Parametrized Thue Equations — A Survey

Clemens Heuberger
Institut fr Mathematik B
Technische Universitt Graz
8010 Graz
Austria
clemens.heuberger@tugraz.at*

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Abstract

We consider families of parametrized Thue equations

$$F_a(X, Y) = \pm 1, \qquad a \in \mathbb{N},$$

where $F_a \in \mathbb{Z}[a][X, Y]$ is a binary irreducible form with coefficients which are polynomials in some parameter a.

We give a survey on known results.

1 Thue Equations

Let $F \in \mathbb{Z}[X, Y]$ be a homogeneous, irreducible polynomial of degree $n \geq 3$ and m be a nonzero integer. Then the Diophantine equation

$$F(X, Y) = m \tag{1}$$

is called a *Thue equation* in honour of A. Thue, who proved in 1909 [57]:

Theorem 1 (Thue). (1) *has only a finite number of solutions* $(x, y) \in \mathbb{Z}^2$.

Thue's proof is based on his approximation theorem: Let α be an algebraic number of degree $n \geq 2$ and $\epsilon > 0$. Then there exists a constant $c_1(\alpha, \epsilon)$, such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c_1(\alpha, \epsilon)}{q^{n/2+1+\epsilon}}.$$

Since this approximation theorem is not effective, Thue's theorem is neither effective.

2 Number of Solutions

We call a solution (x, y) to $F(x, y) = m$ primitive, if x and y are coprime integers. The problem of giving upper bounds (depending on m and the degree n) for the number of primitive solutions goes back to Siegel. Such a bound has first been given by Evertse [14]. An improved version has been given by Bombieri and Schmidt [6]:

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Theorem 2 (Bombieri-Schmidt [6]). *There is an absolute constant c_2 such that for all $n \geq c_2$ the Diophantine equation $F(X,Y) = m$ has at most $215 \cdot n^{1+\omega(m)}$ primitive solutions, where $\omega(m)$ denotes the number of prime factors of m and solutions (x,y) and $(-x,-y)$ are regarded as the same.*

At least for $m = \pm 1$, this result is best possible (up to the constant 215), since the equation

$$X^n + (X-Y)(2X-Y)\dots(nX-Y) = \pm 1$$

has at least the $n+1$ solutions $\pm\{(1,1),\dots,(1,n),(0,1)\}$.

Sharper bounds have been obtained for special classes of Thue equations.

If only k coefficients of $F(X,Y)$ are nonzero, the number of solutions depends on k and m only (and not on n). For $k = 3$, this is proved by Mueller and Schmidt [41]: There are at most $O(m^{2/n})$ solutions. The general case $k \geq 3$ is proved in Mueller and Schmidt [42]: There are at most $O(k^2 m^{2/n}(1+\log m^{1/n}))$ solutions. Thomas [56] gives absolute upper bounds for the number of solutions for $m = 1$ and $k = 3$: If $n \geq 38$, then there are at most 20 solutions (x,y) with $|xy| \geq 2$, where solutions (x,y) and $(-x,-y)$ are only counted once. For smaller n, similar bounds are given.

If only 2 coefficients of $F(X,Y)$ are nonzero, we arrive at the special case $ax^n - by^n = \pm 1$ and we consider only the case $ab \neq 0$, $x > 0$, $y > 0$. This equation has been studied by many authors, starting with Delone [11] and Nagell [43], who proved that there is at most one solution for $n = 3$. Several authors have contributed to this question. Finally, Bennett [4] could prove that there is at most one solution (x,y).

We now consider cubic Thue equations $F(X,Y) = 1$. If the discriminant of F is negative, there are at most 5 solutions, and the cases of 4 and 5 solutions can be listed explicitly. This has been shown independently by Delaunay [10] and Nagell [44] in the 1920's. If the discriminant is positive, there are at most 10 solutions, as it has been proved by Bennett [3]. Okazaki [47] proves that if the discriminant is at least $5.65 \cdot 10^{65}$, then there are at most 7 solutions. It is conjectured by Nagell [45], Pethő [48], and Lippok [35] that there are at most 5 solutions except for five equations (modulo equivalence) which have 6 or 9 solutions. We note that there are two families of cubic Thue equations which have exactly five solutions, cf. items 2 and 3 in the list in Section 4.1.

