^k$ so that $11x_{n-1} < 11 \cdot 10^k = 1100\ldots0$. Thus if $11x_{n-1}$ has $k+2$ digits, its first two digits are 1 and 0; and rubbing out its first digit leaves x_n with at most k digits. Otherwise, $11x_{n-1}$ has at most $k+1$ digits, so rubbing out its first digit still leaves x_n with at most k digits. Therefore the number of digits in each x_n is bounded, so the x_n are bounded as well.

Problem 10.2 Prove that any positive integer less than $n!$ can be represented as a sum of no more than n positive integer divisors of $n!$.

Solution: Fix n, and write $a_k = \frac{n!}{k!}$ for each $k = 1, 2, \ldots, n$. Suppose we have some number m with $a_k \leq m < a_{k-1}$ where $2 \leq k \leq n$. Then consider the number $d = a_k \left\lfloor \frac{m}{a_k} \right\rfloor$. We have $0 \leq m - d < a_k$; and furthermore, since $s = \left\lfloor \frac{m}{a_k} \right\rfloor < \frac{a_{k-1}}{a_k} = k$, we know that $\frac{n!}{d} = \frac{k!}{s}$ is an integer. Thus from m we can subtract d, a factor of $n!$, to obtain a number less than a_k.

Then if we start with any positive integer $m < n! = a_1$, then by subtracting at most one factor of $n!$ from m we can obtain an integer less than a_2; by subtracting at most one more factor of $n!$ we can obtain an integer less than a_3; and so on, so that we can represent m as the sum of at most $n - 1$ positive integer divisors of $n!$.

Problem 10.5 How many 10-digit numbers divisible by 66667 are there whose decimal representation contains only the digits $3, 4, 5,$ and 6?

Solution: Suppose that $66667n$ had 10 digits, all of which were 3, 4, 5, and 6. Then

$$3333333333 \leq 66667n \leq 6666666666 \quad \Rightarrow \quad 50000 \leq n \leq 99999.$$

Now consider the following cases:

(i) $n \equiv 0 \pmod 3$. Then

$$66667n = \frac{2}{3}n \cdot 10^5 + \frac{1}{3}n,$$

the five digits of $2 \cdot \frac{n}{3}$ followed by the five digits of $\frac{n}{3}$. These digits are all 3, 4, 5, or 6 if and only if $\frac{n}{3} = 33333$ and $n = 99999$.

(ii) $n \equiv 1 \pmod 3$. Then

$$66667n = \frac{2}{3}(n-1) \cdot 10^5 + \frac{1}{3}(n+2) + 66666,$$

the five digits of $\frac{2}{3}(n-1)$ followed by the five digits of $\frac{1}{3}(n+2) + 66666$. But $\frac{1}{3}(n+2) + 66666$ must be between 66667 and 99999, so its digits cannot all be 3, 4, 5, or 6. Therefore there are no satisfactory $n \equiv 1 \pmod 3$.

(iii) $n \equiv 2 \pmod 3$. Let $a = \frac{1}{3}(n-2)$. Then

$$66667n = \left(\frac{2}{3}(n-2) + 1\right) \cdot 10^5 + \frac{1}{3}(n-2) + 33334,$$

the five digits of $x = 2a + 1$ followed by the five digits of $y = a + 33334$. The units digits in x and y are between 3 and 6 if and only if the units digit in a is 1 or 2; then the other digits in x and y are all between 3 and 6 if and only if the other digits in a are 2 or 3. Thus there are thirty-two satisfactory a — we can choose each of its five digits from two options — and each a corresponds to a satisfactory $n = 3a + 2$.

Therefore there is exactly one satisfactory $n \equiv 0 \pmod 3$, and thirty-two satisfactory $n \equiv 2 \pmod 3$ — making a total of thirty-three values of n and thirty-three ten-digit numbers.

Problem 10.6 The numbers $1, 2, \ldots, 100$ are arranged in the squares of a 10×10 table in the following way: the numbers $1, \ldots, 10$ are in the bottom row in increasing order, numbers $11, \ldots, 20$ are in the next row in increasing order, and so on. One can choose any number and two of its neighbors in two opposite directions (horizontal, vertical, or diagonal). Then either the number is increased by 2 and its neighbors are decreased by 1, or the number is decreased by 2 and its neighbors are increased by 1. After several such operations the table again contains all the numbers $1, 2, \ldots, 100$. Prove that they are in the original order.

Solution: Label the table entries $a_{11}, a_{12}, \ldots$, with a_{ij} in the ith row and jth column, where the bottom-left corner is in the first row and first column. Also, let $b_{ij} = 10(i-1)+j$ be the number originally in the ith row and jth column.

Observe that

$$P = \sum_{i,j=1}^{10} a_{ij} b_{ij}$$

is invariant — this is because every time entries a_{mn}, a_{pq}, a_{rs} are changed (with $m + r = 2p$ and $n + s = 2q$), P increases or decreases by $b_{mn} - 2b_{pq} + b_{rs}$, or

$$10\left((m-1) + (r-1) - 2(p-1)\right) + (n + s - 2q) = 0.$$

(For example, if entries a_{35}, a_{46}, a_{57} are changed then P changes by $\pm(35 - 2 \cdot 46 + 57) = 0$.)

At first, $P = \sum_{i,j=1}^{10} b_{ij} b_{ij}$; at the end, the entries a_{ij} equal the b_{ij} in some order, and we now have $P = \sum_{i,j=1}^{10} a_{ij} b_{ij}$. By the rearrangement inequality, this is at least $\sum_{i,j=1}^{10} b_{ij} b_{ij}$, with equality only when each $a_{ij} = b_{ij}$.

But we know equality *does* occur since P is invariant. Therefore the a_{ij} do indeed equal the b_{ij} in the same order, and thus the entries $1, 2, \ldots, 100$ appear in their original order.

Problem 10.7 Quadrilateral $ABCD$ is inscribed in circle ω centered at O. The bisector of $\angle ABD$ meets $\overline{AD}$ and ω at points K and M respectively. The bisector of $\angle CBD$ meets $\overline{CD}$ and ω at points L and N respectively. Suppose that $KL \parallel MN$. Prove that the circumcircle of triangle MON goes through the midpoint of $\overline{BD}$.

Solution: All angles are directed modulo $180°$. Let P, Q, R be the midpoints of $\overline{DB}$, $\overline{DA}$, $\overline{DC}$ respectively. Points M and N are the midpoints of arcs AD and DC, respectively; so M, Q, O and N, R, O are collinear, so that $\angle MON = \angle QOR$. But since $\angle DQO = \angle DRO = 90°$, $DQOR$ is cyclic and $\angle QOR = 180° - \angle RDQ = 180° - \angle CDA = \angle ABC$.

In addition, $\angle QPR = \angle QPD + \angle DPR = \angle ABD + \angle DBC = \angle ABC$. Then to prove our claim, it suffices to show that $\triangle PQM \sim \triangle PRN$ (with the same orientation) since then

$$\angle MPN = \angle QPR - \angle QPM + \angle RPN = \angle QPR = \angle ABC = \angle MON$$

so that $MPON$ would be cyclic. (Alternatively, it is possible that triangles PQM and PRN are both degenerate.)

Now, let $a = AB$, $b = BC$, $c = CD$, $D = DA$, $e = AC$, and $f = BD$. Suppose line MN hits lines AD and CD at E and F, respectively. Then

$$\begin{aligned}\angle DEF &= \frac{1}{2}\left(\widehat{DN} + \widehat{AM}\right)\\ &= \frac{1}{2}\left(\widehat{NC} + \widehat{MD}\right)\\ &= \angle EFD,\end{aligned}$$

so $DE = DF$. Then since $KL \parallel EF$, we have $DK = DL$. And by the Angle Bisector Theorem on triangles ABD and CBD, $DK = d \cdot \frac{f}{a+f}$ and $DL = c \cdot \frac{f}{b+f}$, so that

$$\begin{aligned} d(b+f) &= c(a+f)\\ (c-d)f &= bd - ac. \end{aligned} \tag{1}$$

If $c = d$ then we must have $bd = ac$, so $a = b$. But then BD is a diameter of ω so $P = O$ and the claim is obvious. Otherwise $c - d, bd - ac \neq 0$, and $f = \frac{bd-ac}{c-d}$.

Now we prove that $\triangle PQM \sim \triangle PRN$. Observe that

$$\begin{aligned}\angle PQM &= \angle PQD + \angle DQM = \angle BAD + 90^\circ\\ &= \angle BCD + 90^\circ = \angle PRD + \angle DRN = \angle PRN,\end{aligned}$$

and also note that $\angle PQM$ and $\angle PRN$ are both obtuse.

So, we need only prove that

$$\begin{aligned} &\frac{PQ}{QM} = \frac{PR}{RN}\\ \iff\ &\frac{\frac{BA}{2}}{AQ\tan\angle MAQ} = \frac{\frac{BC}{2}}{CR\tan\angle RCN}\\ \iff\ &\frac{\frac{BA}{2}}{\frac{AD}{2}\tan\angle MCD} = \frac{\frac{BC}{2}}{\frac{CD}{2}\tan\angle DAN} \end{aligned}$$

$$\iff \frac{a}{d\tan\angle ACM} = \frac{b}{c\tan\angle NAC}. \qquad (2)$$

Since lines CM, AN are angle bisectors of $\angle ACD$, $\angle DAC$, they intersect at the incenter I of triangle ACD. Also, let T be the point where the incircle of triangle ACD hits $\overline{AC}$. Then $\tan\angle ACM = \frac{IT}{TC} = \frac{IT}{\frac{1}{2}(e+c-d)}$ and $\tan\angle NAC = \frac{IT}{TA} = \frac{IT}{\frac{1}{2}(e+d-c)}$, so $\frac{\tan\angle ACM}{\tan\angle NAC} = \frac{e+d-c}{e+c-d}$. Thus (2) is equivalent to

$$ac(e+c-d) = bd(e+d-c)$$
$$\iff e = \frac{(ac+bd)(c-d)}{bd-ac}.$$

But by Ptolemy's Theorem and (1), we have

$$e = \frac{ac+bd}{f} = \frac{(ac+bd)(c-d)}{bd-ac},$$

as desired. Therefore triangle PQM *is* similar to triangle PRN, $\angle MPN = \angle MON$, and $MPON$ is indeed cyclic!

Problem 11.1 There are 150 red, 150 blue, and 150 green balls flying under the big top in the circus. There are exactly 13 green balls inside every blue one, and exactly 5 blue balls and 19 green balls inside every red one. (A ball is considered to be "inside" another ball even if it is not immediately inside it; for example, if a green ball is inside a blue ball and that blue ball is inside a red ball, then the green ball is also inside the red ball.) Prove that some green ball is not contained in any of the other 449 balls.

Solution: Suppose by way of contradiction that every green ball were in some other ball. Notice that no red ball is inside a blue ball, since then the blue ball would contain at least 19 green balls.

Look at the red balls that are not inside any other red balls — say there are m of them, and throw them away along with all the balls they contain. Then we throw all the red balls and anything inside a red ball, including $19m$ green balls. Also, since no red ball is inside a blue ball, there are still exactly 13 green balls inside each of the remaining blue balls.

Now look at the blue balls left that are not inside any other blue balls — say there are n of them, and throw them away along with all the balls they contain. Then we throw away all the remaining blue balls and the $13n$ more green balls inside.

At this point, all the green balls are gone and hence we must have $150 = 19m + 13n$. Taking this equation modulo 13, we find that $7 \equiv 6m \Rightarrow m \equiv 12 \pmod{13}$. Then $m \geq 12$ and $150 = 19m + 13n \geq 19 \cdot 12 = 228$, a contradiction. Thus our original assumption was false, and some green ball is contained in no other ball.

Problem 11.3 Let $\{a_n\}$ be an arithmetic sequence of positive integers. For every n, let p_n be the largest prime divisor of a_n. Prove that the sequence $\{\frac{a_n}{p_n}\}$ is unbounded.

Solution: Let d be the common difference of the arithmetic sequence and N be an arbitrary number. Choose primes $p < q$ bigger than N and relatively prime to d; then there is some term a_k divisible by pq. Since $p_k \geq q > p$, we have $p \mid \frac{a_k}{p_k}$ so that

$$\frac{a_k}{p_k} \geq p > N.$$

Thus for any N, there exists k such that $\frac{a_k}{p_k} > N$, so the sequence $\{\frac{a_n}{p_n}\}$ is unbounded.

Problem 11.4 All positive integers not exceeding 100 are written on both sides of 50 cards (each number is written exactly once). The cards are put on a table so that Vasya only knows the numbers on the top side of each card. Vasya can choose several cards, turn them upside down, and then find the sum of all 50 numbers now on top. What is the maximum sum Vasya can be sure to obtain or beat?

Solution: The answer is $2525 = \frac{1}{2}(1+2+3+\cdots+100)$. Vasya can always obtain or beat this: if the 50 numbers on top add to this or more, he is done; otherwise, if they add to less, Vasya can flip all of them.

Sadly, this might be the best Vasya can do. Suppose that Vasya has horrible luck and the numbers on top are 26, 27, 28, ..., 75, with sum 2525; and that the numbers on the cards he flips over are, in order, 1, 2, ..., 25, 76, 77, ..., 100 (although of course he might not flip over all of the cards). If he flips over 0 through 25 cards, his sum decreases; and if he flips over more, his sum is at most $1+2+\cdots+25+76+77+\cdots+100 = 2525$. Either way, Vasya cannot obtain a sum of more than 2525, as claimed.

Problem 11.5 Two players play the following game. They in turn write on a blackboard different divisors of 100! (except 1). A player loses if after his turn, the greatest common divisor of the all the numbers written becomes 1. Which of the players has a winning strategy?

Solution: The second player has a winning strategy. Notice that every prime $p < 100$ divides an *even* number of factors of 100!: the factors it divides can be split into disjoint pairs $(k, 97k)$ — or, if $p = 97$, into the pairs $(k, 89k)$. (Note that none of these factors is 1, since 1 is not divisible by p.)

If the first player writes down a prime p, the second player can write down any other number divisible by p; if the first player writes down a composite number, the second player can write down a prime p dividing that number. Either way, from now on the players can write down a new number $q \mid 100!$ without losing if and only if it is divisible by p. Since there are an even number of such q, the second player will write down the last acceptable number and the first player will lose.

