Nine-point circle, pedal circle and cevian circle

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Abstract

This paper contains some results around nine-point circle, pedal circle, cevian circle and their intersections. These are results, solutions that found by many people who interested in geometry. My contribution is just a little. Writing this, I want to make a collection, as detail as I can of these circles. I am just a normal student with great love for geometry, especially plane geometry, I am sure that I can't do this alone. I really want to say thanks to Mr. Tran Quang Hung - my teacher, Nguyen Van Linh, Telv Cohl, Tran Minh Ngoc, Luiz Gonzalez, who inspired me so much. One more thing, now I am seniors of high school, many pressure is coming so this is may be the last great paper in this year. Today, finally I finish this work. I hope all of you love this.

1 Common points of nine-point circle, pedal circle, cevian circle other than Poncelet point

1.1 Nine-point circle and pedal circle

Theorem 1. (Fontene's first theorem) *$\triangle ABC$ and a point P. $\triangle A'B'C'$, $\triangle DEF$ are pedal triangles of the circumcenter and P wrt $\triangle ABC$. EF, FD, DE intersects $B'C', C'A', A'B'$ at X, Y, Z. Then DX, EY, FZ are concurrent at a common point of $\odot(DEF)$ and $\odot(A'B'C')$.*

Proof. Let O be circumcenter of $\triangle ABC$.

Lemma 2. *Orthopole of P is also the anti-Steiner point of OP wrt $\triangle A'B'C'$.*

Let T be the anti-steiner point of OP, A_1, B_1, C_1 are reflections of T in $B'C', C'A', A'B'$. Then A_1, B_1, C_1 lie on OP.

$$(A_1B', A_1C') = (TC', TB') = (A'C', A'B') = (AC, AB) = (AB', AC')$$

$\Rightarrow A_1$ lies on the circle that has diameter OA, so $AA_1 \perp OP$. Similarly, $BB_1, CC_1 \perp OP$. We also have the lines that pass through A_1, B_1, C_1 and pendicular to BC, CA, AB, respectively are concurrent at T. Then T is the orthopole of OP wrt $\triangle ABC$.

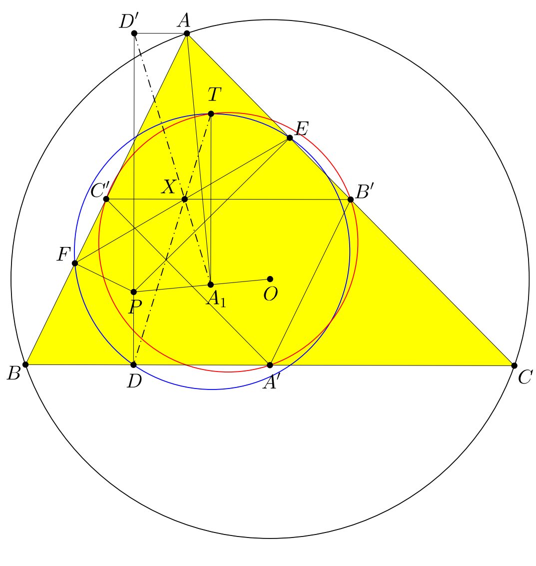

Figure 1

Back to the main proof.
Let D' be the reflection of D in $B'C' \Rightarrow AD' \parallel BC$. Then A', A, E, F, P, A_1 are concyclic.

$$(A_1X, A_1D') = (A_1X, A_1F) + (A_1F, A_1D') = (C'X, C'F) + (AF, AD') = (C'B', C'A) + (BA, BC) = 0$$

$\Rightarrow A_1, D', X$ are collinear. Since D, T are the reflections of D', T in $B'C'$ then DX pass through T.
Cause D', A_1, E, F are concyclic:

$$\overline{XT}.\overline{XD} = \overline{XD'}.\overline{XA_1} = \overline{XE}.\overline{XF}$$

Hence, the pedal circle of P wrt $\triangle ABC$ passes through T.
Similarly, EY, FZ pass through T.

Theorem 3. (Fontene's second theorem) *Given a line that passes through circumcenter of $\triangle ABC$, a point P varies on it then pedal circle of P wrt $\triangle ABC$ always passes through a fixed point*

From the proof of Fontene's first theorem, the pedal circle of P wrt $\triangle ABC$ passes through the orthopole of that line wrt $\triangle ABC$ - which is a fixed point.

Corollary 4. (Nguyen Van Linh) *Let O' be circumcenter of $\triangle DEF$, then O' is orthocenter of $\triangle XYZ$.*

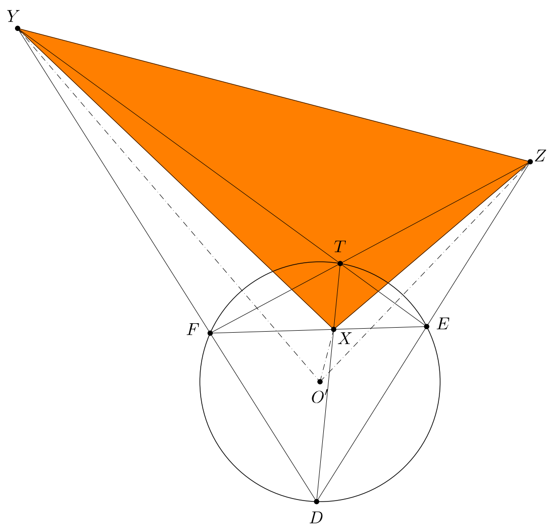

Figure 2

Proof. According to Fontene's first theorem, DX, EY, FZ pass through T on (DEF). So, just simply by Brocard's theorem, O' is orthocenter of $\triangle XYZ$.

Theorem 5. (Fonterne's third's theorem) *Pedal circle of P wrt $\triangle ABC$ is tangent to nine-point circle of $\triangle ABC$ if and only if P, P'(isogonal conjugate of P wrt $\triangle ABC$) and circumcenter of $\triangle ABC$ are collinear.*

Proof. Let T' be anti-Steiner point of OP' wrt $\triangle A'B'C'$ and $\triangle D'E'F'$ is pedal triangle of P' wrt $\triangle ABC$. Since P, P' are isogonal conjugate points wrt $\triangle ABC$ so D, E, F, D', E', F' are concyclic. Hence T' is a common point other than T of $\odot(DEF)$ and nine-point circle of $\triangle ABC$. So pedal circle of P wrt $\triangle ABC$ is tangent to nine-point circle of $\triangle ABC$ if and only if $T \equiv T'$, or equivalently, anti-Steiner point of OP coincides with anti-Steiner point of OP' wrt $\triangle A'B'C' \Leftrightarrow P, O, P'$ are collinear.

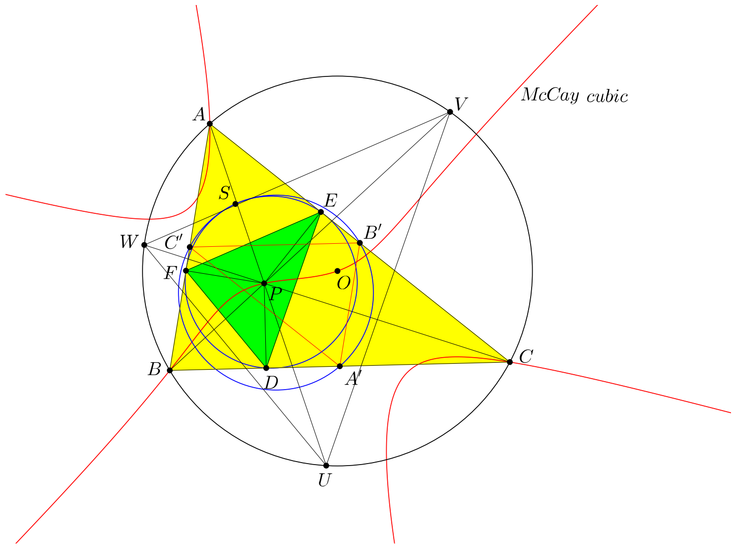

Figure 3. McCay cubic

Note. When P coincides with incenter or excenters, we get the famous Feuerbach's theorem. Furthermore, locus of P that P, O, P' are collinear is McCay cubic, it has barycentric equation:

$$x(c^2y^2 - b^2x^2)(b^2 + c^2 - a^2) + y(a^2z^2 - c^2x^2)(c^2 + a^2 - b^2) + z(b^2x^2 - a^2y^2)(a^2 + b^2 - c^2) = 0$$

This cubic has many interesting properties, such as: pedal triangle and circumcevian triangle of P on McCay cubic wrt $\triangle ABC$ are homothetic. Until now, new properties of McCay cubic are still being found. Because of the framework of this paper, author won't mention in detail so reader can see more properties of McCay cubic in Reference.