Okazaki [46] considers the analogous problem for quartic Thue equations $F(X,Y) = \pm 1$. If all roots of $F(x,1)$ are real and the discriminant is larger than a computable constant c_3, this equation has at most 14 solutions, where solutions (x,y) and $(-x,-y)$ are counted once.

3 Algorithmic Solution of Single Thue Equations

Studying linear forms in logarithms of algebraic numbers, A. Baker could give an effective upper bound for the solutions of such a Thue equation in 1968 [1]:

Theorem 3 (Baker). *Let $\kappa > n+1$ and $(x,y) \in \mathbb{Z}^2$ be a solution of* (1). *Then*

$$\max\{|x|,|y|\} < c_4 e^{\log^\kappa |m|},$$

where $c_4 = c_4(n,\kappa,F)$ is an effectively computable number.

Since that time, these bounds have been improved; Bugeaud and Győry [7] give the following bound:

Theorem 4 (Bugeaud-Győry). *Let $B \geq \max\{|m|, e\}$, α be a root of $F(X,1)$, $K := \mathbb{Q}(\alpha)$, $R := R_K$ the regulator of K and r the unit rank of K. Let $H \geq 3$ be an upper bound for the absolute values of the coefficients of F.*

Then all solutions $(x,y) \in \mathbb{Z}^2$ of (1) *satisfy*

$$\max\{|x|,|y|\} < \exp\Big(c_5 \cdot R \cdot \max\{\log R, 1\} \cdot (R + \log(HB))\Big)$$

and

$$\max\{|x|, |y|\} < \exp\Big(c_6 \cdot H^{2n-2} \cdot \log^{2n-1} H \cdot \log B\Big),$$

with $c_5 = 3^{r+27}(r+1)^{7r+19}n^{2n+6r+14}$ *and* $c_6 = 3^{3(n+9)}n^{18(n+1)}$.

The bounds for the solutions obtained by Baker's method are rather large, thus the solutions practically cannot be found by simple enumeration. For a similar problem Baker and Davenport [2] proposed a method to reduce drastically the bound by using continued fraction reduction. Pethő and Schulenberg [50] replaced the continued fraction reduction by the LLL-algorithm and gave a general method to solve (1) for the totally real case with $m = 1$ and arbitrary n. Tzanakis and de Weger [61] describe the general case. Finally, Bilu and Hanrot [5] were able to replace the LLL-algorithm by the much faster continued fraction method and solve Thue equations up to degree 1000.

4 Families of Thue Equations

We study families of Thue equations

$$F_a(X, Y) = \pm 1, \qquad a \in \mathbb{N} \tag{2}$$

where $F_a \in \mathbb{Z}[a][X, Y]$ is an irreducible binary form of degree of at least 3 with coefficients which are integer polynomials in a. In the investigation of such families usually only two types of solutions appear: Firstly, there are *polynomial solutions* $X(a), Y(a) \in \mathbb{Z}[a]$ which satisfy (2) in $\mathbb{Z}[a]$, and secondly, there occur (sometimes) single solutions for a few small values of the parameter a. However, Lettl [30] points out that the family $X^6 - (a-1)Y^6 = a^2$ does not have any polynomial solution, but there are sporadic solutions for infinitely many values of the parameter a.

The first infinite parametrized families of Thue equations were considered by Thue [58] himself: He proved that the equation

$$(a+1)X^n - aY^n = 1, \qquad X > 0, Y > 0 \tag{3}$$

has only the solution $x = y = 1$ for a suitably large in relation to prime $n \geq 3$. For $n = 3$, the equation (3) has only this solution for $a \geq 386$. Of course, Bennett's result [4] cited in Section 2 implies that this is true for all $n \geq 3$ and $a \geq 1$.

For a description of the techniques used to solve families of Thue equations, we refer to Heuberger [20]. Some automated procedures are presented in [26].