Problem 11.7 A connected graph G has 500 vertices, each with degree 1, 2, or 3. We call a black-and-white coloring of these vertices *interesting* if more than half of the vertices are white but no two white vertices are connected. Prove that it is possible to choose several vertices of G so that in any interesting coloring, more than half of the chosen vertices are black.

Solution: We first give an algorithm to (temporarily) erase edges from G so that our graph consists of "chains" of vertices—sequences of vertices $V_1, \ldots, V_n$ where each V_i is adjacent to V_{i+1}—and possibly one leftover vertex.

First, as long as G still contains any cycle erase an edge from that cycle; this eventually makes G a tree (a connected graph with no cycles). Look at the leaves (vertices with degree 1) that are not part of a chain yet. If all of them are adjacent to a degree-3 vertex, then we must have exactly four unchained vertices left with one central vertex adjacent to the other 3; remove one of the edges, and we are done. Otherwise, one of the leaves is *not* adjacent to a degree-3 vertex. Travel along the graph from this vertex until we reach a degree-3 vertex V, and erase the edge going into V. This creates a new chain, while leaving all the unchained vertices in a smaller, still-connected tree; we can then repeat the algorithm on this tree until we are finished.

Besides our lone vertex, every vertex is in an "odd chain" (a chain with an odd number of vertices) or an "even chain" (a chain with an even number of vertices). For our chosen vertices, in each odd chain pick one vertex adjacent to one of the ends. Even with all the original edges back in place, for any interesting coloring observe that in any of our chains at most every other vertex can be white. Thus in any even chain, at most half the vertices are white. Furthermore, if a chosen vertex is white, then in its odd chain there is at least one more black vertex than white; and if a chosen vertex is black, then there is at most one more white vertex than black.

Suppose there are b odd chains with a black chosen vertex, and w odd chains with a white chosen vertex. If there is a lone vertex, there are at most $1 + b - w$ more white vertices than black in our graph so that $1 + b - w > 0$. But we know $1 + b - w$ must be even since $1+b+w \equiv 500 \pmod{2}$. Then $1+b-w \geq 2$ and $b-w \geq 1$, implying that we have more black chosen vertices than white.

And if there is *no* lone vertex, then there are at most $b - w$ more white vertices than black in our graph. Thus $b - w > 0$ and we still have more black chosen vertices than white. Therefore either way, for every interesting coloring we have more black chosen vertices than white, as desired.

Problem 11.8 Three conjurers show a trick. They give a spectator a pack of cards with numbers $1, 2, \ldots, 2n + 1$ ($n > 6$). The spectator takes one card and arbitrarily distributes the rest evenly between the first and the second conjurers. Without communicating with each other, these conjurers study their cards, each chooses an ordered pair of their cards, and gives these pairs to the third conjurer. The third conjurer studies these four cards and announces which card is taken by the spectator. Explain how such a trick can be done.

Solution: We will have each of the first two conjurers use their ordered pairs to communicate the sums of their card values modulo $2n+1$. With this information, the third conjurer can simply subtract these two sums from $1 + 2 + \cdots + (2n + 1) = (2n + 1)(n + 1) \equiv 0 \pmod{2n + 1}$ to determine the remaining card.

From now on, all entries of ordered pairs are taken modulo $2n + 1$; also, let $(a, b)_k$ denote the ordered pair $(a + k, b + k)$ and say its "difference" is $b - a$ (taken modulo $2n + 1$ between 1 and $2n$).

Let $(0, 2n)_k$, $(0, 1)_k$, $(n, 2)_k$, $(n, 2n-1)_k$, $(4, n+1)_k$, and $(2n-3, n+1)_k$ all represent the sum $k \pmod{2n+1}$. These pairs' differences are $2n, 1, n+3, n-1, n-3, n+5$; because $n > 5$, these differences are all distinct.

If n is odd then let $(1, 2n)_k$, $(2, 2n - 1)_k$, $\ldots$, $(n - 1, n + 2)_k$ also represent the sum $k \pmod{2n+1}$. These pairs' differences are all odd: $2n - 1, 2n - 3, \ldots, 3$. Furthermore, they are all different from the one odd difference, 1, that we found in the last paragraph.

Similarly, if n is even then let $(2n, 1)_k$, $(2n-1, 2)_k$, $\ldots$, $(n+2, n-1)_k$ represent the sum $k \pmod{2n+1}$. These pairs' differences are all even: $2, 4, \ldots, 2n-2$; and they are all different from the one even difference, $2n$, that we found two paragraphs ago.

Note that if two of the assigned pairs $(a_1, b_1)_{k_1}$ and $(a_2, b_2)_{k_2}$ are equal, then their differences must be equal and we must have $b_1 - a_1 \equiv b_2 - a_2 \pmod{2n + 1}$. But because we found that the differences $b - a$ are distinct, we must have $(a_1, b_1) = (a_2, b_2)$ and therefore $k_1 = k_2$ as well. Thus any pair (a, b) is assigned to at most one sum, and our choices are well-defined.

Now, say that one of the first two conjurers has cards whose values sum to $k \pmod{2n + 1}$; suppose by way of contradiction that he could not give any pair $(a, b)_k$ described above. Then, letting $S_k = \{s + k \mid s \in S\}$, he has at most one card from each of the three triples $\{0, 1, 2n\}_k$, $\{2, n, 2n - 1\}_k$, $\{4, n + 1, 2n - 3\}_k$; and he has at most one card from each of the $n-4$ pairs $\{3, 2n-2\}_k$, $\{5, 2n-4\}_k$, $\{6, 2n-5\}_k$, $\ldots$, $\{n - 1, n + 2\}_k$. But these sets partition *all* of $\{0, 1, 2, \ldots, 2n\}$, so the magician must then have at most $3 + (n - 4) = n - 1$ cards—a contradiction. Thus our assumption was false, and both conjurers can indeed communicate the desired sums. This completes the proof.

2
2000 National Contests: Problems

2.1 Belarus

Problem 1 Find all pairs of positive integers (m, n) which satisfy the equality

$$(m-n)^2(n^2-m) = 4m^2n.$$

Problem 2 Let M be the intersection point of the diagonals AC and BD of a convex quadrilateral $ABCD$. The bisector of angle ACD hits ray BA at K. If $MA \cdot MC + MA \cdot CD = MB \cdot MD$, prove that $\angle BKC = \angle CDB$.

Problem 3 An equilateral triangle of side n is divided into n^2 equilateral triangles of side 1 by lines parallel to the sides of the triangle. Each point that is a vertex of at least one of these unit triangles is labeled with a number; exactly one of these points is labeled with -1, all the others with 1's. On each move one can choose a line passing through the side of one of the small triangles and change the signs of the numbers at all the labeled points on this line. Determine all possible initial arrangements (the value of n and the position of the -1) from which one can obtain an arrangement of all 1's using the described operations.

Problem 4 Tom and Jerry play the following game. They alternate putting pawns onto empty squares of an initially empty 25×25 chessboard, with Tom going first. A player wins if after his move, some four pawns are the vertices of a rectangle with sides parallel to the sides of the board. Which player has a winning strategy?

Problem 5

(a) We are given a rectangle $ABCD$. Prove that for any point X in the plane, some three of the segments XA, XB, XC, and XD could be the sides of a triangle.

(b) Is the previous statement for any parallelogram $ABCD$?

Problem 6 Pit and Bill play the following game. Pit writes a number a on a blackboard, then Bill writes a number b, and finally Pit writes a number c. Can Pit choose his numbers so that the three polynomials $x^3 + ax^2 + bx + c$, $x^3 + bx^2 + cx + a$, and $x^3 + cx^2 + ax + b$ have

(a) a common real root?

(b) a common negative root?

Problem 7 How many pairs (n, q) satisfy $\{q^2\} = \left\{\frac{n!}{2000}\right\}$, where n is a positive integer and q is a nonintegral rational number between 0 and 2000?

Problem 8 Let $n \geq 5$ be a positive integer. Define a *sign-sequence* to be a sequence of n numbers all equal to 1 or -1, and let a *move* consist of changing the signs of any five consecutive terms of a sign-sequence. Two sign-sequences are said to be *similar* if one of them can be obtained from the other with a finite number of moves. Find the maximum number m such that there exist m sign-sequences, no two of which are similar to each other.

Problem 9 A line ℓ intersects the lateral sides and diagonals of a trapezoid. It is known that the portion of ℓ between the lateral sides is divided by the diagonals into three equal parts. Does it follow that the line ℓ is parallel to the bases of the trapezoid?

Problem 10 Nine points are marked on a plane, no three of them collinear. Each pair of marked points is connected with a segment. Is it possible to color each segment with one of twelve colors, such that the segments of each color form a triangle?

Problem 11 A vertex of a tetrahedron is called *perfect* if one could construct a triangle with the edges from this vertex as its sides. Find all n such that some tetrahedron has exactly n perfect vertices.

Problem 12

(a) Find all positive integers n such that the equation $(a^a)^n = b^b$ has at least one solution in integers $a, b > 1$.

(b) Find all positive integers a and b which satisfy $(a^a)^5 = b^b$.

Problem 13 The sides of scalene triangle ABC have lengths a, b, and c, measured in meters. Its angles, measured in radians, are α, β, and γ. The set $\{a, b, c, \alpha, \beta, \gamma\}$ contains exactly n distinct elements. Find the minimum possible value of n.

Problem 14 On a 5×7 board, we call two squares *adjacent* if they are distinct and share one common side. Find the minimum number of squares on the board that must be painted so that any unpainted square has exactly one adjacent square which is painted.

Problem 15 We are given triangle ABC with $\angle C = 90°$. Let M be the midpoint of the hypotenuse $\overline{AB}$, H be the foot of the altitude $\overline{CH}$, and P be a point inside the triangle such that $AP = AC$. Prove that $\overline{PM}$ bisects angle BPH if and only if $\angle A = 60°$.

Problem 16 Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n-1)) = f(n+1) - f(n)$$

for all $n \geq 2$?

Problem 17 Five points A, B, C, D, E lie on a sphere of diameter 1, and that $AB = BC = CD = DE = EA = \ell$. Prove that $\ell \leq \sin 72°$.

Problem 18 In a convex polyhedron with m triangular faces, exactly four edges meet at each vertex. Find the minimum possible value of m.

Problem 19 We call two lattice points (a_1, b_1) and (a_2, b_2) in the Cartesian plane *connected* if either $(a_1, b_1) = (-a_2, b_2 \pm 1)$ or $(a_1, b_1) = (a_2 \pm 1, -b_2)$. Prove that it is possible to construct an infinite sequence (m_1, n_1), (m_2, n_2), $\ldots$ of lattice points such that any two consecutive points of the sequence are connected, and each lattice point in the plane appears exactly once in this sequence.

Problem 20 In triangle ABC, $a \neq b$ where $a = BC$ and $b = AC$. Points E and F are on the sides AC and BC, respectively, such that $AE = BF = \frac{ab}{a+b}$. Let M be the midpoint of $\overline{AB}$, N be the midpoint of $\overline{EF}$, and P be the intersection point of $\overline{EF}$ and the bisector of angle ACB. Find $\frac{[CPMN]}{[ABC]}$.

Problem 21 In triangle ABC, let m_a and m_b be the lengths of the medians from the vertices A and B, respectively. Find all real λ such that $m_a + \lambda\, BC = m_b + \lambda\, AC$ implies that $BC = AC$.

Problem 22

(a) Prove that $\{n\sqrt{3}\} > \frac{1}{n\sqrt{3}}$ for every positive integer n, where $\{x\}$ denotes the fractional part of x.

(b) Does there exist a constant $c > 1$ such that $\{n\sqrt{3}\} > \frac{c}{n\sqrt{3}}$ for every positive integer n?

Problem 23 A graph has 15 vertices and n edges. Each edge of the graph is colored either red or blue such that no three vertices A, B, C are connected pairwise with edges of the same color. Determine the largest possible value of n.

Problem 24 Find all functions $f, g, h : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y^3) + g(x^3 + y) = h(xy)$$

for all $x, y \in \mathbb{R}$.

Problem 25 Let $M = \{1, 2, \ldots, 40\}$. Find the smallest positive integer n for which it is possible to partition M into n disjoint subsets such that whenever a, b, and c (not necessarily distinct) are in the same subset, $a \neq b + c$.

Problem 26 A positive integer is called *monotonic* if its digits in base 10, read from left to right, are in nondecreasing order. Prove that for each $n \in \mathbb{N}$, there exists an n-digit monotonic number which is a perfect square.

Problem 27 Given a pair $(\vec{a}, \vec{b})$ of vectors in the plane, a *move* consists of choosing a nonzero integer k and then changing $(\vec{a}, \vec{b})$ to either (i) $(\vec{a}+2k\vec{b}, \vec{b})$ or (ii) $(\vec{a}, \vec{b}+2k\vec{a})$. A *game* consists of applying a finite sequence of moves, alternating between moves of types (i) and (ii), to some initial pair of vectors.

(a) Is it possible to obtain the pair $((1, 0), (2, 1))$ during a game with initial pair $((1, 0), (0, 1))$, if our first move is of type (i)?

(b) Find all pairs $((a, b), (c, d))$ that can be obtained during a game with initial pair $((1, 0), (0, 1))$, where our first move can be of either type.

Problem 28 Prove that

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a + b + c)^3}{3(x + y + z)}$$

for all positive real numbers a, b, c, x, y, z.

Problem 29 Let P be the intersection point of the diagonals AC and BD of the convex quadrilateral $ABCD$ in which $AB = AC = BD$. Let O and I be the circumcenter and incenter of triangle ABP, respectively. Prove that if $O \neq I$, then lines OI and CD are perpendicular.

2.2 Bulgaria

Problem 1 Let $F = x^3y + xy^3$ for some real numbers x and y. If $x^2 + xy + y^2 = 1$,

(a) prove that $F \geq -2$;

(b) find the greatest possible value of F.

Problem 2 A line ℓ is drawn through the orthocenter of acute triangle ABC. Prove that the reflections of ℓ across the sides of the triangle are concurrent.

Problem 3 There are 2000 white balls in a box. There are also unlimited supplies of white, green, and red balls, initially outside the box. During each turn, we can replace two balls in the box with one or two balls as follows: two whites with a green, two reds with a green, two greens with a white and red, a white and green with a red, or a green and red with a white.

(a) After finitely many of the above operations there are three balls left in the box. Prove that at least one of them is a green ball.