Proposition 6. (AoPS) *Simson line of T wrt $\triangle D'E'F'$ is parallel to OP.*

Proof. At first, we introduce a lemma.

Lemma 7. (Telv Cohl) *(O) is a fixed circle and BC is a fixed chord of (O), P is a fixed point. A varies on (O). $\triangle DEF$ is pedal triangle of P wrt $\triangle ABC$, T is orthopole of OP wrt $\triangle ABC$ then $\angle(DF, DT)$ is a fixed when A varies on (O).*

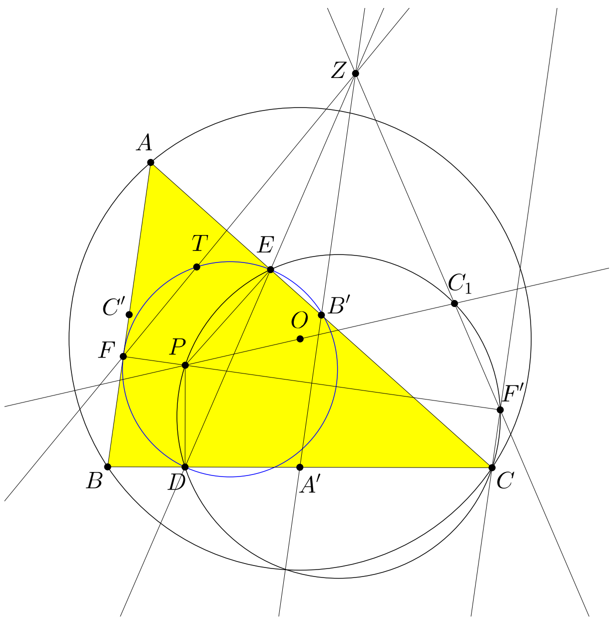

Figure 4

Proof of this lemma is given also by Telv Cohl.
Let C_1, F' be orthogonal projections of C on OP, PF. From the proof of Fontene's first theorem above,

we have: $F'C_1$, $A'B'$, DE, TF are concurrent at Z.

$$\begin{aligned}(FD,FT) &= (FD,FP)+(FP,FT)\\ &= (BD,BP)+(F'Z,F'P)(\triangle ZFF' \text{ is isoceles})\\ &= (BC,BP)+(F'C_1,F'P)\\ &= (BC,BP)+(CC_1,CP)\\ &= (BC,BP)+(OP,PC)+\frac{\pi}{2}\\ &= constant\end{aligned}$$

Now, back to the main problem. Let U,V be orthogonal projections of T on $E'F',F'D'$.

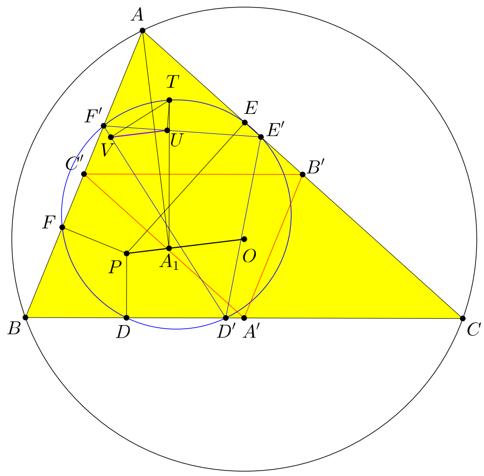

Figure 5

$$\begin{aligned}(UV,UT) &= (E'D',E'T)\\ &= (E'D,E'T)+(E'D',E'E)+(E'E,E'D)\\ &= (FD,FT)+(PC,PE)+(FE,FD)\\ &= (BC,BP)+(OP,PC)+\frac{\pi}{2}+(PC,PE)+(FE,FP)+(FP,FD)\\ &= (BC,BP)+(OP,PC)+(CP,CA)+(AC,AP)+(BP,BC)\\ &= (OP,AP)\end{aligned}$$

Since $AP\perp E'F'$, $TU\perp E'F'$ then $AP\parallel TU$.
$\Longrightarrow UV\parallel OP \Leftrightarrow$ Simson line of T wrt $\triangle D'E'F'$ is parallel to OP.

Proposition 8. (Tran Quang Hung) *A line that passes through D and parallel to PA intersects the A-altitude of $\triangle ABC$ at D''. E'',F'' are determined similarly.*
Prove that the circles that have diameter DD'',EE'',FF'' pass through T.

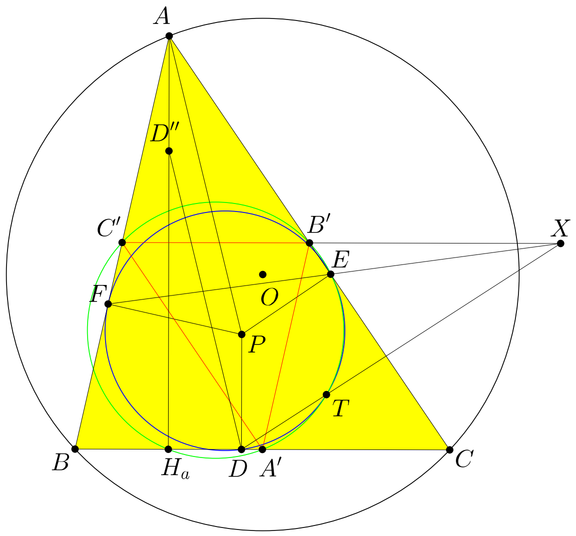

Figure 6

Proof(based on Nguyen Van Linh's). Since $AD'' \parallel PD$, $AP \parallel DD''$ then $APDD''$ is a parallelogram. $\Rightarrow$ Midpoint of AP is reflection of midpoint of DD'' in intersection of AD, PD'', which is midpoint of AD and lies on $B'C'$. $\Rightarrow \odot(DD'')$ is reflection of (PA) in $B'C'$ so $B'C'$ is radical axis of $\odot(DD'')$ and $\odot(AP)$.
Let consider three circles $\odot(DD''), \odot(PA), \odot(DEF)$:
$B'C'$ is radical axis of $\odot(DD'')$ and $\odot(PA)$.
EF is radical axis of $\odot(PA)$ and $\odot(DEF)$.
So X is radical center of $\odot(DD''), \odot(PA), \odot(DEF)$. Since D lies on $\odot(DEF), \odot(DD'')$, XD is radical axis of $\odot(DEF), \odot(DD'')$. Furethermore, from Fontene's first theorem, T, X, D are collinear, T lies on XD so T lies on (DD''). Similarly, $\odot(EE''), \odot(FF'')$ pass through T.

Proposition 9. (Tran Quang Hung) *Given $\triangle ABC$, let P be an arbitrary point, $\triangle DEF$ be the pedal triangle of P wrt $\triangle ABC$. $\mathcal{R}$ is radius of $\odot(DEF)$.*
P' is the isogonal conjugate of P wrt $\triangle ABC$. D_1, E_1, F_1 are reflections of P in D, E, F, PD_1, PE_1, PF_1 intersect $\odot(D_1E_1F_1)$ at $D_2, E_2, F_2 \neq D_1, E_1, F_1$. Then AD_2, BE_2, CF_2 are concurrent at a point on $\odot(P'; 2\mathcal{R})$.

In this problem, we introduce 2 proofs.
Proof 1(by Nguyen Van Linh).

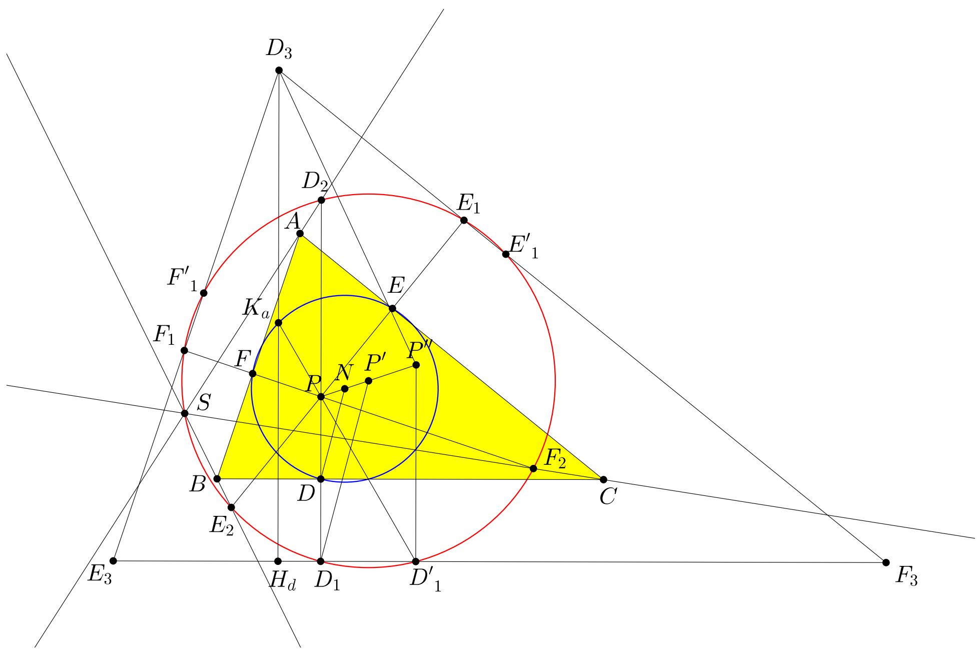

Figure 7

Let N be midpoint of PP'. It is well-known that N is circumcenter of $\triangle DEF$ so $ND = \mathcal{R}$. From Thales's theorem, $P'D_1 = 2ND = 2\mathcal{R}$. Similarly, $P'E_1 = P'F_1 = 2\mathcal{R}$ so D_1, E_1, F_1 lie on $\odot(P', 2\mathcal{R})$.