4.1 Families of Fixed Degree

In 1990, Thomas [53] investigated for the first time a parametrized family of cubic Thue equations of positive discriminant. Since 1990, the following particular families of Thue equations have been studied:

1. $X^3 - (a-1)X^2Y - (a+2)XY^2 - Y^3 = 1$.

   Thomas [53] and Mignotte [36] proved that for $a \geq 4$, the only solutions are $(0, -1)$, $(1, 0)$ and $(-1, +1)$, while for the cases $0 \leq a \leq 4$ there exist some nontrivial solutions, too, which are given explicitly in [53]. For the same form $F_a(X, Y)$, all solutions of the Thue inequality $|F_a(X, Y)| \leq 2a + 1$ have been found by Mignotte, Pethő, and Lemmermeyer [39].

2. $X^3 - aX^2Y - (a+1)XY^2 - Y^3 = X(X+Y)(X-(a+1)Y) - Y^3 = 1$.

   Lee [29] and independently Mignotte and Tzanakis [40] proved that for $a \geq 3.33 \cdot 10^{23}$ there are only the solutions

   $$(1, 0), (0, -1), (1, -1), (-a-1, -1), (1, -a).$$

   Mignotte [37] could prove the same result for all $a \geq 3$.

3. Wakabayashi [66] proved that for $a \geq 1.35 \cdot 10^{14}$, the equation $X^3 - a^2XY^2 + Y^3 = 1$ has exactly the five solutions $(0,1)$, $(1,0)$, $(1,a^2)$, $(\pm a, 1)$.

4. Togbe [60] considered the equation $X^3 - (n^3 - 2n^2 + 3n - 3)X^2Y - n^2XY^2 - Y^3 = \pm 1$. If $n \geq 1$, the only solutions are $(\pm 1, 0)$ and $(0, \pm 1)$.

5. Wakabayashi [64]: $|X^3 + aXY^2 + bY^3| \leq a + |b| + 1$ for arbitrary b and $a \geq 360b^4$ as well as for $b \in \{1,2\}$ and $a \geq 1$. He uses Padé approximations.

6. Thomas [55]: Let b, c be nonzero integers such that the discriminant of $t^3 - bt^2 + ct - 1$ is negative, $\Delta = 4c - b^2 > 0$, and $c \geq \min\{4.2 \times 10^{41} \times |b|^{2.32}, 3.6 \times 10^{41} \times \Delta^{1.1582}\}$. Then the Thue equation $X^3 - bX^2Y + cXY^2 - Y^3 = 1$ only has the trivial solutions $(1,0)$, $(0,-1)$.

7. $X(X - a^{d_2}Y)(X - a^{d_3}Y) \pm Y^3 = 1$.

   This family was investigated by Thomas [54]. He proved that for $0 < d_2 < d_3$ and
   $$a \geq \left(2 \cdot 10^6 \cdot (d_2 + 2d_3)\right)^{4.85/(d_3 - d_2)}$$
   nontrivial solutions cannot exist. He also investigated this family with a^{d_1} and a^{d_2} replaced by monic polynomials in a of degrees d_1 and d_2, respectively (see Theorem 5).

8. $X^4 - aX^3Y - X^2Y^2 + aXY^3 + Y^4 = X(X-Y)(X+Y)(X-aY) + Y^4 = \pm 1$.

   This quartic family was solved by Pethő [49] for large values of a; Mignotte, Pethő, and Roth [38] solved it completely: The only solutions are $\pm\{(0,1), (1,0), (1,1), (1,-1), (a,1), (1,-a)\}$ for $|a| \notin \{2,4\}$. If $|a| = 4$, four more solutions exist. If $|a| = 2$, the family is reducible.

9. $X^4 - aX^3Y - 3X^2Y^2 + aXY^3 + Y^4 = \pm 1$ has been solved for $a \geq 9.9 \cdot 10^{27}$ by Pethő [49].

10. $|bX^4 - aX^3Y - 6bX^2Y^2 + aXY^3 + bY^4| \leq N$.