(b) Is it possible after finitely many operations to have only one ball left in the box?

Problem 4 Solve the equation $\sqrt{x} + \sqrt[3]{x+7} = \sqrt[4]{x+80}$ in real numbers.

Problem 5 The incircle of the isosceles triangle ABC touches the legs AC and BC at points M and N respectively. A line t is drawn tangent to minor arc $\widehat{MN}$, intersecting $\overline{NC}$ and $\overline{MC}$ at points P and Q, respectively. Let T be the intersection point of lines AP and BQ.

(a) Prove that T lies on $\overline{MN}$;

(b) Prove that the sum of the areas of triangles ATQ and BTP is smallest when t is parallel to line AB.

Problem 6 We are given $n \geq 4$ points in the plane such that the distance between any two of them is an integer. Prove that at least $\frac{1}{6}$ of these distances are divisible by 3.

Problem 7 In triangle ABC, $\overline{CH}$ is an altitude, and cevians $\overline{CM}$ and $\overline{CN}$ bisect angles ACH and BCH, respectively. The circumcenter of triangle CMN coincides with the incenter of triangle ABC. Prove that $[ABC] = \frac{AN \cdot BM}{2}$.

Problem 8 Let $a_1, a_2, \ldots$ be a sequence such that $a_1 = 43$, $a_2 = 142$, and $a_{n+1} = 3a_n + a_{n-1}$ for all $n \geq 2$. Prove that

(a) a_n and a_{n+1} are relatively prime for all $n \geq 1$;

(b) for every natural number m, there exist infinitely many natural numbers n such that $a_n - 1$ and $a_{n+1} - 1$ are both divisible by m.

Problem 9 In convex quadrilateral $ABCD$, $\angle BCD = \angle CDA$. The bisector of angle ABC intersects $\overline{CD}$ at point E. Prove that $\angle AEB = 90^\circ$ if and only if $AB = AD + BC$.

Problem 10 Prove that for any two real numbers a and b there exists a real number $c \in (0, 1)$ such that

$$\left| ac + b + \frac{1}{c+1} \right| > \frac{1}{24}.$$

Problem 11 Find all sets S of four distinct points in the plane such that if any two circles k_1 and k_2 have diameters whose endpoints are in S, then k_1 and k_2 intersect at a point in S.

Problem 12 In the coordinate plane, a set of 2000 points $\{(x_1, y_1), (x_2, y_2), \ldots, (x_{2000}, y_{2000})\}$ is called *good* if $0 \leq x_i \leq 83, 0 \leq y_i \leq 1$ for $i = 1, 2, \ldots, 2000$ and $x_i \neq x_j$ when $i \neq j$. Find the largest positive integer n such that, for any good set, some unit square contains at least n of the points in the set.

Problem 13 We are given the acute triangle ABC.

(a) Prove that there exist unique points A_1, B_1, and C_1 on $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, with the following property: If we project any two of the points onto the corresponding side, the midpoint of the projected segment is the third point.

(b) Prove that triangle $A_1B_1C_1$ is similar to the triangle formed by the medians of triangle ABC.

Problem 14 Let $p \geq 3$ be a prime number and $a_1, a_2, \ldots, a_{p-2}$ be a sequence of positive integers such that p does not divide either a_k or $a_k^k - 1$ for all $k = 1, 2, \ldots, p-2$. Prove that the product of some terms of the sequence is congruent to 2 modulo p.

Problem 15 Find all polynomials $P(x)$ with real coefficients such that

$$P(x)P(x+1) = P(x^2)$$

for all real x.

Problem 16 Let D be the midpoint of base AB of the isosceles acute triangle ABC. Choose a point E on $\overline{AB}$, and let O be the circumcenter of triangle ACE. Prove that the line through D perpendicular to $\overline{DO}$, the line through E perpendicular to $\overline{BC}$, and the line through B parallel to $\overline{AC}$ are concurrent.

Problem 17 Let n be a positive integer. A *binary sequence of length* n is a sequence of n integers, all equal to 0 or 1. Let $\mathcal{A}$ be the set of all such sequences, and let $\mathbf{0} \in \mathcal{A}$ be the sequence of all zeroes. The sequence $c = c_1, c_2, \ldots, c_n$ is called the sum $a + b$ of $a = a_1, a_2, \ldots, a_n$ and $b = b_1, b_2, \ldots, b_n$ if $c_i = 0$ when $a_i = b_i$ and $c_i = 1$ when $a_i \neq b_i$. Let $f : \mathcal{A} \to \mathcal{A}$ be a function with $f(\mathbf{0}) = \mathbf{0}$ such that whenever the sequences a and b differ in exactly k terms, the sequences $f(a)$ and $f(b)$ also differ in exactly k terms. Prove that if a, b, and c are sequences from $\mathcal{A}$ such that $a + b + c = \mathbf{0}$, then $f(a) + f(b) + f(c) = \mathbf{0}$.

2.3 Canada

Problem 1 At 12:00 noon, Anne, Beth, and Carmen begin running laps around a circular track of length three hundred meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least one hundred meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners.)

Problem 2 Given a permutation $(a_1, a_2, \ldots, a_{100})$ of the integers 1901, 1902, …, 2000, we form the sequence of partial sums

$$s_1 = a_1,\ s_2 = a_1 + a_2,\ \ldots,\ s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

For how many such permutations will no terms of the corresponding sequence $s_1, s_2, \ldots, s_{100}$ be divisible by three?

Problem 3 Let $a_1, a_2, \ldots, a_{2000}$ be a sequence of integers each lying in the interval $[-1000, 1000]$. Suppose that $\sum_{i=1}^{1000} a_i = 1$. Show that the terms in some nonempty subsequence of $a_1, a_2, \ldots, a_{2000}$ sum to zero.

Problem 4 Let $ABCD$ be a quadrilateral with $\angle CBD = 2\angle ADB$, $\angle ABD = 2\angle CDB$, and $AB = CB$. Prove that $AD = CD$.

Problem 5 Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy (i) $a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0$, (ii) $a_1 + a_2 \leq 100$, and (iii) $a_3 + a_4 + \cdots + a_{100} \leq 100$. Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ for which this maximum is achieved.

2.4 China

Problem 1 In triangle ABC, $BC \leq CA \leq AB$. Let R and r be the circumradius and inradius, respectively, of triangle ABC. As a function of $\angle C$, determine whether $BC + CA - 2R - 2r$ is positive, negative, or zero.

Problem 2 Define the infinite sequence $a_1, a_2, \ldots$ recursively as follows: $a_1 = 0$, $a_2 = 1$, and

$$a_n = \frac{1}{2}na_{n-1} + \frac{1}{2}n(n-1)a_{n-2} + (-1)^n\left(1 - \frac{n}{2}\right)$$

for all $n \geq 3$. Find an explicit formula for

$$f_n = a_n + 2\binom{n}{1}a_{n-1} + 3\binom{n}{2}a_{n-2} + \cdots + n\binom{n}{n-1}a_1.$$

Problem 3 A table tennis club wishes to organize a doubles tournament, a series of matches where in each match one pair of players competes against a pair of two different players. Let a player's *match number* for a tournament be the number of matches he or she participates in. We are given a set $A = \{a_1, a_2, \ldots, a_k\}$ of distinct positive integers all divisible by 6. Find with proof the minimal number of players among whom we can schedule a doubles tournament such that

(i) each participant belongs to at most 2 pairs;

(ii) any two different pairs have at most 1 match against each other;

(iii) if two participants belong to the same pair, they never compete against each other; and

(iv) the set of the participants' match numbers is exactly A.

Problem 4 We are given an integer $n \geq 2$. For any ordered n-tuple of real numbers $A = (a_1, a_2, \ldots, a_n)$, let A's *domination score* be the number of values $k \in \{1, 2, \ldots, n\}$ such that $a_k > a_j$ for all $1 \leq j \leq k$. Consider all permutations $A = (a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$ with domination score 2. Find with proof the arithmetic mean of the first elements a_1 of these permutations.

Problem 5 Find all positive integers n such that there exist integers $n_1, n_2, \ldots, n_k \geq 3$ with

$$n = n_1 n_2 \cdots n_k = 2^{\frac{1}{2^k}(n_1-1)(n_2-1)\cdots(n_k-1)} - 1.$$

Problem 6 An exam paper consists of 5 multiple-choice questions, each with 4 different choices; 2000 students take the test, and each student chooses exactly one answer per question. Find the least possible value of n such that among any n of the students' answer sheets, there exist 4 of them among which no two have exactly the same answers chosen.

2.5 Czech and Slovak Republics

Problem 1 Determine all real numbers p for which the system of equations

$$(x-y)^2 = p^2$$
$$x^3 - y^3 = 16$$

has precisely one solution in real numbers x, y.

Problem 2 Cevians $\overline{AK}$, $\overline{BL}$, and $\overline{CM}$ of triangle ABC intersect at a point U inside the triangle. Prove that if $[AMU] = [KCU] = P$ and $[MBU] = [CLU] = Q$, then $P = Q$.

Problem 3 Find the smallest natural number k such that among any k distinct numbers from the set $\{1, 2, 3, \ldots, 2000\}$, there exist two whose sum or difference equals 667.

Problem 4 Let $P(x)$ be a quadratic polynomial with $P(-2) = 0$. Find all roots of the equation

$$P(x^2 + 4x - 7) = 0,$$

given that the equation has at least one double root.

Problem 5 An isosceles trapezoid $UVST$ is given in which $3ST < 2UV$. Show how to construct an isosceles triangle ABC with base AB so that the points B, C lie on the line VS; the point U lies on the line AB; and the point T is the centroid of the triangle ABC.

Problem 6 Show that

$$\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}} \le \sqrt[3]{2(a+b)\left(\frac{1}{a} + \frac{1}{b}\right)}$$

for all positive real numbers a and b, and determine when equality occurs.

Problem 7 Find all convex quadrilaterals $ABCD$ for which there exists a point E inside the quadrilateral with the following property: Any line which passes through E and intersects sides $\overline{AB}$ and $\overline{CD}$ divides the quadrilateral $ABCD$ into two parts of equal area.

Problem 8 An isosceles triangle ABC is given with base $\overline{AB}$ and altitude $\overline{CD}$. Let E be the intersection of line AP with side BC, and let F be the intersection of line BP with side AC. Point P is chosen on $\overline{CD}$ so that the incircles of triangle ABP and quadrilateral $PECF$ are congruent. Show that the incircles of the triangles ADP and BCP are also congruent.

Problem 9 In the plane are given 2000 congruent triangles of area 1, which are images of a single triangle under different translations. Each of these triangles contains the centroids of all the others. Show that the area of the union of these triangles is less than $\frac{22}{9}$.

Problem 10 For which quadratic functions $f(x)$ does there exist a quadratic function $g(x)$ such that the equation $g(f(x)) = 0$ has four distinct roots in arithmetic progression, which are also real roots of the equation $f(x)g(x) = 0$?

Problem 11 Monica constructed a paper model of a triangular pyramid, the base of which was a right triangle. When she cut the model along the two legs of the base and along a median of one of the faces, upon unfolding it into the plane she obtained a square with side a. Determine the volume of the pyramid.

2.6 Estonia

Problem 1 Five real numbers are given such that, no matter which three of them we choose, the difference between the sum of these three numbers and the sum of the remaining two numbers is positive. Prove that the product of all these 10 differences (corresponding to all the possible triples of chosen numbers) is less than or equal to the product of the squares of these five numbers.

Problem 2 Prove that it is not possible to divide any set of 18 consecutive positive integers into two disjoint sets A and B, such that the product of elements in A equals the product of elements in B.

Problem 3 Let M, N, and K be the points of tangency of the incircle of triangle ABC with the sides of the triangle, and let Q be the center of the circle drawn through the midpoints of $\overline{MN}$, $\overline{NK}$, and $\overline{KM}$. Prove that the incenter and circumcenter of triangle ABC are collinear with Q.

Problem 4 Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$

for all $n \in \mathbb{Z}^+$.

Problem 5 In a triangle ABC we have $AC \neq BC$. Take a point X in the interior of this triangle and let $\alpha = \angle A$, $\beta = \angle B$, $\phi = \angle ACX$, and $\psi = \angle BCX$. Prove that

$$\frac{\sin\alpha \sin\beta}{\sin(\alpha-\beta)} = \frac{\sin\phi \sin\psi}{\sin(\phi-\psi)}$$

if and only if X lies on the median of triangle ABC drawn from the vertex C.

Problem 6 We call an infinite sequence of positive integers an *F-sequence* if every term of this sequence (starting from the third term) equals the sum of the two preceding terms. Is it possible to decompose the set of all positive integers into

(a) a finite;

(b) an infinite

number of F-sequences having no common members?

2.7 Georgia

Problem 1 Do there exist positive integers x and y such that $x^3+2xy+x+2y+1$ and $y^3+2xy+y+2x+1$ are both perfect cubes?

Problem 2 The positive numbers a, b, c satisfy the inequality $abc \geq \frac{1}{64}$. Prove that

$$a^2+b^2+c^2+\frac{1}{4}(a+b+c) \geq \frac{1}{4}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right),$$

and determine when equality occurs.

Problem 3 For any positive integer n, let $a(n)$ denote the product of all its positive divisors.

(a) Prove that $a(400) > 10^{19}$.

(b) Find all solutions of the equation $a(n^3) = n^{60}$ which do not exceed 100.

(c) Find all solutions of the equation $a(n^2) = (a(n))^9$ which do not exceed 2000.

Problem 4 From a point P lying outside a circle ω the tangents $\overline{PA_1}$ and $\overline{PA_2}$ are drawn. Let K be a point inside ω with $PK = PA_1$. Chords $\overline{A_1B_1}$ and $\overline{A_2B_2}$ are drawn in ω through K. Prove that $\overline{B_1B_2}$ is a diameter of ω.

2.8 Hungary

Problem 1 Let H be a set consisting of positive and negative real numbers, with 2000 elements in all. Let N be the number of 4-element subsets of H whose elements have a negative product. How many negative elements should H have in order to maximize N?

Problem 2 In the scalene triangle ABC, let C_1, A_1, and B_1 be the midpoints of sides AB, BC, and CA, respectively. Let B_2 denote the midpoint of the broken-line path leading from A to B to C; define points A_2 and C_2 similarly. Prove that $\overline{A_1A_2}$, $\overline{B_1B_2}$, and $\overline{C_1C_2}$ are concurrent.