$$\begin{aligned} \mathcal{H}_P^2 : \odot(DEF) &\rightarrow \odot(D_1E_1F_1) \\ A, B, C, P' &\rightarrow D_3, E_3, F_3, P'' \end{aligned}$$

$\odot(D_1E_1F_1)$ intersects E_3F_3 at $D'_1 \neq D_1$ then $P''D'_1 \perp E_3F_3$. K_a is a point on the altitude DH_d such that $K_aD_3P''D'_1$ is a parallelogram. J is circumcenter of $\triangle D_3E_3F_3$. Then from problem 9, $\odot(H_dK_aD'_1)$ passes through orthopole S of JP'' wrt $\triangle D_3E_3F_3 \Rightarrow K_aS \perp SD'_1$. Since D'_1D_2 is diameter of $\odot(D_1E_1F_1)$, $D_2S \perp SD'_1$. Therefore, D_2, K_a, S are collinear.
From the homothecy $\mathcal{H}_P^2$, we get $\overrightarrow{PD_2} = \overrightarrow{D'_1P''} = \overrightarrow{K_aD_3}$ then $D_3K_aPD_2$ is a parallelogram. A is midpoint of $PD_3 \Rightarrow A$ is midpoint of K_aD_2, this means A, K_a, D_2 are collinear.
$\Rightarrow A, D_2, K_a, S$ are collinear. So AD_2 passes through S. Similarly, AD_2, BE_2, CF_2 are concurrent at S.
Proof 2. This second proof uses the same notations as the first proof.

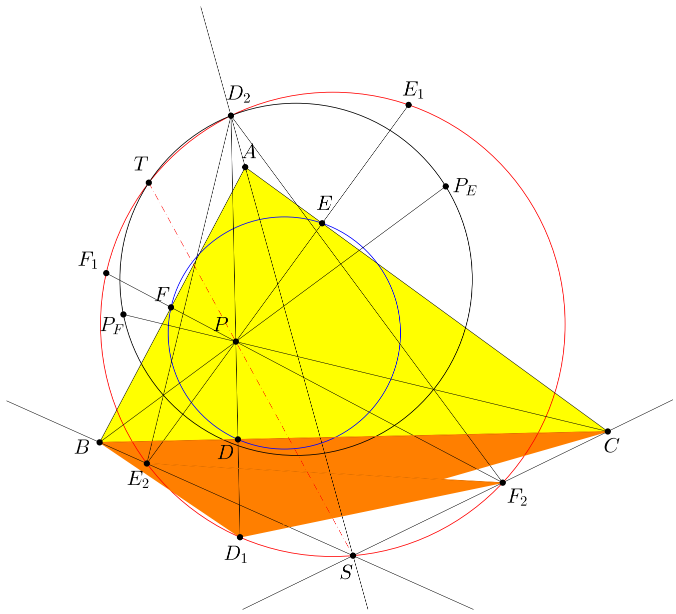

Figure 8

Since A is circumcenter of $\triangle PE_1F_1$ and E_1, F_1, E_2, F_2 are concyclic
$\Rightarrow PA \perp E_2F_2$. Similarly, $PB, PC \perp F_2D_{2,2}E_2$.
$\Rightarrow$ Two orthologic centers of $\triangle ABC$ and $\triangle D_2E_2F_2$ coincide with each other. So from Sondat's theorem, AD_2, BE_2, CF_2 are concurrent. Let S be their concurrency point.

$$(BD_1, BC) = (BC, BP) = (D_2P, D_2F_2) = (D_2D_1, D_2F_2) = (E_2D_1, E_2F_2)$$

$$(CD_1, CB) = (CB, CD) = (D_2E_2, D_2P) = (D_2E_2, D_2D_1) = (F_2E_2, F_2D_1)$$

Hence $\triangle D_1BC$ and $\triangle D_1E_2F_2$ are directly similar.
$\Rightarrow (SE_2, SF_2) = (BE_2, CF_2) = (D_1E_2, D_1F_2) \Rightarrow S$ lies on $\odot(D_2E_2F_2)$.
Remark. Let P_D, P_E, P_F be reflections of P in E_2F_2, F_2D_2, D_2E_2.

The inversion $\mathbf{I}(P, \overline{PD}.\overline{PD_2}) : B, C, D_1, S \rightarrow P_E, P_F, D_2, T$
Since B, C, D_1, S are concyclic, $\odot(D_2P_EP_F)$ passes through T. Similarly, $\odot(E_2P_FP_D), \odot(F_2P_DP_E)$ pass through T. It is well-known that $\odot(D_2P_EP_F)$, $\odot(E_2P_FP_D)$, $\odot(F_2P_DP_E)$ are concurrent at anti-Steiner point of HP wrt $\triangle D_2E_2F_2$ where H is orthocenter of $\triangle D_2E_2F_2$.
Hence T, P, anti-Steiner point of HP wrt $\triangle D_2E_2F_2$ are collinear.

1.2 Cevian circle and Nine-point circle

In this part, we use notation as follow: $\triangle ABC$ and a point P. PA, PB, PC intersects BC, CA, AB at D, E, F; X, Y, Z are the intersections of (BC, EF), (CA, FD), (AB, DE); A', B', C' are midpoint of BC, CA, AB.

Proposition 10. *Let M be Miquel point of the complete quadrilateral $EF, FD, DE, \overline{XYZ}$. Then M is an intersection of $\odot(DEF)$ and $\odot(A'B'C')$.*

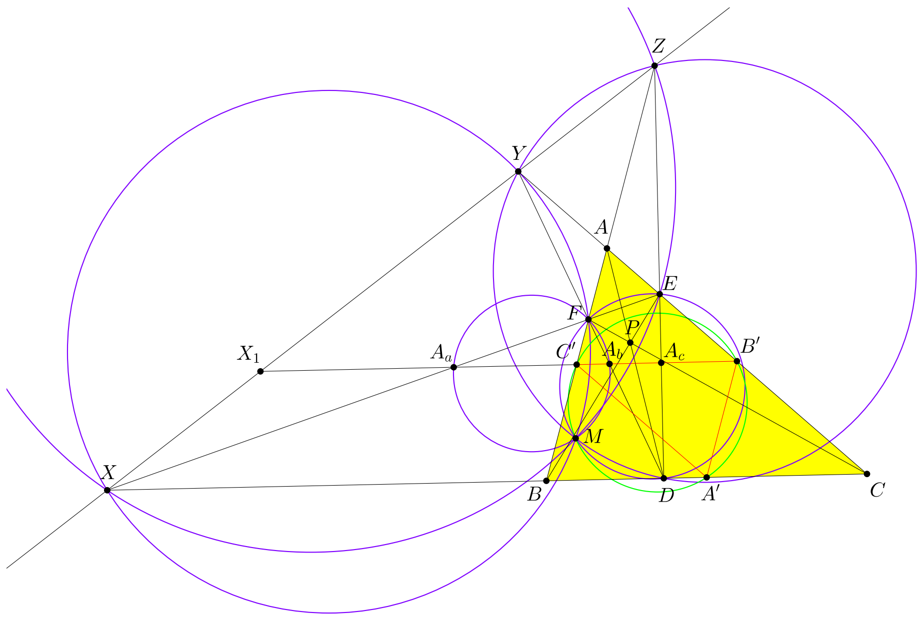

Figure 9

Proof. $B'C'$ intersects $DE, DF, EF, \overline{X, Y, Z}$ at A_c, A_b, A_a, X_1, respectively.
We will show that $\odot(FA_bA_a)$ passes through M by ratio of power. First, from Thales's theorem:

$$\frac{\overline{A_aE}}{\overline{A_aX}} = \frac{\overline{B'E}}{\overline{B'C}}$$