    For $b = 1$ and $N = 1$, this equation has been solved completely by Lettl and Pethő [31]; Chen and Voutier [9] solved it independently by using the hypergeometric method. For the same form binary form $F_{a,b}(X,Y)$, Lettl, Pethő and Voutier [33] proved that $|F_a(X,Y)| \leq 6a + 7$ has only trivial primitive solutions for $a \geq 58$, if $b = 1$. Furthermore, $x^2 + y^2 \leq \max\{25a^2/(64b^2), 4N^2/a\}$ if $a > 308b^4$, cf. Yuan [67].

11. Togbé [59] gives all solutions to $X^4 - a^2X^3Y - (a^3 + 2a^2 + 4a + 2)X^2Y^2 - a^2XY^3 + Y^4 = 1$ for $a \geq 1.191 \cdot 10^{19}$ and $a, a+2, a^2+4$ squarefree.

12. $|X^4 - a^2X^2Y^2 + Y^4| = \left|X^2(X-a)(X+a) + Y^4\right| \leq a^2 - 2$

    This family of Thue inequalities has only trivial solutions with $|y| \leq 1$ for $a \geq 8$ (Wakabayashi [62]).

13. $\left|X^4 + 4aX^3Y + 6aX^2Y^2 + 4a^2XY^3 + a^2Y^4\right| \leq a^2$ has been solved for $a \geq 205$ by Chen and Voutier [8].

14. Dujella and Jadrijević [12], [13] prove that $\left|X^4 - 4cX^3Y + (6c+2)X^2Y^2 + 4cXY^3 + Y^4\right| \leq 6c + 4$ has only trivial solutions for all $c \geq 3$.

15. $X(X-Y)(X-aY)(X-bY) - Y^4 = \pm 1$.

    All solutions of this two-parametric family are known for $10^{2 \cdot 10^{28}} < a + 1 < b \leq a(1 + (\log a)^{-4})$, cf. Pethő and Tichy [51]. The case of $b = a + 1$ has been considered by Heuberger, Pethő and Tichy [23], where all solutions could be determined for all $a \in \mathbb{Z}$.

16. Jadrijević [27] proves that for every $0.5 < s \leq 1$, there is an effectively computable constant $P(s)$ such that if $a \neq 0$ and $\max\{|a|, |b|\} \geq P(s)$ and $\gcd(a,b) \geq \max\{|a|^s, |b|^s\}$, then the equation $X^4 - 2abX^3Y + 2(a^2 - b^2 + 1)X^2Y^2 + 2abXY^3 + Y^4 = 1$ only has trivial solutions. In particular, $P(0.999) = 10^{27}$ and $P(0.501) = 10^{36836}$.

17. Wakabayashi [63] found all solutions of $|X^4 - a^2X^2Y^2 - bY^4| \leq a^2 + b - 1$ for $a \geq 5.3 \cdot 10^{10}b^{6.22}$.
18. $X(X^2 - Y^2)(X^2 - a^2Y^2) - Y^5 = \pm 1$.
    For $a > 3.6 \cdot 10^{19}$, all solutions have been found by Heuberger [18].
19. Gaál and Lettl [15] investigated the family $X^5 + (a-1)X^4Y - (2a^3 + 4a + 4)X^3Y^2 + (a^4 + a^3 + 2a^2 + 4a - 3)X^2Y^3 + (a^3 + a^2 + 5a + 3)XY^4 + Y^5 = \pm 1$ and found all solutions for $|a| \geq 3.3 \cdot 10^{15}$. The remaining cases have been solved in Gaál and Lettl [16].
20. Levesque and Mignotte [34] found all solutions of the equation $X^5 + 2X^4Y + (a+3)X^3Y^2 + (2a+3)X^2Y^3 + (a+1)XY^4 - Y^5 = \pm 1$ for sufficiently large a.
21. $X^6 - 2aX^5Y - (5a+15)X^4Y^2 - 20X^3Y^3 + 5aX^2Y^4 + (2a+6)XY^5 + Y^6 \in \{\pm 1, \pm 27\}$ was investigated by Lettl, Pethő, and Voutier. They found all solutions for $a \geq 89$ by hypergeometric methods [33] and all solutions for $a < 89$ by using Baker's method [32]. In [33], they also proved that $|F_a(X,Y)| \leq 120a + 323$ (for the form $F_a(X,Y)$ considered) has only trivial primitive solutions for $a \geq 89$.
22. $X^8 - 8nX^7Y - 28X^6Y^2 + 56nX^5Y^3 + 70X^4Y^4 - 56nX^3Y^5 - 28nX^2Y^6 + 8nXY^7 + Y^8 = \pm 1$ has only trivial solutions for $n \in \{a \in \mathbb{Z} : a + b\sqrt{2} = (1+\sqrt{2})^{2k+1}, k \in \mathbb{N}\}$ with $n \geq 6.71 \cdot 10^{32}$. (Heuberger, Togbé and Ziegler [26]).