Problem 3 Let a_n denote the closest integer to $\sqrt{n}$. Determine the value of $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}$, where $k = 1999 \cdot 2000$.

Problem 4 Find all positive primes p for which there exist positive integers n, x, y such that $p^n = x^3 + y^3$.

Problem 5 In the tetrahedron $ABCP$, edges PA, PB, PC are pairwise perpendicular. Let S be a sphere such that the circumcircle of ABC is a great circle on S, and let $\overline{XY}$ be the diameter of S perpendicular to plane (ABC). Let S' be the ellipsoid which passes through X and Y, is symmetric about axis $\overline{XY}$, and intersects plane (ABC) in a circle of diameter $XY/\sqrt{2}$. Prove that P lies on S'.

Problem 6 Is there a polynomial f of degree 1999 with integer coefficients, such that $f(n), f(f(n)), f(f(f(n))), \ldots$ are pairwise relatively prime for any integer n?

Problem 7 Let p be a polynomial with odd degree and integer coefficients. Prove that there are only finitely many pairs of integers a, b such that the points $(a, p(a))$ and $(b, p(b))$ are an integral distance apart.

Problem 8 The feet of the angle bisectors of triangle ABC are X, Y, and Z. The circumcircle of triangle XYZ cuts off three segments from lines AB, BC, and CA. Prove that two of these segments' lengths add up to the third segment's length.

Problem 9 Let k and t be relatively prime integers greater than 1. Starting from the permutation $(1, 2, \ldots, n)$ of the numbers $1, 2, \ldots, n$, we may swap two numbers if their difference is either k or t. Prove that we can get any permutation of $1, 2, \ldots, n$ with such steps if and only if $n \geq k + t - 1$.

Problem 10 For any positive integer k, let $e(k)$ denote the number of positive even divisors of k, and let $o(k)$ denote the number of positive odd divisors of k. For all $n \geq 1$, prove that $\sum_{k=1}^{n} e(k)$ and $\sum_{k=1}^{n} o(k)$ differ by at most n.

Problem 11 Given a triangle in the plane, show how to construct a point P inside the triangle which satisfies the following condition: if we drop perpendiculars from P to the sides of the triangle, the feet of the perpendiculars determine a triangle whose centroid is P.

Problem 12 Given a natural number k and more than 2^k different integers, prove that a set S of $k+2$ of these numbers can be selected such that for any positive integer $m \leq k+2$, all the m-element subsets of S have different sums of elements.

2.9 India

Problem 1 Let ABC be a nonequilateral triangle. Suppose there is an interior point P such that the three cevians through P all have the same length λ where $\lambda < \min\{AB, BC, CA\}$. Show that there is another interior point $P' \neq P$ such that the three cevians through P' also are of equal length.

Problem 2 Find all ordered pairs of prime numbers (p, q) such that $p \mid 5^q + 1$ and $q \mid 5^p + 1$.

Problem 3 Determine whether or not it is possible to label each vertex of a cube with a natural number such that two vertices are connected by an edge of the cube if and only if one of their corresponding labels a divides the other label b.

Problem 4 Let ABC be an acute triangle and let $\overline{AD}$ be the altitude from A. Let the internal bisectors of angles B and C meet $\overline{AD}$ at E and F, respectively. If $BE = CF$, prove that $AB = AC$.

Problem 5 Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists an integer m which is divisible by n and whose digits have sum k.

Problem 6 Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be n real numbers such that $\sum_{j=1}^{n} a_j = 0$. Show that

$$na_1a_n + \sum_{j=1}^{n} a_j^2 \leq 0.$$

Problem 7 Let $p > 3$ be a prime number. Let E be the set of all $(p-1)$-tuples $(x_1, x_2, \ldots, x_{p-1})$ such that each $x_i \in \{0, 1, 2\}$ and $x_1 + 2x_2 + \cdots + (p-1)x_{p-1}$ is divisible by p. Show that the number of elements in E is $(3^{p-1} + p - 1)/p$.

Problem 8 Let m, n be positive integers such that $m \leq n^2/4$ and every prime divisor of m is less than or equal to n. Show that m divides $n!$.

Problem 9 Determine whether there exists a sequence $x_1, x_2, \ldots$ of distinct positive real numbers such that $x_{n+2} = \sqrt{x_{n+1}} - \sqrt{x_n}$ for all positive integers n.

Problem 10 Let G be a graph with $n \geq 4$ vertices and m edges. If $m > n(\sqrt{4n-3}+1)/4$ show that G has a 4-cycle.

Problem 11 Suppose $f : \mathbb{Q} \to \{0, 1\}$ is a function with the property that for $x, y \in \mathbb{Q}$, if $f(x) = f(y)$ then $f(x) = f((x+y)/2) = f(y)$. If $f(0) = 0$ and $f(1) = 1$ show that $f(q) = 1$ for all rational numbers q greater than or equal to 1.

Problem 12 Let $n \geq 1$ be an integer. A *Catalan path* from $(0, 0)$ to (n, n) in the xy-plane is a sequence of unit moves either to the right (a move denoted by E) or upwards (a move denoted by N), where the path never crosses above the line $y = x$. A *step* in a Catalan path is the occurrence of two consecutive unit moves of the form EN. For $1 \leq s \leq n$, show that the number of Catalan paths from $(0, 0)$ to (n, n) that contain exactly s steps is

$$\frac{1}{s}\binom{n-1}{s-1}\binom{n}{s-1}.$$

2.10 Iran

Problem 1 Does there exist a natural number N which is a power of 2 whose digits (in base 10) can be permuted to form a different power of 2?

Problem 2 Call two circles in three-dimensional space *pairwise tangent* at a point P if they both pass through P and the lines tangent to each circle at P coincide. Three circles not all lying in a plane are pairwise tangent at three distinct points. Prove that there exists a sphere which passes through the three circles.

Problem 3 We are given a sequence $c_1, c_2, \ldots$ of natural numbers. For any natural numbers m, n with $1 \le m \le \sum_{i=1}^{n} c_i$, we can choose natural numbers $a_1, a_2, \ldots, a_n$ such that

$$m = \sum_{i=1}^{n} \frac{c_i}{a_i}.$$

For each n, find the maximum value of c_n.

Problem 4 Circles C_1 and C_2 with centers O_1 and O_2, respectively, meet at points A and B. Lines O_1B and O_2B intersect C_2 and C_1 at F and E, respectively. The line parallel to EF through B meets C_1 and C_2 at M and N. Prove $MN = AE + AF$.

Problem 5 Two triangles ABC and $A'B'C'$ lie in three-dimensional space. The sides of triangle ABC have lengths greater than or equal to a, and the sides of triangle $A'B'C'$ have lengths greater than or equal to a'. Prove that one can select one vertex from triangle ABC and one vertex from triangle $A'B'C'$ such that the distance between them is at least $\sqrt{\frac{a^2+a'^2}{3}}$.

Problem 6 The function $f : \mathbb{N} \to \mathbb{N}$ is defined recursively with $f(1) = 1$ and

$$f(n+1) = \begin{cases} f(n+2) & \text{if } n = f(f(n) - n + 1) \\ f(n) + 1 & \text{otherwise.} \end{cases}$$

for all $n \ge 1$.

(a) Prove that $f(f(n) - n + 1) \in \{n, n+1\}$.

(b) Find an explicit formula for f.

Problem 7 Let H equal $\{(x, y) \mid y > 0\}$, the upper half of the xy-plane. A *semi-line* is a curve in H which equals $C \cap H$ for some circle C centered on the x-axis; in other words, it is any semi-circle whose "center" is on the x-axis, with its endpoints removed. Let the *interior* of a semi-line S denote the set of points in the interior of the corresponding circle C which are also in H. Given two semi-lines S_1 and S_2 which intersect at a point A, the tangents to S_1 and S_2 at A form an angle α. Then the *bisector* of S_1 and S_2 is the semi-line S_3 passing through A such that the tangent to S_3 at A bisects α and passes through the region common to the interiors of S_1 and S_2. Prove that if three different semi-lines intersect pairwise, then the bisectors of the three pairs of semi-lines pass through a common point.

Problem 8 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

(i) $f(m) = 1$ if and only if $m = 1$;

(ii) if $d = \gcd(m, n)$, then $f(mn) = \frac{f(m)f(n)}{f(d)}$; and

(iii) for every $m \in \mathbb{N}$, we have $f^{2000}(m) = m$.

Problem 9 On a circle are given n points, and $nk+1$ of the chords between these points are drawn where $2k + 1 < n$. Prove that it is possible to select $k+1$ of the chords such that no two of them intersect.

Problem 10 The n tennis players, $A_1, A_2, \ldots, A_n$, participate in a tournament. Any two players play against each other at most once, and $k \leq \frac{n(n-1)}{2}$ matches take place. No draws occur, and in each match the winner adds 1 point to his tournament score while the loser adds 0. For nonnegative integers $d_1, d_2, \ldots, d_n$, prove that it is possible for $A_1, A_2, \ldots, A_n$ to obtain the tournament scores $d_1, d_2, \ldots, d_n$, respectively, if and only if the following conditions are satisfied:

(i) $\sum_{i=1}^{n} d_i = k$.

(ii) For every subset $X \subseteq \{A_1, \ldots, A_n\}$, the number of matches taking place among the players in X is at most $\sum_{A_j \in X} d_j$.

Problem 11 Isosceles triangles $A_3A_1O_2$ and $A_1A_2O_3$ are constructed externally along the sides of a triangle $A_1A_2A_3$ with $O_2A_3 = O_2A_1$ and $O_3A_1 = O_3A_2$. Let O_1 be a point on the opposite side of line A_2A_3 as A_1 with $\angle O_1A_3A_2 = \frac{1}{2}\angle A_1O_3A_2$ and $\angle O_1A_2A_3 = \frac{1}{2}\angle A_1O_2A_3$, and let T be the foot of the perpendicular from O_1 to $\overline{A_2A_3}$. Prove that $A_1O_1 \perp O_2O_3$ and that $\frac{A_1O_1}{O_2O_3} = 2\frac{O_1T}{A_2A_3}$.

Problem 12 Given a circle Γ, a line d is drawn not intersecting Γ. M, N are two points varying on line d such that the circle with diameter $\overline{MN}$ is tangent to Γ. Prove that there exists a point P in the plane such that for any such segment MN, $\angle MPN$ is constant.

Problem 13 Let n be a positive integer. S is a set containing ordered n-tuples of nonnegative integers such that if $(a_1, \ldots, a_n) \in S$, then every $(b_1, \ldots, b_n)$ for which $b_i \leq a_i$ $(1 \leq i \leq n)$ is also in S. Let $h_m(S)$ be the number of n-tuples in S whose sum of components equals m. Show that for some N, h_m is a polynomial in m for all $m \geq N$.

Problem 14 Suppose that a, b, c are real numbers such that for any positive real numbers $x_1, x_2, \ldots, x_n$, we have

$$\left(\frac{\sum_{i=1}^n x_i}{n}\right)^a \cdot \left(\frac{\sum_{i=1}^n x_i^2}{n}\right)^b \cdot \left(\frac{\sum_{i=1}^n x_i^3}{n}\right)^c \geq 1.$$

Prove that the vector (a, b, c) can be represented in the form $p(-2, 1, 0) + q(1, -2, 1)$ for nonnegative real numbers p and q.

2.11 Ireland

Problem 1 The sequence of real numbers $a_1, a_2, \ldots, a_n$ is called a *weak arithmetic progression* of length n if there exist real numbers c and d such that $c + (k-1)d \le a_k < c + kd$ for $k = 1, 2, \ldots, n$.

(a) Prove that if $a_1 < a_2 < a_3$ then a_1, a_2, a_3 is a weak arithmetic progression of length 3.

(b) Let A be a subset of $\{0, 1, 2, 3, \ldots, 999\}$ with at least 730 members. Prove that some ten elements of A form a weak arithmetic progression of length 10.

Problem 2 Let $x \ge 0$, $y \ge 0$ be real numbers with $x + y = 2$. Prove that

$$x^2y^2(x^2 + y^2) \le 2.$$

Problem 3 For each positive integer n, determine with proof all positive integers m such that there exist positive integers $x_1 < x_2 < \cdots < x_n$ with $\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} + \cdots + \frac{n}{x_n} = m$.

Problem 4 Prove that in each set of ten consecutive integers there is one which is relatively prime to each of the other integers.

Problem 5 Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with non-negative real coefficients. Suppose that $p(4) = 2$ and that $p(16) = 8$. Prove that $p(8) \le 4$ and find with proof all such polynomials with $p(8) = 4$.

2.12 Israel

Problem 1 Define $f(n) = n!$. Let

$$a = 0.f(1)f(2)f(3)\ldots.$$

In other words, to obtain the decimal representation of a write the decimal representations of $f(1), f(2), f(3), \ldots$ in a row. Is a rational?

Problem 2 ABC is a triangle whose vertices are lattice points. Two of its sides have lengths which belong to the set $\{\sqrt{17}, \sqrt{1999}, \sqrt{2000}\}$. What is the maximum possible area of triangle ABC?

Problem 3

(a) Do there exist three positive integers a, b, d such that

$$\frac{a^3+b^3}{a^3+d^3} = \frac{2000}{1999}?$$

(b) Do there exist four positive integers a, b, c, d such that

$$\frac{a^3+b^3}{c^3+d^3} = \frac{2000}{999}?$$

Problem 4 The points A, B, C, D, E, F lie on a circle, and the lines AD, BE, CF concur. Let P, Q, R be the midpoints of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$, respectively. Two chords AG, AH are drawn such that $AG \parallel BE$ and $AH \parallel CF$. Prove that triangles PQR and DGH are similar.

Problem 5 A square $ABCD$ is given. A *triangulation* of the square is a partition of the square into triangles such that any two triangles are either disjoint, share only a common vertex, or share only a common side. A *good triangulation* of the square is a triangulation in which all the triangles are acute.

(a) Give an example of a good triangulation of the square.

(b) What is the minimal number of triangles required for a good triangulation?

2.13 Italy

Problem 1 Three odd numbers $a < b < c$ are called *consecutive* if $c - b = b - a = 2$. A positive integer is called *special* if its digits in base 10 are all equal and if it is the sum of the squares of three consecutive odd integers.

(a) Determine all special numbers with 4 digits.

(b) Are there any special numbers with 2000 digits?