Since $(EYAC) = -1$, B' is midpoint of CA then according to Newton's formula: $\overline{B'C}^2 = \overline{B'E}.\overline{E'Y}$

$$\Rightarrow \frac{\overline{B'E}}{\overline{B'C}} = \frac{\overline{B'C}}{\overline{B'Y}} = \frac{\overline{A_bD}}{\overline{A_bY}}$$

$$\frac{\mathcal{P}_{A_a/\odot(DEF)}}{\mathcal{P}_{A_a/\odot(FXY)}} = \frac{\overline{A_aE}.\overline{A_aF}}{\overline{A_aF}.\overline{A_aX}} = \frac{\overline{A_aE}}{\overline{A_aX}}$$

$$\frac{\mathcal{P}_{A_b/\odot(DEF)}}{\mathcal{P}_{A_b/\odot(FXY)}} = \frac{\overline{A_bD}.\overline{A_bF}}{\overline{A_bF}.\overline{A_bY}} = \frac{\overline{A_bD}}{\overline{A_bY}}$$

$$\Rightarrow \frac{\mathcal{P}_{A_a/\odot(DEF)}}{\mathcal{P}_{A_a/\odot(FXY)}} = \frac{\mathcal{P}_{A_b/\odot(DEF)}}{\mathcal{P}_{A_b/\odot(FXY)}}$$

Hence $\odot(DEF), \odot(FXY), \odot(FA_aA_b)$ are coaxial, then $\odot(FA_aA_b)$ passes through M.
Similarly, also by ratio of power, associate with Miquel's theorem, circumcircle of triangles formed by 3 in set of 6 lines $B'C', C'A', A'B', EF, FD, DE$ passes through M. Therefore, $\odot(A'B'C')$ passes through M.

Corollary 11. (Telv Cohl) *Steiner line of M wrt $\triangle DEF$ passes through circumcenter O of $\triangle ABC$.*

Proof. Since circumcircle of triangles formed by 3 in set of 6 lines $B'C', C'A', A'B', EF, FD, DE$ passes through M then Steiner line of M wrt $\triangle DEF$ passes through orthocenters of triangles formed by 3 in those lines, which contain O - orthcenter of $\triangle A'B'C'$.
Remark. Let H_P be orthocenter of $\triangle DEF$ then M is orthopole of OH_P wrt $\triangle ABC$.

Proposition 12. *AH_a is the altitude of $\triangle ABC$.*
$XX_1A_aH_a, DH_aA_bA_c$ are isoceles trapezoid.

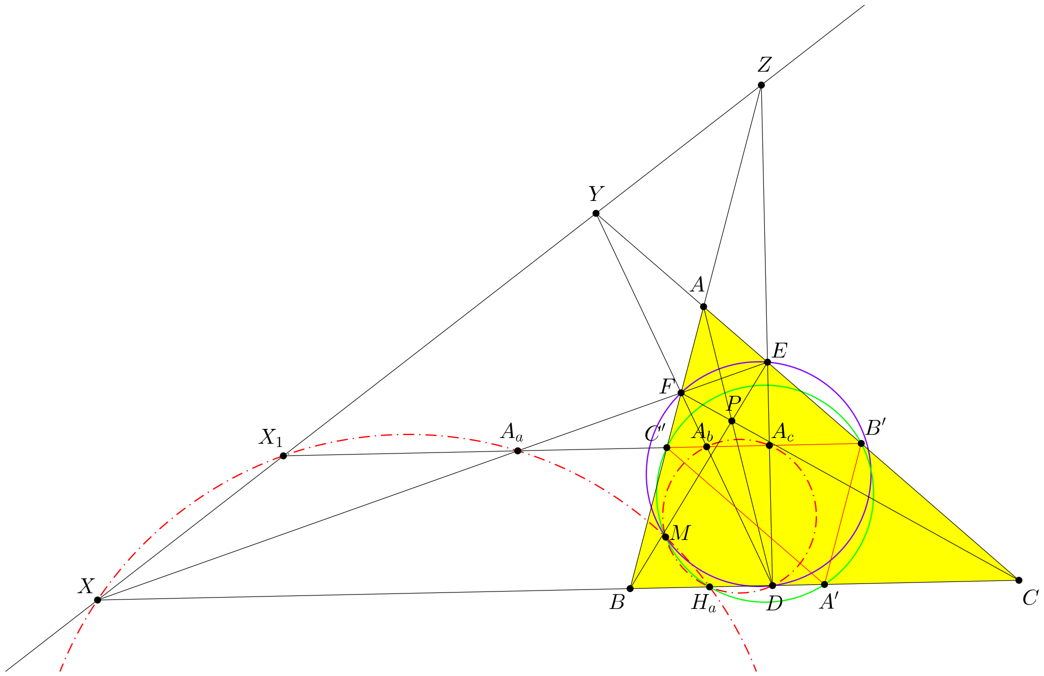

Figure 10

Proof. Since $B'C' \parallel BC$ then we only have to show that $\odot(XX_1A_a)$ and $\odot(DA_bA_c)$ pass through H_a. From the proof of problem 11, $\odot(XX_1A_a), \odot(DA_BA_c)$ pass through M.

$$\frac{\mathcal{P}_{B'/\odot(XX_1A_a)}}{\mathcal{P}_{B'/\odot(DA_bA_c)}} = \frac{\overline{B'X_1}.\overline{B'A_a}}{\overline{B'A_b}.\overline{B'A_c}} = \frac{\overline{B'X_1}}{\overline{B'A_b}} \cdot \frac{\overline{B'A_a}}{\overline{B'A_c}} = \frac{\overline{CX}}{\overline{CD}} \cdot \frac{\overline{CX}}{\overline{CD}} = \frac{CX^2}{CD^2}$$

$$\frac{\mathcal{P}_{C'/\odot(XX_1A_a)}}{\mathcal{P}_{C'/\odot(DA_bA_c)}} = \frac{\overline{C'X_1}.\overline{C'A_a}}{\overline{C'A_b}.\overline{C'A_c}} = \frac{\overline{C'X_1}}{\overline{C'A_b}} \cdot \frac{\overline{C'A_a}}{\overline{C'A_c}} = \frac{\overline{BX}}{\overline{BD}} \cdot \frac{\overline{BX}}{\overline{BD}} = \frac{BX^2}{BD^2}$$

Since $(BCDX) = -1$

$$\Rightarrow \frac{\mathcal{P}_{B'/\odot(XX_1A_a)}}{\mathcal{P}_{B'/\odot(DA_bA_c)}} = \frac{\mathcal{P}_{C'/\odot(XX_1A_a)}}{\mathcal{P}_{C'/\odot(DA_bA_c)}}$$

Hence $\odot(A'B'C'), \odot(DA_BA_c), \odot(XX_1A_a)$ are coaxial.

$$\frac{\overline{A'X}}{\overline{A'D}} = \frac{\mathcal{P}_{B'/\odot(XX_1A_a)}}{\mathcal{P}_{B'/\odot(DA_bA_c)}}$$

This is true, because:

$$\frac{\overline{A'X}}{\overline{A'D}} = \frac{\overline{DX}.\overline{A'D}}{\overline{DA'}.\overline{DX}} = -\frac{\overline{XB}.\overline{XC}}{\overline{DB}.\overline{DC}} = \frac{BX^2}{BD^2} = \frac{\mathcal{P}_{B'/\odot(XX_1A_a)}}{\mathcal{P}_{B'/\odot(DA_bA_c)}}$$

This means the second common point of $\odot(A'B'C'), \odot(DA_BA_c), \odot(XX_1A_a)$ lies on BC. But $\odot(A'B'C')$ intersects BC at A', H_a then H_a lies on $\odot(XX_1A_a), \odot(DA_bA_c)$.

Proposition 13. *(Similar to Fontene's first theorem) $\odot(DEF)$ intersects BC, CA, AB at $D', E', F' \neq D, E, F$.*
$D'A_a, E'B_b, F'C_c$ pass through M.