A more detailed survey on cubic families is contained in Wakabayashi [65].

4.2 Families of Relative Thue Equations

A few families of relative Thue equations have also been solved, i.e., families where the parameters and the solutions are elements of the same imaginary quadratic number field.

So let $D > 0$ be an integer, $k := \mathbb{Q}(\sqrt{-D})$, $\mathfrak{o}_k$ its ring of algebraic integers, and μ a root of unity in $\mathfrak{o}_k$.

1. For $t \in \mathfrak{o}_k$ with $|t| \geq 3.03 \cdot 10^9$, the only solutions $(x,y) \in \mathfrak{o}_k^2$ to $X^3 - (t-1)X^2Y - (t+2)XY^2 - Y^3 = \mu$ satisfy $\max\{|x|, |y|\} \leq 1$ and can be listed explicitly (Heuberger, Pethő, and Tichy[24]).
2. For $t \in \mathfrak{o}_k$ with $|t| > 2.88 \cdot 10^{33}$, the only solutions $(x,y) \in \mathfrak{o}_k^2$ to $X^3 - tX^2Y - (t+1)XY^2 - Y^3 = \mu$ satisfy $\min\{|x|, |y|\} \leq 1$ and can be listed explicitly (Ziegler [68]).
3. For $s, t \in \mathfrak{o}_k$ with $|t| \geq 5.3 \cdot 10^10 |s|^{12.44}$ or $s = 1$ and $|t| > \sqrt{550}$, all solutions $(x,y) \in \mathfrak{o}_k^2$ to $|X^4 - t^2X^2Y^2 + s^2Y^4| \leq |t|^2 - |s|^2 - 2$ are explicitly known (Ziegler [69]).

4.3 Families of Arbitrary Degree

Moreover, some general families of arbitrary degree have been considered. Apart from (3), the investigated general families are of the shape

$$F_a(X,Y) := \prod_{i=1}^{n} (X - p_i(a)Y) - Y^n = \pm 1, \tag{4}$$

where $p_1, \ldots, p_n \in \mathbb{Z}[a]$ are polynomials, which have been called *split families* by E. Thomas [54]. For $i = 1, \ldots, n$ it can easily be seen that $(X,Y) \in \{\pm(p_i, 1), (\pm 1, 0)\}$ are solutions. Thomas conjectured that if

$$p_1 = 0, \qquad \deg p_2 < \cdots < \deg p_n$$

and the polynomials are monic, there are no further solutions for sufficiently large values of the parameter a. In [54] he proved this conjecture for $n = 3$ under some technical hypothesis:

Theorem 5. *Let $u = \pm 1$, $a(t), b(t) \in \mathbb{Z}[t]$ be monic polynomials and $a := \deg a(t)$, $b := \deg b(t)$ with $0 < a < b$. We write $A(t) := a(t)/t^a - 1$ and $B(t) := b(t)/t^b - 1$ and define for $n \geq 1$*

$$W(n) := \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(b \cdot A(n)^j - a \cdot B(n)^j),$$

which can be written in powers of $1/n$ as $W(n) = \sum_{j=1}^{n} w_j n^{-j}$. Further we define $J := \min\{j \in \mathbb{N} : w_j \neq 0\}$.