Problem 2 Let $ABCD$ be a convex quadrilateral, and write $\alpha = \angle DAB$; $\beta = \angle ADB$; $\gamma = \angle ACB$; $\delta = \angle DBC$; and $\epsilon = \angle DBA$. Assuming that $\alpha < 90^\circ$, $\beta + \gamma = 90^\circ$, and $\delta + 2\epsilon = 180^\circ$, prove that

$$(DB + BC)^2 = AD^2 + AC^2.$$

Problem 3 Given a fixed integer $n > 1$, Alberto and Barbara play the following game, starting with the first step and then alternating between the second and third:

- Alberto chooses a positive integer.
- Barbara picks an integer greater than 1 which is a multiple or divisor of Alberto's number, possibly choosing Alberto's number itself.
- Alberto adds or subtracts 1 from Barbara's number.

Barbara wins if she succeeds in picking n by her fiftieth move. For which values of n does she have a winning strategy?

Problem 4 Let $p(x)$ be a polynomial with integer coefficients such that $p(0) = 0$ and $0 \le p(1) \le 10^7$, and such that there exist integers a, b satisfying $p(a) = 1999$ and $p(b) = 2001$. Determine the possible values of $p(1)$.

2.14 Japan

Problem 1 Let O be the origin $(0,0)$ and A be the point $(0, \frac{1}{2})$ in the coordinate plane. Prove there is no finite sequence of points $P_1, P_2, \ldots, P_n$ in the plane, each of whose x- and y- coordinates are both rational numbers, such that

$$OP_1 = P_1P_2 = P_2P_3 = \cdots = P_{n-1}P_n = P_nA = 1.$$

Problem 2 We *shuffle* a line of cards labeled $a_1, a_2, \ldots, a_{3n}$ from left to right by rearranging the cards into the new order

$$a_3, a_6, \ldots, a_{3n}, a_2, a_5, \ldots, a_{3n-1}, a_1, a_4, \cdots, a_{3n-2}.$$

For example, if six cards are labeled $1, 2, \ldots, 6$ from left to right, then shuffling them twice changes their order as follows:

$$1,2,3,4,5,6 \longrightarrow 3,6,2,5,1,4 \longrightarrow 2,4,6,1,3,5.$$

Starting with 192 cards labeled $1, 2, \ldots, 192$ from left to right, is it possible to obtain the order $192, 191, \ldots, 1$ after a finite number of shuffles?

Problem 3 In the plane are given distinct points A, B, C, P, Q, no three of which are collinear. Prove that

$$AB + BC + CA + PQ < AP + AQ + BP + BQ + CP + CQ.$$

Problem 4 Given a natural number $n \geq 3$, prove that there exists a set A_n with the following two properties:

(i) A_n consists of n distinct natural numbers.

(ii) For any $a \in A_n$, the product of all the other elements in A_n has remainder 1 when divided by a.

Problem 5 We are given finitely many lines in the plane. Let an *intersection point* be a point where at least two of these lines meet, and let a *good intersection point* be a point where exactly two of these lines meet. Given that there are at least two intersection points, find the minimum number of good intersection points.

2.15 Korea

Problem 1 Show that given any prime p, there exist integers x, y, z, w satisfying $x^2 + y^2 + z^2 - wp = 0$ and $0 < w < p$.

Problem 2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 - y^2) = (x - y)\,(f(x) + f(y))$$

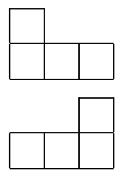
for all $x, y \in \mathbb{R}$.

Problem 3 For a quadrilateral $ABCD$ inscribed in a circle with center O, let P, Q, R, S be the intersections of the exterior angle bisectors of $\angle ABD$ and $\angle ADB$, $\angle DAB$ and $\angle DBA$, $\angle ACD$ and $\angle ADC$, $\angle DAC$ and $\angle DCA$, respectively. Show that the four points P, Q, R, S are concyclic.

Problem 4 Let p be a prime number such that $p \equiv 1 \pmod 4$. Evaluate

$$\sum_{k=1}^{p-1} \left(\left\lfloor \frac{2k^2}{p} \right\rfloor - 2 \left\lfloor \frac{k^2}{p} \right\rfloor \right).$$

Problem 5 Consider the following L-shaped figures, each made of four unit squares:



Let m and n be integers greater than 1. Prove that an $m \times n$ rectangular region can be tiled with such figures if and only if mn is a multiple of 8.

Problem 6 The real numbers a, b, c, x, y, z satisfy $a \geq b \geq c > 0$ and $x \geq y \geq z > 0$. Prove that

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)}$$

is at least $\frac{3}{4}$.

2.16 Lithuania

Problem 1 In the triangle ABC, D is the midpoint of side AB. Point E divides $\overline{BC}$ in the ratio $BE : EC = 2 : 1$. Given that $\angle ADC = \angle BAE$, determine $\angle BAC$.

Problem 2 A competition consisting of several tests has been organized for the three pilots K, L, and M, including a reaction-time test and a running test. In each test, no ties can occur; the first-place pilot in the test is awarded A points, the second-place pilot B points, and the third-place pilot C points for some fixed positive integers $A > B > C$. During the competition, K scores 22 points, and L and M each gather 9 points. If L won the reaction-time test, who took second place in the running test?

Problem 3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equality

$$(x + y)(f(x) - f(y)) = f(x^2) - f(y^2)$$

for all $x, y \in \mathbb{R}$.

Problem 4 Prove that infinitely many 4-tuples (x, y, z, u) of positive integers satisfy the equation $x^2 + y^2 + z^2 + u^2 = xyzu + 6$.

2.17 Mongolia

Problem 1 Let $\mathrm{rad}(1) = 1$, and for $k > 1$ let $\mathrm{rad}(k)$ equal the product of the prime divisors of k. A sequence of natural numbers $a_1, a_2, \ldots$ with arbitrary first term a_1 is defined recursively by the relation $a_{n+1} = a_n + \mathrm{rad}(a_n)$. Show that for any positive integer N, the sequence $a_1, a_2, \ldots$ contains some N consecutive terms in arithmetic progression.

Problem 2 The circles $\omega_1, \omega_2, \omega_3$ in the plane are pairwise externally tangent to each other. Let P_1 be the point of tangency between circles ω_1 and ω_3, and let P_2 be the point of tangency between circles ω_2 and ω_3. A and B, both different from P_1 and P_2, are points on ω_3 such that $\overline{AB}$ is a diameter of ω_3. Line AP_1 intersects ω_1 again at X, line BP_2 intersects ω_2 again at Y, and lines AP_2 and BP_1 intersect at Z. Prove that X, Y, and Z are collinear.

Problem 3 A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(i) $|f(a) - f(b)| \le |a - b|$ for any real numbers $a, b \in \mathbb{R}$.

(ii) $f(f(f(0))) = 0$.

Prove that $f(0) = 0$.

Problem 4 Given a natural number n, find the number of quadruples (x, y, u, v) of natural numbers such that the eight numbers x, y, u, v, $v + x - y$, $x + y - u$, $u + v - y$, and $v + x - u$ are all integers between 1 and n inclusive.

Problem 5 The bisectors of angles A, B, C of a triangle ABC intersect its sides at points A_1, B_1, C_1. Prove that if the quadrilateral $BA_1B_1C_1$ is cyclic, then

$$\frac{BC}{AC + AB} = \frac{AC}{AB + BC} - \frac{AB}{BC + AC}.$$

Problem 6 Which integers can be represented in the form $\frac{(x+y+z)^2}{xyz}$ where x, y, and z are positive integers?

Problem 7 In a country with n towns the cost of travel from the i-th town to the j-th town is x_{ij}. Suppose that the total cost of any route passing through each town exactly once and ending at its starting point does not depend on which route is chosen. Prove that there exist numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that $x_{ij} = a_i + b_j$ for all integers i, j with $1 \le i < j \le n$.

2.18 Poland

Problem 1 Given an integer $n \geq 2$ find the number of solutions of the system of equations

$$\begin{aligned} x_1 + x_n^2 &= 4x_n \\ x_2 + x_1^2 &= 4x_1 \\ &\vdots \\ x_n + x_{n-1}^2 &= 4x_{n-1} \end{aligned}$$

in nonnegative reals $x_1, x_2, \ldots, x_n$.

Problem 2 The sides AC and BC of a triangle ABC have equal length. Let P be a point inside triangle ABC such that $\angle PAB = \angle PBC$ and let M be the midpoint of $\overline{AB}$. Prove that $\angle APM + \angle BPC = 180°$.

Problem 3 A sequence $p_1, p_2, \ldots$ of prime numbers satisfies the following condition: for $n \geq 3$, p_n is the greatest prime divisor of $p_{n-1} + p_{n-2} + 2000$. Prove that the sequence is bounded.

Problem 4 For an integer $n \geq 3$ consider a pyramid with vertex S and the regular n-gon $A_1A_2\ldots A_n$ as a base, such that all the angles between lateral edges and the base equal $60°$. Points $B_2, B_3, \ldots$ lie on $\overline{A_2S}, \overline{A_3S}, \ldots, \overline{A_nS}$, respectively, such that $A_1B_2 + B_2B_3 + \cdots + B_{n-1}B_n + B_nA_1 < 2A_1S$. For which n is this possible?

Problem 5 Given a natural number $n \geq 2$, find the smallest integer k with the following property: Every set consisting of k cells of an $n \times n$ table contains a nonempty subset S such that in every row and in every column of the table, there is an even number of cells belonging to S.

Problem 6 Let P be a polynomial of odd degree satisfying the identity

$$P(x^2 - 1) = P(x)^2 - 1.$$

Prove that $P(x) = x$ for all real x.

2.19 Romania

Problem 1 The sequence $x_1, x_2, \ldots$ is defined recursively by setting $x_1 = 3$ and setting $x_{n+1} = \lfloor x_n\sqrt{2} \rfloor$ for every $n \geq 1$. Find all n for which x_n, x_{n+1}, and x_{n+2} are in arithmetic progression.

Problem 2 Two nonzero complex numbers a and b satisfy

$$a \cdot 2^{|a|} + b \cdot 2^{|b|} = (a+b) \cdot 2^{|a+b|}.$$

Prove that $a^6 = b^6$.

Problem 3 Let f be a third-degree polynomial with rational coefficients, having roots x_1, x_2, and x_3. Prove that if there exist nonzero rational numbers a and b such that $ax_1 + bx_2$ is rational, then x_3 is also a rational number.

Problem 4 A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called *olympic* if it has the following property: given $n \geq 3$ distinct points A_1, A_2, $\ldots$, $A_n \in \mathbb{R}^2$, if $f(A_1) = f(A_2) = \cdots = f(A_n)$ then the points A_1, A_2, $\ldots$, A_n are the vertices of a convex polygon. Let $P \in \mathbb{C}[X]$ be a non-constant polynomial. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = |P(x + iy)|$, is olympic if and only if all the roots of P are equal.

Problem 5 Let $n \geq 2$ be a positive integer. Find the number of functions $f : \{1, 2, \ldots, n\} \to \{1, 2, 3, 4, 5\}$ which have the following property: $|f(k+1) - f(k)| \geq 3$ for $k = 1, 2, \ldots, n-1$.

Problem 6 Let $n \geq 1$ be a positive integer and x_1, x_2, $\ldots$, x_n be real numbers such that $|x_{k+1} - x_k| \leq 1$ for $k = 1, 2, \ldots, n-1$. Show that

$$\sum_{k=1}^{n} |x_k| - \left| \sum_{k=1}^{n} x_k \right| \leq \frac{n^2 - 1}{4}.$$

Problem 7 Let n, k be arbitrary positive integers. Show that there exist positive integers $a_1 > a_2 > a_3 > a_4 > a_5 > k$ such that

$$n = \pm\binom{a_1}{3} \pm \binom{a_2}{3} \pm \binom{a_3}{3} \pm \binom{a_4}{3} \pm \binom{a_5}{3},$$

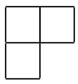
where $\binom{a}{3} = \frac{a(a-1)(a-2)}{6}$.

Problem 8 Let $P_1P_2\cdots P_n$ be a convex polygon in the plane. Assume that for any pair of vertices P_i, P_j, there exists a vertex V of the polygon such that $\angle P_iVP_j = 60^\circ$. Show that $n = 3$.

Problem 9 Show that there exist infinitely many 4-tuples of positive integers (x, y, z, t) such that the four numbers' greatest common divisor is 1 and such that

$$x^3 + y^3 + z^2 = t^4.$$

Problem 10 Consider the following figure, made of three unit squares:



Determine all pairs m, n of positive integers such that a $m \times n$ rectangle can be tiled with such pieces.

Problem 11 Find the least positive integer n such that for all odd integers a, 2^{2000} is a divisor of $a^n - 1$.

Problem 12 Let ABC be an acute triangle and let M be the midpoint of segment BC. Consider the interior point N such that $\angle ABN = \angle BAM$ and $\angle ACN = \angle CAM$. Prove that $\angle BAN = \angle CAM$.

Problem 13 Let $\mathcal{S}$ be the set of interior points of a unit sphere, and let $\mathcal{C}$ be the set of interior points of a unit circle. Find, with proof, whether there exists a function $f : \mathcal{S} \to \mathcal{C}$ such that the distance between $f(A)$ and $f(B)$ is greater than or equal to AB for all points A and B in $\mathcal{S}$.

Problem 14 Let $n \geq 3$ be an odd integer and $m \geq n^2 - n + 1$ be an integer. The sequence of polygons $P_1, P_2, \ldots, P_m$ is defined as follows:

(i) P_1 is a regular polygon with n vertices.

(ii) For $k > 1$, P_k is the regular polygon whose vertices are the midpoints of the sides of P_{k-1}.

Find, with proof, the maximum number of colors which can be used such that for every coloring of the vertices of these polygons, one can find four vertices A, B, C, D which have the same color and form an isosceles trapezoid.

Problem 15 Prove that if p and q are monic polynomials with complex coefficients such that $p(p(x)) = q(q(x))$, then $p(x)$ and $q(x)$ are equal.

2.20 Russia

Problem 1 Sasha tries to determine some positive integer $X \leq 100$. He can choose any two positive integers M and N that are less than 100 and ask the question, "What is the greatest common divisor of the numbers $X+M$ and N?" Prove that Sasha can determine the value of X after 7 questions.