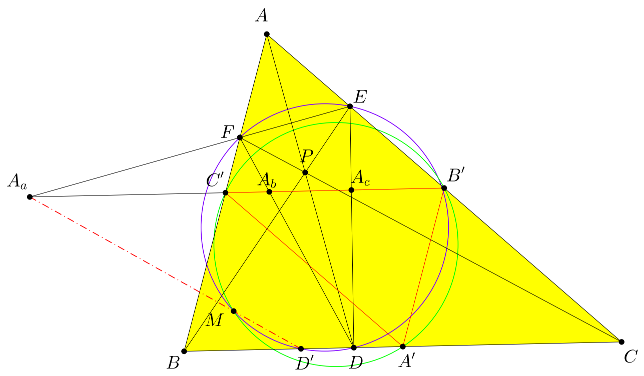

Figure 11

Proof. From symmetry, it is suffice to prove that D', A_a, M are collinear.

$$(MA_a, MD') = (MA_a, MF) + (MF, MD') = (A_bA_a, A_bF) + (DF, DD') = (A_bA_a, DD') = 0$$

Proposition 14. *L_a, L_b, L_c lie on the altitudes of $\triangle ABC$ such that $D'L_aE'L_b, F'L_c$ are pendicular to EF, FD, DE. $\odot(D'L_a), \odot(E'L_b), \odot(F'L_c)$ pass through M.*

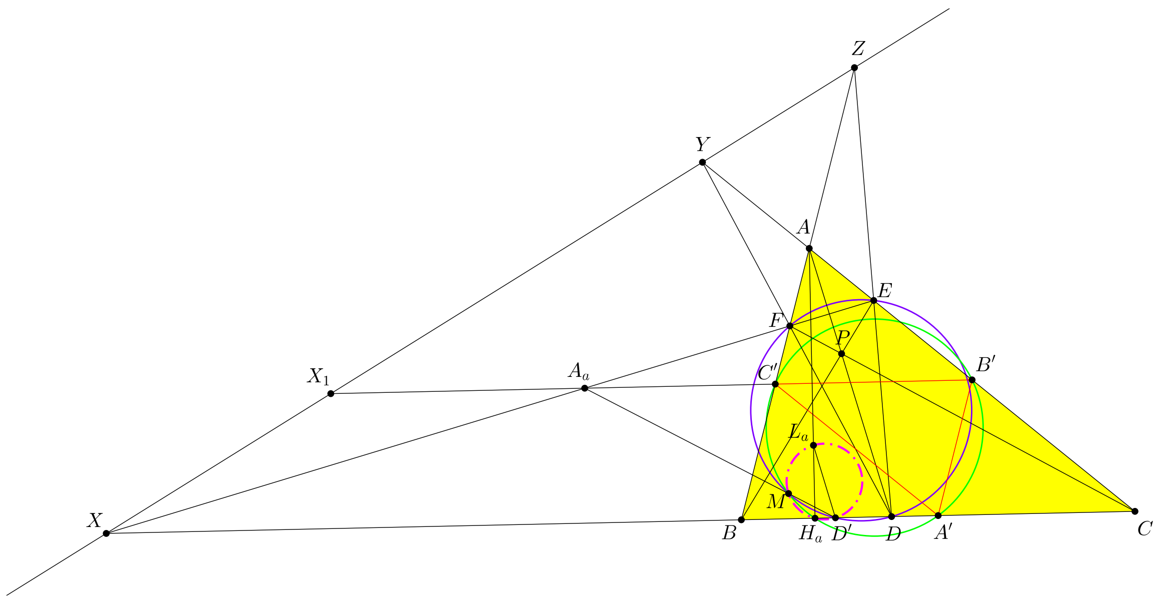

Figure 12

Proof. Like the above proposition, this can be solved easily by angle chasing.

$$(MH_a, MD') = (MH_a, MA_a) = (XH_a, XA_a) = (BC, EF) = (L_aH_a, L_aD')$$

$\Rightarrow M, H_a, D', L_a$ are concyclic.

Proposition 15. *$\odot(DYZ), \odot(EZX), \odot(FXY)$ intersects $\odot(A'B'C')$ at $D_1, E_1, F_1 \neq M$. DD_1, EE_1, FF_1 are concurrent at a point on $\odot(A'B'C')$.*

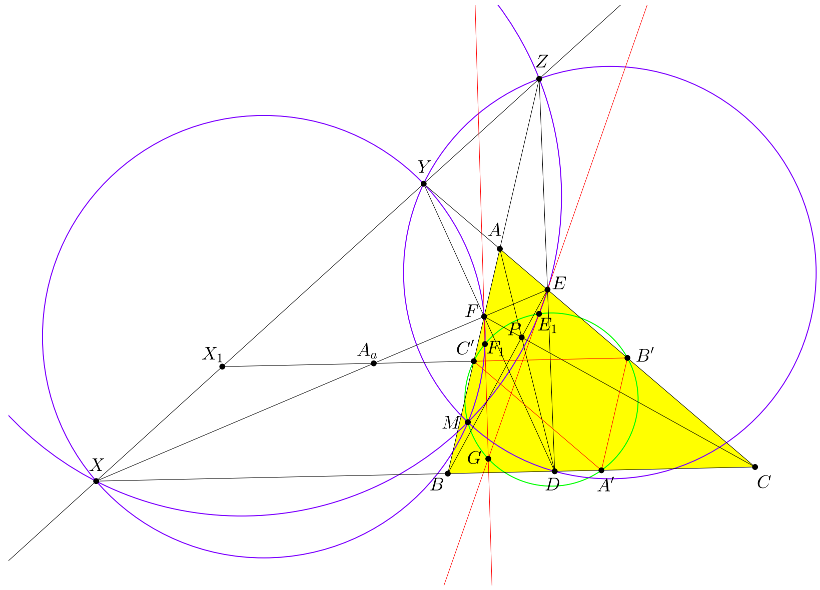

Figure 13

Proof.

$$(EE_1, FF_1) = (EE_1, EX) + (FX, FF_1) = (ME_1, MX) + (MX, MF_1) = (ME_1, MF_1)$$

Therefore, intersection of EE_1, FF_1 lies on $\odot(A'B'C')$. Similarly, DD_1, EE_1, FF_1 are concurrent at a point on $\odot(A'B'C')$.

Proposition 16. *(Similar to Fontene's third theorem) P^* is the cyclocevian conjugate of P wrt $\triangle ABC$. H_P, H_P^* are orthocenters of cevian triangles of P, P^* wrt $\triangle ABC$. Then cevian circle of P wrt $\triangle ABC$ is tangent to nine-point circle of $\triangle ABC$ if and only if H_P, H_P^*, O are collinear.*

Proof. From corollary 16, the common points of $\odot(A'B'C')$, $\odot(DEF)$ are anti-Steiner points of OH_P, OH_P^* wrt $\triangle A'B'C'$ then $\odot(A'B'C')$ is tangent to $\odot(DEF)$ if and only if O, H_P, H_P^* are collinear.
Now we will find locus of P such that its cevian circle is tangent to nine-point circle.
Let D_2, E_2, F_2 be midpoints of DX, EY, FZ. Then the line that passes through D_2, E_2, F_2 is Gauss line of $EF, FD, DE, \overline{X, Y, Z}$.
In barycentric coordinates, let $P(\alpha, \beta, \gamma)$.

$$\frac{\overline{D_2B}}{\overline{D_2C}} = -\frac{\overline{BD_2}.\overline{BC}}{\overline{CD_2}.\overline{CB}} = -\frac{\overline{BDBX}}{\overline{CD}.\overline{CX}} = \frac{DB^2}{DC^2} = \frac{\gamma^2}{\beta^2}$$

Similarly

$$\frac{\overline{E_2C}}{\overline{E_2A}} = \frac{\alpha^2}{\gamma^2} \qquad \frac{\overline{F_2A}}{\overline{F_2B}} = \frac{\beta^2}{\alpha^2}$$

Then the line $\overline{D_2, E_2, F_2}$ has equation:

$$\frac{x}{\alpha^2} + \frac{y}{\beta^2} + \frac{z}{\gamma^2} = 0$$

The cyclocevian conjugate of P has barycentric coordinate:

$$P^*\Big(\frac{1}{\frac{b^2\gamma\alpha}{\gamma+\alpha} + \frac{c^2\alpha\beta}{\alpha+\beta} - \frac{a^2\beta\gamma}{\beta+\gamma}}, \frac{1}{\frac{c^2\alpha\beta}{\alpha+\beta} + \frac{a^2\beta\gamma}{\beta+\gamma} - \frac{b^2\gamma\alpha}{\gamma+\alpha}}, \frac{1}{\frac{a^2\beta\gamma}{\beta+\gamma} + \frac{b^2\gamma\alpha}{\gamma+\alpha} - \frac{c^2\alpha\beta}{\alpha+\beta}}\Big)$$

AP^*, BP^*, CP^* intersect BC, CA, AB at D', E', F', $E'F', F'D', D'E'$ intersect BC, CA, AB at X', Y', Z'.