If $J \neq b - a$ or $J = b - a \wedge 3w_J + 2b + a \neq 0 \wedge 3w_J - 2(b-a) \neq 0$, then there is an effectively computable constant c_7 depending on the coefficients of $a(t)$ and $b(t)$ such that for $n \geq c_7$ the family of Thue equations

$$X(X - a(n)Y)(X - b(n)Y) + uY^3 = \pm 1$$

only has the solutions

$$\pm\{(1,0), (0,u), (a(n)u, u), (b(n)u, u)\}.$$

Halter-Koch, Lettl, Pethő and Tichy [17] considered (4) for $p_1 = 0, p_2 = d_2, \ldots, p_{n-1} = d_{n-1}$ and $p_n = a$, where $d_2, \ldots, d_{n-1}$ are fixed distinct integers. They found all solutions for sufficiently large values of a assuming a conjecture of Lang and Waldschmidt [28]—which is a very sharp bound for linear forms in logarithms of algebraic numbers—:

Theorem 6. *Let $n \geq 3$, $p_1 = 0, p_2 = d_2, \ldots, p_{n-1} = d_{n-1}$ be distinct integers and $p_n = a$. Let $\alpha = \alpha(a)$ be a zero of $P(x) = \prod_{i=1}^{n}(x - p_i) - d$ with $d = \pm 1$ and suppose that the index I of $\langle \alpha - d_1, \ldots, \alpha - d_{n-1} \rangle$ in $\mathfrak{O}^\times$, the group of units of $\mathfrak{O} := \mathbb{Z}[\alpha]$, is bounded by a constant $J = J(d_1, \ldots, d_{n-1}, n)$ for every a from some subset $\Omega \in \mathbb{Z}$. Assume further that the Lang-Waldschmidt conjecture is true. Then for all but finitely many values of $a \in \Omega$ the Diophantine equation*

$$\prod_{i=1}^{n}(x - p_i y) - dy^n = \pm 1$$

has only solutions $(x, y) \in \mathbb{Z}^2$ with $|y| \leq 1$, except for the cases of $n = 3$ and $|d_2| = 1$ or $n = 4$ and $(d_2, d_3) \in \{(1, -1), (\pm 1, \pm 2)\}$, where it has exactly one more solution for every value of a.

If $\mathbb{Q}(\alpha)$ is primitive over $\mathbb{Q}$ — especially if n is prime — then there exists a bound $J = J(d_1, \ldots, d_{n-1}, n)$ for the index I by lower bounds for the regulator of $\mathfrak{O}$ (cf. Pohst and Zassenhaus [52], chapter 5.6, (6.22)). Applying the theory of Hilbertian fields and results on thin sets, primitivity is proved for almost all choices (in the sense of density) of the parameters, cf. [17].

The two exceptional families are those considered under 2 and 8 in the list in Section 4.1.

A similar family has been considered by Heuberger in [19], however, in this case, the result is unconditionally true:

Theorem 7. *Let $n \geq 4$ be an integer, $d_2, \ldots, d_{n-1}$ pairwise distinct integers and a an integral parameter. Furthermore we assume*

$$d_2 + \cdots + d_{n-1} \neq 0 \qquad \text{or} \qquad d_2 \cdots d_{n-1} \neq 0.$$

Let

$$F_a(X, Y) := (X + aY)(X - d_2 Y)(X - d_3 Y) \cdots (X - d_{n-1} Y)(X - aY) - Y^n.$$

Then there exists a (computable) constant c_8 depending only on the degree n and $d_2, \ldots, d_{n-1}$, such that for all $a \geq c_8$, the only solutions $(x, y) \in \mathbb{Z}^2$ of the Diophantine equation

$$F_a(X, Y) = \pm 1$$

are $\pm\{(1, 0), (-a, 1), (d_2, 1), (d_3, 1), \ldots, (d_{n-1}, 1), (a, 1)\}$.