Problem 2 Let I be the center of the incircle ω of an acute-angled triangle ABC. The circle ω_1 with center K passes through the points A, I, C and intersects sides AB and BC at points M and N. Let L be the reflection of K across line MN. Prove that $BL \perp AC$.

Problem 3 There are several cities in a state and a set of roads, each road connecting two cities. It is known that at least 3 roads go out of every city. Prove that there exists a cyclic path (that is, a path where the last road ends where the first road begins) such that the number of roads in the path is not divisible by 3.

Problem 4 Let $x_1, x_2, \ldots, x_n$ be real numbers, satisfying the conditions $-1 < x_1 < x_2 < \cdots < x_n < 1$ and

$$x_1^{13} + x_2^{13} + \cdots + x_n^{13} = x_1 + x_2 + \cdots + x_n.$$

Prove that

$$x_1^{13}y_1 + x_2^{13}y_2 + \cdots + x_n^{13}y_n < x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

for any real numbers $y_1 < y_2 < \cdots < y_n$.

Problem 5 Let $\overline{AA_1}$ and $\overline{CC_1}$ be the altitudes of an acute-angled nonisosceles triangle ABC. The bisector of the acute angle between lines AA_1 and CC_1 intersects sides AB and BC at P and Q, respectively. Let H be the orthocenter of triangle ABC and let M be the midpoint of $\overline{AC}$; and let the bisector of $\angle ABC$ intersect $\overline{HM}$ at R. Prove that $PBQR$ is cyclic.

Problem 6 Five stones which appear identical all have different weights; Oleg knows the weight of each stone. Given any stone x, let $m(x)$ denote its weight. Dmitrii tries to determine the order of the weights of the stones. He is allowed to choose any three stones A, B, C and ask Oleg the question, "Is it true that $m(A) < m(B) < m(C)$?" Oleg then responds "yes" or "no." Can Dmitrii determine the order of the weights with at most nine questions?

Problem 7 Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(x+y) + f(y+z) + f(z+x) \geq 3f(x+2y+3z)$$

for all $x, y, z \in \mathbb{R}$.

Problem 8 Prove that the set of all positive integers can be partitioned into 100 nonempty subsets such that if three positive integers a, b, c satisfy $a + 99b = c$, then two of them belong to the same subset.

Problem 9 Let $ABCDE$ be a convex pentagon on the coordinate plane. Each of its vertices are lattice points. The five diagonals of $ABCDE$ form a convex pentagon $A_1B_1C_1D_1E_1$ inside of $ABCDE$. Prove that this smaller pentagon contains a lattice point on its boundary or within its interior.

Problem 10 Let $a_1, a_2, \ldots, a_n$ be a sequence of nonnegative real numbers. For $1 \leq k \leq n$, let

$$m_k = \max_{1 \leq i \leq k} \frac{a_{k-i+1} + a_{k-i+2} + \cdots + a_k}{i}.$$

Prove that for any $\alpha > 0$, the number of integers k which satisfy $m_k > \alpha$ is less than $\frac{a_1+a_2+\cdots+a_n}{\alpha}$.

Problem 11 Let $a_1, a_2, a_3, \ldots$ be a sequence with $a_1 = 1$ satisfying the recursion

$$a_{n+1} = \begin{cases} a_n - 2 & \text{if } a_n - 2 \notin \{a_1, a_2, \ldots, a_n\} \text{ and } a_n - 2 > 0 \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Prove that for every positive integer k, we have $a_n = k^2 = a_{n-1} + 3$ for some n.

Problem 12 There are black and white checkers on some squares of a $2n \times 2n$ board, with at most one checker on each square. First, we remove every black checker that is in the same column as any white checker. Next, we remove every white checker that is in the same row as any remaining black checker. Prove that for some color, at most n^2 checkers of this color remain.

Problem 13 Let E be a point on the median CD of triangle ABC. Let S_1 be the circle passing through E and tangent to line AB at A, intersecting side AC again at M; let S_2 be the circle passing through E and tangent to line AB at B, intersecting side BC again at N.

Prove that the circumcircle of triangle CMN is tangent to circles S_1 and S_2.

Problem 14 One hundred positive integers, with no common divisor greater than one, are arranged in a circle. To any number, we can add the greatest common divisor of its neighboring numbers. Prove that using this operation, we can transform these numbers into a new set of pairwise coprime numbers.

Problem 15 M is a finite set of real numbers such that given three distinct elements from M, we can choose two of them whose sum also belongs to M. What is the largest number of elements that M can have?

Problem 16 A positive integer n is called *perfect* if the sum of all its positive divisors, excluding n itself, equals n. For example, 6 is perfect since $6 = 1 + 2 + 3$. Prove that

(a) if a perfect number larger than 6 is divisible by 3, then it is also divisible by 9.

(b) if a perfect number larger than 28 is divisible by 7, then it is also divisible by 49.

Problem 17 Circles ω_1 and ω_2 are internally tangent at N. The chords BA and BC of ω_1 are tangent to ω_2 at K and M, respectively. Let Q and P be the midpoints of the arcs AB and BC not containing the point N. Let the circumcircles of triangles BQK and BPM intersect at B and B_1. Prove that BPB_1Q is a parallelogram.

Problem 18 There is a finite set of congruent square cards, placed on a rectangular table with their sides parallel to the sides of the table. Each card is colored in one of k colors. For any k cards of different colors, it is possible to pierce some two of them with a single pin. Prove that all the cards can be pierced by $2k - 2$ pins.

Problem 19 Prove the inequality

$$\sin^n(2x) + (\sin^n x - \cos^n x)^2 \leq 1.$$

Problem 20 The circle ω is inscribed in the quadrilateral $ABCD$, and O is the intersection point of the lines AB and CD. The circle ω_1 is tangent to side BC at K and is tangent to lines AB and CD at points lying outside $ABCD$; the circle ω_2 is tangent to side AD

at L and is also tangent to lines AB and CD at points lying outside $ABCD$. If O, K, L are collinear, prove that the midpoint of side BC, the midpoint of side AD, and the center of ω are collinear.

Problem 21 Every cell of a 100×100 board is colored in one of 4 colors so that there are exactly 25 cells of each color in every column and in every row. Prove that one can choose two columns and two rows so that the four cells where they intersect are colored in four different colors.

Problem 22 The non-zero real numbers a, b satisfy the equation

$$a^2b^2(a^2b^2 + 4) = 2(a^6 + b^6).$$

Prove that a and b are not both rational.

Problem 23 In a country, each road either connects two towns or starts from a town and goes out of the country. Some of the roads are colored in one of three colors. For every town, exactly three of the roads that go out of this town are colored, and the colors of these roads are different. If exactly three of the colored roads go out of the country, prove that the colors of these roads are different.

Problem 24 Find the smallest odd integer n such that some n-gon (not necessarily convex) can be partitioned into parallelograms whose interiors do not overlap.

Problem 25 Two pirates divide their loot, consisting of two sacks of coins and one diamond. They decide to use the following rules. On each turn, one pirate chooses a sack and takes $2m$ coins from it, keeping m for himself and putting the rest into the other sack. The pirates alternate taking turns until no more moves are possible; the pirate who makes the last move takes the diamond. For what initial amounts of coins can the first pirate guarantee that he will obtain the diamond?

Problem 26 The coefficients a and b of an equation $x^2 + ax + b = 0$ and its roots c and d are four different numbers. Given a, b, c, d in some order, is it possible to determine which is a and which is b?

Problem 27 Do there exist coprime integers $a, b, c > 1$ such that $2^a + 1$ is divisible by b, $2^b + 1$ is divisible by c, and $2^c + 1$ is divisible by a?

Problem 28 $2n + 1$ segments are marked on a line. Each of the segments intersects at least n other segments. Prove that one of these segments intersects all the other segments.

Problem 29 The circles S_1 and S_2 intersect at points M and N. Let A and D be points on S_1 and S_2 such that lines AM and AN intersect S_2 at B and C, lines DM and DN intersect S_1 at E and F, and the triples A, E, F and D, B, C lie on opposite sides of line MN. Prove that there is a fixed point O such that for any points A and D that satisfy the condition $AB = DE$, the quadrilateral $AFCD$ is cyclic.

Problem 30 Let the set M consist of the 2000 numbers $10^1 + 1, 10^2 + 1, \ldots, 10^{2000} + 1$. Prove that at least 99% of the elements of M are not prime.

Problem 31 There are 2 counterfeit coins among 5 coins that look identical. Both counterfeit coins have the same weight and the other three real coins have the same weight. The five coins do not all weight the same, but it is unknown whether the weight of each counterfeit coin is more or less than the weight of each real coin. Find the minimal number of weighings needed to find at least one real coin, and describe how to do so. (The balance scale reports the difference between the weights of the objects in two pans.)

Problem 32 Let $ABCD$ be a parallelogram with $\angle A = 60^\circ$. Let O be the circumcenter of triangle ABD. Line AO intersects the external angle bisector of angle BCD at K. Find the value $\frac{AO}{OK}$.

Problem 33 Find the smallest integer n such that an $n \times n$ square can be partitioned into 40×40 and 49×49 squares, with both types of squares present in the partition.

Problem 34 Prove that there exist 10 distinct real numbers a_1, a_2, $\ldots$, a_{10} such that the equation

$$(x - a_1)(x - a_2) \cdots (x - a_{10}) = (x + a_1)(x + a_2) \cdots (x + a_{10})$$

has exactly 5 different real roots.

Problem 35 We are given a cylindrical region in space, whose altitude is 1 and whose base has radius 1. Find the minimal number of balls of radius 1 needed to cover this region.

Problem 36 The sequence $a_1, a_2, \ldots, a_{2000}$ of real numbers satisfies the condition

$$a_1^3 + a_2^3 + \cdots + a_n^3 = (a_1 + a_2 + \cdots + a_n)^2$$

for all n, $1 \le n \le 2000$. Prove that every element of the sequence is an integer.

(The balance scale reports the difference between the weights of the objects in two pans.)

Problem 37 The bisectors $\overline{AD}$ and $\overline{CE}$ of a triangle ABC intersect at I. Let ℓ_1 be the reflection of line AB across line CE, and let ℓ_2 be the reflection of line BC across line AD. If lines ℓ_1 and ℓ_2 intersect at K, prove that $KI \perp AC$.

Problem 38 There are 2000 cities in a country, some pairs of which are connected by a direct airplane flight. For every city A the number of cities connected with A by direct flights equals 1, 2, 4, ..., or 1024. Let $S(A)$ be the number of routes from A to other cities (different from A) with at most one intermediate landing. Prove that the sum of $S(A)$ over all 2000 cities A cannot be equal to 10000.

Problem 39 A heap of balls consists of one thousand 10-gram balls and one thousand 9.9-gram balls. We wish to pick out two heaps of balls with equal numbers of balls in them but different total weights. What is the minimal number of weighings needed to do this?

Problem 40 Let D be a point on side AB of triangle ABC. The circumcircle of triangle BCD intersects line AC at C and M, and the circumcircle of triangle CMN intersects line BC at C and N. Let O be the center of the circumcircle of triangle CMN. Prove that $OD \perp AB$.

Problem 41 Every cell of a 200×200 table is colored black or white. The difference between the number of black and white cells is 404. Prove that some 2×2 square contains an odd number of white cells.

Problem 42 Is there a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x + y) + \sin x + \sin y| < 2$$

for all $x, y \in \mathbb{R}$?

Problem 43 For any integer $a_0 > 5$, consider the sequence $a_0, a_1, a_2, \ldots,$ where

$$a_{n+1} = \begin{cases} a_n^2 - 5 & \text{if } a_n \text{ is odd} \\ \frac{a_n}{2} & \text{if } a_n \text{ is even} \end{cases}$$

for all $n \geq 0$. Prove that this sequence is not bounded.

Problem 44 Let ℓ_a, ℓ_b, ℓ_c, and ℓ_d be the external angle bisectors of angles DAB, ABC, BCD, and CDA, respectively. The pairs of lines ℓ_a and ℓ_b, ℓ_b and ℓ_c, ℓ_c and ℓ_d, ℓ_d and ℓ_a intersect at points K, L, M, N, respectively. Suppose that the perpendiculars to line AB passing through K, to line BC passing through L, and to line CD passing through M are concurrent. Prove that $ABCD$ can be inscribed in a circle.

Problem 45 There are 2000 cities in a country, and each pair of cities is connected by either no roads or exactly one road. A *cyclic path* is a collection of roads such that each city is at the end of either 0 or 2 roads in the path. For every city, there at most N cyclic paths which both pass through this city and contain an odd number of roads. Prove that the country can be separated into $2N+2$ republics such that any two cities from the same republic are not connected by a road.

Problem 46 Prove the inequality

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \leq \frac{2}{\sqrt{1+xy}}$$

for $0 \leq x, y \leq 1$.

Problem 47 The incircle of triangle ABC touches side AC at K. A second circle S with the same center intersects all the sides of the triangle. Let E and F be the intersection points on $\overline{AB}$ and $\overline{BC}$ closer to B; let B_1 and B_2 be the intersection points on $\overline{AC}$ with B_1 closer to A. Finally, let P be the intersection point of segments B_2E and B_1F. Prove that points B, K, P are collinear.

Problem 48 Each of the numbers $1, 2, \ldots, N$ is colored black or white. We are allowed to simultaneously change the colors of any three numbers in arithmetic progression. For which numbers N can we always make all the numbers white?

2.21 Taiwan

Problem 1 Find all possible pairs (x, y) of positive integers such that

$$y^{x^2} = x^{y+2}.$$

Problem 2 In an acute triangle ABC, $AC > BC$ and M is the midpoint of $\overline{AB}$. Let altitudes $\overline{AP}$ and $\overline{BQ}$ meet at H, and let lines AB and PQ meet at R. Prove that the two lines RH and CM are perpendicular.

Problem 3 Let $S = \{1, 2, \ldots, 100\}$, and let $\mathcal{P}$ denote the family of all 49-element subsets T of S. Each set T in $\mathcal{P}$ is labeled with some number from S. Show that there exists a 50-element subset M of S such that for each $x \in M$, the set $M \setminus \{x\}$ is not labeled with x.

Problem 4 Let $\phi(k)$ denote the number of positive integers n satisfying $\gcd(n, k) = 1$ and $n \leq k$. Suppose that $\phi(5^m - 1) = 5^n - 1$ for some positive integers m, n. Prove that $\gcd(m, n) > 1$.