D_3, E_3, F_3 are midpoints of $D'X', E'Y', F'Z'$ then similarly, the line $\overline{D_3, E_3, F_3}$ has equation:

$$x(\frac{b^2\gamma\alpha}{\gamma+\alpha} + \frac{c^2\alpha\beta}{\alpha+\beta} - \frac{a^2\beta\gamma}{\beta+\gamma})^2 + y(\frac{c^2\alpha\beta}{\alpha+\beta} + \frac{a^2\beta\gamma}{\beta+\gamma} - \frac{b^2\gamma\alpha}{\gamma+\alpha})^2 + z(\frac{a^2\beta\gamma}{\beta+\gamma} + \frac{b^2\gamma\alpha}{\gamma+\alpha} - \frac{c^2\alpha\beta}{\alpha+\beta})^2 = 0$$

Since Gauss line and Steiner line of a complete quadrilateral are pendicular so O, H_P, H_P^* are collinear if and only if $\overline{D_2, E_2, F_2}$ and $\overline{D_3, E_3, F_3}$ are parallel:

$$\begin{vmatrix} \frac{(\frac{c^2\alpha\beta}{\alpha+\beta}+\frac{a^2\beta\gamma}{\beta+\gamma}-\frac{b^2\gamma\alpha}{\gamma+\alpha})^2}{\frac{1}{\beta^2}} & \frac{(\frac{a^2\beta\gamma}{\beta+\gamma}+\frac{b^2\gamma\alpha}{\gamma+\alpha}-\frac{c^2\alpha\beta}{\alpha+\beta})^2}{\frac{1}{\gamma^2}} \end{vmatrix}$$

$$+\begin{vmatrix} \frac{(\frac{a^2\beta\gamma}{\beta+\gamma}+\frac{b^2\gamma\alpha}{\gamma+\alpha}-\frac{c^2\alpha\beta}{\alpha+\beta})^2}{\frac{1}{\gamma^2}} & \frac{(\frac{b^2\gamma\alpha}{\gamma+\alpha}+\frac{c^2\alpha\beta}{\alpha+\beta}-\frac{a^2\beta\gamma}{\beta+\gamma})^2}{\frac{1}{\alpha^2}} \end{vmatrix}$$

$$+\begin{vmatrix} \frac{(\frac{b^2\gamma\alpha}{\gamma+\alpha}+\frac{c^2\alpha\beta}{\alpha+\beta}-\frac{a^2\beta\gamma}{\beta+\gamma})^2}{\frac{1}{\alpha^2}} & \frac{(\frac{c^2\alpha\beta}{\alpha+\beta}+\frac{a^2\beta\gamma}{\beta+\gamma}-\frac{b^2\gamma\alpha}{\gamma+\alpha})^2}{\frac{1}{\beta^2}} \end{vmatrix}=0$$

$$\Leftrightarrow (\frac{b^2\gamma\alpha}{\gamma+\alpha}+\frac{c^2\alpha\beta}{\alpha+\beta}-\frac{a^2\beta\gamma}{\beta+\gamma})^2(\frac{1}{\beta^2}-\frac{1}{\gamma^2})+(\frac{c^2\alpha\beta}{\alpha+\beta}+\frac{a^2\beta\gamma}{\beta+\gamma}-\frac{b^2\gamma\alpha}{\gamma+\alpha})^2(\frac{1}{\gamma^2}-\frac{1}{\alpha^2})+(\frac{a^2\beta\gamma}{\beta+\gamma}+\frac{b^2\gamma\alpha}{\gamma+\alpha}-\frac{c^2\alpha\beta}{\alpha+\beta})^2(\frac{1}{\alpha^2}-\frac{1}{\beta^2})=0$$

$$\Leftrightarrow \frac{b^2c^2\alpha^2\beta\gamma}{(\gamma+\alpha)(\alpha+\beta)}(\frac{1}{\beta^2}-\frac{1}{\gamma^2})+\frac{c^2a^2\beta^2\gamma\alpha}{(\alpha+\gamma)(\beta+\gamma)}(\frac{1}{\gamma^2}-\frac{1}{\alpha^2})+\frac{a^2b^2\gamma^2\alpha\beta}{(\gamma+\alpha)(\beta+\gamma)}(\frac{1}{\alpha^2}-\frac{1}{\beta^2})=0$$

$$\Leftrightarrow b^2c^2\alpha^3(\beta+\gamma)^2(\beta-\gamma)+c^2a^2\beta^3(\gamma+\alpha)^2(\gamma-\alpha)+a^2b^2\gamma^3(\alpha+\beta)^2(\alpha-\beta)=0$$

Hence the locus is a sextic:

$$b^2c^2x^3(y+z)^2(y-z)+c^2a^2y^3(z+x)^2(z-x)+a^2b^2z^3(x+y)^2(x-y)=0$$

This sextic is the isotomic of anticomplement of Grebe cubic(see [14]):

$$a^2x(c^2y^2-b^2z^2)+b^2y(a^2z^2-c^2x^2)+c^2z(b^2x^2-a^2y^2)=0$$

1.3 Intersection of pedal circle and cevian circle

Proposition 17. (Luiz Gonzalez, Telv Cohl) *Given $\triangle ABC$ and a point P. $\triangle DEF$ and $\triangle XYZ$ are pedal triangle, cevian triangle of P wrt $\triangle ABC$, respectively. H_P is orthocenter of $\triangle XYZ$. Then anti-Steiner point of PH_P wrt $\triangle XYZ$ is a common point of $\odot(DEF)$ and $\odot(XYZ)$.*

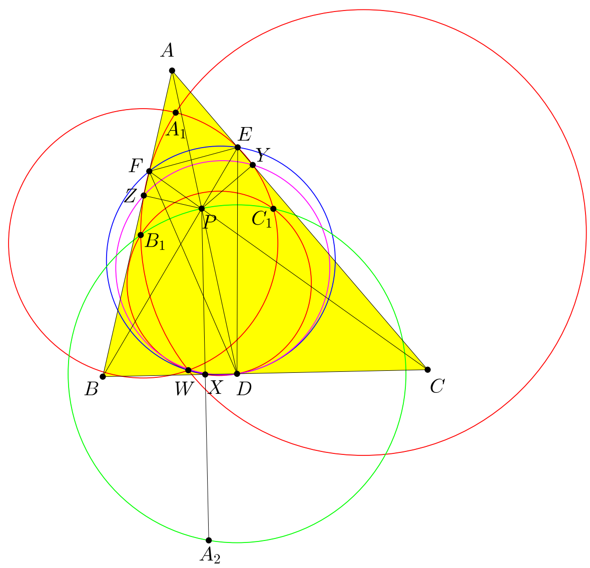

Figure 14

Proof. A_1, B_1, C_1 are reflections of P in EF, FD, DE; A_2 is the reflection of P in BC.
It is obvious that $DA_2 = DP = DB_1 = DC_1$ so A_2, B_1, C_1, P are concyclic.

$$(DE, DF, DX, DP) = -1 \Longrightarrow (PC_1, PB_1, PA_2, \perp PD) = -1$$

Therefore $PB_1A_2C_1$ is a harmonic quadrilateral so D, X, B_1, C_1 are concyclic.
It is well-known that $\odot(DB_1C_1), \odot(EC_1A_1), \odot(FA_1B_1)$ are concurrent at a point W - anti-Steiner point of PH_P wrt $\triangle DEF$.

$$\begin{aligned}(WE, WF) &= (WE, WA_1) + (WA_1, WF)\\ &= (YE, YA_1) + (ZA_1, ZF)\\ &= (AC, YA_1) + (ZA_1, AB)\\ &= (AC, AB) + (A_1Z, A_1Y)\\ &= (AC, AB) + (PB, PC)\\ &= (CA, CP) + (BP, BA)\\ &= (DE, DP) + (DP, DF)\\ &= (DE.DF)\end{aligned}$$

So W lies on $\odot(DEF)$ and $\odot(XYZ)$.

2 Poncelet point

2.1 A proof of Poncelet point's problem

Theorem 18. (Randy Hutson). *$\triangle ABC$ and a point P. Prove that pedal circle, cevian circle of P wrt $\triangle ABC$ and nine-point circle of $\triangle ABC$ are concurrent.*

Proof(based on Tran Minh Ngoc's). A', B', C' are midpoints of BC, CA, AB.
$\triangle DEF$ and $\triangle XYZ$ are pedal triangle and cevian triangle of P wrt $\triangle ABC$.
By using the inversion that has center P, we get new problem:
$\triangle ABC$ and a point P. $\triangle DEF$ is the antipedal triangle of P wrt $\triangle ABC$.
A', B', C' are the points on $\odot(PBC), \odot(PCA), \odot(PAB)$ such that $PBA'C$, $PCB'A$, $PAC'B$ are harmonic quadrilateral.
AP, BP, CP intersect $\odot(PBC), \odot(PCA), \odot(PAB)$ at $D, E, F \neq P$. Prove that $\odot(DEF)$, $\odot(XYZ)$, $\odot(A'B'C')$ are concurrent.
Let D', E', F' be the reflections of P in EF, FD, DE. A_1, B_1, C_1 are midpoints of PA', PB', PC'. O_a, O_b, O_c are circumcenters of $\triangle PBC$, $\triangle PCA$, $\triangle PAB$.