In [25], Heuberger and Tichy considered a multivariate version of (4):

Theorem 8. *Let $n \geq 4$, $r \geq 1$, $p_i \in \mathbb{Z}[A_1, \ldots, A_r]$ for $1 \leq i \leq n$. We make the following assumptions on the polynomials p_i:*

$$\deg p_1 < \cdots < \deg p_{n-2} < \deg p_{n-1} = \deg p_n,$$
$$\mathrm{LH}(p_n) = \mathrm{LH}(p_{n-1}), \text{ but } p_n \neq p_{n-1}.$$

Furthermore we suppose that for $p \in \{p_1, \ldots, p_n, p_n - p_{n-1}\}$, there exist positive constants t_p, c_p such that

$$|(\mathrm{LH}(p))(a_1, \ldots, a_r)| \geq c_p \cdot (\min_k a_k)^{\deg p} \qquad \text{for } a_1, \ldots, a_r \geq t_p.$$

Let

$$F_{a_1,\ldots,a_r}(X, Y) := \prod_{i=1}^{n} (X - p_i(a_1, \ldots, a_r)Y) - Y^n.$$

For every constant $C > 1$ there is a constant t_0 such that for all integers $a_1, \ldots, a_r$ satisfying $t_0 \leq \min_k a_k$ and

$$\max_k a_k \leq C \cdot \min_k a_k,$$

the Diophantine equation

$$F_{a_1,\ldots,a_r}(x, y) = \pm 1$$

considered for $x, y \in \mathbb{Z}$ only has the solutions $\{(\pm 1, 0)\} \cup \{\pm(p_i(a_1, \ldots, a_r), 1) : 1 \leq i \leq n\}$.

In Heuberger [21] Thomas' conjecture is proved under some technical hypothesis:

Theorem 9. *Let $n \in \mathbb{N}$, $n \geq 3$ and $p_i \in \mathbb{Z}[a]$ be monic polynomials for $i = 1, \ldots, n$. We write*

$$p_i(a) = a^{d_i} + k_i a^{d_i - 1} + \text{ terms of lower degree}, \qquad i = 2, \ldots, n,$$

allow $p_1 = 0$ and assume

$$d_1 < d_2 < \cdots < d_{n-1} < d_n \qquad \text{and} \qquad n + d_2 \geq 4.$$

Let

$$\delta_i := \begin{cases} 1 & \text{if } d_i - d_{i-1} = 1, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad e := \sum_{i=2}^{n} d_i.$$

If $\delta_4 = 1$ or

$$(e - d_2 + 2d_3)(k_2 - \delta_2) + (-e - 2d_2 + d_3)k_3 + (d_3 - d_2) \sum_{i=4}^{n} k_i \notin \{2\delta_3, -(e + d_3)\delta_3\}, \tag{5}$$

then there is a (computable) constant $c_9 = c_9(p_1, \ldots, p_n)$ depending on the coefficients of the polynomials p_i such that for all integers $a \geq c_9$ the Diophantine equation

$$F_a(X, Y) := \prod_{i=1}^{n} (X - p_i(a)Y) - Y^n = \pm 1$$

only has the solutions

$$(\pm 1, 0) \text{ and } \pm (p_i(a), 1), 1 \leq i \leq n.$$

In [21], there is also a version with a stronger technical hypothesis than that in (5). For $n = 3$, that version improves Theorem 5.

Especially there are only trivial solutions if

$$\max(\deg p_1, 0) < \deg p_2 < \deg p_3 < \cdots < \deg p_n$$
$$\max(\deg p_1, 0) + \deg p_2 + \ldots + \deg p_n < 15.$$

In Heuberger [22], an explicit constant c_9 for Theorem 9 is given:

$$c_9 = \exp\left(1.01(n+1)(n-1)!(n-1)^{n-2} \exp\bigl(1.04(n-2)(nd_n - n + 3)\bigr) \binom{nd_n - 1}{n - 3} (2P + 1)^{nd_n}\right),$$

where $d_j = \deg p_j$ and P is an upper bound for the absolute values of the coefficients of the p_j, $j = 1, \ldots, n$.

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