Problem 5 Let $A = \{1, 2, \ldots, n\}$, where n is a positive integer. A subset of A is *connected* if it is a nonempty set which consists of one element or of consecutive integers. Determine the greatest integer k for which A contains k distinct subsets $A_1, A_2, \ldots, A_k$ such that the intersection of any two distinct sets A_i and A_j is connected.

Problem 6 Let $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be defined recursively by $f(1) = 0$ and

$$f(n) = \max_{1 \leq j \leq \lfloor \frac{n}{2} \rfloor} \{f(j) + f(n-j) + j\}$$

for all $n \geq 2$. Determine $f(2000)$.

2.22 Turkey

Problem 1 Find the number of ordered quadruples (x, y, z, w) of integers with $0 \leq x, y, z, w \leq 36$ such that

$$x^2 + y^2 \equiv z^3 + w^3 \pmod{37}.$$

Problem 2 Given a circle with center O, the two tangent lines from a point S outside the circle touch the circle at points P and Q. Line SO intersects the circle at A and B, with B closer to S. Let X be an interior point of minor arc PB, and let line OS intersect lines QX and PX at C and D, respectively. Prove that

$$\frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB}.$$

Problem 3 For any two positive integers n and p, prove that there are exactly $(p+1)^{n+1} - p^{n+1}$ functions

$$f : \{1, 2, \ldots, n\} \to \{-p, -p+1, \ldots, p\}$$

such that $|f(i) - f(j)| \leq p$ for all $i, j \in \{1, 2, \ldots, n\}$.

Problem 4 Find all sequences $a_1, a_2, \ldots, a_{2000}$ of real numbers such that $\sum_{n=1}^{2000} a_n = 1999$ and such that $\frac{1}{2} < a_n < 1$ and $a_{n+1} = a_n(2 - a_n)$ for all $n \geq 1$.

Problem 5 In an acute triangle ABC with circumradius R, altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ have lengths h_1, h_2, h_3, respectively. If t_1, t_2, t_3 are the lengths of the tangents from A, B, C, respectively, to the circumcircle of triangle DEF, prove that

$$\sum_{i=1}^{3} \left(\frac{t_i}{\sqrt{h_i}} \right)^2 \leq \frac{3}{2} R.$$

Problem 6

(a) Prove that for each positive integer n, the number of ordered pairs (x, y) of integers satisfying

$$x^2 - xy + y^2 = n$$

is finite and divisible by 6.

(b) Find all ordered pairs (x, y) of integers satisfying

$$x^2 - xy + y^2 = 727.$$

Problem 7 Given a triangle ABC, the internal and external bisectors of angle A intersect line BC at points D and E, respectively. Let F be the point (different from A) where line AC intersects the circle ω with diameter $\overline{DE}$. Finally, draw the tangent at A to the circumcircle of triangle ABF, and let it hit ω at A and G. Prove that $AF = AG$.

Problem 8 Let $P(x) = x + 1$ and $Q(x) = x^2 + 1$. We form all sequences of ordered pairs $(x_1, y_1), (x_2, y_2), \ldots$ with $(x_1, y_1) = (1, 3)$ and

$$(x_{k+1}, y_{k+1}) \in \{(P(x_k), Q(y_k)),\ (Q(x_k), P(y_k))\}$$

for each positive integer k. Find all positive integers n such that $x_n = y_n$ in at least one of these sequences.

Problem 9 Show that it is possible to cut any triangular prism of infinite length with a plane such that the resulting intersection is an equilateral triangle.

Problem 10 Given a square $ABCD$, the points M, N, K, L are chosen on sides AB, BC, CD, DA, respectively, such that lines MN and LK are parallel and such that the distance between lines MN and LK equals AB. Show that the circumcircles of triangles ALM and NCK intersect each other, while those of triangles LDK and MBN do not.

Problem 11 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x + y) - f(x) - f(y)| \leq 1$ for all $x, y \in \mathbb{R}$. Show that there exists a function $g : \mathbb{R} \to \mathbb{R}$ with $|f(x) - g(x)| \leq 1$ for all $x \in \mathbb{R}$, and with $g(x + y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$.

2.23 Ukraine

Problem 1 Let n numbers greater than 1 be given. During each step, we replace any two numbers a, b with the number $\frac{ab\sqrt{2}}{a+b}$. Prove that after $n-1$ steps, the remaining number is at least $\frac{1}{\sqrt{n}}$.

Problem 2 Acute triangle PNK is inscribed in a circle with diameter $\overline{NM}$. Let A be the intersection point of $\overline{MN}$ and $\overline{PK}$, and let H be a point on minor arc PN. The circumcircle of triangle PAH intersects lines MN and PN again at points B and D, respectively; the circle with diameter $\overline{BN}$ intersects lines PN and NK at points F and Q, respectively. Let C be the intersection point of lines MN and FQ, and let E be the intersection point different from D of line CD with the circumcircle of triangle PAH. Prove that the points H, E, N are collinear.

Problem 3 Let $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ be the altitudes of acute triangle ABC. Let A_2, B_2, C_2 be the tangency points of the incircle of triangle $A_1B_1C_1$ with sides B_1C_1, C_1A_1, A_1B_1, respectively. Prove that the lines AA_2, BB_2, CC_2 are concurrent.

Problem 4 Do there exist positive integers m, n such that $\frac{m^2+1}{n^2-1}$ is an integer?

2.24 United Kingdom

Problem 1 Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N. Prove that the triangles MNP and MNQ have equal areas.

Problem 2 Given that x, y, z are positive real numbers satisfying $xyz = 32$, find the minimum value of

$$x^2 + 4xy + 4y^2 + 2z^2.$$

Problem 3 Find positive integers a and b such that

$$\left(\sqrt[3]{a} + \sqrt[3]{b} - 1\right)^2 = 49 + 20\sqrt[3]{6}.$$

Problem 4

(a) Find a set A of ten positive integers such that no six distinct elements of A have a sum which is divisible by 6.

(b) Is it possible to find such a set if “ten” is replaced by “eleven”?

2.25 United States of America

Problem 1 Call a real-valued function f *very convex* if

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x-y|$$

holds for all real numbers x and y. Prove that no very convex function exists.

Problem 2 Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP}+\frac{1}{BQ}+\frac{1}{CR}\right) - \frac{3}{\min\{AP, BQ, CR\}} = \frac{6}{r},$$

where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA, respectively. Prove that all triangles in S are isosceles and similar to one another.

Problem 3 A game of solitaire is played with R red cards, W white cards, and B blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand. Find, as a function of R, W, and B, the minimal total penalty a player can amass and all the ways in which this minimum can be achieved.

Problem 4 Find the smallest positive integer n such that if n unit squares of a 1000×1000 unit-square board are colored, then there will exist three colored unit squares whose centers form a right triangle with legs parallel to the edges of the board.

Problem 5 Let $A_1A_2A_3$ be a triangle and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exist circles ω_2, ω_3, $\ldots$, ω_7 such that for $k = 2, 3, \ldots, 7$, ω_k is externally tangent to ω_{k-1} and passes through A_k and A_{k+1}, where $A_{n+3} = A_n$ for all $n \geq 1$. Prove that $\omega_7 = \omega_1$.

Problem 6 Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_i a_j, b_i b_j\} \le \sum_{i,j=1}^{n} \min\{a_i b_j, a_j b_i\}.$$

2.26 Vietnam

Problem 1 Given a real number $c > 2$, a sequence $x_1, x_2, \ldots$ of real numbers is defined recursively by $x_1 = 0$ and

$$x_{n+1} = \sqrt{c - \sqrt{c + x_n}}$$

for all $n \geq 1$. Prove that the sequence $x_1, x_2, \ldots$ is defined for all n and has a finite limit.

Problem 2 Two circles ω_1 and ω_2 are given in the plane, with centers O_1 and O_2, respectively. Let M_1' and M_2' be two points on ω_1 and ω_2, respectively, such that the lines O_1M_1' and O_2M_2' intersect. Let M_1 and M_2 be points on ω_1 and ω_2, respectively, such that when measured clockwise the angles $\angle M_1'OM_1$ and $\angle M_2'OM_2$ are equal.

(a) Determine the locus of the midpoint of $\overline{M_1M_2}$.

(b) Let P be the point of intersection of lines O_1M_1 and O_2M_2. The circumcircle of triangle M_1PM_2 intersects the circumcircle of triangle O_1PO_2 at P and another point Q. Prove that Q is fixed, independent of the locations of M_1 and M_2.

Problem 3 Given the polynomial

$$P(x) = x^3 - 9x^2 + 24x - 97,$$

prove that for each positive integer n there exists a positive integer a_n for which $P(a_n)$ is divisible by 3^n.

Problem 4 Given an angle $\alpha \in (0, \pi)$, find a quadratic polynomial of the form $f(x) = x^2 + ax + b$ such that for every $n \geq 3$, the polynomial

$$P_n(x) = x^n \sin\alpha - x \sin n\alpha + \sin(n-1)\alpha$$

is divisible by $f(x)$.

Problem 5 Suppose that all circumcircles of the four faces of a tetrahedron have congruent radii. Show that any two opposite edges of the tetrahedron are congruent.

Problem 6 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$x^2 f(x) + f(1-x) = 2x - x^4$$

for all $x \in \mathbb{R}$.

Problem 7 Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at two points P and Q. The common tangent of $\mathcal{C}_1$ and $\mathcal{C}_2$ closer to P than to Q touches $\mathcal{C}_1$ and $\mathcal{C}_2$ at A and B, respectively. The tangent to $\mathcal{C}_1$ at P intersects $\mathcal{C}_2$ at E (distinct from P) and the tangent to $\mathcal{C}_2$ at P intersects $\mathcal{C}_1$ at F (distinct from P). Let H and K be two points on the rays AF and BE, respectively, such that $AH = AP, BK = BP$. Prove that the five points A, H, Q, K, B lie on the same circle.

Problem 8 Given a positive integer k, let $x_1 = 1$ and define the sequence $x_1, x_2, \ldots$ of positive integers recursively as follows: for each integer $n \geq 1$, let x_{n+1} be the smallest positive integer not belonging to the set $\{x_1, x_2, \ldots, x_n, x_1 + k, x_2 + 2k, \ldots, x_n + nk\}$. Show that there exists a real number a such that

$$x_n = \lfloor an \rfloor$$

for all $n = 1, 2, \ldots$.

Problem 9 Let a, b, c be pairwise relatively prime positive integers. The positive integer n is said to be *stubborn* if it cannot be written in the form

$$n = bcx + cay + abz$$

for any positive integers x, y, z. Determine, as a function of a, b, and c, the number of stubborn integers.

Problem 10 Let $\mathbb{R}^+$ denote the set of positive real numbers, and let $a, r > 1$ be real numbers.

(a) Suppose that $f : \mathbb{R}^+ \to \mathbb{R}$ is a function satisfying the following conditions:

  (i) $f(f(x)) \leq ax^r f\left(\frac{x}{a}\right)$ for all $x > 0$.

  (ii) $f(x) < 2^{2000}$ for all $x < \dfrac{1}{2^{2000}}$.

  Prove that $f(x) \leq x^r a^{1-r}$ for all $x > 0$.

(b) Construct a function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying condition (i) such that $f(x) > x^r a^{1-r}$ for all $x > 0$.

3
2000 Regional Contests: Problems

3.1 Asian Pacific Mathematical Olympiad

Problem 1 Compute the sum

$$S = \sum_{i=0}^{101} \frac{x_i^3}{1 - 3x_i + 3x_i^2}$$

where $x_i = \frac{i}{101}$ for $i = 0, 1, \ldots, 101$.

Problem 2 We are given an arrangement of nine circular slots along three sides of a triangle: one slot at each corner, and two more along each side. Each of the numbers $1, 2, \ldots, 9$ is to be written into exactly one of these circles, so that

(i) the sums of the four numbers on each side of the triangle are equal;

(ii) the sums of the squares of the four numbers on each side of the triangle are equal.

Find all ways in which this can be done.

Problem 3 Let ABC be a triangle with median $\overline{AM}$ and angle bisector $\overline{AN}$. Draw the perpendicular to line NA through N, hitting lines MA and BA at Q and P, respectively. Also let O be the point where the perpendicular to line BA through P meets line AN. Prove that $QO \perp BC$.

Problem 4 Let n, k be positive integers with $n > k$. Prove that

$$\frac{1}{n+1} \cdot \frac{n^n}{k^k(n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k(n-k)^{n-k}}.$$

Problem 5 Given a permutation $(a_0, a_1, \ldots, a_n)$ of the sequence $0, 1, \ldots, n$, a transposition of a_i with a_j is called *legal* if $a_i = 0$, $i > 0$, and $a_{i-1} + 1 = a_j$. The permutation $(a_0, a_1, \ldots, a_n)$ is called *regular* if after finitely many legal transpositions it becomes $(1, 2, \ldots, n, 0)$. For which numbers n is the permutation $(1, n, n-1, \ldots, 3, 2, 0)$ regular?

3.2 Austrian-Polish Mathematics Competition

Problem 1 Determine all polynomials $P(x)$ with real coefficients such that for some positive integer n, the equality

$$\sum_{k=1}^{2n+1}(-1)^k\left\lfloor\frac{k}{2}\right\rfloor P(x+k)=0$$

holds for infinitely many real numbers x.

Problem 2 We are given a $1\times1\times1$ unit cube with opposite faces $ABCD$ and $EFGH$, where $\overline{AE}$, $\overline{BF}$, $\overline{CG}$, and $\overline{DH}$ are edges of the cube. X is a point on the incircle of square $ABCD$, and Y is a point on the circumcircle of triangle BDG. Find the minimum possible value of XY.

Problem 3 For each positive integer $n\geq 3$, find all n-tuples $(x_1,x_2,\ldots,x_n)$ of real numbers that satisfy the following system of equations:

$$\begin{aligned} x_n^3 &= x_1+x_2+1\\ x_1^3 &= x_2+x_3+1\\ &\vdots\\ x_{n-1}^3 &= x_n+x_1+1. \end{aligned}$$

Problem 4 Find all positive integers N whose only prime divisors are 2 and 5, such that the number $N+25$ is a perfect square.

Problem 5 For which integers $n\geq 5$ is it possible to color the vertices of a regular n-gon using at most 6 colors such that any 5 consecutive vertices have different colors?