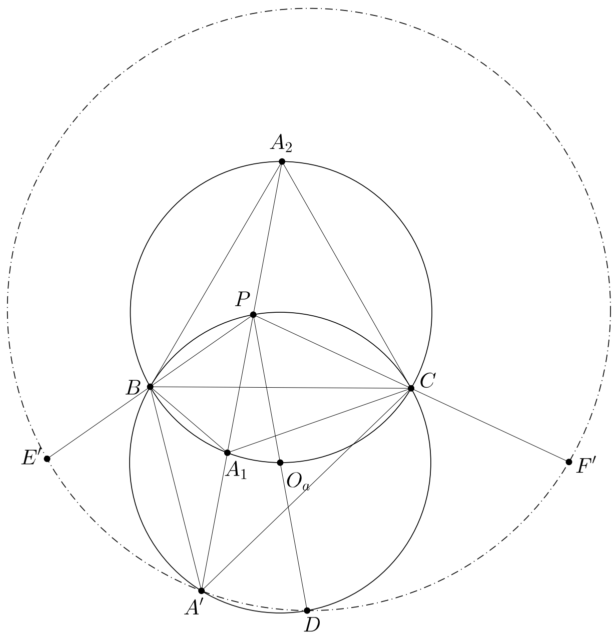

Figure 15

$PBA'C$ is a harmonic quadrilateral, then tangent lines of $\odot(PBC)$ at B, C intersect each other at a point A_2 on PA'.
$\Rightarrow B, C, O_a, A_1$ lie on a circle that has diameter O_aA_2. From the homothety $\mathcal{H}_{(P,2)}$, E', F', D, A' are concyclic.

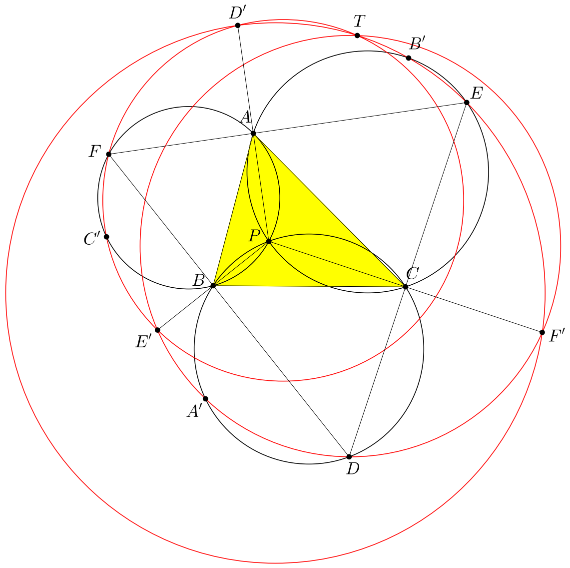

Figure 16

Since D', E', F' are reflections of P in EF, FD, DE, then $\odot(DE'F'), \odot(EF'D'), \odot(FD'E')$ are concurrent at antisteiner point T of P wrt $\triangle DEF$.
We show that $\odot(A'B'C'), \odot(XYZ)$ pass through T.

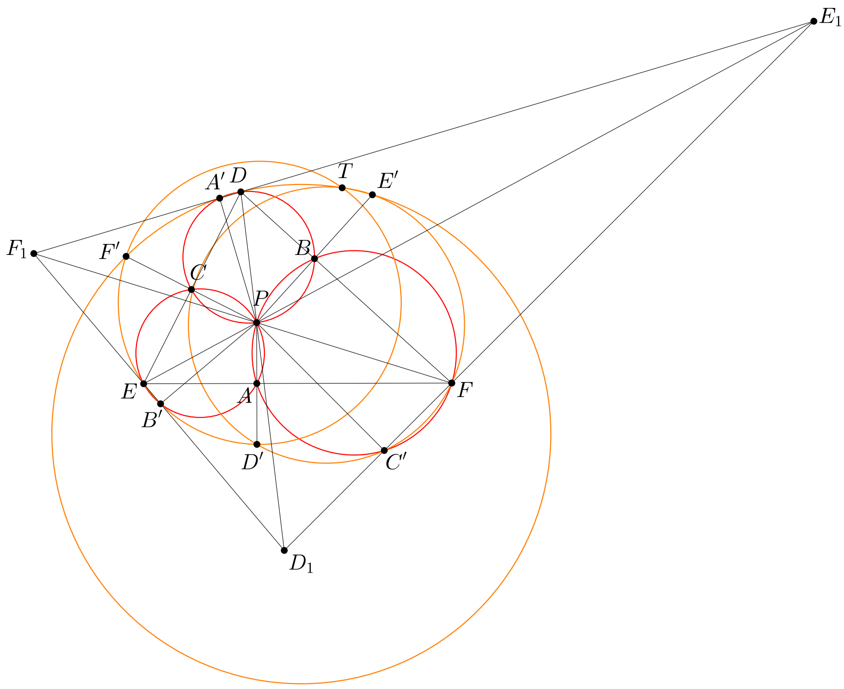

Figure 17

$B'CPA, C'APB$ are harmonic quadrilateral, then:

$$E(B'PCA) = F(C'PBA) = -1 \Leftrightarrow E(D_1PDF) = F(D_1PDE)$$

Hence D, P, D_1 are collinear.

$$\begin{aligned}(A'B', A'C') &= (A'B', A'P) + (A'P, A'C') \\ &= (F_1D_1, F_1P) + (E_1P, E_1D_1) \\ &= (D_1F_1, D_1E_1) + (PE_1, PF_1) \\ &= (D_1F_1, D_1E_1) + (PE, PF)\end{aligned}$$

$$\begin{aligned}(TB', TC') &= (TB', TD') + (TD', TC') \\ &= (EB', ED') + (FD', FC') \\ &= (D_1F_1, ED') + (FD', D_1E_1) \\ &= (D_1F_1, D_1E_1) + (D'F, D'E) \\ &= (D_1F_1, D_1E_1) + (PE, PF)\end{aligned}$$

$\Rightarrow (A'B', A'C') = (TB', TC')$, then A', B', C', T are concyclic.

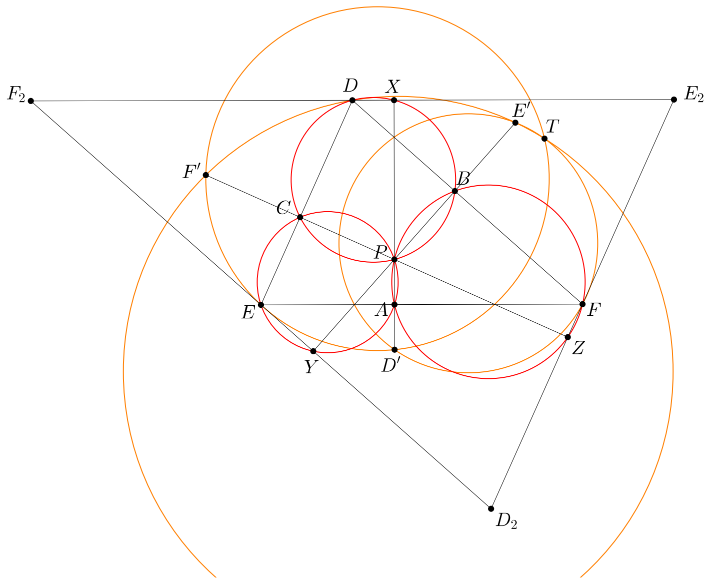

Figure 18

Let $\triangle D_2E_2F_2$ be the anticomplementary triangle of $\triangle DEF$. It is obvious that X, Y, Z are orthogonal projections of P on E_2F_2, F_2D_2, D_2E_2. Let I be circumcenter of $\triangle D_2E_2F_2$, then according to Fontene's theorem, $\odot(XYZ)$ passes through orthopole of IP wrt $\triangle D_2E_2F_2$. But, orthopole of IP wrt $\triangle D_2E_2F_2$ is also the anti-Steiner point of IP wrt $\triangle DEF$. This implies that $\odot(XYZ)$ passes through T.
Hence $\odot(A'B'C'), \odot(DEF), \odot(XYZ)$ are concurrent.