Problem 6 Let the *3-cross* be the solid made up of one central unit cube with six other unit cubes attached to its faces, such as the solid made of the seven unit cubes centered at $(0,0,0)$, $(\pm1,0,0)$, $(0,\pm1,0)$, and $(0,0,\pm1)$. Prove or disprove that the space can be tiled with 3-crosses in such a way that no two of them share any interior points.

Problem 7 In the plane the triangle $A_0B_0C_0$ is given. Consider all triangles ABC satisfying the following conditions: (i) lines AB, BC, and CA pass through points C_0, A_0, and B_0, respectively; (ii) $\angle ABC = \angle A_0B_0C_0$, $\angle BCA = \angle B_0C_0A_0$, and $\angle CAB = \angle C_0A_0B_0$. Find the locus of the circumcenter of all such triangles ABC.

Problem 8 We are given a set of 27 distinct points in the plane, no three collinear. Four points from this set are vertices of a unit square; the other 23 points lie inside this square. Prove that there exist three distinct points X, Y, Z in this set such that $[XYZ] \leq \frac{1}{48}$.

Problem 9 For all real numbers $a, b, c \geq 0$ such that $a + b + c = 1$, prove that

$$2 \leq (1 - a^2)^2 + (1 - b^2)^2 + (1 - c^2)^2 \leq (1 + a)(1 + b)(1 + c)$$

and determine when equality occurs for each of the two inequalities.

3.3 Balkan Mathematical Olympiad

Problem 1 Let E be a point inside nonisosceles acute triangle ABC lying on median $\overline{AD}$, and drop perpendicular $\overline{EF}$ to line BC. Let M be an arbitrary point on segment EF, and let N and P be the orthogonal projections of M onto lines AC and AB, respectively. Prove that the angle bisectors of $\angle PMN$ and $\angle PEN$ are parallel.

Problem 2 Find the maximum number of $1 \times 10\sqrt{2}$ rectangles one can remove from a 50×90 rectangle by using cuts parallel to the edges of the original rectangle.

Problem 3 Call a positive integer r a *perfect power* if it is of the form $r = t^s$ for some integers s, t greater than 1. Show that for any positive integer n there exists a set S of n distinct perfect powers, such that for any nonempty subset T of S, the arithmetic mean of the elements in T is *also* a perfect power.

3.4 Czech-Slovak Match

Problem 1 A triangle ABC with incircle k is given. Circle k_a passes through B and C and is orthogonal to k; circles k_b and k_c are defined similarly. (Two circles are said to be *orthogonal* if they intersect and their tangents at any common point are perpendicular.) Let k_a and k_b intersect again at C', and define A' and B' similarly. Show that the circumradius of triangle $A'B'C'$ equals half the radius of k.

Problem 2 Let $P(x)$ be a polynomial with integer coefficients. Show that the polynomial

$$Q(x) = P(x^4)P(x^3)P(x^2)P(x) + 1$$

has no integer roots.

Problem 3 Let $ABCD$ be an isosceles trapezoid with bases AB and CD. The incircle of the triangle BCD touches side CD at a point E. Let F be the point on the internal bisector of $\angle DAC$ such that $EF \perp CD$. The circumcircle of triangle ACF intersects the line CD at two points C and G. Show that triangle AFG is isosceles.

3.5 Mediterranean Mathematical Competition

Problem 1 We are given n different positive numbers $a_1, a_2, \ldots, a_n$ and the set $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, where each $\sigma_i \in \{-1, 1\}$. Prove that there exist a permutation $(b_1, b_2, \ldots, b_n)$ of $(a_1, a_2, \ldots, a_n)$ and a set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ where each $\beta_i \in \{-1, 1\}$, such that the sign of $\sum_{j=1}^{i} \beta_j b_j$ equals the sign of σ_i for all $1 \leq i \leq n$.

Problem 2 In the convex quadrilteral $ABCD$, $AC = BD$. Outwards along its sides are constructed equilateral triangles WAB, XBC, YCD, ZDA with centroids S_1, S_2, S_3, S_4, respectively. Prove that $S_1S_3 \perp S_2S_4$ if and only if $AC \perp BD$.

Problem 3 For a positive integer $n \geq 2$, let $c_1, \ldots, c_n$ and $b_1, \ldots, b_n$ be positive real numbers. Prove that the equation

$$\sum_{i=1}^{n} c_i \sqrt{x_i - b_i} = \frac{1}{2} \sum_{i=1}^{n} x_i$$

has exactly one solution if and only if

$$\sum_{i=1}^{n} c_i^2 = \sum_{i=1}^{n} b_i.$$

Problem 4 P, Q, R, S are the midpoints of sides BC, CD, DA, AB, respectively, of convex quadrilateral $ABCD$. Prove that

$$4(AP^2 + BQ^2 + CR^2 + DS^2) \leq 5(AB^2 + BC^2 + CD^2 + DA^2).$$

3.6 Nordic Mathematical Contest

Problem 1 In how many ways can the number 2000 be written as a sum of three positive integers $a_1 \le a_2 \le a_3$?

Problem 2 People $P_1, P_2, \ldots, P_n$, sitting around a table in that order, have $m+n-1$, $m+n-2$, $\ldots$, m coins, respectively. P_i gives P_{i+1} exactly i coins for $i = 1, 2, \ldots$ in that order (where $P_{i+n} = P_i$ for all i) until one person no longer has enough coins to continue. At this moment, it turns out that some person has exactly five times as many coins as one of his neighbors. Determine m and n.

Problem 3 In triangle ABC, internal angle bisectors $\overline{AD}$ and $\overline{CE}$ meet at I. If $ID = IE$, prove that either triangle ABC is isosceles or $\angle ABC = 60^\circ$.

Problem 4 The function $f : [0,1] \to \mathbb{R}$ satisfies $f(0) = 0$, $f(1) = 1$, and

$$\frac{1}{2} \le \frac{f(z) - f(y)}{f(y) - f(x)} \le 2$$

for all $0 \le x < y < z \le 1$ with $z - y = y - x$. Prove that

$$\frac{1}{7} \le f\left(\frac{1}{3}\right) \le \frac{4}{7}.$$

3.7 St. Petersburg City Mathematical Olympiad (Russia)

Problem 1 Do there exist four quadratic polynomials such that if you put them in any order, there exists a number such that the values of the polynomials at that number, in the chosen order, are strictly increasing?

Problem 2 Let S_1 and S_2 be two nonintersecting circles. A common external tangent meets S_1 and S_2 at A and B, respectively. Let S_3 be a circle passing through A and B, and let C and D be its second intersections with S_1 and S_2, respectively. Let K be the point where the tangents to S_1 and S_2 at C and D, respectively, meet. Prove that $KC = KD$.

Problem 3 On a 1001×1001 checkerboard, call two (unit) squares *adjacent* if they share an edge. Several squares are chosen, no two adjacent, such that the number of squares adjacent to chosen squares is less than the number of chosen squares. How many squares have been chosen?

Problem 4 Let S be a set of 1000 positive integers. For each nonempty subset A of B, let $g(A)$ be the greatest common divisor of the elements in A. Is it possible that $g(A_1) \neq g(A_2)$ for any two distinct subsets A_1, A_2 of B?

Problem 5 Let $\overline{AA_1}, \overline{BB_1}, \overline{CC_1}$ be the altitudes of an acute triangle ABC. The points A_2 and C_2 on line A_1C_1 are such that line CC_1 bisects $\overline{A_2B_1}$ and line AA_1 bisects $\overline{C_2B_1}$. Lines A_2B_1 and AA_1 meet at K, and lines C_2B_1 and CC_1 meet at L. Prove that lines KL and AC are parallel.

Problem 6 One hundred points are chosen in the coordinate plane. Show that at most $2025 = 45^2$ rectangles with vertices among these points have sides parallel to the axes.

Problem 7 Find all pairs of distinct positive integers a, b such that $b^2 + a \mid a^2 + b$ and $b^2 + a$ is a power of a prime.

Problem 8 In a country of 2000 airports, there are initially no airlines. Two airlines take turns introducing new nonstop flights, and given any two cities only one airline may offer flights between them. Each airline attempts to introduce enough flights so that if any airport is shut down, it can still offer trips from any airport to any other airport, possibly with transfers. Which airline can ensure that it achieves this goal first?

Problem 9 We are given several monic quadratic polynomials, all with the same discriminant. The sum of any two of the polynomials has distinct real roots. Show that the sum of all of the polynomials also has distinct real roots.

Problem 10 Let a and b be distinct positive integers greater than 1 such that a^2+b-1 is divisible by b^2+a-1. Prove that b^2+a-1 has at least two distinct prime factors.

Problem 11 On an infinite checkerboard are placed 111 nonoverlapping *corners*, L-shaped figures made of 3 unit squares. The collection has the following property: for any corner, the 2×2 square containing it is entirely covered by the corners. Prove that one can remove between 1 and 110 of the corners so that the property will be preserved.

Problem 12 We are given distinct positive integers $a_1, a_2, \ldots, a_{20}$. The set of pairwise sums $\{a_i + a_j \mid 1 \le i \le j \le 20\}$ contains 201 elements. What is the smallest possible number of elements in the set $\{|a_i - a_j| \mid 1 \le i < j \le 20\}$, the set of (positive) differences between the integers?

Problem 13 Let $ABCD$ be an isoceles trapezoid with bases AD and BC. An arbitrary circle tangent to lines AB and AC intersects $\overline{BC}$ at M and N. Let X and Y be the intersections closer to D of the incircle of triangle BCD with $\overline{DM}$ and $\overline{DN}$, respectively. Show that line XY is parallel to line AD.

Problem 14 In each square of a chessboard is written a positive real number such that the sum of the numbers in each row is 1. It is known that for any eight squares, no two in the same row or column, the product of the numbers in these squares is no greater than the product of the numbers on the main diagonal. Prove that the sum of the numbers on the main diagonal is at least 1.

Problem 15 Is it possible to draw finitely many segments in three-dimensional space such that any two segments either share an endpoint or do not intersect, any endpoint of a segment is the endpoint of exactly two other segments, and any closed polygon made from these segments has at least 30 sides?

Problem 16 Does there exist a quadratic polynomial f with positive coefficients such that for every positive real number x, the equality $\lfloor f(x) \rfloor = f(\lfloor x \rfloor)$ holds?

Problem 17 What is the smallest number of weighings on a balance scale needed to identify the individual weights of a set of objects known to weigh $1, 3, 3^2, \ldots, 3^{26}$ in some order? (The balance scale reports the difference between the weights of the objects in two pans.)

Problem 18 The line ℓ is tangent to the circumcircle of acute triangle ABC at B. Let K be the projection of the orthocenter of ABC onto ℓ, and let L be the midpoint of side AC. Show that triangle BKL is isosceles.

Problem 19 Two points move within a vertical 1×1 square at the same constant speed. Each travels in a straight path except when it hits a wall, in which case it reflects off the wall so that its angle of incidence equals its angle of reflection. Show that a spider, moving at the same speed as the balls, can descend straight down on a string from the top edge of the square to the bottom so that while the spider is within in the square, neither the spider nor its string is touching one of the balls.

Problem 20 Let $n \geq 3$ be an integer. Prove that for positive numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\frac{x_n x_1}{x_2} + \frac{x_1 x_2}{x_3} + \cdots + \frac{x_{n-1} x_n}{x_1} \geq x_1 + x_2 + \cdots + x_n.$$

Problem 21 In the plane is given a convex n-gon $\mathcal{P}$ with area less than 1. For each point X in the plane, let $F(X)$ denote the area of the union of all segments joining X to points of $\mathcal{P}$. Show that the set of points X such that $F(X) = 1$ is a convex polygon with at most $2n$ sides.

Problem 22 What is the smallest number of unit segments that can be erased from the interior of a 2000×3000 rectangular grid so that no smaller rectangle remains intact?

Problem 23 Let x, y, z, t be pairwise relatively prime positive integers such that $xy + yz + zt = xt$. Prove that the sum of the squares of some two of these numbers equals twice the sum of the squares of the other two.

Problem 24 Let $\overline{AA_1}$ and $\overline{CC_1}$ be altitudes of acute triangle ABC. The line through the incenters of triangles AA_1C and AC_1C meets lines AB and BC at X and Y, respectively. Prove that $BX = BY$.

Problem 25 Does there exist a 30-digit number such that the number obtained by taking any five of its consecutive digits is divisible by 13?

Problem 26 One hundred volleyball teams play in a round-robin tournament, where each pair of teams plays against each other exactly once. Each game of the tournament is played at a different time, and no game ends in a draw. It turns out that in each match, the two teams playing the match have the same number of victories up to that point. If the minimum number of games won by any team is m, find all possible values of m.

Problem 27 Let $ABCD$ be a convex quadrilateral, and M and N the midpoints of $\overline{AD}$ and $\overline{BC}$, respectively. Suppose A, B, M, N lie on a circle such that $\overline{AB}$ is tangent to the circumcircle of triangle BMC. Prove that $\overline{AB}$ is also tangent to the circumcircle of triangle AND.

Problem 28 Let $n \geq 3$ be a positive integer. For all positive numbers $a_1, a_2, \ldots, a_n$, show that

$$\tfrac{a_1+a_2}{2}\,\tfrac{a_2+a_3}{2} \cdots \tfrac{a_n+a_1}{2} \leq \tfrac{a_1+a_2+a_3}{2\sqrt{2}}\,\tfrac{a_2+a_3+a_4}{2\sqrt{2}} \cdots \tfrac{a_n+a_1+a_2}{2\sqrt{2}}.$$

Problem 29 A connected graph is said to be 2-connected if after removing any single vertex, the graph remains connected. Prove that given any 2-connected graph in which the degree of every vertex is greater than 2, it is possible to remove a vertex (and all edges adjacent to that vertex) so that the remaining graph is still 2-connected.

Problem 30 Let m be a positive integer. Prove that there exist infinitely many prime numbers p such that $m + p^3$ is composite.

Problem 31 The perpendicular bisectors of sides AB and BC of nonequilateral triangle ABC meet lines BC and AB at A_1 and C_1, respectively. Let the bisectors of angles A_1AC and C_1CA meet at B', and define C' and A' analogously. Prove that the points A', B', C' lie on a line passing through the circumcenter of triangle ABC.

Problem 32 Is it possible to select 102 17-element subsets of a 102-element set, such that the intersection of any two of the subsets has at most 3 elements?