Note that, in the inverse problem, the concurrency of $\odot(DE'F')$, $\odot(EF'D')$, $\odot(FD'E')$, $\odot(DEF)$, $\odot(A'B'C')$ was found by Nguyen Van Linh. That is the main idea of the proof by Tran Minh Ngoc.
Another proof by Michael Rolinek and Le Anh Dung was published to Forum Geometricorum.

2.2 Problems

Proposition 19. *Nine-point circles of $\triangle ABC$, $\triangle PBC$, $\triangle PCA$, $\triangle PAB$, pedal circles of P, A, B, C wrt $\triangle ABC$, $\triangle PBC$, $\triangle PCA$, $\triangle PAB$, respectively and cevian circle of P wrt $\triangle ABC$ are concurrent.*

The above proof implies this property.

Proposition 20. *P', P^* are isogonal conjugate and cyclocevian conjugate of P wrt $\triangle ABC$.*
i) Poncelet point of A, B, C, P' is T.
ii) Poncelet point of A, B, C, P^* is M.

Proposition 21. (Luiz Gonzalez and Cosmin Pohoata) *Given $\triangle ABC$ and a point P. $\triangle DEF$ is pedal triangle of P wrt $\triangle ABC$. PD, PE, PF intersect $\odot(DEF)$ at X, Y, Z. A_1, B_1, C_1 are midpoints of PA, PB, PC.*
Then XA_1, YB_1, ZC_1 pass through Poncelet point of A, B, C, P.

This is a homothetic corollary of proposition 9.

Proposition 22. *Poncelet point is center of rectangular hyperbola that passes through A, B, C, P, H where H is orthocenter of $\triangle ABC$.*

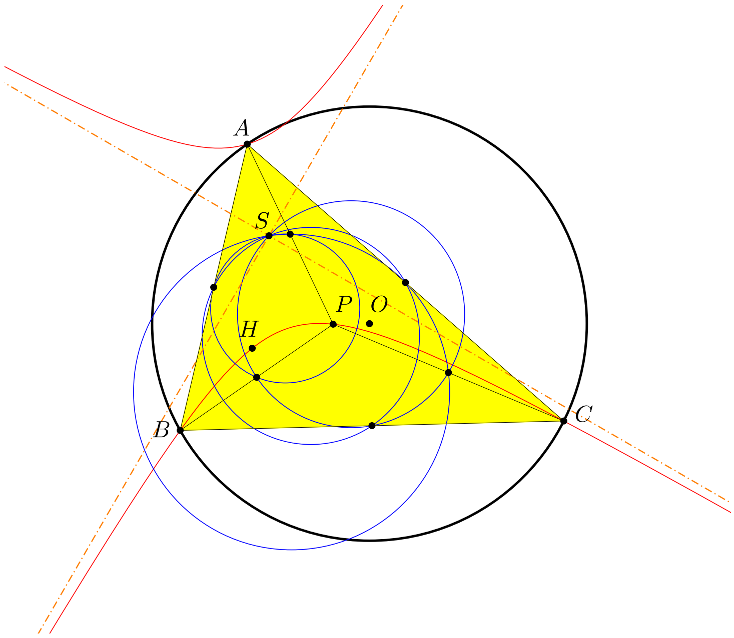

Figure 19

Proof. From Feuerbach's conic theorem, $\triangle ABC$ is inscribed in rectangular hyperbola $\mathcal{H}$ then nine-point circle of $\triangle ABC$ passes through center of $\mathcal{H}$, then nine-point circle of $\triangle ABC$, $\triangle PBC$, $\triangle PCA$, $\triangle PAB$ are concurrent at center of the rectangular hyperbola passes through A, B, C, H, P.
To end this paper, I present a hard and difficult problem.

Proposition 23. (Francesco Sala) *P, Q are isogonal conjugate wrt $\triangle ABC$. X is Poncelet point of A, B, C, P.*
$\triangle Q_AQ_BQ_C$ is pedal triangle of Q wrt $\triangle ABC$. PX intersects $\odot(Q_AQ_BQ_C)$ at Y.
Then Steiner line of Y wrt $\triangle Q_AQ_BQ_C$ is parallel to orthotransversal of P wrt $\triangle ABC$.

Proof. An important step in the solution of this problem is using the following lemma.

Lemma 24. (Telv Cohl) *Let H_Q be orthocenter of $\triangle Q_AQ_BQ_C$. Then $H_QQ \perp$ orthotransversal of P wrt $\triangle ABC$*

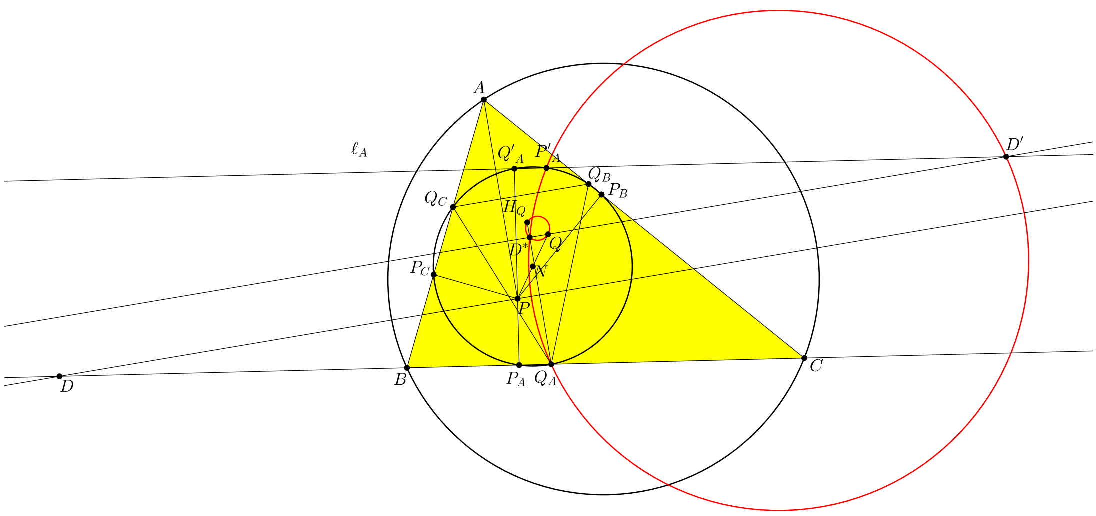

Figure 20

Telv Cohl also proved this lemma.
Let the orthotransversal intersects BC, CA, AB at D, E, F.
Let $P'_A, P'_B, P'_C, Q'_A, Q'_B, Q'_C$ be the antipode of $P_A, P_B, P_C, Q_A, Q_B, Q_C$ in $\odot(P_AP_BP_C)$.
Let ℓ_A be the line through Q'_A and parallel to BC (define ℓ_B and ℓ_C similarly).

Let D' be a point on ℓ_A such that $QD' \parallel Q_BQ_C$. PD' intersects H_QQ_A at D^*
$\Rightarrow D', D^*, Q_A, P'_A$ are concyclic, then $\overline{QD'}.\overline{QD^*} = \overline{QQ_A}.\overline{QP'_A} = \mathcal{P}_{Q/\odot(Q_AQ_BQ_C)}$. Similarly, we have:

$$\overline{QE'}.\overline{QE^*} = \overline{QF'}.\overline{QF^*} = \mathcal{P}_{Q/\odot(Q_AQ_BQ_C)}$$

Since D^*, E^*, F^* lie on $\odot(H_PP)$ so by inversion, D', E', F' are collinear on the line that pendicular to H_QQ.
Furthermore, D' is reflection of D in center N of $\triangle P_AP_BP_C$ so $QH_Q \perp$ orthotransversal of P wrt $\triangle ABC$.
Back to the main problem
A_1, B_1, C_1 are midpoints of PA, PB, PC then by proposition 21, $A_1Q'_A, B_1Q'_B, C_1Q'_C$ pass through X.
$\triangle P_AP_BP_C$ is circumcevian triangle of P wrt $\triangle Q'_AQ'_BQ'_C$, then according to remark in proposition 9, Y is anti-Steiner point of PH'_Q wrt $\triangle Q'_AQ'_BQ'_C$ where H'_Q is orthocenter of $\triangle Q'_AQ'_BQ'_C$.
Since $\triangle Q_AQ_BQ_C$ is reflection of $\triangle Q'_AQ'_BQ'_C$ in center of $\odot(P_AP_BP_C)$ then Y is reflection of anti-Steiner point of H_QQ wrt $\triangle Q_AQ_BQ_C$ in center of $\odot(P_AP_BP_C)$. Hence Steiner line of Y wrt $\triangle Q_AQ_BQ_C$ is pendicular to H_QQ.
In the last words of this paper, I want to say, around these circles, these intersections still have so many interesting problems. I tried my best to collect the problems that in my capability. I hope that we can return to this topic in other time.

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